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Weak convergence rates for Euler-type approximations of semilinear stochastic evolution equations with nonlinear diffusion coefficients

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Abstract

Strong convergence rates for time-discrete numerical approximations of semilinear stochastic evolution equations (SEEs) with smooth and regular nonlinearities are well understood in the literature. Weak convergence rates for time-discrete numerical approximations of such SEEs have been investigated since about 12 years and are far away from being well understood: roughly speaking, no essentially sharp weak convergence rates are known for time-discrete numerical approximations of parabolic SEEs with nonlinear diffusion coefficient functions; see Remark 2.3 in [A. Debussche, *Weak approximation of stochastic partial differential equations: the nonlinear case*, Math. Comp. **80** (2011), no. 273, 89–117] for details. In the recent article [D. Conus, A. Jentzen & R. Kurniawan, *Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients*, arXiv:1408.1108] the weak convergence problem emerged from Debussche's article has been solved in the case of spatial spectral Galerkin approximations for semilinear SEEs with nonlinear diffusion coefficient functions. In this article we overcome the problem emerged from Debussche's article in the case of a class of time-discrete Euler-type approximation methods (including exponential and linear-implicit Euler approximations as special cases) and, in particular, we establish essentially sharp weak convergence rates for linear-implicit Euler approximations of semilinear SEEs with nonlinear diffusion coefficient functions. Key ingredients of our approach are applications of a mild Itô type formula and the use of suitable semilinear integrated counterparts of the time-discrete numerical approximation processes.

Contents

1 Introduction

This article studies weak convergence rates for time-discrete numerical approximations of semilinear stochastic evolution equations (SEEs). We first review a few weak convergence results from the literature and then present the main weak convergence result obtained in this article. This introductory section is based on Section 1 of Conus et al. [11]. For finite dimensional stochastic ordinary differential equations (SODEs) with smooth and regular nonlinearities both strong and numerically weak convergence rates of time-discrete numerical approximations are well understood in the literature; see, e.g., the monographs Kloeden & Platen [27] and Milstein [34]. The situation is different in the case of possibly infinite dimensional semilinear stochastic evolution equations (SEEs). While strong convergence rates for time-discrete numerical approximations of semilinear SEEs with smooth and regular nonlinearities are well understood in the literature, weak convergence rates for time-discrete numerical approximations of such SEEs have been investigated since about 12 years and are far away from being well understood: roughly speaking, no essentially sharp weak convergence rates are known for time-discrete numerical approximations of parabolic SEEs with nonlinear diffusion coefficient functions; see Remark 2.3 in Debussche [17] for details. In this article we overcome the problem emerged from Debussche's article in the case of a class of time-discrete Euler-type approximation methods for SEEs (including exponential and linear-implicit Euler approximations as special cases) and, in particular, we establish essentially sharp weak convergence rates for linear-implicit Euler approximations of semilinear SEEs with nonlinear diffusion coefficient functions. To illustrate the weak convergence problem emerged from Debussche's article and our solution to this problem we consider the following setting as a special case of our general setting in Section 8 below. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, let $A\colon D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with $\sup\big(\mathrm{Re}(\mathrm{spectrum}(A))\big) < 0$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., Theorem and Definition 2.5.32 in [25]), let $\iota \in \mathbb{R}$, $\xi \in H_\iota$, $\gamma \in [0, \frac{1}{2}]$, and let $F\colon H_\iota \to (\cap_{r<\iota-\gamma} H_r)$ and $B\colon H_\iota \to \mathrm{Lin}(U, \cap_{r<\iota-\gamma/2} H_r)$ be functions with the property that $\forall\, r \in (-\infty, \iota-\gamma)\colon \big[\big(H_\iota \ni v \mapsto F(v) \in H_r\big) \in C_b^5(H_\iota, H_r)\big]$, with the property that $\forall\, r \in (-\infty, \iota - \frac{\gamma}{2}), v \in H_\iota\colon \big[\big(U \ni u \mapsto B(v)u \in H_r\big) \in HS(U, H_r)\big]$, and with the property that $\forall\, r \in (-\infty, \iota - \frac{\gamma}{2})\colon \big[\big(H_\iota \ni v \mapsto [U \ni u \mapsto B(v)u \in H_r] \in HS(U, H_r)\big) \in C_b^5(H_\iota, HS(U, H_r))\big]$, where for two $\mathbb{R}$-vector spaces V_1 and V_2 we denote by $\mathrm{Lin}(V_1, V_2)$ the set of all linear mappings from V_1 to V_2 and where for two $\mathbb{R}$-Banach spaces $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ we denote by $C_b^5(V_1, V_2)$ the set of all five times continuously Fréchet differentiable functions from V_1 to V_2 which have globally bounded derivatives (see Subsection 1.2 below for more details). The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [12], Theorem 4.3 in Brzeźniak [8], Theorem 6.2 in Van Neerven et al. [40]) the existence of a continuous mild solution process $X\colon [0,T] \times \Omega \to H_\iota$ of the SEE

$$dX_t = [AX_t + F(X_t)]\, dt + B(X_t)\, dW_t, \qquad t \in [0,T], \qquad X_0 = \xi. \tag{1}$$

As an example for (1), we think of $H = U = L^2((0,1);\mathbb{R})$ being the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$ and A being a linear differential operator on H. In particular, in Subsection 1.1.1 we formulate the continuous version of *the parabolic Anderson model* as an example of (1) (in that case the parameter γ, which controls the regularity of the operators F and B, satisfies $\gamma = \frac{1}{2}$) and in Subsection 1.1.2 we formulate *a fourth-order stochastic partial differential equation* as an example of (1) (in that example we have $\gamma = \frac{1}{4}$). In this work we are interested in the analysis of numerical approximations of (1). For example, let $Y^N\colon \{0,1,\ldots,N\} \times \Omega \to H_\iota$, $N \in \mathbb{N}$, be stochastic processes with the property that for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$ it holds $\mathbb{P}$-a.s. that

$$Y_0^N = \xi, \qquad Y_{n+1}^N = \big(\mathrm{Id}_H - \tfrac{T}{N}A\big)^{-1}\Big(Y_n + F(Y_n)\tfrac{T}{N} + \textstyle\int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B(Y_n)\, dW_s\Big). \tag{2}$$

The stochastic processes Y^N, $N \in \mathbb{N}$, are referred to as linear-implicit Euler approximations of (1).

Strong convergence rates for numerical approximations for SEEs of the form (1) are well understood. Weak convergence rates for numerical approximations of SEEs of the form (1) have been investigated since about 12 years; cf. [39, 20, 16, 18, 19, 21, 17, 29, 5, 43, 33, 30, 4, 7, 3, 32, 6, 31, 28, 42, 11, 41, 2]. Except for Debussche & De Bouard [16], Debussche [17], Andersson & Larsson [4], and Conus et al. [11], all of the above cited weak convergence results assume, beside further assumptions, that the considered SEE is driven by additive noise. In Debussche & De Bouard [16] weak convergence rates for the nonlinear Schrödinger equation, whose dominant linear operator generates a group (see Section 2 in [16]) instead of only a semigroup as in the general setting of the SEE (1), are analyzed. The method of proof in Debussche & De Bouard [16] strongly exploits this property of the nonlinear Schrödinger equation (see Section 5.2 in [16]). Therefore, the method of proof in [16] can, in general, not be used to establish weak convergence rates for the SEE (1). In Debussche's seminal article [17], essentially sharp weak convergence rates for *linear-implicit Euler approximations* (see (2)) of SEEs of the form (1) are established under the hypothesis that $\iota = 0$ and that the second derivative of the diffusion coefficient B satisfies the smoothing property that there exists an $L \in \mathbb{R}$ such that for all $x, v, w \in H$ it holds that[1]

$$\|B''(x)(v,w)\|_{L(H)} \leq L \, \|v\|_{H_{-1/4}} \, \|w\|_{H_{-1/4}} . \tag{3}$$

The article Andersson & Larsson [4] also assumes (3) but establishes weak convergence rates for spatial approximations. As pointed out in Remark 2.3 in Debussche [17], assumption (3) is a serious restriction for SEEs of the form (1). Roughly speaking, assumption (3) imposes that the second derivative of the diffusion coefficient function vanishes and thus that the diffusion coefficient function is affine linear. Remark 2.3 in Debussche [17] also asserts that assumption (3) is crucial in the weak convergence proof in [17], that assumption (3) is used in an essential way in Lemma 4.5 in [17] and that Lemma 4.5 in [17], in turn, is used at many points in the weak convergence proof in [17]. Debussche's article naturally suggests the problem of establishing essentially sharp weak convergence rates in the case where Debussche's assumption (3) is not satisfied. In Conus [11] essentially sharp weak convergence rates have been established without imposing Debussche's assumption (3) in the case of spatial spectral Galerkin approximations. To the best of our knowledge, it remained an open problem to establish essentially sharp weak convergence rates for time-discrete numerical approximations of the SEE (1) without imposing Debussche's assumption (3). In this article we overcome this problem in the case of a class of time-discrete Euler-type approximation methods for SEEs (including exponential and linear-implicit Euler approximations as special cases) and, in particular, we establish essentially sharp weak convergence rates for linear-implicit Euler approximations of semilinear SEEs with nonlinear diffusion coefficient functions. This is the subject of the following result, Theorem 1.1. Theorem 1.1 follows[2] immediately from Corollary 8.2 and Subsection 1.5.2.

Theorem 1.1. *Assume the setting in the first paragraph of Section 1 and let $\varphi \in C_b^5(H_\iota, \mathbb{R})$. Then for every $\varepsilon \in (0,\infty)$ there exists a $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ it holds that*

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_N^N)\big]\big| \leq C \cdot N^{-(1-\gamma-\varepsilon)}. \tag{4}$$

Let us add a few comments regarding Theorem 1.1. First, we would like to emphasize that in the general setting of Theorem 1.1, the weak convergence rate established in Theorem 1.1 can *essentially not be improved*. More specifically, in Corollary 9.8 in Section 9 below we give for every $\iota \in \mathbb{R}$, $\gamma \in [0, \frac{1}{2}]$ examples of $A\colon D(A) \subseteq H \to H$, $F\colon H_\iota \to (\cap_{r<\iota-\gamma} H_r)$, $B\colon H_\iota \to \operatorname{Lin}(U, \cap_{r<\iota-\gamma/2} H_r)$ and $\varphi \in C_b^5(H_\iota, \mathbb{R})$ with $\forall\, r \in (-\infty, \iota-\gamma)\colon \big[\big(H_\iota \ni v \mapsto F(v) \in H_r\big) \in C_b^5(H_\iota, H_r)\big]$, $\forall\, r \in (-\infty, \iota - \frac{\gamma}{2}), v \in H_\iota\colon \big[\big(U \ni u \mapsto B(v)u \in H_r\big) \in HS(U, H_r)\big]$, $\forall\, r \in (-\infty, \iota - \frac{\gamma}{2})\colon \big[\big(H_\iota \ni v \mapsto [U \ni u \mapsto B(v)u \in H_r] \in HS(U,H_r)\big) \in C_b^5(H_\iota, HS(U,H_r))\big]$ such that there exists a $C \in (0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_N^N)\big]\big| \geq C \cdot N^{-(1-\gamma)}. \tag{5}$$

[1]Assumption (3) above slighlty differs from the original assumption in [17] as we believe that there is a small typo in equation (2.5) in [17]; see inequality (4.3) in the proof of Lemma 4.5 in [17] for details.

[2]with $H = H_\iota$ in the notation of Corollary 8.2

In addition, we emphasize that in the setting of Theorem 1.1 it is well known that for every $\varepsilon \in (0,\infty)$ there exists a $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ it holds that

$$\big(\mathbb{E}\big[\|X_T - Y_N^N\|_{H_\iota}^2\big]\big)^{1/2} \leq C \cdot N^{-(\frac{1-\gamma}{2}-\varepsilon)}. \tag{6}$$

The weak convergence rate $1-\gamma-\varepsilon$ established in Theorem 1.1 is thus *twice the well known strong convergence rate* $\frac{1-\gamma-\varepsilon}{2}$ in (6). Next we add that Theorem 1.1 is, to the best of our knowledge, the first result in the literature which establishes the essentially sharp weak convergence rate $1-\gamma-\varepsilon$ for time-discrete numerical approximations of the continuous version of *the parabolic Anderson model* (see Subsection 1.1.1 for details).

In the following we briefly outline a few key ideas in the proof of Theorem 1.1 (and Corollary 8.2 respectively). For simplicity we restrict ourself to the case $\iota = 0$. Our proof of Theorem 1.1 is partially based on the proof of the weak convergence result in Conus et al. [11]. The first step in the proof of Theorem 1.1 is to rewrite the time-discrete stochastic processes Y^N, $N \in \mathbb{N}$, (see (2)) as appropriate time-continuous stochastic processes (see (8) below). More formally, let $\lfloor \cdot \rfloor_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be the mappings with the property that for all $h \in (0,\infty)$, $t \in \mathbb{R}$ it holds that $\lfloor t \rfloor_h = \max\big((-\infty,t] \cap \{0,h,-h,2h,-2h,\dots\}\big)$, let $S^N \colon \{(t_1,t_2) \in [0,T]^2 \colon t_1 \leq t_2\} \to L(H)$, $N \in \mathbb{N}$, be the mappings with the property that for all $N \in \mathbb{N}$, $(t_1,t_2) \in [0,T]^2$ with $t_1 \leq t_2$ it holds that

$$S^N_{t_1,t_2} = \Big(\mathrm{Id}_H - (t_1 - \lfloor t_1 \rfloor_{T/N})\,A\Big)\Big(\mathrm{Id}_H - (t_2 - \lfloor t_2 \rfloor_{T/N})\,A\Big)^{-1}\Big(\mathrm{Id}_H - \tfrac{T}{N}\,A\Big)^{-(\lfloor t_2 \rfloor_{T/N} - \lfloor t_1 \rfloor_{T/N})\,N/T} \tag{7}$$

(cf., e.g., (142) in Da Prato et al. [12]), and let $\tilde{Y}^N \colon [0,T] \times \Omega \to H$, $N \in \mathbb{N}$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes with the property that for all $N \in \mathbb{N}$, $t \in [0,T]$ it holds P-a.s. that

$$\begin{aligned}\tilde{Y}^N_t &= S^N_{0,t}\,\xi \\ &+ \int_0^t S^N_{s,t}\,\big(\mathrm{Id}_H - (s - \lfloor s \rfloor_{T/N})\,A\big)^{-1} F\big(\tilde{Y}^N_{\lfloor s \rfloor_{T/N}}\big)\,ds + \int_0^t S^N_{s,t}\,\big(\mathrm{Id}_H - (s - \lfloor s \rfloor_{T/N})\,A\big)^{-1} B\big(\tilde{Y}^N_{\lfloor s \rfloor_{T/N}}\big)\,dW_s\end{aligned} \tag{8}$$

(cf. (143) in Da Prato et al. [12]). Note that for all $N \in \mathbb{N}$, $n \in \{0, \frac{T}{N}, \frac{2T}{N}, \dots, T\}$ it holds P-a.s. that $\tilde{Y}^N_{nT/N} = Y^N_n$. Moreover, recall that the solution process X of the SEE (1) satisfies that for all $t \in [0,T]$ it holds P-a.s. that

$$X_t = e^{tA}\,\xi + \int_0^t e^{(t-s)A} F(X_s)\,ds + \int_0^t e^{(t-s)A} B(X_s)\,dW_s. \tag{9}$$

The next key step in the proof of Theorem 1.1 is to use the idea in Conus et al. [11] to plug an appropriate process in between $\mathbb{E}\big[\varphi(X_T)\big]$ and $\mathbb{E}\big[\varphi(Y^N_N)\big] = \mathbb{E}\big[\varphi(\tilde{Y}^N_T)\big]$. More formally, we use the triangle inequality to obtain that for all $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\tilde{Y}^N_T)\big]\big| \leq \big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}^N_T)\big]\big| + \big|\mathbb{E}\big[\varphi(\bar{Y}^N_T)\big] - \mathbb{E}\big[\varphi(\tilde{Y}^N_T)\big]\big| \tag{10}$$

where $\bar{Y}^N \colon [0,T] \times \Omega \to H$, $N \in \mathbb{N}$, are appropriate stochastic processes so that it is in some sense not so difficult anymore to estimate $\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}^N_T)\big]\big|$ and $\big|\mathbb{E}\big[\varphi(\bar{Y}^N_T)\big] - \mathbb{E}\big[\varphi(\tilde{Y}^N_T)\big]\big|$ for $N \in \mathbb{N}$. The main difficulty and also a key difference of the proof of Theorem 1.1 in this article to the proof of the weak convergence result in Conus et al. [11] is the appropriate choice of the processes $\bar{Y}^N$, $N \in \mathbb{N}$, which we put in between. In the case of Theorem 1.1 it turns out to be rather useful to choose $\bar{Y}^N \colon [0,T] \times \Omega \to H$, $N \in \mathbb{N}$, such that for every $N \in \mathbb{N}$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$\bar{Y}^N_t = e^{tA}\,\xi + \int_0^t e^{(t-s)A}\,F\big(\tilde{Y}^N_{\lfloor s \rfloor_{T/N}}\big)\,ds + \int_0^t e^{(t-s)A}\,B\big(\tilde{Y}^N_{\lfloor s \rfloor_{T/N}}\big)\,dW_s, \tag{11}$$

cf. (11) with (8) and (9). In the remainder of this article we refer to $\bar{Y}^N$, $N \in \mathbb{N}$, as *semilinear integrated counterparts* of (8). In Proposition 6.4 (a key result of this article) in Subsection 6.3 the terms $\big|\mathbb{E}\big[\varphi(\bar{Y}_T^N)\big] - \mathbb{E}\big[\varphi(\tilde{Y}_T^N)\big]\big|$, $N \in \mathbb{N}$, in (10) are estimated in an appropriate way by using the mild Itô formula; see Theorem 1 in Da Prato et al. [12]. More precisely, in Section 5 we generalize the mild Itô formula in Theorem 1 in Da Prato et al. [12] so that it applies also in the case of stopping times instead of deterministic time points; see Theorem 5.3 and Corollary 5.5 in Subsection 5.3. We then use Corollary 5.5 to derive in Proposition 5.8 in Subsection 5.5 an estimate for the expectation of a smooth function composed with an appropriate type of stochastic process which we call a mild Itô process; see Definition 1 in Da Prato et al. [12] and, e.g., Definition 5.1 in Subsection 5.3 below. Next we recall that we have rewritten the time-discrete numerical approximation processes Y^N, $N \in \mathbb{N}$, (see (2)) as the time-continuous stochastic processes $\tilde{Y}^N$, $N \in \mathbb{N}$, (see (8)) and we emphasize that $\tilde{Y}^N$, $N \in \mathbb{N}$, are mild Itô processes; see (142)–(146) in Da Prato et al. [12]. This allows us to apply the mild Itô formula and so also Proposition 5.8 to $\tilde{Y}^N$, $N \in \mathbb{N}$. Thereby we obtain an appropriate estimate for the terms $\big|\mathbb{E}\big[\varphi(\bar{Y}_T^N)\big] - \mathbb{E}\big[\varphi(\tilde{Y}_T^N)\big]\big|$, $N \in \mathbb{N}$, in Proposition 6.4 in Subsection 6.3. The mild Itô formula has also been used in Conus et al. [11] to establish weak convergence rates for spatial spectral Galerkin approximations. In this work the analysis is more involved than in Conus et al. [11] as the numerical approximation processes $\tilde{Y}^N$, $N \in \mathbb{N}$, are not solution processes of SEEs of the form (1) but merely mild Itô processes with two-parameter evolution families (see Subsection 1.4 and Section 6 below for more details). For the estimation of the terms $\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}_T^N)\big]\big|$, $N \in \mathbb{N}$, we use (as it is often the case in the case weak convergence analysis; see, e.g., Rößler [38], Debussche [17] and Conus et al. [11]) the Kolmogorov backward equation associated to (1) and we also use again the mild Itô formula and its consequences respectively (see Section 6 and Section 7 for details). Combining these estimates with the in some sense non-standard mollification procedure in Conus et al. [11] will allow us to complete the proof of Theorem 1.1 (see Section 8 for details).

1.1 Examples

In this section we illustrate Theorem 1.1 by two simple examples. In Subsection 1.1.1 we apply Theorem 1.1 to the continuous version of *the parabolic Anderson model* and in Subsection 1.1.2 we apply Theorem 1.1 to a linear Cahn-Hilliard-Cook type equation.

1.1.1 Parabolic Anderson model

Let $H = L^2((0,1);\mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, let $T, \kappa, \delta, \nu \in (0,\infty)$, $\xi \in H$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_H-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, let $A\colon D(A) \subseteq H \to H$ be the Laplacian with Dirichlet boundary conditions on H multiplied by ν, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [25]), let $B \in C(H, HS(H, H_{-1/4-\delta}))$ satisfy that for all $v \in H$, $u \in C([0,1],\mathbb{R})$, $x \in (0,1)$ it holds that $\big(B(v)u\big)(x) = \kappa \cdot v(x) \cdot u(x)$, and let $Y^N\colon \{0,1,\ldots,N\} \times \Omega \to H$, $N \in \mathbb{N}$, be stochastic processes which satisfy that for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$ it holds $\mathbb{P}$-a.s. that $Y_0^N = \xi$ and $Y_{n+1}^N = \big(\mathrm{Id}_H - \tfrac{T}{N}A\big)^{-1}\big(Y_n + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B(Y_n)\,dW_s\big)$. The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [12], Theorem 4.3 in Brzeźniak [8], Theorem 6.2 in Van Neerven et al. [40]) the existence of an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted continuous stochastic process $X\colon [0,T]\times\Omega \to H$ which satisfies that for all $t \in [0,T]$ it holds P-a.s. that $X_t = e^{tA}\,\xi + \int_0^t e^{(t-s)A}B(X_s)\,dW_s$. The stochastic process X is thus a solution process (of the continuous version) of *the parabolic Anderson model*

$$dX_t(x) = \nu\,\tfrac{\partial^2}{\partial x^2}X_t(x)\,dt + \kappa\,X_t(x)\,dW_t(x), \qquad X_t(0) = X_t(1) = 0, \qquad X_0(x) = \xi(x) \tag{12}$$

for $x \in (0,1)$, $t \in [0,T]$ (cf., e.g., Carmona & Molchanov [10]). Theorem 1.1 applies here with $\gamma = \frac{1}{2}$. More precisely, Theorem 1.1 proves that for all $\varphi \in C_b^5(H,\mathbb{R})$, $\varepsilon \in (0,\infty)$ there exists a $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_N^N)\big]\big| \leq C \cdot N^{-(1/2-\varepsilon)}. \tag{13}$$

1.1.2 A linear Cahn-Hilliard-Cook type equation

Let $H = L^2((0,1);\mathbb{R})$ be the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0,1)$ to $\mathbb{R}$, let $T,\kappa,\delta \in (0,\infty)$, $\xi \in H$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_H-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, let $\mathcal{A}\colon D(\mathcal{A}) \subseteq H \to H$ be the Laplacian with Neumann boundary conditions on H, let $A\colon D(A) \subseteq H \to H$ be the linear operator with the property that $D(A) = D(\mathcal{A}^2) = \{v \in D(\mathcal{A})\colon \mathcal{A}v \in D(\mathcal{A})\}$ and with the property that for all $v \in D(A)$ it holds that $Av = -\mathcal{A}^2 v - \mathcal{A}v - v$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [25]), let $B \in C(H, HS(H, H_{-1/8-\delta}))$ satisfy that for all $v \in H$, $u \in C([0,1],\mathbb{R})$, $x \in (0,1)$ it holds that $\big(B(v)u\big)(x) = \kappa \cdot v(x) \cdot u(x)$, and let $Y^N\colon \{0,1,\ldots,N\} \times \Omega \to H$, $N \in \mathbb{N}$, be stochastic processes which satisfy that for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$ it holds $\mathbb{P}$-a.s. that $Y_0^N = \xi$ and $Y_{n+1}^N = \big(\mathrm{Id}_H - \tfrac{T}{N}A\big)^{-1}\big(Y_n + Y_n\,\tfrac{T}{N} + \int_{\frac{nT}{N}}^{\frac{(n+1)T}{N}} B(Y_n)\,dW_s\big)$. The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [12], Theorem 4.3 in Brzeźniak [8], Theorem 6.2 in Van Neerven et al. [40]) the existence of an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted continuous stochastic process $X\colon [0,T] \times \Omega \to H$ which satisfies that for all $t \in [0,T]$ it holds P-a.s. that $X_t = e^{tA}\,\xi + \int_0^t e^{(t-s)A} X_s\,ds + \int_0^t e^{(t-s)A} B(X_s)\,dW_s$. The stochastic process X is thus a solution process of the *linear Cahn-Hilliard-Cook type equation*

$$\begin{aligned} dX_t(x) &= \big[-\tfrac{\partial^4}{\partial x^4} X_t(x) - \tfrac{\partial^2}{\partial x^2} X_t(x)\big]\,dt + \kappa\, X_t(x)\,dW_t(x), \\ X_t'(0) &= X_t'(1) = X_t^{(3)}(0) = X_t^{(3)}(1) = 0, \qquad X_0(x) = \xi(x) \end{aligned} \tag{14}$$

for $x \in (0,1)$, $t \in [0,T]$. Theorem 1.1 applies here with $\gamma = \frac{1}{4}$. More precisely, Theorem 1.1 proves that for all $\varphi \in C_b^5(H,\mathbb{R})$, $\varepsilon \in (0,\infty)$ there exists a $C \in \mathbb{R}$ such that for all $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_N^N)\big]\big| \leq C \cdot N^{-(3/4-\varepsilon)}. \tag{15}$$

1.2 Notation

Throughout this article the following notation is used. By $\mathbb{N} = \{1,2,3,\ldots\}$ the set of natural numbers is denoted and by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ the union of $\{0\}$ and the set of natural numbers is denoted. Moreover, for a set A we denote by $\mathrm{Id}_A\colon A \to A$ the identity mapping on A, that is, it holds for all $a \in A$ that $\mathrm{Id}_A(a) = a$. Furthermore, for a set A we denote by $\mathcal{P}(A)$ the power set of A. Let $\mathcal{E}_r\colon [0,\infty) \to [0,\infty)$, $r \in (0,\infty)$, be the functions with the property that for all $x \in [0,\infty)$, $r \in (0,\infty)$ it holds that $\mathcal{E}_r(x) = \big[\sum_{n=0}^\infty \frac{x^{2n}\,\Gamma(r)^n}{\Gamma(nr+1)}\big]^{1/2}$ (cf. Chapter 7 in [22] and Chapter 3 in [25]). In addition, let $(\cdot)^+, (\cdot)^-\colon \mathbb{R} \to [0,\infty)$ be the functions with the property that for all $a \in \mathbb{R}$ it holds that $a^+ = \max\{a,0\}$ and $a^- = \max\{-a,0\}$. Moreover, for a number $k \in \mathbb{N}_0$ and normed $\mathbb{R}$-vector spaces $(E_i, \|\cdot\|_{E_i})$, $i \in \{1,2\}$, let $|\cdot|_{\mathrm{Lip}^k(E_1,E_2)}, \|\cdot\|_{\mathrm{Lip}^k(E_1,E_2)}\colon C^k(E_1,E_2) \to [0,\infty]$ be the mappings with the property that for all $f \in C^k(E_1,E_2)$ it holds that

$$|f|_{\mathrm{Lip}^k(E_1,E_2)} = \sup_{\substack{x,y\in E_1,\\ x\neq y}} \frac{\big\|f^{(k)}(x) - f^{(k)}(y)\big\|_{L^{(k)}(E_1,E_2)}}{\|x-y\|_{E_1}}, \tag{16}$$

$$\|f\|_{\mathrm{Lip}^k(E_1,E_2)} = \|f(0)\|_{E_2} + \sum_{l=0}^{k} |f|_{\mathrm{Lip}^l(E_1,E_2)}. \tag{17}$$

Furthermore, for a number $k \in \mathbb{N}_0$ and normed $\mathbb{R}$-vector spaces $(E_i, \|\cdot\|_{E_i})$, $i \in \{1,2\}$, let $\operatorname{Lip}^k(E_1, E_2)$ be the set given by $\operatorname{Lip}^k(E_1,E_2) = \big\{ f \in C^k(E_1,E_2) \colon \|f\|_{\operatorname{Lip}^k(E_1,E_2)} < \infty \big\}$. In addition, for a natural number $k \in \mathbb{N}$ and normed $\mathbb{R}$-vector spaces $(E_i, \|\cdot\|_{E_i})$, $i \in \{1,2\}$, let $|\cdot|_{C_b^k(E_1,E_2)}, \|\cdot\|_{C_b^k(E_1,E_2)} \colon C^k(E_1,E_2) \to [0,\infty]$ be the mappings with the property that for all $f \in C^k(E_1,E_2)$ it holds that

$$|f|_{C_b^k(E_1,E_2)} = \sup_{x \in E_1} \big\| f^{(k)}(x) \big\|_{L^k(E_1,E_2)}, \quad \|f\|_{C_b^k(E_1,E_2)} = \|f(0)\|_{E_2} + \sum_{l=1}^{k} |f|_{C_b^l(E_1,E_2)} \tag{18}$$

and let $C_b^k(E_1,E_2)$ be the set given by $C_b^k(E_1,E_2) = \big\{ f \in C^k(E_1,E_2) \colon \|f\|_{C_b^k(E_1,E_2)} < \infty \big\}$. Moreover, for a normed $\mathbb{R}$-vector space $(U, \|\cdot\|_U)$ and a linear operator $A \colon D(A) \subseteq U \to U$ we denote by $\operatorname{spectrum}(A) \subseteq \mathbb{C}$ the spectrum of A. For sets A and B we denote by $\mathbb{M}(A,B)$ the set of all mappings from A to B. In addition, for measurable spaces $(\Omega_i, \mathcal{F}_i)$, $i \in \{1,2\}$, we denote by $\mathcal{M}(\mathcal{F}_1, \mathcal{F}_2)$ the set of all $\mathcal{F}_1/\mathcal{F}_2$-measurable mappings. For two separable $\mathbb{R}$-Hilbert spaces $(\check{H}, \langle\cdot,\cdot\rangle_{\check{H}}, \|\cdot\|_{\check{H}})$ and $(\hat{H}, \langle\cdot,\cdot\rangle_{\hat{H}}, \|\cdot\|_{\hat{H}})$ let $\mathcal{S}(\hat{H},\check{H})$ be the strong sigma algebra on $L(\hat{H},\check{H})$ given by $\mathcal{S}(\hat{H},\check{H}) = \sigma_{L(\hat{H},\check{H})}(\cup_{v\in\hat{H}} \cup_{\mathcal{A}\in\mathcal{B}(\check{H})} \{A \in L(\hat{H},\check{H}) \colon Av \in \mathcal{A}\})$ (see, e.g., Section 1.2 in Da Prato & Zabczyk [13]). Finally, let $\lfloor\cdot\rfloor_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, and $\lceil\cdot\rceil_h \colon \mathbb{R} \to \mathbb{R}$, $h \in (0,\infty)$, be the mappings with the property that for all $t \in \mathbb{R}$, $h \in (0,\infty)$ it holds that

$$\lfloor t \rfloor_h = \max\big((-\infty, t] \cap \{0, h, -h, 2h, -2h, \dots\} \big), \tag{19}$$

$$\lceil t \rceil_h = \min\big([t, \infty) \cap \{0, h, -h, 2h, -2h, \dots\} \big). \tag{20}$$

1.3 General setting

Throughout this article the following setting is frequently used. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, and $(V, \langle\cdot,\cdot\rangle_V, \|\cdot\|_V)$ be separable $\mathbb{R}$-Hilbert spaces, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $A \colon D(A) \subseteq H \to H$ be a generator of a strongly continuous analytic semigroup with the property that $\sup\big(\operatorname{Re}(\operatorname{spectrum}(A))\big) < 0$ (cf., e.g., Theorem 11.31 in Renardy & Rogers [37]), let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (cf., e.g., Theorem and Definition 2.5.32 in [25] and Section 11.4.2 in Renardy & Rogers [37]), let $T \in (0,\infty)$, let $\angle = \{(t_1,t_2) \in [0,T]^2 \colon t_1 < t_2\}$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, and let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$.

1.4 Evolution family setting

In Sections 6, 7 and 8 below the following setting is also frequently used. Assume the setting in Section 1.3, let $h \in (0,\infty)$, $(C_r)_{r\in\mathbb{R}} \subseteq [1,\infty)$, $(C_{r,\rho})_{r,\rho\in\mathbb{R}} \subseteq [1,\infty)$, $(C_{r,\tilde{r},\rho})_{r,\tilde{r},\rho\in\mathbb{R}} \subseteq [1,\infty)$, $R \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}(L(H_{-1}))\big)$, $S \in \mathcal{M}\big(\mathcal{B}(\angle), \mathcal{B}(L(H_{-1}))\big)$ satisfy that for all $t_1, t_2, t_3 \in [0,T]$ with $t_1 < t_2 < t_3$ it holds that $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$ and that for all $(s,t) \in \angle$, $r, \rho \in [0,1)$, $\tilde{r} \in [-1, 1-r)$ it holds that $S_{s,t}(H) \subseteq H$, $S_{s,t}\, R_s(H_{-r}) \subseteq H_{\tilde{r}}$, $\|e^{tA}\|_{L(H,H_\rho)} \leq C_\rho\, t^{-\rho}$, $\|e^{tA} - \operatorname{Id}_H\|_{L(H,H_{-\rho})} \leq C_\rho\, t^\rho$, $\|S_{s,t}\|_{L(H)} \leq C_0$, $\|S_{s,t}\, R_s\|_{L(H_{-r},H)} \leq C_r\, (t-s)^{-r}$, $\|S_{s,t} - e^{(t-s)A}\|_{L(H,H_{-r})} \leq C_{-r,\rho}\, h^\rho\, (t-s)^{-(\rho-r)^+}$, and $\|S_{s,t}\, R_s - e^{(t-s)A}\|_{L(H_{-r},H_{\tilde{r}})} \leq C_{r,\tilde{r},\rho}\, h^\rho\, (t-s)^{-(\rho+r+\tilde{r})^+}$.

