QMC integration for lognormal-parametric, elliptic PDEs: local supports imply product weights.

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QMC integration for lognormal-parametric, elliptic PDEs: local supports imply product weights. *

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Abstract

We analyze convergence rates of quasi-Monte Carlo (QMC) quadratures for countably-parametric solutions of linear, elliptic partial differential equations (PDE) in divergence form with log-Gaussian diffusion coefficient, based on the error bounds in [James A. Nichols and Frances Y. Kuo: Fast CBC construction of randomly shifted lattice rules achieving $\mathcal{O}(N^{-1+\delta})$ convergence for unbounded integrands over $\mathbb{R}^s$ in weighted spaces with POD weights. J. Complexity, 30(4):444-468, 2014].

We prove, for representations of the Gaussian random field PDE input with locally supported basis functions, and for continuous, piecewise polynomial Finite Element discretizations in the physical domain error bounds in weighted spaces with *product weights* that exploit localization of supports. The convergence rate $\mathcal{O}(N^{-1+\delta})$ (independent of the parameter space dimension s) is achieved under weak summability conditions on the expansion coefficients.

1 Introduction

A particular quasi-Monte Carlo (QMC for short) quadrature for the approximation of the mean field of (output functions of) the solution of lognormal diffusion problems is analyzed. The lognormal diffusion problem under consideration is an elliptic partial differential equation (PDE for short) with lognormal stochastic diffusion coefficient a and with deterministic right hand side f. For a bounded Lipschitz domain $D \subset \mathbb{R}^d$, we thus consider

$$-\nabla \cdot (a\nabla u) = f \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{1}$$

Let $\Omega := \mathbb{R}^{\mathbb{N}}$ and define a Gaussian product measure on Ω by

$$\mu(\mathrm{d}\boldsymbol{y}) := \bigotimes_{j \geq 1} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_j^2}{2}} \mathrm{d}y_j, \quad \boldsymbol{y} \in \Omega.$$

The triplet $(\Omega, \bigotimes_{j\geq1} \mathcal{B}(\mathbb{R}), \mu)$ is a probability space, cp. for example [2, Example 2.3.5]. We suppose that the Gaussian random field $Z = \log(a) : \Omega \to L^\infty(D)$ is (formally) represented in the following way

$$Z := \sum_{j \geq 1} y_j \psi_j, \tag{2}$$

where $(\psi_j)_{j\geq1}$ is a function system of real-valued, bounded, and measurable functions. In particular, with respect to μ the sequence $\boldsymbol{y} = (y_j)_{j\geq1}$ has independent and identically distributed

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(i.i.d. for short) components and for every $j \geq 1$, y_j is standard normally distributed. That is to say, $y_j \sim \mathcal{N}(0,1)$, i.i.d. for $j \in \mathbb{N}$. The lognormal coefficient a in (1) is formally given by

$$a := \exp(Z) . \tag{3}$$

For a Banach space B and a strongly measurable mapping $F : \Omega \to B$ that is μ-integrable, let the expectation with respect to μ be denoted by the Bochner integral

$$\mathbb{E}(F) = \int_\Omega F(\boldsymbol{y})\mu(\mathrm{d}\boldsymbol{y}). \tag{4}$$

The lognormal-parametric PDE in (1) is a prominent class of elliptic PDEs with unbounded random coefficients, which was considered in [3, 13, 16, 15, 21]. Specifically, we are interested in approximations of (4) with QMC quadrature. The integrands $F := G(u)$ are linear, continuous functionals $G : H_0^1(D) \to \mathbb{R}$ of the solution u of (1). The evaluation of integrands F requires the solutions of PDEs for realizations of u, which in general are solved numerically. Approximate integrand evaluation through Galerkin Finite Element (FE for short) discretization introduces an error, which is controlled by dimension-independent error bounds.

The assumptions for the QMC convergence theory in [13] on the functions $(\psi_j)_{j \geq 1}$ relied on the p-summability of their $L^\infty(D)$-norms: in [13], it was assumed that for some $p \in (0,1]$

$$\sum_{j \geq 1} \|\psi_j\|_{L^\infty(D)}^p < \infty. \tag{5}$$

In this paper, we aim to extend the QMC convergence theory of [13] by analyzing consequences for the QMC weights due to accounting for possible locality of the supports of $(\psi_j)_{j \geq 1}$. Similar to what has been shown for N-term convergence rates in [1], in certain cases this can imply significant gains in the convergence rate. In the analysis in [13, 19], product and order dependent (POD for short) QMC weights were essential. In this manuscript, we analyze, as in the case of affine-parametetric coefficients, cp. [11], convergence rates of first order, randomly shifted lattice rules from [23] for function systems $(\psi_j)_{j \geq 1}$ used in the representation (2) which have local supports in D. This is motivated, on the one hand, by the complexity of QMC rules according to product weights scaling linearly with the dimension of integration, cp. [24, 25, 12]. On the other hand, systems of locally supported ψ_j may afford better local resolution in D.

Convergence in $L^q(\Omega; L^\infty(D))$, $q \in [1, \infty)$, of the series in (2) will be shown under the assumption that there exists a positive sequence $(b_j)_{j \geq 1}$ such that

$$K := \left\| \sum_{j \geq 1} \frac{|\psi_j|}{b_j} \right\|_{L^\infty(D)} < \infty \tag{A1}$$

and that $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (0, \infty)$. The sequence $(b_j)_{j \geq 1}$ will enter the construction of QMC integration rules via the product weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$. These are defined by $\gamma_\emptyset = 1$ and

$$\gamma_{\mathfrak{u}} := \prod_{j \in \mathfrak{u}} b_j^\rho, \quad \emptyset \neq \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty, \tag{6}$$

where $\rho > 0$ is a constant. If $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for $p \in (2/3, 2)$, we obtain with a randomly shifted lattice QMC quadrature rule with product weights (6) and Gaussian weight functions a convergence rate of $\mathcal{O}(N^{-1/4 - 1/(2p) + \varepsilon})$ for sufficiently small $\varepsilon > 0$ [1]. In the case that $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for $p \in (2/3, 1]$, a randomly shifted lattice QMC quadrature rule with exponential weight

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[1]Here and throughout, all constants implied in $\mathcal{O}(\cdot)$ are independent of the integration dimension s.

functions and product weights (6) has a convergence rate of $\mathcal{O}(N^{-1/p+1/2})$. In either case, the implied constants are independent of N, which is the number of sample points, and of s, which is the integration-dimension.

In Section 2, we review results from [23] on QMC quadrature required in the following. In Section 3, we show integrability and approximation of the lognormal diffusion coefficient, which is applied in Section 5 to estimate the error that is introduced by truncating the expansion of the Gaussian random field. Existence and uniqueness is shown in Section 4. The main parametric regularity estimates are discussed in Section 6, which result in convergence rates of the exact solution in Section 7. Section 8 addresses the impact of a FE discretization in D. Section 9 discusses a particular choice of basis for representation of Gaussian random fields in D, and verifies that this representation satisfies the previously made conditions in our QMC convergence rate analysis. Finally, Section 10 presents some conclusions and generalizations.

2 QMC integration of Gaussian random fields

We recapitulate elements from randomly shifted lattice rules and weighted Sobolev spaces that are necessary for the QMC convergence theory, cp. [23, Theorem 8].

We seek to approximate with a QMC quadrature s-dimensional integrals of the form

$$I_s(F) := \int_{\mathbb{R}^s} F(\boldsymbol{y}) \prod_{j=1}^{s} \phi(y_j) \mathrm{d}\boldsymbol{y}, \tag{7}$$

where $s \in \mathbb{N}$ and ϕ is the standard normal density function, i.e.,

$$\phi(y) := \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad y \in \mathbb{R}.$$

These intgrals arise by truncation of expansions of random quantities in particular function systems that are random inputs for PDEs. The integrand F will be the composition of a linear functional $G(\cdot)$ with the solution u. An N-point QMC quadrature for the s-dimensional integral in (7) is an equal-weight quadrature rule and denoted by $Q_{s,N}^{\boldsymbol{\Delta}}$. Here, $\boldsymbol{\Delta} \sim \mathcal{U}((0,1)^s)$ denotes a random shift (see, e.g., [7] and the references there). For every $s, N \in \mathbb{N}$, let us define

$$Q_{s,N}^{\boldsymbol{\Delta}}(F) := \frac{1}{N} \sum_{i=0}^{N-1} F(\boldsymbol{y}^{(i)}),$$

with particularly chosen points $\{\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(N-1)}\} \subset \mathbb{R}^s$.

The s-dimensional cumulative distribution function Φ_s corresponding to the probability density ϕ is defined by

$$\Phi_s(\boldsymbol{y}) := \int_{\boldsymbol{y}' \leq \boldsymbol{y}} \prod_{j=1}^{s} \phi(y_j') \, \mathrm{d}\boldsymbol{y}', \quad \boldsymbol{y} \in \mathbb{R}^s,$$

where $\boldsymbol{y}' \leq \boldsymbol{y}$ is understood as $y_j' \leq y_j$ for $j = 1, \ldots, s$. In the case that $s = 1$ we omit the subscript. For *randomly shifted lattice rules* the points are obtained by

$$\boldsymbol{y}^{(i)} := \Phi_s^{-1}\left(\left\{\frac{(i+1)\boldsymbol{z}}{N} + \boldsymbol{\Delta}\right\}\right), \quad i = 0, \ldots, N-1, \tag{8}$$

where $\boldsymbol{z}$ is a generating vector, and for every $c \in (0, \infty)$, $\{c\} \in [0, 1)$ denotes the fractional part of c. We refer to the surveys [7, 18] for further details and references.

The integrands in (7) that are under consideration in this paper belong to weighted, unanchored Sobolev spaces. The error analysis of randomly shifted lattice rules involves spaces

of type $\mathcal{W}_{s,\boldsymbol{\gamma}}$, which require *weight functions* for their definition. In this paper we consider two particular kinds of weight functions being centered, generally unnormalized Gaussian density functions with variance $\alpha_{\mathrm{g}} > 1$ and exponentially decaying weight functions depending on $\alpha_{\exp} > 0$. Specifically, let us define the weight functions $(w_{\mathrm{g}}, j)_{j \geq 1}$ and $(w_{\exp,j})_{j \geq 1}$ by

$$w_{\mathrm{g},j}^2(y) := e^{-\frac{y^2}{2\alpha_{\mathrm{g}}}}, \quad y \in \mathbb{R}, j \in \mathbb{N}, \tag{9}$$

and

$$w_{\exp,j}^2(y) := e^{-\alpha_{\exp}|y|}, \quad y \in \mathbb{R}, j \in \mathbb{N}. \tag{10}$$

For a Hilbert space H and for a collection of positive weights $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$, define the weighted Sobolev space $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; H)$ as a Bochner space of strongly measurable functions from $\mathbb{R}^s$ taking values in the separable Hilbert space H that have finite $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; H)$-norm. Here, for finite parameter dimension $s \in \mathbb{N}$, the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; H)$-norm of *unanchored, mixed first order partial derivatives* is defined by

$$\|F\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s;H)} :=$$
$$\left(\sum_{\mathfrak{u} \subseteq \{1:s\}} \gamma_{\mathfrak{u}}^{-1} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left\| \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \partial^{\mathfrak{u}} F(\boldsymbol{y}) \prod_{j \in \{1:s\}\setminus\mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}_{\{1:s\}\setminus\mathfrak{u}} \right\|_H^2 \prod_{j \in \mathfrak{u}} w_j^2(y_j) \mathrm{d}\boldsymbol{y}_{\mathfrak{u}} \right)^{1/2}, \tag{11}$$

where $(w_j)_{j \geq 1}$ denotes either of the weight functions defined in (9) or in (10) and where the inner integral is understood as a Bochner integral (cp. for example [29, Chapter V.5]). In the case that $H = \mathbb{R}$, we simply write $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$. We used the notation that $\{1 : s\}$ denotes the set of integers $\{1, \dots, s\}$ and for $\boldsymbol{y} \in \mathbb{R}^s$ and $\mathfrak{u} \subset \{1 : s\}$, $\boldsymbol{y}_{\mathfrak{u}}$ denotes the coordinates $(y_j)_{j \in \mathfrak{u}}$ of $\boldsymbol{y}$. We recall a version of [23, Theorem 8] for our choices of weight functions in (9) and in (10).

