

# Numerical Analysis of Lognormal Diffusions on the Sphere

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# NUMERICAL ANALYSIS OF LOGNORMAL DIFFUSIONS ON THE SPHERE

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ABSTRACT. Numerical solutions of stationary diffusion equations on the unit sphere with isotropic lognormal diffusion coefficients are considered. Hölder regularity in  $L^p$  sense for isotropic Gaussian random fields is obtained and related to the regularity of the driving lognormal coefficients. This yields regularity in  $L^p$  sense of the solution to the diffusion problem in Sobolev spaces. Convergence rate estimates of multilevel Monte Carlo Finite and Spectral Element discretizations of these problems are then deduced. Specifically, a convergence analysis is provided with convergence rate estimates in terms of the number of Monte Carlo samples of the solution to the considered diffusion equation and in terms of the total number of degrees of freedom of the spatial discretization, and with bounds for the total work required by the algorithm in the case of Finite Element discretizations. The obtained convergence rates are solely in terms of the decay of the angular power spectrum of the (logarithm) of the diffusion coefficient. Numerical examples confirm the presented theory.

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## 1. INTRODUCTION

In the present paper, we are concerned with the existence, regularity, and approximation of solutions of elliptic partial differential equations (PDEs for short) with stochastic coefficients on the unit sphere  $\mathbb{S}^2$ . In particular, we are interested in PDEs with *isotropic lognormal random field* coefficients  $a$ , i.e.,  $T = \log a$  is an isotropic Gaussian random field (iGRF for short) on  $\mathbb{S}^2$ . For a given smooth, deterministic source term  $f$ , and for a positive random field  $a$  taking values in  $C^0(\mathbb{S}^2)$ , we consider the stochastic elliptic problem

$$(1) \quad -\nabla_{\mathbb{S}^2} \cdot (a \nabla_{\mathbb{S}^2} u) = f \quad \text{on } \mathbb{S}^2.$$

Since  $\partial\mathbb{S}^2 = \emptyset$  (as boundary of a manifold), no boundary conditions are required for the well-posedness of (1). The regularity and integrability of solutions in terms of the random field  $a$  as well as error and convergence rate analysis of Finite Element and Spectral Galerkin discretizations on  $\mathbb{S}^2$  combined with multilevel Monte Carlo (MLMC for short) sampling are the purpose of the present paper.

While the combined Finite Element MLMC discretization of PDEs with random input data has received considerable attention in recent years (see, for example, [6, 15] and the survey [13] originating from Heinrich [19]), the invariance properties of the particular geometry  $\mathbb{S}^2$  entail several specific consequences in the numerical analysis which allow more precise convergence results. Specifically, as we showed in [27, 20], the geometric setting of  $\mathbb{S}^2$  allows for an essentially sharp characterization of Hölder regularity exponents of realizations of  $a$  in terms of the angular power spectrum of the Karhunen–Loève expansion of the Gaussian random field  $T = \log a$ . Furthermore,  $\partial\mathbb{S}^2 = \emptyset$  implies the absence of corner singularities. We are therefore able to obtain elliptic regularity estimates in Sobolev scales, cp. [17], as well as Schauder estimates of classical elliptic regularity theory as presented for example in [12] and elaborated in detail for the presently considered PDE (1) in [20]. Based on these we derive explicit convergence rate bounds of discretizations of (1). Particularly, we obtain convergence rates with respect to the mesh width of Finite Element discretizations and to the spectral degree of Spectral Galerkin discretizations on  $\mathbb{S}^2$  solely in terms of the decay of the angular power spectrum of the Gaussian random field  $T = \log a$ . These convergence rates are, in the Finite Element case, bounded by the polynomial degree of the basis functions. We confine our error analysis to sufficiently smooth source terms  $f$  in (1), which yields that the lack of smoothness of solutions is caused by the roughness of the lognormal random coefficients  $a$ .

Throughout the paper, we employ standard notation. We denote in particular by  $H^s(\mathbb{S}^2)$  Sobolev spaces of square integrable functions of (not necessarily integer) order  $s$  on  $\mathbb{S}^2$ . By  $\nabla_{\mathbb{S}^2}$ ,  $\nabla_{\mathbb{S}^2} \cdot$ , and by  $\Delta_{\mathbb{S}^2} = \nabla_{\mathbb{S}^2} \cdot \nabla_{\mathbb{S}^2}$  we denote the spherical gradient, the spherical divergence, and the Laplace–Beltrami operator on  $\mathbb{S}^2$ , respectively.

The outline of the paper is as follows: In Section 2 we recapitulate basic properties of iGRFs from [29, 5]. We introduce standard notation and classical results from the differential geometry of surfaces as required in the ensuing developments. We also review results on the Hölder regularity of realizations of the random field from our earlier work [27], and relate the Hölder exponent to the angular power spectrum. We develop Hölder regularity here in the  $L^p$  sense. In Section 3 we review and establish basic results on existence, uniqueness, integrability, and regularity of solutions to the stochastic partial differential equation (SPDE for short) (1). In Section 4 we present isoparametric Finite Element (FE for short) discretizations of the SPDE (1) on  $\mathbb{S}^2$  and establish a priori estimates on their convergence. Particular attention is given to the dependence of the convergence rate on the Hölder regularity of the

random field  $a$ . In Section 4, we prove convergence rate estimates for two families of discretizations of (1). Section 4.1 is devoted to the analysis of Finite Element discretizations, while Section 4.2 to the convergence analysis of Spectral Galerkin discretizations. In Section 5 we address the convergence of multilevel Monte Carlo methods for either variant of the Galerkin discretizations. Numerical examples that confirm the presented theory are presented in Section 6. Finally, some lengthy proofs are given in the appendix.

## 2. ISOTROPIC GAUSSIAN AND LOGNORMAL RANDOM FIELDS

In this section we introduce isotropic Gaussian random fields on the unit sphere  $\mathbb{S}^2$  and their properties. We focus in particular on Karhunen–Loève expansions of these random fields and their regularity in terms of Hölder continuity and  $L^p$  integrability. Furthermore, similar results are presented for spectral approximations as well as the corresponding lognormal random fields. The section is based on results from [29] and [27] and follows closely the master’s thesis [20] of one of the authors.

Let the unit sphere  $\mathbb{S}^2$  in  $\mathbb{R}^3$  be given by

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3, \|x\|_{\mathbb{R}^3} = 1\},$$

where  $\|\cdot\|_{\mathbb{R}^3}$  denotes the Euclidean norm on  $\mathbb{R}^3$ . Consider the compact metric space  $(\mathbb{S}^2, d)$  with geodesic metric given by

$$d(x, x') := \arccos(\langle x, x' \rangle_{\mathbb{R}^3})$$

for every  $x, x' \in \mathbb{S}^2$ , where  $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$  denotes the corresponding Euclidean inner product. Furthermore, let  $(\Omega, \mathcal{A}, \mathbb{P})$  be a probability space and  $T$  a 2-weakly iGRF on  $\mathbb{S}^2$ . Then, by [29, Theorem 5.13],  $T$  admits an expansion with respect to the *surface spherical harmonic functions*  $\mathcal{Y} := (Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$  as mappings  $Y_{\ell m} : [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{C}$ , which are given by

$$Y_{\ell m}(\vartheta, \varphi) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \vartheta) e^{im\varphi}$$

for  $\ell \in \mathbb{N}_0$ ,  $m = 0, \dots, \ell$ , and by

$$Y_{\ell m} := (-1)^m \overline{Y_{\ell - m}}$$

for  $\ell \in \mathbb{N}$  and  $m = -\ell, \dots, -1$ . Here  $(P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$  denote the *associated Legendre functions* which are given by

$$P_{\ell m}(\rho) := (-1)^m (1 - \rho^2)^{m/2} \frac{\partial^m}{\partial \rho^m} P_{\ell}(\rho)$$

for  $\ell \in \mathbb{N}_0$ ,  $m = 0, \dots, \ell$ , and  $\rho \in [-1, 1]$ , where  $(P_{\ell}, \ell \in \mathbb{N}_0)$  are the *Legendre polynomials* given by Rodrigues’ formula (see, e.g., [37])

$$P_{\ell}(\rho) := 2^{-\ell} \frac{1}{\ell!} \frac{\partial^{\ell}}{\partial \rho^{\ell}} (\rho^2 - 1)^{\ell}$$

for all  $\ell \in \mathbb{N}_0$  and  $\rho \in [-1, 1]$ . This expansion of  $T$  converges in  $L^2(\Omega \times \mathbb{S}^2)$  as well as for every  $x \in \mathbb{S}^2$  in  $L^2(\Omega)$  and is given by (see, e.g., [27, Corollary 2.5])

$$T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m},$$

where  $\mathbb{A} := (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$  is a sequence of complex-valued, centered, Gaussian random variables with the following properties:

- (1)  $\mathbb{A}_+ := (a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$  is a sequence of independent, complex-valued Gaussian random variables.
- (2) The elements of  $\mathbb{A}_+$  with  $m > 0$  satisfy  $\operatorname{Re} a_{\ell m}$  and  $\operatorname{Im} a_{\ell m}$  are independent and  $\mathcal{N}(0, A_\ell/2)$  distributed.
- (3) The elements of  $\mathbb{A}_+$  with  $m = 0$  are real-valued and the elements  $\operatorname{Re} a_{\ell 0}$  are  $\mathcal{N}(0, A_\ell)$  distributed for  $\ell \in \mathbb{N}$  while  $\operatorname{Re} a_{00}$  is  $\mathcal{N}(\mathbb{E}(T)2\sqrt{\pi}, A_0)$  distributed.
- (4) The elements of  $\mathbb{A}$  with  $m < 0$  are deduced from those of  $\mathbb{A}_+$  by the formulae

$$\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell -m}, \quad \operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell -m}.$$

Here  $(A_\ell, \ell \in \mathbb{N}_0)$  is called the *angular power spectrum*.

In what follows we set  $Y_{\ell m}(y) := Y_{\ell m}(\vartheta, \varphi)$  for  $y \in \mathbb{S}^2$ , where we identify (with a slight abuse of notation) Cartesian and angular coordinates by  $y := (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ , and we do not separate indices for doubly sub- or superscripted functions and coefficients by a comma, with the understanding that the reader will recognize double indices as such. Furthermore, we denote by  $\sigma$  the *Lebesgue measure on the sphere* which admits the representation

$$d\sigma(y) = \sin \vartheta \, d\vartheta \, d\varphi$$

for  $y \in \mathbb{S}^2$ ,  $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ .

We define the *spherical Laplacian*, also called *Laplace–Beltrami operator*, in terms of spherical coordinates similarly to [29, Section 3.4.3] by

$$\Delta_{\mathbb{S}^2} := (\sin \vartheta)^{-1} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + (\sin \vartheta)^{-2} \frac{\partial^2}{\partial \varphi^2}.$$

It is well-known (see, e.g., [30, Theorem 2.13]) that the spherical harmonic functions  $\mathcal{Y}$  are the eigenfunctions of  $-\Delta_{\mathbb{S}^2}$  with eigenvalues  $(\ell(\ell+1), \ell \in \mathbb{N}_0)$ , i.e.,

$$-\Delta_{\mathbb{S}^2} Y_{\ell m} = \ell(\ell+1) Y_{\ell m}$$

for all  $\ell \in \mathbb{N}_0$ ,  $m = -\ell, \dots, \ell$ . Furthermore, it is shown in [30, Theorem 2.42] that  $L^2(\mathbb{S}^2; \mathbb{C})$  has the direct sum decomposition

$$L^2(\mathbb{S}^2; \mathbb{C}) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^2),$$

where the spaces  $(\mathcal{H}_\ell, \ell \in \mathbb{N}_0)$  are spanned by spherical harmonic functions

$$\mathcal{H}_\ell(\mathbb{S}^2) := \operatorname{span}\{Y_{\ell m}, m = -\ell, \dots, \ell\},$$

i.e.,  $\mathcal{H}_\ell(\mathbb{S}^2)$  denotes the space of eigenfunctions of  $-\Delta_{\mathbb{S}^2}$  that correspond to the eigenvalue  $\ell(\ell+1)$  for  $\ell \in \mathbb{N}_0$ . Let us denote by  $L^2(\mathbb{S}^2)$  the subspace of all real-valued functions of  $L^2(\mathbb{S}^2; \mathbb{C})$ . Then, every real-valued function  $f$  in  $L^2(\mathbb{S}^2; \mathbb{C})$  admits a spherical harmonics series expansion

$$(2) \quad f = \sum_{\ell=0}^{+\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m},$$

and the coefficients satisfy (cp., e.g., [29, Remark 3.37])

$$f_{\ell m} = (-1)^m \overline{f_{\ell -m}},$$

i.e.,  $f$  can be represented in  $L^2(\mathbb{S}^2)$  by the series expansion

$$(3) \quad f = \sum_{\ell=0}^{+\infty} \left( f_{\ell 0} Y_{\ell 0} + 2 \sum_{m=1}^{\ell} (\operatorname{Re} f_{\ell m} \operatorname{Re} Y_{\ell m} - \operatorname{Im} f_{\ell m} \operatorname{Im} Y_{\ell m}) \right).$$

We shall be partly concerned with spectral approximations by truncation of the spherical harmonics expansion (2). To state results on convergence rates of such truncations, we introduce for any truncation levels  $L_1 < L_2 \in \mathbb{N}_0$  the spaces

$$(4) \quad \mathcal{H}_{L_1:L_2} := \bigoplus_{\ell=L_1}^{L_2} \mathcal{H}_{\ell} \subset L^2(\mathbb{S}^2; \mathbb{C})$$

and identify  $\mathcal{H}_{L:L} := \mathcal{H}_L$  for any  $L \in \mathbb{N}_0$ . Evidently,  $\mathcal{H}_{0:L}$  is a space of finite dimension that satisfies for  $L \in \mathbb{N}$  that

$$(5) \quad L^2 \leq N_L := \dim(\mathcal{H}_{0:L}) = (L+1)^2 \leq 4L^2,$$

and thus, in particular, is closed. For  $L \in \mathbb{N}_0$ , we denote by  $\Pi_L : L^2(\mathbb{S}^2; \mathbb{C}) \rightarrow \mathcal{H}_{0:L}$  the projector on  $\mathcal{H}_{0:L}$  given by the truncated Karhunen–Loève series (2), i.e., for  $f \in L^2(\mathbb{S}^2; \mathbb{C})$ ,

$$(6) \quad \Pi_L f := \sum_{\ell=0}^L \sum_{|m| \leq \ell} f_{\ell m} Y_{\ell m}.$$

To characterize the decay of the coefficients in the expansion (2) and, accordingly, also convergence rates of the projections  $\Pi_L$  in (6), we introduce for a smoothness index  $s \in \mathbb{R}$  and  $q \in (1, +\infty)$  the Sobolev spaces on  $\mathbb{S}^2$  as

$$H_q^s(\mathbb{S}^2) := (\operatorname{Id} - \Delta_{\mathbb{S}^2})^{-s/2} L^q(\mathbb{S}^2).$$

Then, for every  $f \in H_q^s(\mathbb{S}^2)$ ,

$$\|f\|_{H_q^s(\mathbb{S}^2)} := \|(\operatorname{Id} - \Delta_{\mathbb{S}^2})^{s/2} f\|_{L^q(\mathbb{S}^2)}$$

defines a norm on  $H_q^s(\mathbb{S}^2)$ , where for  $s < 0$ , the elements of  $H_q^s(\mathbb{S}^2)$  have to be understood as distributions (cp. [36, Definition 4.1]). The positive definiteness of this norm is implied by [38, Theorem XI.2.5]. For more details on these spaces, we refer the reader to [36, 38]. In the case  $q = 2$  we omit  $q$  in our notation and simply write  $H^s(\mathbb{S}^2)$ . In this setting  $H^0(\mathbb{S}^2) := L^2(\mathbb{S}^2)$  is identified with its dual space  $H^0(\mathbb{S}^2)^*$  and  $H^s(\mathbb{S}^2)^* = H^{-s}(\mathbb{S}^2)$  for every  $s > 0$ . Since the norm on  $H_q^s(\mathbb{S}^2)$  is well-defined for every  $s \in \mathbb{R}$  and every  $q \in (1, +\infty)$ , we obtain that

$$(7) \quad (\operatorname{Id} - \Delta_{\mathbb{S}^2})^{s/2} : H_q^t(\mathbb{S}^2) \rightarrow H_q^{t-s}(\mathbb{S}^2)$$

is bounded and surjective for every  $t \in \mathbb{R}$ .

Since  $\mathcal{Y}$  diagonalizes  $-\Delta_{\mathbb{S}^2}$  and therefore

$$(8) \quad (\operatorname{Id} - \Delta_{\mathbb{S}^2})^{s/2} Y_{\ell m} = (1 + \ell(\ell + 1))^{s/2} Y_{\ell m}$$

for every  $Y_{\ell m} \in \mathcal{Y}$  by the spectral mapping theorem, cp. [32, Theorem 10.33(a)] applied to the bounded inverse of  $(\operatorname{Id} - \Delta_{\mathbb{S}^2})$  on  $L^2(\mathbb{S}^2)$ , we obtain the following approximation result of the operator  $(\operatorname{Id} - \Pi_L)$ .

**Proposition 2.1.** *For every  $-\infty < s \leq t < +\infty$  and for every  $f \in H^t(\mathbb{S}^2)$ ,*

$$\|f - \Pi_L f\|_{H^s(\mathbb{S}^2)} \leq L^{-(t-s)} \|f\|_{H^t(\mathbb{S}^2)} \leq 2^{t-s} N_L^{-(t-s)/2} \|f\|_{H^t(\mathbb{S}^2)}$$

for every  $L \in \mathbb{N}_0$ .

*Proof.* Let  $-\infty < s \leq t < +\infty$ , and  $f \in H^t(\mathbb{S}^2)$ . Then, for  $L \in \mathbb{N}$ , it holds by (2) and (8) that

$$\begin{aligned} \|f - \Pi_L f\|_{H^s(\mathbb{S}^2)}^2 &= \sum_{\ell=L+1}^{+\infty} \sum_{m=-\ell}^{\ell} |f_{\ell m}|^2 (1 + \ell(\ell + 1))^s \\ &\leq \sum_{\ell=L+1}^{+\infty} \sum_{m=-\ell}^{\ell} |f_{\ell m}|^2 (1 + \ell(\ell + 1))^s \left( \frac{\ell(\ell + 1)}{L(L + 1)} \right)^{t-s} \\ &\leq L^{-2(t-s)} \sum_{\ell=L+1}^{+\infty} \sum_{m=-\ell}^{\ell} |f_{\ell m}|^2 (1 + \ell(\ell + 1))^t \leq L^{-2(t-s)} \|f\|_{H^t(\mathbb{S}^2)}^2. \end{aligned}$$

The relation between  $L$  and  $N_L$  in (5) implies the assertion.  $\square$

Let us next introduce Hölder spaces on  $\mathbb{S}^2$ . For  $\iota \in \mathbb{N}_0$ , we denote by  $C^\iota(\mathbb{S}^2)$  the space of  $\iota$ -times continuously differentiable functions taking values in  $\mathbb{R}$  and, for  $\gamma \in (0, 1)$ , by  $C^{\iota, \gamma}(\mathbb{S}^2) \subset C^\iota(\mathbb{S}^2)$  the subspace of functions whose  $\iota$ -th derivative is Hölder continuous with exponent  $\gamma$ . We identify  $C^{\iota, 0}(\mathbb{S}^2)$  with  $C^\iota(\mathbb{S}^2)$ . The Hölder spaces satisfy the Sobolev embedding that

$$H_q^s(\mathbb{S}^2) \subset C^{\iota, \gamma}(\mathbb{S}^2)$$

is continuously embedded for  $s - 2/q \geq \iota + \gamma$ ,  $\gamma \neq 0$ , which is stated for  $\mathbb{S}^2$  in Theorem A.2.