1.5 Examples of evolution families

In this subsection we provide two examples of evolution families which satisfy the assumptions in Subsection 1.4.

1.5.1 Exponential Euler approximations

Assume the setting in Section 1.3 and let $S \in \mathcal{M}\big(\mathcal{B}(\angle), \mathcal{B}(L(H_{-1}))\big)$ and $R^h \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}(L(H_{-1}))\big)$, $h \in (0,\infty)$, satisfy that for all $h \in (0,\infty)$, $t \in [0,T]$, $(t_1,t_2) \in \angle$ it holds that $S_{t_1,t_2} = e^{(t_2-t_1)A}$ and $R^h_t = e^{(t-\lfloor t \rfloor_h)A}$. Then it is well-known (see, e.g., Lemma 11.36 in Renardy & Rogers [37]) that there exist real numbers $(C_r)_{r\in\mathbb{R}} \subseteq [1,\infty)$ such that for all $h \in (0,\infty)$, $t_1,t_2,t_3 \in [0,T]$ $(s,t) \in \angle$, $r,\rho \in [0,1)$, $\tilde{r} \in [-1,1-r)$ with $t_1 < t_2 < t_3$ it holds that $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$, $S_{s,t}(H) \subseteq H$, $S_{s,t}\, R^h_s(H_{-r}) \subseteq H_{\tilde{r}}$, $\|e^{tA}\|_{L(H,H_\rho)} \leq C_\rho\, t^{-\rho}$, $\|e^{tA} - \mathrm{Id}_H\|_{L(H,H_{-\rho})} \leq C_\rho\, t^\rho$, $\|S_{s,t}\|_{L(H)} \leq C_0$, $\|S_{s,t}\, R^h_s\|_{L(H_{-r},H)} \leq C_r\,(t-s)^{-r}$, $\|S_{s,t} - e^{(t-s)A}\|_{L(H,H_{-r})} = 0$, and $\|S_{s,t}\, R^h_s - e^{(t-s)A}\|_{L(H_{-r},H_{\tilde{r}})} \leq C_{r+\tilde{r}+\rho}\, h^\rho\,(t-s)^{-(\rho+r+\tilde{r})^+}$.

1.5.2 Linear-implicit Euler approximations

Assume the setting in Section 1.3 and let $S^h \in \mathcal{M}\big(\mathcal{B}(\angle), \mathcal{B}(L(H_{-1}))\big)$, $h \in (0,\infty)$, and $R^h \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}(L(H_{-1}))\big)$, $h \in (0,\infty)$, satisfy that for all $h \in (0,\infty)$, $t \in [0,T]$, $(t_1,t_2) \in \angle$ it holds that $S^h_{t_1,t_2} = \big(\mathrm{Id}_H - (t_1 - \lfloor t_1 \rfloor_h)\,A\big)\big(\mathrm{Id}_H - (t_2 - \lfloor t_2 \rfloor_h)\,A\big)^{-1}\big(\mathrm{Id}_H - hA\big)^{-(\lfloor t_2 \rfloor_h - \lfloor t_1 \rfloor_h)/h}$ and $R^h_t = \big(\mathrm{Id}_H - (t - \lfloor t \rfloor_h)\,A\big)^{-1}$. Then there exist real numbers $(C_r)_{r\in\mathbb{R}} \subseteq [1,\infty)$, $(C_{r,\rho})_{r,\rho\in\mathbb{R}} \subseteq [1,\infty)$, $(C_{r,\tilde{r},\rho})_{r,\tilde{r},\rho\in\mathbb{R}} \subseteq [1,\infty)$ such that for all $h \in (0,\infty)$, $t_1,t_2,t_3 \in [0,T]$, $(s,t) \in \angle$, $r,\rho \in [0,1)$, $\tilde{r} \in [-1,1-r)$ with $t_1 < t_2 < t_3$ it holds that $S^h_{t_2,t_3} S^h_{t_1,t_2} = S^h_{t_1,t_3}$, $S^h_{s,t}(H) \subseteq H$, $S^h_{s,t}\, R^h_s(H_{-r}) \subseteq H_{\tilde{r}}$, $\|e^{tA}\|_{L(H,H_\rho)} \leq C_\rho\, t^{-\rho}$, $\|e^{tA} - \mathrm{Id}_H\|_{L(H,H_{-\rho})} \leq C_\rho\, t^\rho$, $\|S^h_{s,t}\|_{L(H)} \leq C_0$, $\|S^h_{s,t}\, R^h_s\|_{L(H_{-r},H)} \leq C_r\,(t-s)^{-r}$, $\|S^h_{s,t} - e^{(t-s)A}\|_{L(H,H_{-r})} \leq C_{-r,\rho}\, h^\rho\,(t-s)^{-(\rho-r)^+}$, and $\|S^h_{s,t}\, R^h_s - e^{(t-s)A}\|_{L(H_{-r},H_{\tilde{r}})} \leq C_{r,\tilde{r},\rho}\, h^\rho\,(t-s)^{-(\rho+r+\tilde{r})^+}$.

1.6 Acknowledgements

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2 Strong a priori estimates for SPDEs

In this section we establish in Proposition 2.1 below an a priori estimate (see (22)) for an appropriate class of stochastic processes (see (21)) which includes solution processes of certain SEEs as special cases. The proof of Proposition 2.1 uses the generalized Gronwall lemma in Chapter 7 in Henry [22] (see, e.g., also Corollary 3.4.6 in [25]). Related estimates can, e.g., be found in Proposition 2.5 in Andersson & Jentzen [1].

2.1 Setting

Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T \in (0,\infty)$, $p \in [2,\infty)$, $\vartheta \in [0,1)$, $\mathbf{y}, \mathbf{z} \in [0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, let $X\colon [0,T]\times\Omega \to H$ be a stochastic process with $\sup_{s\in[0,T]} \|X_s\|_{L^p(\mathbb{P};H)} < \infty$, and for every $t \in (0,T]$ let $Y^t\colon [0,t]\times\Omega \to H$ and $Z^t\colon [0,t]\times\Omega \to HS(U,H)$ be $(\mathcal{F}_s)_{s\in[0,t]}$-predictable stochastic processes which satisfy that for all $s \in (0,t)$ it holds that

$$\|Y^t_s\|_{L^p(\mathbb{P};H)} \leq \frac{\mathbf{y}\sup_{u\in[0,s]} \|X_u\|_{L^p(\mathbb{P};H)}}{(t-s)^\vartheta} \qquad \text{and} \qquad \|Z^t_s\|_{L^p(\mathbb{P};HS(U,H))} \leq \frac{\mathbf{z}\sup_{u\in[0,s]} \|X_u\|_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta/2}}. \tag{21}$$

2.2 A strong a priori estimate

Proposition 2.1 (A strong a priori estimate)**.** *Assume the setting in Section 2.1. Then it holds for all $t \in [0,T]$ that $\mathbb{P}\big(\int_0^t \|Y_s^t\|_H + \|Z_s^t\|^2_{HS(U,H)}\,ds < \infty\big) = 1$ and it holds that*

$$\begin{split}
&\sup_{t\in[0,T]} \|X_t\|_{L^p(\mathbb{P};H)} \leq \sqrt{2}\,\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\mathbf{y}\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\right]\\
&\cdot \sup_{t\in[0,T]}\left\| X_t - \left[\int_0^t Y_s^t\,ds + \int_0^t Z_s^t\,dW_s\right]\right\|_{L^p(\mathbb{P};H)} \leq \left[1 + \tfrac{\mathbf{y}\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\mathbf{z}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}}{\sqrt{2\,(1-\vartheta)}}\right]\\
&\cdot \sqrt{2}\,\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\mathbf{y}\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\right] \sup_{t\in[0,T]} \|X_t\|_{L^p(\mathbb{P};H)} < \infty.
\end{split} \tag{22}$$

Proof. We first observe that (21), Hölder's inequality, and the assumption that $\sup_{s\in[0,T]} \|X_s\|_{L^p(\mathbb{P};H)} < \infty$ imply that for all $t \in [0,T]$ it holds that

$$\begin{split}
&\int_0^t \|Y_s^t\|_{L^p(\mathbb{P};H)}\,ds \leq \mathbf{y}\int_0^t \frac{\sup_{v\in[0,s]}\|X_v\|_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta}}\,ds\\
&\leq \mathbf{y}\left[\frac{t^{(1-\vartheta)}}{(1-\vartheta)}\int_0^t \frac{\sup_{v\in[0,s]}\|X_v\|^2_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta}}\,ds\right]^{1/2} < \infty.
\end{split} \tag{23}$$

In addition, we note that (21) and again the assumption that $\sup_{s\in[0,T]}\|X_s\|_{L^p(\mathbb{P};H)} < \infty$ show that for all $t\in[0,T]$ it holds that

$$\left[\tfrac{p\,(p-1)}{2}\int_0^t \|Z_s^t\|^2_{L^p(\mathbb{P};HS(U,H))}\,ds\right]^{1/2} \leq \mathbf{z}\left[\tfrac{p\,(p-1)}{2}\int_0^t \frac{\sup_{v\in[0,s]}\|X_v\|^2_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta}}\,ds\right]^{1/2} < \infty. \tag{24}$$

Combining (23)–(24) and the assumption that $p \geq 2$ proves that for all $t\in[0,T]$ it holds that $\int_0^t \|Y_s^t\|_{L^1(\mathbb{P};H)} + \|Z_s^t\|^2_{L^2(\mathbb{P};HS(U,H))}\,ds < \infty$. This, in turn, shows that for all $t\in[0,T]$ it holds $\mathbb{P}$-a.s. that

$$\int_0^t \|Y_s^t\|_H + \|Z_s^t\|^2_{HS(U,H)}\,ds < \infty. \tag{25}$$

It thus remains to prove (22) to complete the proof of Proposition 2.1. For this observe that (23)–(25) and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] imply that for all $t\in[0,T]$ it holds that

$$\begin{split}
&\left\|\int_0^t Y_s^t\,ds\right\|_{L^p(\mathbb{P};H)} + \left\|\int_0^t Z_s^t\,dW_s\right\|_{L^p(\mathbb{P};H)}\\
&\leq \left[\frac{\mathbf{y}\,t^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} + \mathbf{z}\sqrt{\frac{p\,(p-1)}{2}}\right]\left[\int_0^t \frac{\sup_{v\in[0,s]}\|X_v\|^2_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta}}\,ds\right]^{1/2}.
\end{split} \tag{26}$$

Next we observe (cf., e.g., [24]) that for all $t,u\in[0,T]$ with $t \leq u$ it holds that

$$\begin{split}
&\int_0^t \frac{\sup_{v\in[0,s]}\big\|X_v\big\|^2_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta}}\,ds = \int_{u-t}^u \frac{\sup_{v\in[0,s-u+t]}\big\|X_v\big\|^2_{L^p(\mathbb{P};H)}}{(u-s)^{\vartheta}}\,ds\\
&\leq \int_{u-t}^u \frac{\sup_{v\in[0,s]}\big\|X_v\big\|^2_{L^p(\mathbb{P};H)}}{(u-s)^{\vartheta}}\,ds \leq \int_0^u \frac{\sup_{v\in[0,s]}\big\|X_v\big\|^2_{L^p(\mathbb{P};H)}}{(u-s)^{\vartheta}}\,ds.
\end{split} \tag{27}$$

Moreover, we note that the Minkowski inequality ensures that for all $t\in[0,T]$ it holds that

$$\begin{split}
&\|X_t\|_{L^p(\mathbb{P};H)}\\
&\leq \left\|X_t - \left[\int_0^t Y_s^t\,ds + \int_0^t Z_s^t\,dW_s\right]\right\|_{L^p(\mathbb{P};H)} + \left\|\int_0^t Y_s^t\,ds\right\|_{L^p(\mathbb{P};H)} + \left\|\int_0^t Z_s^t\,dW_s\right\|_{L^p(\mathbb{P};H)}.
\end{split} \tag{28}$$

Combining (26)–(28) with the fact that $\forall\, a,b\in\mathbb{R}\colon\ (a+b)^2\leq 2a^2+2b^2$ proves that for all $u\in[0,T]$ it holds that

$$\sup_{t\in[0,u]}\|X_t\|^2_{L^p(\mathbb{P};H)}\leq 2\sup_{t\in[0,T]}\left\|X_t-\left[\int_0^t Y_s^t\,ds+\int_0^t Z_s^t\,dW_s\right]\right\|^2_{L^p(\mathbb{P};H)} \tag{29}$$

$$+2\left[\frac{\mathbf{y}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}+\mathbf{z}\sqrt{\frac{p\,(p-1)}{2}}\right]^2\int_0^u\frac{\sup_{t\in[0,s]}\|X_t\|^2_{L^p(\mathbb{P};H)}}{(u-s)^{\vartheta}}\,ds. \tag{30}$$

This and the assumption that $\sup_{s\in[0,T]}\|X_s\|_{L^p(\mathbb{P};H)}<\infty$ together with the generalized Gronwall lemma in Chapter 7 in Henry [22] (see, e.g., also Corollary 3.4.6 in [25]) proves the first inequality in (22). In the next step we note that (26) implies that

$$\begin{aligned}
&\sup_{t\in[0,T]}\left\|X_t-\left[\int_0^t Y_s^t\,ds+\int_0^t Z_s^t\,dW_s\right]\right\|_{L^p(\mathbb{P};H)}\\
&\leq\sup_{t\in[0,T]}\|X_t\|_{L^p(\mathbb{P};H)}+\sup_{t\in[0,T]}\left[\left\|\int_0^t Y_s^t\,ds\right\|_{L^p(\mathbb{P};H)}+\left\|\int_0^t Z_s^t\,dW_s\right\|_{L^p(\mathbb{P};H)}\right]\\
&\leq\left[1+\frac{\mathbf{y}\,T^{(1-\vartheta)}}{(1-\vartheta)}+\mathbf{z}\sqrt{\frac{p\,(p-1)\,T^{(1-\vartheta)}}{2\,(1-\vartheta)}}\right]\sup_{t\in[0,T]}\|X_t\|_{L^p(\mathbb{P};H)}.
\end{aligned} \tag{31}$$

This proves the second inequality in (22). The third inequality in (22) is an immediate consequence of the assumption that $\sup_{s\in[0,T]}\|X_s\|_{L^p(\mathbb{P};H)}<\infty$. The proof of Proposition 2.1 is thus completed. □

3 Strong perturbations for SPDEs

In this section we prove in Corollary 3.1 a perturbation estimate (see (33)) for an appropriate class of stochastic processes which includes solution processes of certain SEEs as special cases. Corollary 3.1 follows immediately from Proposition 2.1 in Section 2. Corollary 3.1 extends the perturbation estimate in Proposition 2.5 in Andersson & Jentzen [1]. Further related strong perturbation estimates for SEEs can, e.g., be found in Hutzenthaler & Jentzen [23].

3.1 Setting

Let $(H,\langle\cdot,\cdot\rangle_H,\|\cdot\|_H)$ and $(U,\langle\cdot,\cdot\rangle_U,\|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces, let $T\in(0,\infty)$, $p\in[2,\infty)$, $\vartheta\in[0,1)$, $\mathbf{y},\mathbf{z}\in[0,\infty)$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, let $X,\bar{X}\colon[0,T]\times\Omega\to H$ be stochastic processes with $\sup_{s\in[0,T]}\|X_s-\bar{X}_s\|_{L^p(\mathbb{P};H)}<\infty$, and for every $t\in(0,T]$ let $Y^t,\bar{Y}^t\colon[0,t]\times\Omega\to H$, $Z^t,\bar{Z}^t\colon[0,t]\times\Omega\to HS(U,H)$ be $(\mathcal{F}_s)_{s\in[0,t]}$-predictable stochastic processes such that it holds $\mathbb{P}$-a.s. that $\int_0^t\|Y_s^t\|_H+\|\bar{Y}_s^t\|_H+\|Z_s^t\|^2_{HS(U,H)}+\|\bar{Z}_s^t\|^2_{HS(U,H)}\,ds<\infty$ and such that for all $s\in(0,t)$ it holds that

$$\|Y_s^t-\bar{Y}_s^t\|_{L^p(\mathbb{P};H)}\leq\tfrac{\mathbf{y}\sup_{u\in[0,s]}\|X_u-\bar{X}_u\|_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta}},\quad\|Z_s^t-\bar{Z}_s^t\|_{L^p(\mathbb{P};HS(U,H))}\leq\tfrac{\mathbf{z}\sup_{u\in[0,s]}\|X_u-\bar{X}_u\|_{L^p(\mathbb{P};H)}}{(t-s)^{\vartheta/2}}. \tag{32}$$

3.2 Strong perturbation estimates

In the next result, Corollary 3.1, a certain strong perturbation estimate is presented. Corollary 3.1 is an immediate consequence of Proposition 2.1. Corollary 3.1 is an extension of the perturbation estimate in Proposition 2.5 in Andersson & Jentzen [1].

Corollary 3.1 (A strong perturbation estimate)**.** *Assume the setting in Section 3.1. Then*

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{L^p(\mathbb{P};H)} \leq \sqrt{2}\,\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\mathbf{y}\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\right] \\
&\cdot \sup_{t\in[0,T]} \left\| X_t - \left[\int_0^t Y_s^t\,ds + \int_0^t Z_s^t\,dW_s\right] + \left[\int_0^t \bar{Y}_s^t\,ds + \int_0^t \bar{Z}_s^t\,dW_s\right] - \bar{X}_t \right\|_{L^p(\mathbb{P};H)} \\
&\leq \sqrt{2}\,\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\mathbf{y}\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\right]\left[1 + \tfrac{\mathbf{y}\,T^{(1-\vartheta)}}{(1-\vartheta)} + \mathbf{z}\sqrt{\tfrac{p\,(p-1)\,T^{(1-\vartheta)}}{2\,(1-\vartheta)}}\right] \\
&\cdot \sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{L^p(\mathbb{P};H)} < \infty.
\end{aligned}
\tag{33}
$$

As an application of Corollary 3.1 we establish in the next result, Corollary 3.2, an a priori estimate for the difference of two numerical approximation processes with possibly different initial values. Corollary 3.2 is an extension of Corollary 2.6 in Andersson & Jentzen [1].

Corollary 3.2. *Assume the setting in Section 3.1, let $S \in \mathbb{M}([0,T], L(H))$, and assume that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that*

$$
X_t = S_t\,X_0 + \int_0^t Y_s^t\,ds + \int_0^t Z_s^t\,dW_s, \qquad \bar{X}_t = S_t\,\bar{X}_0 + \int_0^t \bar{Y}_s^t\,ds + \int_0^t \bar{Z}_s^t\,dW_s. \tag{34}
$$

Then

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|X_t - \bar{X}_t\|_{L^p(\mathbb{P};H)} \\
&\leq \sqrt{2}\left[\sup_{t\in[0,T]} \|S_t\|_{L(H)}\right] \|X_0 - \bar{X}_0\|_{L^p(\mathbb{P};H)}\,\mathcal{E}_{(1-\vartheta)}\left[\tfrac{\mathbf{y}\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + \mathbf{z}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\right].
\end{aligned}
\tag{35}
$$

4 Strong convergence of mollified solutions for SPDEs

In this section we establish in Proposition 4.3 below an elementary a priori bound on the difference between a certain stochastic process and a mollified version of this process. In the proof of Proposition 4.3 we use Corollary 3.1 from Section 3 above. Results related to Proposition 4.3 can, e.g., be found in Proposition 4.1 in Conus et al. [11] and in Lemma 2.8 in Andersson & Jentzen [1].

4.1 Setting

Assume the setting in Section 1.3, let $p \in [2,\infty)$, $\vartheta \in [0,1)$, $\Pi \in \mathcal{M}\big(\mathcal{B}([0,T]), \mathcal{B}([0,T])\big)$, $(C_r)_{r\in[0,1]} \subseteq [1,\infty)$, $F \in \operatorname{Lip}^0(H, H_{-\vartheta})$, $B \in \operatorname{Lip}^0(H, HS(U, H_{-\vartheta/2}))$, $L \in \mathcal{M}\big(\mathcal{B}(\angle), \mathcal{B}(L(H_{-1}))\big)$ satisfy that for all $t \in [0,T]$ it holds that $\Pi(t) \leq t$ and that for all $(s,t) \in \big(\angle \cap (0,T]^2\big)$, $\rho \in [0,1)$ it holds that $L_{0,t}(H) \subseteq H$, $L_{s,t}(H_{-\rho}) \subseteq H$, $\|e^{tA}\|_{L(H)} \leq C_0$, $\|e^{tA} - \mathrm{Id}_H\|_{L(H_\rho,H)} \leq C_\rho\,t^\rho$, and $\|L_{s,t}\|_{L(H_{-\rho},H)} \leq C_\rho\,(t-s)^{-\rho}$, and let $Y^\kappa \colon [0,T] \times \Omega \to H$, $\kappa \in [0,T]$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy that for all $\kappa \in [0,T]$ it holds that $\sup_{t\in[0,T]} \|Y^\kappa_{\Pi(t)}\|_{L^p(\mathbb{P};H)} < \infty$ and which satisfy that for all $\kappa \in [0,T]$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that $Y_0^\kappa = Y_0^0$ and

$$
Y_t^\kappa = L_{0,t}\,Y_0^\kappa + \int_0^t L_{s,t}\,e^{\kappa A} F(Y^\kappa_{\Pi(s)})\,ds + \int_0^t L_{s,t}\,e^{\kappa A} B(Y^\kappa_{\Pi(s)})\,dW_s. \tag{36}
$$

4.2 A priori bounds for the non-mollified process

In this subsection we establish two elementary and essentially well-known a priori bounds for the processes Y^κ, $\kappa \in [0,T]$, from Subsection 4.1. The first a priori bound is presented in Lemma 4.1 and the second a priori bound is given in Proposition 4.2 below.

Lemma 4.1. *Assume the setting in Section 4.1 and let* $\kappa \in [0,T]$. *Then* $\sup_{t\in[0,T]} \|Y_t^\kappa\|_{L^p(\mathbb{P};H)} < \infty$.

Proof. We observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] ensures that for all $t \in [0,T]$ it holds that

$$\begin{aligned}
&\|Y_t^\kappa\|_{L^p(\mathbb{P};H)} \le \|L_{0,t}\, Y_0^\kappa\|_{L^p(\mathbb{P};H)} + \int_0^t \|L_{s,t}\, e^{\kappa A} F(Y^\kappa_{\Pi(s)})\|_{L^p(\mathbb{P};H)}\, ds \\
&+ \left[\tfrac{p\,(p-1)}{2} \int_0^t \|L_{s,t}\, e^{\kappa A} B(Y^\kappa_{\Pi(s)})\|^2_{L^p(\mathbb{P};HS(U,H))}\, ds\right]^{1/2} \\
&\le C_0\, \|Y_0^\kappa\|_{L^p(\mathbb{P};H)} + \int_0^t \tfrac{C_0\, C_\vartheta\, \|F(Y^\kappa_{\Pi(s)})\|_{L^p(\mathbb{P};H_{-\vartheta})}}{(t-s)^\vartheta}\, ds \\
&\quad + \left[\tfrac{p\,(p-1)}{2} \int_0^t \tfrac{|C_0|^2\, |C_{\vartheta/2}|^2\, \|B(Y^\kappa_{\Pi(s)})\|^2_{L^p(\mathbb{P};HS(U,H_{-\vartheta/2}))}}{(t-s)^\vartheta}\, ds\right]^{1/2} \\
&\le \left[C_0 + \tfrac{C_0\, C_\vartheta\, T^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-\vartheta)} + \tfrac{C_0\, C_{\vartheta/2} \sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta}}\right] \\
&\quad \cdot \sup_{s\in[0,T]} \|\max\{1, \|Y^\kappa_{\Pi(s)}\|_H\}\|_{L^p(\mathbb{P};\mathbb{R})}.
\end{aligned} \tag{37}$$

This and the fact that $\sup_{t\in[0,T]} \|\max\{1, \|Y^\kappa_{\Pi(t)}\|_H\}\|_{L^p(\mathbb{P};\mathbb{R})} \le 1 + \sup_{t\in[0,T]} \|Y^\kappa_{\Pi(t)}\|_{L^p(\mathbb{P};H)} < \infty$ complete the proof of Lemma 4.1. □

In the next result, Proposition 4.2, an a priori bound for the process Y^0 is established. The proof of Proposition 4.2 uses Corollary 3.1 and Lemma 4.1 above.

Proposition 4.2 (An a priori bound for the non-mollified process)**.** *Assume the setting in Section 4.1. Then*

$$\begin{aligned}
&\sup_{t\in[0,T]} \|Y_t^0\|_{L^p(\mathbb{P};H)} \le \sqrt{2} \Bigg[\sup_{t\in(0,T]} \max\{1, \|L_{0,t}\|_{L(H)}\}\, \|Y_0^0\|_{L^p(\mathbb{P};H)} \\
&+ \tfrac{C_\vartheta\, T^{(1-\vartheta)}\, \|F(0)\|_{H_{-\vartheta}}}{(1-\vartheta)} + \tfrac{C_{\vartheta/2}\sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, \|B(0)\|_{HS(U,H_{-\vartheta/2})}}{\sqrt{2(1-\vartheta)}} \Bigg] \\
&\cdot \mathcal{E}_{(1-\vartheta)}\left[\tfrac{\sqrt{2}\, C_\vartheta\, T^{(1-\vartheta)}\, |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + C_{\vartheta/2}\sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right] < \infty.
\end{aligned} \tag{38}$$

Proof. Throughout this proof let $\tilde{L}\colon \{(t_1,t_2) \in [0,T]^2 \colon t_1 \le t_2\} \to L(H_{-1})$ be the mapping with the property that for all $(t_1,t_2) \in \angle$, $v \in H_{-1}$ it holds that $\tilde{L}_{t_1,t_2} v = L_{t_1,t_2} v$ and with the property that for all $t \in [0,T]$ it holds that $\tilde{L}_{t,t} = \mathrm{Id}_{H_{-1}}$. Combining Corollary 3.1 and Lemma 4.1 shows[3] that

$$\begin{aligned}
&\sup_{t\in[0,T]} \left\|Y_t^0\right\|_{L^p(\mathbb{P};H)} \le \sqrt{2} \sup_{t\in[0,T]} \left\| \tilde{L}_{0,t}\, Y_0^0 + \int_0^t \tilde{L}_{s,t}\, F(0)\, ds + \int_0^t \tilde{L}_{s,t}\, B(0)\, dW_s \right\|_{L^p(\mathbb{P};H)} \\
&\cdot \mathcal{E}_{(1-\vartheta)}\left[\tfrac{\sqrt{2}\, T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} C_\vartheta\, |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})} + \sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right].
\end{aligned} \tag{39}$$

Combining (39) with the triangle inequality and the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] completes the proof of Proposition 4.2. □

[3] with $\bar{X}_t = 0$, $\bar{Y}_s^t = \tilde{L}_{s,t} F(0)$, $\bar{Z}_s^t = \tilde{L}_{s,t} B(0)$ for $s \in (0,t)$, $t \in (0,T]$ in the notation of Corollary 3.1

4.3 A strong convergence result

Proposition 4.3 (A bound on the difference between the mollified and the non-mollified processes)**.** *Assume the setting in Section 4.1 and let* $\kappa \in [0, T]$*,* $\rho \in [0, \frac{1-\vartheta}{2})$*. Then*

$$\sup_{t\in[0,T]} \left\|Y_t^0 - Y_t^\kappa\right\|_{L^p(\mathbb{P};H)} \leq \tfrac{2\,\kappa^\rho}{T^\rho} \Bigg[\sup\nolimits_{t\in(0,T]} \max\!\left\{1, \|L_{0,t}\|_{L(H)}\right\} \max\!\left\{1, \|Y_0^0\|_{L^p(\mathbb{P};H)}\right\} + \tfrac{C_\rho\, C_\vartheta\, C_{\rho+\vartheta}\, T^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-\vartheta-\rho)} + \tfrac{C_\rho\, C_{\vartheta/2}\, C_{\rho+\vartheta/2} \sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2\,(1-\vartheta-2\rho)}} \Bigg]^2 \cdot \left| \mathcal{E}_{(1-\vartheta)}\!\left[\tfrac{\sqrt{2}\,T^{(1-\vartheta)}\,C_0\,C_\vartheta}{\sqrt{1-\vartheta}} |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})} + \sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\; C_0\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right] \right|^2. \tag{40}$$

Proof. First of all, we observe that Lemma 4.1 allows us to apply Corollary 3.1 to obtain[4] that

$$\begin{aligned} &\sup_{t\in[0,T]} \left\|Y_t^0 - Y_t^\kappa\right\|_{L^p(\mathbb{P};H)} \\ &\leq \mathcal{E}_{(1-\vartheta)}\!\left[C_\vartheta\, |e^{\kappa A} F|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \tfrac{\sqrt{2}\,T^{(1-\vartheta)}}{\sqrt{1-\vartheta}} + C_{\vartheta/2}\, |e^{\kappa A} B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \sqrt{p\,(p-1)\,T^{(1-\vartheta)}} \right] \\ &\cdot \sqrt{2} \sup_{t\in[0,T]} \left\| \int_0^t L_{s,t} \left(\mathrm{Id}_H - e^{\kappa A}\right) F(Y^0_{\Pi(s)})\, ds + \int_0^t L_{s,t} \left(\mathrm{Id}_H - e^{\kappa A}\right) B(Y^0_{\Pi(s)})\, dW_s \right\|_{L^p(\mathbb{P};H)}. \end{aligned} \tag{41}$$

Moreover, we observe that for all $t \in (0, T]$ it holds that

$$\begin{aligned} &\left\| \int_0^t L_{s,t} \left(\mathrm{Id}_H - e^{\kappa A}\right) F(Y^0_{\Pi(s)})\, ds \right\|_{L^p(\mathbb{P};H)} \leq \int_0^t \frac{C_\rho\, C_{\rho+\vartheta}\, \kappa^\rho}{(t-s)^{(\rho+\vartheta)}} \|F(Y^0_{\Pi(s)})\|_{L^p(\mathbb{P};H_{-\vartheta})}\, ds \\ &\leq \tfrac{C_\rho\, C_{\rho+\vartheta}\, t^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \sup_{s\in[0,T]} \max\!\left\{1, \|Y_s^0\|_{L^p(\mathbb{P};H)}\right\} \kappa^\rho. \end{aligned} \tag{42}$$

In addition, the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] implies that for all $t \in (0, T]$ it holds that

$$\begin{aligned} &\left\| \int_0^t L_{s,t} \left(\mathrm{Id}_H - e^{\kappa A}\right) B(Y^0_{\Pi(s)})\, dW_s \right\|_{L^p(\mathbb{P};H)} \\ &\leq \left[\tfrac{p\,(p-1)}{2} \int_0^t \frac{|C_\rho\, C_{\rho+\vartheta/2}\, \kappa^\rho|^2}{(t-s)^{(2\rho+\vartheta)}} \|B(Y^0_{\Pi(s)})\|^2_{L^p(\mathbb{P};HS(U,H_{-\vartheta/2}))}\, ds \right]^{1/2} \\ &\leq \tfrac{C_\rho\, C_{\rho+\vartheta/2} \sqrt{p\,(p-1)\, t^{(1-\vartheta-2\rho)}}}{\sqrt{2-2\vartheta-4\rho}} \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \sup_{s\in[0,T]} \max\!\left\{1, \|Y_s^0\|_{L^p(\mathbb{P};H)}\right\} \kappa^\rho. \end{aligned} \tag{43}$$

Putting (42) and (43) into (41) yields that

$$\begin{aligned} &\sup_{t\in[0,T]} \left\|Y_t^0 - Y_t^\kappa\right\|_{L^p(\mathbb{P};H)} \leq \sqrt{2}\, \kappa^\rho \sup_{t\in[0,T]} \max\!\left\{1, \|Y_t^0\|_{L^p(\mathbb{P};H)}\right\} \\ &\cdot \mathcal{E}_{(1-\vartheta)}\!\left[\tfrac{\sqrt{2}\,T^{(1-\vartheta)}\,C_0\,C_\vartheta}{\sqrt{1-\vartheta}} |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})} + \sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\; C_0\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right] \\ &\cdot \left[\tfrac{C_\rho\, C_{\rho+\vartheta}\, T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} + \tfrac{C_\rho\, C_{\rho+\vartheta/2} \sqrt{p\,(p-1)\, T^{(1-\vartheta-2\rho)}}}{\sqrt{2-2\vartheta-4\rho}} \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right]. \end{aligned} \tag{44}$$

---

[4] with $\bar{X}_t = Y_t^\kappa$, $\bar{Y}_s^t = L_{s,t}\, e^{\kappa A} F(Y^\kappa_{\Pi(s)})$, $\bar{Z}_s^t = L_{s,t}\, e^{\kappa A} B(Y^\kappa_{\Pi(s)})$ for $s \in (0, t)$, $t \in (0, T]$ in the notation of Corollary 3.1

Combining Proposition 4.2 and (44) proves that

$$\begin{aligned}
&\left\|Y_T^0 - Y_T^\kappa\right\|_{L^p(\mathbb{P};H)} \leq 2\,\kappa^\rho \left[\sup_{t\in(0,T]} \max\big\{1, \|L_{0,t}\|_{L(H)}\big\} \max\{1, \|Y_0^0\|_{L^p(\mathbb{P};H)}\} \right.\\
&\left. + \frac{C_\vartheta\, T^{(1-\vartheta)}\, \|F(0)\|_{H_{-\vartheta}}}{(1-\vartheta)} + \frac{C_{\vartheta/2}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\, \|B(0)\|_{HS(U,H_{-\vartheta/2})}}{\sqrt{2-2\vartheta}} \right]\\
&\cdot \left[\frac{C_\rho\, C_{\rho+\vartheta}\, T^{(1-\vartheta-\rho)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-\vartheta-\rho)} + \frac{C_\rho\, C_{\rho+\vartheta/2}\sqrt{p\,(p-1)\,T^{(1-\vartheta-2\rho)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta-4\rho}} \right]\\
&\cdot \left| \mathcal{E}_{(1-\vartheta)} \left[\frac{\sqrt{2}\, T^{(1-\vartheta)}\, C_0\, C_\vartheta\, |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + \sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\, C_0\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right] \right|^2 .
\end{aligned} \tag{45}$$

Hence, we obtain that

$$\begin{aligned}
&\left\|Y_T^0 - Y_T^\kappa\right\|_{L^p(\mathbb{P};H)} \leq \tfrac{2\,\kappa^\rho}{T^\rho} \left[\sup_{t\in(0,T]} \max\big\{1, \|L_{0,t}\|_{L(H)}\big\} \max\{1, \|Y_0^0\|_{L^p(\mathbb{P};H)}\} \right.\\
&\left. + \frac{C_\vartheta\, T^{(1-\vartheta)}\, \|F(0)\|_{H_{-\vartheta}}}{(1-\vartheta)} + \frac{C_{\vartheta/2}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\, \|B(0)\|_{HS(U,H_{-\vartheta/2})}}{\sqrt{2-2\vartheta}} \right]\\
&\cdot \left[\frac{C_\rho\, C_{\rho+\vartheta}\, T^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-\vartheta-\rho)} + \frac{C_\rho\, C_{\rho+\vartheta/2}\sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta-4\rho}} \right]\\
&\cdot \left| \mathcal{E}_{(1-\vartheta)} \left[\frac{\sqrt{2}\, T^{(1-\vartheta)}\, C_0\, C_\vartheta\, |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + \sqrt{p\,(p-1)\,T^{(1-\vartheta)}}\, C_0\, C_{\vartheta/2}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right] \right|^2 .
\end{aligned} \tag{46}$$

This implies (40). The proof of Proposition 4.3 is thus completed. □

5 Mild stochastic calculus

In Theorem 1 in Da Prato et al. [12] a new – somehow mild – Itô type formula has been proposed and this formula has been called mild Itô formula. The mild Itô formula suggested in Theorem 1 in Da Prato et al. [12] has been proved from the deterministic starting time point $t_0 \in [0,\infty)$ to the deterministic end time point $t \in [t_0,\infty)$. In Theorem 5.3 in this section we generalize this mild Itô formula by allowing the end time point $t \in [t_0,\infty)$ to be a stopping time. We then use Theorem 5.3 to derive a mild Dynkin-type formula in Corollary 5.6. This mild Dynkin-type formula, in turn, is used in Proposition 5.8 below to derive suitable estimates for expectations of compositions of smooth functions and mild Itô processes. Proposition 5.8 is used intensively in the proof of the weak convergence result in Theorem 1.1 (see Section 6 for details).