Theorem 1 *Let* $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subseteq \{1:s\}}$ *be some product weights,* $s \in \mathbb{N}$ *the truncation level, and* $(w_j)_{j \geq 1}$ *be either of the weight functions defined in (9) or in (10). Then, a randomly shifted lattice rule with N points can be constructed in $\mathcal{O}(sN \log N)$ operations using the fast CBC algorithm of [25, 24] such that for every $F \in \mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)$ and for every $\lambda \in (1/(2r), 1]$ there holds the error bound*

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(F) - Q_{s,N}^{\boldsymbol{\Delta}}(F)|^2)} \leq \left((\varphi(N))^{-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \prod_{j \in \mathfrak{u}} \rho(\lambda) \right)^{1/(2\lambda)} \|F\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s)},$$

where Euler's totient function is denoted by $\varphi(\cdot)$. For weight functions $(w_{\mathrm{g},j})_{j \geq 1}$ defined in (9),

$$\rho(\lambda) = 2 \left(\frac{4\sqrt{2\pi}\alpha_{\mathrm{g}}^2}{\pi^{2-1/\alpha_{\mathrm{g}}}(2\alpha_{\mathrm{g}} - 1)} \right)^{\lambda} \zeta(2r\lambda) \quad and \quad r = 1 - \frac{1}{2\alpha_{\mathrm{g}}}$$

and for weight functions $(w_{\exp,j})_{j \geq 1}$ defined in (10)

$$\rho(\lambda) = 2 \left(\frac{\sqrt{2\pi} \exp(\alpha_{\exp}^2/(4\delta))}{\pi^{2-2\delta}(1 - \delta)\delta} \right)^{\lambda} \zeta(2r\lambda) \quad and \quad r = 1 - \delta \quad for \ any \ \delta \in \left(0, \frac{1}{2} \right).$$

This result is [23, Theorem 8]. The value of the first factor in the expression $\rho(\lambda)$ and the value of r that correspond to the weight functions $(w_{\mathrm{g},j})_{j \geq 1}$ and $(w_{\exp,j})_{j \geq 1}$ are derived in [20, Example 4 and Example 5] respectively.

3   Lognormal random fields

The Gaussian random field Z is in (2) formally defined as the limit of a series expansion with i.i.d. standard normally distributed coefficients. For any Banach space $(B, \|\cdot\|_B)$ and every $q \in [1, \infty)$, let $L^q(\Omega; B)$ denote the space of all strongly measurable mappings $X : \Omega \to B$ such that $\|X\|_B^q$ is μ-integrable. We note that the measurability of $\|X\|_B$ follows from the strong measurability of X in B. We investigate convergence of the series in (2) in the following theorem. In its proof and in what follows, for $s \in \mathbb{N}$, we define partial sums of (2)

$$Z^s := \sum_{j=1}^{s} y_j \psi_j \ . \tag{12}$$

Theorem 2 *Let the assumption in* (**A1**) *be satisfied for some* $p_0 \in (0, \infty)$ *and for some* $K \in (0, \infty)$. *Then, the Gaussian random field* Z *is well defined and for every* $q \in [1, \infty)$, $Z \in L^q(\Omega; L^\infty(D))$. *Moreover, for every* $\varepsilon \in (0, 1)$ *there exists* $C_{q,\varepsilon} > 0$ *such that*

$$\|Z - Z^s\|_{L^q(\Omega; L^\infty(D))} \leq C_{q,\varepsilon} \sup_{j > s} \left\{ b_j^{1-\varepsilon} \right\},$$

where for $r \in \mathbb{N}$ *such that* $r \geq \max\{p_0/(2\varepsilon), \lceil q/2 \rceil\}$, *the constant* $C_{q,\varepsilon}$ *is given by*

$$C_{q,\varepsilon} := K \|(b_j)_{j \geq 1}\|_{\ell^{p_0}(\mathbb{N})}^{\varepsilon} \frac{\sqrt{2}}{\pi^{1/4}} \sqrt{r}.$$

Proof. For a sequence of i.i.d. standard normally distributed variables $(y_j)_{j \geq 1}$ and for every $s' \in \mathbb{N}$, the finite sum $Z^{s'} = \sum_{j=1}^{s'} y_j \psi_j$ is weakly measurable, i.e., for every $\ell \in (L^\infty(D))^*$, $\ell(Z^s) = \sum_{j=1}^{s} y_j \ell(\psi_j)$ is measurable as a finite sum of real valued random variables. Since the $\text{span}\{\psi_j : j \in \{1 : s\}\}$ is finite dimensional, Z^s is separably valued in $\text{span}\{\psi_j : j \in \{1 : s\}\} \subset L^\infty(D)$. Pettis' theorem (e.g. cp. [29, Theorem V.4]) implies that Z^s is also strongly measurable in the space $L^\infty(D)$. Let $s < s_1 \in \mathbb{N}$ be arbitrary. We observe that

$$\|Z^{s_1} - Z^s\|_{L^q(\Omega; L^\infty(D))} \leq \left\| \sum_{s < j \leq s_1} \frac{|\psi_j|}{b_j} \right\|_{L^\infty(D)} \left\| \sup_{s < j \leq s_1} \{b_j |y_j|\} \right\|_{L^q(\Omega)} .$$

We set $q' := \lceil q/2 \rceil$ such that $2q' = 2\lceil q/2 \rceil$ is the smallest even natural number that is greater or equal than q. We pick $r \in \mathbb{N}$ such that $2\varepsilon r \geq p_0$ and such that $r \geq q'$ and conclude with the Jensen inequality for concave functions and with the norm estimate $\|\cdot\|_{\ell^\infty(\{s+1:s_1\})} \leq \|\cdot\|_{\ell^{2r}(\{s+1:s_1\})}$

$$\mathbb{E}\left(\left(\sup_{s < j \leq s_1} \{b_j |y_j|\} \right)^{2q'} \right) \leq \left(\sup_{s < j \leq s_1} \left\{ b_j^{1-\varepsilon} \right\} \right)^{2q'} \mathbb{E}\left(\left(\sum_{s < j \leq s_1} b_j^{2\varepsilon r} |y_j|^{2r} \right)^{q'/r} \right)$$

$$\leq \left(\sup_{s < j \leq s_1} \left\{ b_j^{1-\varepsilon} \right\} \right)^{2q'} \left(\mathbb{E}\left(\sum_{s < j \leq s_1} b_j^{2\varepsilon r} |y_j|^{2r} \right) \right)^{q'/r}$$

$$\leq \left(\sup_{s < j \leq s_1} \left\{ b_j^{1-\varepsilon} \right\} \right)^{2q'} \left(\sum_{s < j \leq s_1} b_j^{2\varepsilon r} \frac{(2r)!}{2^r r!} \right)^{q'/r}$$

$$\leq \left(\sup_{s < j \leq s_1} \left\{ b_j^{1-\varepsilon} \right\} \right)^{2q'} \|(b_j)_{j \geq 1}\|_{\ell^{p_0}(\mathbb{N})}^{2\varepsilon q'} \left(\frac{(2r)!}{2^r r!} \right)^{q'/r} ,$$

5

where we used the fact that for a random variable $X \sim \mathcal{N}(0,1)$, $\mathbb{E}(X^{2r}) = (2r)!/(2^r r!)$. The assumption $(b_j)_{j\geq 1} \in \ell^{p_0}(\mathbb{N})$ implies that $(Z^{s'})_{s' \geq 1}$ is a Cauchy sequence in $L^q(\Omega; L^\infty(D))$, which is a Banach space (cp. [8, Theorem III.6.6]). Define $\widetilde{Z}$ to be the unique limit of $(Z^{s'})_{s' \geq 1}$ in $L^q(\Omega; L^\infty(D))$. The continuous embedding $L^{q_2}(\Omega; L^\infty(D)) \subset L^{q_1}(\Omega; L^\infty(D))$, for every $q_1 \leq q_2 \in [1, \infty)$, implies that the limit $\widetilde{Z}$ does not depend on q. We denote this limit by Z. As an element of $L^q(\Omega; L^\infty(D))$, Z is a $L^\infty(D)$-valued μ-equivalence class.

The Stirling bounds $\sqrt{2\pi} n^{n+1/2} e^{-n} \leq n! \leq e n^{n+1/2} e^{-n}$, for every $n \in \mathbb{N}$, cp. [9, (9.5) and (9.8)], imply the assertion of the proposition with

$$\left(\frac{(2r)!}{2^r r!} \right)^{1/(2r)} \leq 2^{1/2} \left(\frac{e}{\sqrt{\pi}} \right)^{1/(2r)} \sqrt{\frac{r}{e}} \, .$$

$\square$

For the partial sum Z^s in (12), we define

$$a^s := \exp(Z^s) \, , \quad \text{for every} \quad s \in \mathbb{N} \, .$$

Proposition 3 *Let the assumption in* (**A1**) *be satisfied for some* $p_0 \in (0, \infty)$ *and for* $K \in (0, \infty)$. *Then, for every* $q \in [1, \infty)$, $a \in L^q(\Omega; L^\infty(D))$ *and there exists a constant* $C > 0$ *such that for every* $s \in \mathbb{N}$

$$\|a^s\|_{L^q(\Omega; L^\infty(D))} \leq C \, .$$

Proof. While the space $L^\infty(D)$ is not separable, the strong measurability of $a = \exp(Z)$ in $L^\infty(D)$ follows because the composition with the exponential function is a continuous mapping from $L^\infty(D)$ to $L^\infty(D)$. The proof of this proposition is based on an application of Fernique's theorem (e.g. cp. [2, Theorem 2.8.5] or [6, Theorem 2.7]). We verify the conditions in order to apply Fernique's theorem. Our approach is similar to the proof of [15, Proposition 2.11]. We detail the argument for the convenience of the reader. We claim that for every $\ell \in (L^\infty(D))^*$, $\ell(Z)$ is centered, normally distributed, i.e., the law of Z is a centered Gaussian measure on $L^\infty(D)$. Indeed, for arbitrary $s' \in \mathbb{N}$, $\ell(Z^{s'}) \sim \mathcal{N}(0, \sum_{j=1}^{s'} \ell(\psi_j)^2)$. Since

$$\sum_{j=1}^{s'} \ell(\psi_j)^2 \leq \left(\sum_{j=1}^{s'} \ell(\psi_j) \right)^2 = \left(\ell \left(\sum_{j=1}^{s'} \psi_j \right) \right)^2 \leq \|\ell\|_{(L^\infty(D))^*}^2 K^2 \sup_{j \geq 1} \{b_j^2\} \, ,$$

the monotone sequence $\sum_{j=1}^{s'} \ell(\psi_j)^2$ indexed by $s' \in \mathbb{N}$ is bounded and hence has a finite limit that we denote by σ_ℓ^2. This implies that for fixed $\ell \in (L^\infty(D))^*$, the characteristic functions of the random variables $(\ell(Z^{s'}) : s' \in \mathbb{N})$ converge pointwise to the characteristic function of a $\mathcal{N}(0, \sigma_\ell^2)$ distributed random variable as $s' \to \infty$. Since $\ell(Z^{s'})$ converges to $\ell(Z)$ as $s' \to \infty$ in particular in the L^2-sense by Theorem 2 and thus also in distribution, Lévy's continuity theorem (e.g. cp. [22, Theorem IV.13.2.B]) implies that $\ell(Z) \sim \mathcal{N}(0, \sigma_\ell^2)$ and we conclude that the law of Z is a Gaussian measure on $L^\infty(D)$, which is one of the conditions of Fernique's theorem.