As final functional analytical ingredient, we need  $L^p$  spaces on the probability space with values in a Banach space to consider integrability of iGRFs as Hölder-space-valued random variables. Therefore, let  $(B, \|\cdot\|_B)$  denote a Banach space. For  $p \in [1, +\infty)$ , the *Bochner space*  $L^p(\Omega; B)$  consists of all strongly  $B$ -measurable functions  $X : \Omega \rightarrow B$  such that  $\|X\|_B$  is in  $L^p(\Omega)$ , i.e.,

$$\|X\|_{L^p(\Omega; B)}^p := \left( \mathbb{E}(\|X\|_B^p) \right) < +\infty.$$

Then  $(L^p(\Omega; B), \|\cdot\|_{L^p(\Omega; B)})$  is a Banach space by [9, Theorem III.6.6]. To connect the already introduced convergence of Karhunen–Loève expansions of iGRF with Bochner spaces, we observe that  $L^2(\Omega \times \mathbb{S}^2)$  and  $L^2(\Omega; L^2(\mathbb{S}^2))$  are isometrically isomorphic, i.e., the Karhunen–Loève expansion also converges in  $L^2(\Omega; L^2(\mathbb{S}^2))$  and  $\|T\|_{L^2(\Omega; L^2(\mathbb{S}^2))}^2 = \sum_{\ell=0}^{+\infty} A_\ell \frac{2\ell+1}{4\pi}$  is finite. For more details on the functional analytical setting and measurability, the reader is referred to Appendix A.

Let us now return to the isotropic Gaussian random field  $T$  and assume from here on that

$$(9) \quad \sum_{\ell=0}^{+\infty} A_\ell \ell^{1+\beta} < +\infty$$

for some  $\beta > 0$ . It was shown in [27, Theorem 4.6] that this condition yields the existence of a modification of  $T$  that is in  $C^{\iota, \gamma}(\mathbb{S}^2)$  for all  $\iota + \gamma < \beta/2$ , which we consider from now on without loss of generality. The purpose of the following theorem is to show strong measurability,  $L^p$ -integrability, and approximation of this iGRF. We remark that the proof of the theorem just requires a continuous modification of  $T$ , which exists by [27, Theorem 4.5], and therefore recovers [27, Theorem 4.6] from [27, Theorem 4.5] with a possibly different modification. This follows since  $L^p$  integrability holds only if  $T \in C^{\iota, \gamma}(\mathbb{S}^2)$   $\mathbb{P}$ -a.s..

**Theorem 2.2.** *Let  $T$  be a continuous iGRF that satisfies (9) for some  $\beta > 0$ . Then, for every  $p \in [1, +\infty)$ ,  $\iota \in \mathbb{N}_0$ , and  $\gamma \in (0, 1)$  with  $\iota + \gamma < \beta/2$ , it holds that  $T \in L^p(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))$ .*

Furthermore, there exists a constant  $C_{p,\iota,\gamma}$ , which is independent of  $(A_\ell, \ell \in \mathbb{N}_0)$  such that for every  $L \in \mathbb{N}_0$ ,

$$\|T - \Pi_L T\|_{L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))} \leq C_{p,\iota,\gamma} \left( \sum_{\ell > L} A_\ell \ell^{1+\beta} \right)^{1/2}.$$

*Proof.* It suffices to prove the theorem for  $p$  even, i.e., for  $p = 2p'$  and  $p' \in \mathbb{N}$ . The result for all remaining  $p \in [1, +\infty)$  follows then by Hölder's inequality. We set  $T^L := \Pi_L T$  and show first that  $(T^L, L \in \mathbb{N}_0)$  is a Cauchy sequence in  $L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$ . The smoothness of the spherical harmonics implies with Pettis' theorem (see Theorem A.4) that  $T^L$  is strongly measurable in every function space that contains  $C^\infty(\mathbb{S}^2)$ ,  $L \in \mathbb{N}_0$ . In particular,  $T^L$  is strongly  $B$ -measurable,  $B \in \{C^{\iota,\gamma}(\mathbb{S}^2), H_q^{\beta/2}(\mathbb{S}^2), q \in (1, +\infty)\}$  for every  $L \in \mathbb{N}_0$ . With the identity  $\sum_{|m| \leq \ell} |Y_{\ell m}(x)|^2 = (2\ell + 1)/(4\pi)$  (cp. [31, Theorem 2.4.5]) and the Karhunen–Loève expansion, we observe that  $\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x)$  is  $\mathcal{N}(0, (2\ell + 1)/(4\pi) A_\ell)$ -distributed for every  $\ell \in \mathbb{N}$  and  $x \in \mathbb{S}^2$  as well as that  $(\sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(x), \ell \in \mathbb{N}_0)$  is a sequence of independent random variables for every fixed  $x \in \mathbb{S}^2$ . Hence, for  $L_1 > L_2 \in \mathbb{N}_0$ , we obtain that

$$\begin{aligned} \|T^{L_1} - T^{L_2}\|_{L^{2p'}(\Omega; H_{2p'}^{\beta/2}(\mathbb{S}^2))}^{2p'} &= \int_{\mathbb{S}^2} \mathbb{E} \left( \left( \sum_{\ell=L_2+1}^{L_1} \sum_{m=-\ell}^{\ell} a_{\ell m} (1 + \ell(\ell + 1))^{\beta/4} Y_{\ell m} \right)^{2p'} \right) d\sigma \\ &= \frac{(2p')!}{2^{p'} p'!} |\mathbb{S}^2| \left( \sum_{\ell=L_2+1}^{L_1} A_\ell \frac{2\ell + 1}{4\pi} (1 + \ell(\ell + 1))^{\beta/2} \right)^{p'} < +\infty, \end{aligned}$$

where we applied Fubini's theorem and the fact that moments of centered Gaussian random variables satisfy  $\mathbb{E}(X^{2p'}) = (2p')!/(2^{p'} p'!) \mathbb{E}(X^2)^{p'}$ . Finiteness follows since (9) holds.

This implies especially with the Sobolev embedding (cp. Theorem A.2) that there exists a constant  $C$  such that

$$\|T^{L_1} - T^{L_2}\|_{L^{2p'}(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))} \leq C \left( 4\pi \frac{(2p')!}{2^{p'} p'!} \right)^{1/(2p')} \left( \sum_{\ell=L_2+1}^{L_1} A_\ell \frac{2\ell + 1}{4\pi} (1 + \ell(\ell + 1))^{\beta/2} \right)^{1/2}$$

for  $\beta/2 - 1/p' \geq \iota + \gamma$  and therefore that  $(T^L, L \in \mathbb{N}_0)$  is a Cauchy sequence in  $L^{2p'}(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$  that converges due to completeness. Furthermore, the result extends by Hölder's inequality to  $L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$  for every  $p \leq 2p'$ . Since  $L^p$  limits are  $\mathbb{P}$ -almost surely unique and we know by the properties of the Karhunen–Loève expansion that  $(T^L, L \in \mathbb{N}_0)$  converges to  $T$  in  $L^2(\Omega; L^2(\mathbb{S}^2))$ ,  $T \in L^{2p'}(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$  holds also due to the assumed continuity.

For given  $p \geq 1$ , we choose  $p' \in \mathbb{N}$  such that  $p \leq 2p'$  and  $\beta/2 - 1/p' \geq \iota + \gamma$  for fixed  $\iota$  and  $\gamma$ . This implies that there exists a constant  $C_{p'}$ , i.e.,  $C_{p,\iota,\gamma}$ , such that

$$\|T^{L_1} - T^{L_2}\|_{L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))} \leq C_{p,\iota,\gamma} \left( \sum_{\ell=L_2+1}^{L_1} A_\ell \ell^{1+\beta} \right)^{1/2}.$$

We obtain the claim by taking the limit  $L_1 \rightarrow +\infty$ .  $\square$

Let us continue with the properties of the corresponding isotropic lognormal random fields  $a := \exp(T)$  given by  $a(x) := \exp(T(x))$  for every  $x \in \mathbb{S}^2$ . These will be of interest as diffusion coefficients of the elliptic operators in our considered SPDEs. For the approximation of these lognormal random fields, we set similarly  $a^L := \exp(\Pi_L T)$  for every  $L \in \mathbb{N}_0$ . Then, the



properties of  $T$  and  $T^L$  shown in Theorem 2.2 imply similar results for  $a$  and  $a^L$ , which are stated in the following theorem.

**Theorem 2.3.** *Let  $a = \exp(T)$  be an isotropic lognormal RF such that  $T$  is a continuous iGRF and satisfies (9) for some  $\beta > 0$ . Then, for every  $p \in [1, +\infty)$ ,  $\iota \in \mathbb{N}_0$ , and for  $\gamma \in (0, 1)$  satisfying  $\iota + \gamma < \beta/2$ , and for every  $L \in \mathbb{N}_0$ , it holds that  $a, a^L \in L^p(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))$ , where the  $L^p(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))$ -norm of  $a^L$  can be bounded independently of  $L$  and the same stays true for  $a, a^L \in L^p(\Omega; C^0(\mathbb{S}^2))$ . Furthermore, for every  $\varepsilon \in (0, \beta)$ , there exists a constant  $C_{p, \varepsilon}$  such that for every  $L \in \mathbb{N}_0$ , it holds that*

$$\|a - a^L\|_{L^p(\Omega; C^0(\mathbb{S}^2))} \leq C_{p, \varepsilon} \left( \sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \right)^{\frac{1}{2}}.$$

*Proof.* We observe first that the composition with the exponential function is a continuous mapping from  $C^{\iota, \gamma}(\mathbb{S}^2)$  into itself and  $T$  is strongly  $C^{\iota, \gamma}(\mathbb{S}^2)$ -measurable by Theorem 2.2. Then, the inequality

$$\|\exp(v)\|_{C^{\iota, \gamma}(\mathbb{S}^2)} \leq C_{\iota, \gamma} \|\exp(v)\|_{C^0(\mathbb{S}^2)} (1 + \|v\|_{C^{\iota, \gamma}(\mathbb{S}^2)}^{\iota+1}),$$

which follows in a similar way as the proof of [23, Theorem A.8] and which is proven in Lemma A.1, implies strong  $C^{\iota, \gamma}(\mathbb{S}^2)$ -measurability of  $a = \exp(T)$  and of  $a^L = \exp(\Pi_L T)$  for every  $L \in \mathbb{N}_0$ . The Cauchy–Schwarz inequality then implies that there exists a constant  $C$  that does not depend on  $T$  such that

$$\|a\|_{L^p(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))} = \|\exp(T)\|_{L^p(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))} \leq C \|\exp(T)\|_{L^{2p}(\Omega; C^0(\mathbb{S}^2))} (1 + \|T\|_{L^{2p(\iota+1)}(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))}^{\iota+1}).$$

The second term in the product is bounded by Theorem 2.2, while the boundedness of the first one is a consequence of Fernique’s theorem, which is proven in a similar way as [5, Proposition 3.10] and can be found for iGRFs on  $\mathbb{S}^2$  in Proposition B.1.

The second assertion about  $a^L$  is proven completely analogously and the  $L^p(\Omega; C^{\iota, \gamma}(\mathbb{S}^2))$ -norm of  $a^L$  can be bounded independently of  $L$  due to Theorem 2.2 and the independence of  $L$  in the  $L^p(\Omega; C^0(\mathbb{S}^2))$ -norm, which is also part of Proposition B.1.

For the proof of the third claim, note that the fundamental theorem of calculus implies for arbitrary  $t, s \in \mathbb{R}$  that  $|\exp(t) - \exp(s)| \leq (\exp(t) + \exp(s))|t - s|$ , which yields with the Cauchy–Schwarz inequality that

$$\|a - a^L\|_{L^p(\Omega; C^0(\mathbb{S}^2))} \leq (\|a\|_{L^{2p}(\Omega; C^0(\mathbb{S}^2))} + \|a^L\|_{L^{2p}(\Omega; C^0(\mathbb{S}^2))}) \|T - \Pi_L T\|_{L^{2p}(\Omega; C^0(\mathbb{S}^2))}.$$

Therefore, the third assertion follows with Theorem 2.2.  $\square$

In the following and especially in the analysis of (1), the properties of the minimum and the maximum of a random field are of major interest. Therefore, we define for  $a = \exp(T)$ , where  $T$  is a continuous iGRF  $T$ , the random variables

$$\hat{a} := \max_{x \in \mathbb{S}^2} a(x) \quad \text{and} \quad \check{a} := \min_{x \in \mathbb{S}^2} a(x),$$

and similarly for  $L \in \mathbb{N}_0$

$$\hat{a}^L := \max_{x \in \mathbb{S}^2} a^L(x) \quad \text{and} \quad \check{a}^L := \min_{x \in \mathbb{S}^2} a^L(x).$$

Here we recall that  $a^L = \exp(\Pi_L T)$ . Since

$$\|\check{a}^{-1}\|_{L^p(\Omega)} = \left\| \left( \min_{x \in \mathbb{S}^2} a(x) \right)^{-1} \right\|_{L^p(\Omega)} = \left\| \max_{x \in \mathbb{S}^2} \exp(-T(x)) \right\|_{L^p(\Omega)} = \|\exp(-T)\|_{L^p(\Omega; C^0(\mathbb{S}^2))}$$

and

$$\|\hat{a}\|_{L^p(\Omega)} = \|\exp(T)\|_{L^p(\Omega; C^0(\mathbb{S}^2))},$$

these are elements of  $L^p(\Omega)$ ,  $p \in [1, +\infty)$  by Theorem 2.3, which is summarized in the following corollary.

**Corollary 2.4.** *Let  $T$  be a continuous iGRF, then  $\hat{a}$ ,  $\check{a}^{-1}$ ,  $\hat{a}^L$ , and  $(\check{a}^L)^{-1}$  are in  $L^p(\Omega)$  for every  $p \in [1, +\infty)$  and every  $L \in \mathbb{N}_0$ , where the  $L^p(\Omega)$ -norm of  $\hat{a}^L$  and  $(\check{a}^L)^{-1}$  can be bounded independently of  $L$ .*

### 3. EXISTENCE, UNIQUENESS, AND REGULARITY OF SOLUTIONS

Having introduced the analytic and approximation properties of the random source of interest, we are now in state to come back to the SPDE of interest

$$(1) \quad -\nabla_{\mathbb{S}^2} \cdot (a \nabla_{\mathbb{S}^2} u) = f,$$

where  $a = \exp(T)$  is an isotropic lognormal random field such that the iGRF  $T$  is continuous and satisfies (9) for some  $\beta > 0$  and  $f$  is a deterministic source term which has at least  $H^{-1}(\mathbb{S}^2)$  regularity.

In what follows we first introduce the variational framework in which we consider solutions before we show existence, uniqueness, and regularity of solutions where the latter depends on the regularity of  $a$  and  $f$ . We derive similar results for the SPDEs corresponding to the approximate random fields  $a^L$ .

We observe first that solutions of the SPDE on the closed, compact submanifold  $\mathbb{S}^2$  of  $\mathbb{R}^3$  without boundary may exhibit nonuniqueness since  $-\nabla_{\mathbb{S}^2} \cdot (a \nabla_{\mathbb{S}^2})$  might have a nontrivial kernel, i.e., a constant  $u$  is a solution of the homogeneous equation.

Therefore, we shall work in factor spaces of function spaces which are orthogonal (in  $L^2(\mathbb{S}^2)$ ) to constants. The closed subspace of  $H^1(\mathbb{S}^2)$  that consists of all  $v \in H^1(\mathbb{S}^2)$  whose inner product with 1 satisfies  $(v, 1) = 0$  is denoted by  $H^1(\mathbb{S}^2)/\mathbb{R}$ . For every  $v \in H^1(\mathbb{S}^2)/\mathbb{R}$ ,

$$\|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}} := \|\nabla_{\mathbb{S}^2} v\|_{L^2(\mathbb{S}^2)}$$

defines a norm on  $H^1(\mathbb{S}^2)/\mathbb{R}$  due to the second Poincaré inequality

$$\|v\|_{L^2(\mathbb{S}^2)} \leq \frac{1}{\sqrt{2}} \|\nabla_{\mathbb{S}^2} v\|_{L^2(\mathbb{S}^2)},$$

which is proven considering the Reyleigh quotient and the spectrum of  $-\Delta_{\mathbb{S}^2}$  (for details, see [20, Lemma 8.3]). Since  $H^1(\mathbb{S}^2)/\mathbb{R}$  is a closed linear subspace of  $H^1(\mathbb{S}^2)$  and the norm  $\|\cdot\|_{H^1(\mathbb{S}^2)/\mathbb{R}}$  is induced by the inner product  $(\nabla_{\mathbb{S}^2} \cdot, \nabla_{\mathbb{S}^2} \cdot)$ ,  $H^1(\mathbb{S}^2)/\mathbb{R}$  is a Hilbert space.

Let us consider the variational formulation of (1) in  $H^1(\mathbb{S}^2)/\mathbb{R}$  with right hand side  $f \in H^{-1}(\mathbb{S}^2)$  such that  $f(1) = 0$ : find a strongly  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurable mapping  $u$  such that

$$(10) \quad (a \nabla_{\mathbb{S}^2} u, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}.$$

Moreover, we want to show that this mapping  $u : \Omega \rightarrow H^1(\mathbb{S}^2)/\mathbb{R}$  is  $L^p$ -integrable. To this end, let us fix this right hand side  $f$ . In what follows let us first recall the deterministic existence and uniqueness theory and derive the results in such a form that they are suitable for the stochastic framework. These will then be applied to (10). Therefore, let

$$C_+^0(\mathbb{S}^2) := \{\tilde{a} \in C^0(\mathbb{S}^2), \min_{x \in \mathbb{S}^2} \tilde{a}(x) > 0\}.$$

and consider the corresponding deterministic variational problem for  $\tilde{a} \in C_+^0(\mathbb{S}^2)$  with right hand side  $f \in H^{-1}(\mathbb{S}^2)$  such that  $f(1) = 0$ : find  $u \in H^1(\mathbb{S}^2)/\mathbb{R}$  such that

$$(11) \quad (\tilde{a}\nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2}v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}.$$

Since the bilinear form  $(\tilde{a}\nabla_{\mathbb{S}^2}\cdot, \nabla_{\mathbb{S}^2}\cdot)$  is continuous and coercive on the space  $H^1(\mathbb{S}^2)/\mathbb{R} \times H^1(\mathbb{S}^2)/\mathbb{R}$ , i.e.,

$$(12) \quad (\tilde{a}\nabla_{\mathbb{S}^2}v, \nabla_{\mathbb{S}^2}w) \leq \|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}}\|w\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \quad \forall v, w \in H^1(\mathbb{S}^2)/\mathbb{R}$$

and

$$(13) \quad \|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}}^2 \leq \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\tilde{a}\nabla_{\mathbb{S}^2}v, \nabla_{\mathbb{S}^2}v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R},$$

existence and uniqueness of a solution  $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$  to (11) as well as the estimate

$$(14) \quad \|\tilde{u}\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)},$$

are implied by the Lax–Milgram lemma, where we used that

$$\sup_{0 \neq v \in H^1(\mathbb{S}^2)/\mathbb{R}} |f(v)|/\|v\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)}.$$

The difference of two solutions with respect to different coefficients  $\tilde{a}$  and the same right hand side  $f$  can be estimated with a version of Strang’s second lemma. This is made precise in the following lemma, where the variational formulation (11) is also considered with respect to subspaces of  $H^1(\mathbb{S}^2)/\mathbb{R}$  to be suitable for approximations in Section 4. The proof for  $H^1(\mathbb{S}^2)/\mathbb{R}$  can be found in [20, Proposition 8.6] (with a different norm on  $f$ ) which also applies for proper, closed subspaces of  $H^1(\mathbb{S}^2)/\mathbb{R}$ .