5.1 Setting

Throughout this section we assume the following setting. Let $t_0 \in [0,\infty)$, $T \in (t_0,\infty)$, $\angle = \{(t_1,t_2) \in [t_0,T]^2 \colon t_1 < t_2\}$, let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t)_{t\in[t_0,T]})$ be a stochastic basis, let $(W_t)_{t\in[t_0,T]}$ be a cylindrical Id_U-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[t_0,T]}$, let $(\check{H}, \langle\cdot,\cdot\rangle_{\check{H}}, \|\cdot\|_{\check{H}})$, $(\tilde{H}, \langle\cdot,\cdot\rangle_{\tilde{H}}, \|\cdot\|_{\tilde{H}})$, $(\hat{H}, \langle\cdot,\cdot\rangle_{\hat{H}}, \|\cdot\|_{\hat{H}})$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, and $(V, \langle\cdot,\cdot\rangle_V, \|\cdot\|_V)$ be separable $\mathbb{R}$-Hilbert spaces with $\check{H} \subseteq \tilde{H} \subseteq \hat{H}$ continuously and densely, and let $\mathbb{U} \subseteq U$ be an orthonormal basis of U.

5.2 Mild Itô processes

For the convenience of the reader we recall the notion of a mild Itô process; see Definition 1 in Da Prato et al. [12].

Definition 5.1 (Mild Itô process)**.** Assume the setting in Section 5.1, let $S\colon \angle \to L(\hat{H}, \check{H})$ be a $\mathcal{B}(\angle)/\mathcal{S}(\hat{H}, \check{H})$-measurable mapping such that for all $t_1, t_2, t_3 \in [t_0, T]$ with $t_1 < t_2 < t_3$ it holds that $S_{t_2,t_3} S_{t_1,t_2} = S_{t_1,t_3}$, and let $Y\colon [t_0, T] \times \Omega \to \hat{H}$, $Z\colon [t_0, T] \times \Omega \to HS(U, \hat{H})$, and $X\colon [t_0, T] \times \Omega \to \check{H}$ be $(\mathcal{F}_t)_{t \in [t_0,T]}$-predictable stochastic processes such that for all $t \in (t_0, T]$ it holds $\mathbb{P}$-a.s. that $\int_{t_0}^{t} \|S_{s,t} Y_s\|_{\check{H}} + \|S_{s,t} Z_s\|^2_{HS(U,\check{H})} \, ds < \infty$ and

$$X_t = S_{t_0,t} \, X_{t_0} + \int_{t_0}^{t} S_{s,t} \, Y_s \, ds + \int_{t_0}^{t} S_{s,t} \, Z_s \, dW_s. \tag{47}$$

Then we call X a mild Itô process (with evolution family S, mild drift Y, and mild diffusion Z).

Lemma 5.2 (Regularization of mild Itô processes)**.** *Assume the setting in Section 5.1 and let $X\colon [t_0, T] \times \Omega \to \check{H}$ be a mild Itô process with evolution family $S\colon \angle \to L(\hat{H}, \check{H})$, mild drift $Y\colon [t_0, T] \times \Omega \to \hat{H}$, and mild diffusion $Z\colon [t_0, T] \times \Omega \to HS(U, \hat{H})$. Then*

(i) there exists an up to indistinguishability unique continuous stochastic process $\bar{X}\colon [t_0, T] \times \Omega \to \check{H}$ with $\forall \, t \in [t_0, T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T} X_t\big) = 1$

(ii) and for all continuous stochastic process $\bar{X}\colon [t_0, T] \times \Omega \to \check{H}$ with $\forall \, t \in [t_0, T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T} X_t\big) = 1$ and all $t \in [t_0, T]$ it holds that $\bar{X}$ is $(\mathcal{F}_s)_{s \in [t_0,T]}$-predictable, it holds $\mathbb{P}$-a.s. that $\bar{X}_T = X_T$ and it holds $\mathbb{P}$-a.s. that

$$\bar{X}_t = S_{t_0,T} \, X_{t_0} + \int_{t_0}^{t} S_{s,T} \, Y_s \, ds + \int_{t_0}^{t} S_{s,T} \, Z_s \, dW_s. \tag{48}$$

Proof. The assumption that X is a mild Itô process, in particular, ensures that it holds $\mathbb{P}$-a.s. that $\int_{t_0}^{T} \|S_{s,T} Y_s\|_{\check{H}} + \|S_{s,T} Z_s\|^2_{HS(U,\check{H})} \, ds < \infty$. This implies that there exists a continuous stochastic process $\bar{X}\colon [t_0, T] \times \Omega \to \check{H}$ such that for all $t \in [t_0, T]$ it holds $\mathbb{P}$-a.s. that

$$\bar{X}_t = S_{t_0,T} \, X_{t_0} + \int_{t_0}^{t} S_{s,T} \, Y_s \, ds + \int_{t_0}^{t} S_{s,T} \, Z_s \, dW_s. \tag{49}$$

Next observe that Definition 5.1 ensures that for all $t \in (t_0, T)$ it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} & S_{t_0,T} \, X_{t_0} + \int_{t_0}^{t} S_{s,T} \, Y_s \, ds + \int_{t_0}^{t} S_{s,T} \, Z_s \, dW_s \\ & = S_{t,T} \left(S_{t_0,t} \, X_{t_0} + \int_{t_0}^{t} S_{s,t} \, Y_s \, ds + \int_{t_0}^{t} S_{s,t} \, Z_s \, dW_s \right) = S_{t,T} \, X_t. \end{aligned} \tag{50}$$

This establishes that for all $t \in [t_0, T)$ it holds $\mathbb{P}$-a.s. that

$$S_{t_0,T} \, X_{t_0} + \int_{t_0}^{t} S_{s,T} \, Y_s \, ds + \int_{t_0}^{t} S_{s,T} \, Z_s \, dW_s = S_{t,T} \, X_t. \tag{51}$$

Combining (51) with (49) shows that there exists a continuous stochastic process $\bar{X}\colon [t_0, T] \times \Omega \to \check{H}$ such that for all $t \in [t_0, T)$ it holds $\mathbb{P}$-a.s. that

$$\bar{X}_t = S_{t_0,T} \, X_{t_0} + \int_{t_0}^{t} S_{s,T} \, Y_s \, ds + \int_{t_0}^{t} S_{s,T} \, Z_s \, dW_s = S_{t,T} X_t. \tag{52}$$

Moreover, observe that for all *continuous* stochastic processes $\bar{X}, \bar{Y}\colon [0, T] \times \Omega \to \check{H}$ with $\forall \, t \in [t_0, T)\colon \mathbb{P}\big(\bar{X}_t = \bar{Y}_t\big) = 1$ it holds that $\mathbb{P}\big(\forall \, t \in [t_0, T]\colon \bar{X}_t = \bar{Y}_t\big) = 1$. Combining this with (52) proves Item (i). Item (ii) is an immediate consequence from (52) and Item (i). The proof of Lemma 5.2 is thus completed. ◻

5.3 Mild Itô formula for stopping times

Theorem 5.3 (Mild Itô formula)**.** *Assume the setting in Section 5.1, let* $X\colon [t_0,T]\times\Omega\to\tilde{H}$ *be a mild Itô process with evolution family* $S\colon \angle \to L(\hat{H},\check{H})$*, mild drift* $Y\colon [t_0,T]\times\Omega\to\hat{H}$*, and mild diffusion* $Z\colon [t_0,T]\times\Omega\to HS(U,\hat{H})$*, let* $\bar{X}\colon [t_0,T]\times\Omega\to\check{H}$ *be a continuous stochastic process with* $\forall\, t\in[t_0,T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big)=1$ *(see Lemma 5.2), let* $r\in[t_0,T)$*,* $\varphi=(\varphi(t,x))_{t\in[r,T],\,x\in\check{H}}\in C^{1,2}([r,T]\times\check{H},V)$ *and let* $\tau\colon\Omega\to[r,T]$ *be an* $(\mathcal{F}_t)_{t\in[r,T]}$*-stopping time. Then it holds* $\mathbb{P}$*-a.s. that*

$$\int_r^T \big\|(\tfrac{\partial}{\partial x}\varphi)(s,S_{s,T}X_s)S_{s,T}Y_s\big\|_V + \big\|(\tfrac{\partial}{\partial x}\varphi)(s,S_{s,T}X_s)S_{s,T}Z_s\big\|^2_{HS(U,V)}\,ds<\infty, \tag{53}$$

$$\int_r^T \big\|(\tfrac{\partial}{\partial t}\varphi)(s,S_{s,T}X_s)\big\|_V + \|(\tfrac{\partial^2}{\partial x^2}\varphi)(s,S_{s,T}X_s)\|_{L^{(2)}(\check{H},V)}\,\|S_{s,T}Z_s\|^2_{HS(U,\check{H})}\,ds<\infty, \tag{54}$$

$$\begin{aligned}\varphi(\tau,\bar{X}_\tau)&=\varphi(r,S_{r,T}X_r)+\int_r^\tau(\tfrac{\partial}{\partial t}\varphi)(s,S_{s,T}X_s)\,ds+\int_r^\tau(\tfrac{\partial}{\partial x}\varphi)(s,S_{s,T}X_s)\,S_{s,T}\,Y_s\,ds\\&+\int_r^\tau(\tfrac{\partial}{\partial x}\varphi)(s,S_{s,T}X_s)\,S_{s,T}\,Z_s\,dW_s+\tfrac12\sum_{u\in\mathbb{U}}\int_r^\tau(\tfrac{\partial^2}{\partial x^2}\varphi)(s,S_{s,T}X_s)\,(S_{s,T}Z_su,S_{s,T}Z_su)\,ds.\end{aligned} \tag{55}$$

Proof of Theorem 5.3. First of all, we note that Theorem 1 in Da Prato et al. [12] establishes (53) and (54). It thus remains to prove (55). For this let $\varphi_{1,0}\colon[r,T]\times\check{H}\to V$, $\varphi_{0,1}\colon[r,T]\times\check{H}\to L(\check{H},V)$, $\varphi_{0,2}\colon[r,T]\times\check{H}\to L^{(2)}(\check{H},V)$ be the functions with the property that for all $t\in[r,T]$, $x,v_1,v_2\in\check{H}$ it holds that $\varphi_{1,0}(t,x)=\big(\tfrac{\partial}{\partial t}\varphi\big)(t,x)$, $\varphi_{0,1}(t,x)\,v_1=\big(\tfrac{\partial}{\partial x}\varphi\big)(t,x)\,v_1$, and $\varphi_{0,2}(t,x)(v_1,v_2)=\big(\tfrac{\partial^2}{\partial x^2}\varphi\big)(t,x)(v_1,v_2)$. Then note that Item (ii) of Lemma 5.2 and the standard Itô formula in Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9] show that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}\varphi(\tau,\bar{X}_\tau)&=\varphi(r,\bar{X}_r)+\int_r^\tau\varphi_{1,0}(s,\bar{X}_s)\,ds+\int_r^\tau\varphi_{0,1}(s,\bar{X}_s)\,S_{s,T}\,Y_s\,ds\\&+\int_r^\tau\varphi_{0,1}(s,\bar{X}_s)\,S_{s,T}\,Z_s\,dW_s+\tfrac12\sum_{u\in\mathbb{U}}\int_r^\tau\varphi_{0,2}(s,\bar{X}_s)\,(S_{s,T}\,Z_s\,u,S_{s,T}\,Z_s\,u)\,ds.\end{aligned} \tag{56}$$

Combining this with Lemma 1 in Da Prato et al. [12] and with the fact that $\forall\,t\in[t_0,T)\colon\mathbb{P}\big(\bar{X}_t=S_{t,T}\,X_t\big)=1$ shows that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}\varphi(\tau,\bar{X}_\tau)&=\varphi(r,S_{r,T}X_r)+\int_r^\tau\varphi_{1,0}(s,S_{s,T}X_s)\,ds+\int_r^\tau\varphi_{0,1}(s,S_{s,T}X_s)\,S_{s,T}\,Y_s\,ds\\&+\int_r^\tau\varphi_{0,1}(s,S_{s,T}X_s)\,S_{s,T}\,Z_s\,dW_s+\tfrac12\sum_{u\in\mathbb{U}}\int_r^\tau\varphi_{0,2}(s,S_{s,T}X_s)\,(S_{s,T}Z_su,S_{s,T}Z_su)\,ds.\end{aligned} \tag{57}$$

The proof of Theorem 5.3 is thus completed. □

Definition 5.4 (Extended mild Kolmogorov operators)**.** Assume the setting in Section 5.1, let $S\colon\angle\to L(\hat{H},\check{H})$ be a $\mathcal{B}(\angle)/\mathcal{S}(\hat{H},\check{H})$-measurable mapping such that for all $t_1,t_2,t_3\in[t_0,T]$ with $t_1<t_2<t_3$ it holds that $S_{t_2,t_3}S_{t_1,t_2}=S_{t_1,t_3}$, and let $(t_1,t_2)\in\angle$. Then we denote by $\mathcal{L}^S_{t_1,t_2}\colon C^2(\check{H},V)\to C(\tilde{H}\times\hat{H}\times HS(U,\hat{H}),V)$ the function with the property that for all $\varphi\in C^2(\check{H},V)$, $x\in\tilde{H}$, $y\in\hat{H}$, $z\in HS(U,\hat{H})$ it holds that

$$\big(\mathcal{L}^S_{t_1,t_2}\varphi\big)(x,y,z)=\varphi'(S_{t_1,t_2}\,x)\,S_{t_1,t_2}\,y+\tfrac12\sum_{u\in\mathbb{U}}\varphi''(S_{t_1,t_2}\,x)(S_{t_1,t_2}\,z\,u,S_{t_1,t_2}\,z\,u). \tag{58}$$

The next corollary of Theorem 5.3 specialises Theorem 5.3 to the case where $r=t_0$ and where the test function $(\varphi(t,x))_{t\in[t_0,T],\,x\in\check{H}}\in C^{1,2}([t_0,T]\times\check{H},V)$ depends on $x\in\check{H}$ only.

Corollary 5.5. *Assume the setting in Section 5.1, let* $X\colon [t_0,T]\times\Omega\to\tilde{H}$ *be a mild Itô process with evolution family* $S\colon \angle \to L(\hat{H},\check{H})$*, mild drift* $Y\colon [t_0,T]\times\Omega\to\hat{H}$*, and mild diffusion* $Z\colon [t_0,T]\times\Omega\to HS(U,\hat{H})$*, let* $\bar{X}\colon [t_0,T]\times\Omega\to\check{H}$ *be a continuous stochastic process with* $\forall\, t\in[t_0,T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big) = 1$ *(see Lemma 5.2), let* $\varphi\in C^2(\check{H},V)$*, and let* $\tau\colon\Omega\to[t_0,T]$ *be an* $(\mathcal{F}_t)_{t\in[t_0,T]}$*-stopping time. Then it holds* $\mathbb{P}$*-a.s. that* $\int_{t_0}^T \|\varphi'(S_{s,T}X_s)S_{s,T}Y_s\|_V + \|\varphi'(S_{s,T}X_s)S_{s,T}Z_s\|^2_{HS(U,V)} + \|\varphi''(S_{s,T}X_s)\|_{L^{(2)}(\check{H},V)}\,\|S_{s,T}Z_s\|^2_{HS(U,\check{H})}\,ds<\infty$ *and it holds* $\mathbb{P}$*-a.s. that*

$$\varphi(\bar{X}_\tau) = \varphi(S_{t_0,T}X_{t_0}) + \int_{t_0}^{\tau} (\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,ds + \int_{t_0}^{\tau} \varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\,dW_s. \tag{59}$$

5.4 Mild Dynkin-type formula

Under suitable additional assumptions (see Corollary 5.6 below), the stochastic integral in (59) is integrable and centered. This is the subject of the following result.

Corollary 5.6 (Mild Dynkin-type formula)**.** *Assume the setting in Section 5.1, let* $X\colon [t_0,T]\times\Omega\to\tilde{H}$ *be a mild Itô process with evolution family* $S\colon\angle\to L(\hat{H},\check{H})$*, mild drift* $Y\colon[t_0,T]\times\Omega\to\hat{H}$*, and mild diffusion* $Z\colon[t_0,T]\times\Omega\to HS(U,\hat{H})$*, let* $\bar{X}\colon[t_0,T]\times\Omega\to\check{H}$ *be a continuous stochastic process with* $\forall\, t\in[t_0,T)\colon\mathbb{P}\big(\bar{X}_t=S_{t,T}X_t\big)=1$ *(see Lemma 5.2), let* $\varphi\in C^2(\check{H},V)$*, and let* $\tau\colon\Omega\to[t_0,T]$ *be an* $(\mathcal{F}_t)_{t\in[t_0,T]}$*-stopping time with the property that* $\mathbb{E}\big[|\int_{t_0}^\tau\|\varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\|^2_{HS(U,V)}\,ds|^{1/2}\big] + \min\big\{\mathbb{E}\big[\|\varphi(S_{t_0,T}X_{t_0})+\int_{t_0}^\tau(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,ds\|_V\big],\ \mathbb{E}\big[\|\varphi(\bar{X}_\tau)\|_V\big]\big\}<\infty$*. Then* $\mathbb{E}\big[\|\varphi(\bar{X}_\tau)\|_V + \|\varphi(S_{t_0,T}X_{t_0})+\int_{t_0}^\tau(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,ds\|_V\big]<\infty$ *and*

$$\mathbb{E}\big[\varphi(\bar{X}_\tau)\big] = \mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0}) + \textstyle\int_{t_0}^\tau(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,ds\big]. \tag{60}$$

Corollary 5.6 is an immediate consequence of Corollary 5.5 and, e.g., of the Burkholder-Davis-Gundy inequality in Problem 3.29 in Karatzas & Shreve [26].

5.5 Weak estimates for terminal values of mild Itô processes

Proposition 5.7. *Assume the setting in Section 5.1, let* $X\colon[t_0,T]\times\Omega\to\tilde{H}$ *be a mild Itô process with evolution family* $S\colon\angle\to L(\hat{H},\check{H})$*, mild drift* $Y\colon[t_0,T]\times\Omega\to\hat{H}$*, and mild diffusion* $Z\colon[t_0,T]\times\Omega\to HS(U,\hat{H})$*, let* $\varphi\in C^2(\check{H},V)$*, and assume that* $\big\{\|\varphi(S_{t_0,T}\,X_{t_0}+\int_{t_0}^\tau S_{s,T}\,Y_s\,ds + \int_{t_0}^\tau S_{s,T}\,Z_s\,dW_s)\|_V\colon (\mathcal{F}_t)_{t\in[t_0,T]}\text{-stopping time } \tau\colon\Omega\to[t_0,T]\big\}$ *is uniformly* $\mathbb{P}$*-integrable. Then it holds that* $\mathbb{E}\big[\|\varphi(X_T)\|_V+\|\varphi(S_{t_0,T}X_{t_0})\|_V\big]<\infty$ *and*

$$\big\|\mathbb{E}\big[\varphi(X_T)\big]\big\|_V \leq \big\|\mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0})\big]\big\|_V + \int_{t_0}^{T}\mathbb{E}\Big[\big\|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\big\|_V\Big]\,ds. \tag{61}$$

Proof. First of all, we observe that the assumption that the set $\big\{\|\varphi(S_{t_0,T}\,X_{t_0}+\int_{t_0}^\tau S_{s,T}\,Y_s\,ds+\int_{t_0}^\tau S_{s,T}\,Z_s\,dW_s)\|_V\colon(\mathcal{F}_t)_{t\in[t_0,T]}\text{-stopping time }\tau\colon\Omega\to[t_0,T]\big\}$ is uniformly $\mathbb{P}$-integrable ensures that $\mathbb{E}\big[\|\varphi(X_T)\|_V+\|\varphi(S_{t_0,T}X_{t_0})\|_V\big]<\infty$. It thus remains to prove (61). For this let $\tau_n\colon\Omega\to[t_0,T]$, $n\in\mathbb{N}$, be the functions with the property that for all $n\in\mathbb{N}$ it holds that

$$\tau_n = \inf\left(\{T\}\cup\left\{t\in[t_0,T]\colon \int_{t_0}^{t}\|\varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\|^2_{HS(U,V)}\,ds\geq n\right\}\right) \tag{62}$$

and let $\bar{X}\colon[t_0,T]\times\Omega\to\check{H}$ be a continuous stochastic process with the property that $\forall\, t\in[t_0,T)\colon\mathbb{P}\big(\bar{X}_t=S_{t,T}X_t\big)=1$. Note that Item (i) of Lemma 5.2 ensures that $\bar{X}$ does indeed exist. Moreover, observe that for all $n\in\mathbb{N}$ it holds that τ_n is an $(\mathcal{F}_t)_{t\in[t_0,T]}$-stopping time. Next note that Corollary 5.5 shows that it holds $\mathbb{P}$-a.s. that $\int_{t_0}^T\|\varphi'(S_{s,T}X_s)\,S_{s,T}\,Z_s\|^2_{HS(U,V)}\,ds<\infty$. This, in turn, establishes that it holds $\mathbb{P}$-a.s. that $\lim_{n\to\infty}\tau_n=T$. In addition, note that

Item (ii) of Lemma 5.2 together with the assumption that the set $\big\{\|\varphi(S_{t_0,T}\,X_{t_0} + \int_{t_0}^{\tau} S_{s,T}\,Y_s\,ds + \int_{t_0}^{\tau} S_{s,T}\,Z_s\,dW_s)\|_V \colon (\mathcal{F}_t)_{t\in[t_0,T]}\text{-stopping time } \tau\colon \Omega \to [t_0,T]\big\}$ is uniformly $\mathbb{P}$-integrable ensures that the set $\{\|\varphi(\bar{X}_{\tau_n})\|_V \colon n \in \mathbb{N}\}$ is uniformly $\mathbb{P}$-integrable. This and (62) establish that for all $n \in \mathbb{N}$ it holds that $\mathbb{E}\big[\|\varphi(\bar{X}_{\tau_n})\|_V\big] + \mathbb{E}\big[\int_0^{\tau_n} \|\varphi'(S_{s,T}S_{s,T}Z_s)\|^2_{HS(U,V)}\,ds\big] < \infty$. We can thus apply Corollary 5.6 to obtain that for all $n \in \mathbb{N}$ it holds that

$$\mathbb{E}\big[\varphi(\bar{X}_{\tau_n})\big] = \mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0}) + \textstyle\int_{t_0}^{\tau_n} (\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\,ds\big]. \tag{63}$$

The triangle inequality hence proves that

$$\limsup_{n\to\infty} \big\|\mathbb{E}\big[\varphi(\bar{X}_{\tau_n})\big]\big\|_V \le \big\|\mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0})\big]\|_V + \textstyle\int_{t_0}^{T} \mathbb{E}\big[\|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\|_V\big]\,ds. \tag{64}$$

This together with the uniform $\mathbb{P}$-integrability of $\{\|\varphi(\bar{X}_{\tau_n})\|_V \colon n \in \mathbb{N}\}$ proves (61). The proof of Proposition 5.7 is thus completed. □

Proposition 5.8 (Test functions with at most polynomial growth)**.** *Assume the setting in Section 5.1, let $X\colon [t_0,T]\times\Omega \to \tilde{H}$ be a mild Itô process with evolution family $S\colon \angle \to L(\hat{H},\check{H})$, mild drift $Y\colon [t_0,T]\times\Omega \to \hat{H}$, and mild diffusion $Z\colon [t_0,T]\times\Omega \to HS(U,\hat{H})$, and let $p \in [0,\infty)$, $\varphi \in C^2(\check{H},V)$ satisfy $\sup_{x\in\check{H}}\big[\|\varphi(x)\|_V(1+\|x\|^p_{\check{H}})^{-1}\big] < \infty$ and $\|S_{t_0,T}X_{t_0}\|_{\check{H}} + \int_{t_0}^{T}\|S_{s,T}Y_s\|_{\check{H}}\,ds + \big[\int_{t_0}^{T}\|S_{s,T}Z_s\|^2_{HS(U,\check{H})}\,ds\big]^{1/2} \in L^p(\mathbb{P};\mathbb{R})$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_V + \|\varphi(S_{t_0,T}X_{t_0})\|_V\big] < \infty$ and*

$$\big\|\mathbb{E}\big[\varphi(X_T)\big]\big\|_V \le \big\|\mathbb{E}\big[\varphi(S_{t_0,T}X_{t_0})\big]\big\|_V + \int_{t_0}^{T} \mathbb{E}\Big[\big\|(\mathcal{L}^S_{s,T}\varphi)(X_s,Y_s,Z_s)\big\|_V\Big]\,ds. \tag{65}$$

Proof. Throughout this proof let $\bar{X}\colon [t_0,T]\times\Omega \to \check{H}$ be a continuous stochastic process with $\forall\, t\in[t_0,T)\colon \mathbb{P}\big(\bar{X}_t = S_{t,T}X_t\big) = 1$. Item (i) of Lemma 5.2 ensures that $\bar{X}$ does indeed exist. In addition, we observe that Item (ii) of Lemma 5.2 also implies that for all $t \in [t_0,T]$ it holds $\mathbb{P}$-a.s. that

$$\begin{split}
\|\varphi(\bar{X}_t)\|_V &\le \left[\sup_{x\in\check{H}} \frac{\|\varphi(x)\|_V}{(1+\|x\|^p_{\check{H}})}\right]\big(1+\|\bar{X}_t\|^p_{\check{H}}\big)\\
&\le 3^p \left[\sup_{x\in\check{H}} \frac{\|\varphi(x)\|_V}{(1+\|x\|^p_{\check{H}})}\right]\left(1+\|S_{t_0,T}X_{t_0}\|^p_{\check{H}} + \left|\int_{t_0}^{T}\|S_{s,T}Y_s\|_{\check{H}}\,ds\right|^p + \left\|\int_{t_0}^{t} S_{s,T}\,Z_s\,dW_s\right\|^p_{\check{H}}\right).
\end{split} \tag{66}$$

Moreover, e.g., the Burkholder-Davis-Gundy inequality in Problem 3.29 in Karatzas & Shreve [26] shows that there exists a real number $C \in [0,\infty)$ such that

$$\mathbb{E}\left[\sup_{t\in[t_0,T]}\left\|\int_{t_0}^{t} S_{s,T}\,Z_s\,dW_s\right\|^p_{\check{H}}\right] \le C\,\mathbb{E}\left[\left|\int_{t_0}^{T}\|S_{s,T}\,Z_s\|^2_{HS(U,\check{H})}\,ds\right|^{p/2}\right]. \tag{67}$$

Combining (66) and (67) yields that there exists a real number $C \in [0,\infty)$ such that

$$\begin{split}
&\mathbb{E}\left[\sup_{t\in[t_0,T]}\|\varphi(\bar{X}_t)\|_V\right]\\
&\le C\left(1+\mathbb{E}\big[\|S_{t_0,T}\,X_{t_0}\|^p_{\check{H}}\big] + \mathbb{E}\left[\left|\int_{t_0}^{T}\|S_{s,T}\,Y_s\|_{\check{H}}\,ds\right|^p\right] + \mathbb{E}\left[\left|\int_{t_0}^{T}\|S_{s,T}\,Z_s\|^2_{HS(U,\check{H})}\,ds\right|^{p/2}\right]\right).
\end{split} \tag{68}$$

In the next step we combine (68) with the assumption that $\|S_{t_0,T}X_{t_0}\|_{\check{H}} + \int_{t_0}^{T}\|S_{s,T}Y_s\|_{\check{H}}\,ds + \big[\int_{t_0}^{T}\|S_{s,T}Z_s\|^2_{HS(U,\check{H})}\,ds\big]^{1/2} \in L^p(\mathbb{P};\mathbb{R})$ to obtain that $\mathbb{E}\big[\sup_{t\in[t_0,T]}\|\varphi(\bar{X}_t)\|_V\big] < \infty$. Item (ii) of Lemma 5.2 hence proves that $\mathbb{E}\big[\sup_{t\in[t_0,T]}\|\varphi(S_{t_0,T}X_{t_0} + \int_{t_0}^{t} S_{s,t}Y_s\,ds + \int_{t_0}^{t} S_{s,t}Z_s\,dW_s)\|_V\big] < \infty$. Combining this with Proposition 5.7 completes the proof of Proposition 5.8. □

6 Weak temporal regularity and analysis of the weak distance between Euler-type approximations of SPDEs and their semilinear integrated counterparts

In this section we establish a weak temporal regularity result in Proposition 6.2 below. In addition, we prove a weak approximation result in Proposition 6.4 below. The proofs of Proposition 6.2 and Proposition 6.4 use Proposition 5.8 which, in turn, is established by an application of the mild Itô formula.

6.1 Setting

Assume the setting in Section 1.4, let $\vartheta \in [0,1)$, $F \in \operatorname{Lip}^0(H, H_{-\vartheta})$, $B \in \operatorname{Lip}^0(H, HS(U, H_{-\vartheta/2}))$, $p \in [2,\infty)$, let $(B^b)_{b \in \mathbb{U}} \subseteq C(H, H_{-\vartheta/2})$ be the functions with the property that for all $v \in H$, $b \in \mathbb{U}$ it holds that $B^b(v) = B(v)\, b$, let $\varsigma_{F,B} \in \mathbb{R}$ be a real number given by $\varsigma_{F,B} = \max\{1, \|F\|_{\operatorname{Lip}^0(H,H_{-\vartheta})}, \|B\|^2_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\}$, let $Y, \bar{Y} \colon [0,T] \times \Omega \to H$ be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that $\|Y_0\|_{L^p(\mathbb{P};H)} < \infty$, such that $\bar{Y}_0 = Y_0$, and such that for all $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$Y_t = S_{0,t}\, Y_0 + \int_0^t S_{s,t}\, R_s\, F(Y_{\lfloor s \rfloor_h})\, ds + \int_0^t S_{s,t}\, R_s\, B(Y_{\lfloor s \rfloor_h})\, dW_s, \tag{69}$$

$$\bar{Y}_t = e^{tA}\, \bar{Y}_0 + \int_0^t e^{(t-s)A}\, F(Y_{\lfloor s \rfloor_h})\, ds + \int_0^t e^{(t-s)A}\, B(Y_{\lfloor s \rfloor_h})\, dW_s, \tag{70}$$

and let $(K_r)_{r\in[0,\infty)} \subseteq [0,\infty]$ be extended real numbers which satisfy that for all $r \in [0,\infty)$ it holds that $K_r = \sup_{s,t\in[0,T]} \mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^r, \|Y_t\|_H^r\}\big]$.

6.2 Weak temporal regularity of semilinear integrated Euler-type approximations

In Proposition 6.2 below we establish a weak temporal regularity result for the process $\bar{Y}$ in Subsection 6.1. The proof of Proposition 6.2 uses the following elementary result.