We will treat the case $s < \infty$ first. By Theorem 2, there exists an upper bound C of the $L^2(\Omega; L^\infty)$-norm of the Gaussian random fields Z and Z^s, that is independent of s. The existence of this uniform upper bound C is the main ingredient of the remaining argument. Let in the following $X \in \{Z, Z^s\}$ be arbitrary. Let $\kappa_1 \in (1/(1 + \exp(-2)), 1)$ and set $\kappa_2 := C/\sqrt{1 - \kappa_1}$ and conclude with the Chebychev inequality that

$$1 - \mu(\|X\|_{L^\infty(D)} \leq \kappa_2) = \mu(\|X\|_{L^\infty(D)} > \kappa_2) \leq \frac{\mathbb{E}(\|X\|_{L^\infty(D)}^2)}{\kappa_2^2} \leq \frac{C^2}{\kappa_2^2} = 1 - \kappa_1.$$

6

Hence, $\mu(\|X\|_{L^\infty(D)} \leq \kappa_2) \geq \kappa_1 > 1/(1 + \exp(-2)) > 1/2$. Let us set $\lambda := (1 - \kappa_1)/(32C^2)$, which implies that $32\lambda\kappa_2^2 \leq 1$. Thus, by the monotonicity of the logarithm

$$\log\left(\frac{1 - \mu(\|X\|_{L^\infty(D)} \leq \kappa_2)}{\mu(\|X\|_{L^\infty(D)} \leq \kappa_2)}\right) + 32\lambda\kappa_2^2 \leq \log\left(\frac{1 - \kappa_1}{\kappa_1}\right) \leq -1.$$

This is the second requirement in order to apply [6, Theorem 2.7]. Since Z^s is in particular a Gaussian measure on the separable Banach space $\mathrm{span}\{\psi_j : j \in \{1 : s\}\}$ with respect to the $L^\infty(D)$-norm, [6, Theorem 2.7] implies that

$$\mathbb{E}(\exp(\lambda\|Z^s\|_{L^\infty(D)}^2)) \leq \exp(16\lambda\kappa_2^2) + \frac{\exp(2)}{\exp(2) - 1}. \tag{13}$$

Since κ_2 and λ do not depend on s (because C does not), the upper bound in (13) is uniform with respect to s. For every $x \in \mathbb{R}$, $qx \leq \lambda x^2 + q^2/(4\lambda)$ is concluded from $0 \leq (\sqrt{\lambda}x - q/(2\sqrt{\lambda}))^2$, which yields the second assertion of the proposition, i.e.,

$$\mathbb{E}(\|\exp(Z^s)\|_{L^\infty(D)}^q) \leq \mathbb{E}(\exp(\lambda\|Z^s\|_{L^\infty(D)}))\exp\left(\frac{q^2}{4\lambda}\right). \tag{14}$$

The case of Z (corresponding formally to the case of $s = \infty$) is treated separately. Since $L^\infty(D)$ is not separable, [6, Theorem 2.7] is not applicable. We argue with [2, Theorem 2.8.5] instead. To this end, we define

$$\widehat{\lambda} := \frac{1}{24\kappa_2^2}\log\left(\frac{\mu(\|Z\|_{L^\infty(D)} \leq \kappa_2)}{1 - \mu(\|Z\|_{L^\infty(D)} \leq \kappa_2)}\right),$$

which is strictly positive because $\kappa_1 > 1/2$. Then, [2, Theorem 2.8.5] is applicable and we obtain

$$\mathbb{E}(\exp(\widehat{\lambda}\|Z\|_{L^\infty(D)}^2)) < \infty,$$

which implies as above, cp. (14), that $\mathbb{E}(\|\exp(Z)\|_{L^\infty(D)}^q) < \infty$. □

We note that the line of argument in the second paragraph of the proof seems to originate with the proof of [3, Proposition 3.10].

Remark 4 *The property that $a = \exp(Z) \in L^q(\Omega; L^\infty(D))$, for every $q \in [1, \infty)$, also holds under weaker summability assumptions on $(b_j)_{j\geq 1}$, cp. [1, Theorem 2.1] that was proven with a different approach. However the membership of $(b_j)_{j\geq 1}$ in $\ell^p(\mathbb{N})$ for a certain range of p (as assumed in Proposition 3) seems indispensible for the considered QMC rules to be applicable, cp. Section 7. Also, our argument yields bounds of truncated expansions of Gaussian random fields that are uniform in s.*

Proposition 5 *Let the assumption in* (**A1**) *be satisfied for some $p_0 \in (0, \infty)$ and for $K \in (0, \infty)$. Then, for every $q \in [1, \infty)$ and every $\varepsilon \in (0, 1)$ there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$*

$$\|a - a^s\|_{L^q(\Omega; L^\infty(D))} \leq C \sup_{j>s}\left\{b_j^{1-\varepsilon}\right\},$$

Proof. The fundamental theorem of calculus implies that for every $t_1, t_2 \in \mathbb{R}$, $|e^{t_2} - e^{t_1}| \leq (e^{t_2} + e^{t_1})|t_2 - t_1|$. Thus, by the Cauchy–Schwarz inequality

$$\|a - a^s\|_{L^q(\Omega; L^\infty(D))} \leq \|a + a^s\|_{L^{2q}(\Omega; L^\infty(D))}\|Z - Z^s\|_{L^{2q}(\Omega; L^\infty(D))}.$$

The assertion follows with the triangle inequality, Theorem 2, and Proposition 3. □

Our ensuing analysis of the solution to (1) will require the following random variables:

$$a_{\min} := \operatorname*{ess\,inf}_{x \in D} a(x), \ a_{\max} := \|a\|_{L^\infty(D)}, \ a_{\min}^s := \operatorname*{ess\,inf}_{x \in D} a^s(x), \ a_{\max}^s := \|a^s\|_{L^\infty(D)} .$$

Here, $s \in \mathbb{N}$ is arbitrary, finite.

Corollary 6 *Let the assumption of Proposition 3 be satisfied. Then, for every $q \in [1, \infty)$, $a_{\min}^{-1} \in L^q(\Omega; L^\infty(D))$ and there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$*

$$\left\| \frac{1}{a_{\min}^s} \right\|_{L^q(\Omega; L^\infty(D))} \leq C.$$

4 Existence and uniqueness

In this paper we are interested in mean field approximations. We consider the solution to (1) as a μ-equivalence class taking values in $V := H_0^1(D)$. The existence and uniqueness of the solution to (1) is well known, cp. [3, Proposition 2.4]; we review the basic results, following the presentation in [15, Section 3.1].

Since the right hand side in (1) is deterministic, we are interested in the data-to-solution map $\mathcal{S}_f$ that maps a (realization of the) diffusion coefficient $\widehat{a} \in L^\infty(D)$ to the solution $\widehat{u} \in V$ for fixed right hand side $f \in V^*$. In what follows, we fix $f \in V^*$ unless explicitly stated otherwise. For every $\widehat{a} \in L_+^\infty(D) := \{ \widetilde{a} \in L^\infty(D) : \operatorname{ess\,inf}_{x \in D} \widetilde{a}(x) > 0 \}$, consider the deterministic diffusion equation problem: find a unique $\widehat{u} \in V$ such that

$$\int_D \widehat{a} \, \nabla \widehat{u} \cdot \nabla v \, \mathrm{d}x = f(v), \quad \forall v \in V. \tag{15}$$

For such $\widehat{a}$, the bilinear form $(w, v) \mapsto \int_D \widehat{a} \, \nabla w \cdot \nabla v \, \mathrm{d}x$ is continuous and coercive on $V \times V$, since by $\widehat{a} \in L_+^\infty(D)$

$$\left| \int_D \widehat{a} \, \nabla w \cdot \nabla v \, \mathrm{d}x \right| \leq \|\widehat{a}\|_{L^\infty(D)} \|w\|_V \|v\|_V, \quad \forall w, v \in V,$$

and

$$\int_D \widehat{a} \, \nabla w \cdot \nabla w \, \mathrm{d}x \geq \operatorname*{ess\,inf}_{x \in D} \{\widehat{a}(x)\} \|w\|_V^2, \quad \forall w \in V.$$

The Lax–Milgram lemma implies that the problem in (15) is well posed. Thus, for every fixed $f \in V^*$ the mapping

$$\mathcal{S}_f : L_+^\infty(D) \to V : \widehat{a} \mapsto \widehat{u}$$

is well defined. Moreover, the Lax–Milgram lemma implies that for every $\widehat{a} \in L_+^\infty(D)$

$$\|\mathcal{S}_f(\widehat{a})\|_V \leq \frac{1}{\operatorname{ess\,inf}_{x \in D} \{\widehat{a}(x)\}} \|f\|_{V^*}. \tag{16}$$

Also it is well known that $\mathcal{S}_f : L_+^\infty(D) \to V$ is Lipschitz continuous, which can be shown by the second Strang lemma (see (18) ahead or [15] for details).