**Lemma 3.1.** *Let  $V \subset H^1(\mathbb{S}^2)/\mathbb{R}$  be a closed, not necessarily strict subspace of  $H^1(\mathbb{S}^2)/\mathbb{R}$  endowed with the  $H^1(\mathbb{S}^2)/\mathbb{R}$ -norm. For  $\tilde{a}_1, \tilde{a}_2 \in C_+^0(\mathbb{S}^2)$ , let  $\tilde{u}_1, \tilde{u}_2 \in V$  satisfy*

$$(\tilde{a}_i \nabla_{\mathbb{S}^2} \tilde{u}_i, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in V$$

for  $i = 1, 2$ . Then,

$$\|\tilde{u}_1 - \tilde{u}_2\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \sqrt{\frac{3}{2}} \frac{\|f\|_{H^{-1}(\mathbb{S}^2)}}{(\min_{x \in \mathbb{S}^2} \tilde{a}_1(x))(\min_{x \in \mathbb{S}^2} \tilde{a}_2(x))} \|\tilde{a}_1 - \tilde{a}_2\|_{C^0(\mathbb{S}^2)}.$$

Let us denote the solution map that maps the coefficient  $\tilde{a} \in C_+^0(\mathbb{S}^2)$  to the respective unique solution  $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$  of (11) by

$$(15) \quad \Phi_f : C_+^0(\mathbb{S}^2) \rightarrow H^1(\mathbb{S}^2)/\mathbb{R},$$

then we obtain the following proposition as a direct consequence of the previous lemma.

**Proposition 3.2.**  $\Phi_f : C_+^0(\mathbb{S}^2) \rightarrow H^1(\mathbb{S}^2)/\mathbb{R}$  is continuous.

We now state the well-posedness of the weak formulation of the SPDE (10).

**Theorem 3.3.** *Let  $a = \exp(T)$  be an isotropic lognormal RF such that the iGRF  $T$  is continuous and satisfies (9) for some  $\beta > 0$ . Then, there exists  $u \in L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$  for every  $p \in [1, +\infty)$  such that  $u$  is in this sense the unique solution of (10).*

*Proof.* Since  $a$  takes values in  $C_+^0(\mathbb{S}^2)$ , we set  $u := \Phi_f(a)$ , which solves (10) uniquely. The continuity of  $\Phi_f$  in Proposition 3.2 implies strong  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurability (cp. Lemma A.5) and  $L^p$ -integrability follows with (14) and Corollary 2.4.  $\square$

Since the computation of the random coefficient  $a = \exp(T)$  does not seem to be feasible in general due to the infinite Karhunen–Loève expansion of  $T$ , we consider solutions with respect to the coefficients  $(a^L, L \in \mathbb{N}_0)$  in what follows and analyze the convergence of the resulting sequence of solutions in  $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ ,  $p \in [1, +\infty)$ . For every  $L \in \mathbb{N}_0$ , we consider the variational problem: find a strongly  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurable mapping  $u^L$  such that

$$(16) \quad (a^L \nabla_{\mathbb{S}^2} u^L, \nabla_{\mathbb{S}^2} v) = f(v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}.$$

This is a special case of Theorem 3.3, which implies existence, uniqueness, and  $L^p$ -integrability of a solution  $u^L$ . It is clear from Corollary 2.4 that the  $L^p$ -norm can be bounded uniformly in  $L$ . We state the result for further use in the following corollary.

**Corollary 3.4.** *Let the assumptions of Theorem 3.3 be satisfied. For every  $L \in \mathbb{N}_0$ , there exists a unique  $u^L$  such that  $u^L$  solves (16) as well as its  $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ -norm is finite for every  $p \in [1, +\infty)$  and can be bounded uniformly in  $L$ .*

We conclude the part on existence and uniqueness of solutions with a convergence result that the sequence of solutions  $(u^L, L \in \mathbb{N}_0)$  of (16) converges in  $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$  to the solution  $u$  of (10).

**Proposition 3.5.** *Let the assumptions of Theorem 3.3 be satisfied. Furthermore, let  $u$  be the unique solution of (10) and  $(u^L, L \in \mathbb{N}_0)$  be the sequence of unique solutions of (16). Then, for every  $p \in [1, +\infty)$  and  $\varepsilon \in (0, \beta)$ , there exists a constant  $C_{p,\varepsilon}$  such that for every  $L \in \mathbb{N}_0$ , it holds that*

$$\|u - u^L\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_{p,\varepsilon} \left( \sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \right)^{1/2}.$$

*Proof.* For every  $L \in \mathbb{N}_0$ , a twofold application of Hölder's inequality implies the claim with Lemma 3.1, Corollary 2.4, Theorem 2.3, and (14), i.e., there exists a constant  $C_{p,\varepsilon}$  such that for every  $L \in \mathbb{N}_0$ , it holds that

$$\begin{aligned} \|u - u^L\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} &\leq \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)} \|1/\tilde{a}\|_{L^{3p}(\Omega)} \|1/\tilde{a}^L\|_{L^{3p}(\Omega)} \|a - a^L\|_{L^{3p}(\Omega; C^0(\mathbb{S}^2))} \\ &\leq C_{p,\varepsilon} \left( \sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \right)^{1/2}. \end{aligned} \quad \square$$

Since the goal of this manuscript is to derive high order approximations of the solution  $u$  of (10) with Finite Element and Spectral Methods, higher order regularity of  $u$  is essential. In what follows we show that  $u$  takes values in  $H^{1+s}(\mathbb{S}^2)$  for  $s > 0$  such that the range of  $s$  is only limited by the regularity of  $a$  and the right hand side  $f$ . As before we first consider the regularity of the solution  $\tilde{u}$  of the deterministic problem (11) in terms of the solution map (15) before applying it to the stochastic framework. We remark that the domain of  $\Phi_f$  reflects the regularity of the coefficient  $\tilde{a}$  while the range of  $\Phi_f$  reflects the regularity of the respective solution  $\tilde{u}$ .

**Proposition 3.6.** *Let  $\iota \in \mathbb{N}_0$ ,  $\gamma \in (0, 1)$ , and  $s \in [0, +\infty)$  satisfy  $s < \iota + \gamma$ . If  $f \in H^{-1+s}(\mathbb{S}^2)$ , then*

$$\Phi_f : C^{\iota,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2) \rightarrow H^{1+s}(\mathbb{S}^2)$$

is continuous with respect to the topology of  $C^{\iota,\gamma}(\mathbb{S}^2)$ .

Moreover the  $H^{1+s}(\mathbb{S}^2)$ -norm can be bounded by the following recursion. For  $s < 1$ , it holds that

$$\|\Phi_f(\tilde{a})\|_{H^{1+s}(\mathbb{S}^2)} \leq C \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)} \|1/\tilde{a}\|_{C^0(\mathbb{S}^2)}^2 \|f\|_{H^{-1+s}(\mathbb{S}^2)}.$$

If  $s \geq 1$ , then for every  $n \in \{0, \dots, \lfloor s \rfloor - 1\}$ , there exists a constant  $C > 0$  such that for every  $\tilde{a} \in C^{\iota,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$ ,

$$\begin{aligned} & \|\Phi_f(\tilde{a})\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \\ & \leq C \|1/\tilde{a}\|_{C^{n,\gamma}(\mathbb{S}^2)} (\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a})\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}) \end{aligned}$$

where  $\{s\}$  denotes the fractional part of  $s$ .

While the base case for  $s < 0$  is proven by the translation of results on domains in Euclidean space in [6], higher order regularity is shown by induction with a perturbation argument. The detailed proof can be found in Appendix C.

The proposition transfers to the stochastic framework and enables us to prove the main result of this section to obtain higher order approximations in the following Section 4.

**Theorem 3.7.** *Let  $a = \exp(T)$  be an isotropic lognormal RF such that the iGRF  $T$  is continuous and satisfies (9) for some  $\beta > 0$ . Furthermore, let  $u$  be the solution of (10) and  $(u^L, L \in \mathbb{N}_0)$  be the sequence of solutions of (16). Then, for every  $s \in [0, \beta/2)$  and  $L \in \mathbb{N}_0$ , it holds that  $u, u^L \in L^p(\Omega; H^{1+s}(\mathbb{S}^2))$  for every  $p \in [1, +\infty)$ , if  $f \in H^{-1+s}(\mathbb{S}^2)$ . Moreover the  $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$ -norm of  $u^L$  can be bounded uniformly in  $L$ .*

*Proof.* Let us write  $s = \lfloor s \rfloor + \{s\}$ , where  $\{s\} \in [0, 1)$  is the fractional part of  $s$ , and then set  $\iota := \lfloor s \rfloor \in \mathbb{N}_0$  and choose  $\gamma \in (\{s\}, \min\{\beta/2 - \iota, 1\})$ , which implies that  $s < \iota + \gamma$ . We deduce that  $a, a^L \in L^{p'}(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$  for every  $L \in \mathbb{N}_0, p' \in [1, +\infty)$ , from Theorem 2.3. In particular, these RFs are strongly  $C^{\iota,\gamma}(\mathbb{S}^2)$ -measurable and positive. Hence, by the continuity of the solution map  $\Phi_f$  from Proposition 3.6, the mappings  $u = \Phi_f(a)$  and  $u^L = \Phi_f(a^L)$  are strongly  $H^{1+s}$ -measurable for every  $L \in \mathbb{N}_0$  (cp. Lemma A.5).

The boundedness of the  $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$ -norm will be proved inductively. As a base case we apply the base case estimate of the  $H^{1+\{s\}}(\mathbb{S}^2)$ -norm of  $u$  from Proposition 3.6 and use the Cauchy–Schwarz inequality to obtain that there exists a constant  $C > 0$  (independent of  $u, a$ , and  $f$ ) such that

$$\|u\|_{L^p(\Omega; H^{1+\{s\}}(\mathbb{S}^2))} \leq C \|f\|_{H^{-1+\{s\}}(\mathbb{S}^2)} \|a\|_{L^{2p}(\Omega; C^{0,\gamma}(\mathbb{S}^2))} \|\tilde{a}^{-2}\|_{L^{2p}(\Omega)}.$$

We infer from Theorem 2.3 and Corollary 2.4 that the right hand side of the previous inequality is finite. Let us assume as induction hypothesis that the  $L^p(\Omega; H^{1+n+\{s\}}(\mathbb{S}^2))$ -norm of  $u$  is finite for every  $n \in \{0, 1, \dots, \lfloor s \rfloor - 1\}$ , which we just established for  $n = 0$ . Let  $n \in \{0, 1, \dots, \lfloor s \rfloor - 1\}$  and let us apply the recursion formula on the  $H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$ -norm from Proposition 3.6 and apply the Cauchy–Schwarz inequality twice to obtain that there exists a constant  $C$  that is independent of  $u, a$ , and  $f$  such that

$$\begin{aligned} & \|u\|_{L^p(\Omega; H^{1+(n+1)+\{s\}}(\mathbb{S}^2))} \\ & \leq C \|1/a\|_{L^{3p}(\Omega; C^{n,\gamma}(\mathbb{S}^2))} (\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} + \|a\|_{L^{3p}(\Omega; C^{n+1,\gamma}(\mathbb{S}^2))} \|u\|_{L^{3p}(\Omega; H^{1+n+\{s\}}(\mathbb{S}^2))}). \end{aligned}$$

Since  $1/a = \exp(-T)$ , Theorem 2.3 is applicable to  $-T$ , which satisfies (9) in the same way as  $T$  does. Hence, the  $L^{3p}(\Omega; C^{n,\gamma}(\mathbb{S}^2))$ -norm of  $1/a$  is finite. The induction hypothesis, Theorem 2.3, and Corollary 2.4 imply that the right hand side of the previous inequality is finite.

This completes the induction. We conclude that the  $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$ -norm of  $u$  is finite. The proof for  $u^L$ ,  $L \in \mathbb{N}_0$ , is analogous. The uniform boundedness of the  $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$ -norm of  $u^L$  in  $L \in \mathbb{N}_0$  is implied by Theorem 2.3 and Corollary 2.4.  $\square$

#### 4. DISCRETIZATION

**4.1. Finite Element Methods.** In Proposition 3.5 we analyzed the error that occurs when we consider the solution  $u^L = \Phi_f(a^L)$  to the SPDE (10) with respect to the approximate isotropic lognormal RF  $a^L = \exp(\Pi_L T)$  for  $L \in \mathbb{N}_0$ , where  $a^L = \exp(\Pi_L T)$  can be simulated via the truncated Karhunen–Loève expansion of the iGRF  $T$  for every  $L \in \mathbb{N}_0$ . In this section we aim at a spatial discretization to numerically simulate realizations of  $u^L$ ,  $L \in \mathbb{N}_0$ , with a *Galerkin* Finite Element Method and analyze the error in the  $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ -norm for  $p \in [1, +\infty)$ .

We review basic results on the deterministic theory of FEs on  $\mathbb{S}^2$  as required in the ensuing analysis. FEs on surfaces to approximate solutions of elliptic PDEs appear to have been first introduced in [10]. There, first order convergence estimates are obtained using affine approximations of the surface. Higher order estimates are shown in [8], where also an FE Method is defined on the surface so as to avoid a surface approximation error. We refer to [8, Section 2.6] for details.

Given a regular, quasiuniform triangulation  $\mathcal{T}$  of  $\mathbb{S}^2$  into parametric, curvilinear triangles  $K \in \mathcal{T}$  of mesh width  $h > 0$  (which we indicate by tagging  $\mathcal{T}$  with the subscript  $h$ , i.e., by writing  $\mathcal{T}_h$ ), we define  $S^k(\mathbb{S}^2, \mathcal{T}_h)$  to be the space of continuous, piecewise parametric polynomials of degree  $k \geq 1$  on the triangulation  $\mathcal{T}_h$  of  $\mathbb{S}^2$  and equip it with the  $H^1(\mathbb{S}^2)$ -norm. To approximate functions in  $H^1(\mathbb{S}^2)/\mathbb{R}$  we define the subspace of  $S^k(\mathbb{S}^2, \mathcal{T}_h)$  of functions that have zero average, i.e.,

$$V^{h,k} := \{v^h \in S^k(\mathbb{S}^2, \mathcal{T}_h), (v^h, 1) = 0\}.$$

Then,  $V^{h,k} \subset H^1(\mathbb{S}^2)/\mathbb{R}$  and we equip it with the  $H^1(\mathbb{S}^2)/\mathbb{R}$ -norm. The FE spaces  $S^k(\mathbb{S}^2, \mathcal{T}_h)$  and  $V^{h,k}$ ,  $h > 0$ , are of finite dimension such that  $\dim(S^k(\mathbb{S}^2, \mathcal{T}_h)) = \dim(V^{h,k}) + 1$ . Also it holds that the degrees of freedom  $N_h := \dim(V^{h,k}) = \mathcal{O}(h^{-2})$  as  $h \rightarrow 0$  for fixed polynomial degree  $k \in \mathbb{N}$ . We refer to [33, Chapter 4] for details and remark that we will tag elements of  $V^{h,k}$  respectively  $S^k(\mathbb{S}^2, \mathcal{T}_h)$  only with the mesh width  $h$  keeping in mind that they implicitly also depend on the polynomial degree  $k$  of the FE space, i.e., let  $v^h \in V^{h,k}$ .

For every  $\tilde{a} \in C_+^0(\mathbb{S}^2)$ ,  $h > 0$ , and  $k \in \mathbb{N}$ , we consider the variational formulation of the deterministic, elliptic PDE (11) over the finite dimensional space  $V^{h,k}$ : find a *Galerkin FE solution*  $\tilde{u}^h \in V^{h,k}$  such that

$$(17) \quad (\tilde{a} \nabla_{\mathbb{S}^2} \tilde{u}^h, \nabla_{\mathbb{S}^2} v^h) = f(v^h) \quad \forall v^h \in V^{h,k}.$$

The conformity of the FE Method, i.e.,  $V^{h,k} \subset H^1(\mathbb{S}^2)/\mathbb{R}$ , implies with (12) and (13) that the bilinear form  $(\tilde{a} \nabla_{\mathbb{S}^2} \cdot, \nabla_{\mathbb{S}^2} \cdot)$  on  $V^{h,k} \times V^{h,k}$  is continuous and coercive with coercivity constant  $(\min_{x \in \mathbb{S}^2} \tilde{a}(x))^{-1}$  which is independent of  $h$  and of  $k$ .

Hence, by the Lax–Milgram lemma, the Galerkin approximation  $\tilde{u}^h \in V^{h,k}$  exists and is the unique solution of (17). Also  $\tilde{u}^h$  satisfies the estimate in (14) uniformly in  $h > 0$ , i.e.,

$$(18) \quad \|\tilde{u}^h\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)}.$$

As in the previous section we introduce a solution mapping  $\Phi_f^{h,k}$  that maps the coefficient  $\tilde{a} \in C_+^0(\mathbb{S}^2)$  to the respective unique Galerkin FE solution  $\tilde{u}^h \in V^{h,k}$  by

$$\Phi_f^{h,k} : C_+^0(\mathbb{S}^2) \rightarrow V^{h,k}.$$

Continuity follows as in Proposition 3.2 with Lemma 3.1 and is stated in the following proposition.

**Proposition 4.1.**  $\Phi_f^{h,k} : C_+^0(\mathbb{S}^2) \rightarrow V^{h,k}$  is continuous for every  $h > 0$  and  $k \in \mathbb{N}$ .

Functions in  $H^{1+s}(\mathbb{S}^2)$  and in particular solutions to (11) can be approximated in  $S^k(\mathbb{S}^2, \mathcal{T}_h)$ ,  $s, h > 0$ , and  $k \in \mathbb{N}$ , cp. [8, Proposition 2.7]. We will phrase this in terms of the solution mappings  $\Phi_f$  and  $\Phi_f^{h,k}$ ,  $h > 0$  and  $k \in \mathbb{N}$ , in the following proposition. The proof uses this well-known approximation property of  $S^k(\mathbb{S}^2, \mathcal{T}_h)$ ,  $h > 0$  and  $k \in \mathbb{N}$ , in  $H^1(\mathbb{S}^2)$  in combination with C ea's lemma. For details, we refer the reader to Appendix D.

**Proposition 4.2.** Let  $k \in \mathbb{N}$  be the polynomial degree of the FE spaces  $V^{h,k}$ ,  $h > 0$ , and let  $\iota \in \mathbb{N}_0$  and  $\gamma \in (0, 1)$ . For every  $s \in (0, \iota + \gamma)$  such that  $f \in H^{-1+s}(\mathbb{S}^2)$ , there exists a constant  $C_s$  such that for every  $h > 0$  and every  $\tilde{a} \in C^{\iota, \gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$ , it holds that

$$\|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq C_s \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \|\Phi_f(\tilde{a})\|_{H^{1+s}(\mathbb{S}^2)} h^{\min\{s, k\}}.$$

Since the mappings  $\Phi_f^{h,k}$ ,  $h > 0, k \in \mathbb{N}$ , are continuous due to Proposition 4.1, the introduced theory on Galerkin FE Methods is applicable to our stochastic framework. Indeed, for every  $L \in \mathbb{N}_0$ ,  $h > 0$ , and  $k \in \mathbb{N}$ , the problem to find a strongly  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurable  $u^{L,h}$  such that

$$(19) \quad (a^L \nabla_{\mathbb{S}^2} u^{L,h}, \nabla_{\mathbb{S}^2} v^h) = f(v^h) \quad \forall v^h \in V^{h,k}$$

admits a unique solution by setting  $u^{L,h} := \Phi_f^{h,k}(a^L)$ , where we omit  $k$  in our notation of the solution. The strong  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurability of  $u^{L,h}$  follows from the strong  $C^0(\mathbb{S}^2)$ -measurability of  $a^L$  and the continuity of  $\Phi_f^{h,k}$  with Lemma A.5. Moreover Corollary 2.4 implies with (18) that for every  $p \in [1, +\infty)$ , there exists a constant  $C_p$  such that for every  $L \in \mathbb{N}_0$  and every  $h > 0$ , it holds that

$$(20) \quad \|u^{L,h}\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq \|1/\tilde{a}^L\|_{L^p(\Omega)} \sqrt{\frac{3}{2}} \|f\|_{H^{-1}(\mathbb{S}^2)} \leq C_p.$$

With the given properties of the Galerkin Finite Elements, we are now able to prove the extension of Proposition 3.5 to space discretizations.