Lemma 6.1. *Assume the setting in Section 6.1. Then*

$$\begin{aligned} &\sup_{r\in[0,p]} K_r = K_p \\ &\leq \left[C_0 \max\{1, \|Y_0\|_{L^p(\mathbb{P};H)}\} + \frac{C_\vartheta\, \|F\|_{\operatorname{Lip}^0(H,H_{-\vartheta})}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \frac{C_{\vartheta/2} \sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, \|B\|_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta}} \right]^{2p} \\ &\cdot 2^{(\frac{p}{2}+1)} \left| \mathcal{E}_{(1-\vartheta)} \left[\frac{\sqrt{2}\, C_\vartheta\, T^{(1-\vartheta)}\, |F|_{\operatorname{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + C_{\vartheta/2} \sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, |B|_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right] \right|^p < \infty. \end{aligned} \tag{71}$$

Proof. First of all, we observe that the equality in (71) follows from the fact that for all $x \in H$, $r, s \in [0,\infty)$ with $r \leq s$ it holds that $\max\{1, \|x\|_H^r\} \leq \max\{1, \|x\|_H^s\}$. Moreover, we note that the second inequality in (71) is an immediate consequence from the assumption that $\|Y_0\|_{L^p(\mathbb{P};H)} < \infty$. It thus remains to prove the first inequality in (71). For this, we observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] ensures that for all $k \in$

$\{1, 2, \ldots, \lfloor T \rfloor_h/h\}$ it holds that

$$
\begin{aligned}
&\|Y_{kh}\|_{L^p(\mathbb{P};H)} \\
&\leq \|S_{0,kh}\, Y_0\|_{L^p(\mathbb{P};H)} + \left\| \int_0^{kh} S_{s,kh}\, R_s\, F(Y_{\lfloor s \rfloor_h})\, ds \right\|_{L^p(\mathbb{P};H)} + \left\| \int_0^{kh} S_{s,kh}\, R_s\, B(Y_{\lfloor s \rfloor_h})\, dW_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq C_0\, \|Y_0\|_{L^p(\mathbb{P};H)} + \int_0^{kh} \|S_{s,kh}\, R_s\, F(Y_{\lfloor s \rfloor_h})\|_{L^p(\mathbb{P};H)}\, ds \\
&+ \left[\tfrac{p\,(p-1)}{2} \int_0^{kh} \|S_{s,kh}\, R_s\, B(Y_{\lfloor s \rfloor_h})\|^2_{L^p(\mathbb{P};HS(U,H))}\, ds \right]^{1/2} \\
&\leq C_0\, \|Y_0\|_{L^p(\mathbb{P};H)} + C_\vartheta\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \left[\max_{j \in \{0,1,\ldots,k-1\}} \big\| \max\{1, \|Y_{jh}\|_H\} \big\|_{L^p(\mathbb{P};\mathbb{R})} \right] \int_0^{kh} \tfrac{1}{(kh-s)^\vartheta}\, ds \\
&+ C_{\vartheta/2}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \left[\max_{j \in \{0,1,\ldots,k-1\}} \big\| \max\{1, \|Y_{jh}\|_H\} \big\|_{L^p(\mathbb{P};\mathbb{R})} \right] \left[\tfrac{p\,(p-1)}{2} \int_0^{kh} \tfrac{1}{(kh-s)^\vartheta}\, ds \right]^{1/2} \\
&\leq \left[C_0 + \tfrac{C_\vartheta\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, |kh|^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{C_{\vartheta/2}\, \sqrt{p\,(p-1)\, |kh|^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta}} \right] \\
&\cdot \max_{j \in \{0,1,\ldots,k-1\}} \big\| \max\{1, \|Y_{jh}\|_H\} \big\|_{L^p(\mathbb{P};\mathbb{R})}
\end{aligned}
\tag{72}
$$

This and the assumption that $\|Y_0\|_{L^p(\mathbb{P};H)} < \infty$ allow us to conclude inductively that

$$
\sup_{t \in [0,T]} \|Y_{\lfloor t \rfloor_h}\|_{L^p(\mathbb{P};H)} = \max_{k \in \{0,1,\ldots,\lfloor T \rfloor_h/h\}} \|Y_{kh}\|_{L^p(\mathbb{P};H)} < \infty.
$$

We can hence apply Proposition 4.2 to obtain[5] that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|Y_t\|_{L^p(\mathbb{P};H)} \leq \sqrt{2} \\
&\cdot \left[C_0\, \|Y_0\|_{L^p(\mathbb{P};H)} + \tfrac{C_\vartheta\, T^{(1-\vartheta)} \|F(0)\|_{H_{-\vartheta}}}{(1-\vartheta)} + C_{\vartheta/2} \sqrt{\tfrac{p\,(p-1)\, T^{(1-\vartheta)}}{(2-2\vartheta)}} \|B(0)\|_{HS(U,H_{-\vartheta/2})} \right] \\
&\cdot \mathcal{E}_{(1-\vartheta)} \left[\tfrac{\sqrt{2}\, C_\vartheta\, T^{(1-\vartheta)}\, |F|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + C_{\vartheta/2} \sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, |B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \right].
\end{aligned}
\tag{73}
$$

Next we note that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] shows that

$$
\begin{aligned}
&\sup_{t \in [0,T]} \|\bar{Y}_t\|_{L^p(\mathbb{P};H)} \leq \sup_{t \in [0,T]} \big\| \max\{1, \|Y_t\|_H\} \big\|_{L^p(\mathbb{P};\mathbb{R})} \\
&\cdot \left[C_0 + \tfrac{C_\vartheta\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{C_{\vartheta/2}\, \sqrt{p\,(p-1)\, T^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta}} \right].
\end{aligned}
\tag{74}
$$

Moreover, we observe that for all $s, t \in [0,T]$ it holds that

$$
\begin{aligned}
\mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^p, \|Y_t\|_H^p\} \big] &\leq \mathbb{E}\big[\|\bar{Y}_s\|_H^p \big] + \mathbb{E}\big[\max\{1, \|Y_t\|_H^p\} \big] \\
&\leq \sup_{u \in [0,T]} \|\bar{Y}_u\|^p_{L^p(\mathbb{P};H)} + \sup_{u \in [0,T]} \big\| \max\{1, \|Y_u\|_H\} \big\|^p_{L^p(\mathbb{P};\mathbb{R})}.
\end{aligned}
\tag{75}
$$

This together with (73) and (74) proves the first inequality in (71). The proof of Lemma 6.1 is thus completed. ☐

[5] with $\kappa = 0$, $L_{0,t} = S_{0,t}$, $L_{s,t} = S_{s,t}\, R_s$, $\Pi(s) = \lfloor s \rfloor_h$ for $(s,t) \in (\angle \cap (0,T]^2)$ in the notation of Proposition 4.2

Proposition 6.2. *Assume the setting in Section 6.1 and let* $\eta \in [0,\infty)$, $q \in [0,\infty) \cap (-\infty, p-3]$, $\rho \in [0, 1-\vartheta)$, $\psi = (\psi(x,y))_{x,y\in H} \in C^2(H \times H, V)$ *satisfy that for all* $x_1, x_2, y \in H$, $i, j \in \{0,1,2\}$ *with* $i + j \leq 2$ *it holds that*

$$\big\| \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_1,y) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_2,y) \big\|_{L^{(i+j)}(H,V)} \leq \eta \max\{1, \|x_1\|_H^q, \|x_2\|_H^q, \|y\|_H^q\} \, \|x_1 - x_2\|_H \,.$$

Then for all $(s,t) \in \angle$ *it holds that* $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\|_V\big] < \infty$ *and*

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V \leq \eta \, |C_0|^{(q+1)} \, |C_\rho|^2 \, \varsigma_{F,B} \, K_{q+3} \, (t-s)^\rho \\ &\cdot \left[\tfrac{2^\rho}{t^\rho} + \tfrac{\left(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\right) s^{(1-\vartheta-\rho)} + \left(C_\vartheta + \frac{1}{2}|C_{\vartheta/2}|^2\right) |t-s|^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right]. \end{aligned} \tag{76}$$

Proof. Throughout this proof let $(g_r)_{r\in[0,\infty)} \subseteq C(H,\mathbb{R})$ be the functions with the property that for all $r \in [0,\infty)$, $x \in H$ it holds that $g_r(x) = \max\{1, \|x\|_H^r\}$ and let $\psi_{1,0} \colon H \times H \to L(H,V)$, $\psi_{0,1} \colon H \times H \to L(H,V)$, $\psi_{2,0} \colon H \times H \to L^{(2)}(H,V)$, $\psi_{0,2} \colon H \times H \to L^{(2)}(H,V)$, $\psi_{1,1} \colon H \times H \to L^{(2)}(H,V)$ be the functions with the property that for all $x, y, v_1, v_2 \in H$ it holds that $\psi_{1,0}(x,y)\, v_1 = \big(\frac{\partial}{\partial x}\psi(x,y)\big)(v_1)$ and

$$\psi_{0,1}(x,y)\, v_1 = \big(\tfrac{\partial}{\partial y}\psi(x,y)\big)(v_1), \qquad \psi_{2,0}(x,y)(v_1,v_2) = \big(\tfrac{\partial^2}{\partial x^2}\psi(x,y)\big)(v_1,v_2), \tag{77}$$

$$\psi_{0,2}(x,y)(v_1,v_2) = \big(\tfrac{\partial^2}{\partial y^2}\psi(x,y)\big)(v_1,v_2), \qquad \psi_{1,1}(x,y)(v_1,v_2) = \big(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\psi(x,y)\big)(v_1,v_2). \tag{78}$$

Next we observe that Lemma 6.1 and the assumption that $q \leq p-3$ ensure that $K_{q+1} \leq K_{q+3} < \infty$. Combining this with the fact that

$$\forall\, x_1, x_2, y \in H \colon \|\psi(x_1,y) - \psi(x_2,y)\|_V \leq 2\,\eta \max\big\{1, \|x_1\|_H^{q+1}, \|x_2\|_H^{q+1}, \|y\|_H^{q+1}\big\} \tag{79}$$

shows that for all $(s,t) \in \angle$ it holds that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\|_V\big] < \infty$. It thus remains to prove (76). To do so, we make use of a consequence of the mild Itô formula in Corollary 5.5 above. More formally, an application of Proposition 5.8 shows[6] that for all $(s,t) \in \angle$ it holds that $\mathbb{E}\big[\|\psi(e^{(t-s)A}\,\bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\|_V\big] < \infty$ and

$$\begin{aligned} &\big\|\mathbb{E}\big[\psi(\bar{Y}_t, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V \leq \big\|\mathbb{E}\big[\psi(e^{(t-s)A}\,\bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V \\ &+ \int_s^t \mathbb{E}\big[\big\|\psi_{1,0}(e^{(t-r)A}\,\bar{Y}_r, Y_s)\, e^{(t-r)A} F(Y_{\lfloor r \rfloor_h})\big\|_V\big]\, dr \\ &+ \int_s^t \mathbb{E}\left[\left\|\tfrac{1}{2}\sum_{b\in\mathbb{U}} \psi_{2,0}(e^{(t-r)A}\,\bar{Y}_r, Y_s)\big(e^{(t-r)A}\,B^b(Y_{\lfloor r \rfloor_h}), e^{(t-r)A}\,B^b(Y_{\lfloor r \rfloor_h})\big)\right\|_V\right] dr. \end{aligned} \tag{80}$$

In the following we establish suitable estimates for the three summands appearing on right hand side of (80). Combining these estimates with (80) will then allow us to establish (76). We begin with the second and the third summands on the right hand side of (80). We note that the assumption that $\forall\, x_1, x_2, y \in H$:

$$\|\psi(x_1,y) - \psi(x_2,y)\|_V \leq \eta \max\{1, \|x_1\|_H^q, \|x_2\|_H^q, \|y\|_H^q\} \, \|x_1 - x_2\|_H \tag{81}$$

implies that $\forall\, x, y \in H \colon \|\psi_{1,0}(x,y)\|_{L(H,V)} \leq \eta \max\{1, \|x\|_H^q, \|y\|_H^q\}$. This, in turn, proves that for all $(r,t) \in \angle$, $u, v, w \in H$ it holds that

$$\begin{aligned} &\big\|\psi_{1,0}(e^{(t-r)A}\,u, v)\, e^{(t-r)A}\,F(w)\big\|_V \\ &\leq \eta\, |C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\} \, \|e^{(t-r)A}\|_{L(H,H_\vartheta)} \, \|F(w)\|_{H_{-\vartheta}} \\ &\leq \frac{\eta\, |C_0|^q \, C_\vartheta \max\{1, \|u\|_H^q, \|v\|_H^q\} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \, g_1(w)}{(t-r)^\vartheta}. \end{aligned} \tag{82}$$

[6]with $t_0 = s$, $T = t$, $\check{H} = H \times H \times H$, $p = q + 1$, and $\varphi(x,y,z) = \psi(x,y) - \psi(z,y)$ for $(x,y,z) \in \check{H}$ in the notation of Proposition 5.8

Next we observe that the assumption that $\forall\, x_1, x_2, y \in H$:

$$\|\psi_{1,0}(x_1, y) - \psi_{1,0}(x_2, y)\|_{L(H,V)} \leq \eta \max\{1, \|x_1\|_H^q, \|x_2\|_H^q, \|y\|_H^q\} \|x_1 - x_2\|_H \tag{83}$$

shows that $\forall\, x, y \in H$: $\|\psi_{2,0}(x, y)\|_{L^{(2)}(H,V)} \leq \eta \max\{1, \|x\|_H^q, \|y\|_H^q\}$. This, in turn, proves that for all $(r, t) \in \angle$, $u, v, w \in H$ it holds that

$$\begin{aligned} &\tfrac{1}{2} \sum_{b \in \mathbb{U}} \big\| \psi_{2,0}\big(e^{(t-r)A} u, v\big)\big(e^{(t-r)A} B^b(w), e^{(t-r)A} B^b(w)\big)\big\|_V \\ &\leq \eta\, |C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\, \|e^{(t-r)A} B(w)\|^2_{HS(U,H)} \\ &\leq \frac{\eta\, |C_0|^q\, \frac{1}{2}\, |C_{\vartheta/2}|^2 \max\{1, \|u\|_H^q, \|v\|_H^q\}\, \|B\|^2_{\mathrm{Lip}^0(H, HS(U, H_{-\vartheta/2}))}\, g_2(w)}{(t-r)^{\vartheta}}. \end{aligned} \tag{84}$$

Furthermore, we note that Hölder's inequality implies that for all $r, l \in (0, \infty)$, $s, t \in [0, T]$ it holds that

$$\begin{aligned} &\mathbb{E}\big[\max\big\{1, \|\bar{Y}_s\|_H^r, \|Y_t\|_H^r\big\}\, g_l(Y_{\lfloor s \rfloor_h})\big] \\ &\leq \left(\sup_{u,v \in [0,T]} \big\|\max\big\{1, \|\bar{Y}_u\|_H^r, \|Y_v\|_H^r\big\}\big\|_{L^{1+l/r}(\mathbb{P};\mathbb{R})}\right) \left(\sup_{u \in [0,T]} \big\|\max\big\{1, \|Y_u\|_H^l\big\}\big\|_{L^{1+r/l}(\mathbb{P};\mathbb{R})}\right) \\ &\leq |K_{r+l}|^{\frac{1}{1+l/r}}\, |K_{r+l}|^{\frac{1}{1+r/l}} = K_{r+l}. \end{aligned} \tag{85}$$

This and the fact that for all $l \in [0, \infty)$ it holds that $\sup_{s \in [0,T]} \mathbb{E}\big[g_l(Y_{\lfloor s \rfloor_h})\big] \leq K_l$ prove that for all $r, l \in [0, \infty)$, $s, t \in [0, T]$ it holds that

$$\mathbb{E}\big[\max\big\{1, \|\bar{Y}_s\|_H^r, \|Y_t\|_H^r\big\}\, g_l(Y_{\lfloor s \rfloor_h})\big] \leq K_{r+l}. \tag{86}$$

Combining (82), (84), and (86) implies that for all $(s, t) \in \angle$ it holds that

$$\begin{aligned} &\int_s^t \mathbb{E}\big[\big\|\psi_{1,0}(e^{(t-r)A} \bar{Y}_r, Y_s)\, e^{(t-r)A} F(Y_{\lfloor r \rfloor_h})\big\|_V\big]\, dr \\ &+ \int_s^t \mathbb{E}\left[\left\| \tfrac{1}{2} \sum_{b \in \mathbb{U}} \psi_{2,0}(e^{(t-r)A} \bar{Y}_r, Y_s)\big(e^{(t-r)A} B^b(Y_{\lfloor r \rfloor_h}), e^{(t-r)A} B^b(Y_{\lfloor r \rfloor_h})\big) \right\|_V\right] dr \\ &\leq \eta\, |C_0|^q \left(C_\vartheta\, \|F\|_{\mathrm{Lip}^0(H, H_{-\vartheta})} + \tfrac{1}{2}\, |C_{\vartheta/2}|^2\, \|B\|^2_{\mathrm{Lip}^0(H, HS(U, H_{-\vartheta/2}))}\right) K_{q+2} \int_s^t \frac{1}{(t-r)^{\vartheta}}\, dr \\ &\leq \frac{\eta\, |C_0|^q\, \big(C_\vartheta + \frac{1}{2} |C_{\vartheta/2}|^2\big)\, \varsigma_{F,B}\, K_{q+2}\, (t-s)^{(1-\vartheta)}}{(1-\vartheta)}. \end{aligned} \tag{87}$$

Inequality (87) provides us an appropriate estimate for the second and the third summand on the right hand side of (80). It thus remains to provide a suitable estimate for the first summand on the right hand side of (80). For this we will employ Proposition 5.8 again and this will allow us to obtain an appropriate upper bound for $\big\|\mathbb{E}\big[\psi(e^{(t-s)A} \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V$ for $(s, t) \in \angle$. More formally, let $\tilde{F}_{r,s,t}\colon H \times H \times H \to V$, $r \in [0, s)$, $s \in (0, t)$, $t \in (0, T]$, be the functions with the property that for all $t \in (0, T]$, $s \in (0, t)$, $r \in [0, s)$, $u, v, w \in H$ it holds that

$$\begin{aligned} \tilde{F}_{r,s,t}(u, v, w) &= \psi_{1,0}\big(e^{(t-r)A} u, S_{r,s} v\big)\, e^{(t-r)A} F(w) - \psi_{1,0}\big(e^{(s-r)A} u, S_{r,s} v\big)\, e^{(s-r)A} F(w) \\ &\quad + \big[\psi_{0,1}\big(e^{(t-r)A} u, S_{r,s} v\big) - \psi_{0,1}\big(e^{(s-r)A} u, S_{r,s} v\big)\big]\, S_{r,s}\, R_r\, F(w) \end{aligned} \tag{88}$$

and let $\tilde{B}_{r,s,t}\colon H \times H \times H \to V$, $r \in [0, s)$, $s \in (0, t)$, $t \in (0, T]$, be the functions with the property that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\tilde{B}_{r,s,t}(u,v,w) = \tfrac{1}{2} \sum_{b \in \mathbb{U}} \psi_{2,0}\big(e^{(t-r)A}\, u, S_{r,s}\, v\big)\big(e^{(t-r)A}\, B^b(w), e^{(t-r)A}\, B^b(w)\big) \\
&- \tfrac{1}{2} \sum_{b \in \mathbb{U}} \psi_{2,0}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\big(e^{(s-r)A}\, B^b(w), e^{(s-r)A}\, B^b(w)\big) \\
&+ \tfrac{1}{2} \sum_{b \in \mathbb{U}} \big[\psi_{0,2}\big(e^{(t-r)A}\, u, S_{r,s}\, v\big) - \psi_{0,2}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\big]\big(S_{r,s}\, R_r\, B^b(w), S_{r,s}\, R_r\, B^b(w)\big) \\
&+ \sum_{b \in \mathbb{U}} \psi_{1,1}\big(e^{(t-r)A}\, u, S_{r,s}\, v\big)\big(e^{(t-r)A}\, B^b(w), S_{r,s}\, R_r\, B^b(w)\big) \\
&- \sum_{b \in \mathbb{U}} \psi_{1,1}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\big(e^{(s-r)A}\, B^b(w), S_{r,s}\, R_r\, B^b(w)\big).
\end{aligned} \tag{89}$$

An application of Proposition 5.8 then shows[7] that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\, \bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V \leq \mathbb{E}\big[\big\|\psi\big(e^{tA}\, Y_0, S_{0,s}\, Y_0\big) - \psi\big(e^{sA}\, Y_0, S_{0,s}\, Y_0\big)\big\|_V\big] \\
&+ \int_0^s \mathbb{E}\big[\|\tilde{F}_{r,s,t}\big(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h}\big)\|_V\big]\, dr + \int_0^s \mathbb{E}\big[\|\tilde{B}_{r,s,t}\big(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h}\big)\|_V\big]\, dr.
\end{aligned} \tag{90}$$

In the next step we estimate the summands on the right hand side of (90). We observe that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\big\|\psi\big(e^{tA}\, Y_0, S_{0,s}\, Y_0\big) - \psi\big(e^{sA}\, Y_0, S_{0,s}\, Y_0\big)\big\|_V \\
&\leq \eta \max\big\{1, \|e^{tA}\, Y_0\|_H^q, \|S_{0,s}\, Y_0\|_H^q\big\}\, \|e^{tA}\, Y_0 - e^{sA}\, Y_0\|_H \\
&\leq \eta\, |C_0|^q\, g_q(Y_0)\, \|e^{tA} - e^{sA}\|_{L(H)}\, \|Y_0\|_H \leq \eta\, |C_0|^q\, g_{q+1}(Y_0)\, \tfrac{|C_\rho|^2\,(t-s)^\rho}{s^\rho}.
\end{aligned} \tag{91}$$

This and the fact that $\mathbb{E}\big[g_{q+1}(Y_0)\big] \leq K_{q+1}$ imply that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\mathbb{E}\big[\|\psi\big(e^{tA}\, Y_0, S_{0,s}\, Y_0\big) - \psi\big(e^{sA}\, Y_0, S_{0,s}\, Y_0\big)\|_V\big] \leq \eta\, |C_0|^q\, K_{q+1}\, \tfrac{|C_\rho|^2}{s^\rho}\,(t-s)^\rho\,. \tag{92}$$

Inequality (92) provides us an appropriate estimate for the first summand on the right hand side of (90). In the next step we establish a suitable bound for the second summand on the right hand side of (90). Note that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\big\|\psi_{1,0}\big(e^{(t-r)A}\, u, S_{r,s}\, v\big)\, e^{(t-r)A}\, F(w) - \psi_{1,0}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\, e^{(s-r)A}\, F(w)\big\|_V \\
&\leq \big\|\big[\psi_{1,0}\big(e^{(t-r)A}\, u, S_{r,s}\, v\big) - \psi_{1,0}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\big]\, e^{(t-r)A}\, F(w)\big\|_V \\
&+ \big\|\psi_{1,0}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\, e^{(s-r)A}\, \big(e^{(t-s)A} - \mathrm{Id}_H\big)\, F(w)\big\|_V \\
&\leq \eta\, \max\big\{1, \|e^{(t-r)A}u\|_H^q, \|e^{(s-r)A}u\|_H^q, \|S_{r,s}v\|_H^q\big\}\, \big\|\big[e^{(t-r)A} - e^{(s-r)A}\big]\, u\big\|_H\, \|e^{(t-r)A}F(w)\|_H \\
&+ \eta\, \max\big\{1, \|e^{(s-r)A}u\|_H^q, \|S_{r,s}v\|_H^q\big\}\, \big\|e^{(s-r)A}\, \big(e^{(t-s)A} - \mathrm{Id}_H\big)\, F(w)\big\|_H \\
&\leq \frac{\eta\, |C_0|^q \max\big\{1, \|u\|_H^q, \|v\|_H^q\big\}\, |C_\rho|^2\, (t-s)^\rho\, \|u\|_H\, C_\vartheta\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, g_1(w)}{(s-r)^\rho\, (t-r)^\vartheta} \\
&+ \frac{\eta\, |C_0|^q \max\big\{1, \|u\|_H^q, \|v\|_H^q\big\}\, C_{\rho+\vartheta}\, C_\rho\, (t-s)^\rho\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, g_1(w)}{(s-r)^{(\rho+\vartheta)}},
\end{aligned} \tag{93}$$

$$\begin{aligned}
&\big\|\big[\psi_{0,1}\big(e^{(t-r)A}\, u, S_{r,s}\, v\big) - \psi_{0,1}\big(e^{(s-r)A}\, u, S_{r,s}\, v\big)\big]\, S_{r,s}\, R_r\, F(w)\big\|_V \\
&\leq \eta\, |C_0|^q \max\big\{1, \|u\|_H^q, \|v\|_H^q\big\}\, \|e^{(t-r)A}\, u - e^{(s-r)A}\, u\|_H\, \|S_{r,s}R_r\|_{L(H_{-\vartheta},H)}\, \|F(w)\|_{H_{-\vartheta}} \\
&\leq \frac{\eta\, |C_0|^q \max\big\{1, \|u\|_H^q, \|v\|_H^q\big\}\, |C_\rho|^2\, (t-s)^\rho\, \|u\|_H\, C_\vartheta\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, g_1(w)}{(s-r)^{(\rho+\vartheta)}} \\
&\leq \frac{\eta\, |C_0|^q\, |C_\rho|^2\, C_\vartheta \max\big\{1, \|u\|_H^{q+1}, \|v\|_H^{q+1}\big\}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, g_1(w)\, (t-s)^\rho}{(s-r)^{(\rho+\vartheta)}}.
\end{aligned} \tag{94}$$

[7] with $t_0 = 0$, $T = s$, $\check{H} = H \times H$, $p = q+1$, and $\varphi(x,y) = \psi(e^{(t-s)A}x, y) - \psi(x,y)$ for $(x,y) \in \check{H}$ in the notation of Proposition 5.8

Inequalities (93) and (94) prove that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\|\tilde{F}_{r,s,t}(u,v,w)\|_V \\
&\leq \eta\,|C_0|^q \left[\frac{|C_\rho|^2\,C_\vartheta}{(s-r)^\rho\,(t-r)^\vartheta} + \frac{C_{\rho+\vartheta}\,C_\rho}{(s-r)^{(\rho+\vartheta)}} + \frac{|C_\rho|^2\,C_\vartheta}{(s-r)^{(\rho+\vartheta)}}\right] \\
&\quad \cdot \max\bigl\{1, \|u\|_H^{q+1}, \|v\|_H^{q+1}\bigr\}\,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\,g_1(w)\,(t-s)^\rho \\
&\leq \eta\,|C_0|^q \left[\frac{C_\rho\,(2\,C_\rho C_\vartheta + C_{\rho+\vartheta})}{(s-r)^{(\rho+\vartheta)}}\right] \max\bigl\{1, \|u\|_H^{q+1}, \|v\|_H^{q+1}\bigr\}\,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\,g_1(w)\,(t-s)^\rho.
\end{aligned} \tag{95}$$

This and (86) prove that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\int_0^s \mathbb{E}\bigl[\|\tilde{F}_{r,s,t}(\bar{Y}_r, Y_r, Y_{\lfloor r\rfloor_h})\|_V\bigr]\,dr \\
&\leq \frac{\eta\,|C_0|^q\,C_\rho\,\bigl(2\,C_\rho\,C_\vartheta + C_{\rho+\vartheta}\bigr)}{(1-\vartheta-\rho)}\,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\,K_{q+2}\,(t-s)^\rho\,s^{(1-\vartheta-\rho)}.
\end{aligned} \tag{96}$$

Next we provide an appropriate bound for the third summand on the right hand side of (90). Observe that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\Bigl\|\sum_{b\in\mathbb{U}} \psi_{2,0}\bigl(e^{(t-r)A}\,u, S_{r,s}\,v\bigr)\bigl(e^{(t-r)A}\,B^b(w), e^{(t-r)A}\,B^b(w)\bigr) \\
&- \sum_{b\in\mathbb{U}} \psi_{2,0}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigl(e^{(s-r)A}\,B^b(w), e^{(s-r)A}\,B^b(w)\bigr)\Bigr\|_V \\
&\leq \sum_{b\in\mathbb{U}} \bigl\|\bigl[\psi_{2,0}\bigl(e^{(t-r)A}\,u, S_{r,s}\,v\bigr) - \psi_{2,0}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigr]\bigl(e^{(t-r)A}\,B^b(w), e^{(t-r)A}\,B^b(w)\bigr)\bigr\|_V \\
&+ \sum_{b\in\mathbb{U}} \bigl\|\psi_{2,0}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigl((e^{(t-r)A} + e^{(s-r)A})\,B^b(w), e^{(s-r)A}\,(e^{(t-s)A} - \mathrm{Id}_H)\,B^b(w)\bigr)\bigr\|_V \\
&\leq \frac{\eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\,|C_\rho|^2\,(t-s)^\rho\,\|u\|_H\,|C_{\vartheta/2}|^2\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)}{(s-r)^\rho\,(t-r)^\vartheta} \\
&+ \frac{\eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\,2\,C_{\vartheta/2}\,C_{\rho+\vartheta/2}\,C_\rho\,(t-s)^\rho\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)}{(s-r)^{(\rho+\vartheta)}},
\end{aligned} \tag{97}$$

$$\begin{aligned}
&\Bigl\|\sum_{b\in\mathbb{U}} \bigl[\psi_{0,2}\bigl(e^{(t-r)A}\,u, S_{r,s}\,v\bigr) - \psi_{0,2}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigr]\bigl(S_{r,s}\,R_r\,B^b(w), S_{r,s}\,R_r\,B^b(w)\bigr)\Bigr\|_V \\
&\leq \frac{\eta\,|C_0|^q \max\bigl\{1, \|u\|_H^q, \|v\|_H^q\bigr\}\,|C_\rho|^2\,(t-s)^\rho\,\|u\|_H\,|C_{\vartheta/2}|^2\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)}{(s-r)^{(\rho+\vartheta)}},
\end{aligned} \tag{98}$$

$$\begin{aligned}
&\Bigl\|\sum_{b\in\mathbb{U}} \psi_{1,1}\bigl(e^{(t-r)A}\,u, S_{r,s}\,v\bigr)\bigl(e^{(t-r)A}\,B^b(w), S_{r,s}\,R_r\,B^b(w)\bigr) \\
&- \sum_{b\in\mathbb{U}} \psi_{1,1}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigl(e^{(s-r)A}\,B^b(w), S_{r,s}\,R_r\,B^b(w)\bigr)\Bigr\|_V \\
&\leq \sum_{b\in\mathbb{U}} \bigl|\bigl[\psi_{1,1}\bigl(e^{(t-r)A}\,u, S_{r,s}\,v\bigr) - \psi_{1,1}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigr]\bigl(e^{(t-r)A}\,B^b(w), S_{r,s}\,R_r\,B^b(w)\bigr)\bigr| \\
&\quad + \sum_{b\in\mathbb{U}} \bigl|\psi_{1,1}\bigl(e^{(s-r)A}\,u, S_{r,s}\,v\bigr)\bigl(e^{(s-r)A}\,(e^{(t-s)A} - \mathrm{Id}_H)\,B^b(w), S_{r,s}\,R_r\,B^b(w)\bigr)\bigr| \\
&\leq \frac{\eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\,|C_\rho|^2\,(t-s)^\rho\,\|u\|_H\,|C_{\vartheta/2}|^2\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)}{(s-r)^{(\rho+\vartheta/2)}\,(t-r)^{\vartheta/2}} \\
&\quad + \frac{\eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\,C_{\rho+\vartheta/2}\,C_\rho\,(t-s)^\rho\,C_{\vartheta/2}\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)}{(s-r)^{(\rho+\vartheta)}}.
\end{aligned} \tag{99}$$

Inequalities (97)–(99) imply that for all $t \in (0,T]$, $s \in (0,t)$, $r \in [0,s)$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\|\tilde{B}_{r,s,t}(u,v,w)\|_V \leq \eta\,|C_0|^q \max\{1, \|u\|_H^{q+1}, \|v\|_H^{q+1}\}\,(t-s)^\rho\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)\\
&\cdot\left[\tfrac{\frac{1}{2}|C_\rho|^2\,|C_{\vartheta/2}|^2}{(s-r)^\rho\,(t-r)^\vartheta} + \tfrac{\frac{1}{2}\,2\,C_{\vartheta/2}\,C_{\rho+\vartheta/2}\,C_\rho}{(s-r)^{(\rho+\vartheta)}} + \tfrac{\frac{1}{2}|C_\rho|^2\,|C_{\vartheta/2}|^2}{(s-r)^{(\rho+\vartheta)}} + \tfrac{|C_\rho|^2\,|C_{\vartheta/2}|^2}{(s-r)^{(\rho+\vartheta/2)}\,(t-r)^{\vartheta/2}} + \tfrac{C_{\rho+\vartheta/2}\,C_\rho\,C_{\vartheta/2}}{(s-r)^{(\rho+\vartheta)}}\right]\\
&\leq \eta\,|C_0|^q \max\{1, \|u\|_H^{q+1}, \|v\|_H^{q+1}\}\,(t-s)^\rho\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)\\
&\cdot\frac{2\,C_\rho\,C_{\vartheta/2}\,\big(C_\rho\,C_{\vartheta/2} + C_{\rho+\vartheta/2}\big)}{(s-r)^{(\rho+\vartheta)}}.
\end{aligned}\tag{100}$$

This and (86) prove that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\int_0^s \big\|\mathbb{E}\big[\tilde{B}_{r,s,t}(\bar{Y}_r, Y_r, Y_{\lfloor r \rfloor_h})\big]\big\|_V\,dr\\
&\leq \eta\,|C_0|^q\,K_{q+3}\,(t-s)^\rho\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,\frac{2\,C_\rho\,C_{\vartheta/2}\,\big(C_\rho\,C_{\vartheta/2} + C_{\rho+\vartheta/2}\big)\,s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}.
\end{aligned}\tag{101}$$

Combining (90) with the estimates (92), (96), and (101) yields that for all $t \in (0,T]$, $s \in (0,t)$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\,\bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V \leq \eta\,|C_0|^q\,K_{q+1}\,\tfrac{|C_\rho|^2}{s^\rho}\,(t-s)^\rho\\
&+ \frac{\eta\,|C_0|^q\,C_\rho\,(2\,C_\rho C_\vartheta + C_{\rho+\vartheta})}{(1-\vartheta-\rho)}\,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\,K_{q+2}\,(t-s)^\rho\,s^{(1-\vartheta-\rho)}\\
&+ \frac{\eta\,|C_0|^q\,2\,C_\rho\,C_{\vartheta/2}\,\big(C_\rho\,C_{\vartheta/2} + C_{\rho+\vartheta/2}\big)}{(1-\vartheta-\rho)}\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,K_{q+3}\,(t-s)^\rho\,s^{(1-\vartheta-\rho)}\\
&\leq \eta\,|C_0|^q\,\varsigma_{F,B}\,K_{q+3}\,(t-s)^\rho\left[\tfrac{|C_\rho|^2}{s^\rho} + \tfrac{C_\rho\big(2\,C_\rho C_\vartheta + C_{\rho+\vartheta} + 2\,C_\rho\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\big)\,s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right]\\
&\leq \eta\,|C_\rho|^2\,|C_0|^q\,\varsigma_{F,B}\,K_{q+3}\,(t-s)^\rho\left[\tfrac{1}{s^\rho} + \tfrac{\big(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\big)\,s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right].
\end{aligned}\tag{102}$$