Let in the following a denote the lognormal random field in Proposition 3. The weak (or variational) formulation of the parametric, elliptic PDE (1) for fixed, deterministic $f \in V^*$ reads: find a strongly measurable V-valued mapping $u : \Omega \to V$ such that

$$\int_D a \, \nabla u \cdot \nabla v \, \mathrm{d}x = f(v), \quad \forall v \in V . \tag{17}$$

8

Due to Proposition 3, a is strongly measurable in $L^\infty(D)$ and by Corolloary 6, $a_{\min} > 0$ μ-almost surely (a.s. for short). Hence, a takes values in $L^\infty_+(D)$ and $u := \mathcal{S}_f(a)$ is the unique solution to (17), where we recall that uniqueness is meant as V-valued μ-equivalence class. The strong measurability in V of u is deduced from the strong measurability of a and the continuity of $\mathcal{S}_f$. By (16) and Corollary 6, for every $q \in [1, \infty)$ there holds

$$\|u\|_{L^q(\Omega;V)} \le \left\|\frac{1}{a_{\min}}\right\|_{L^q(\Omega)} \|f\|_{V^*} < \infty\ .$$

5   Dimension truncation

In applications of QMC integration a finite dimensional integration domain is required, which in our case will be $\mathbb{R}^s$ for $s \in \mathbb{N}$. Truncation of the series in (2) will introduce a truncation error. For every $s \in \mathbb{N}$, $u^s := \mathcal{S}_f(a^s)$ uniquely solves

$$\int_D a^s\, \nabla u^s \cdot \nabla v\, \mathrm{d}x = f(v), \quad \forall v \in V.$$

Proposition 7 *Let the assumption in* (**A1**) *be satisfied for some* $p_0 \in (0, \infty)$ *and for* $K > 0$. *Let* $G(\cdot) \in V^*$ *and* $\varepsilon \in (0,1)$ *be arbitrary. There exists a constant* C *independent of* f *and of* $G(\cdot)$ *such that for every* $s \in \mathbb{N}$

$$|\mathbb{E}(G(u)) - I_s(G(u^s))| \le C\|G(\cdot)\|_{V^*}\|f\|_{V^*} \sup_{j > s}\left\{b_j^{1-\varepsilon}\right\}.$$

Proof. The second Strang lemma implies that

$$\|u - u^s\|_V \le \frac{1}{a_{\min}\, a_{\min}^s}\|a - a^s\|_{L^\infty(D)}\|f\|_{V^*}\ . \tag{18}$$

By the linearity and continuity of $G(\cdot)$ and by the Hölder inequality

$$|\mathbb{E}(G(u)) - I_s(G(u^s))| \le \|G(\cdot)\|_{V^*}\|u - u^s\|_{L^1(\Omega;V)}$$

$$\le \|G(\cdot)\|_{V^*}\left\|\frac{1}{a_{\min}}\right\|_{L^3(\Omega)}\left\|\frac{1}{a_{\min}^s}\right\|_{L^3(\Omega)}\|a - a^s\|_{L^3(\Omega;L^\infty(D))}\|f\|_{V^*}.$$

The assertion follows with Proposition 5 and Corollary 6.   □

Realizations of the Gaussian random field Z^s can be obtained from Gaussian vectors $\boldsymbol{y} \in \mathbb{R}^s$. Specifically, one realization of Z^s requires s draws of independent, standard normally distributed random variables which results in a vector $(y_1, \ldots, y_s)^\top \in \mathbb{R}^s$. Since the support of the s-dimensional multivariate Gaussian measure on $\mathbb{R}^s$ with covariance equal to the identity is $\mathbb{R}^s$, the whole of $\mathbb{R}^s$ is the parameter set. We denote realizations of Z^s by $Z^s(\boldsymbol{y}) := \sum_{j=1}^s y_j\psi_j$, where $\boldsymbol{y} = (y_1, \ldots, y_s)^\top \in \mathbb{R}^s$ is the particular realization of the i.i.d. standard normally distributed coefficient sequence $(y_j)_{1 \le j \le s}$. Moreover, for every $s \in \mathbb{R}^s$, Z^s also denotes the respective mapping from $\mathbb{R}^s$ to $L^\infty(D)$. Similarly, for every $s \in \mathbb{R}^s$, a^s also denotes the respective mapping from $\mathbb{R}^s$ to $L^\infty_+(D)$, $a_{\min}^s$ and $a_{\max}^s$ also denote the respective mappings from $\mathbb{R}^s$ to $(0, \infty)$, and u^s also denotes the respective mapping from $\mathbb{R}^s$ to V.

6 Parametric regularity

By the definition of the weighted Sobolev norm in (11), it is crucial for the QMC convergence analysis to derive estimates of the mixed partial derivatives $\partial^{\mathfrak{u}} u^s$, $\mathfrak{u} \subset \{1 : s\}$, in order to bound the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; V)$-norm of u^s uniformly in the parameter dimension s.

Bounds on the parametric partial derivatives of the solution u^s have been proven in [16, 13, 1]. It is well known that for every $\mathbf{0} \neq \boldsymbol{\tau} \in \mathbb{N}_0^s$ and for every $\boldsymbol{y} \in \mathbb{R}^s$ there holds

$$\int_D a^s(\boldsymbol{y}) \nabla \partial^{\boldsymbol{\tau}} u^s(\boldsymbol{y}) \cdot \nabla v \, \mathrm{d}x = -\int_D \sum_{\boldsymbol{\nu} \leq \boldsymbol{\tau}, \boldsymbol{\nu} \neq \boldsymbol{\tau}} \binom{\boldsymbol{\tau}}{\boldsymbol{\nu}} \prod_{j \in \mathrm{supp}(\boldsymbol{\tau})} \psi_j^{\tau_j - \nu_j} a^s(\boldsymbol{y}) \nabla \partial^{\boldsymbol{\nu}} u^s(\boldsymbol{y}) \cdot \nabla v \, \mathrm{d}x, \quad \forall v \in V,$$

(19)

cp. for example [1, Lemma 3.1] and see also [16, (3.6)]. The arguments in [16, 13] rely on global bounds of the functions $(\psi_j)_{j \geq 1}$. Specifically, the $L^\infty(D)$-norm of the functions $(\psi_j)_{j \geq 1}$ was (in these references) taken inside the summation over multiindices in (19). This way information of locality of the support of the functions $(\psi_j)_{j \geq 1}$ is lost. For the quantitative analysis of parametric regularity, we introduce for every $s \in \mathbb{N}$, for every $\boldsymbol{y} \in \mathbb{R}^s$ and every $v \in V$ the parametrized energy norm $\|v\|_{a^s(\boldsymbol{y})}$ by

$$\|v\|_{a^s(\boldsymbol{y})} := \sqrt{\int_D a^s(\boldsymbol{y}) \, |\nabla v|^2 \, \mathrm{d}x} \ .$$

For every $\boldsymbol{y} \in \mathbb{R}^s$ and every $v \in V$ there holds

$$(a_{\min}^s(\boldsymbol{y}))^{1/2} \|v\|_V \leq \|v\|_{a^s(\boldsymbol{y})} \leq (a_{\max}^s(\boldsymbol{y}))^{1/2} \|v\|_V \ . \tag{20}$$

The following proposition was proven with an approach that accounts for possible locality of the supports. We state a version of first order mixed partial derivatives and truncated dimension.

Proposition 8 *[1, Theorem 4.1] Assume that there exists a positive sequence $(\rho_j)_{j \geq 1}$ such that*

$$\left\| \sum_{j \geq 1} \rho_j |\psi_j| \right\|_{L^\infty(D)} < \log(2) \ .$$

Then, there exists a constant $C > 0$ such that for every $s \in \mathbb{N}$ and every $\boldsymbol{y} \in \mathbb{R}^s$

$$\sum_{\mathfrak{u} \subset \{1:s\}} \|\partial^{\mathfrak{u}} u^s(\boldsymbol{y})\|_{a^s(\boldsymbol{y})}^2 \prod_{j \in \mathfrak{u}} \rho_j^2 \leq C \|u^s(\boldsymbol{y})\|_{a^s(\boldsymbol{y})}^2 \ .$$

We extend the parametric regularity estimates that are given in Proposition 8 in order to obtain estimates that are suitable to yield dimension independent convergence rates of randomly shifted lattice rules.

Theorem 9 *Let the assumption in (**A1**) be satisfied for some $K > 0$. Let $(\boldsymbol{w}_j)_{j \geq 1}$ be either of the weight functions defined in (9) and (10). Let $\kappa \in (0, \log(2)/K)$ be fixed and $p' \in (0, 1)$. There exists a constant $C > 0$ such that for every $s \in \mathbb{N}$, and for positive $\boldsymbol{\gamma}$,*

$$\|u^s\|_{\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; V)}^2 \leq C \|f\|_{V^*}^2 \int_{\mathbb{R}^s} \frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \sup_{\mathfrak{u} \subset \{1:s\}} \left\{ \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j \in \{1:s\} \setminus \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}$$

$$\times \sup_{\mathfrak{u} \subset \{1:s\}} \gamma_{\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} b_j^{2p'} \ .$$

Proof. We obtain with the Jensen inequality, for any $s \in \mathbb{N}$,

$$\|u^s\|^2_{\mathcal{W}_\gamma(\mathbb{R}^s;V)}$$

$$\leq \sum_{\mathfrak{u} \subset \{1:s\}} \frac{1}{\gamma_\mathfrak{u}} \int_{\mathbb{R}^s} \|\partial^\mathfrak{u} u(\boldsymbol{y})\|^2_V \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi_j(y_j) \prod_{j \in \mathfrak{u}} w_j^2(y_j) \mathrm{d}\boldsymbol{y}$$

$$\leq \int_{\mathbb{R}^s} \sum_{\mathfrak{u} \subset \{1:s\}} \frac{\kappa^{2|\mathfrak{u}|}}{\gamma_\mathfrak{u} \prod_{j \in \mathfrak{u}} b_j^{2(1-p')}} \|\partial^\mathfrak{u} u(\boldsymbol{y})\|^2_V \sup_{\mathfrak{u} \subset \{1:s\}} \left\{ \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}.$$

In the present setting, the assumption of Proposition 8 is satisfied by the sequence $(\rho_j)_{j\geq 1} = (\kappa/b_j)_{j\geq 1}$. Hence, by the Hölder inequality and Proposition 8, and using (20) we obtain the following bound

$$\|u^s\|^2_{\mathcal{W}_\gamma(\mathbb{R}^s;V)} \leq \int_{\mathbb{R}^s} \frac{1}{a^s_{\min}(\boldsymbol{y})} \left(\sum_{\mathfrak{u} \subset \{1:s\}} \|\partial^\mathfrak{u} u^s(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})} \gamma_\mathfrak{u}^{-1} \prod_{j \in \mathfrak{u}} \frac{\kappa^2}{b_j^{2(1-p')}} \right)$$

$$\times \sup_{\mathfrak{u} \subset \{1:s\}} \prod_{j \in \mathfrak{u}} \left\{ \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}$$

$$\leq \int_{\mathbb{R}^s} \frac{1}{a^s_{\min}(\boldsymbol{y})} C \|u^s(\boldsymbol{y})\|^2_{a^s(\boldsymbol{y})} \sup_{\mathfrak{u} \subset \{1:s\}} \left\{ \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}$$

$$\times \sup_{\mathfrak{u} \subset \{1:s\}} \gamma_\mathfrak{u}^{-1} \prod_{j \in \mathfrak{u}} b_j^{2p'}$$

$$\leq C \|f\|^2_{V^*} \int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u} \subset \{1:s\}} \left\{ \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}$$

$$\times \sup_{\mathfrak{u} \subset \{1:s\}} \gamma_\mathfrak{u}^{-1} \prod_{j \in \mathfrak{u}} b_j^{2p'},$$

where we have used that $\|u^s(\boldsymbol{y})\|_{a^s(\boldsymbol{y})} \leq \|f\|_{V^*}/\sqrt{a^s_{\min}(\boldsymbol{y})}$. $\qquad \square$

Corollary 10 *Under the assumption of Theorem 9, there exists a finite constant C such that for every $s \in \mathbb{N}$ and for every $G(\cdot) \in V^*$ holds for $F = G(u^s)$*

$$\|F\|_{\mathcal{W}_\gamma(\mathbb{R}^s)} \leq C \|G(\cdot)\|_{V^*} \|f\|_{V^*} \sqrt{\int_{\mathbb{R}^s} \frac{1}{(a^s_{\min}(\boldsymbol{y}))^2} \sup_{\mathfrak{u} \subset \{1:s\}} \prod_{j \in \mathfrak{u}} \left(\frac{b_j^{2(1-p')}}{\kappa^2} \right) w_j^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}}$$

$$\times \sup_{\mathfrak{u} \subset \{1:s\}} \gamma_\mathfrak{u}^{-1/2} \prod_{j \in \mathfrak{u}} b_j^{p'}.$$

7 QMC analysis for the exact solution

In this section we show dimension-independent convergence rates for QMC integration of (functionals of) the parametric solution $u^s(\boldsymbol{y})$, which are obtained from the parametric regularity bounds shown in Section 6. The cases of Gaussian and exponential weight functions in the norm (11) will be treated separately, since the ensuing analysis suggests that the convergence rates hold under different summability assumptions on the sequence $(b_j)_{j\geq 1}$. In this section we assume that the integrand functions can be evaluated exactly. Ahead, in Section 8, the

additional discretization error that arises by single-level Petrov-Galerkin discretizations of the parametric PDE (17) is taken into account.