**Theorem 4.3.** Let the assumptions of Theorem 3.7 be satisfied. Let  $u = \Phi_f(a)$  be the unique solution of (10) and for every  $h > 0$ , let  $u^{L,h} = \Phi_f^{h,k}(a^L)$  be the unique Galerkin FE solution of (19) for  $k \in \mathbb{N}$ . Then, for every  $s \in (0, \beta/2)$  such that  $f \in H^{-1+s}(\mathbb{S}^2)$  and every  $p \in [1, +\infty)$ , there exists a constant  $C_{p,s}$  such that for every  $h > 0$  and every  $L \in \mathbb{N}_0$ , it holds that

$$\|u - u^{L,h}\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_{p,s} (L^{-s} + h^{\min\{s, k\}}).$$

*Proof.* Let us set  $u^L := \Phi_f(a^L)$  for every  $L \in \mathbb{N}_0$ . A twofold application of Hölder's inequality implies with Proposition 4.2 that there exists a constant  $C_s$  such that for every  $L \in \mathbb{N}_0$  and every  $h > 0$ , it holds that

$$\|u^L - u^{L,h}\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \|a^L\|_{L^{3p}(\Omega; C^0(\mathbb{S}^2))} \|1/\check{a}^L\|_{L^{3p}(\Omega)} \|u^L\|_{L^{3p}(\Omega; H^{1+s}(\mathbb{S}^2))} h^{\min\{s,k\}}.$$

Due to Theorem 2.3, Corollary 2.4, and Theorem 3.7 there exists a constant  $\hat{C}_{p,s}$  such that for every  $L \in \mathbb{N}_0$ , it holds that

$$C_s \|a^L\|_{L^{3p}(\Omega; C^0(\mathbb{S}^2))} \|1/\check{a}^L\|_{L^{3p}(\Omega)} \|u^L\|_{L^{3p}(\Omega; H^{1+s}(\mathbb{S}^2))} \leq C_{p,s}.$$

Let  $\varepsilon := \beta - 2s \in (0, \beta)$ . We apply the triangle inequality and conclude with Proposition 3.5 that there exists a constant that we also denote by  $\hat{C}_{p,s}$  such that for every  $L \in \mathbb{N}_0$  and every  $h > 0$ , it holds that

$$\|u - u^{L,h}\|_{L^p(\Omega; H^1(\mathbb{S}^2))} \leq \hat{C}_{p,s} \left( \sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \right)^{\frac{1}{2}} + \hat{C}_{p,s} h^{\min\{s,k\}}.$$

We further bound

$$\sum_{\ell > L} A_\ell \ell^{1+\varepsilon} \leq (L^{-1})^{\beta-\varepsilon} \sum_{\ell > L} A_\ell \ell^{1+\beta} \leq (L^{-1})^{2s} \sum_{\ell \geq 0} A_\ell \ell^{1+\beta}.$$

Since  $\sum_{\ell \geq 0} A_\ell \ell^{1+\beta} < +\infty$  by assumption, we conclude the proof of the theorem.  $\square$

**4.2. Spectral Methods.** In Theorem 4.3 we established a rate of convergence for Galerkin approximations of the stochastic solution in subspaces  $V^{h,k}$  of continuous, piecewise polynomial functions on a quasiuniform triangulation  $\mathcal{T}_h$  on  $\mathbb{S}^2$ . The obtained bound for the convergence rate in Theorem 4.3 indicated an asymptotic convergence order  $N_h^{-\min\{s,k\}/2}$  as  $N_h = \dim(V^{h,k}) \rightarrow +\infty$ , i.e., the convergence rate is limited by the regularity of the solutions (as expressed in the Sobolev scale parameter  $s \geq 0$ ) and by the polynomial degree  $k \in \mathbb{N}$  of the Finite Elements used in the discretization. If, in particular, the Sobolev regularity of the solution is high, i.e., if  $s > 0$  is large, the convergence of the Galerkin FE approximations  $u^{L,h}$  defined in (19) is limited by the order  $k$  of the used Finite Elements. Spectral Elements do not have this drawback.

To introduce them, we recall the space  $\mathcal{H}_{0:L^u} \subset H^1(\mathbb{S}^2)$  spanned by spherical harmonics of order at most  $L^u$  defined in (4). Since we are interested in a conforming method, we restrict ourselves to the functions that are orthogonal to constants as in the FE case, i.e., we consider  $\mathcal{H}_{1:L^u}$  as Spectral Element spaces,  $L^u \in \mathbb{N}$ . In the following the index  $L^a$  refers to the degree of the approximation of  $a$  and  $L^u$  refers to the degree of the Spectral Element space. Its dimension is  $N_{L^u} := \dim(\mathcal{H}_{1:L^u}) = \mathcal{O}((L^u)^2)$  as  $L^u \rightarrow +\infty$ , and is also referred to as degrees of freedom. Let  $a = \exp(T)$  be an isotropic lognormal RF that results from a continuous iGRF  $T$  satisfying (9) for some  $\beta > 0$ . Similarly to (19), for every  $L^a, L^u \in \mathbb{N}_0$ , we define a Galerkin approximation as the solution of the problem to find a strongly  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurable  $u^{L^a, L^u}$  that takes values in  $\mathcal{H}_{1:L^u}$  such that

$$(a^{L^a} \nabla_{\mathbb{S}^2} u^{L^a, L^u}, \nabla_{\mathbb{S}^2} v^{L^u}) = f(v^{L^u}) \quad \forall v^{L^u} \in \mathcal{H}_{1:L^u}.$$

The coercivity of the bilinear form  $(a^{L^a} \nabla_{\mathbb{S}^2} \cdot, \nabla_{\mathbb{S}^2} \cdot)$  implies that  $u^{L^a, L^u}$  exists and is unique, since  $\mathcal{H}_{1:L^u} \subset H^1(\mathbb{S}^2)/\mathbb{R}$  is a closed subspace. Strong  $H^1(\mathbb{S}^2)/\mathbb{R}$ -measurability of  $u^{L^a, L^u}$  follows in the same way as in Section 4.1.



Let us conclude this subsection with the spectral version of Theorem 4.3 which expresses the convergence rate just in terms of the Sobolev regularity of the solution of the original problem.

**Theorem 4.4.** *Let the assumptions of Theorem 3.7 be satisfied. For every  $s \in (0, \beta/2)$  such that  $f \in H^{-1+s}(\mathbb{S}^2)$  and for every  $p \in [1, +\infty)$ , there exists a constant  $C_{p,s}$  such that for every  $L^a, L^u \in \mathbb{N}_0$ , it holds that*

$$\|u - u^{L^a, L^u}\|_{L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_{p,s} \left( (L^a)^{-s} + (L^u)^{-s} \right).$$

*Proof.* The proof is similar to that of Theorem 4.3. We use the approximation result Proposition 3.5, the quasioptimality, the regularity result Theorem 3.7, and the approximation property of  $\mathcal{H}_{0:L^u}$  in Proposition 2.1 to conclude the assertion.  $\square$

## 5. MLMC CONVERGENCE ANALYSIS

In this section we aim at approximating the expectation of the solution of (10)  $\mathbb{E}(u)$ . So far we established for FE approximations of  $u$  in Section 4.1 that the constructed double indexed sequence  $(u^{L,h}, L \in \mathbb{N}_0, h > 0)$  converges to  $u$  in  $L^p(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$  for every  $p \in [1, +\infty)$  with a particular convergence rate, cp. Theorem 4.3. The remaining part of the numerical analysis is to approximate  $\mathbb{E}(u^{L,h})$  for  $L \in \mathbb{N}_0$  and  $h > 0$  with a sampling method. To this end, we apply an MLMC estimator in order to reduce the computational cost that a conventional Monte Carlo simulation would incur.

The error analysis of MLMC discretizations is standard, by now, and our development is analogous to those carried out in [14, 2, 1]. In particular, in [14] the error from truncating a Karhunen–Loève expansion of the Gaussian random field was considered. In contrast to the situation there, we will benefit in our analysis from the knowledge of the properties of iGRFs and of the behavior of their Karhunen–Loève expansions that we developed in Section 2. This relieves us from additional assumptions on the Karhunen–Loève eigenfunctions, on the behavior of the truncated Karhunen–Loève expansion, and on the iGRF itself, apart from summability assumptions on the angular power spectrum.

We introduce the usual Monte Carlo (MC) estimator and the MLMC estimator in a general setting. Let  $(V, \|\cdot\|_V)$  be a separable Hilbert space. For every  $v \in L^2(\Omega; V)$ , let  $(\hat{v}_i, i \in \mathbb{N})$  be a sequence of independent, identically distributed random variables in  $L^2(\Omega; V)$  such that they are independent from  $v$  and have the same law as  $v$ . For every  $M \in \mathbb{N}$ , the MC estimator  $E_M(v)$  of  $v$  is then defined by

$$E_M(v) := \frac{1}{M} \sum_{i=1}^M \hat{v}_i.$$

It is well-known that for every  $v \in L^2(\Omega; V)$  and every  $M \in \mathbb{N}$ , it holds that

$$(21) \quad \|\mathbb{E}(v) - E_M(v)\|_{L^2(\Omega; V)}^2 = \frac{1}{M} \|v - \mathbb{E}(v)\|_{L^2(\Omega; V)}^2 = \frac{1}{M} (\|v\|_{L^2(\Omega; V)}^2 - \|\mathbb{E}(v)\|_V^2).$$

For every  $L^2(\Omega; V)$ -valued sequence  $(v^j, j \in \mathbb{N}_0)$ , we consider a finite telescoping sum expansion with the convention that  $v^{-1} = 0$ , i.e., for every  $J' \in \mathbb{N}_0$ , it holds that

$$v^{J'} = \sum_{j=0}^{J'} v^j - v^{j-1},$$

and define for every  $\mathbb{N}$ -valued sequence  $(M_j, j = 0, \dots, J)$ ,  $J \in \mathbb{N}_0$ , the MLMC estimator  $E^J$  of  $v^J$  by

$$(22) \quad E^J(v^J) := \sum_{j=0}^J E_{M_j}(v^j - v^{j-1})$$

such that the MC estimators  $(E_{M_j}(v^j - v^{j-1}), j = 0, \dots, J)$  are independent.

In the following lemma we express the error introduced by the MLMC estimator in terms of the errors of the approximations and the numbers of samples chosen on each level.

**Lemma 5.1.** *For every  $L^2(\Omega; V)$ -valued sequence  $(v^j, j \in \mathbb{N}_0)$  and every integer-valued  $(M_j, j = 0, \dots, J)$  sequence with finite  $J \in \mathbb{N}_0$ , the MLMC estimator  $E^J(v^J)$  satisfies that*

$$\| \mathbb{E}(v^J) - E^J(v^J) \|_{L^2(\Omega; V)}^2 = \sum_{j=0}^J \frac{1}{M_j} \left( \|v^j - v^{j-1}\|_{L^2(\Omega; V)}^2 - \| \mathbb{E}(v^j - v^{j-1}) \|_V^2 \right).$$

*Proof.* The independence of the MC estimators in (22) on the different levels  $(M_j, j = 0, \dots, J)$  and (21) imply that

$$\begin{aligned} \| \mathbb{E}(v^J) - E^J(v^J) \|_{L^2(\Omega; V)}^2 &= \left\| \sum_{j=0}^J \mathbb{E}(v^j - v^{j-1}) - E_{M_j}(v^j - v^{j-1}) \right\|_{L^2(\Omega; V)}^2 \\ &= \sum_{j=0}^J \| \mathbb{E}(v^j - v^{j-1}) - E_{M_j}(v^j - v^{j-1}) \|_{L^2(\Omega; V)}^2 \\ &= \sum_{j=0}^J \frac{1}{M_j} \left( \|v^j - v^{j-1}\|_{L^2(\Omega; V)}^2 - \| \mathbb{E}(v^j - v^{j-1}) \|_V^2 \right). \quad \square \end{aligned}$$

After having computed the error introduced by an MLMC estimator, we are now in state to compute the overall error of the full discretization in terms of the regularity of the solution, the approximation of the iGRF, the FE discretization, and the sample sizes.

**Theorem 5.2.** *Let the assumptions of Theorem 4.3 be satisfied and let  $u$  be the unique solution to (10). Consider for any increasing  $\mathbb{N}$ -valued sequence  $(L_j, j \in \mathbb{N}_0)$  and decreasing positive sequence  $(h_j, j \in \mathbb{N}_0)$  the corresponding sequence of FE solutions  $(u^{L_j, h_j}, j \in \mathbb{N}_0)$  to (19), i.e., for fixed  $k \in \mathbb{N}$  and for every  $j \in \mathbb{N}_0$ ,  $u^{L_j, h_j}$  satisfies*

$$(a^{L_j} \nabla_{\mathbb{S}^2} u^{L_j, h_j}, \nabla_{\mathbb{S}^2} v^{h_j}) = f(v^{h_j}) \quad \forall v^{h_j} \in V^{h_j, k}.$$

*Then, for every  $s \in (0, \beta/2)$ , there exists a constant  $C_s$  such that for every  $\mathbb{N}$ -valued sequence  $(M_j, j = 0, \dots, J)$ ,  $J \in \mathbb{N}_0$ , it holds that*

$$\| \mathbb{E}(u) - E^J(u^{L_j, h_j}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \left( \frac{1}{M_0} + \sum_{j=1}^J \frac{L_{j-1}^{-2s} + h_{j-1}^{2 \min\{s, k\}}}{M_j} + L_J^{-2s} + h_J^{2 \min\{s, k\}} \right)^{1/2}.$$

*Proof.* Theorem 4.3 implies that there exists a constant  $\hat{C}_s$  independent of  $(L_j, j \in \mathbb{N}_0)$ ,  $(h_j, j \in \mathbb{N}_0)$ , and  $J$  such that for every  $j = 1, \dots, J$ , it holds that

$$\begin{aligned} &\| u^{L_j, h_j} - u^{L_{j-1}, h_{j-1}} \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \\ &\leq \| u - u^{L_j, h_j} \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} + \| u - u^{L_{j-1}, h_{j-1}} \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \end{aligned}$$

$$\leq \hat{C}_s(L_j^{-s} + h_j^{\min\{s,k\}} + L_{j-1}^{-s} + h_{j-1}^{\min\{s,k\}}) \leq 2\hat{C}_s(L_{j-1}^{-s} + h_{j-1}^{\min\{s,k\}}),$$

where we apply that  $(L_j, j \in \mathbb{N}_0)$  is increasing and  $(h_j, j \in \mathbb{N}_0)$  is decreasing and recall that the elements of  $(u^{L_j, h_j}, j \in \mathbb{N}_0)$  depend on the polynomial degree  $k$  of  $V^{h,k}$ . Another implication of Theorem 4.3 is that for the same constant  $\hat{C}_s$ , it holds that

$$\|\mathbb{E}(u) - \mathbb{E}(u^{L_J, h_J})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \leq \|u - u^{L_J, h_J}\|_{L^1(\Omega; H^1(\mathbb{S}^2))} \leq \hat{C}_s(L_J^{-s} + h_J^{\min\{s,k\}}),$$

and due to (20) there exists a constant  $\hat{C}$  that is independent of  $L_0$  and  $h_0$  such that

$$\|u^{L_0, h_0}\|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq \hat{C}.$$

Hence, we conclude the claim of this theorem with the triangle inequality, Lemma 5.1, and with the elementary inequality that  $(r_1 + r_2)^2 \leq 2(r_1^2 + r_2^2)$  for every  $r_1, r_2 \in \mathbb{R}$ . Specifically, for  $C_s := 4 \max\{\hat{C}_s, \hat{C}\}$ , it holds that

$$\begin{aligned} & \| \mathbb{E}(u) - E^J(u^{L_J, h_J}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \\ & \leq \| \mathbb{E}(u) - \mathbb{E}(u^{L_J, h_J}) \|_{H^1(\mathbb{S}^2)/\mathbb{R}} + \| \mathbb{E}(u^{L_J, h_J}) - E^J(u^{L_J, h_J}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \\ & \leq \sqrt{2} \left( \hat{C}_s^2 (L_J^{-s} + h_J^{\min\{s,k\}})^2 + \frac{\hat{C}^2}{M_0} + 4\hat{C}_s^2 \sum_{j=1}^J \frac{(L_{j-1}^{-s} + h_{j-1}^{\min\{s,k\}})^2}{M_j} \right)^{1/2} \\ & \leq C_s \left( \frac{1}{M_0} + \sum_{j=1}^J \frac{(L_{j-1}^{-2s} + h_{j-1}^{2\min\{s,k\}})}{M_j} + L_J^{-2s} + h_J^{2\min\{s,k\}} \right)^{1/2}. \quad \square \end{aligned}$$

We remark that Theorem 5.2 also covers the convergence analysis of the usual Monte Carlo estimator by the choice  $J = 0$ .

It is natural to require

$$(23) \quad h_j = O(2^{-j} h_0), \quad j \in \mathbb{N}_0,$$

for some initial mesh width  $h_0 > 0$ . Generally, one attempts to equilibrate the error contributions of the approximations of the noise, in space, and of the expectation. From Theorem 4.3 or Theorem 5.2 we see that to equilibrate the error contributions from the truncation of the Karhunen–Loève expansion of the continuous iGRF and the error contribution from the Galerkin FE approximation we need to choose the increasing sequence  $(L_j, j \in \mathbb{N}_0)$  *comparably* to  $(h_j^{-1}, j \in \mathbb{N}_0)$ , i.e., there exists a constant  $C$  with  $C^{-1}h_j \leq (L_j)^{-1} \leq Ch_j$  for every  $j \in \mathbb{N}_0$ . Hence, we consider

$$(24) \quad L_j := \left\lceil \frac{h_0}{h_j} \right\rceil L_0, \quad j \in \mathbb{N}_0,$$

for some initial truncation level  $L_0 \in \mathbb{N}$ . Under our only assumption that the angular power spectrum of the continuous iGRF satisfies (9) for some  $\beta > 0$  we obtained with Theorem 3.7 that the unique solution  $u$  to (10) is in  $L^p(\Omega; H^{1+s}(\mathbb{S}^2))$  for every  $s \in [0, \beta/2)$  and every  $p \in [1, +\infty)$ . To determine the sample sizes for a given  $\beta > 0$ , we fix  $s \in (0, \beta/2)$  such that  $s \leq k$ , where  $k$  denotes the polynomial degree of the FE space. A possible choice of the sample numbers  $(M_j, j = 0, \dots, J)$ ,  $J \in \mathbb{N}_0$ , in the MLMC estimator is to equilibrate the error contributions of the MLMC estimator across the discretization levels according to

Theorem 5.2. This leads to the following choice: for a given maximal discretization level  $J \in \mathbb{N}_0$ , we set

$$(25) \quad M_0 = \lceil h_J^{-2s} \kappa \rceil \quad \text{and} \quad M_j = \left\lceil \left( \frac{h_{j-1}}{h_j} \right)^{2s} j^{1+\varepsilon} \kappa \right\rceil$$

for  $j = 1, \dots, J$ , a scaling factor  $\kappa \geq 2^{-2s}$  (allow  $\kappa > 0$  if  $J = 0$ ), and a positive constant  $\varepsilon > 0$ . If  $s > k$  we make the same choices as in (25) with  $s$  replaced by  $k$ .

**Corollary 5.3.** *Let  $J \in \mathbb{N}_0$  and  $\varepsilon > 0$  be fixed. Assume that the conditions of Theorem 5.2 are satisfied for some  $\beta > 0$  and choose  $(L_j, h_j, M_j, j = 0, \dots, J)$  according to (24), (23), and (25). Denote by  $\zeta$  the Riemann zeta function. Then, for every  $s \in (0, \beta/2)$ , there exists  $C_s > 0$  such that*

$$\| \mathbb{E}(u) - E^J(u^{L_J, h_J}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \left( \zeta(1 + \varepsilon) \frac{1}{\kappa} + 1 \right)^{1/2} h_J^{\min\{s, k\}}.$$

If for  $\eta_1 > 0$  and  $\eta_2 \geq 0$ , the work to compute one sample of  $u^{L_j, h_j}$  is comparable to  $h_j^{-2\eta_1} \log^{\eta_2}(h_j^{-2})$ ,  $j = 0, \dots, J$ , then the total work to compute  $E^J(u^{L_J, h_J})$  satisfies

$$\mathcal{W}_J = \begin{cases} \mathcal{O}(h_J^{-2\min\{s, k\}} \kappa) = \mathcal{O}(2^{2\min\{s, k\}J} \kappa) & \min\{s, k\} > \eta_1 \\ \mathcal{O}(h_J^{-2\eta_1} \max\{J, 1\}^{\eta_2+2+\varepsilon} \kappa) = \mathcal{O}(2^{2J\eta_1} \max\{J, 1\}^{\eta_2+2+\varepsilon} \kappa) & \min\{s, k\} \leq \eta_1 \end{cases},$$

where the contributions of  $h_0^{-1}, L_0, \eta_1, \eta_2$  are absorbed into the Landau symbols.