In addition, we note that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\,\bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V\\
&\leq \eta\,\mathbb{E}\big[\max\{1, \|e^{(t-s)A}\bar{Y}_s\|_H^q, \|\bar{Y}_s\|_H^q, \|Y_s\|_H^q\}\,\|e^{(t-s)A} - \mathrm{Id}_H\|_{L(H)}\,\|\bar{Y}_s\|_H\big]\\
&\leq \eta\,|C_0|^{(q+1)}\,\mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^q, \|Y_s\|_H^q\}\,\|\bar{Y}_s\|_H\big]\\
&\leq \eta\,|C_0|^{(q+1)}\,\mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^{q+1}, \|Y_s\|_H^{q+1}\}\big] \leq \eta\,|C_0|^{(q+1)} K_{q+1}.
\end{aligned}\tag{103}$$

Combining this with (102) proves that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\psi(e^{(t-s)A}\,\bar{Y}_s, Y_s) - \psi(\bar{Y}_s, Y_s)\big]\big\|_V\\
&\leq \eta\,|C_\rho|^2\,|C_0|^{(q+1)}\,\varsigma_{F,B}\,K_{q+3}\left[\min\Big\{1, \tfrac{(t-s)^\rho}{s^\rho}\Big\} + \tfrac{(t-s)^\rho\big(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\big)\,s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right]\\
&= \eta\,|C_\rho|^2\,|C_0|^{(q+1)}\,\varsigma_{F,B}\,K_{q+3}\\
&\cdot\left[\mathbb{1}_{[\frac{t}{2},T]}(s)\cdot\tfrac{(t-s)^\rho}{s^\rho} + \mathbb{1}_{[0,\frac{t}{2})}(s)\cdot\tfrac{(t-s)^\rho}{(t-s)^\rho} + \tfrac{(t-s)^\rho\big(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\big)\,s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right]\\
&\leq \eta\,|C_\rho|^2\,|C_0|^{(q+1)}\,\varsigma_{F,B}\,K_{q+3}\left[\tfrac{(t-s)^\rho}{(t/2)^\rho} + \tfrac{(t-s)^\rho\big(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\big)\,s^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right].
\end{aligned}\tag{104}$$

Combining this, (87), and (80) establishes that for all $(s,t) \in \angle$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\psi\big(\bar{Y}_t, Y_s\big) - \psi\big(\bar{Y}_s, Y_s\big)\big]\big\|_V \leq \eta\,|C_0|^{(q+1)}\,|C_\rho|^2\,\varsigma_{F,B}\,K_{q+3}\,(t-s)^\rho\\
&\cdot\left[\big|\tfrac{2}{t}\big|^\rho + \tfrac{\big(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\big)\,s^{(1-\vartheta-\rho)} + \big(C_\vartheta + \frac{1}{2}|C_{\vartheta/2}|^2\big)\,|t-s|^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\right].
\end{aligned}\tag{105}$$

The proof of Proposition 6.2 is thus completed. □

6.3 Analysis of the weak distance between Euler-type approximations and their semilinear integrated counterparts

Lemma 6.3 (Analysis of the analytically weak distance between Euler-type approximations and their semilinear integrated counterparts). *Assume the setting in Section 6.1 and let $\rho \in [0,1)$, $\varrho \in \big[0, 1 - [\frac{1+\vartheta}{2} - \rho]^+\big)$, $t \in (0,T]$. Then*

$$\|Y_t - \bar{Y}_t\|_{L^p(\mathbb{P};H_{-\rho})} \le |K_p|^{\frac{1}{p}}\, h^{\varrho} \cdot \left[\tfrac{C_{-\rho,\varrho}}{t^{(\varrho-\rho)^+}} + \tfrac{C_{\vartheta,-\rho,\varrho}\, t^{1-(\vartheta+\varrho-\rho)^+} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-(\vartheta+\varrho-\rho)^+)} + \tfrac{C_{\vartheta/2,-\rho,\varrho} \sqrt{p\,(p-1)\, t^{1-(\vartheta+2\varrho-2\rho)^+}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2(\vartheta+2\varrho-2\rho)^+}} \right]. \tag{106}$$

Proof. First of all, we observe that

$$\begin{aligned} \big\|Y_t - \bar{Y}_t\big\|_{L^p(\mathbb{P};H_{-\rho})} &\le \left\| \int_0^t \big(S_{s,t}\, R_s - e^{(t-s)A}\big)\, F(Y_{\lfloor s \rfloor_h})\, ds \right\|_{L^p(\mathbb{P};H_{-\rho})} \\ &+ \left\| \int_0^t \big(S_{s,t}\, R_s - e^{(t-s)A}\big)\, B(Y_{\lfloor s \rfloor_h})\, dW_s \right\|_{L^p(\mathbb{P};H_{-\rho})} + \big\|\big(S_{0,t} - e^{tA}\big)\, Y_0\big\|_{L^p(\mathbb{P};H_{-\rho})}. \end{aligned} \tag{107}$$

Next we note that

$$\begin{aligned} \big\|(S_{0,t} - e^{tA})Y_0\big\|_{L^p(\mathbb{P};H_{-\rho})} &\le \|S_{0,t} - e^{tA}\|_{L(H,H_{-\rho})}\, \|Y_0\|_{L^p(\mathbb{P};H)} \le \frac{C_{-\rho,\varrho}}{t^{(\varrho-\rho)^+}}\, \|Y_0\|_{L^p(\mathbb{P};H)}\, h^{\varrho} \\ &\le \frac{C_{-\rho,\varrho}}{t^{(\varrho-\rho)^+}}\, |K_p|^{\frac{1}{p}}\, h^{\varrho} \end{aligned} \tag{108}$$

and

$$\begin{aligned} &\left\| \int_0^t \big(S_{s,t}\, R_s - e^{(t-s)A}\big)\, F(Y_{\lfloor s \rfloor_h})\, ds \right\|_{L^p(\mathbb{P};H_{-\rho})} \le \int_0^t \big\|(S_{s,t}\, R_s - e^{(t-s)A}) F(Y_{\lfloor s \rfloor_h})\big\|_{L^p(\mathbb{P};H_{-\rho})}\, ds \\ &\le \int_0^t \frac{C_{\vartheta,-\rho,\varrho}\, h^{\varrho}\, \|F(Y_{\lfloor s \rfloor_h})\|_{L^p(\mathbb{P};H_{-\vartheta})}}{(t-s)^{(\vartheta+\varrho-\rho)^+}}\, ds \le \frac{C_{\vartheta,-\rho,\varrho}\, t^{1-(\vartheta+\varrho-\rho)^+}}{(1-(\vartheta+\varrho-\rho)^+)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}\, |K_p|^{\frac{1}{p}}\, h^{\varrho}. \end{aligned} \tag{109}$$

Moreover, observe that the Burkholder-Davis-Gundy type inequality in Lemma 7.7 in Da Prato & Zabczyk [13] proves that

$$\begin{aligned} &\left\| \int_0^t (S_{s,t}\, R_s - e^{(t-s)A}) B\big(Y_{\lfloor s \rfloor_h}\big)\, dW_s \right\|_{L^p(\mathbb{P};H_{-\rho})} \\ &\le \left[\frac{p\,(p-1)}{2} \int_0^t \big\|\big(S_{s,t}\, R_s - e^{(t-s)A}\big)\, B(Y_{\lfloor s \rfloor_h})\big\|^2_{L^p(\mathbb{P};HS(U,H_{-\rho}))}\, ds \right]^{1/2} \\ &\le \left[\frac{p\,(p-1)}{2} \int_0^t \frac{|C_{\vartheta/2,-\rho,\varrho}|^2\, h^{2\varrho}\, \|B(Y_{\lfloor s \rfloor_h})\|^2_{L^p(\mathbb{P};HS(U,H_{-\vartheta/2}))}}{(t-s)^{(\vartheta+2\varrho-2\rho)^+}}\, ds \right]^{1/2} \\ &\le \frac{C_{\vartheta/2,-\rho,\varrho} \sqrt{p\,(p-1)\, t^{1-(\vartheta+2\varrho-2\rho)^+}}}{\sqrt{2-2(\vartheta+2\varrho-2\rho)^+}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\, |K_p|^{\frac{1}{p}}\, h^{\varrho}. \end{aligned} \tag{110}$$

Combining (107)–(110) completes the proof of Lemma 6.3. □

Proposition 6.4 (Weak distance between Euler-type approximations and their semilinear integrated counterparts). *Assume the setting in Section 6.1 and let $\eta \in [0,\infty)$, $q \in [0,\infty) \cap (-\infty, p-3]$, $\rho \in [0, 1-\vartheta)$, $\psi = (\psi(x,y))_{x,y \in H} \in C^2(H \times H, V)$ satisfy that for all $x, y_1, y_2 \in H$, $i, j \in \mathbb{N}_0$ with $i + j \le 2$ it holds that*

$$\big\|\big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_1) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_2)\big\|_{L^{(i+j)}(H,V)} \le \eta \max\{1, \|x\|_H^q, \|y_1\|_H^q, \|y_2\|_H^q\}\, \|y_1 - y_2\|_H.$$

Then for all $t \in (0,T]$ it holds that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_t) - \psi(\bar{Y}_t, \bar{Y}_t)\|_V\big] < \infty$ and

$$
\begin{aligned}
&\big\|\mathbb{E}\big[\psi\big(\bar{Y}_t, Y_t\big) - \psi\big(\bar{Y}_t, \bar{Y}_t\big)\big]\big\|_V \leq \eta\, |C_0|^q\, \varsigma_{F,B}\, K_{q+3}\, h^\rho \\
&\cdot \Bigg[\tfrac{C_{0,\rho}}{t^\rho} + \tfrac{t^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\Bigg(C_{\vartheta,0,\rho} + 2\, C_{\vartheta/2}\, C_{\vartheta/2,0,\rho} + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_{0,\rho} + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_\rho \\
&\cdot \Bigg[C_{-\rho,\rho} + \frac{C_{\vartheta,-\rho,\rho}\, t^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-\vartheta)} + \frac{C_{\vartheta/2,-\rho,\rho} \sqrt{(q+3)(q+2)\, t^{(1-\vartheta)}}\|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta}}\Bigg]\Bigg)\Bigg].
\end{aligned}
\tag{111}
$$

Proof. Throughout this proof let $(g_r)_{r\in[0,\infty)} \subseteq C(H,\mathbb{R})$ be the functions with the property that for all $r \in [0,\infty)$, $x \in H$ it holds that $g_r(x) = \max\{1, \|x\|_H^r\}$ and let $\psi_{1,0}\colon H \times H \to L(H,V)$, $\psi_{0,1}\colon H \times H \to L(H,V)$, $\psi_{2,0}\colon H \times H \to L^{(2)}(H,V)$, $\psi_{0,2}\colon H \times H \to L^{(2)}(H,V)$, $\psi_{1,1}\colon H \times H \to L^{(2)}(H,V)$ be the functions with the property that for all $x, y, v_1, v_2 \in H$ it holds that $\psi_{1,0}(x,y)\, v_1 = \big(\frac{\partial}{\partial x}\psi(x,y)\big)(v_1)$ and

$$
\psi_{0,1}(x,y)\, v_1 = \big(\tfrac{\partial}{\partial y}\psi(x,y)\big)(v_1), \qquad \psi_{2,0}(x,y)(v_1,v_2) = \big(\tfrac{\partial^2}{\partial x^2}\psi(x,y)\big)(v_1,v_2), \tag{112}
$$

$$
\psi_{0,2}(x,y)(v_1,v_2) = \big(\tfrac{\partial^2}{\partial y^2}\psi(x,y)\big)(v_1,v_2), \qquad \psi_{1,1}(x,y)(v_1,v_2) = \big(\tfrac{\partial}{\partial y}\tfrac{\partial}{\partial x}\psi(x,y)\big)(v_1,v_2). \tag{113}
$$

Next we observe that Lemma 6.1 and the assumption that $q \leq p-3$ ensure that $K_{q+1} \leq K_{q+3} < \infty$. Combining this with the fact that

$$
\forall\, x, y_1, y_2 \in H\colon \|\psi(x,y_1) - \psi(x,y_2)\|_V \leq 2\,\eta \max\big\{1, \|x\|_H^{q+1}, \|y_1\|_H^{q+1}, \|y_2\|_H^{q+1}\big\} \tag{114}
$$

shows that for all $t \in (0,T]$ it holds that $\mathbb{E}\big[\|\psi(\bar{Y}_t, Y_t) - \psi(\bar{Y}_t, \bar{Y}_t)\|_V\big] < \infty$. It thus remains to prove (111). To do so, we make use of a consequence of the mild Itô formula in Corollary 5.5, that is, we will apply Proposition 5.8 above. For this let $\tilde{F}_{s,t}\colon H \times H \times H \to V$, $(s,t) \in \angle$, be the functions with the property that for all $(s,t) \in \angle$, $u, v, w \in H$ it holds that

$$
\begin{aligned}
\tilde{F}_{s,t}(u,v,w) &= \big[\psi_{1,0}\big(e^{(t-s)A}\, u, S_{s,t}\, v\big) - \psi_{1,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big]\, e^{(t-s)A}\, F(w) \\
&\quad + \psi_{0,1}\big(e^{(t-s)A}\, u, S_{s,t}\, v\big)\, S_{s,t}\, R_s\, F(w) - \psi_{0,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\, e^{(t-s)A}\, F(w)
\end{aligned}
\tag{115}
$$

and let $\tilde{B}_{s,t}\colon H \times H \times H \to V$, $(s,t) \in \angle$, be the functions with the property that for all $(s,t) \in \angle$, $u, v, w \in H$ it holds that

$$
\begin{aligned}
&\tilde{B}_{s,t}(u,v,w) \\
&= \tfrac{1}{2} \sum_{b\in\mathbb{U}} \big[\psi_{2,0}\big(e^{(t-s)A}\, u, S_{s,t}\, v\big) - \psi_{2,0}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big]\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \\
&+ \tfrac{1}{2} \sum_{b\in\mathbb{U}} \psi_{0,2}\big(e^{(t-s)A}\, u, S_{s,t}\, v\big)\big(S_{s,t}\, R_s\, B^b(w), S_{s,t}\, R_s\, B^b(w)\big) \\
&- \tfrac{1}{2} \sum_{b\in\mathbb{U}} \psi_{0,2}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big) \\
&+ \sum_{b\in\mathbb{U}} \psi_{1,1}\big(e^{(t-s)A}\, u, S_{s,t}\, v\big)\big(e^{(t-s)A}\, B^b(w), S_{s,t}\, R_s\, B^b(w)\big) \\
&- \sum_{b\in\mathbb{U}} \psi_{1,1}\big(e^{(t-s)A}\, u, e^{(t-s)A}\, u\big)\big(e^{(t-s)A}\, B^b(w), e^{(t-s)A}\, B^b(w)\big).
\end{aligned}
\tag{116}
$$

An application of Proposition 5.8 then shows[8] that for all $t \in (0,T]$ it holds that

$$
\begin{aligned}
&\big\|\mathbb{E}\big[\psi\big(\bar{Y}_t, Y_t\big) - \psi\big(\bar{Y}_t, \bar{Y}_t\big)\big]\big\|_V \leq \mathbb{E}\big[\|\psi\big(e^{tA}\, Y_0, S_{0,t}\, Y_0\big) - \psi\big(e^{tA}\, Y_0, e^{tA}\, Y_0\big)\|_V\big] \\
&+ \int_0^t \mathbb{E}\Big[\big\|\tilde{F}_{s,t}\big(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h}\big)\big\|_V\Big] + \mathbb{E}\Big[\big\|\tilde{B}_{s,t}\big(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h}\big)\big\|_V\Big]\, ds.
\end{aligned}
\tag{117}
$$

[8]with $t_0 = 0$, $T = t$, $\check{H} = H \times H$, $p = q+1$, and $\varphi(x,y) = \psi(x,y) - \psi(x,x)$ for $(x,y) \in \check{H}$ in the notation of Proposition 5.8

In the following we establish suitable estimates for the two summands on the right hand side of (117). We begin with the first summand on the right hand side of (117). Observe that for all $t \in (0,T]$ it holds that

$$\begin{aligned} &\big\|\psi\big(e^{tA} Y_0, S_{0,t} Y_0\big) - \psi\big(e^{tA} Y_0, e^{tA} Y_0\big)\big\|_V \\ &\leq \eta \max\{1, \|S_{0,t} Y_0\|_H^q, \|e^{tA} Y_0\|_H^q\} \|(S_{0,t} - e^{tA}) Y_0\|_H \\ &\leq \eta \, |C_0|^q \, g_q(Y_0) \, \|S_{0,t} - e^{tA}\|_{L(H)} \, \|Y_0\|_H \leq \frac{\eta \, |C_0|^q \, C_{0,\rho} \, g_{q+1}(Y_0) \, h^\rho}{t^\rho}. \end{aligned} \tag{118}$$

This and the fact that $\mathbb{E}\big[g_{q+1}(Y_0)\big] \leq K_{q+1}$ imply that for all $t \in (0,T]$ it holds that

$$\mathbb{E}\big[\|\psi\big(e^{tA} Y_0, S_{0,t} Y_0\big) - \psi\big(e^{tA} Y_0, e^{tA} Y_0\big)\|_V\big] \leq \frac{\eta \, |C_0|^q \, C_{0,\rho} \, K_{q+1} \, h^\rho}{t^\rho}. \tag{119}$$

Now we will estimate the second summand on the right hand side of (117). Observe that for all $(s,t) \in \angle$, $u, v, w \in H$ it holds that

$$\begin{aligned} &\big\|\big[\psi_{1,0}\big(e^{(t-s)A} u, S_{s,t} v\big) - \psi_{1,0}\big(e^{(t-s)A} u, e^{(t-s)A} u\big)\big] \, e^{(t-s)A} \, F(w)\big\|_V \\ &\leq \eta \max\big\{1, \|S_{s,t} v\|_H^q, \|e^{(t-s)A} u\|_H^q\big\} \|S_{s,t} v - e^{(t-s)A} u\|_H \, \|e^{(t-s)A} F(w)\|_H \\ &\leq \frac{\eta \, |C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\} \big[\|(S_{s,t} - e^{(t-s)A}) v\|_H + \|e^{(t-s)A}(v-u)\|_H\big] \, C_\vartheta \, \|F(w)\|_{H_{-\vartheta}}}{(t-s)^\vartheta} \\ &\leq \frac{\eta \, |C_0|^q \, C_\vartheta \max\big\{1, \|u\|_H^q, \|v\|_H^q\big\} \, \|F\|_{\mathrm{Lip}^0(H, H_{-\vartheta})} \, g_1(w)}{(t-s)^\vartheta} \\ &\quad \cdot \left[\|S_{s,t} - e^{(t-s)A}\|_{L(H)} \, \|v\|_H + \frac{C_\rho \|v-u\|_{H_{-\rho}}}{(t-s)^\rho}\right]. \end{aligned} \tag{120}$$

Moreover, we note that the assumption that $\forall x, y_1, y_2 \in H$:

$$\|\psi(x, y_1) - \psi(x, y_2)\|_V \leq \eta \max\big\{1, \|x\|_H^q, \|y_1\|_H^q, \|y_2\|_H^q\big\} \, \|y_1 - y_2\|_H \tag{121}$$

implies that for all $x, y \in H$ it holds that $\|\psi_{0,1}(x,y)\|_{L(H,V)} \leq \eta \max\{1, \|x\|_H^q, \|y\|_H^q\}$. This, in turn, proves that for all $(s,t) \in \angle$, $u, v, w \in H$ it holds that

$$\begin{aligned} &\big\|\psi_{0,1}\big(e^{(t-s)A} u, S_{s,t} v\big) \, S_{s,t} \, R_s \, F(w) - \psi_{0,1}\big(e^{(t-s)A} u, e^{(t-s)A} u\big) \, e^{(t-s)A} \, F(w)\big\|_V \\ &\leq \big\|\psi_{0,1}\big(e^{(t-s)A} u, S_{s,t} v\big)\big[S_{s,t} \, R_s - e^{(t-s)A}\big] F(w)\big\|_V \\ &+ \big\|\big[\psi_{0,1}\big(e^{(t-s)A} u, S_{s,t} v\big) - \psi_{0,1}\big(e^{(t-s)A} u, e^{(t-s)A} u\big)\big] \, e^{(t-s)A} \, F(w)\big\|_V \\ &\leq \eta \, \max\big\{1, \|S_{s,t} v\|_H^q, \|e^{(t-s)A} u\|_H^q\big\} \, \big\|\big[S_{s,t} \, R_s - e^{(t-s)A}\big] F(w)\big\|_H \\ &+ \eta \, \max\big\{1, \|S_{s,t} v\|_H^q, \|e^{(t-s)A} u\|_H^q\big\} \, \|S_{s,t} v - e^{(t-s)A} u\|_H \, \|e^{(t-s)A} F(w)\|_H \\ &\leq \eta \, |C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\} \, \|F\|_{\mathrm{Lip}^0(H, H_{-\vartheta})} \, g_1(w) \\ &\cdot \left[\|S_{s,t} \, R_s - e^{(t-s)A}\|_{L(H_{-\vartheta}, H)} + \frac{C_\vartheta}{(t-s)^\vartheta}\left[\|S_{s,t} - e^{(t-s)A}\|_{L(H)} \, \|v\|_H + \frac{C_\rho \, \|v-u\|_{H_{-\rho}}}{(t-s)^\rho}\right]\right]. \end{aligned} \tag{122}$$

Inequalities (120) and (122) imply that for all $(s,t) \in \angle$, $u, v, w \in H$ it holds that

$$\begin{aligned} \|\tilde{F}_{s,t}(u,v,w)\|_V &\leq \eta \, |C_0|^q \max\big\{1, \|u\|_H^q, \|v\|_H^q\big\} \, \|F\|_{\mathrm{Lip}^0(H, H_{-\vartheta})} \, g_1(w) \\ &\cdot \left[\left[\frac{C_{\vartheta,0,\rho} + 2 \, C_\vartheta \, C_{0,\rho} \, \|v\|_H}{(t-s)^{(\rho+\vartheta)}}\right] h^\rho + \frac{2 \, C_\vartheta \, C_\rho \, \|v-u\|_{H_{-\rho}}}{(t-s)^{(\rho+\vartheta)}}\right]. \end{aligned} \tag{123}$$

Next we observe that for all $(s,t) \in \angle$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\sum_{b\in\mathbb{U}} \big\| \big[\psi_{2,0}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big) - \psi_{2,0}\big(e^{(t-s)A}\,u, e^{(t-s)A}\,u\big)\big]\big(e^{(t-s)A}\,B^b(w), e^{(t-s)A}\,B^b(w)\big)\big\|_V \\
&\leq \frac{\eta\,|C_0|^q\,|C_{\vartheta/2}|^2 \max\{1, \|u\|_H^q, \|v\|_H^q\}\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w)}{(t-s)^\vartheta} \\
&\quad\cdot \left[\|S_{s,t} - e^{(t-s)A}\|_{L(H)}\,\|v\|_H + \frac{C_\rho\,\|v-u\|_{H_{-\rho}}}{(t-s)^\rho}\right],
\end{aligned} \tag{124}$$

$$\begin{aligned}
&\sum_{b\in\mathbb{U}} \big\|\psi_{0,2}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big)\big(S_{s,t}\,R_s\,B^b(w), S_{s,t}\,R_s\,B^b(w)\big) \\
&- \psi_{0,2}\big(e^{(t-s)A}\,u, e^{(t-s)A}\,u\big)\big(e^{(t-s)A}\,B^b(w), e^{(t-s)A}\,B^b(w)\big)\big\|_V \\
&\leq \sum_{b\in\mathbb{U}} \big\|\psi_{0,2}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big)\big([S_{s,t}\,R_s + e^{(t-s)A}]\,B^b(w), [S_{s,t}\,R_s - e^{(t-s)A}]\,B^b(w)\big)\big\|_V \\
&+ \sum_{b\in\mathbb{U}} \big\|\big[\psi_{0,2}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big) - \psi_{0,2}\big(e^{(t-s)A}\,u, e^{(t-s)A}\,u\big)\big]\big(e^{(t-s)A}\,B^b(w), e^{(t-s)A}\,B^b(w)\big)\big\|_V \\
&\leq \eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\Big[\|S_{s,t}\,v - e^{(t-s)A}\,u\|_H\,\|e^{(t-s)A}\,B(w)\|^2_{HS(U,H)} \\
&+ \|(S_{s,t}\,R_s + e^{(t-s)A})\,B(w)\|_{HS(U,H)}\,\|(S_{s,t}\,R_s - e^{(t-s)A})\,B(w)\|_{HS(U,H)}\Big] \\
&\leq \eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w) \\
&\quad\cdot \Bigg[\|S_{s,t}\,R_s + e^{(t-s)A}\|_{L(H_{-\vartheta/2},H)}\,\|S_{s,t}\,R_s - e^{(t-s)A}\|_{L(H_{-\vartheta/2},H)} \\
&\quad + \frac{|C_{\vartheta/2}|^2}{(t-s)^\vartheta}\left[\|S_{s,t} - e^{(t-s)A}\|_{L(H)}\,\|v\|_H + \frac{C_\rho\,\|v-u\|_{H_{-\rho}}}{(t-s)^\rho}\right]\Bigg],
\end{aligned} \tag{125}$$

and

$$\begin{aligned}
&\sum_{b\in\mathbb{U}} \big\|\psi_{1,1}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big)\big(e^{(t-s)A}\,B^b(w), S_{s,t}\,R_s\,B^b(w)\big) \\
&- \psi_{1,1}\big(e^{(t-s)A}\,u, e^{(t-s)A}\,u\big)\big(e^{(t-s)A}\,B^b(w), e^{(t-s)A}\,B^b(w)\big)\big\|_V \\
&\leq \sum_{b\in\mathbb{U}} \big\|\psi_{1,1}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big)\big(e^{(t-s)A}\,B^b(w), [S_{s,t}\,R_s - e^{(t-s)A}]\,B^b(w)\big)\big\|_V \\
&+ \sum_{b\in\mathbb{U}} \big\|\big[\psi_{1,1}\big(e^{(t-s)A}\,u, S_{s,t}\,v\big) - \psi_{1,1}\big(e^{(t-s)A}\,u, e^{(t-s)A}\,u\big)\big]\big(e^{(t-s)A}\,B^b(w), e^{(t-s)A}\,B^b(w)\big)\big\|_V \\
&\leq \eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\Big[\|S_{s,t}\,v - e^{(t-s)A}\,u\|_H\,\|e^{(t-s)A}\,B(w)\|^2_{HS(U,H)} \\
&+ \|e^{(t-s)A}\,B(w)\|_{HS(U,H)}\,\|[S_{s,t}\,R_s - e^{(t-s)A}]\,B(w)\|_{HS(U,H)}\Big] \\
&\leq \eta\,|C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w) \\
&\cdot \left[\frac{C_{\vartheta/2}\,\|S_{s,t}\,R_s - e^{(t-s)A}\|_{L(H_{-\vartheta/2},H)}}{(t-s)^{\vartheta/2}} + \frac{|C_{\vartheta/2}|^2}{(t-s)^\vartheta}\left[\|S_{s,t} - e^{(t-s)A}\|_{L(H)}\,\|v\|_H + \frac{C_\rho\,\|v-u\|_{H_{-\rho}}}{(t-s)^\rho}\right]\right].
\end{aligned} \tag{126}$$

Combining (124)–(126) implies that for all $(s,t) \in \angle$, $u,v,w \in H$ it holds that

$$\begin{aligned}
&\|\tilde{B}_{s,t}(u,v,w)\|_V \\
&\leq 2\,\eta\,|C_0|^q\,C_{\vartheta/2} \max\{1, \|u\|_H^q, \|v\|_H^q\}\,\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\,g_2(w) \\
&\cdot \left[\left[\frac{C_{\vartheta/2,0,\rho} + C_{\vartheta/2}\,C_{0,\rho}\,\|v\|_H}{(t-s)^{(\rho+\vartheta)}}\right] h^\rho + \frac{C_{\vartheta/2}\,C_\rho\,\|v-u\|_{H_{-\rho}}}{(t-s)^{(\rho+\vartheta)}}\right].
\end{aligned} \tag{127}$$

Next observe that (123) and (127) show that for all $(s,t) \in \angle$, $u, v, w \in H$ it holds that

$$\|\tilde{F}_{s,t}(u,v,w)\|_V + \|\tilde{B}_{s,t}(u,v,w)\|_V \leq \eta\, |C_0|^q \max\{1, \|u\|_H^q, \|v\|_H^q\}\, \varsigma_{F,B}\, g_2(w) \cdot \left[\left[\frac{C_{\vartheta,0,\rho} + 2\, C_{\vartheta/2}\, C_{\vartheta/2,0,\rho} + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_{0,\rho}}{(t-s)^{(\rho+\vartheta)}} \right] g_1(v)\, h^\rho + \frac{2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_\rho\, \|v-u\|_{H_{-\rho}}}{(t-s)^{(\rho+\vartheta)}} \right]. \tag{128}$$

In addition, note that Hölder's inequality ensures that for all $r \in (0,\infty)$, $s \in [0,T]$ it holds that

$$\begin{split} &\mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^r, \|Y_s\|_H^r\}\, g_2(Y_{\lfloor s \rfloor_h})\big] \\ &\leq \left(\sup_{u,v \in [0,T]} \big\| \max\{1, \|\bar{Y}_u\|_H^r, \|Y_v\|_H^r\} \big\|_{L^{1+2/r}(\mathbb{P};\mathbb{R})} \right) \left(\sup_{u \in [0,T]} \big\| \max\{1, \|Y_u\|_H^2\} \big\|_{L^{1+r/2}(\mathbb{P};\mathbb{R})} \right) \\ &\leq |K_{r+2}|^{\frac{1}{1+2/r}}\, |K_{r+2}|^{\frac{1}{1+r/2}} = K_{r+2}, \end{split} \tag{129}$$

$$\begin{split} \mathbb{E}\big[g_2(Y_{\lfloor s \rfloor_h})\, \|Y_s - \bar{Y}_s\|_{H_{-\rho}}\big] &\leq \|g_2(Y_{\lfloor s \rfloor_h})\|_{L^{3/2}(\mathbb{P};\mathbb{R})}\, \|Y_s - \bar{Y}_s\|_{L^3(\mathbb{P};H_{-\rho})} \\ &\leq |K_3|^{2/3} \left(\sup_{u \in [0,T]} \big\|Y_u - \bar{Y}_u\big\|_{L^3(\mathbb{P};H_{-\rho})} \right), \end{split} \tag{130}$$

and

$$\begin{split} &\mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^r, \|Y_s\|_H^r\}\, g_2(Y_{\lfloor s \rfloor_h})\, \|Y_s - \bar{Y}_s\|_{H_{-\rho}}\big] \\ &\leq \|\max\{1, \|\bar{Y}_s\|_H^r, \|Y_s\|_H^r\}\|_{L^{1+3/r}(\mathbb{P};\mathbb{R})}\, \|g_2(Y_{\lfloor s \rfloor_h})\|_{L^{(r+3)/2}(\mathbb{P};\mathbb{R})}\, \|Y_s - \bar{Y}_s\|_{L^{r+3}(\mathbb{P};H_{-\rho})} \\ &\leq |K_{r+3}|^{\frac{r+2}{r+3}} \left(\sup_{u \in [0,T]} \big\|Y_u - \bar{Y}_u\big\|_{L^{r+3}(\mathbb{P};H_{-\rho})} \right). \end{split} \tag{131}$$

Combining (128)–(131) with Lemma 6.3 and the fact that $1 - (\frac{1+\vartheta}{2} - \rho)^+ > \rho$ yields that for all $t \in (0,T]$ it holds that

$$\begin{split} &\int_0^t \mathbb{E}\Big[\big\|\tilde{F}_{s,t}(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h})\big\|_V\Big] + \mathbb{E}\Big[\big\|\tilde{B}_{s,t}(\bar{Y}_s, Y_s, Y_{\lfloor s \rfloor_h})\big\|_V\Big]\, ds \leq \tfrac{\eta\, |C_0|^q\, \varsigma_{F,B}\, K_{q+3}\, h^\rho\, t^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \\ &\cdot \Bigg[C_{\vartheta,0,\rho} + 2\, C_{\vartheta/2}\, C_{\vartheta/2,0,\rho} + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_{0,\rho} + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_\rho \\ &\cdot \left(C_{-\rho,\rho} + \tfrac{C_{\vartheta,-\rho,\rho}\, t^{(1-\vartheta)}\, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{(1-\vartheta)} + \tfrac{C_{\vartheta/2,-\rho,\rho}\, \sqrt{(q+3)\,(q+2)\, t^{(1-\vartheta)}}\, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}}{\sqrt{2-2\vartheta}} \right) \Bigg]. \end{split} \tag{132}$$

Putting (119) and (132) into (117) proves (111). This finishes the proof of Proposition 6.4. □

7 Weak convergence rates for Euler-type approximations of SPDEs with mollified nonlinearities

In this section we use the results of Section 6 and the Kolmogorov backward equation associated to an SEE to establish weak convergence rates for temporal numerical approximations of SEEs with mollified nonlinearities; see Corollary 7.5 and Corollary 7.6 below. Some of the arguments in this section are similar to some of the arguments in Section 3 in Conus et al. [11].