Theorem 11 *[Gaussian weight functions] Let assumption* (**A1**) *be satisfied for $K > 0$ and for $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $p \in (2/3, 2)$. For some $\varepsilon \in (0, 3/4 - 1/(2p))$ such that $\varepsilon \leq 1/(2p) - 1/4$ set $p' = p/4 + 1/2 - \varepsilon p$. Let $(w_{g,j})_{j\geq 1}$ be the weight functions defined in (9) with*

$$\alpha_g \in \left(\frac{p}{2(p - p')}, \frac{p}{p - 2(1 - p')} \right). \tag{21}$$

Define the product weights

$$\gamma_{\mathfrak{u}} := \prod_{j\in\mathfrak{u}} b_j^{2p'}, \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty. \tag{22}$$

Let $s \in \mathbb{N}$ and $G(\cdot) \in V^*$ be given. Then, for every $N \in \mathbb{N}$ a randomly shifted lattice rule with N points can be constructed in $\mathcal{O}(sN\log N)$ operations using the fast CBC algorithm of [25, 24] such that the root-mean square error over all random shifts can be estimated as follows: there exists a constant $C > 0$ that is independent of s and N such that

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|I_s(G(u^s)) - Q_{s,N}^{\mathbf{\Delta}}(G(u^s))|^2)} \leq C \left(\varphi(N) \right)^{-(1/(2p)+1/4-\varepsilon)}.$$

Proof. The assertion of the theorem will follow by Theorem 1 once the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; V)$-norm of u^s has been bounded independently of s, which in turn will be deduced from the bound in Theorem 9 and in Corollary 10. To this end, fix $\kappa \in (0, \log(2)/K)$. Since $p > 2(1 - p')$ is implied by $(3/4 - 1/(2p)) > \varepsilon$, thus $q := p/(2(1 - p')) > 1$. From the Jensen inequality we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^s} \frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w_{g,j}^2(y_j) \prod_{j\in\{1:s\}\backslash\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \\
&\leq \int_{\mathbb{R}^s} \left(\left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right)^q \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p}{\kappa^{2q}} w_{g,j}^{2q}(y_j) \prod_{j\in\{1:s\}\backslash\mathfrak{u}} \phi(y_j)^q \right)^{1/q} \mathrm{d}\boldsymbol{y} \\
&\leq \left(\int_{\mathbb{R}^s} \left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right)^q \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p}{\kappa^{2q}} w_{g,j}^{2q}(y_j)\phi(y_j)^{-q} \prod_{j\in\{1:s\}} \phi(y_j)\mathrm{d}\boldsymbol{y} \right)^{1/q} \\
&= \left(\sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p}{\kappa^{2q}} \int_{\mathbb{R}^s} \left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right)^q \prod_{j\in\mathfrak{u}} w_{g,j}^{2q}(y_j)\phi(y_j)^{-q} \prod_{j\in\{1:s\}} \phi(y_j)\mathrm{d}\boldsymbol{y} \right)^{1/q}.
\end{aligned}
\tag{23}
$$

Here, we inserted the factor $1 = \prod_{j\in\{1:s\}} \phi(y_j)\phi(y_j)^{-1}$ and we moved factors under the exponent $1/q$ to move the exponent $1/q$ outside of the integral with the Jensen inequality.

The parameter $\alpha_g > 1$ of the weight functions $(w_{g,j})_{j\geq 1}$ is chosen such that $\alpha_g < q/(q-1)$, which implies that $1 > (1 - 1/\alpha_g)q$. The function $x \mapsto x/(x-1)$ is strictly decreasing on $(1, \infty)$. Thus, there exists $q' > q$ such that $\alpha_g < q'/(q'-1)$ and therefore also $1 > (1 - 1/\alpha_g)q'$. Since $\int_{\mathbb{R}} \exp(-y^2/(2\sigma^2))\mathrm{d}y = \sqrt{2\pi}\sigma$ for every $\sigma > 0$, it holds that

$$\int_{\mathbb{R}} w_{g,j}^{2q'}(y)\phi(y)^{-q'}\phi(y)\mathrm{d}y = \frac{1}{\sqrt{2\pi}} \alpha_g^{-q'/2} \int_{\mathbb{R}} e^{-\frac{y^2}{2}\left(1 - \left(1 - \frac{1}{\alpha_g}\right)q'\right)} \mathrm{d}y = \alpha_g^{-q'/2} \frac{\alpha_g}{\alpha_g - (\alpha_g - 1)q'} =: C'.$$

The Hölder inequality applied with $q'/q > 1$ and conjugate $q'/(q'-q)$ results in

$$\int_{\mathbb{R}^s} \left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right)^q \prod_{j \in \mathfrak{u}} w_{\mathrm{g},j}^{2q}(y_j) \phi(y_j)^{-q} \prod_{j \in \{1:s\}} \phi(y_j) \mathrm{d}\boldsymbol{y}$$

$$\leq \left(\int_{\mathbb{R}^s} \left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right)^{qq'/(q'-q)} \prod_{j \in \{1:s\}} \phi(y_j) \mathrm{d}\boldsymbol{y} \right)^{(q'-q)/q'} (C')^{q/q'}$$

$$= \left(\mathbb{E} \left(\left(\frac{1}{(a_{\min}^s)^2} \right)^{qq'/(q'-q)} \right) \right)^{(q'-q)/q'} (C')^{q/q'} =: C'',$$

where the bound C'' is independent of s by Corollary 6 and by the Cauchy–Schwarz inequality. Together with (23) and [19, Lemma 6.3], we obtain that

$$\int_{\mathbb{R}^s} \frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \sup_{\mathfrak{u} \subset \{1:s\}} \left\{ \prod_{j \in \mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w_{\mathrm{g},j}^2(y_j) \prod_{j \in \{1:s\} \setminus \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y} \leq \left(C'' \sum_{\mathfrak{u} \subset \{1:s\}} \prod_{j \in \mathfrak{u}} \frac{b_j^p}{\kappa^{2q}} \right)^{1/q}$$

$$\leq (C'')^{1/q} \exp \left(\frac{1}{q} \sum_{j \geq 1} \frac{b_j^p}{\kappa^{2q}} \right),$$

which bound is independent of s and finite by the assumption $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$. Then, by Corollary 10, there exists a constant C independently of s such that for our chosen weights

$$\|G(u^s)\|_{\mathcal{W}_{s,\gamma}(\mathbb{R}^s)} \leq C \|G(\cdot)\|_{V^*} \|f\|_{V^*}.$$

The parameter α_{g} of the weight functions $(w_{\mathrm{g},j})_{j \geq 1}$ is chosen such that $\alpha_{\mathrm{g}} > p/(2(p-p'))$, which implies that $\lambda > 1/(2r)$, where $\lambda := p/(2p')$ and $r := 1 - 1/(2\alpha_{\mathrm{g}})$. Also note that $\varepsilon \leq 1/(2p) - 1/4$ implies $\lambda \leq 1$. We recall from Theorem 1

$$\rho(\lambda) := 2 \left(\frac{4\sqrt{2\pi}\alpha_{\mathrm{g}}^2}{\pi^{2-1/\alpha_{\mathrm{g}}}(2\alpha_{\mathrm{g}} - 1)} \right)^\lambda \zeta(2r\lambda)^\lambda .$$

The two conditions on the parameter α_{g} of the weight functions, that $\alpha_{\mathrm{g}} < q/(q-1)$ and that $\alpha_{\mathrm{g}} > p/(2(p-p'))$, are compatible, since

$$\frac{p}{2(p-p')} < \frac{q}{q-1} = \frac{p}{p - 2(1 - p')}$$

is implied by

$$p' < \frac{p}{4} + \frac{1}{2}.$$

Note that $p > p/4 + 1/2 > p'$ implies that α_{g} is well defined. Since product weights are considered, [19, Lemma 6.3] implies with the assumption $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ that

$$\sum_{\emptyset \neq \mathfrak{u} \subset \{1:s\}} \gamma_{\mathfrak{u}}^\lambda \rho(\lambda)^{|\mathfrak{u}|} \leq \sum_{\mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty} \prod_{j \in \mathfrak{u}} b_j^p \rho(\lambda)^{|\mathfrak{u}|} \leq \exp \left(\sum_{j \geq 1} b_j^p \rho(\lambda) \right) < \infty,$$

which bound is uniform in s. The assertion of the theorem follows with Theorem 1 applied with the choices $\lambda = p/(2p')$ and $p/(2(p-p')) < \alpha_{\mathrm{g}} < p/(p - 2(1 - p'))$. The convergence rate resulting from Theorem 1 is $1/(2\lambda) = p'/p = 1/(2p) + 1/4 - \varepsilon$. $\qquad \square$

Remark 12 *In Theorem 11, the case $p = 2$ does not seem accessible with the present argument, since in Theorem 1 neither of the choices $\lambda > 1$ nor $\alpha_{\mathrm{g}} = 1$ are permitted.*

Theorem 13 *[Exponential weight functions]*
Let assumption (**A1**) *be satisfied for $K > 0$ and for $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for $p \in (2/3, 1]$. Let $(w_{\exp,j})_{j \geq 1}$ be the weight functions defined in (10) with $\alpha_{\exp} > 2K \sup_{j \geq 1}\{b_j\}$. Define $p' := 1 - p/2 \in [1/2, 2/3)$. Let $s \in \mathbb{N}$ and $G(\cdot) \in V^*$ be given and define product weights*

$$\gamma_{\mathfrak{u}} := \prod_{j \in \mathfrak{u}} b_j^{2p'} \quad \mathfrak{u} \subset \mathbb{N}, |\mathfrak{u}| < \infty . \tag{24}$$

Then, for every $N \in \mathbb{N}$ a randomly shifted lattice rule with N points can be constructed in $\mathcal{O}(sN \log N)$ operations using the fast CBC algorithm of [25, 24] such that the root-mean square error over all random shifts can be estimated independently of s and N, i.e., there exists a constant $C > 0$ that is independent of s and N such that

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}(|I_s(G(u^s)) - Q_{s,N}^{\boldsymbol{\Delta}}(G(u^s))|^2)} \leq C\,(\varphi(N))^{-1/p+1/2} .$$