*Proof.* Let  $s_0 := \min\{s, k\}$ . The error estimate follows from the choices of the values for  $(L_j, h_j, M_j, j = 0, \dots, J)$  by Theorem 5.2, i.e., we conclude that

$$\begin{aligned} & \| \mathbb{E}(u) - E^J(u^{L_J, h_J}) \|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \\ & \leq \hat{C}_s \left( \frac{1}{M_0} + \sum_{j=1}^J \frac{L_{j-1}^{-2s} + h_{j-1}^{2s_0}}{M_j} + L_J^{-2s} + h_J^{2s_0} \right)^{1/2} \\ & \leq \hat{C}_s \left( \frac{1}{\kappa} + \frac{1}{\kappa} \left( \frac{1}{(L_0 h_0)^{2s_0}} + 1 \right) \zeta(1 + \varepsilon) + \frac{1}{(L_0 h_0)^{2s_0}} + 1 \right)^{1/2} h_J^{s_0}, \end{aligned}$$

where  $\hat{C}_s$  is the constant from Theorem 5.2. Since  $\zeta(1 + \varepsilon) > 1$  for every  $\varepsilon > 0$ , we obtain the claimed estimate with  $C_s := \hat{C}_s \sqrt{(L_0 h_0)^{-2s_0} + 2}$ . To prove the bound on the computational work, we insert the values for  $M_j$  and  $h_j$  and obtain

$$\begin{aligned} \mathcal{W}_J & \leq C_1 \left( M_0 h_0^{-2\eta_1} \log^{\eta_2}(h_0^{-2}) + \sum_{j=1}^J M_j (h_j^{-2\eta_1} \log^{\eta_2}(h_j^{-2}) + h_{j-1}^{-2\eta_1} \log^{\eta_2}(h_{j-1}^{-2})) \right) \\ & \leq C_2 \kappa \left( 2^{2s_0 J} + \sum_{j=1}^J 2^{2s_0(J-j+1)+2j\eta_1} j^{\eta_2+1+\varepsilon} \right), \end{aligned}$$

where the constant  $C_2 > 0$  depends on  $C_1 > 0, L_0, h_0, \eta_1$ , and  $\eta_2$ . If  $s_0 \leq \eta_1$ , then  $\mathcal{W}_J = \mathcal{O}(2^{2J\eta_1} J^{\eta_2+2+\varepsilon} \kappa) = \mathcal{O}(h_J^{-2\eta_1} J^{\eta_2+2+\varepsilon} \kappa)$ . In the other case that  $s_0 > \eta_1$ , it follows with the fact that  $\sum_{j \geq 1} \rho^j j^{\eta_2+1+\varepsilon} < +\infty$  for every  $\rho \in (0, 1)$  that

$$\mathcal{W}_J \leq C_2 2^{2s_0 J} \kappa \left( 1 + \sum_{j=1}^J 2^{-2j(s_0 - \eta_1)} j^{\eta_2+1+\varepsilon} \right) = \mathcal{O}(2^{2s_0 J} \kappa) = \mathcal{O}(h_J^{-2s_0} \kappa),$$

which finishes the proof of the corollary.  $\square$

Note that the choices  $(M_j, j = 0, \dots, J)$ ,  $J \in \mathbb{N}_0$  in (25) depend on the regularity of the solution  $u$  of (10). However, the closer  $s$  is to  $\beta/2$  the harder it should be to observe the convergence behavior that is theoretically guaranteed by Theorem 5.2, because constants may become arbitrarily large. We conclude the theoretical part of the paper with several remarks on the convergence bounds.

*Remark 5.4.* The proof of Theorem 5.2 is not restricted to the considered FE Methods above. If the conditions of Theorem 4.4 are satisfied with  $\beta > 0$ , an analogous argument implies the respective statement in the case of Spectral Methods, i.e., for every  $s \in (0, \beta/2)$ , there exists a constant  $C_s > 0$  such that for  $J \in \mathbb{N}_0$ ,

$$\|\mathbb{E}(u) - E^J(u^{L_j^a, L_j^u})\|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \left( \frac{1}{M_0} + \sum_{j=1}^J \frac{1}{M_j} L_{j-1}^{-2s} + L_J^{-2s} \right)^{1/2},$$

where the degrees of  $a^{L_j^a}$  and of  $\mathcal{H}_{1:L_j^u}$ ,  $j \in \mathbb{N}_0$ , are chosen as increasing sequences that define  $L_j := \min\{L_j^a, L_j^u\}$ ,  $j \in \mathbb{N}_0$ . As in the FE case the number of samples to equilibrate the MC errors on the levels can be chosen  $M_0 := \lceil L_0^{2s} \kappa \rceil$  and  $M_j := \lceil (L_j/L_{j-1})^{2s} j^{1+\varepsilon} \kappa \rceil$ ,  $j = 1, \dots, J$ , for a positive constant  $\varepsilon > 0$  and  $\kappa \geq (L_J/L_{J-1})^{-2s}$  (allow  $\kappa > 0$  if  $J = 0$ ). Hence, there exists  $C_s > 0$  such that

$$\|\mathbb{E}(u) - E^J(u^{L_j^a, L_j^u})\|_{L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})} \leq C_s \left( \zeta(1 + \varepsilon) \frac{1}{\kappa} + 1 \right)^{1/2} L_J^{-s}.$$

*Remark 5.5.* For smooth source terms  $f \in C^\infty(\mathbb{S}^2) = \bigcap_{s' > 0} H^{-1+s'}(\mathbb{S}^2)$ , the convergence rate of the MLMC estimator for Spectral Methods is given by  $s$  without further restrictions, cp. Theorem 4.4 and Remark 5.4. The decay of the angular power spectrum of the underlying iGRF  $T$  in terms of  $\beta > 0$  in (9) is the only constraint on the convergence rate  $s$  since  $s < \beta/2$ . For Finite Element Methods, the convergence rate is additionally bounded by the polynomial degree  $k$  of the Finite Element space, cp. Theorem 5.2.

We conclude that we have essentially determined the achievable convergence rates of MLMC FE and Spectral Methods solely with the decay of the angular power spectrum of the underlying iGRF in the stochastic operator, which in the FE case are bounded by the polynomial degree of the basis functions.

*Remark 5.6.* There exists an algorithm to compute samples of an iGRF that has a complexity behaving as  $O(N \log^2(N))$ , cp. [18], where  $N$  is the number of sample points of a quadrature to compute stiffness matrices. The number of sample points is comparable to the degrees of freedom of the spatial discretization. In the FE case, iterative solvers such as multigrid, cp. [4], suggest to have a complexity that is linear in the degrees of freedom, where here the resulting linear systems do not render the classical theory, since condition numbers of system matrices may be close to degenerate due to the lognormal diffusion coefficient. In the setting of Corollary 5.3, this would allow for  $\eta_1 = 1$  and  $\eta_2 = 2$ .

*Remark 5.7.* Since the degrees of freedom of either of the considered spatial discretizations relate to the discretization parameter with  $N_{h,J} = \dim(V^{h,k}) = O(h_J^{-2})$  and  $N_{L^u} = \dim(\mathcal{H}_{1:L^u}) = O((L^u)^2)$ , cp. Section 4, respective convergence estimates and work bounds from Corollary 5.3 and Remark 5.4 in the degrees of freedom are implied.

*Remark 5.8.* A decrease of the choices of samples  $(M_j, j = 0, \dots, J)$  in (25), i.e., if  $\kappa < 1$ , will increase the MC error contribution in Theorem 5.2 basically by the inverted square root of  $\kappa$  due to a larger MC error contribution. For instance, applying [26, Theorem 1] in our setting yields sample numbers scaled by a factor of  $2^{-2\min\{s,k\}}$  for  $j = 1, \dots, J$  sacrificing an increase in the corresponding constant  $C_2$  (in the notation of [26, Theorem 1]) of the error estimate. This constant will be scaled by a factor of  $2^{2\min\{s,k\}}$ .

## 6. NUMERICAL EXPERIMENTS

We consider here the test problem with smooth right hand side  $f = Y_{10}$ , i.e.,  $\ell = 1$  and  $m = 0$ , and angular power spectrum given by

$$A_\ell = (1 + \ell)^{-\alpha}, \quad \ell \in \mathbb{N}_0,$$

for  $\alpha \in (2, +\infty)$ . Since

$$\sum_{\ell \geq 0} A_\ell \ell^{1+\beta} < +\infty$$

for every  $\beta < \alpha - 2$ , Theorem 2.3 implies that the respective lognormal random field  $a \in L^p(\Omega; C^{\iota,\gamma}(\mathbb{S}^2))$ ,  $p \in [1, +\infty)$ , for every  $\iota \in \mathbb{N}_0$  and  $\gamma \in (0, 1)$  such that  $\iota + \gamma < (\alpha - 2)/2$ . For a given number of levels  $J \in \mathbb{N}_0$ , we study the error  $\mathbb{E}(u) - E^J(u^{L_j, h_j})$  in the  $L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ -norm. The sample numbers per level are chosen according to (25). Also the truncation levels are chosen as mentioned in Section 5, i.e.,  $L_j = \lceil h_0/h_j \rceil L_0$ ,  $j = 0, \dots, J$ , for some fixed  $L_0 \in \mathbb{N}$ . We therefore expect by Corollary 5.3 to observe a convergence rate of  $O(h_j^{\min\{s,k\}})$  for any  $s < (\alpha - 2)/2$ , where  $k$  is the polynomial degree of the ansatz functions.

The implementation of the MLMC estimator in (22) for the FE method presented in Section 4.1 builds on the structures of the boundary element C++ library BETL, cp. [22]. The geometrical error that occurs when surfaces are polynomially approximated can in the case of  $\mathbb{S}^2$  be avoided with the correction formula in [8, Equation (2.12)] for the gradient and with the correction formula in [8, Equation (2.10)] for the surface measure. The FE spaces result from refining an inscribed initial affine approximation of  $\mathbb{S}^2$ . New vertices are not projected to  $\mathbb{S}^2$  to obtain nested FE spaces. FE spaces on  $\mathbb{S}^2$  result by lifting functions to  $\mathbb{S}^2$ , cp. [8, Sections 2.4 and 2.5]. This is feasible in the case of  $\mathbb{S}^2$ , because a *signed distance function* is explicitly known. A signed distance function of  $\mathbb{S}^2$  maps points in  $\mathbb{R}^3 \setminus \{0\}$  to their distances to  $\mathbb{S}^2$  multiplied by a negative sign if (by convention) they are inside of the closed surface  $\mathbb{S}^2$ .

The evaluation of the truncated spherical harmonics series (6) is implemented with the SHTns library, cp. [34]. This implementation has a larger complexity of  $O(N^{3/2})$  assuming that  $N \sim L^2$  is the number of grid points compared to  $O(N \log(N)^2)$ , which is the complexity of the algorithm presented in [18]. However, the available implementation of the latter algorithm seems to require significant amounts of memory, cp. [34], which limits the truncation level  $L$ . Also, SHTns outperforms amongst others the available implementation of the algorithm in [18] in measured computing time as demonstrated in [34] and in particular allows for higher truncation levels. The linear systems are solved with the Intel MKL version of the PARDISO solver (see also [35]). The evaluation of the MLMC estimator is parallelized with the gMLQMC library, cp. [11], which allows for generic sampling. The parallelization in gMLQMC uses the Boost.MPI library. As a pseudo random number generator, we use the Mersenne Twister implementation from the C++11 standard. We will use matplotlib, cp. [24], to visualize our data (see Figure 1).

We present numerical results for first order FE, i.e.,  $k = 1$ , for  $\alpha = 3$  and  $\alpha = 4$ , which have a theoretical convergence rate of essentially 0.5 and 1, respectively. The sample numbers on each level are chosen as in (25) with  $\kappa = 1$ ,  $\varepsilon = 0.2$ , and  $L_0 = 2$ . As reference for  $\mathbb{E}(u)$  we use the average of 10 realizations of the MLMC FE estimator with one further level of refinement and parameter choices  $\kappa = 40$  and  $L_0 = 5$ . The  $L^2(\Omega; H^1(\mathbb{S}^2)/\mathbb{R})$ -norm is approximated by the square root of the average over 20 realizations of  $\|\mathbb{E}(u) - E^J(u^J)\|_{H^1(\mathbb{S}^2/\mathbb{R})}^2$ . In Figure 1 we observe convergence rates that our theoretical analysis predicts, since for the border line cases of convergence rates equal to 0.5 and 1, constants in the error bounds may become arbitrarily large. The empirical rates have been computed with least squares taking into account the five data points corresponding to finer spatial grids.

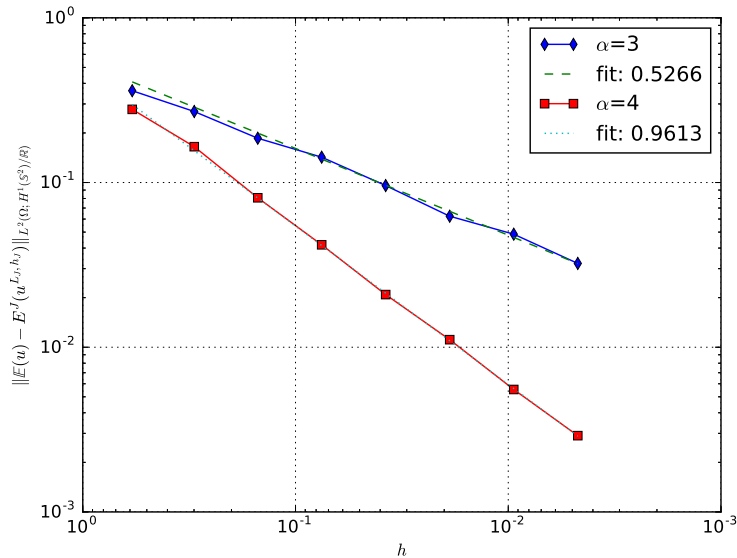


Figure 1: Convergence of MLMC with  $\alpha = 3, 4$  and  $\kappa = 1$ .

#### APPENDIX A. MEASURE THEORY AND FUNCTIONAL ANALYSIS

The purpose of this appendix is to collect the measure theoretical and functional analytical background of our analysis in the main text. While results on domains in Euclidean space are well-known in the literature, the corresponding results on the unit sphere are not explicitly available. Therefore, we derive these missing results for the unit sphere in what follows, which include properties of Hölder and Sobolev spaces and especially a Sobolev embedding theorem on  $\mathbb{S}^2$ .

One way to translate results from Euclidean space to manifolds such as the unit sphere is to use an atlas and show the invariance of the results under a change of atlas. This will be of frequent use in what follows. Therefore, let  $\{(U_i, \eta_i), i \in \mathcal{I}\}$  be a finite  $C^\infty$  atlas of  $\mathbb{S}^2$ , where  $\{U_i, i \in \mathcal{I}\}$  is a finite open cover of  $\mathbb{S}^2$  and  $\{\eta_i : U_i \rightarrow \eta_i(U_i) \subset \mathbb{R}^2, i \in \mathcal{I}\}$  are the respective coordinate charts, which are sometimes also simply called *coordinates*. Furthermore, let  $g$  be

the metric tensor which is expressed for any  $x_0 \in \mathbb{S}^2$  locally in the coordinates  $\{\eta_i, i \in \mathcal{I}\}$  as

$$g_{k\ell}(x_0) := \left\langle \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k}, \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^\ell} \right\rangle_{\mathbb{R}^3}$$

for  $k, \ell = 1, 2$ , where  $\hat{x}_0 = \eta_i(x_0)$  and  $i \in \mathcal{I}$  is such that  $x_0 \in U_i$ . The matrix  $g(x_0)$  induces an inner product on the *tangent space*  $T_{x_0}\mathbb{S}^2$  at  $x_0$  in the basis  $\frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k}$ ,  $k = 1, 2$ , i.e., for  $v = \sum_{k=1}^2 v^k \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k}$ ,  $w = \sum_{k=1}^2 w^k \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^k} \in T_{x_0}\mathbb{S}^2$ , it holds that  $\langle v, w \rangle_{\mathbb{R}^3} = \sum_{k,\ell=1}^2 g_{k\ell}(x_0) v^k w^\ell$ . We denote the components of the inverse of  $g$  at any arbitrarily chosen  $x_0 \in \mathbb{S}^2$  by  $g^{k\ell}(x_0) := (g^{-1}(x_0))_{k\ell}$  for  $k, \ell = 1, 2$  and further introduce  $|g|(x_0) := \det(g(x_0))$ . The *spherical gradient*  $\nabla_{\mathbb{S}^2}$  and the *spherical divergence*  $\nabla_{\mathbb{S}^2} \cdot$  are locally expressed in terms of  $g$ , i.e., for any  $x_0 \in \mathbb{S}^2$ ,  $i \in \mathcal{I}$  such that for any  $x_0 \in U_i$  and  $\hat{x}_0 = \eta_i(x_0)$ ,

$$\nabla_{\mathbb{S}^2} f(x_0) := \sum_{k,\ell=1}^2 g^{k\ell}(x_0) \frac{\partial (f \circ \eta_i^{-1})(\hat{x}_0)}{\partial \hat{x}^k} \frac{\partial \eta_i^{-1}(\hat{x}_0)}{\partial \hat{x}^\ell}$$

and

$$\nabla_{\mathbb{S}^2} \cdot Z(x_0) := \frac{1}{\sqrt{|g|(x_0)}} \sum_{\ell=1}^2 \frac{\partial}{\partial \hat{x}^\ell} ((\sqrt{|g|} Z^\ell) \circ \eta_i^{-1})(\hat{x}_0),$$

where  $f : \mathbb{S}^2 \rightarrow \mathbb{R}$  is a function and  $Z = \sum_{\ell=1}^2 Z^\ell \frac{\partial \eta_i^{-1}}{\partial \hat{x}^\ell}$  a vector field, cp. [25, Equations (3.1.17), (3.1.19)]. We define the *spherical Laplacian*, also called *Laplace–Beltrami operator*, by

$$\Delta_{\mathbb{S}^2} := \nabla_{\mathbb{S}^2} \cdot \nabla_{\mathbb{S}^2},$$

and we denote by  $\sigma$  the *Lebesgue measure on the sphere* which admits for every  $i \in \mathcal{I}$  the local representation

$$d\sigma(x) = \sqrt{|g|(x)} d\hat{x}^1 d\hat{x}^2$$

on  $U_i$  by [25, Equation (3.3.8)], where  $x \in U_i$  and  $\hat{x} = \eta_i(x)$ . These definitions are valid for general coordinates and therefore generalize the respective expressions given in Section 2. Note that for any  $x_0 \in \mathbb{S}^2$ , the inner product that is induced in  $T_{x_0}\mathbb{S}^2$  by  $g(x_0)$  does not depend on the choice of the coordinates  $\{\eta_i, i \in \mathcal{I}\}$ , cp. [25, Equations (1.4.4), (1.4.5)]. For further details, the reader is referred to [25, Sections 1.4 and 3.1].