7.1 Setting

Assume the setting in Section 1.4, let $\vartheta \in [0, \frac{1}{2})$, $F \in C_b^5(H, H_1)$, $B \in C_b^5(H, HS(U, H_1))$, $\varphi \in C_b^5(H, V)$, let $(B^b)_{b\in\mathbb{U}} \subseteq C(H, H)$ be the functions with the property that for all $v \in H$, $b \in \mathbb{U}$ it holds that $B^b(v) = B(v)\, b$, let $\varsigma_{F,B} \in \mathbb{R}$ be a real number given by $\varsigma_{F,B} = \max\big\{1, \|F\|^3_{C_b^3(H,H_{-\vartheta})}, \|B\|^6_{C_b^3(H,HS(U,H_{-\vartheta/2}))}\big\}$, let $X, Y : [0,T] \times \Omega \to H$, $\bar{Y} : [0,T] \times \Omega \to H_1$, and $X^x : [0,T]\times\Omega \to H$, $x \in H$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes such that for all $x \in H$ it holds that $\sup_{t\in[0,T]} \big[\|X_t\|_{L^5(\mathbb{P};H)} + \|X_t^x\|_{L^5(\mathbb{P};H)}\big] < \infty$, $X_0^x = x$, $\bar{Y}_0 \in L^5(\mathbb{P}; H_1)$, and $Y_0 = X_0 = \bar{Y}_0$ and such that for all $x \in H$, $t \in (0, T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}\, X_0 + \int_0^t e^{(t-s)A}\, F(X_s)\, ds + \int_0^t e^{(t-s)A}\, B(X_s)\, dW_s, \tag{133}$$

$$X_t^x = e^{tA}\, x + \int_0^t e^{(t-s)A}\, F(X_s^x)\, ds + \int_0^t e^{(t-s)A}\, B(X_s^x)\, dW_s, \tag{134}$$

$$Y_t = S_{0,t}\, Y_0 + \int_0^t S_{s,t}\, R_s\, F(Y_{\lfloor s\rfloor_h})\, ds + \int_0^t S_{s,t}\, R_s\, B(Y_{\lfloor s\rfloor_h})\, dW_s, \tag{135}$$

$$\bar{Y}_t = e^{tA}\, \bar{Y}_0 + \int_0^t e^{(t-s)A}\, F(Y_{\lfloor s\rfloor_h})\, ds + \int_0^t e^{(t-s)A}\, B(Y_{\lfloor s\rfloor_h})\, dW_s, \tag{136}$$

let $u : [0,T] \times H \to V$ be the function with the property that for all $x \in H$, $t \in [0,T]$ it holds that $u(t,x) = \mathbb{E}\big[\varphi(X^x_{T-t})\big]$, let $c_{\delta_1,\dots,\delta_k} \in [0,\infty]$, $\delta_1, \dots, \delta_k \in \mathbb{R}$, $k \in \{1,2,3,4\}$, be the extended real numbers with the property that for all $k \in \{1,2,3,4\}$, $\delta_1,\dots,\delta_k \in \mathbb{R}$ it holds that

$$c_{\delta_1,\delta_2,\dots,\delta_k} = \sup_{t\in[0,T)} \sup_{x\in H} \sup_{v_1,\dots,v_k \in H\setminus\{0\}} \left[\frac{\big\|\big(\frac{\partial^k}{\partial x^k} u\big)(t,x)(v_1,\dots,v_k)\big\|_V}{(T-t)^{(\delta_1+\ldots+\delta_k)}\, \|v_1\|_{H_{\delta_1}} \cdot \ldots \cdot \|v_k\|_{H_{\delta_k}}} \right], \tag{137}$$

let $\tilde{c}_{\delta_1,\delta_2,\delta_3,\delta_4} \in [0,\infty]$, $\delta_1,\delta_2,\delta_3,\delta_4 \in \mathbb{R}$, be the extended real numbers with the property that for all $\delta_1,\dots,\delta_4 \in \mathbb{R}$ it holds that

$$\begin{aligned} &\tilde{c}_{\delta_1,\delta_2,\delta_3,\delta_4} \\ &= \sup_{t\in[0,T)} \sup_{\substack{x_1,x_2\in H,\\ x_1\neq x_2}} \sup_{v_1,\dots,v_4\in H\setminus\{0\}} \left[\frac{\big\|\big(\big(\frac{\partial^4}{\partial x^4}u\big)(t,x_1) - \big(\frac{\partial^4}{\partial x^4}u\big)(t,x_2)\big)(v_1,\dots,v_4)\big\|_V}{(T-t)^{(\delta_1+\ldots+\delta_4)}\, \|x_1-x_2\|_H\, \|v_1\|_{H_{\delta_1}} \cdot \ldots \cdot \|v_4\|_{H_{\delta_4}}} \right], \end{aligned} \tag{138}$$

let $(K_r)_{r\in[0,\infty)} \subseteq [0,\infty]$ be the extended real numbers which satisfy that for all $r \in [0,\infty)$ it holds that $K_r = \sup_{s,t\in[0,T]} \mathbb{E}\big[\max\{1, \|\bar{Y}_s\|_H^r, \|Y_t\|_H^r\}\big]$, and let $u_{1,0} : [0,T]\times H \to V$ and $u_{0,k} : [0,T]\times H \to L^{(k)}(H,V)$, $k \in \{1,2,3,4\}$, be the functions with the property that for all $t \in [0,T]$, $x \in H$, $k \in \{1,2,3,4\}$, $v_1,\dots,v_k \in H$ it holds that $u_{1,0}(t,x) = (\frac{\partial}{\partial t}u)(t,x)$ and $u_{0,k}(t,x)(v_1,\dots,v_k) = \big((\frac{\partial^k}{\partial x^k}u)(t,x)\big)(v_1,\dots,v_k)$.

7.2 Weak convergence rates for semilinear integrated Euler-type approximations of SPDEs with mollified nonlinearities

Lemma 7.1. *Assume the setting in Section 7.1 and let $t \in [0,T)$, $\psi = (\psi(x,y))_{x,y\in H} \in \mathbb{M}(H \times H, V)$, $\phi \in \mathbb{M}(H,V)$ satisfy that for all $x, y \in H$ it holds that $\psi(x,y) = u_{0,1}(t,x)F(y)$ and $\phi(x) = \psi(x,x)$. Then it holds that $\psi \in C^3(H \times H, V)$, $\phi \in C^3(H,V)$ and for all $x, x_1, x_2, y, y_1, y_2 \in H$ it holds that*

$$\begin{aligned} &\max_{i,j\in\mathbb{N}_0,\, i+j\leq 2} \big\|\big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_1,y) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_2,y)\big\|_{L^{(i+j)}(H,V)} \\ &\leq \tfrac{\|x_1-x_2\|_H}{(T-t)^{\vartheta}}\, \|F\|_{C_b^2(H,H_{-\vartheta})} \big[c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0}\big] \max\{1, \|y\|_H\}, \end{aligned} \tag{139}$$

$$
\begin{aligned}
&\max_{i,j\in\mathbb{N}_0,\, i+j\le 2} \big\| \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_1) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_2) \big\|_{L^{(i+j)}(H,V)} \\
&\le \tfrac{\|y_1-y_2\|_H}{(T-t)^\vartheta} \, \|F\|_{C^3_b(H,H_{-\vartheta})} \big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} \big],
\end{aligned} \tag{140}
$$

$$
\begin{aligned}
&\max_{i\in\{0,1,2\}} \big\| \phi^{(i)}(x_1) - \phi^{(i)}(x_2) \big\|_{L^{(i)}(H,V)} \\
&\le \tfrac{3\,\|x_1-x_2\|_H}{(T-t)^\vartheta} \, \|F\|_{C^3_b(H,H_{-\vartheta})} \big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0} \big] \max\{1, \|x_1\|_H, \|x_2\|_H\}.
\end{aligned} \tag{141}
$$

Proof. First, we note that the assumption that $F \in \operatorname{Lip}^4(H, H_1)$ and the fact that $\big(H \ni x \mapsto u_{0,1}(t,x) \in L(H,V)\big) \in C^3(H, L(H,V))$ ensure that $\psi \in C^3(H \times H, V)$ and $\phi \in C^3(H,V)$. Next we observe that for all $x, y, v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \le 1$ it holds that

$$
\big\| \big(\tfrac{\partial}{\partial x}\psi\big)(x,y)\, v_1 \big\|_V = \big\| u_{0,2}(t,x)\big(F(y), v_1\big) \big\|_V \le \tfrac{c_{-\vartheta,0}}{(T-t)^\vartheta} \, \|F(y)\|_{H_{-\vartheta}}, \tag{142}
$$

$$
\big\| \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,y)\, (v_1,v_2) \big\|_V = \big\| u_{0,3}(t,x)\big(F(y), v_1, v_2\big) \big\|_V \le \tfrac{c_{-\vartheta,0,0}}{(T-t)^\vartheta} \, \|F(y)\|_{H_{-\vartheta}}, \tag{143}
$$

$$
\big\| \big(\tfrac{\partial^3}{\partial x^3}\psi\big)(x,y)\, (v_1,v_2,v_3) \big\|_V = \big\| u_{0,4}(t,x)\big(F(y), v_1, v_2, v_3\big) \big\|_V \le \tfrac{c_{-\vartheta,0,0,0}}{(T-t)^\vartheta} \, \|F(y)\|_{H_{-\vartheta}}, \tag{144}
$$

$$
\big\| \big(\tfrac{\partial}{\partial y}\psi\big)(x,y)\, v_1 \big\|_V = \| u_{0,1}(t,x)\, F'(y)\, v_1 \|_V \le \tfrac{c_{-\vartheta}}{(T-t)^\vartheta} \, \|F'(y)\|_{L(H,H_{-\vartheta})}, \tag{145}
$$

$$
\big\| \big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,y)(v_1,v_2) \big\|_V = \big\| u_{0,1}(t,x)\big(F''(y)(v_1,v_2)\big) \big\|_V \le \tfrac{c_{-\vartheta}}{(T-t)^\vartheta} \, \|F''(y)\|_{L^{(2)}(H,H_{-\vartheta})}, \tag{146}
$$

$$
\begin{aligned}
&\big\| \big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,y)(v_1,v_2,v_3) \big\|_V = \big\| u_{0,1}(t,x)\big(F^{(3)}(y)(v_1,v_2,v_3)\big) \big\|_V \\
&\le \tfrac{c_{-\vartheta}}{(T-t)^\vartheta} \, \|F^{(3)}(y)\|_{L^{(3)}(H,H_{-\vartheta})},
\end{aligned} \tag{147}
$$

$$
\big\| \big(\tfrac{\partial^2}{\partial x \partial y}\psi\big)(x,y)(v_1,v_2) \big\|_V = \big\| u_{0,2}(t,x)\big(F'(y)\, v_1, v_2\big) \big\|_V \le \tfrac{c_{-\vartheta,0}}{(T-t)^\vartheta} \, \|F'(y)\|_{L(H,H_{-\vartheta})}, \tag{148}
$$

$$
\begin{aligned}
&\big\| \big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,y)(v_1,v_2,v_3) \big\|_V = \big\| u_{0,3}(t,x)\big(F'(y)\, v_1, v_2, v_3\big) \big\|_V \\
&\le \tfrac{c_{-\vartheta,0,0}}{(T-t)^\vartheta} \, \|F'(y)\|_{L(H,H_{-\vartheta})},
\end{aligned} \tag{149}
$$

$$
\begin{aligned}
&\big\| \big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,y)(v_1,v_2,v_3) \big\|_V = \big\| u_{0,2}(t,x)\big(F''(y)(v_1,v_2), v_3\big) \big\|_V \\
&\le \tfrac{c_{-\vartheta,0}}{(T-t)^\vartheta} \, \|F''(y)\|_{L^{(2)}(H,H_{-\vartheta})}.
\end{aligned} \tag{150}
$$

Combining (142)–(144) and (148)–(150) with the fundamental theorem of calculus in Banach spaces proves (139). Moreover, combining (145)–(150) with the fundamental theorem of calculus in Banach spaces shows (140). It thus remains to prove (141). For this we observe that (142)–(150) ensure that for all $x, v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \le 1$ it holds that

$$
\begin{aligned}
&\big\| \phi'(x)\, v_1 \big\|_V \le \big\| \big(\tfrac{\partial}{\partial x}\psi\big)(x,x)\, v_1 \big\|_V + \big\| \big(\tfrac{\partial}{\partial y}\psi\big)(x,x)\, v_1 \big\|_V \\
&\le \tfrac{c_{-\vartheta,0}\, \|F(x)\|_{H_{-\vartheta}} + c_{-\vartheta}\, \|F'(x)\|_{L(H,H_{-\vartheta})}}{(T-t)^\vartheta} \le \tfrac{[c_{-\vartheta} + c_{-\vartheta,0}]}{(T-t)^\vartheta} \, \|F\|_{C^1_b(H,H_{-\vartheta})} \max\{1, \|x\|_H\},
\end{aligned} \tag{151}
$$

$$
\begin{aligned}
&\big\| \phi''(x)\, (v_1,v_2) \big\|_V \\
&\le \big\| \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x)\, (v_1,v_2) \big\|_V + 2 \, \big\| \big(\tfrac{\partial^2}{\partial x \partial y}\psi\big)(x,x)\, (v_1,v_2) \big\|_V + \big\| \big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,x)\, (v_1,v_2) \big\|_V \\
&\le \tfrac{c_{-\vartheta,0,0}\, \|F(x)\|_{H_{-\vartheta}} + 2\, c_{-\vartheta,0}\, \|F'(x)\|_{L(H,H_{-\vartheta})} + c_{-\vartheta}\, \|F''(x)\|_{L^{(2)}(H,H_{-\vartheta})}}{(T-t)^\vartheta} \\
&\le \tfrac{2\,[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0}]}{(T-t)^\vartheta} \, \|F\|_{C^2_b(H,H_{-\vartheta})} \max\{1, \|x\|_H\},
\end{aligned} \tag{152}
$$

$$
\begin{aligned}
&\big\| \phi^{(3)}(x)\, (v_1,v_2,v_3) \big\|_V \le \big\| \big(\tfrac{\partial^3}{\partial x^3}\psi\big)(x,x)\, (v_1,v_2,v_3) \big\|_V + 3 \, \big\| \big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,x)\, (v_1,v_2,v_3) \big\|_V \\
&+ 3 \, \big\| \big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,x)\, (v_1,v_2,v_3) \big\|_V + \big\| \big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,x)\, (v_1,v_2,v_3) \big\|_V \\
&\le \tfrac{c_{-\vartheta,0,0,0}\, \|F(x)\|_{H_{-\vartheta}} + 3\, c_{-\vartheta,0,0}\, \|F'(x)\|_{L(H,H_{-\vartheta})} + 3\, c_{-\vartheta,0}\, \|F''(x)\|_{L^{(2)}(H,H_{-\vartheta})} + c_{-\vartheta}\, \|F^{(3)}(x)\|_{L^{(3)}(H,H_{-\vartheta})}}{(T-t)^\vartheta} \\
&\le \tfrac{3\,[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0}]}{(T-t)^\vartheta} \, \|F\|_{C^3_b(H,H_{-\vartheta})} \max\{1, \|x\|_H\}.
\end{aligned} \tag{153}
$$

Combining (151)–(153) with the fundamental theorem of calculus in Banach spaces establishes (141). The proof of Lemma 7.1 is thus completed. □

Lemma 7.2. *Assume the setting in Section 7.1 and let* $t \in [0,T)$, $\psi = (\psi(x,y))_{x,y\in H} \in \mathbb{M}(H \times H, V)$, $\phi \in \mathbb{M}(H,V)$ *satisfy that for all* $x,y \in H$ *it holds that* $\psi(x,y) = \sum_{b\in\mathbb{U}} u_{0,2}(t,x)\big(B^b(y), B^b(y)\big)$ *and* $\phi(x) = \psi(x,x)$. *Then it holds that* $\psi \in C^2(H\times H, V)$, $\phi \in C^2(H,V)$ *and for all* $x, x_1, x_2, y, y_1, y_2 \in H$ *it holds that*

$$\begin{split} &\max_{i,j\in\mathbb{N}_0,\, i+j\leq 2} \big\| \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_1,y) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x_2,y)\big\|_{L^{(i+j)}(H,V)} \leq \tfrac{2\,\|x_1-x_2\|_H}{(T-t)^\vartheta} \\ &\cdot \|B\|^2_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \big[c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|y\|_H^2\}, \end{split} \tag{154}$$

$$\begin{split} &\max_{i,j\in\mathbb{N}_0,\, i+j\leq 2} \big\| \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_1) - \big(\tfrac{\partial^{(i+j)}}{\partial x^i \partial y^j}\psi\big)(x,y_2)\big\|_{L^{(i+j)}(H,V)} \leq \tfrac{6\,\|y_1-y_2\|_H}{(T-t)^\vartheta} \\ &\cdot \|B\|^2_{C^3_b(H,HS(U,H_{-\vartheta/2}))} \big[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|y_1\|_H, \|y_2\|_H\}, \end{split} \tag{155}$$

$$\begin{split} &\max_{i\in\{0,1,2\}} \big\|\phi^{(i)}(x_1) - \phi^{(i)}(x_2)\big\|_{L^{(i)}(H,V)} \leq \tfrac{8\,\|x_1-x_2\|_H}{(T-t)^\vartheta}\, \|B\|^2_{C^3_b(H,HS(U,H_{-\vartheta/2}))} \\ &\cdot \big[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|x_1\|_H^2, \|x_2\|_H^2\}. \end{split} \tag{156}$$

Proof. First of all, we note that the assumption that $B \in \operatorname{Lip}^4(H, HS(U,H_1))$ and the fact that $\big(H \ni x \mapsto u_{0,2}(t,x) \in L^{(2)}(H,V)\big) \in C^2(H, L^{(2)}(H,V))$ ensure that $\psi \in C^2(H\times H, V)$, $\phi \in C^2(H,V)$, and $\big(H\times H \ni (x,y) \mapsto (\tfrac{\partial^2}{\partial y^2}\psi)(x,y) \in L^{(2)}(H,V)\big) \in C^2(H, L^{(2)}(H,V))$. Next we observe that for all $x, x_1, x_2, y, v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \leq 1$ it holds that

$$\begin{split} \big\|\big(\tfrac{\partial}{\partial x}\psi\big)(x,y)\, v_1\big\|_V &\leq \sum_{b\in\mathbb{U}} \big\|u_{0,3}(t,x)\big(B^b(y), B^b(y), v_1\big)\big\|_V \\ &\leq \tfrac{c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^\vartheta}\, \|B(y)\|^2_{HS(U,H_{-\vartheta/2})}, \end{split} \tag{157}$$

$$\begin{split} &\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,y)\,(v_1,v_2)\big\|_V \leq \sum_{b\in\mathbb{U}} \big\|u_{0,4}(t,x)\big(B^b(y), B^b(y), v_1, v_2\big)\big\|_V \\ &\leq \tfrac{c_{-\vartheta/2,-\vartheta/2,0,0}}{(T-t)^\vartheta}\, \|B(y)\|^2_{HS(U,H_{-\vartheta/2})}, \end{split} \tag{158}$$

$$\begin{split} &\big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_1,y)\,(v_1,v_2) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,y)\,(v_1,v_2)\big\|_V \\ &\leq \sum_{b\in\mathbb{U}} \big\|\big(u_{0,4}(t,x_1) - u_{0,4}(t,x_2)\big)\big(B^b(y), B^b(y), v_1, v_2\big)\big\|_V \\ &\leq \tfrac{\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\,\|x_1-x_2\|_H}{(T-t)^\vartheta}\, \|B(y)\|^2_{HS(U,H_{-\vartheta/2})}, \end{split} \tag{159}$$

$$\begin{split} \big\|\big(\tfrac{\partial}{\partial y}\psi\big)(x,y)\, v_1\big\|_V &\leq 2\sum_{b\in\mathbb{U}} \big\|u_{0,2}(t,x)\big(B^b(y), (B^b)'(y)\, v_1\big)\big\|_V \\ &\leq \tfrac{2\,c_{-\vartheta/2,-\vartheta/2}}{(T-t)^\vartheta}\, \|B(y)\|_{HS(U,H_{-\vartheta/2})}\, \|B'(y)\|_{L(H,HS(U,H_{-\vartheta/2}))}, \end{split} \tag{160}$$

$$\begin{split} &\big\|\big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,y)(v_1,v_2)\big\|_V \\ &\leq 2\sum_{b\in\mathbb{U}} \big\|u_{0,2}(t,x)\big((B^b)'(y)\, v_1, (B^b)'(y)\, v_2\big) + u_{0,2}(t,x)\big(B^b(y), (B^b)''(y)(v_1,v_2)\big)\big\|_V \\ &\leq \tfrac{2\,c_{-\vartheta/2,-\vartheta/2}}{(T-t)^\vartheta}\, \big(\|B'(y)\|^2_{L(H,HS(U,H_{-\vartheta/2}))} + \|B(y)\|_{HS(U,H_{-\vartheta/2})}\, \|B''(y)\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))}\big), \end{split} \tag{161}$$

$$\begin{split} &\big\|\big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_V \\ &\leq 2\sum_{b\in\mathbb{U}} \big\|u_{0,2}(t,x)\big((B^b)'(y)\, v_2, (B^b)''(y)(v_1,v_3)\big) \\ &\quad + u_{0,2}(t,x)\big((B^b)'(y)\, v_1, (B^b)''(y)(v_2,v_3)\big) \\ &\quad + u_{0,2}(t,x)\big((B^b)'(y)\, v_3, (B^b)''(y)(v_1,v_2)\big) \\ &\quad + u_{0,2}(t,x)\big(B^b(y), (B^b)^{(3)}(y)(v_1,v_2,v_3)\big)\big\|_V \\ &\leq \tfrac{2\,c_{-\vartheta/2,-\vartheta/2}}{(T-t)^\vartheta}\, \big(3\,\|B'(y)\|_{L(H,HS(U,H_{-\vartheta/2}))}\, \|B''(y)\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \\ &\quad + \|B(y)\|_{HS(U,H_{-\vartheta/2})}\, \|B^{(3)}(y)\|_{L^{(3)}(H,HS(U,H_{-\vartheta/2}))}\big), \end{split} \tag{162}$$

$$\begin{aligned} &\big\|\big(\tfrac{\partial^2}{\partial x \partial y}\psi\big)(x,y)(v_1,v_2)\big\|_V \leq 2\sum_{b\in\mathbb{U}}\big\|u_{0,3}(t,x)\big(B^b(y),(B^b)'(y)\,v_1,v_2\big)\big\|_V \\ &\leq \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^\vartheta}\,\|B(y)\|_{HS(U,H_{-\vartheta/2})}\,\|B'(y)\|_{L(H,HS(U,H_{-\vartheta/2}))}, \end{aligned} \tag{163}$$

$$\begin{aligned} &\big\|\big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_V \\ &\leq 2\sum_{b\in\mathbb{U}}\big\|u_{0,4}(t,x)\big(B^b(y),(B^b)'(y)\,v_1,v_2,v_3\big)\big\|_V \\ &\leq \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0,0}}{(T-t)^\vartheta}\,\|B(y)\|_{HS(U,H_{-\vartheta/2})}\,\|B'(y)\|_{L(H,HS(U,H_{-\vartheta/2}))}, \end{aligned} \tag{164}$$

$$\begin{aligned} &\big\|\big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,y)(v_1,v_2,v_3)\big\|_V \\ &\leq 2\sum_{b\in\mathbb{U}}\big\|u_{0,3}(t,x)\big((B^b)'(y)\,v_1,(B^b)'(y)\,v_2,v_3\big)+u_{0,3}(t,x)\big(B^b(y),(B^b)''(y)(v_1,v_2),v_3\big)\big\|_V \\ &\leq \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0}}{(T-t)^\vartheta}\,\big(\|B'(y)\|^2_{L(H,HS(U,H_{-\vartheta/2}))}+\|B(y)\|_{HS(U,H_{-\vartheta/2})}\,\|B''(y)\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))}\big). \end{aligned} \tag{165}$$

Combining (157)–(159) and (163)–(165) with the fundamental theorem of calculus in Banach spaces proves (154). Moreover, combining (160)–(165) with the fundamental theorem of calculus in Banach spaces establishes (155). It thus remains to prove (156). For this we observe that (157)–(165) ensure that for all $x, v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \leq 1$ it holds that

$$\begin{aligned} &\big\|\phi'(x)\,v_1\big\|_V \leq \big\|\big(\tfrac{\partial}{\partial x}\psi\big)(x,x)\,v_1\big\|_V+\big\|\big(\tfrac{\partial}{\partial y}\psi\big)(x,x)\,v_1\big\|_V \\ &\leq \frac{c_{-\vartheta/2,-\vartheta/2,0}\,\|B(x)\|^2_{HS(U,H_{-\vartheta/2})}+2\,c_{-\vartheta/2,-\vartheta/2}\,\|B(x)\|_{HS(U,H_{-\vartheta/2})}\,\|B'(x)\|_{L(H,HS(U,H_{-\vartheta/2}))}}{(T-t)^\vartheta} \\ &\leq \tfrac{2\,[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}]}{(T-t)^\vartheta}\,\|B\|^2_{C^1_b(H,HS(U,H_{-\vartheta/2}))}\,\max\{1,\|x\|^2_H\}, \end{aligned} \tag{166}$$

$$\begin{aligned} &\big\|\phi''(x)\,(v_1,v_2)\big\|_V \\ &\leq \big\|\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x)\,(v_1,v_2)\big\|_V+2\,\big\|\big(\tfrac{\partial^2}{\partial x \partial y}\psi\big)(x,x)\,(v_1,v_2)\big\|_V+\big\|\big(\tfrac{\partial^2}{\partial y^2}\psi\big)(x,x)\,(v_1,v_2)\big\|_V \\ &\leq \frac{c_{-\vartheta/2,-\vartheta/2,0,0}\,\|B(x)\|^2_{HS(U,H_{-\vartheta/2})}+4\,c_{-\vartheta/2,-\vartheta/2,0}\,\|B(x)\|_{HS(U,H_{-\vartheta/2})}\,\|B'(x)\|_{L(H,HS(U,H_{-\vartheta/2}))}}{(T-t)^\vartheta} \\ &\quad+\frac{2\,c_{-\vartheta/2,-\vartheta/2}\,\big(\|B'(x)\|^2_{L(H,HS(U,H_{-\vartheta/2}))}+\|B(x)\|_{HS(U,H_{-\vartheta/2})}\,\|B''(x)\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))}\big)}{(T-t)^\vartheta} \\ &\leq \tfrac{4\,[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}]}{(T-t)^\vartheta}\,\|B\|^2_{C^2_b(H,HS(U,H_{-\vartheta/2}))}\,\max\{1,\|x\|^2_H\}. \end{aligned} \tag{167}$$

In the next step we observe that (162), (164), (165), and the fact that $\big(H \ni x \mapsto \phi''(x)-\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x) \in L^{(2)}(H,V)\big) \in C^1(H,L^{(2)}(H,V))$ show that for all $x, x_1, x_2, v_1, v_2, v_3 \in H$ with $\|v_1\|_H, \|v_2\|_H, \|v_3\|_H \leq 1$ it holds that

$$\begin{aligned} &\big\|\tfrac{\partial}{\partial x}\big(\phi''(x)-\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x,x)\big)(v_1,v_2,v_3)\big\|_V \leq 2\,\big\|\big(\tfrac{\partial^3}{\partial x^2 \partial y}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_V \\ &+3\,\big\|\big(\tfrac{\partial^3}{\partial x \partial y^2}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_V+\big\|\big(\tfrac{\partial^3}{\partial y^3}\psi\big)(x,x)\,(v_1,v_2,v_3)\big\|_V \\ &\leq \frac{4\,c_{-\vartheta/2,-\vartheta/2,0,0}\,\|B(x)\|_{HS(U,H_{-\vartheta/2})}\,\|B'(x)\|_{L(H,HS(U,H_{-\vartheta/2}))}}{(T-t)^\vartheta} \\ &+\frac{6\,c_{-\vartheta/2,-\vartheta/2,0}\,\big(\|B'(x)\|^2_{L(H,HS(U,H_{-\vartheta/2}))}+\|B(x)\|_{HS(U,H_{-\vartheta/2})}\,\|B''(x)\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))}\big)}{(T-t)^\vartheta} \\ &+\frac{6\,c_{-\vartheta/2,-\vartheta/2}\,\big(\|B'(x)\|_{L(H,HS(U,H_{-\vartheta/2}))}\,\|B''(x)\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))}+\|B(x)\|_{HS(U,H_{-\vartheta/2})}\,\|B^{(3)}(x)\|_{L^{(3)}(H,HS(U,H_{-\vartheta/2}))}\big)}{(T-t)^\vartheta} \\ &\leq \tfrac{6\,[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}]}{(T-t)^\vartheta}\,\|B\|^2_{C^3_b(H,HS(U,H_{-\vartheta/2}))}\,\max\{1,\|x\|_H\}. \end{aligned} \tag{168}$$

In addition, we combine (159) and (164) with the fundamental theorem of calculus in Banach spaces to obtain that for all $x_1, x_2, v_1, v_2 \in H$ with $\|v_1\|_H, \|v_2\|_H \leq 1$ it holds that

$$\begin{aligned}
&\big\|\big(\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_1,x_1) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_2)\big)(v_1,v_2)\big\|_V \\
&\leq \big\|\big(\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_1,x_1) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_1)\big)(v_1,v_2)\big\|_V \\
&+ \big\|\big(\big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_1) - \big(\tfrac{\partial^2}{\partial x^2}\psi\big)(x_2,x_2)\big)(v_1,v_2)\big\|_V \\
&\leq \tfrac{\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\,\|x_1-x_2\|_H}{(T-t)^{\vartheta}}\, \|B(x_1)\|^2_{HS(U,H_{-\vartheta/2})} \\
&+ \tfrac{2\,c_{-\vartheta/2,-\vartheta/2,0,0}\,\|x_1-x_2\|_H}{(T-t)^{\vartheta}}\, \|B\|^2_{C^1_b(H,HS(U,H_{-\vartheta/2}))}\, \max\{1, \|x_1\|_H, \|x_2\|_H\} \\
&\leq \tfrac{2\,\|x_1-x_2\|_H}{(T-t)^{\vartheta}}\, \|B\|^2_{C^1_b(H,HS(U,H_{-\vartheta/2}))}\, \big[c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big] \max\{1, \|x_1\|^2_H, \|x_2\|^2_H\}.
\end{aligned} \tag{169}$$

Combining (166)–(169) with the fundamental theorem of calculus in Banach spaces finally yields (156). The proof of Lemma 7.2 is thus completed. □

Lemma 7.3 (Weak convergence of semilinear integrated Euler-type approximations of SPDEs with mollified nonlinearities)**.** *Assume the setting in Section 7.1 and let $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_V + \|\varphi(\bar{Y}_T)\|_V\big] < \infty$ and*

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}_T)\big]\big\|_V \leq \tfrac{5\,|C_0|^3\,|C_\rho|^2\,C_{0,\rho}\,T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\, \varsigma_{F,B}\, K_5\, h^\rho \\
&\cdot \Big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big] \\
&\cdot \Bigg[2^{(\rho+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2} + C_{\vartheta,0,\rho} + 2\,C_{\vartheta/2}\,C_{\vartheta/2,0,\rho} \\
&+ 3\,(|C_{\vartheta/2}|^2 + C_\vartheta) + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\,C_\rho\Big[C_{-\rho,\rho} + \tfrac{C_{\vartheta,-\rho,\rho}\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\,C_{\vartheta/2,-\rho,\rho}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg].
\end{aligned} \tag{170}$$

Proof. First of all, we observe that the assumption that $\sup_{t\in[0,T]} \|X_t\|_{L^5(\mathbb{P};H)} < \infty$ implies that $\mathbb{E}\big[\|\varphi(X_T)\|_V\big] < \infty$. Moreover, combining the assumption that $Y_0 \in L^5(\mathbb{P}; H_1)$ with Lemma 6.1 proves that $K_5 < \infty$. This shows, in particular, that we have that $\sup_{s\in[0,T]} \mathbb{E}\big[\|\varphi(\bar{Y}_T)\|_V + \|\bar{Y}_s\|_{H_1} + \int_0^T \|u_{0,1}(t,\bar{Y}_t)\,B(Y_{\lfloor t\rfloor_h})\|^2_{HS(U,V)}\,dt\big] < \infty$. This and the standard Itô formula in Theorem 2.4 in Brzeźniak, Van Neerven, Veraar & Weis [9] prove that

$$\begin{aligned}
\mathbb{E}\big[\varphi(\bar{Y}_T)\big] - \mathbb{E}\big[\varphi(X_T)\big] &= \mathbb{E}\big[u(T,\bar{Y}_T) - u(0,\bar{Y}_0)\big] \\
&= \int_0^T \mathbb{E}\big[u_{1,0}(t,\bar{Y}_t) + u_{0,1}(t,\bar{Y}_t)\big(A\bar{Y}_t + F(Y_{\lfloor t\rfloor_h})\big)\big]\,dt \\
&+ \frac{1}{2}\sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[u_{0,2}(t,\bar{Y}_t)\big(B^b(Y_{\lfloor t\rfloor_h}), B^b(Y_{\lfloor t\rfloor_h})\big)\big]\,dt.
\end{aligned} \tag{171}$$

Exploiting the fact that u is a solution of the Kolmogorov backward equation associated to $X^x\colon [0,T]\times\Omega \to H$, $x \in H$, and φ (cf., e.g., Theorem 7.5.1 in Da Prato & Zabczyk [14]) hence shows that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(\bar{Y}_T)\big] - \mathbb{E}\big[\varphi(X_T)\big] \\
&= \int_0^T \mathbb{E}\big[u_{0,1}(t,\bar{Y}_t)\,F(Y_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_t)\,F(\bar{Y}_t)\big]\,dt \\
&+ \frac{1}{2}\sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[u_{0,2}(t,\bar{Y}_t)\big(B^b(Y_{\lfloor t\rfloor_h}), B^b(Y_{\lfloor t\rfloor_h})\big) - u_{0,2}(t,\bar{Y}_t)\big(B^b(\bar{Y}_t), B^b(\bar{Y}_t)\big)\big]\,dt.
\end{aligned} \tag{172}$$