Proof. The assertion of the theorem will follow from Theorem 1 once the $\mathcal{W}_{\boldsymbol{\gamma}}(\mathbb{R}^s; V)$-norm of u^s has been bounded independently of s. This, in turn, will be shown using Theorem 9. Let $\kappa \in (0, \log(2)/K)$ be fixed. The choice $p' = 1 - p/2$ implies that $2(1 - p') = p$ and we obtain

$$\int_{\mathbb{R}^s} \frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \sup_{\mathfrak{u} \subset \{1:s\}} \left\{ \prod_{j \in \mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w_{\exp,j}^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}$$

$$\leq \int_{\mathbb{R}^s} \left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right) \sum_{\mathfrak{u} \subset \{1:s\}} \prod_{j \in \mathfrak{u}} \frac{b_j^p}{\kappa^2} w_{\exp,j}^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}$$

$$= \sum_{\mathfrak{u} \subset \{1:s\}} \prod_{j \in \mathfrak{u}} \frac{b_j^p}{\kappa^2} \int_{\mathbb{R}^s} \left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right) \prod_{j \in \mathfrak{u}} w_{\exp,j}^2(y_j) \prod_{j \in \{1:s\} \backslash \mathfrak{u}} \phi(y_j) \mathrm{d}\boldsymbol{y}.$$

We observe that for every $\boldsymbol{y} \in \mathbb{R}^s$,

$$\left(\frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \right) \leq e^{2\|Z(\boldsymbol{y})\|_{L^\infty(D)}} \leq e^{2K \sup_{j \in \{1:s\}}\{|y_j||b_j|\}} \leq e^{2K \sum_{j \in \{1:s\}} |y_j||b_j|},$$

which allows for an upper bound of the integrand that is in product form to separate the integrals. Since the parameter $\alpha_{\exp}$ of the weight functions satisfies that $\alpha_{\exp} > 2K\|(b_j)_{j \geq 1}\|_{\ell^\infty(\mathbb{N})}$, we obtain that for every $j \in \{1:s\}$

$$\int_{\mathbb{R}} e^{2K|y_j||b_j|} w_j^2(y_j) \mathrm{d}y_j = \frac{1}{\alpha_{\exp} - 2Kb_j}$$

and (as in [13, Eq.(4.15)])

$$1 \leq \int_{\mathbb{R}} e^{2K|y_j||b_j|} \phi(y_j) \mathrm{d}y_j = 2\exp\left(\frac{(2Kb_j)^2}{2} \right) \Phi(2Kb_j) \leq \exp\left(\frac{(2Kb_j)^2}{2} + \frac{4Kb_j}{\sqrt{2\pi}} \right) .$$

Here, we used the bound $\Phi(y) \leq 1/2\exp(2y/\sqrt{2\pi})$ for every $y \geq 0$, which can be shown by an affine approximation of Φ and the elementary bound $1 + x \leq e^x$ for every $x \in [0, \infty)$ (we refer to [13, p. 355] for details). By the assumption that $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N}) \subset \ell^1(\mathbb{N})$, for every $\mathfrak{u} \subset \{1:s\}$ holds

$$\prod_{j \in \mathfrak{u}} \exp\left(\frac{(2Kb_j)^2}{2} + \frac{4Kb_j}{\sqrt{2\pi}} \right) \leq \exp\left(\sum_{j \geq 1} \frac{(2Kb_j)^2}{2} + \frac{4Kb_j}{\sqrt{2\pi}} \right) =: C < \infty .$$

14

We conclude with [19, Lemma 6.3] and the assumption $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ that

$$\int_{\mathbb{R}^s} \frac{1}{(a_{\min}^s(\boldsymbol{y}))^2} \sup_{\mathfrak{u}\subset\{1:s\}} \left\{ \prod_{j\in\mathfrak{u}} \frac{b_j^{2(1-p')}}{\kappa^2} w_{\exp,j}^2(y_j) \prod_{j\in\{1:s\}\setminus\mathfrak{u}} \phi(y_j) \right\} \mathrm{d}\boldsymbol{y}$$

$$\leq C \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \frac{b_j^p/\kappa^2}{\alpha_{\exp} - 2Kb_j}$$

$$\leq C \exp\left(\sum_{j\geq 1} \frac{b_j^p/\kappa^2}{\alpha_{\exp} - 2Kb_j} \right) < \infty.$$

By Theorem 9 we obtain for our choice (24) of product weights $\boldsymbol{\gamma}$

$$\|G(u^s)\|_{\mathcal{W}_{s,\boldsymbol{\gamma}}(\mathbb{R}^s)} \leq \sqrt{C} \exp\left(\frac{1}{2} \sum_{j\geq 1} \frac{b_j^p/\kappa^2}{\alpha_{\exp} - 2Kb_j} \right) < \infty.$$

Here, the constant $C > 0$ is independent of the integration dimension s. The assertion now follows similarly as in the proof of Theorem 11 from Theorem 1. We have chosen the weight functions defined in (10) with $\lambda = p/(2p')$ and $\delta < 1 - 1/(2\lambda)$. We note that by the assumption $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ and [19, Lemma 6.3], for every $s \in \mathbb{N}$,

$$\sum_{\mathfrak{u}\subset\{1:s\}} \gamma_{\mathfrak{u}}^{\lambda} \rho(\lambda)^{|\mathfrak{u}|} = \sum_{\mathfrak{u}\subset\{1:s\}} \prod_{j\in\mathfrak{u}} \left(b_j^p \rho(\lambda) \right) \leq \exp\left(\sum_{j\geq 1} b_j^p \rho(\lambda) \right) < \infty.$$

$\square$

The QMC convergence rate bounds in Theorems 11 and 13 are also applicable for globally supported functions $(\psi_j)_{j\geq 1}$ as studied in [13]. The product structure of the QMC weight sequences $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u}\subset\mathbb{N},|\mathfrak{u}|<\infty}$ considered here entails stronger summability conditions on the sequence $(b_j)_{j\in\mathbb{N}}$ to achieve a prescribed, dimension-independent convergence rate.

Corollary 14 *Under the assumption that (5) is satisfied for some $p \in (2/5, 2/3)$. Define the sequence $(b_j)_{j\geq 1}$ by $b_j := \|\psi_j\|_{L^\infty(D)}^{1-p}$, $j \geq 1$. Then,*

1. *a randomly shifted lattice QMC rule based on Gaussian weight functions with product weights converges with rate $1/(2p) - 1/4 - \varepsilon$ for $\varepsilon > 0$ sufficiently small.*

2. *a randomly shifted lattice QMC rule based on exponential weight functions with product weights for $p \in (2/5, 1/2]$ converges with rate $1/p - 3/2$.*

We remark that in [13], for exponential weight functions, globally supported $(\psi_j)_{j\geq 1}$ and for summability exponent $p \in (2/3, 1)$, the dimension-independent convergence rate $1/p - 1/2$ in terms of N was established for a randomly shifted lattice rule with *product and order dependent weights*, from [23]. For such weights, however, the fast CBC construction of QMC rules has cost which increases quadratically w.r. to the quadrature dimension s, whereas fast CBC constructions for product weights scale linearly w.r. to s.

8 Combined QMC Finite Element discretization

In general, the exact evaluation of the solution of (1) is not possible as required for the computation of $Q_{s,N}^{\Delta}(G(u^s))$ for a functional $G(\cdot) \in V^*$. We approximate the solution by a Galerkin FE method. For simplicity we introduce the assumption that

$$D \subset \mathbb{R}^d \text{ is a bounded polyhedron with plane faces.} \qquad (\mathbf{A2})$$

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of nested, shape regular simplicial triangulations of the polygonal resp. polyhedral domain D, where h is the maximal diameter of all elements in $\mathcal{T}_h$. Let V_h denote all continuous piecewise polynomal functions of total degree $r \geq 1$, that vanish on ∂D. Thus, $V_h \subset V$ is a subspace such that $\dim(V_h) = \mathcal{O}(h^{-d})$ as $h \to 0$. The deterministic Galerkin discretization reads: for every $\widehat{a} \in L_+^\infty(D)$ find $\widehat{u}^h \in V_h$ such that

$$\int_D \widehat{a} \, \nabla \widehat{u}^h \cdot \nabla v^h \, \mathrm{d}x = f(v), \quad \forall v^h \in V_h. \qquad (25)$$

From the discussion in Section 4 we know that the problem in (25) is well posed. Similar to Section 4, we denote by $\mathcal{S}_f^h$ the discretized data-to-solution map that maps a (realization of the) diffusion coefficient $\widehat{a} \in L_+^\infty(D)$ to the FE solution $\widehat{u}^h \in V_h$ for fixed right hand side $f \in V^*$. We note that $\mathcal{S}_f^h : L_+^\infty(D) \to V_h \subset V$ is continuous. This implies that the FE solution

$$u^{s,h} := \mathcal{S}_f^h(a^s)$$

is strongly measurable in V for every $h > 0$ and $s \geq 1$. It is the unique solution to the s-parametric, deterministic variational problem

$$\int_D a^s \, \nabla u^{s,h} \cdot \nabla v^h \, \mathrm{d}x = f(v), \quad \forall v^h \in V_h$$

as a V_h-valued μ-equivalence class; see also [15, Section 4.1] for details.

Let $C^t(\overline{D})$, $t \geq 0$, denote the Hölder spaces such that for $t \in \mathbb{N}$, $C^t(\overline{D})$ is the space of t-times continuously differentiable functions on $\overline{D}$ with bounded derivatives on $\overline{D}$. Regularity of solutions to (15) in Sobolev scales accounting for singularities due to re-entrant corners has been studied for $d = 2$ in [26, 14], where in [26, Lemma 5.2] the explicit dependence of the constant in the error bound has been tracked: let $t \in (0,1)$, $\tau \in (0, \max\{t, \pi/\beta_{\max}\}) \backslash \{1/2\}$, and assume that $f \in H^{-1+\tau}(D)$ and $\widehat{a} \in C^t(\overline{D}) \cap L_+^\infty(D)$, then $\mathcal{S}_f(\widehat{a}) \in H^{1+\tau}(D)$ and there exists a constant C such that for every $f \in H^{-1+\tau}(D)$ and for every $\widehat{a} \in C^t(\overline{D}) \cap L_+^\infty(D)$

$$\|\mathcal{S}_f(\widehat{a})\|_{H^{1+\tau}(D)} \leq C \frac{\|\widehat{a}\|_{L^\infty(D)}}{(\operatorname{ess\,inf}_{x \in D} \widehat{a}(x))^4} \|\widehat{a}\|_{C^t(\overline{D})}^2 \|f\|_{H^{-1+\tau}(D)}, \qquad (26)$$

where $\beta_{\max}$ is the maximal opening angle of the interior tangent cones to ∂D with vertex in the corner points of D. Under ($\mathbf{A2}$), for $d = 2$, in Sobolev scales the regularity of the inverse of the Dirichlet Laplacean $(-\Delta)^{-1} : V^* \to V$ is limited by the maximal interior angle $\beta_{\max}$ of D such that $(-\Delta)^{-1} : H^{-1+\tau}(D) \to H^{1+\tau}(D) \cap V$ is bounded for every $\tau \in [0, \pi/\beta_{\max})$, cp. [14, Section 5].