Furthermore, let  $\Psi = \{\Psi_i, i \in \mathcal{I}\}$  be a  $C^\infty$  partition of unity, which is subordinate to  $\{U_i, i \in \mathcal{I}\}$ , i.e.,  $\text{supp}(\Psi_i) \subset U_i$  for every  $i \in \mathcal{I}$ . The support of a function is the closure of the points, where the function is non-zero. We infer from [40, Theorem 7.4.5] and [16, Theorem 3.9] that Sobolev spaces on  $\mathbb{S}^2$  can equivalently to Section 2 be characterized via pullbacks with respect to general coordinates, i.e.,  $v \in H_q^s(\mathbb{S}^2)$  if and only if  $(v\Psi_i) \circ \eta_i^{-1} \in H_q^s(\mathbb{R}^2)$  for every  $i \in \mathcal{I}$ , where  $v\Psi_i$  has to be understood as pointwise multiplication, and

$$v \mapsto \left( \sum_{i \in \mathcal{I}} \|(v\Psi_i) \circ \eta_i^{-1}\|_{H_q^s(\mathbb{R}^2)}^q \right)^{1/q}$$

is an equivalent norm on  $H_q^s(\mathbb{S}^2)$ , where  $H_q^s(\mathbb{R}^2)$  denote the usual Bessel potential spaces on  $\mathbb{R}^2$ , which are equal to the Sobolev–Slobodeckij spaces for  $q = 2$  with equivalent norms, cp. [41, Definition 2.3.1(d), Theorem 2.3.2(d), Equation 4.4.1(8)]. More precisely, [40, Theorem 7.4.5] implies that  $H_q^s(\mathbb{S}^2)$  can be equivalently characterized via pullbacks with respect to the geodesic normal coordinates. In [16, Theorem 3.9] it is shown that the characterization of Sobolev spaces on manifolds with bounded geometry, e.g.,  $\mathbb{S}^2$ , via pullbacks with respect to



arbitrary coordinates does not depend on the coordinates and that different coordinates lead to equivalent norms. We remark that a function like  $(v\Psi_i) \circ \eta_i^{-1}$  on  $\eta_i(U_i)$  can be extended smoothly by zero to all of  $\mathbb{R}^2$ , since  $\Psi_i \circ \eta_i^{-1}$  is smooth and compactly supported in  $\eta_i(U_i)$ . For details on the geodesic normal coordinates, which are sometimes also called (Riemannian) normal coordinates (cp. [25, Definition 1.4.4]), we refer the reader to [16, Example 3], while a detailed description of Bessel potential spaces can be found in [39, Chapter 2].

Finally, we equip the Hölder spaces  $C^{\iota,\gamma}(\mathbb{S}^2)$ ,  $\iota \in \mathbb{N}_0$ ,  $\gamma \in [0, 1)$ , that were introduced in Section 2 with the norm  $\|\cdot\|_{C^{\iota,\gamma}(\mathbb{S}^2)}$  given by

$$\|v\|_{C^{\iota,\gamma}(\mathbb{S}^2)} := \max_{i \in \mathcal{I}} \|(v\Psi_i) \circ \eta_i^{-1}\|_{C^{\iota,\gamma}(\mathbb{R}^2)}$$

for every  $v \in C^{\iota,\gamma}(\mathbb{S}^2)$ . This norm is well-defined, since different choices of atlases and partitions of unity will lead to equivalent norms (cp. [20, Proposition 6.9]).

A nice and convenient property of the regularity of the exponential function in terms of Hölder norm bounds that will be introduced as the following lemma. This lemma is proven by an induction argument using the fact that Hölder spaces are algebras, i.e., the product of functions is an element of the same Hölder space and the product is continuous.

**Lemma A.1.** *Let  $\iota \in \mathbb{N}_0$  and  $\gamma \in (0, 1)$ , then there exists a constant  $C_{\iota,\gamma}$  such that for every  $v \in C^{\iota,\gamma}(\mathbb{S}^2)$*

$$(26) \quad \|\exp(v)\|_{C^{\iota,\gamma}(\mathbb{S}^2)} \leq C_{\iota,\gamma} \|\exp(v)\|_{C^0(\mathbb{S}^2)} (1 + \|v\|_{C^{\iota,\gamma}(\mathbb{S}^2)}^{\iota+1}).$$

*Proof.* Generally, this proof is inspired by [23, Theorem A.8], but it achieves a specific result not explicitly available in that theorem.

Let  $D \subset \mathbb{R}^2$  be a bounded, convex open domain. The first step is to prove the estimate for Hölder spaces over Euclidean domains, i.e.,  $C^{\iota,\gamma}(\overline{D})$ . We set  $g := \exp$  and recall that the derivative  $g'$  is again equal to  $g$ .

For convenience, we will omit the set  $\overline{D}$  if the context is clear. For  $\iota = 0$ , it is easily seen that for every  $\tilde{v} \in C^{0,\gamma}(\overline{D})$ , it holds that  $\|g(\tilde{v})\|_{C^{0,\gamma}} \leq \|g(\tilde{v})\|_{C^0} (1 + \|\tilde{v}\|_{C^{0,\gamma}})$ , cp. the proof of [23, Theorem A.8], which is the base case of an induction argument to the following induction hypothesis:

Let the estimate in (26) be satisfied for Hölder spaces over the Euclidean set  $\overline{D}$ , i.e., for functions  $\tilde{v} \in C^{n,\gamma}(\overline{D})$  for every  $n \in \{0, \dots, \iota - 1\}$ . We directly perform the induction step from  $n = \iota - 1$  to  $n + 1 = \iota$ . For  $\tilde{v}_1, \tilde{v}_2 \in C^{\iota-1,\gamma}(\overline{D})$ , the product estimate  $\|\tilde{v}_1 \tilde{v}_2\|_{C^{\iota,\gamma}} \leq \hat{C}_{\iota,\gamma} \|\tilde{v}_1\|_{C^{\iota-1,\gamma}} \|\tilde{v}_2\|_{C^{\iota-1,\gamma}}$  holds by [23, Theorem A.7], which implies with the chain rule from calculus and the induction hypothesis that

$$(27) \quad \begin{aligned} \|g(\tilde{v})\|_{C^{\iota,\gamma}} &= \|g(\tilde{v})\|_{C^0} + \sum_{j=1,2} \|\partial_{x^j}(g \circ \tilde{v})\|_{C^{\iota-1,\gamma}} \\ &\leq \|g(\tilde{v})\|_{C^0} + \hat{C}_{\iota-1,\gamma} \sum_{j=1,2} \|g'(\tilde{v})\|_{C^{\iota-1,\gamma}} \|\partial_{x^j} \tilde{v}\|_{C^{\iota-1,\gamma}} \\ &\leq \|g(\tilde{v})\|_{C^0} + \hat{C}_{\iota-1,\gamma} \|g(\tilde{v})\|_{C^{\iota-1,\gamma}} \|\tilde{v}\|_{C^{\iota,\gamma}} \\ &\leq \|g(\tilde{v})\|_{C^0} + \hat{C}_{\iota-1,\gamma} \tilde{C}_{\iota-1,\gamma} \|g(\tilde{v})\|_{C^0} (1 + \|\tilde{v}\|_{C^{\iota-1,\gamma}}^{\iota}) \|\tilde{v}\|_{C^{\iota,\gamma}} \\ &\leq \tilde{C}_{\iota,\gamma} \|g(\tilde{v})\|_{C^0} (1 + \|\tilde{v}\|_{C^{\iota,\gamma}}^{\iota+1}), \end{aligned}$$

and finishes the induction step. Note that for convenience we used  $\partial_{x^j} := \partial/\partial x^j$ ,  $j = 1, 2$ .

Next let  $\{(U_i, \eta_i), i \in \mathcal{I}\}$  be a finite  $C^\infty$  atlas and  $\{\Psi_i, i \in \mathcal{I}\}$  a  $C^\infty$  partition of unity subordinate to  $\{U_i, i \in \mathcal{I}\}$ . We fix  $j \in \mathcal{I}$  and choose another  $C^\infty$  partition of unity  $\{\hat{\Psi}_i, i \in \mathcal{I}\}$  subordinate to  $\{U_i, i \in \mathcal{I}\}$  such that  $\hat{\Psi}_j = 1$  on  $\text{supp}(\Psi_j)$ . We can assume that  $\bar{D} := \text{supp}(\Psi_j \circ \eta_j^{-1})$  and  $\text{supp}(\hat{\Psi}_j \circ \eta_j^{-1})$  are convex and observe with (27) that

$$\begin{aligned}
(28) \quad & \|(\exp(v)\Psi_j) \circ \eta_j^{-1}\|_{C^{\iota, \gamma}(\mathbb{R}^2)} = \|(\exp(v)\Psi_j) \circ \eta_j^{-1}\|_{C^{\iota, \gamma}(\bar{D})} \\
& \leq \hat{C}_{\iota, \gamma} \|\exp(v) \circ \eta_j^{-1}\|_{C^{\iota, \gamma}(\bar{D})} \|\Psi_j \circ \eta_j^{-1}\|_{C^{\iota, \gamma}(\bar{D})} \\
& \leq C_{\iota, \gamma} \|\exp(v) \circ \eta_j^{-1}\|_{C^0(\bar{D})} (1 + \|v \circ \eta_j^{-1}\|_{C^{\iota, \gamma}(\bar{D})}^{\iota+1}) \\
& \leq C_{\iota, \gamma} \|(\exp(v)\hat{\Psi}_j) \circ \eta_j^{-1}\|_{C^0(\mathbb{R}^2)} (1 + \|(v\hat{\Psi}_j) \circ \eta_j^{-1}\|_{C^{\iota, \gamma}(\mathbb{R}^2)}^{\iota+1}).
\end{aligned}$$

We apply that different  $C^\infty$  partitions of unity result in equivalent norms on  $C^{\iota, \gamma}(\mathbb{S}^2)$  and conclude the estimate of the lemma by taking the maximum over  $j$  on both sides of (28).  $\square$

As in Euclidean space, Sobolev spaces can be embedded into Hölder spaces (see e.g. [41, Theorem 4.6.1(e)]) which is made precise in the following Sobolev embedding theorem on  $\mathbb{S}^2$ .

**Theorem A.2** (Sobolev embedding theorem). *If  $s \in (0, +\infty)$ ,  $q \in (1, +\infty)$ ,  $\iota \in \mathbb{N}_0$ , and  $\gamma \in (0, 1)$  satisfy  $s - 2/q \geq \iota + \gamma$ , then the embedding  $H_q^s(\mathbb{S}^2) \subset C^{\iota, \gamma}(\mathbb{S}^2)$  is continuous.*

Furthermore,  $H_q^s(\mathbb{S}^2)$ -norms of products of functions can be bounded by a combination of Hölder and Sobolev norms, which is made in the following proposition. In the proof, the estimate for domains in Euclidean space in [39, Theorem 3.3.2] is translated to  $\mathbb{S}^2$ .

**Proposition A.3.** *Let  $q \in (1, +\infty)$  and let  $\iota \in \mathbb{N}_0$ ,  $\gamma \in (0, 1)$ , and  $s \in \mathbb{R}$  be such that  $|s| < \iota + \gamma$ . If  $v \in C^{\iota, \gamma}(\mathbb{S}^2)$  and  $w \in H_q^s(\mathbb{S}^2)$ , then  $vw \in H_q^s(\mathbb{S}^2)$ . Moreover the following product estimate holds: there exists a constant  $C_{\iota, \gamma}$  such that for every  $v \in C^{\iota, \gamma}(\mathbb{S}^2)$  and every  $w \in H_q^s(\mathbb{S}^2)$ ,*

$$\|vw\|_{H_q^s(\mathbb{S}^2)} \leq C_{\iota, \gamma} \|v\|_{C^{\iota, \gamma}(\mathbb{S}^2)} \|w\|_{H_q^s(\mathbb{S}^2)}.$$

The proofs of Theorem A.2 and Proposition A.3 follow with a localization argument as applied in the second paragraph of the proof of Lemma A.1.

Let us conclude this appendix with some facts about measurability of Banach-space-valued random variables, which includes our framework of Sobolev and Hölder spaces. Therefore, consider a Banach space  $(B, \|\cdot\|_B)$  with dual space  $B^*$  and  $X : \Omega \rightarrow B$ . We recall that  $X$  is called *weakly measurable* if for every  $\mathcal{G} \in B^*$ , the real-valued function  $\mathcal{G}(X)$  is measurable. Furthermore,  $X$  is called *countably-valued* if  $X$  assumes at most a countable set of values in  $B$  on countably many, disjoint measurable subsets. It is *strongly measurable* if there exists a sequence of countably-valued mappings  $(X_n, n \in \mathbb{N})$ ,  $X_n : \Omega \rightarrow B$ , such that  $\lim_{n \rightarrow +\infty} X_n(\omega) = X(\omega)$  in  $B$  for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , and we say that  $X$  is called  *$\mathbb{P}$ -almost separably-valued* if there exists a measurable set  $N$  with  $\mathbb{P}(N) = 0$  such that the set  $\{X(\omega), \omega \in \Omega \setminus N\}$  is separable in  $B$  (cp. [21, Definitions 3.5.3 and 3.5.4]). A well-known result on the connection of strong and weak measurability is Pettis' theorem (see, e.g., [21, Theorem 3.5.3]).

**Theorem A.4** (Pettis' theorem). *A  $B$ -valued mapping on  $\Omega$  is strongly measurable if and only if it is weakly measurable and  $\mathbb{P}$ -almost separably-valued.*

The following lemma is the generalization to Banach spaces of the well-known property that real-valued random variables under continuous mappings are random variables, i.e., measurable. It is a direct consequence of the definition of strong measurability.

**Lemma A.5.** *Let  $B_1, B_2$  be Banach spaces and let  $\varphi : B_1 \rightarrow B_2$  be continuous. If  $f : \Omega \rightarrow B_1$  is strongly  $B_1$ -measurable, then  $\varphi \circ f$  is strongly  $B_2$ -measurable.*

Since we consider in this manuscript measurability with respect to different Banach spaces, we write for clarity  $B$ -measurable where necessary. We remark that a mapping  $X : \Omega \rightarrow B$  is Bochner integrable if and only if it is strongly  $B$ -measurable and the real-valued function  $\|X\|_B$  is integrable, cp. [21, Theorem 3.7.4]. The strong  $B$ -measurability of  $X$  implies the measurability of  $\|X\|_B$ .

## APPENDIX B. INTEGRABILITY OF CONTINUOUS LOGNORMAL RFS

Integrability of a lognormal random field in terms of  $L^p(\Omega)$ -norms is a consequence of Fernique's theorem. While this was performed for random fields on domains in Euclidean space in [5, Proposition 3.10], we derive the corresponding result on spheres in this appendix in the following proposition.

**Proposition B.1.** *Let  $p \in [1, +\infty)$  and let  $T$  be a continuous iGRF and satisfy (9) for some  $\beta > 0$ . Then, the lognormal random fields  $\exp(T)$  and  $\exp(\Pi_L T)$  are in  $L^p(\Omega; C^0(\mathbb{S}^2))$  for all  $L \in \mathbb{N}_0$  and the  $L^p(\Omega; C^0(\mathbb{S}^2))$ -norm of  $\exp(\Pi_L T)$  can be bounded independently of  $L$ .*

*Proof.* It will be sufficient to prove the case that  $T$  is centered, i.e.,  $\mathbb{E}(T) = 0 \in C^0(\mathbb{S}^2)$ . By the definition of an iGRF  $\mathbb{E}(T)$  is a constant function on  $\mathbb{S}^2$ , which implies  $\|\exp(T)\|_{L^p(\Omega; C^0(\mathbb{S}^2))} = \|\exp(T - \mathbb{E}(T))\|_{L^p(\Omega; C^0(\mathbb{S}^2))} \exp(\mathbb{E}(T))$ . Hence, the general case can be reduced to the case of a centered, continuous iGRF. So in the following we can assume that  $T$  is centered.

The idea of the proof is to apply Fernique's theorem, cp. [7, Theorem 2.7], on the separable Banach space  $C^0(\mathbb{S}^2)$ . Therefore, we have to establish that the law of  $T$  is a centered (symmetric) Gaussian measure on  $C^0(\mathbb{S}^2)$ , i.e., for every  $\mathcal{G} \in C^0(\mathbb{S}^2)^*$ , the dual space of  $C^0(\mathbb{S}^2)$ , there exists  $\sigma_{\mathcal{G}} \in [0, +\infty)$  such that  $\mathcal{G}(T) \sim \mathcal{N}(0, \sigma_{\mathcal{G}}^2)$ . This is the first requirement in order to apply [7, Theorem 2.7]. We remark that in [7] the term 'symmetric' Gaussian measure is used instead of centered meaning the same. For every  $L \in \mathbb{N}_0$ ,  $\Pi_L T$  has the finite real expansion according to (3)

$$\Pi_L T = \sum_{\ell=0}^L \left( a_{\ell 0} Y_{\ell 0} + 2 \sum_{m=1}^{\ell} (\operatorname{Re} a_{\ell m} \operatorname{Re} Y_{\ell m} - \operatorname{Im} a_{\ell m} \operatorname{Im} Y_{\ell m}) \right).$$

From the properties of the Karhunen–Loève expansion we deduce that  $\{a_{\ell 0}, \operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \dots, \ell\}$  are independent real-valued random variables. Additionally this corollary implies that  $a_{\ell 0} \sim \mathcal{N}(0, A_{\ell})$  and  $\operatorname{Re} a_{\ell m}, \operatorname{Im} a_{\ell m} \sim \mathcal{N}(0, A_{\ell}/2)$  for  $\ell \in \mathbb{N}_0$  and  $m = 1, \dots, \ell$ . Let  $L \in \mathbb{N}_0$  and  $\mathcal{G} \in C^0(\mathbb{S}^2)^*$  be arbitrary. Hence,

$$\mathcal{G}(\Pi_L T) = \sum_{\ell=0}^L \left( a_{\ell 0} \mathcal{G}(Y_{\ell 0}) + 2 \sum_{m=1}^{\ell} (\operatorname{Re} a_{\ell m} \mathcal{G}(\operatorname{Re} Y_{\ell m}) - \operatorname{Im} a_{\ell m} \mathcal{G}(\operatorname{Im} Y_{\ell m})) \right) \sim \mathcal{N}(0, \sigma_{\mathcal{G}, L}^2)$$

and therefore the characteristic function  $\varphi_{\mathcal{G}, L}$  of  $\mathcal{G}(\Pi_L T)$  is given by

$$\lambda \mapsto \varphi_{\mathcal{G}, L}(\lambda) := \exp\left(-\frac{1}{2} \lambda^2 \sigma_{\mathcal{G}, L}^2\right),$$

where

$$\sigma_{\mathcal{G}, L}^2 = \sum_{\ell=0}^L A_{\ell} \left( \mathcal{G}(Y_{\ell 0})^2 + 2 \sum_{m=1}^{\ell} (\mathcal{G}(\operatorname{Re} Y_{\ell m})^2 + \mathcal{G}(\operatorname{Im} Y_{\ell m})^2) \right).$$

Thus,  $\Pi_L T$  is a centered Gaussian measure on  $C^0(\mathbb{S}^2)$  for every  $L \in \mathbb{N}_0$ . The next step is to show that the sequence  $(\sigma_{\mathcal{G},L}^2, L \in \mathbb{N}_0)$  is uniformly bounded. The Riesz representation theorem for  $C^0(\mathbb{S}^2)$  (cp. [3, Theorem 7.10.4]) and [3, Theorem 3.1.1, Remark 3.1.5] imply that there exist a finite, positive measure  $\nu$  on  $(\mathbb{S}^2, \mathcal{B}(\mathbb{S}^2))$  and a measurable function  $g$  satisfying  $|g(x)| = 1$  for every  $x \in \mathbb{S}^2$  such that  $\mathcal{G}(v) = \int_{\mathbb{S}^2} v g d\nu$  for every  $v \in C^0(\mathbb{S}^2)$ , which implies with the Cauchy–Schwarz inequality that for every  $v \in C^0(\mathbb{S}^2)$ ,

$$\mathcal{G}(v)^2 = \left( \int_{\mathbb{S}^2} v g d\nu \right)^2 \leq \|v\|_{L^2(\mathbb{S}^2, \nu)}^2 \|g\|_{L^2(\mathbb{S}^2, \nu)}^2 = \|v\|_{L^2(\mathbb{S}^2, \nu)}^2 \nu(\mathbb{S}^2).$$

This implies with the identity  $\sum_{|m| \leq \ell} |Y_{\ell m}(x)|^2 = (2\ell + 1)/(4\pi)$  (cp. [31, Theorem 2.4.5]) that

$$\begin{aligned} \mathcal{G}(Y_{\ell 0})^2 + 2 \sum_{m=1}^{\ell} (\mathcal{G}(\operatorname{Re} Y_{\ell m})^2 + \mathcal{G}(\operatorname{Im} Y_{\ell m})^2) \\ \leq \left( \int_{\mathbb{S}^2} Y_{\ell 0}^2 + 2 \sum_{m=1}^{\ell} ((\operatorname{Re} Y_{\ell m})^2 + (\operatorname{Im} Y_{\ell m})^2) d\nu \right) \nu(\mathbb{S}^2) \\ = \left( \int_{\mathbb{S}^2} \sum_{m=-\ell}^{\ell} |Y_{\ell m}|^2 d\nu \right) \nu(\mathbb{S}^2) = \nu(\mathbb{S}^2)^2 \frac{2\ell + 1}{4\pi}. \end{aligned}$$

Summing the previous inequality over  $\ell$  implies with the finiteness of  $\sum_{\ell \geq 0} A_{\ell} \frac{2\ell + 1}{4\pi}$  that  $(\sigma_{\mathcal{G},L}^2, L \in \mathbb{N}_0)$  is uniformly bounded in  $L$ . Hence, there exists a unique  $\sigma_{\mathcal{G}} \in [0, +\infty)$  such that  $\sigma_{\mathcal{G},L}^2 \rightarrow \sigma_{\mathcal{G}}^2$  as  $L \rightarrow +\infty$ . Thus,  $\lim_{L \rightarrow +\infty} \varphi_{\mathcal{G},L}(\lambda) = \exp(-1/2 \lambda^2 \sigma_{\mathcal{G}}^2) =: \varphi_{\mathcal{G}}(\lambda)$  for every  $\lambda \in \mathbb{R}$ . The  $L^2(\Omega; C^0(\mathbb{S}^2))$ -convergence of  $\Pi_L T \rightarrow T$ , which is implied by Theorem 2.2, yields that  $\mathcal{G}(\Pi_L T) \rightarrow \mathcal{G}(T)$  in  $L^2(\Omega)$  and thus in distribution. Lévy's continuity theorem, cp. [28, Theorem IV.13.2.B], implies that  $\mathcal{G}(T) \sim \mathcal{N}(0, \sigma_{\mathcal{G}}^2)$  and we conclude that the law of  $T$  is a centered (symmetric) Gaussian measure on  $C^0(\mathbb{S}^2)$ .