The triangle inequality hence shows that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\bar{Y}_T)\big]\big\|_V \\
&\leq \int_0^T \big\|\mathbb{E}\big[u_{0,1}(t,\bar{Y}_t)\,F(Y_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(Y_{\lfloor t\rfloor_h})\big]\big\|_V\,dt \\
&+ \int_0^T \big\|\mathbb{E}\big[u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(Y_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(\bar{Y}_{\lfloor t\rfloor_h})\big]\big\|_V\,dt \\
&+ \int_0^T \big\|\mathbb{E}\big[u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(\bar{Y}_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_t)\,F(\bar{Y}_t)\big]\big\|_V\,dt \qquad (173)\\
&+ \tfrac{1}{2}\int_0^T \Bigg\|\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_t)\big(B^b\big(Y_{\lfloor t\rfloor_h}\big),B^b\big(Y_{\lfloor t\rfloor_h}\big)\big) - \sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(Y_{\lfloor t\rfloor_h}\big),B^b\big(Y_{\lfloor t\rfloor_h}\big)\big)\Bigg]\Bigg\|_V\,dt \\
&+ \tfrac{1}{2}\int_0^T \Bigg\|\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(Y_{\lfloor t\rfloor_h}\big),B^b\big(Y_{\lfloor t\rfloor_h}\big)\big) - \sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big),B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big)\big)\Bigg]\Bigg\|_V\,dt \\
&+ \tfrac{1}{2}\int_0^T \Bigg\|\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big),B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big)\big) - \sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_t)\big(B^b\big(\bar{Y}_t\big),B^b\big(\bar{Y}_t\big)\big)\Bigg]\Bigg\|_V\,dt.
\end{aligned}$$

In the next step we combine Lemma 7.1 and Lemma 7.2 with Proposition 6.2 to obtain that for all $t\in(0,T)$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[u_{0,1}(t,\bar{Y}_t)\,F(Y_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(Y_{\lfloor t\rfloor_h})\big]\big\|_V \\
&+ \big\|\mathbb{E}\big[u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(\bar{Y}_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_t)\,F(\bar{Y}_t)\big]\big\|_V \\
&+ \tfrac{1}{2}\Bigg\|\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_t)\big(B^b\big(Y_{\lfloor t\rfloor_h}\big),B^b\big(Y_{\lfloor t\rfloor_h}\big)\big) - \sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(Y_{\lfloor t\rfloor_h}\big),B^b\big(Y_{\lfloor t\rfloor_h}\big)\big)\Bigg]\Bigg\|_V \\
&+ \tfrac{1}{2}\Bigg\|\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big),B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big)\big) - \sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_t)\big(B^b\big(\bar{Y}_t\big),B^b\big(\bar{Y}_t\big)\big)\Bigg]\Bigg\|_V \\
&\leq \tfrac{|C_0|^3\,|C_\rho|^2}{(T-t)^\vartheta}\,K_5\,h^\rho\,\max\big\{1,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})},\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\big\} \qquad (174)\\
&\cdot\Bigg[4\,\big[c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}+c_{-\vartheta,0,0,0}\big]\,\|F\|_{C_b^3(H,H_{-\vartheta})} \\
&+ 5\,\big[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\big]\,\|B\|^2_{C_b^3(H,HS(U,H_{-\vartheta/2}))}\Bigg] \\
&\cdot\Bigg[\tfrac{2^\rho}{t^\rho}+\tfrac{\left(2\,C_\vartheta+C_{\rho+\vartheta}+2\,|C_{\vartheta/2}|^2+2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}\right)|\lfloor t\rfloor_h|^{(1-\vartheta-\rho)}+\left(C_\vartheta+\frac{1}{2}|C_{\vartheta/2}|^2\right)(t-\lfloor t\rfloor_h)^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\Bigg].
\end{aligned}$$

In addition, we combine Lemma 7.1 and Lemma 7.2 with Proposition 6.4 and the fact that $\forall\, t\in[h,T]\colon \lfloor t\rfloor_h > t/2$ to obtain that for all $t\in(0,T)$ it holds that

$$\begin{aligned}
&\big\|\mathbb{E}\big[u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(Y_{\lfloor t\rfloor_h}) - u_{0,1}(t,\bar{Y}_{\lfloor t\rfloor_h})\,F(\bar{Y}_{\lfloor t\rfloor_h})\big]\big\|_V \\
&+ \tfrac{1}{2}\Bigg\|\mathbb{E}\Bigg[\sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(Y_{\lfloor t\rfloor_h}\big),B^b\big(Y_{\lfloor t\rfloor_h}\big)\big) - \sum_{b\in\mathbb{U}} u_{0,2}(t,\bar{Y}_{\lfloor t\rfloor_h})\big(B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big),B^b\big(\bar{Y}_{\lfloor t\rfloor_h}\big)\big)\Bigg]\Bigg\|_V \\
&\leq \tfrac{C_0\,C_{0,\rho}}{(T-t)^\vartheta}\,K_4\,h^\rho\,\max\big\{1,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})},\|B\|^2_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\big\} \\
&\cdot\max\big\{1,\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})},\|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\big\}\,\Big(\big[c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}\big]\,\|F\|_{C_b^3(H,H_{-\vartheta})} \qquad (175)\\
&+ 3\,\big[c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}\big]\,\|B\|^2_{C_b^3(H,HS(U,H_{-\vartheta/2}))}\Big) \\
&\cdot\Bigg[\tfrac{2^\rho}{t^\rho}+\tfrac{|\lfloor t\rfloor_h|^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)}\Bigg(C_{\vartheta,0,\rho}+2\,C_{\vartheta/2}\,C_{\vartheta/2,0,\rho}+2\,(|C_{\vartheta/2}|^2+C_\vartheta)+2\,(|C_{\vartheta/2}|^2+C_\vartheta)\,C_\rho \\
&\cdot\Big[C_{-\rho,\rho}+\tfrac{C_{\vartheta,-\rho,\rho}\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\sqrt{6}\,C_{\vartheta/2,-\rho,\rho}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\Bigg)\Bigg].
\end{aligned}$$

Combining (173)–(175) proves that

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(\bar{Y}_T)\big]\big\|_V \leq 5\,|C_0|^3\,|C_\rho|^2\,C_{0,\rho}\,\varsigma_{F,B}\,K_5\,h^\rho \int\limits_0^T \tfrac{1}{(T-t)^\vartheta\,t^\rho}\,dt \\
&\cdot\Big[c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}+c_{-\vartheta,0,0,0}+c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0}+c_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big] \\
&\cdot\Bigg[2^{(\rho+1)}+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(2\,C_\vartheta+C_{\rho+\vartheta}+2\,|C_{\vartheta/2}|^2+2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}+C_{\vartheta,0,\rho}+2\,C_{\vartheta/2}\,C_{\vartheta/2,0,\rho} \\
&+3\,(|C_{\vartheta/2}|^2+C_\vartheta)+2\,(|C_{\vartheta/2}|^2+C_\vartheta)\,C_\rho\Big[C_{-\rho,\rho}+\tfrac{C_{\vartheta,-\rho,\rho}\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\sqrt{6}\,C_{\vartheta/2,-\rho,\rho}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg].
\end{aligned} \tag{176}$$

This and Lemma 3.1.6 in [25] show[9] (170). The proof of Lemma 7.3 is thus completed. □

7.3 Weak convergence rates for Euler-type approximations of SPDEs with mollified nonlinearities

The next result, Corollary 7.4, provides a bound for the weak distance of the numerical approximation and its semilinear integrated counterpart. Corollary 7.4 is an immediate consequence of Proposition 6.4 and of Lemma 7.3.

Corollary 7.4 (Weak distance between Euler-type approximations of SPDEs with mollified nonlinearities and their semilinear integrated counterparts)**.** *Assume the setting in Section 7.1 and let $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(\bar{Y}_T)\|_V+\|\varphi(Y_T)\|_V\big]<\infty$ and*

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(\bar{Y}_T)\big]-\mathbb{E}\big[\varphi(Y_T)\big]\big\|_V \leq \tfrac{C_{0,\rho}}{T^\rho}\,\|\varphi\|_{\mathrm{Lip}^2(H,V)}\,K_3\,h^\rho\,\varsigma_{F,B} \\
&\cdot\Bigg[1+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(C_{\vartheta,0,\rho}+2\,C_{\vartheta/2}\,C_{\vartheta/2,0,\rho}+2\,(|C_{\vartheta/2}|^2+C_\vartheta) \\
&+2\,(|C_{\vartheta/2}|^2+C_\vartheta)\,C_\rho\Big[C_{-\rho,\rho}+\tfrac{C_{\vartheta,-\rho,\rho}\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{C_{\vartheta/2,-\rho,\rho}\sqrt{3\,T^{(1-\vartheta)}}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg].
\end{aligned} \tag{177}$$

The next result is a direct consequence of the triangle inequality, of Corollary 7.4 and of Lemma 7.3.

Corollary 7.5 (Weak convergence of Euler-type approximations of SPDEs with mollified nonlinearities)**.** *Assume the setting in Section 7.1 and let $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_V+\|\varphi(Y_T)\|_V\big]<\infty$ and*

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(Y_T)\big]\big\|_V \leq \tfrac{5\,|C_0|^3\,|C_\rho|^2\,C_{0,\rho}\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-\rho)\,T^\rho}\,\varsigma_{F,B}\,K_5\,h^\rho \\
&\cdot\Bigg[2^{(\rho+1)}+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(2\,C_\vartheta+C_{\rho+\vartheta}+2\,|C_{\vartheta/2}|^2+2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2}+C_{\vartheta,0,\rho}+2\,C_{\vartheta/2}\,C_{\vartheta/2,0,\rho} \\
&+3\,(|C_{\vartheta/2}|^2+C_\vartheta)+2\,(|C_{\vartheta/2}|^2+C_\vartheta)\,C_\rho\Big[C_{-\rho,\rho}+\tfrac{C_{\vartheta,-\rho,\rho}\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\sqrt{6}\,C_{\vartheta/2,-\rho,\rho}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg] \\
&\cdot\Big[\|\varphi\|_{\mathrm{Lip}^2(H,V)}+c_{-\vartheta}+c_{-\vartheta,0}+c_{-\vartheta,0,0}+c_{-\vartheta,0,0,0}+c_{-\vartheta/2,-\vartheta/2}+c_{-\vartheta/2,-\vartheta/2,0} \\
&+c_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big].
\end{aligned} \tag{178}$$

[9]with $x = 1-\vartheta$ and $y = 1-\rho$ in the notation of Lemma 3.1.6 in [25]

In the next result, Corollary 7.6, we use Lemma 6.1 to estimate the real number K_5 on the right hand side of (178). For the formulation of Corollary 7.6 we recall that for all $x \in [0,\infty)$, $\theta \in [0,1)$ it holds that $\mathcal{E}_{1-\theta}(x) = \big[\sum_{n=0}^{\infty} \frac{x^{2n}\,\Gamma(1-\theta)^n}{\Gamma(n(1-\theta)+1)}\big]^{1/2}$ (see Section 1.2).

Corollary 7.6 (Weak convergence of Euler-type approximations of SPDEs with mollified nonlinearities)**.** *Assume the setting in Section 7.1 and let $\theta \in [0,1)$, $\rho \in [0, 1-\vartheta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_V + \|\varphi(Y_T)\|_V\big] < \infty$ and*

$$
\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_V \leq \tfrac{57\,|C_0|^3\,|C_\rho|^2\,C_{0,\rho}\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-\rho)\,T^\rho}\,\varsigma_{F,B}\,h^\rho \\
&\cdot \left[C_0\max\{1,\|X_0\|_{L^5(\mathbb{P};H)}\} + \frac{C_\theta\,T^{(1-\theta)}\,\|F\|_{\mathrm{Lip}^0(H,H_{-\theta})}}{(1-\theta)} + C_{\theta/2}\sqrt{\tfrac{10\,T^{(1-\theta)}}{(1-\theta)}}\,\|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\theta/2}))}\right]^{10} \\
&\cdot \left|\mathcal{E}_{(1-\theta)}\left[\frac{\sqrt{2}\,C_\theta\,T^{(1-\theta)}\,|F|_{\mathrm{Lip}^0(H,H_{-\theta})}}{\sqrt{1-\theta}} + 2\,C_{\theta/2}\sqrt{5\,T^{(1-\theta)}}\,|B|_{\mathrm{Lip}^0(H,HS(U,H_{-\theta/2}))}\right]\right|^5 \\
&\cdot \Bigg[2^{(\rho+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\bigg(2\,C_\vartheta + C_{\rho+\vartheta} + 2\,|C_{\vartheta/2}|^2 + 2\,C_{\rho+\vartheta/2}\,C_{\vartheta/2} + C_{\vartheta,0,\rho} + 2\,C_{\vartheta/2}\,C_{\vartheta/2,0,\rho} \\
&+ 3\,(|C_{\vartheta/2}|^2 + C_\vartheta) + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\,C_\rho\Big[C_{-\rho,\rho} + \tfrac{C_{\vartheta,-\rho,\rho}\,T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\,C_{\vartheta/2,-\rho,\rho}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\bigg)\Bigg] \\
&\cdot \Big[\|\varphi\|_{\mathrm{Lip}^2(H,V)} + c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta,0,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} \\
&+ c_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}_{-\vartheta/2,-\vartheta/2,0,0}\Big].
\end{aligned}
\tag{179}
$$

8 Weak convergence rates for Euler-type approximations of SPDEs

In this section we use Corollary 7.6 in Section 7 and the somehow non-standard mollification procedure in Conus et al. [11] to establish in Corollary 8.2 weak convergence rates for temporal numerical approximations of a certain class of SEEs. Corollary 8.2, in turn, implies Theorem 1.1 in the introduction. The arguments in this section are quite similar to the arguments in Section 5 in Conus et al. [11].

8.1 Setting

Assume the setting in Section 1.4, assume that $h \leq T$, let $\theta \in [0,1)$, $\vartheta \in [0, 1/2) \cap [0,\theta]$, $F \in C_b^5(H, H_{-\theta})$, $B \in C_b^5(H, HS(U, H_{-\theta/2}))$, $\varphi \in C_b^5(H,V)$, let $\varsigma_{F,B} \in \mathbb{R}$ be a real number given by $\varsigma_{F,B} = \max\big\{1, \|F\|^3_{C_b^3(H,H_{-\theta})}, \|B\|^6_{C_b^3(H,HS(U,H_{-\theta/2}))}\big\}$, let $X, Y\colon [0,T]\times\Omega \to H$ and $X^{\kappa,x}\colon [0,T]\times\Omega \to H$, $\kappa \in [0,T]$, $x \in H$, be $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes which satisfy that for all $\kappa \in [0,T]$, $x \in H$ it holds that $\sup_{t\in[0,T]}\big[\|X_t\|_{L^5(\mathbb{P};H)} + \|X_t^{\kappa,x}\|_{L^5(\mathbb{P};H)}\big] < \infty$, $X_0^{\kappa,x} = x$, and $Y_0 = X_0$ and which satisfy that for all $\kappa \in [0,T]$, $x \in H$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{tA}X_0 + \int_0^t e^{(t-s)A}\,F(X_s)\,ds + \int_0^t e^{(t-s)A}\,B(X_s)\,dW_s, \tag{180}$$

$$X_t^{\kappa,x} = e^{tA}\,x + \int_0^t e^{(\kappa+t-s)A}\,F(X_s^{\kappa,x})\,ds + \int_0^t e^{(\kappa+t-s)A}\,B(X_s^{\kappa,x})\,dW_s, \tag{181}$$

$$Y_t = S_{0,t}\,Y_0 + \int_0^t S_{s,t}\,R_s\,F(Y_{\lfloor s\rfloor_h})\,ds + \int_0^t S_{s,t}\,R_s\,B(Y_{\lfloor s\rfloor_h})\,dW_s, \tag{182}$$

let $u^{(\kappa)}\colon [0,T]\times H \to V$, $\kappa \in [0,T]$, be the functions with the property that for all $\kappa, t \in [0,T]$, $x \in H$ it holds that $u^{(\kappa)}(t,x) = \mathbb{E}\big[\varphi(X_{T-t}^{\kappa,x})\big]$, let $c^{(\kappa)}_{\delta_1,\dots,\delta_k} \in [0,\infty]$, $\delta_1,\dots,\delta_k \in \mathbb{R}$, $k \in \{1,2,3,4\}$,

$\kappa \in [0,T]$, be the extended real numbers with the property that for all $\kappa \in [0,T]$, $k \in \{1,2,3,4\}$, $\delta_1, \dots, \delta_k \in \mathbb{R}$ it holds that

$$c^{(\kappa)}_{\delta_1,\delta_2,\dots,\delta_k} = \sup_{t\in[0,T)} \sup_{x\in H} \sup_{v_1,\dots,v_k \in H\backslash\{0\}} \left[\frac{\left\| \left(\frac{\partial^k}{\partial x^k} u^{(\kappa)} \right)(t,x)(v_1,\dots,v_k) \right\|_V}{(T-t)^{(\delta_1+\ldots+\delta_k)} \, \|v_1\|_{H_{\delta_1}} \cdot \ldots \cdot \|v_k\|_{H_{\delta_k}}} \right], \tag{183}$$

and let $\tilde{c}^{(\kappa)}_{\delta_1,\delta_2,\delta_3,\delta_4} \in [0,\infty]$, $\delta_1,\delta_2,\delta_3,\delta_4 \in \mathbb{R}$, $\kappa \in [0,T]$, be the extended real numbers with the property that for all $\kappa \in [0,T]$, $\delta_1,\delta_2,\delta_3,\delta_4 \in \mathbb{R}$ it holds that

$$\begin{aligned} &\tilde{c}^{(\kappa)}_{\delta_1,\delta_2,\delta_3,\delta_4} \\ &= \sup_{t\in[0,T)} \sup_{\substack{x_1,x_2\in H,\\ x_1\neq x_2}} \sup_{v_1,\dots,v_4\in H\backslash\{0\}} \left[\frac{\left\| \left(\left(\frac{\partial^4}{\partial x^4} u^{(\kappa)} \right)(t,x_1) - \left(\frac{\partial^4}{\partial x^4} u^{(\kappa)} \right)(t,x_2) \right)(v_1,\dots,v_4) \right\|_V}{(T-t)^{(\delta_1+\ldots+\delta_4)} \, \|x_1-x_2\|_H \, \|v_1\|_{H_{\delta_1}} \cdot \ldots \cdot \|v_4\|_{H_{\delta_4}}} \right]. \end{aligned} \tag{184}$$

8.2 Weak convergence result

Proposition 8.1. *Assume the setting in Section 8.1 and let $r \in [0, 1-\vartheta)$, $\rho \in (0, 1-\theta)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_V + \|\varphi(Y_T)\|_V\big] < \infty$ and*

$$\begin{aligned} &\left\| \mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big] \right\|_V \leq \Big[57 \left| \max\{T, \tfrac{1}{T}\} \right|^{(r+3(\theta-\vartheta))} |C_0|^{20} \Big] \, h^{\frac{\rho\, r}{(\rho+6(\theta-\vartheta))}} \\ &\cdot \left[\max\{1, \|X_0\|_{L^5(\mathbb{P};H)}\} + \tfrac{C_\theta \, C_{\rho/2+\theta} \, T^{(1-\theta)} \, \|F\|_{C^1_b(H,H_{-\theta})}}{(1-\theta-\rho/2)} + \tfrac{C_{\theta/2} \, C_{(\rho+\theta)/2} \sqrt{10\, T^{(1-\theta)}} \, \|B\|_{C^1_b(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta-\rho}} \right]^{10} \\ &\cdot \left| \mathcal{E}_{(1-\theta)} \left[\tfrac{\sqrt{2}\, C_0 \, C_\theta \, T^{(1-\theta)} \, |F|_{C^1_b(H,H_{-\theta})}}{\sqrt{1-\theta}} + 2\, C_0 \, C_{\theta/2} \sqrt{5\, T^{(1-\theta)}} \, |B|_{C^1_b(H,HS(U,H_{-\theta/2}))} \right] \right|^5 \\ &\cdot \Bigg[2^{(r+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)} \Big(2\, C_\vartheta + C_{r+\vartheta} + 2\, |C_{\vartheta/2}|^2 + 2\, C_{r+\vartheta/2} \, C_{\vartheta/2} + C_{\vartheta,0,r} + 2\, C_{\vartheta/2} \, C_{\vartheta/2,0,r} \\ &+ 3\, (|C_{\vartheta/2}|^2 + C_\vartheta) + 2\, (|C_{\vartheta/2}|^2 + C_\vartheta) \, C_r \Big[C_{-r,r} + \tfrac{C_{\vartheta,-r,r} \, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\, C_{\vartheta/2,-r,r} \, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} \Big] \Big) \Bigg] \\ &\cdot \Bigg[\tfrac{|C_{\rho/2}|^2}{T^{\rho/2}} \, |\varphi|_{C^1_b(H,V)} + \tfrac{|C_0|^3 \, |C_r|^2 \, C_{0,r} \, |C_{\theta-\vartheta}|^3 \, |C_{(\theta-\vartheta)/2}|^6 \max\{1, T^{(1-\vartheta)}\} \, \varsigma_{F,B}}{(1-\vartheta-r)\, T^r} \Big(\|\varphi\|_{C^3_b(H,V)} + \sup_{\kappa\in(0,T]} \big[c^{(\kappa)}_{-\vartheta} \\ &+ c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} \big] \Big) \Bigg] < \infty. \end{aligned} \tag{185}$$

Proof. First of all, we note that there exist up to modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $\hat{Y}^{\kappa,\delta}\colon [0,T]\times\Omega \to H$, $\kappa,\delta \in [0,T]$, and $\hat{X}^{\kappa,\delta}\colon [0,T]\times\Omega \to H$, $\kappa,\delta \in [0,T]$, which satisfy that for all $\kappa,\delta \in [0,T]$ it holds that $\sup_{t\in[0,T]} \|\hat{X}^{\kappa,\delta}_t\|_{L^5(\mathbb{P};H)} < \infty$ and $\hat{X}^{\kappa,\delta}_0 = \hat{Y}^{\kappa,\delta}_0 = e^{\delta A} X_0$ and which satisfy that for all $\kappa,\delta \in [0,T]$, $t \in (0,T]$ it holds $\mathbb{P}$-a.s. that

$$\hat{X}^{\kappa,\delta}_t = e^{tA} \hat{X}^{\kappa,\delta}_0 + \int_0^t e^{(\kappa+t-s)A} F(\hat{X}^{\kappa,\delta}_s) \, ds + \int_0^t e^{(\kappa+t-s)A} B(\hat{X}^{\kappa,\delta}_s) \, dW_s, \tag{186}$$

$$\hat{Y}^{\kappa,\delta}_t = S_{0,t} \, \hat{Y}^{\kappa,\delta}_0 + \int_0^t S_{s,t} \, R_s \, e^{\kappa A} F(\hat{Y}^{\kappa,\delta}_{\lfloor s \rfloor_h}) \, ds + \int_0^t S_{s,t} \, R_s \, e^{\kappa A} B(\hat{Y}^{\kappa,\delta}_{\lfloor s \rfloor_h}) \, dW_s \tag{187}$$

(see, e.g., Proposition 3 in Da Prato et al. [12], Theorem 4.3 in Brzeźniak [8], Theorem 6.2 in Van Neerven et al. [40]). In the next step we combine Lemma 6.1 with the fact that $\forall\, \kappa,\delta \in [0,T]\colon \|\hat{Y}^{\kappa,\delta}_0\|_{L^5(\mathbb{P};H)} < \infty$ to obtain that for all $\kappa,\delta \in [0,T]$ it holds that $\sup_{t\in[0,T]} \|\hat{Y}^{\kappa,\delta}_t\|_{L^5(\mathbb{P};H)} <$

∞. This, the fact that $\forall\,\kappa,\delta\in[0,T]\colon \sup_{t\in[0,T]}\|\hat{X}_t^{\kappa,\delta}\|_{L^5(\mathbb{P};H)}<\infty$ and the assumption that $\varphi\in\operatorname{Lip}^4(H,V)$ ensure that for all $\kappa,\delta\in[0,T]$ it holds that

$$\mathbb{E}\big[\|\varphi(\hat{X}_T^{\kappa,\delta})\|_V+\|\varphi(\hat{Y}_T^{\kappa,\delta})\|_V\big]<\infty. \tag{188}$$

This proves, in particular, that $\mathbb{E}\big[\|\varphi(X_T)\|_V+\|\varphi(Y_T)\|_V\big]<\infty$. It thus remains to show (185). For this we observe that the triangle inequality ensures that for all $\kappa,\delta\in[0,T]$ it holds that

$$\begin{split}\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big]\big\|_V &\le \big\|\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big]-\mathbb{E}\big[\varphi(\hat{X}_T^{\kappa,\delta})\big]\big\|_V\\ &+\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{\kappa,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{\kappa,\delta})\big]\big\|_V+\big\|\mathbb{E}\big[\varphi(\hat{Y}_T^{\kappa,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big]\big\|_V.\end{split} \tag{189}$$

In the following we provide suitable bounds for the three summands on the right hand side of (189). For the first and the third summand on the right hand side of (189) we observe that Proposition 4.3 together with the fact that $\forall\,\kappa,\delta\in[0,T]\colon \sup_{t\in[0,T]}\|\hat{Y}_{\lfloor t\rfloor_h}^{\kappa,\delta}\|_{L^2(\mathbb{P};H)}\le\sup_{t\in[0,T]}\|\hat{Y}_t^{\kappa,\delta}\|_{L^2(\mathbb{P};H)}<\infty$ shows that for all $\kappa,\delta\in[0,T]$ it holds that

$$\begin{split}&\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big]-\mathbb{E}\big[\varphi(\hat{X}_T^{\kappa,\delta})\big]\big\|_V+\big\|\mathbb{E}\big[\varphi(\hat{Y}_T^{\kappa,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big]\big\|_V\le\tfrac{4\,|C_{\rho/2}|^2}{T^{\rho/2}}\,|\varphi|_{C_b^1(H,V)}\,\kappa^{\frac{\rho}{2}}\\ &\cdot\left[C_0\max\{1,\|e^{\delta A}X_0\|_{L^2(\mathbb{P};H)}\}+\tfrac{C_\theta\,C_{\rho/2+\theta}\,T^{(1-\theta)}\,\|F\|_{C_b^1(H,H_{-\theta})}}{(1-\theta-\rho/2)}+\tfrac{C_{\theta/2}\,C_{(\rho+\theta)/2}\sqrt{T^{(1-\theta)}}\,\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^2\\ &\cdot\left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\,T^{(1-\theta)}\,C_0\,C_\theta}{\sqrt{1-\theta}}|F|_{C_b^1(H,H_{-\theta})}+\sqrt{2\,T^{(1-\theta)}}\,C_0\,C_{\theta/2}\,|B|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^2.\end{split} \tag{190}$$

Next we bound the second summand on the right hand side of (189). For this we note that for all $\kappa\in(0,T]$ it holds that

$$\begin{split}&\max\big\{1,\|e^{\kappa A}F\|^3_{C_b^3(H,H_{-\vartheta})},\|e^{\kappa A}B\|^6_{C_b^3(H,HS(U,H_{-\vartheta/2}))}\big\}\\ &\le|C_{\theta-\vartheta}|^3\,|C_{(\theta-\vartheta)/2}|^6\,\varsigma_{F,B}\max\big\{1,\kappa^{-3(\theta-\vartheta)}\big\}.\end{split} \tag{191}$$

This and Corollary 7.6 show that for all $\kappa,\delta\in(0,T]$ it holds that

$$\begin{split}&\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{\kappa,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{\kappa,\delta})\big]\big\|_V\\ &\le\tfrac{57\,|C_0|^3\,|C_r|^2\,C_{0,r}\,|C_{\theta-\vartheta}|^3\,|C_{(\theta-\vartheta)/2}|^6\max\{1,T^{(1-\vartheta)}\}}{(1-\vartheta-r)\,T^r}\,\varsigma_{F,B}\max\big\{1,\kappa^{-3(\theta-\vartheta)}\big\}\,h^r\\ &\cdot\left[C_0\max\{1,\|e^{\delta A}X_0\|_{L^5(\mathbb{P};H)}\}+\tfrac{C_\theta\,T^{(1-\theta)}\,\|e^{\kappa A}F\|_{C_b^1(H,H_{-\theta})}}{(1-\theta)}+\tfrac{C_{\theta/2}\sqrt{10\,T^{(1-\theta)}}\,\|e^{\kappa A}B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta}}\right]^{10}\\ &\cdot\left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\,C_\theta\,T^{(1-\theta)}\,|e^{\kappa A}F|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}}+2\,C_{\theta/2}\sqrt{5\,T^{(1-\theta)}}\,|e^{\kappa A}B|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^5\\ &\cdot\Bigg[2^{(r+1)}+\tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)}\bigg(2\,C_\vartheta+C_{r+\vartheta}+2\,|C_{\vartheta/2}|^2+2\,C_{r+\vartheta/2}\,C_{\vartheta/2}+C_{\vartheta,0,r}+2\,C_{\vartheta/2}\,C_{\vartheta/2,0,r}\\ &+3\,(|C_{\vartheta/2}|^2+C_\vartheta)+2\,(|C_{\vartheta/2}|^2+C_\vartheta)\,C_r\left[C_{-r,r}+\tfrac{C_{\vartheta,-r,r}\,T^{(1-\vartheta)}}{(1-\vartheta)}+\tfrac{\sqrt{6}\,C_{\vartheta/2,-r,r}\,T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\right]\bigg)\Bigg]\\ &\cdot\Big[\|\varphi\|_{C_b^3(H,V)}+c^{(\kappa)}_{-\vartheta}+c^{(\kappa)}_{-\vartheta,0}+c^{(\kappa)}_{-\vartheta,0,0}+c^{(\kappa)}_{-\vartheta,0,0,0}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2}+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0}\\ &+c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}+\tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\Big].\end{split} \tag{192}$$

Plugging (190) and (192) into (189) then shows that for all $\kappa,\delta\in(0,T]$ it holds that

$$\big\|\mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big]-\mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big]\big\|_V\le\max\Big\{4\,\kappa^{\frac{\rho}{2}},57\max\big\{1,\kappa^{-3(\theta-\vartheta)}\big\}\,h^r\Big\}\,|C_0|^{20}$$

$$\cdot \left[\max\{1, \|X_0\|_{L^5(\mathbb{P};H)}\} + \frac{C_\theta \, C_{\rho/2+\theta} \, T^{(1-\theta)} \, \|F\|_{C_b^1(H,H_{-\theta})}}{(1-\theta-\rho/2)} + \frac{C_{\theta/2} \, C_{(\rho+\theta)/2} \sqrt{10\, T^{(1-\theta)}} \, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta-\rho}} \right]^{10}$$
$$\cdot \left| \mathcal{E}_{(1-\theta)} \left[\frac{\sqrt{2}\, C_0 \, C_\theta \, T^{(1-\theta)} \, |F|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + 2\, C_0 \, C_{\theta/2} \sqrt{5\, T^{(1-\theta)}} \, |B|_{C_b^1(H,HS(U,H_{-\theta/2}))} \right] \right|^5$$
$$\cdot \Bigg[2^{(r+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)} \Big(2\, C_\vartheta + C_{r+\vartheta} + 2\, |C_{\vartheta/2}|^2 + 2\, C_{r+\vartheta/2} \, C_{\vartheta/2} + C_{\vartheta,0,r} + 2\, C_{\vartheta/2} \, C_{\vartheta/2,0,r} \tag{193}$$
$$+ 3\, (|C_{\vartheta/2}|^2 + C_\vartheta) + 2\, (|C_{\vartheta/2}|^2 + C_\vartheta) \, C_r \Big[C_{-r,r} + \tfrac{C_{\vartheta,-r,r} \, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\, C_{\vartheta/2,-r,r} \, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} \Big] \Big) \Bigg]$$
$$\cdot \Bigg[\tfrac{|C_{\rho/2}|^2}{T^{\rho/2}} \, |\varphi|_{C_b^1(H,V)} + \tfrac{|C_0|^3 \, |C_r|^2 \, C_{0,r} \, |C_{\theta-\vartheta}|^3 \, |C_{(\theta-\vartheta)/2}|^6 \max\{1, T^{(1-\vartheta)}\}}{(1-\vartheta-r)\, T^r} \, \varsigma_{F,B} \Big[\|\varphi\|_{C_b^3(H,V)} + c^{(\kappa)}_{-\vartheta} + c^{(\kappa)}_{-\vartheta,0}$$
$$+ c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} \Big] \Bigg].$$

In addition, we observe that

$$\begin{aligned}
&\inf_{\kappa \in (0,T]} \max\Big\{ 4\, \kappa^{\frac{\rho}{2}}, 57 \max\big\{1, \kappa^{-3(\theta-\vartheta)}\big\} \, h^r \Big\} \\
&\leq \max\Big\{ 4 \Big[\min\{1,T\} \, \big|\tfrac{h}{T}\big|^{\frac{2r}{(\rho+6(\theta-\vartheta))}} \Big]^{\frac{\rho}{2}}, 57 \max\Big\{ 1, \Big[\min\{1,T\} \, \big|\tfrac{h}{T}\big|^{\frac{2r}{(\rho+6(\theta-\vartheta))}} \Big]^{-3(\theta-\vartheta)} \Big\} \, h^r \Big\} \\
&= \max\Big\{ 4 \Big[\min\{1,T\} \, \big|\tfrac{h}{T}\big|^{\frac{2r}{(\rho+6(\theta-\vartheta))}} \Big]^{\frac{\rho}{2}}, 57\, h^r \Big[\min\{1,T\} \, \big|\tfrac{h}{T}\big|^{\frac{2r}{(\rho+6(\theta-\vartheta))}} \Big]^{-3(\theta-\vartheta)} \Big\} \\
&= \max\Bigg\{ \frac{4\, |\min\{1,T\}|^{\frac{\rho}{2}}}{T^{\frac{r\rho}{(\rho+6(\theta-\vartheta))}}}, \frac{57\, T^{\frac{6(\theta-\vartheta)r}{(\rho+6(\theta-\vartheta))}}}{|\min\{1,T\}|^{3(\theta-\vartheta)}} \Bigg\} \, h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}} \\
&\leq 57 \max\Big\{ \tfrac{1}{|\min\{1,T\}|^r}, \tfrac{|\max\{1,T\}|^r}{|\min\{1,T\}|^{3(\theta-\vartheta)}} \Big\} \, h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}} \leq \frac{57\, h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}}}{\big|\min\{T, \frac{1}{T}\}\big|^{(r+3(\theta-\vartheta))}}.
\end{aligned} \tag{194}$$