We impose the hypothesis (see Proposition 18 ahead for a class of instances) that for some $t > 0$, a and a^s are strongly measurable in $C^t(\overline{D})$, for every $s \in \mathbb{N}$. Moreover, we assume that for every $q \in [1, \infty)$ there exists a constant C such that for every $s \in \mathbb{N}$

$$\|a^s\|_{L^q(\Omega; C^t(\overline{D}))} \leq C. \qquad (\mathbf{A3})$$

Proposition 15 *Let the assumption in* (**A1**) *be satisfied for some* $p_0 \in (0, \infty)$ *and let the assumption in* (**A2**) *and in* (**A3**) *hold for some* $t > 0$. *Let* $f \in H^{-1+\tau}(D)$ *and let* $G(\cdot) \in H^{-1+\tau'}(D)$ *for* $\tau, \tau' \in (0, \max\{t, \pi/\beta_{\max}\}) \setminus \{1/2 + \mathbb{N}_0\}$. *For every* $q \in [1, \infty)$ *there exists a constant* C *independent of* $h > 0$ *such that for every* $s \geq 1$

$$\|G(u^s) - G(u^{s,h})\|_{L^q(\Omega)} \leq Ch^{\min\{\tau, r\} + \min\{\tau', r\}} .$$

Proof. The first part of the proof follows similarly as respective arguments that resulted in [15, Theorem 3.11].

We decompose $\tau = \lfloor \tau \rfloor + \{\tau\}$, where $\{\tau\}$ is the fractional part, and show by induction on $n \in \{0, \ldots, \lfloor \tau \rfloor\}$ that the $L^q(\Omega; H^{1+n+\{\tau\}}(D))$-norm of $\mathcal{S}_f(a^s)$ can be uniformly bounded in s for every $q \in [1, \infty)$. The base case, i.e., $n = 0$, follows by (26) and (**A3**) with a twofold application of the Cauchy–Schwarz inequality. For $n \in \{1, \ldots, \lfloor \tau \rfloor\}$, $t > 1$ and thus a^s takes values in $C^1(\overline{D})$. Let us assume the statement holds for $n - 1$ as induction hypothesis. Hence, the equation (1) can be reformulated as

$$-\Delta \mathcal{S}_f(a^s) = \frac{1}{a^s} (f + \nabla a^s \cdot \nabla \mathcal{S}_f(a^s)) =: \widetilde{f}$$

with equality in V^*. Since for a constant C that is independent of a^s and f

$$\|\widetilde{f}\|_{H^{-1+n+\{\tau\}}(D)} \leq C \left(\|1/a^s\|_{C^t(\overline{D})} \left(\|f\|_{H^{-1+n+\{\tau\}}(D)} + \|a^s\|_{C^t(\overline{D})} \|\mathcal{S}_f(a^s)\|_{H^{n+\{\tau\}}(D)} \right) \right),$$

where we used that the pointwise product of functions in $C^{\widetilde{t}}(\overline{D})$ with functions in $H^{\widetilde{\tau}}(D)$ is continuous for all $0 \leq \widetilde{\tau} < \widetilde{t}$, cp. [14, Theorem 1.4.1.1]. This implies with the induction hypothesis and a twofold application of the Cauchy–Schwarz inequality that the $L^q(\Omega; H^{-1+n+\{\tau\}}(D))$-norm of $\widetilde{f}$ is bounded uniformly in s for every $q \in [1, \infty)$. Since $(-\Delta)^{-1} : H^{-1+n+\{\tau\}}(D) \to H^{1+n+\{\tau\}}(D) \cap V$ is bounded the induction step is completed and thus the $L^q(\Omega; H^{1+\tau}(D))$-norm of $\mathcal{S}_f(a^s)$ is bounded uniformly in s for every $q \in [1, \infty)$. Note that the strong measurability of $\mathcal{S}_f(a^s)$ in $H^{1+\tau}(D)$ follows, since $\mathcal{S}_f : C^t(\overline{D}) \cap L_+^\infty(D) \to H^{1+\tau}(D)$ is continuous, which can be shown with the estimate in (26) and a perturbation argument with respect to the diffusion coefficient; see the proof of [15, Proposition 3.10] for details. Verbatim, it holds that for every $q \in [1, \infty)$, the $L^q(\Omega; H^{1+\tau'}(D))$-norm of $\mathcal{S}_G(a^s)$ can be bounded by a constant which is independent of s. By the Aubin–Nitsche lemma, cp. [4, Theorem 3.2.4 and (3.2.23)],

$$|G(u^s) - G(u^{s,h})| \leq \|a^s\|_{L^\infty(D)} \|\mathcal{S}_f(a^s) - \mathcal{S}_f^h(a^s)\|_V \|\mathcal{S}_G(a^s) - \mathcal{S}_G^h(a^s)\|_V,$$

which implies with the approximation property of V_h in V, cp. [4, Theorem 3.2.1] (which can be interpolated to non-integer Sobolev scales), Céa's lemma, and the Hölder inequality that for every $q \in [1, \infty)$

$$\|G(u^s) - G(u^{s,h})\|_{L^q(\Omega)} \leq C \left\| \frac{(a_{\max}^s)^3}{(a_{\min}^s)^2} \right\|_{L^{3q}(\Omega)} \|\mathcal{S}_f(a^s)\|_{L^{3q}(\Omega; H^{1+\tau}(D))} \|\mathcal{S}_G(a^s)\|_{L^{3q}(\Omega; H^{1+\tau'}(D))}$$
$$\times h^{\min\{\tau, r\} + \min\{\tau', r\}},$$

where the constant C is due to the approximation property. The assertion of the proposition follows by Proposition 3 and Corollary 6 with the Cauchy–Schwarz inequality and by the fact shown above that the $L^{3q}(\Omega; H^{1+\tau}(D))$-norm and the $L^{3q}(\Omega; H^{1+\tau'}(D))$-norm of $\mathcal{S}_f(a^s)$ and respectively of $\mathcal{S}_G(a^s)$ can be bounded uniformly with respect to s. □

Remark 16 *In Proposition 15, the cases* $\tau, \tau' \in \{1/2 + \mathbb{N}_0\}$ *are permitted if* $f \in H^{-1+\tau+\varepsilon}(D)$, *respectively if* $G(\cdot) \in H^{-1+\tau'+\varepsilon}(D)$, *for some* $\varepsilon > 0$.

Theorem 17 *Let the assumption in* (**A1**) *be satisfied with* $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ *for some* $p \in (2/3, 2)$ *and let the assumption in* (**A2**) *and in* (**A3**) *be satisfied for some* $t > 0$. *Let* $f \in H^{-1+\tau}(D)$ *and let* $G \in H^{-1+\tau'}(D)$ *for* $\tau, \tau' \in (0, \max\{t, \pi/\beta_{\max}\}) \setminus \{1/2 + \mathbb{N}_0\}$ *such that* $\max\{\tau, \tau'\} \leq r$. *The error incurred in the approximation* $Q_{s,n}^{\mathbf{\Delta}}(G(u^{s,h}))$ *with the N-point randomly shifted lattice rule* $Q_{s,N}^{\mathbf{\Delta}}$ *applied to the s-variate, dimensionally truncated integral* $I_s(G(u^{s,h}))$ *satisfies:*

1. *For* $p \in (2/3, 2)$ *and* $\varepsilon \in (0, 3/4 - 1/(2p))$ *such that* $\varepsilon \leq 1/(2p) - 1/4$, *with Gaussian weight functions* $(w_{g,j})_{j\geq 1}$ *defined in* (9) *with* α_g *as in* (21) *the error is bounded by*

$$
\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}(G(u)) - Q_{s,N}^{\mathbf{\Delta}}(G(u^{s,h}))|^2)} \leq C\left((\varphi(N))^{-1/4 - 1/(2p) + \varepsilon} + \sup_{j>s}\left\{b_j^{1-\varepsilon}\right\} + h^{\tau+\tau'}\right).
$$
(27)

2. *For* $p \in (2/3, 1]$ *and* $\varepsilon \in (0, 1)$, *with exponential weight functions* $(w_{\exp,j})_{j\geq 1}$ *defined in* (10) *with* $\alpha_{\exp} > 2K \sup_{j\geq 1}\{b_j\}$ *the error is bounded by*

$$
\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}(G(u)) - Q_{s,N}^{\mathbf{\Delta}}(G(u^{s,h}))|^2)} \leq C\left((\varphi(N))^{-1/p + 1/2} + \sup_{j>s}\left\{b_j^{1-\varepsilon}\right\} + h^{\tau+\tau'}\right).
$$
(28)

The constant C in the error bounds (27) *and* (28) *is independent of N, s, and h.*

Note that $(\varphi(N))^{-1} \leq N^{-1} \cdot (e^{\hat{\gamma}} \log \log N + 3/\log \log N)$, for every $N \geq 3$, where $\hat{\gamma} \approx 0.5772$ is the Euler–Mascheroni constant.

Proof. By the definition of the QMC points in (8), $\{\boldsymbol{y}^{(0)}, \ldots, \boldsymbol{y}^{(N-1)}\}$ are identically $\mathcal{N}(0,1)$-distributed. We observe that by the triangle inequality, for every square integrable function F with respect to the s-dimensional normal distribution with covariance being the identity,

$$
\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|Q_{s,N}^{\mathbf{\Delta}}(F)|^2)} \leq \frac{1}{N}\sum_{i=0}^{N-1}\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|F(\boldsymbol{y}^{(i)})|^2)} = \sqrt{\int_{\mathbb{R}^s}|F(y)|^2\prod_{j\in\{1:s\}}\phi(y_j)\mathrm{d}\boldsymbol{y}}.
$$

Thus, by the triangle inequality,

$$
\begin{aligned}
\sqrt{\mathbb{E}^{\mathbf{\Delta}}(|\mathbb{E}(G(u)) - Q_{s,N}^{\mathbf{\Delta}}(G(u^{s,h}))|^2)} &\leq |\mathbb{E}(G(u)) - I_s(G(u^s))| \\
&\quad + \sqrt{\mathbb{E}^{\mathbf{\Delta}}(|I_s(G(u^s)) - Q_{s,N}^{\mathbf{\Delta}}(G(u^s))|^2)} \\
&\quad + \|G(u^s) - G(u^{s,h})\|_{L^2(\Omega)}.
\end{aligned}
$$

The assertion now follows with Proposition 7, Proposition 15, and by Theorem 11 for Gaussian weight functions and respectively by Theorem 13 for exponential weight functions. $\square$

9 Multiresolution representation of Gaussian random fields

In the polyhedral domain D, cp. the assumption (**A2**), consider an isotropic multiresolution analysis (MRA) $\mathbf{\Psi} = \{\psi_\lambda : \lambda \in \nabla\}$ whose members ψ_λ are indexed by $\lambda \in \nabla$, and are obtained from one or from a finite number of generating elements ψ by translation and scaling, i.e.,

$$
\psi_\lambda(x) = 2^{d|\lambda|/2}\psi(2^{|\lambda|}x - k), \quad k \in \nabla_{|\lambda|},
$$
(29)

where the index set $\nabla_{|\lambda|}$ is of cardinality $\mathcal{O}(2^{d|\lambda|})$, and where $\mathrm{diam}\,\mathrm{supp}(\psi_\lambda) = \mathcal{O}(2^{-|\lambda|})$. The scaling in (29) by the factor $2^{d|\lambda|/2}$ refers to a normalization in $L^2(D)$, i.e., $\|\psi_\lambda\|_{L^2(D)} \sim \|\psi\|_{L^2(D)}$,