We infer from Theorem 2.2 that there exists an upper bound  $K$  of the  $L^2(\Omega; C^0(\mathbb{S}^2))$ -norm of  $T$  and of  $\Pi_L T$ ,  $L \in \mathbb{N}_0$ , which is uniform in  $L$ . Let in the following  $X \in \{T, \Pi_L T, L \in \mathbb{N}_0\}$ . We choose  $x_0 \in [1/(1 + \exp(-2)), 1)$ , which implies that  $\log((1 - x_0)/x_0) \leq -2$ , and set  $r_0 := K/\sqrt{1 - x_0}$ . We use the Chebychev inequality to obtain that

$$1 - \mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0) = \mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} > r_0) \leq \frac{\mathbb{E}(\|X\|_{C^0(\mathbb{S}^2)}^2)}{r_0^2} \leq \frac{K^2}{r_0^2} = 1 - x_0,$$

which implies that  $\mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0) \geq x_0$ . We choose  $\lambda > 0$  such that  $\lambda \leq (1 - x_0)/(32K^2)$ , which implies that  $32\lambda r_0^2 \leq 1$ , and arrive with the monotonicity of the logarithm at the inequality

$$\log\left(\frac{1 - \mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0)}{\mathbb{P}(\|X\|_{C^0(\mathbb{S}^2)} \leq r_0)}\right) + 32\lambda r_0^2 \leq \log\left(\frac{1 - x_0}{x_0}\right) + 32\lambda r_0^2 \leq -1.$$

This is the second requirement for [7, Theorem 2.7]. Since  $X$  is a centered Gaussian measure on  $C^0(\mathbb{S}^2)$ , [7, Theorem 2.7] implies that

$$\mathbb{E}(\exp(\lambda \|X\|_{C^0(\mathbb{S}^2)}^2)) \leq \exp(16\lambda r_0^2) + \frac{\exp(2)}{\exp(2) - 1},$$

which is a bound that is independent of  $L$ , because the choices of  $r_0$  and  $\lambda$  do not depend on  $L$  due to the uniformity of the bound  $K$ . Since  $0 \leq (\sqrt{\lambda}x - p/(2\sqrt{\lambda}))^2$  implies that  $px \leq \lambda x^2 + p^2/(4\lambda)$  for every  $x \in \mathbb{R}$ , we conclude that

$$\mathbb{E}(\|\exp(X)\|_{C^0(\mathbb{S}^2)}^p) \leq \mathbb{E}(\exp(p\|X\|_{C^0(\mathbb{S}^2)})) \leq \mathbb{E}(\exp(\lambda\|X\|_{C^0(\mathbb{S}^2)}^2)) \exp\left(\frac{p^2}{4\lambda}\right),$$

which finishes the proof of the proposition.  $\square$

### APPENDIX C. HIGHER ORDER REGULARITY OF SOLUTIONS

In this appendix we present the proof of Proposition 3.6, which we divide for better readability into one lemma and two propositions. Let us start with the  $H^{1+s}(\mathbb{S}^2)$ -regularity of the solution for  $s \in [0, 1)$ . This is derived with a classical regularity estimate, which in the case of domains in Euclidean space is due to Hackbusch, cp. [17, Theorem 9.1.8] (see also [12]). Here we transfer the problem to Euclidean space and back with an atlas and a partition of unity.

**Lemma C.1.** *For some  $0 \leq s < \gamma < 1$ , let  $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$ ,  $f \in H^{-1+s}(\mathbb{S}^2)$ , and  $\tilde{a} \in C^{0,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$  satisfy the variational problem (11) then  $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$  and there exists a constant  $C$ , which is independent of  $\tilde{u}$ ,  $f$ , and  $\tilde{a}$ , such that*

$$\|\tilde{u}\|_{H^{1+s}(\mathbb{S}^2)} \leq C \left( \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)} \|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) + \|\tilde{u}\|_{H^1(\mathbb{S}^2)} \right).$$

*Proof.* Let  $\{(\eta_j, U_j), j \in \mathcal{I}\}$  be a  $C^\infty$ -atlas and  $\{\Psi_j, j \in \mathcal{I}\}$  be a subordinate,  $C^\infty$  partition of unity. Let us fix  $i \in \mathcal{I}$ . We observe with the product rule, i.e.,  $\nabla \cdot (vW) = \nabla v \cdot W + v \nabla \cdot W$  for scalar and vector fields  $v$  and  $W$ , the divergence theorem, cp. [31, Equation (2.4.185)], and (11) that  $\tilde{u}\Psi_i$  satisfies for every  $v \in H^1(\mathbb{S}^2)/\mathbb{R}$  that

$$\begin{aligned} (\tilde{a} \nabla_{\mathbb{S}^2}(\tilde{u}\Psi_i), \nabla_{\mathbb{S}^2}v) &= (\tilde{a} \nabla_{\mathbb{S}^2}\tilde{u}, \nabla_{\mathbb{S}^2}(v\Psi_i)) - (\tilde{a} \nabla_{\mathbb{S}^2}\tilde{u} \cdot \nabla_{\mathbb{S}^2}\Psi_i, v) + (\tilde{a}\tilde{u} \nabla_{\mathbb{S}^2}\Psi_i, \nabla_{\mathbb{S}^2}v) \\ &= f(v\Psi_i - \frac{1}{|\mathbb{S}^2|} \int v\Psi_i d\sigma) - (\tilde{a} \nabla_{\mathbb{S}^2}\tilde{u} \cdot \nabla_{\mathbb{S}^2}\Psi_i, v) + (\tilde{a}\tilde{u} \nabla_{\mathbb{S}^2}\Psi_i, \nabla_{\mathbb{S}^2}v) \\ &= f(v\Psi_i) - \frac{1}{|\mathbb{S}^2|} f(1)(\Psi_i, v) - (\tilde{a} \nabla_{\mathbb{S}^2}\tilde{u} \cdot \nabla_{\mathbb{S}^2}\Psi_i, v) + (\tilde{a}\tilde{u} \nabla_{\mathbb{S}^2}\Psi_i, \nabla_{\mathbb{S}^2}v), \end{aligned}$$

where we remark that for every  $v \in H^1(\mathbb{S}^2)/\mathbb{R}$ , it holds that  $v\Psi_i - 1/|\mathbb{S}^2| \int v\Psi_i d\sigma \in H^1(\mathbb{S}^2)/\mathbb{R}$ . Let  $V_i := \eta_i(U_i)$  and let  $D \subset \mathbb{R}^2$  with smooth boundary be such that  $\text{supp}(\Psi_i \circ \eta_i^{-1}) \subset\subset D \subset\subset V_i$ . We recall that for two functions  $w_1, w_2 : \mathbb{S}^2 \rightarrow \mathbb{R}$ , the first fundamental form of their gradients satisfies with respect to the coordinate chart  $\eta_i$  that on  $V_i$  it holds that

$$(\nabla_{\mathbb{S}^2}w_1 \cdot \nabla_{\mathbb{S}^2}w_2) \circ \eta_i^{-1} = \sum_{k,\ell=1}^2 g^{k\ell} \circ \eta_i^{-1} \frac{\partial(w_1 \circ \eta_i^{-1})}{\partial x^k} \frac{\partial(w_2 \circ \eta_i^{-1})}{\partial x^\ell}.$$

Furthermore, there exists a constant  $\lambda_g > 0$  such that for every  $y \in U_i$ ,  $\sum_{k,\ell=1}^2 g^{k\ell}(y) \xi_k \xi_\ell \geq \lambda_g \sum_{k=1}^2 \xi_k^2$  for every  $\xi \in T_y \mathbb{S}^2$ . We also recall that with respect to the coordinate chart  $\eta_i$  it holds that  $d\sigma(y) = \sqrt{|g|(y)} dx$ , where  $|g|(y) = \det(g(y))$  and  $y = \eta_i^{-1}(x)$ , and  $|g|(y) > 0$  for every  $x \in V_i$ . We choose  $\chi \in C^\infty(\mathbb{R}^2)$  such that  $\chi = 1$  on  $\text{supp}(\Psi_i \circ \eta_i^{-1})$  and  $\chi = 0$  on the complement of  $D$ . We define the matrix-valued function

$$A := \begin{cases} ((\sqrt{|g|}\tilde{a}g^{-1}) \circ \eta_i^{-1}) \chi + \min_{y \in U_i} \{ \sqrt{|g|(y)} \tilde{a}(y) \} \lambda_g (1 - \chi) \text{Id}_{\mathbb{R}^2} & \text{on } V_i \\ \min_{y \in U_i} \{ \sqrt{|g|}\tilde{a} \} \lambda_g (1 - \chi) \text{Id}_{\mathbb{R}^2} & \text{else} \end{cases}$$

and the functions

$$b := \begin{cases} ((\sqrt{|g|}\tilde{a}) \circ \eta_i^{-1}) \chi \sum_{k,l=1}^2 g^{kl} \circ \eta_i^{-1} \frac{\partial(\tilde{u} \circ \eta_i^{-1})}{\partial x^k} \frac{\partial(\Psi_i \circ \eta_i^{-1})}{\partial x^l} & \text{on } V_i \\ \chi & \text{else} \end{cases},$$

$$c := \begin{cases} ((\sqrt{|g|}\tilde{a}\tilde{u}) \circ \eta_i^{-1}) \chi \sum_{k=1}^2 (g^{k1}, g^{k2})^\top \circ \eta_i^{-1} \frac{\partial(\Psi_i \circ \eta_i^{-1})}{\partial x^k} & \text{on } V_i \\ (\chi, \chi)^\top & \text{else} \end{cases}.$$

We use these three functions to define the functional  $F$  for every  $w \in H^1(\mathbb{R}^2)$  by

$$(29) \quad w \mapsto F(w) := f((w\chi) \circ \eta_i) \Psi_i - \frac{1}{|\mathbb{S}^2|} f(1)(\Psi_i, (w\chi) \circ \eta_i) - \int_{\mathbb{R}^2} bw \, dx + \int_{\mathbb{R}^2} c \cdot \nabla w \, dx.$$

We observe that for every  $w \in H^1(\mathbb{R}^2)$ , the function  $((w\chi) \circ \eta_i)$  can be extended to a function  $\tilde{w} \in H^1(\mathbb{S}^2)/\mathbb{R}$ , which then satisfies that

$$F(w) = f(\tilde{w}\Psi_i) - \frac{1}{|\mathbb{S}^2|} f(1)(\Psi_i, \tilde{w}) - (\tilde{a} \nabla_{\mathbb{S}^2} \tilde{u} \cdot \nabla_{\mathbb{S}^2} \Psi_i, \tilde{w}) + (\tilde{a}\tilde{u} \nabla_{\mathbb{S}^2} \Psi_i, \nabla_{\mathbb{S}^2} \tilde{w})$$

and

$$\int_{V_i} A \nabla((u\Psi_i) \circ \eta_i^{-1}) \cdot \nabla w \, dx = (\tilde{a} \nabla_{\mathbb{S}^2}(u\Psi_i), \nabla_{\mathbb{S}^2} \tilde{w}),$$

where we used that  $\chi = 1$  on  $\text{supp}(\Psi_i \circ \eta_i^{-1})$ . Since  $\text{supp}(\Psi_i \circ \eta_i^{-1}) \subset V_i$ , we obtain that

$$(30) \quad \int_{\mathbb{R}^2} A \nabla((u\Psi_i) \circ \eta_i^{-1}) \cdot \nabla w \, dx = F(w) \quad \forall w \in H^1(\mathbb{R}^2).$$

We now aim to prove finiteness of the  $H^{-1+s}(\mathbb{R}^2)$ -norm of  $F$  and to find a suitable bound. Let  $\{\hat{\Psi}_j, j \in \mathcal{I}\}$  be another partition of unity subordinate to the open cover  $\{U_j, j \in \mathcal{I}\}$  such that  $\hat{\Psi}_i \circ \eta_i^{-1} = 1$  on  $\text{supp}(\chi) \supset \text{supp}(\Psi_i \circ \eta_i^{-1})$ , which necessarily implies that  $\hat{\Psi}_j = 0$  on  $\text{supp}(\Psi_i)$  for every  $j \neq i$ . Thus we obtain with the characterization of the  $H^{1-s}(\mathbb{S}^2)$ -norm on chart domains, the partition of unity property of  $\{\hat{\Psi}_j, j \in \mathcal{I}\}$ , Proposition A.3, and [39, Theorem 3.3.2(ii)] that there are constants  $C_1, C_2, C_3$  such that for every  $w \in H^{1-s}(\mathbb{R}^2)$ , it holds that

$$\begin{aligned} |f(((w\chi) \circ \eta_i) \Psi_i)| &\leq C_1 \|f\|_{H^{-1+s}(\mathbb{S}^2)} \|(((w\chi) \circ \eta_i) \hat{\Psi}_i) \circ \eta_i^{-1}\|_{H^{1-s}(V_i)} \|\Psi_i\|_{C^1(\mathbb{S}^2)} \\ &\leq C_2 \|f\|_{H^{-1+s}(\mathbb{S}^2)} \|w\chi\|_{H^{1-s}(V_i)} \|\hat{\Psi}_i \circ \eta_i^{-1}\|_{C^1(\bar{V}_i)} \\ &\leq C_3 \|f\|_{H^{-1+s}(\mathbb{S}^2)} \|w\|_{H^{1-s}(\mathbb{R}^2)} \|\chi\|_{C^1(\bar{V}_i)}. \end{aligned}$$

The fourth summand in the definition of  $F$  in (29) can be written in a distributional sense as  $w \mapsto -\int_{\mathbb{R}^2} (\nabla \cdot c) w \, dx$ , where we applied that  $c$  is compactly supported in  $V_i$ . Note that for  $\ell = 1, 2$  and  $s \in \mathbb{R}$ , the linear operators  $\frac{\partial}{\partial x^\ell} : H^s(\mathbb{R}^2) \rightarrow H^{s-1}(\mathbb{R}^2)$  are bounded. Hence, we conclude as in the proof of Proposition A.3 with [39, Theorem 3.3.2(ii)] and the property that  $\chi = 1$  on  $\text{supp}(\hat{\Psi}_i \circ \eta_i^{-1})$  that there exist constants  $C_1, C_2, C_3, C_4$  such that

$$\begin{aligned} \|\nabla \cdot c\|_{H^{-1+s}(\mathbb{R}^2)} &\leq C_1 \|c\|_{H^s(\mathbb{R}^2)} \leq C_2 \|(\tilde{a} \circ \eta_i^{-1}) \chi\|_{C^{0,\gamma}(\bar{V}_i)} \|\Psi_i \circ \eta_i^{-1}\|_{C^1(\bar{V}_i)} \|(\tilde{u} \circ \eta_i^{-1}) \chi\|_{H^s(V_i)} \\ &\leq C_3 \|(\tilde{a} \hat{\Psi}_i) \circ \eta_i^{-1}\|_{C^{0,\gamma}(\bar{V}_i)} \|(\tilde{u} \hat{\Psi}_i) \circ \eta_i^{-1}\|_{H^s(V_i)} \|\chi\|_{C^{0,\gamma}(\bar{V}_i)}^2 \\ &\leq C_4 \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)} \|\tilde{u}\|_{H^s(\mathbb{S}^2)}, \end{aligned}$$

where we applied that derivatives of smooth compactly supported functions, e.g.,  $\Psi_i \circ \eta_i^{-1}$  and  $\chi$ , are bounded. Their norms have been included into the constants appearing in the above

inequalities. The  $H^{-1+s}(\mathbb{R}^2)$ -norm of the third summand in (29) can be treated similarly, i.e., there exists a constant  $C$  such that  $\|b\|_{H^{-1+s}(\mathbb{R}^2)} \leq C\|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)}$ . The second summand in (29) poses no difficulty. Hence, we conclude that  $F \in H^{-1+s}(\mathbb{R}^2)$  and that there exists a constant  $C$ , which is independent of  $\tilde{a}$ ,  $\tilde{u}$ , and  $f$ , such that

$$(31) \quad \|F\|_{H^{-1+s}(\mathbb{R}^2)} \leq C(\|f\|_{H^{-1+s}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^s(\mathbb{S}^2)}).$$

We observe that for every  $\xi \in \mathbb{R}^2$ , it holds that  $\xi^\top A \xi \geq \lambda_g \min_{x \in V_i} \sqrt{|g(x)|} \min_{x \in \mathbb{S}^2} \tilde{a}(x) \xi^\top \xi$  on  $\mathbb{R}^2$ . Since the matrix-valued function  $A$  is constant on the complement of  $V_i$ , we observe that there exists a constant  $C$  such that

$$(32) \quad \sup_{x,y \in \mathbb{R}^2, x \neq y} \frac{\|A(x) - A(y)\|_{\mathbb{R}^{2 \times 2}}}{\|x - y\|_{\mathbb{R}^2}^\gamma} \leq C\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}.$$

We are now in the situation to apply the regularity estimate in [6, Lemma 3.2] to the problem in (30), which implies that  $(u\Psi_i) \circ \eta_i^{-1} \in H^{1+s}(\mathbb{R}^2)$ . Also it implies together with the estimates in (31) and in (32) that there exist constants  $C_1, C_2, C_3$  such that

$$\begin{aligned} \|(\tilde{u}\Psi_i) \circ \eta_i^{-1}\|_{H^{1+s}(\mathbb{R}^2)} &\leq C_1 \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|(\tilde{u}\Psi_i) \circ \eta_i^{-1}\|_{H^1(\mathbb{R}^2)} + \|F\|_{H^{-1+s}(\mathbb{R}^2)}) \\ &\quad + C_1 \|(\tilde{u}\Psi_i) \circ \eta_i^{-1}\|_{H^1(\mathbb{R}^2)} \\ &\leq C_2 \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) \\ &\quad + \|\tilde{a}\|_{C^0(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^s(\mathbb{S}^2)} + C_1 \|\tilde{u}\|_{H^1(\mathbb{S}^2)} \\ &\leq C_3 \left( \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) + \|\tilde{u}\|_{H^1(\mathbb{S}^2)} \right), \end{aligned}$$

where the first inequality is the estimate from [6, Lemma 3.2] applied to our setting.

This argument can be repeated for all remaining  $i \in \mathcal{I}$ , which implies that  $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$ , and therefore we can establish the previous estimate for every  $i \in \mathcal{I}$ . Hence, we sum this squared estimate over all  $i \in \mathcal{I}$  and take the square root. We maximize the constants over the finite index set  $\mathcal{I}$  which establishes the estimate claimed in the lemma.  $\square$

It remains to bound the  $H^1(\mathbb{S}^2)$ -norm in the previous lemma with the bound obtained from the Lax–Milgram lemma to obtain the following proposition.

**Proposition C.2.** *For some  $0 \leq s < \gamma < 1$ , let  $\tilde{u} \in H^1(\mathbb{S}^2)/\mathbb{R}$ ,  $f \in H^{-1+s}(\mathbb{S}^2)$ , and  $\tilde{a} \in C^{0,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$  satisfy (11), then,  $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$  and there exists a constant  $C$ , which is independent of  $\tilde{u}$ ,  $f$ , and  $\tilde{a}$ , such that*

$$\|\Phi_f(\tilde{a})\|_{H^{1+s}(\mathbb{S}^2)} = \|\tilde{u}\|_{H^{1+s}(\mathbb{S}^2)} \leq C\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|1/\tilde{a}\|_{C^0(\mathbb{S}^2)}^2\|f\|_{H^{-1+s}(\mathbb{S}^2)}.$$

*Proof.* From Lemma C.1 we readily conclude that  $\tilde{u} \in H^{1+s}(\mathbb{S}^2)$ . Also this lemma implies with the  $H^1(\mathbb{S}^2)/\mathbb{R}$ -estimate in (14) that there exist constants  $C_1, C_2$  such that

$$\begin{aligned} \|\tilde{u}\|_{H^{1+s}(\mathbb{S}^2)} &\leq C_1 \left( \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|\tilde{u}\|_{H^1(\mathbb{S}^2)} + \|f\|_{H^{-1+s}(\mathbb{S}^2)}) + \|\tilde{u}\|_{H^1(\mathbb{S}^2)} \right) \\ &\leq C_2 \|\tilde{a}\|_{C^{0,\gamma}(\mathbb{S}^2)}\|1/\tilde{a}\|_{C^0(\mathbb{S}^2)}^2\|f\|_{H^{-1+s}(\mathbb{S}^2)}, \end{aligned}$$

where we applied that  $\|\tilde{a}\|_{C^0(\mathbb{S}^2)}/(\min_{x \in \mathbb{S}^2} \tilde{a}(x)) \geq 1$  and  $1/(\min_{x \in \mathbb{S}^2} \tilde{a}(x))^2 = \|1/\tilde{a}\|_{C^0(\mathbb{S}^2)}^2$  before summarizing the resulting terms.  $\square$

This finishes the proof of the base case in Proposition 3.6. In the following we show recursively higher order regularity with the known theory for the operator  $\text{Id} - \Delta_{\mathbb{S}^2}$  presented in Section 2 to analyze the domain and the respective range of  $\Phi_f$  more precisely.

**Proposition C.3.** *Let  $\iota \in \mathbb{N}_0$ ,  $\gamma \in (0, 1)$ , and  $s \in [0, +\infty)$  satisfy  $s < \iota + \gamma$ . If  $f \in H^{-1+s}(\mathbb{S}^2)$ , then it holds that*

$$\Phi_f : C^{\iota, \gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2) \rightarrow H^{1+s}(\mathbb{S}^2)$$

is continuous with respect to the topology of  $C^{\iota, \gamma}(\mathbb{S}^2)$ .

Moreover if  $s \geq 1$ , then for every  $n \in \{0, \dots, \lfloor s \rfloor - 1\}$ , there exists a constant  $C > 0$  such that for every  $\tilde{a} \in C^{\iota, \gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$ ,

$$\begin{aligned} \|\Phi_f(\tilde{a})\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \\ \leq C \|1/\tilde{a}\|_{C^{n, \gamma}(\mathbb{S}^2)} (\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{n+1, \gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a})\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}) \end{aligned}$$

where  $\{s\}$  denotes the fractional part of  $s$ .

*Proof.* The case  $s \in [0, 1)$  will serve as a base case for an induction argument. There the case  $s = 0$  is already known from Proposition 3.2. So let  $s \in (0, 1)$  and assume that  $\iota = 0$  and  $\gamma \in (s, 1)$ . From Proposition C.2 we infer that  $\Phi_f(\tilde{a}) \in H^{1+s}(\mathbb{S}^2)$ , which establishes the claimed domain and range of  $\Phi_f$ . To prove the continuity of  $\Phi_f$  let  $(\tilde{a}_j, j \in \mathbb{N}_0)$  be a sequence in  $C^{0, \gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$  such that  $\|\tilde{a}_j - \tilde{a}_0\|_{C^{0, \gamma}(\mathbb{S}^2)} \rightarrow 0$  as  $j \rightarrow +\infty$ . We observe that for every  $j \in \mathbb{N}$ , it holds that

$$(33) \quad (\tilde{a}_0 \nabla_{\mathbb{S}^2} (\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)), \nabla_{\mathbb{S}^2} v) = -(\tilde{a}_0 - \tilde{a}_j) \nabla_{\mathbb{S}^2} \Phi_f(\tilde{a}_j), \nabla_{\mathbb{S}^2} v) \quad \forall v \in H^1(\mathbb{S}^2)/\mathbb{R}.$$

Since  $\Phi_f(\tilde{a}_j) \in H^{1+s}(\mathbb{S}^2)$ ,  $j \in \mathbb{N}$ , we obtain with Proposition A.3 that there exist constants  $C_1, C_2$  such that

$$\begin{aligned} \|\nabla_{\mathbb{S}^2} \cdot ((\tilde{a}_0 - \tilde{a}_j) \nabla_{\mathbb{S}^2} \Phi_f(\tilde{a}_j))\|_{H^{-1+s}(\mathbb{S}^2)} &\leq C_1 \|(\tilde{a}_0 - \tilde{a}_j) \nabla_{\mathbb{S}^2} \Phi_f(\tilde{a}_j)\|_{H^s(\mathbb{S}^2)} \\ &\leq C_2 \|\tilde{a}_0 - \tilde{a}_j\|_{C^{0, \gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a}_j)\|_{H^{1+s}(\mathbb{S}^2)}. \end{aligned}$$

Hence, Proposition C.2 applied to the setting in (33) implies that there exists a constant  $C$ , which is independent of  $(\tilde{a}_j, j \in \mathbb{N}_0)$  and  $f$ , such that for every  $j \in \mathbb{N}$ , it holds that

$$\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+s}(\mathbb{S}^2)} \leq C \frac{\|\tilde{a}_0\|_{C^{0, \gamma}(\mathbb{S}^2)}}{(\min_{x \in \mathbb{S}^2} \tilde{a}_0(x))^2} \frac{\|\tilde{a}_j\|_{C^{0, \gamma}(\mathbb{S}^2)}}{(\min_{x \in \mathbb{S}^2} \tilde{a}_j(x))^2} \|f\|_{H^{-1+s}(\mathbb{S}^2)} \|\tilde{a}_0 - \tilde{a}_j\|_{C^{0, \gamma}(\mathbb{S}^2)}.$$

We have  $\|\tilde{a}_0 - \tilde{a}_j\|_{C^0(\mathbb{S}^2)} =: \epsilon_j \rightarrow 0$  as  $j \rightarrow +\infty$ , which implies that  $\epsilon_j \leq 1/2 \min_{x \in \mathbb{S}^2} \tilde{a}_0(x)$  for every  $j$  that are sufficiently large, i.e.,  $j > j_0$  for some  $j_0 \in \mathbb{N}$ . Since  $\tilde{a}_j(x') \geq \min_{x \in \mathbb{S}^2} \tilde{a}_0(x) - \epsilon_j$  for every  $x' \in \mathbb{S}^2$ , we obtain that  $1/\min_{x \in \mathbb{S}^2} \tilde{a}_j(x) \leq 2/\min_{x \in \mathbb{S}^2} \tilde{a}_0(x)$  for every  $j > j_0$ . Since  $\|\tilde{a}_j\|_{C^{0, \gamma}(\mathbb{S}^2)}$  and  $(\min_{x \in \mathbb{S}^2} \tilde{a}_j(x))^{-2}$  can be bounded independently of  $j$ , it follows that  $\|\Phi_f(\tilde{a}_j) - \Phi_f(\tilde{a}_0)\|_{H^{1+s}(\mathbb{S}^2)} \rightarrow 0$  as  $j \rightarrow +\infty$ , i.e.,  $\Phi_f : C^{0, \gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2) \rightarrow H^{1+s}(\mathbb{S}^2)$  is continuous.

For  $s \geq 1$ , it must hold that  $\iota \geq 1$ . Since  $\tilde{a} \in C^{\iota, \gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$  and  $\tilde{u} := \Phi_f(\tilde{a})$ , for any  $w \in H^1(\mathbb{S}^2)$ , we take  $w/\tilde{a} - 1/|\mathbb{S}^2| \int_{\mathbb{S}^2} w/\tilde{a} d\sigma \in H^1(\mathbb{S}^2)/\mathbb{R}$  as a test function and thus rewrite the PDE in (11) as

$$(\tilde{u}, w) + \left(f, \frac{w}{\tilde{a}}\right) = (\tilde{u}, w) + \left(\tilde{a} \nabla_{\mathbb{S}^2} \tilde{u}, \nabla_{\mathbb{S}^2} \left(\frac{w}{\tilde{a}}\right)\right) = (\tilde{u}, w) + (\nabla_{\mathbb{S}^2} \tilde{u}, \nabla_{\mathbb{S}^2} w) - \left(\frac{\nabla_{\mathbb{S}^2} \tilde{a} \cdot \nabla_{\mathbb{S}^2} \tilde{u}}{\tilde{a}}, w\right),$$

where we applied that  $(f, 1) = 0$ . Hence, for every  $w \in H^1(\mathbb{S}^2)$ , it holds that

$$(\tilde{u}, w) + (\nabla_{\mathbb{S}^2} \tilde{u}, \nabla_{\mathbb{S}^2} w) = \left(\frac{f + \nabla_{\mathbb{S}^2} \tilde{a} \cdot \nabla_{\mathbb{S}^2} \tilde{u}}{\tilde{a}}, w\right) + (\tilde{u}, w),$$



which is stated with equality in  $H^{-1}(\mathbb{S}^2)$  as

$$(34) \quad (\text{Id} - \Delta_{\mathbb{S}^2})\tilde{u} = \frac{f + \nabla_{\mathbb{S}^2}\tilde{a} \cdot \nabla_{\mathbb{S}^2}\tilde{u}}{\tilde{a}} + \tilde{u} =: F.$$

We observe with (7) that  $(\text{Id} - \Delta_{\mathbb{S}^2})^{-1}$  is a linear and bounded operator from  $H^r(\mathbb{S}^2)$  to  $H^{r+2}(\mathbb{S}^2)$  for every  $r \in \mathbb{R}$ . The claim is now shown by induction. Let us write  $s = \lfloor s \rfloor + \{s\}$ , where  $\{s\} \in [0, 1)$  is the fractional part of  $s$ , and assume as induction hypothesis that  $\Phi_f : C^{n,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2) \rightarrow H^{1+n+\{s\}}(\mathbb{S}^2)$  is continuous for every  $n \in \{0, 1, \dots, \lfloor s \rfloor - 1\}$ , which we already showed for  $n = 0$ . Let  $n \in \{0, 1, \dots, \lfloor s \rfloor - 1\}$  and let  $\tilde{a} \in C^{n+1,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$ . Since by our induction hypothesis  $\tilde{u} = \Phi_f(\tilde{a}) \in H^{1+n+\{s\}}(\mathbb{S}^2)$ , we conclude with Proposition A.3 that the right hand side  $F$  in (34) is in  $H^{1+(n-1)+\{s\}}$ . The fact that  $(\text{Id} - \Delta_{\mathbb{S}^2})^{-1}$  is a linear and bounded operator from  $H^{1+(n-1)+\{s\}}(\mathbb{S}^2)$  to  $H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$  implies that  $\tilde{u} = \Phi_f(\tilde{a}) \in H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$ . Moreover it implies with Proposition A.3 a regularity estimate for  $\tilde{u} = \Phi_f(\tilde{a})$ , i.e., there exist constants  $C_1, C_2, C_3$  that are independent of  $\tilde{a}$  and  $f$  such that

$$\begin{aligned} & \|\Phi_f(\tilde{a})\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \\ & \leq C_1 \|F\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} \\ & \leq C_2 \left( \|1/\tilde{a}\|_{C^{n,\gamma}(\mathbb{S}^2)} (\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a})\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}) \right. \\ & \quad \left. + \|\Phi_f(\tilde{a})\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} \right) \\ & \leq C_2 \|1/\tilde{a}\|_{C^{n,\gamma}(\mathbb{S}^2)} (\|f\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} + \|\tilde{a}\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a})\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}), \end{aligned}$$

where the last inequality holds since

$$\|\Phi_f(\tilde{a})\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} \leq \|\Phi_f(\tilde{a})\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}$$

and  $\|1/\tilde{a}\|_{C^{n,\gamma}(\mathbb{S}^2)} \|\tilde{a}\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \geq 1$ . This is the desired recursion formula and implies the claimed domain and range of  $\Phi_f$ . To prove continuity of  $\Phi_f$  let  $(\tilde{a}_j, j \in \mathbb{N}_0)$  be a sequence in  $C^{n+1,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2)$  such that  $\|\tilde{a}_j - \tilde{a}_0\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \rightarrow 0$  as  $j \rightarrow +\infty$  and let  $(\tilde{u}_j = \Phi_f(\tilde{a}_j), j \in \mathbb{N}_0)$  be the sequence of respective solutions. The same manipulations that showed (34) imply with (33) that

$$(\text{Id} - \Delta_{\mathbb{S}^2})(\tilde{u}_0 - \tilde{u}_j) = \frac{\nabla_{\mathbb{S}^2} \cdot ((\tilde{a}_0 - \tilde{a}_j)\nabla_{\mathbb{S}^2}\tilde{u}_j) + \nabla_{\mathbb{S}^2}\tilde{a}_0 \cdot \nabla_{\mathbb{S}^2}(\tilde{u}_0 - \tilde{u}_j)}{\tilde{a}_0} + (\tilde{u}_0 - \tilde{u}_j).$$

Similar estimates as for  $\Phi_f(\tilde{a})$  above imply that

$$\begin{aligned} & \|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \\ & \leq C' \left( \|1/\tilde{a}_0\|_{C^{n,\gamma}(\mathbb{S}^2)} (\|\tilde{a}_0 - \tilde{a}_j\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a}_0)\|_{H^{1+n+\{s\}}(\mathbb{S}^2)} \right. \\ & \quad \left. + \|\tilde{a}_0\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+n+\{s\}}(\mathbb{S}^2)}) \right. \\ & \quad \left. + \|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+(n-1)+\{s\}}(\mathbb{S}^2)} \right). \end{aligned}$$

Since by our induction hypothesis  $\Phi_f : C^{n,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2) \rightarrow H^{1+n+\{s\}}(\mathbb{S}^2)$  is continuous,  $\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{n+\{s\}}(\mathbb{S}^2)} \leq \|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+n+\{s\}}(\mathbb{S}^2)} \rightarrow 0$  as  $j \rightarrow +\infty$ , which implies with  $\|\tilde{a}_0 - \tilde{a}_j\|_{C^{n+1,\gamma}(\mathbb{S}^2)} \rightarrow 0$  as  $j \rightarrow +\infty$  that  $\|\Phi_f(\tilde{a}_0) - \Phi_f(\tilde{a}_j)\|_{H^{1+(n+1)+\{s\}}(\mathbb{S}^2)} \rightarrow 0$  as  $j \rightarrow +\infty$ , i.e.,  $\Phi_f : C^{n+1,\gamma}(\mathbb{S}^2) \cap C_+^0(\mathbb{S}^2) \rightarrow H^{1+(n+1)+\{s\}}(\mathbb{S}^2)$  is continuous. This finishes the induction and the proof of the proposition.  $\square$

We finish this appendix by remarking that the  $H^{1+s}(\mathbb{S}^2)$ -regularity for every  $s < \beta/2$  of the solution can also be deduced from higher order Hölder regularity, which is implied by Schauder estimates, cp. [12, Chapters 6 and 8], applied to pullbacks of the solution to the chart domains. Specifically, the continuous embedding  $C^{\iota,\gamma}(\mathbb{S}^2) \subset H^{s'}(\mathbb{S}^2)$  for  $\iota \in \mathbb{N}_0, \gamma \in (0, 1)$ , and  $s' \geq 0$  such that  $\iota + \gamma > s'$ , which is an immediate consequence of Proposition A.3, would imply  $H^{1+s}(\mathbb{S}^2)$ -regularity. Since the explicit dependence of the coefficients of the elliptic operator in these estimates is analyzed in [20, Section 8.2],  $L^p$ -integrability could also be deduced.

#### APPENDIX D. FINITE ELEMENT CONVERGENCE FOR ELLIPTIC PDES

*Proof of Proposition 4.2.* We observe that Proposition 3.6 implies that  $\Phi_f(\tilde{a}) \in H^{1+s}(\mathbb{S}^2)$ . The approximation property for integer orders, cp. [8, Proposition 2.7], implies by interpolation that for every  $h$  and  $k$ , there exists an interpolation operator  $I^{h,k}$  which is, for every  $s > 0$ , continuous from  $H^{1+s}(\mathbb{S}^2) \rightarrow S^k(\mathbb{S}^2, \mathcal{T}_h)$  and a constant  $C_s > 0$  such that for every  $h > 0$  and for every function  $v \in H^{1+s}(\mathbb{S}^2)$ , it holds that

$$(35) \quad \|v - I^{h,k}v\|_{H^1(\mathbb{S}^2)} \leq C_s h^{\min\{s,k\}} \|v\|_{H^{1+s}(\mathbb{S}^2)},$$

where  $C_s > 0$  is independent of  $h$  but depends on  $s$ . The coercivity and Galerkin orthogonality imply in the usual fashion that for every  $v^h \in V^{h,k}$ , it holds that

$$(36) \quad \begin{aligned} \|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}}^2 &\leq \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\tilde{a} \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})), \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a}))) \\ &= \frac{1}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} (\tilde{a} \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})), \nabla_{\mathbb{S}^2}(\Phi_f(\tilde{a}) - v^h)) \\ &\leq \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)/\mathbb{R}}. \end{aligned}$$

When we equip  $H^1(\mathbb{S}^2)/\mathbb{R}$  with the  $H^1(\mathbb{S}^2)$ -norm, the orthogonal decomposition  $H^1(\mathbb{S}^2) = H^1(\mathbb{S}^2)/\mathbb{R} \oplus \text{span}\{1\}$  holds, which implies with (36) that

$$\begin{aligned} \|\Phi_f(\tilde{a}) - \Phi_f^{h,k}(\tilde{a})\|_{H^1(\mathbb{S}^2)/\mathbb{R}} &\leq \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \inf_{v^h \in V^{h,k}} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)/\mathbb{R}} \\ &\leq \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \inf_{v^h \in V^{h,k}} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)} \\ &= \frac{\|\tilde{a}\|_{C^0(\mathbb{S}^2)}}{\min_{x \in \mathbb{S}^2} \tilde{a}(x)} \inf_{v^h \in S^k(\mathbb{S}^2, \mathcal{T}_h)} \|\Phi_f(\tilde{a}) - v^h\|_{H^1(\mathbb{S}^2)}, \end{aligned}$$

where we also used that  $\Phi_f(\tilde{a}) \in H^1(\mathbb{S}^2)/\mathbb{R}$ . Now the claim follows with (35).  $\square$

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