$\lambda \in \nabla$. For suitable, sufficiently smooth families of wavelets it can be shown that for every $q \in [1, \infty)$ and every $t \geq 0$, there exists a constant C such that

$$\left\| \sum_{\lambda \in \nabla} c_\lambda \psi_\lambda \right\|_{B^t_{q,q}(D)} \leq C \left(\sum_{\ell \geq 0} 2^{tq\ell} 2^{(q/2-1)d\ell} \sum_{k \in \nabla_\ell} |c_{\ell,k}|^q \right)^{1/q}, \tag{30}$$

cp. for example for the case of orthonormal wavelets [28, Theorem 4.23]. However, in this manuscript we adopt for the $(\psi_\lambda)_{\lambda \in \nabla}$ a pointwise normalization, such that for some $\widehat{\alpha} > 0$ and $\sigma > 0$ at our disposal,

$$\|\psi_\lambda\|_{L^\infty(D)} \simeq \sigma 2^{-\widehat{\alpha}|\lambda|}, \quad \lambda \in \nabla. \tag{31}$$

With the scaling (31), the norm estimate in (30) then reads that for every $q \in [1, \infty)$ and every $t \geq 0$, there exists a constant C such that

$$\left\| \sum_{\lambda \in \nabla} c_\lambda \psi_\lambda \right\|_{B^t_{q,q}(D)} \leq C \left(\sum_{\ell \geq 0} 2^{tq\ell} 2^{-(d+\widehat{\alpha}q)\ell} \sum_{k \in \nabla_\ell} |c_{\ell,k}|^q \right)^{1/q}. \tag{A4}$$

We assume that there exists a suitable enumeration of elements of the index set ∇, i.e., a bijective mapping $j : \nabla \to \mathbb{N}$, which we denote by $j(\lambda)$, $\lambda \in \nabla$, such that $|j^{-1}(s_1)| \leq |j^{-1}(s_2)|$ for positive integers $s_1 \leq s_2$. The amount of overlap of the supports at refinement level $|\lambda|$ is assumed to be bounded by an absolute multiple M times $2^{-|\lambda|}$ such that

$$|\{\lambda \in \nabla : |\lambda| = \ell, \psi_\lambda(x) \neq 0\}| \leq M, \quad \text{for all} \quad x \in D, \ell \geq 0.$$

For given $\widehat{\alpha} > 0$ we define the sequence $(b_j)_{j \geq 1}$ for $\widehat{\beta} < \widehat{\alpha}$ by

$$b_{j(\lambda)} = b_\lambda := 2^{-\widehat{\beta}|\lambda|}, \quad \lambda \in \nabla. \tag{32}$$

We observe that $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$. This sequence satisfies (A1), i.e.,

$$\left\| \sum_{\lambda \in \nabla} \frac{|\psi_\lambda|}{b_\lambda} \right\|_{L^\infty(D)} \leq \left\| \sum_{\ell \geq 0} \sum_{k \in \nabla_\ell} \frac{|\psi_{\ell,k}|}{b_{\ell,k}} \right\|_{L^\infty(D)} \leq \sigma M \sum_{\ell \geq 0} 2^{-(\widehat{\alpha}-\widehat{\beta})\ell} = \frac{\sigma M}{1 - 2^{-(\widehat{\alpha}-\widehat{\beta})}} < \infty.$$

Proposition 18 *Let* $(\psi_j)_{j \geq 1}$ *satisfy the scaling in* (31) *for some* $\widehat{\alpha} > 0$ *and let* (**A4**) *hold. For every* $t \in (0, \widehat{\alpha})$ *and every* $q \in [1, \infty)$, $Z \in L^q(\Omega; C^t(\overline{D}))$ *and for every* $\varepsilon \in (0, \widehat{\alpha} - t)$ *there exists a constant* C *such that for every* $s \in \mathbb{N}$,

$$\|Z - Z^s\|_{L^q(\Omega; C^t(\overline{D}))} \leq C \sup_{\ell \geq |j^{-1}(s)|} \left\{ 2^{-(\widehat{\alpha}-t-\varepsilon)\ell} \right\}.$$

Proof. A sequence $(b_j)_{j \geq 1}$ can be defined by (32) for some $0 < \widehat{\beta} < \widehat{\alpha}$. Since $b_j \sim j^{-\widehat{\beta}/d}$, $j \geq 1$, $(b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$, for every $p > d/\widehat{\beta}$. Hence, by Theorem 2, $Z = \lim_{s' \to \infty} Z^{s'}$ with convergence in the $L^q(\Omega; L^\infty(D))$-norm, which equals the $L^q(\Omega; C^0(\overline{D}))$-norm. Since $(\psi_j)_{j \geq 1}$ are continuous on $\overline{D}$, $Z \in L^q(\Omega; C^0(\overline{D}))$.

Let $t' \in (t, t + \varepsilon)$. We set $q' := \lceil q/2 \rceil$ such that $2q' = 2\lceil q/2 \rceil$ is the smallest even natural number that is greater or equal than q and pick $r \in \mathbb{N}$ such that $r \geq q'$ and such that $r > d/(2(t'-t))$, which implies that $t' - d/(2r) > t$. By the continuous embedding $B^{t'}_{2r,2r}(D) \subset C^t(\overline{D})$ using $t' - d/(2r) > t$, cp. [27, Theorem 1.107], $\psi_j \in C^t(\overline{D})$, $j \geq 1$. Since $Z^{s'}$ is separably valued in $C^t(\overline{D})$, it is strongly measurable in $C^t(\overline{D})$ by Pettis' theorem (e.g. cp. [29, Theorem V.4]) for

every $s' \geq 1$ (arguing as in the proof of Theorem 2). Also by the same embedding and (**A4**) it follows similarly as in the proof of Theorem 2 that

$$\|Z^{s'} - Z^s\|^{2r}_{L^{2r}(\Omega;C^t(\overline{D}))} \leq C \sum_{j(\ell,k) \in \{s+1:s'\}} 2^{-(\widehat{\alpha}-t')2r\ell} 2^{-d\ell} \mathbb{E}(|y_{\ell,k}|^{2r})$$

$$\leq C' \sum_{\ell \geq |j^{-1}(s)|} 2^{-(\widehat{\alpha}-t')2r\ell} \frac{(2r)!}{2^r r!}$$

$$\leq C' \frac{(2r)!}{2^r r!} \sup_{\ell \geq |j^{-1}(s)|} \left\{ 2^{-(\widehat{\alpha}-t-\varepsilon)2r\ell} \right\} \sum_{\ell \geq 0} 2^{-(t+\varepsilon-t')2r\ell} < \infty,$$

where we used that the $\#(\nabla_\ell) = \mathcal{O}(2^{d\ell})$. Note that $(y_{\ell,k} : \ell \geq 0, k \in \nabla_\ell)$ is a sequence of i.i.d. $\mathcal{N}(0,1)$-distributed random variables. We observe that $(Z^{s'})_{s' \geq 1}$ is a Cauchy sequence in $L^q(\Omega;C^t(\overline{D}))$ with limit $\widetilde{Z}$. Since limits in $L^q(\Omega;C^0(\overline{D}))$ are unique and since the embedding $C^t(\overline{D}) \subset C^0(\overline{D})$ is continuous, $Z = \widetilde{Z}$. □

The following proposition will give conditions for a class of systems $(\psi_j)_{j \geq 1}$ such that the resulting lognormal random fields satisfy the assumption in (**A3**).

Proposition 19 *Let* $(\psi_j)_{j \geq 1}$ *satisfy the scaling in* (31) *for some* $\widehat{\alpha} > 0$ *and let* (**A4**) *hold. For every* $t \in (0,\widehat{\alpha})$ *and every* $q \in [1,\infty)$, $a \in L^q(\Omega;C^t(\overline{D}))$ *and there exists a constant* C *such that for every* $s \in \mathbb{N}$,

$$\|a^s\|_{L^q(\Omega;C^t(\overline{D}))} \leq C.$$

Proof. Without loss of generality, let us assume that $t \notin \mathbb{Z}$. Hölder norms of compositions with the exponential function have been estimated in [15, Lemma 2.13]. We recall from its proof the following estimate [15, (15)]: there exists a constant C such that for every $v \in C^t(\overline{D})$,

$$\|\exp(v)\|_{C^t(\overline{D})} \leq C \|\exp(v)\|_{C^0(\overline{D})} \left(1 + \|v\|^{\lceil t \rceil}_{C^t(\overline{D})} \right).$$

This estimate follows by induction (cp. the proof of [15, Lemma 2.13]) based on the facts that $\|\exp(v)\|_{C^{t'}(\overline{D})} \leq \|\exp(v)\|_{C^0(\overline{D})}(1 + \|v\|_{C^{t'}(\overline{D})})$ for every $v \in C^{t'}(\overline{D})$, $t' \in (0,1)$, and that there exists a constant C' such that for every $w,v \in C^t(\overline{D})$, $\|wv\|_{C^t(\overline{D})} \leq C' \|w\|_{C^t(\overline{D})} \|w\|_{C^t(\overline{D})}$. The first estimate is easily seen, the second estimate is for example due to [5, Theorem 16.28].

The assertion follows now with an application of the Cauchy–Schwarz inequality, Proposition 3, and Proposition 18, where the strong measurability of $a = \exp(Z)$ in $C^t(\overline{D})$ follows since the composition with the exponential function is a continuous mapping from $C^t(\overline{D})$ to $C^t(\overline{D})$. □

10 Conclusions and generalizations

We extended and refined the QMC error analysis for the parametric, deterministic solutions of the linear elliptic partial differential equation (1) with log-Gaussian coefficient a as given in (2), (3). In particular, we considered locally supported functions $(\psi_j)_{j \geq 1}$ in (2). The assumed local support of the ψ_j and p-summability implied dimension-independent convergence rates of randomly shifted lattice rule quadratures of the parametric solution of (1) - (3) *with product weight sequences*. The present results constitute an extension of the convergence rate bounds in [13], wherein the global supports of the $(\psi_j)_{j \geq 1}$ implied POD weight sequences for the QMC quadratures. As a byproduct of the present analysis, we also obtained dimension-independent convergence rate estimates for globally supported $(\psi_j)_{j \geq 1}$ as in [13], and with exponential weight

functions. The use of product weights, however, entails stronger summability conditions on $(\|\psi_j\|_{L^\infty(D)})_{j\geq 1}$ than those in [13] in order to achieve a certain convergence rate (Corollary 14, item 2.). This drawback may be offset by the linear with respect to dimension s scaling construction cost for the QMC quadrature rules. In the case of Gaussian weight functions in the norm (9), however, in Theorem 11, $(b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for $2/3 < p < 2$ was admissible, for locally supported ψ_j, which constitutes a refinement over the error bounds in [13].

The present work addressed only the *single-level* QMC Finite-Element algorithm, where the same FE space is employed for PDE discretization in all QMC points. The principal results of the present paper, Theorems 11 and 13, allow for *multi-level* extensions of the presently proposed algorithms, which can be designed and analyzed along the lines of [17]. These extensions, as well as detailed numerical experiments, will be developed in the forthcoming work [10].

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