Combining (193) and (194) yields that for all $\delta \in (0,T]$ it holds that

$$\big\| \mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big] - \mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big] \big\|_V \leq \Big[57 \, \big|\max\{T, \tfrac{1}{T}\}\big|^{(r+3(\theta-\vartheta))} \, |C_0|^{20} \Big] \, h^{\frac{\rho r}{(\rho+6(\theta-\vartheta))}} \tag{195}$$
$$\cdot \left[\max\{1, \|X_0\|_{L^5(\mathbb{P};H)}\} + \frac{C_\theta \, C_{\rho/2+\theta} \, T^{(1-\theta)} \, \|F\|_{C_b^1(H,H_{-\theta})}}{(1-\theta-\rho/2)} + \frac{C_{\theta/2} \, C_{(\rho+\theta)/2} \sqrt{10\, T^{(1-\theta)}} \, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta-\rho}} \right]^{10}$$
$$\cdot \left| \mathcal{E}_{(1-\theta)} \left[\frac{\sqrt{2}\, C_0 \, C_\theta \, T^{(1-\theta)} \, |F|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + 2\, C_0 \, C_{\theta/2} \sqrt{5\, T^{(1-\theta)}} \, |B|_{C_b^1(H,HS(U,H_{-\theta/2}))} \right] \right|^5$$
$$\cdot \Bigg[2^{(r+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-r)} \Big(2\, C_\vartheta + C_{r+\vartheta} + 2\, |C_{\vartheta/2}|^2 + 2\, C_{r+\vartheta/2} \, C_{\vartheta/2} + C_{\vartheta,0,r} + 2\, C_{\vartheta/2} \, C_{\vartheta/2,0,r}$$
$$+ 3\, (|C_{\vartheta/2}|^2 + C_\vartheta) + 2\, (|C_{\vartheta/2}|^2 + C_\vartheta) \, C_r \Big[C_{-r,r} + \tfrac{C_{\vartheta,-r,r} \, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\, C_{\vartheta/2,-r,r} \, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}} \Big] \Big) \Bigg]$$
$$\cdot \Bigg[\tfrac{|C_{\rho/2}|^2}{T^{\rho/2}} \, |\varphi|_{C_b^1(H,V)} + \tfrac{|C_0|^3 \, |C_r|^2 \, C_{0,r} \, |C_{\theta-\vartheta}|^3 \, |C_{(\theta-\vartheta)/2}|^6 \, \max\{1, T^{(1-\vartheta)}\}}{(1-\vartheta-r)\, T^r} \, \varsigma_{F,B} \Big[\|\varphi\|_{C_b^3(H,V)} + \sup_{\kappa \in (0,T]} \Big[c^{(\kappa)}_{-\vartheta}$$
$$+ c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} \Big] \Big] \Bigg].$$

In the next step we note that Corollary 3.2 together with Lebesgue's dominated convergence theorem yields that $\lim_{\delta \to 0} \mathbb{E}\big[\varphi(\hat{X}_T^{0,\delta})\big] = \mathbb{E}\big[\varphi(X_T)\big]$ and $\lim_{\delta \to 0} \mathbb{E}\big[\varphi(\hat{Y}_T^{0,\delta})\big] = \mathbb{E}\big[\varphi(Y_T)\big]$. Combining this with inequality (195) proves the first inequality in (185). The second inequality in (185) follows from Andersson & Jentzen [1]. The proof of Proposition 8.1 is thus completed. □

Corollary 8.2. *Assume the setting in Section 8.1 and let $\rho \in (0, 1-\theta) \cap (6(\theta-\vartheta), \infty)$. Then it holds that $\mathbb{E}\big[\|\varphi(X_T)\|_V + \|\varphi(Y_T)\|_V\big] < \infty$ and*

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_V \leq \left[\tfrac{57\,|C_0|^{20}}{|\min\{T,\frac{1}{T}\}|^{3(\rho+\theta)}}\right] h^{(\rho-6(\theta-\vartheta))} \\
&\cdot \left[\max\{1, \|X_0\|_{L^5(\mathbb{P};H)}\} + \tfrac{C_\theta\, C_{\rho/2+\theta}\, T^{(1-\theta)}\, \|F\|_{C_b^1(H,H_{-\theta})}}{(1-\theta-\rho/2)} + \tfrac{C_{\theta/2}\, C_{(\rho+\theta)/2} \sqrt{10\, T^{(1-\theta)}}\, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10} \\
&\cdot \left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\, C_0\, C_\theta\, T^{(1-\theta)}\, |F|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + 2\, C_0\, C_{\theta/2} \sqrt{5\, T^{(1-\theta)}}\, |B|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^5 \\
&\cdot \Bigg[2^{(\rho+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\Big(2\, C_\vartheta + C_{\rho+\vartheta} + 2\, |C_{\vartheta/2}|^2 + 2\, C_{\rho+\vartheta/2}\, C_{\vartheta/2} + C_{\vartheta,0,\rho} + 2\, C_{\vartheta/2}\, C_{\vartheta/2,0,\rho} \\
&+ 3\,(|C_{\vartheta/2}|^2 + C_\vartheta) + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_\rho\Big[C_{-\rho,\rho} + \tfrac{C_{\vartheta,-\rho,\rho}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\, C_{\vartheta/2,-\rho,\rho}\, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\Big)\Bigg] \\
&\cdot \Bigg[\tfrac{|C_{\rho/2}|^2}{T^{\rho/2}}\, |\varphi|_{C_b^1(H,V)} + \tfrac{|C_0|^3\, |C_\rho|^2\, C_{0,\rho}\, |C_{\theta-\vartheta}|^3\, |C_{(\theta-\vartheta)/2}|^6 \max\{1, T^{(1-\vartheta)}\}\, \varsigma_{F,B}}{(1-\vartheta-\rho)\, T^\rho}\Big(\|\varphi\|_{C_b^3(H,V)} + \sup_{\kappa \in (0,T]}\big[c^{(\kappa)}_{-\vartheta} \\
&+ c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\big]\Big)\Bigg] < \infty.
\end{aligned} \tag{196}$$

Proof. First of all, we apply[10] Proposition 8.1 to obtain that $\mathbb{E}\big[\|\varphi(X_T)\|_V + \|\varphi(Y_T)\|_V\big] < \infty$ and

$$\begin{aligned}
&\big\|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(Y_T)\big]\big\|_V \leq \left[\tfrac{57\,|C_0|^{20}}{|\min\{T,\frac{1}{T}\}|^{(\rho+3(\theta-\vartheta))}}\right] h^{\frac{\rho^2}{(\rho+6(\theta-\vartheta))}} \\
&\cdot \left[\max\{1, \|X_0\|_{L^5(\mathbb{P};H)}\} + \tfrac{C_\theta\, C_{\rho/2+\theta}\, T^{(1-\theta)}\, \|F\|_{C_b^1(H,H_{-\theta})}}{(1-\theta-\rho/2)} + \tfrac{C_{\theta/2}\, C_{(\rho+\theta)/2} \sqrt{10\, T^{(1-\theta)}}\, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\theta-\rho}}\right]^{10} \\
&\cdot \left|\mathcal{E}_{(1-\theta)}\left[\tfrac{\sqrt{2}\, C_0\, C_\theta\, T^{(1-\theta)}\, |F|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + 2\, C_0\, C_{\theta/2} \sqrt{5\, T^{(1-\theta)}}\, |B|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^5 \\
&\cdot \Bigg[2^{(\rho+1)} + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\Big(2\, C_\vartheta + C_{\rho+\vartheta} + 2\, |C_{\vartheta/2}|^2 + 2\, C_{\rho+\vartheta/2}\, C_{\vartheta/2} + C_{\vartheta,0,\rho} + 2\, C_{\vartheta/2}\, C_{\vartheta/2,0,\rho} \\
&+ 3\,(|C_{\vartheta/2}|^2 + C_\vartheta) + 2\,(|C_{\vartheta/2}|^2 + C_\vartheta)\, C_\rho\Big[C_{-\rho,\rho} + \tfrac{C_{\vartheta,-\rho,\rho}\, T^{(1-\vartheta)}}{(1-\vartheta)} + \tfrac{\sqrt{6}\, C_{\vartheta/2,-\rho,\rho}\, T^{(1-\vartheta)/2}}{\sqrt{1-\vartheta}}\Big]\Big)\Bigg] \\
&\cdot \Bigg[\tfrac{|C_{\rho/2}|^2}{T^{\rho/2}}\, |\varphi|_{C_b^1(H,V)} + \tfrac{|C_0|^3\, |C_\rho|^2\, C_{0,\rho}\, |C_{\theta-\vartheta}|^3\, |C_{(\theta-\vartheta)/2}|^6 \max\{1, T^{(1-\vartheta)}\}\, \varsigma_{F,B}}{(1-\vartheta-\rho)\, T^\rho}\Big(\|\varphi\|_{C_b^3(H,V)} + \sup_{\kappa \in (0,T]}\big[c^{(\kappa)}_{-\vartheta} \\
&+ c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta,0,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} + \tilde{c}^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\big]\Big)\Bigg] < \infty.
\end{aligned} \tag{197}$$

Next we note that

$$\begin{aligned}
h^{\frac{\rho^2}{(\rho+6(\theta-\vartheta))}} &= h^{\rho\left[\frac{1}{1+6(\theta-\vartheta)/\rho} - 1 + \frac{6(\theta-\vartheta)}{\rho}\right]}\, h^{\rho\left[1-\frac{6(\theta-\vartheta)}{\rho}\right]} \\
&\leq |\max\{1,T\}|^{\rho\left[\frac{1}{1+6(\theta-\vartheta)/\rho} - 1 + \frac{6(\theta-\vartheta)}{\rho}\right]}\, h^{(\rho-6(\theta-\vartheta))} \leq |\max\{1,T\}|^{\rho}\, h^{(\rho-6(\theta-\vartheta))}.
\end{aligned} \tag{198}$$

Plugging (198) into (197) implies (196). This completes the proof of Corollary 8.2. □

[10]with $r = \rho$ in the notation of Proposition 8.1

9 Lower bounds for weak errors of Euler-type approximations for SPDEs

In this section a few specific lower bounds for weak approximation errors of temporal numerical approximations are established in the case of concrete example SEEs. A few specific lower bounds for weak approximation errors of spatial spectral Galerkin approximations can be found in Section 6 in Conus et al. [11]. Lower bounds for strong approximation errors for examples of SEEs and for whole classes of SEEs can be found in [15, 35, 36] and the references mentioned therein. The article [36] and Section 5 in [11] study exclusively parabolic SEEs driven by additive noise. The papers [15, 35] investigate parabolic SEEs driven by possibly non-additive noise. In this section we consider exclusively parabolic SEEs driven by additive noise.

9.1 Setting

Throughout Section 9 the following setting is frequently used. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space with $H \neq \{0\}$, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, let $h \in (0,\infty)$, $T \in \{h, 2h, 3h, \dots\}$, $\beta \in [0, \frac{1}{2})$, $\angle = \{(t_1,t_2) \in [0,T]^2 \colon t_1 < t_2\}$, let $\lambda, \mu \colon \mathbb{H} \to \mathbb{R}$ be functions such that $\sup_{b\in\mathbb{H}} \lambda_b < 0$ and $\sum_{b\in\mathbb{H}} |\mu_b|^2 |\lambda_b|^{-2\beta} < \infty$, let $A \colon D(A) \subseteq H \to H$ be a linear operator such that $D(A) = \{v \in H \colon \sum_{b\in\mathbb{H}} |\lambda_b \langle b, v\rangle_H|^2 < \infty\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{b\in\mathbb{H}} \lambda_b \langle b, v\rangle_H\, b$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [25])), let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_H-Wiener process w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$, and let $B \in HS(H, H_{-\beta})$ satisfy that for all $v \in H$ it holds that $Bv = \sum_{b\in\mathbb{H}} \mu_b \langle b, v\rangle_H\, b$. The above assumptions ensure that there exist $X, Y_1, Y_2 \in \mathcal{M}(\mathcal{F}, \mathcal{B}(H))$ which satisfy that it holds $\mathbb{P}$-a.s. that $X = \int_0^T e^{(T-s)A}\, B\, dW_s$, $Y_1 = \int_0^T e^{(T-\lfloor s\rfloor_h)A}\, B\, dW_s$, and $Y_2 = \int_0^T \big(\mathrm{Id}_H - hA\big)^{-(T-\lfloor s\rfloor_h)/h}\, B\, dW_s$.

9.2 Variance estimates for Euler-type approximations of SPDEs

Lemma 9.1 (Variance estimates for exponential Euler approximations)**.** *Assume the setting in Section 9.1 and let $b \in \mathbb{H}$. Then it holds that $\mathbb{E}\big[|\langle b, X\rangle_H|^2 + |\langle b, Y_1\rangle_H|^2\big] < \infty$ and*

$$\mathrm{Var}\big(\langle b, X\rangle_H\big) - \mathrm{Var}\big(\langle b, Y_1\rangle_H\big) \geq \frac{|\mu_b|^2 \left(1 - e^{-2|\lambda_b|T}\right) h}{4\, e^{|\lambda_b|h}} \geq 0. \tag{199}$$

Proof. First, observe that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned} \langle b, Y_1\rangle_H &= \left\langle b, \int_0^T e^{(T-\lfloor s\rfloor_h)A} B\, dW_s \right\rangle_H = \int_0^T \left\langle e^{(T-\lfloor s\rfloor_h)A}\, b, B\, dW_s\right\rangle_H \\ &= \int_0^T e^{-|\lambda_b|(T-\lfloor s\rfloor_h)} \langle b, B\, dW_s\rangle_H = \mu_b \int_0^T e^{-|\lambda_b|(T-\lfloor s\rfloor_h)} \langle b, dW_s\rangle_H . \end{aligned} \tag{200}$$

This shows that $\mathbb{E}\big[|\langle b, X\rangle_H|^2 + |\langle b, Y_1\rangle_H|^2\big] < \infty$. It thus remains to prove (199). For this we combine (200), Itô's isometry, the fact that $\forall\, s \in [0,T] \colon T - \lfloor T - s\rfloor_h = \lceil s\rceil_h$, and, e.g., Lemma 6.1 in [11] to obtain that

$$\begin{aligned} &\mathrm{Var}\big(\langle b, X\rangle_H\big) - \mathrm{Var}\big(\langle b, Y_1\rangle_H\big) = |\mu_b|^2 \int_0^T e^{-2|\lambda_b|s} - e^{-2|\lambda_b|\lceil s\rceil_h}\, ds \\ &= |\mu_b|^2 \int_0^T e^{-2|\lambda_b|\lceil s\rceil_h} \left(e^{2|\lambda_b|(\lceil s\rceil_h - s)} - 1\right) ds = |\mu_b|^2 \left(\sum_{k=1}^{T/h} e^{-2|\lambda_b|kh}\right) \int_0^h \left(e^{2|\lambda_b|s} - 1\right) ds \\ &= |\mu_b|^2 \left(\frac{1 - e^{-2|\lambda_b|T}}{1 - e^{-2|\lambda_b|h}}\right) \int_0^h \left(e^{-2|\lambda_b|s} - e^{-2|\lambda_b|h}\right) dt. \end{aligned} \tag{201}$$

Moreover, note that

$$\int_0^h \left(e^{-2|\lambda_b|s} - e^{-2|\lambda_b|h}\right) ds \geq \int_0^{h/2} \left(e^{-2|\lambda_b|s} - e^{-2|\lambda_b|h}\right) ds \geq \frac{h}{2} \left(e^{-|\lambda_b|h} - e^{-2|\lambda_b|h}\right). \tag{202}$$

Combining (201) and (202) yields that

$$\mathrm{Var}\big(\langle b, X\rangle_H\big) - \mathrm{Var}\big(\langle b, Y_1\rangle_H\big) \geq \frac{|\mu_b|^2\, e^{-|\lambda_b|h}\left(1 - e^{-2|\lambda_b|T}\right)}{2} \left(\frac{1 - e^{-|\lambda_b|h}}{1 - e^{-2|\lambda_b|h}}\right) h. \tag{203}$$

This and the fact that $\forall\, x \in [0,1)\colon {}^{(1-x)}/_{(1-x^2)} = {}^{1}/_{(1+x)} \geq {}^{1}/_{2}$ finish the proof of Lemma 9.1. ☐

Lemma 9.2 (Variance estimates for linear-implicit Euler approximations)**.** *Assume the setting in Section 9.1 and let $b \in \mathbb{H}$. Then it holds that $\mathbb{E}\big[|\langle b, X\rangle_H|^2 + |\langle b, Y_2\rangle_H|^2\big] < \infty$ and*

$$\mathrm{Var}\big(\langle b, X\rangle_H\big) - \mathrm{Var}\big(\langle b, Y_2\rangle_H\big) \geq \frac{|\mu_b|^2\left(1 - e^{-2|\lambda_b|T}\right) h}{4\,(1 + h|\lambda_b|)} \geq \frac{|\mu_b|^2\left(1 - e^{-2|\lambda_b|T}\right) h}{4\, e^{|\lambda_b|h}} \geq 0. \tag{204}$$

Proof. First, we observe that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}
&\langle b, Y_2\rangle_H \\
&= \left\langle b, \int_0^T \left(\mathrm{Id}_H - hA\right)^{-(T - \lfloor s\rfloor_h)/h} B\, dW_s\right\rangle_H = \int_0^T \left\langle \left(\mathrm{Id}_H - hA\right)^{-(T - \lfloor s\rfloor_h)/h} b, B\, dW_s\right\rangle_H \\
&= \int_0^T (1 + h|\lambda_b|)^{-(T - \lfloor s\rfloor_h)/h} \langle b, B\, dW_s\rangle_H = \mu_b \int_0^T (1 + h|\lambda_b|)^{-(T - \lfloor s\rfloor_h)/h} \langle b, dW_s\rangle_H.
\end{aligned} \tag{205}$$

This shows that $\mathbb{E}\big[|\langle b, X\rangle_H|^2 + |\langle b, Y_2\rangle_H|^2\big] < \infty$. It thus remains to prove (204). For this we combine (205), Itô's isometry, the fact that $\forall\, s \in [0,T]\colon T - \lfloor T - s\rfloor_h = \lceil s\rceil_h$, and, e.g., Lemma 6.1 in [11] to obtain that

$$\begin{aligned}
&\mathrm{Var}\big(\langle b, X\rangle_H\big) - \mathrm{Var}\big(\langle b, Y_2\rangle_H\big) = |\mu_b|^2 \left[\frac{1 - e^{-2|\lambda_b|T}}{2|\lambda_b|} - \int_0^T (1 + h|\lambda_b|)^{-2\lceil s\rceil_h/h}\, ds\right] \\
&= |\mu_b|^2 \left[\frac{1 - e^{-2|\lambda_b|T}}{2\,|\lambda_b|} - h \sum_{k=1}^{T/h} (1 + h|\lambda_b|)^{-2k}\right] = |\mu_b|^2 \left[\frac{1 - e^{-2|\lambda_b|T}}{2\,|\lambda_b|} - \frac{\left[1 - (1 + h|\lambda_b|)^{-2T/h}\right]}{|\lambda_b|\,(2 + h|\lambda_b|)}\right].
\end{aligned} \tag{206}$$

The fact that $\forall\, x \in [0,\infty)\colon (1 + x)^{-1} \geq e^{-x}$ hence yields that

$$\begin{aligned}
\mathrm{Var}\big(\langle b, X\rangle_H\big) - \mathrm{Var}\big(\langle b, Y_2\rangle_H\big) &\geq |\mu_b|^2\left(1 - e^{-2|\lambda_b|T}\right)\left[\frac{1}{2\,|\lambda_b|} - \frac{1}{|\lambda_b|\,(2 + h|\lambda_b|)}\right] \\
&= \frac{|\mu_b|^2\left(1 - e^{-2|\lambda_b|T}\right) h}{2\,(2 + h|\lambda_b|)}.
\end{aligned} \tag{207}$$

This implies (204). The proof of Lemma 9.2 is thus completed. ☐

9.3 Lower bounds for the squared norm as the test function

Proposition 9.3. *Assume the setting in Section 9.1, let $b\colon \mathbb{N} \to \mathbb{H}$ be a bijective function, and let $c, \rho \in (0,\infty)$, $\delta \in \mathbb{R}$, $i \in \{1,2\}$ satisfy that for all $n \in \mathbb{N}$ it holds that $\lambda_{b_n} = -c\, n^\rho$ and $\mu_{b_n} = |\lambda_{b_n}|^\delta$. Then it holds that $B \in \cap_{r \in (-\infty, -\frac{1}{2}[1/\rho + 2\delta])} HS(H, H_r)$ and*

$$\mathbb{E}\big[\|X\|_H^2\big] - \mathbb{E}\big[\|Y_i\|_H^2\big] \geq \frac{\left(1 - e^{-2cT}\right)\left(1 - e^{-1}\right) T^{(1/\rho + 2\delta)^+}\, c^{2\delta}\, h^{(1 - [1/\rho + 2\delta]^+)}}{4^{(1 + \rho\delta^-)}\, e^{2^\rho e c T}\, (\rho + (1 + 2\rho\delta)^-)} > 0. \tag{208}$$

Proof. First of all, we observe that for all $r \in (-\infty, -\frac{1}{2}[1/\rho + 2\delta])$ it holds that $2\rho(r+\delta) < -1$ and

$$\|B\|_{HS(H,H_r)} = \sum_{n=1}^{\infty} |\mu_{b_n}|^2 \, |\lambda_{b_n}|^{2r} = \sum_{n=1}^{\infty} |\lambda_{b_n}|^{2(r+\delta)} = \sum_{n=1}^{\infty} c^{2(r+\delta)} \, n^{2\rho(r+\delta)} < \infty. \tag{209}$$

This proves that $B \in \cap_{r\in(-\infty,-\frac{1}{2}[1/\rho+2\delta])} HS(H, H_r)$. It thus remains to prove (208). For this we note that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^{2\rho\delta}}{e^{cn^\rho h}} &= \sum_{n=0}^{\infty} \int_n^{n+1} \frac{(n+1)^{2\rho\delta}}{e^{c(n+1)^\rho h}} \, dx \geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{(n+1)^{2\rho\delta}}{e^{c(n+1)^\rho h}} \, dx \\ &\geq \sum_{n=1}^{\infty} \int_n^{n+1} \frac{2^{-2\rho\delta^-} \, x^{2\rho\delta}}{e^{2^\rho c x^\rho h}} \, dx = \frac{1}{4^{\rho\delta^-}} \int_1^{\infty} \frac{x^{2\rho\delta}}{e^{2^\rho c x^\rho h}} \, dx. \end{aligned} \tag{210}$$

Combining (210) with Lemma 9.1 and Lemma 9.2 ensures that

$$\begin{aligned} &\mathbb{E}\big[\|X\|_H^2\big] - \mathbb{E}\big[\|Y_i\|_H^2\big] = \sum_{n=1}^{\infty} \Big[\operatorname{Var}\big(\langle b_n, X\rangle_H\big) - \operatorname{Var}\big(\langle b_n, Y_i\rangle_H\big)\Big] \\ &\geq \frac{c^{2\delta}\left(1 - e^{-2cT}\right) h}{4} \sum_{n=1}^{\infty} \frac{n^{2\rho\delta}}{e^{cn^\rho h}} \geq \frac{c^{2\delta}\left(1 - e^{-2cT}\right) h}{4^{(1+\rho\delta^-)}} \int_1^{\infty} \frac{x^{2\rho\delta}}{e^{2^\rho c x^\rho h}} \, dx \\ &= \frac{\left(1 - e^{-2cT}\right) h^{(1-2\delta-1/\rho)}}{2^{(3+2\rho\delta^+)} \, \rho \, c^{1/\rho}} \int_{2^\rho ch}^{\infty} \frac{x^{(1/\rho+2\delta-1)}}{e^x} \, dx \\ &\geq \frac{\left(1 - e^{-2cT}\right) h^{(1-2\delta-1/\rho)}}{2^{(3+2\rho\delta^+)} \, \rho \, c^{1/\rho}} \max\left\{ \int_{2^\rho ch}^{2^\rho ech} \frac{x^{(1/\rho+2\delta-1)}}{e^x} \, dx, \int_{2^\rho cT}^{2^\rho ecT} \frac{x^{(1/\rho+2\delta-1)}}{e^x} \, dx \right\} \\ &\geq \frac{\left(1 - e^{-2cT}\right) h^{(1-2\delta-1/\rho)}}{2^{(3+2\rho\delta^+)} \, e^{2^\rho ecT} \, \rho \, c^{1/\rho}} \max\left\{ \int_{2^\rho ch}^{2^\rho ech} x^{(1/\rho+2\delta-1)} \, dx, \int_{2^\rho cT}^{2^\rho ecT} x^{(1/\rho+2\delta-1)} \, dx \right\}. \end{aligned} \tag{211}$$

Next we observe that the fact that $\forall\, x \in (0,\infty)\colon \frac{(e^x-1)}{x} \geq 1$ implies that for all $r, q \in (0,\infty)$ it holds that

$$\int_r^{er} x^{-1} \, dx = 1, \qquad \int_r^{er} x^{(q-1)} \, dx = \frac{r^q \left(e^q - 1\right)}{q} \geq r^q, \qquad \int_r^{er} x^{(-q-1)} \, dx = \frac{r^{-q}\left(1 - e^{-q}\right)}{q}. \tag{212}$$

This and the fact that $\forall\, x \in (0,\infty)\colon \frac{(1-e^{-(1+x)})}{(1+x)} \leq \frac{(1-e^{-x})}{x} \leq 1$ ensure that for all $r \in (0,\infty)$, $q \in \mathbb{R}$ it holds that

$$\int_r^{er} x^{(q-1)} \, dx \geq \frac{r^q \left(1 - e^{-1-q^-}\right)}{(1+q^-)}. \tag{213}$$

In the next step we combine (211) and (213) to obtain that

$$\begin{aligned} &\mathbb{E}\big[\|X\|_H^2\big] - \mathbb{E}\big[\|Y_i\|_H^2\big] \\ &\geq \frac{\left(1 - e^{-2cT}\right)\left(1 - e^{-1-(1/\rho+2\delta)^-}\right) h^{(1-2\delta-1/\rho)}}{2^{(3+2\rho\delta^+)} \, e^{2^\rho ecT} \, \rho \, c^{1/\rho} \left(1 + (1/\rho + 2\delta)^-\right)} \max\left\{ \left[2^\rho ch\right]^{(1/\rho+2\delta)}, \left[2^\rho cT\right]^{(1/\rho+2\delta)} \right\} \\ &\geq \frac{\left(1 - e^{-2cT}\right)\left(1 - e^{-1-(1/\rho+2\delta)^-}\right) c^{2\delta} \, h^{(1-2\delta-1/\rho)}}{4^{(1+\rho\delta^-)} \, e^{2^\rho ecT} \left(\rho + (1+2\rho\delta)^-\right)} \max\{h^{(1/\rho+2\delta)}, T^{(1/\rho+2\delta)}\}. \end{aligned} \tag{214}$$

This together with the fact that $\max\{h^{(1/\rho+2\delta)}, T^{(1/\rho+2\delta)}\} = T^{(1/\rho+2\delta)^+} \, h^{-(1/\rho+2\delta)^-}$ implies (208). The proof of Proposition 9.3 is thus completed. ☐

The next result, Corollary 9.4, specialises Proposition 9.3 to the case where $c = \pi^2$ and $\rho = 2$ (Laplacian with Dirichlet boundary conditions on $(0,1)$) and slightly further estimates the right hand side of (208).

Corollary 9.4. *Assume the setting in Section 9.1, let $b\colon \mathbb{N} \to \mathbb{H}$ be a bijective function, and let $\delta \in \mathbb{R}$, $i \in \{1, 2\}$ satisfy that for all $n \in \mathbb{N}$ it holds that $\lambda_{b_n} = -\pi^2 n^2$ and $\mu_{b_n} = |\lambda_{b_n}|^{\delta}$. Then it holds that $B \in \cap_{r \in (-\infty, -\delta - 1/4)} HS(H, H_r)$ and*

$$\mathbb{E}\big[\|X\|_H^2\big] - \mathbb{E}\big[\|Y_i\|_H^2\big] \geq \left[\frac{\left(1-e^{-T}\right)\left(1-e^{-1}\right) T^{(1/2+2\delta)^+} \pi^{4\delta}}{4^{(1+2\delta^-)}\, e^{12\pi^2 T}\, (3+4\delta^-)}\right] h^{\min\{1/2 - 2\delta, 1\}} > 0. \tag{215}$$

9.4 Lower bounds for a specific regular test function

Lemma 9.5. *Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be a separable $\mathbb{R}$-Hilbert space with $H \neq \{0\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathbb{H} \subseteq H$ be an orthonormal basis of H, let $\varphi \in \mathbb{M}(H, \mathbb{R})$ satisfy that for all $v \in H$ it holds that $\varphi(v) = \exp(-\|v\|_H^2)$, and let $X, Y \in L^2(\mathbb{P}; H)$ be such that $\langle b, X\rangle_H$, $b \in \mathbb{H}$, is a family of independent centered Gaussian random variables, such that $\langle b, Y\rangle_H$, $b \in \mathbb{H}$, is a family of independent centered Gaussian random variables, and such that for all $b \in \mathbb{H}$ it holds that $\operatorname{Var}(\langle b, X\rangle_H) \geq \operatorname{Var}(\langle b, Y\rangle_H)$. Then it holds that $\varphi \in C_b^5(H, \mathbb{R})$ and*

$$\mathbb{E}\big[\varphi(Y)\big] - \mathbb{E}\big[\varphi(X)\big] \geq \frac{\big(\mathbb{E}[\|X\|_H^2] - \mathbb{E}[\|Y\|_H^2]\big)}{\exp\big(6\,\mathbb{E}[\|X\|_H^2]\big)}. \tag{216}$$

Proof. First of all, we observe that it is well-known that $\varphi \in C_b^5(H, \mathbb{R})$ (see, e.g., (97)–(102) in [11]). Next we assume w.l.o.g. that $\mathbb{H}$ is a finite set ((216) in the case where $\mathbb{H}$ is an infinite set follows from Lebesgue's dominated convergence theorem and (216) in the case where $\mathbb{H}$ is a finite set). Next we note that the assumption that $\langle b, X\rangle_H$, $b \in \mathbb{H}$, is a family of independent centered Gaussian random variables, the assumption that $\langle b, Y\rangle_H$, $b \in \mathbb{H}$, is a family of independent centered Gaussian random variables, and, e.g., (103) in Conus et al. [11] imply that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(Y)\big] - \mathbb{E}\big[\varphi(X)\big] \\
&= \prod_{b\in\mathbb{H}} \big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]^{-1/2} - \prod_{b\in\mathbb{H}} \big[1 + 2\operatorname{Var}\big(\langle b, X\rangle_H\big)\big]^{-1/2} \\
&= \left(\prod_{b\in\mathbb{H}} \big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]^{-1/2}\right)\left(1 - \left[\prod_{b\in\mathbb{H}} \frac{\big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}{\big[1 + 2\operatorname{Var}\big(\langle b, X\rangle_H\big)\big]}\right]^{1/2}\right) \\
&= \mathbb{E}\big[\varphi(Y)\big]\left(1 - \left[\prod_{b\in\mathbb{H}} \frac{\big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}{\big[1 + 2\operatorname{Var}\big(\langle b, X\rangle_H\big)\big]}\right]^{1/2}\right).
\end{aligned} \tag{217}$$

Moreover, Jensen's inequality shows that

$$\mathbb{E}\big[\varphi(Y)\big] \geq \exp\big(-\mathbb{E}[\|Y\|_H^2]\big) \geq \exp\big(-\mathbb{E}[\|X\|_H^2]\big)\,. \tag{218}$$

Next we observe that the facts that $\forall\, b \in \mathbb{H}\colon 2\left[\operatorname{Var}(\langle b, X\rangle_H) - \operatorname{Var}(\langle b, Y\rangle_H)\right] \geq 0$ and $\forall\, n \in \mathbb{N}\colon \forall\, x_1, \ldots, x_n \in [0, \infty)\colon \prod_{k=1}^n [1 + x_k] \geq 1 + \sum_{k=1}^n x_k$ prove that

$$\begin{aligned}
&1 - \left[\prod_{b\in\mathbb{H}} \frac{\big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}{\big[1 + 2\operatorname{Var}\big(\langle b, X\rangle_H\big)\big]}\right]^{1/2} = 1 - \left[\prod_{b\in\mathbb{H}} \frac{\big[1 + 2\operatorname{Var}\big(\langle b, X\rangle_H\big)\big]}{\big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]^{-1/2} \\
&= 1 - \left[\prod_{b\in\mathbb{H}} \left[1 + \frac{2\left[\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)\right]}{\big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]\right]^{-1/2} \\
&\geq 1 - \left[1 + \sum_{b\in\mathbb{H}} \frac{2\left[\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)\right]}{\big[1 + 2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]^{-1/2}.
\end{aligned} \tag{219}$$

In addition, we note that the fundamental theorem of calculus ensures that $\forall\, x \in [0,\infty)\colon 1 - [1+x]^{-1/2} = \frac{1}{2}\int_0^x [1+y]^{-3/2}\,dy \geq \frac{1}{2}x\,[1+x]^{-3/2}$. Hence, we obtain that $\forall\, x \in [0,\infty)\colon 1 - [1+2x]^{-1/2} \geq x\,[1+2x]^{-3/2}$. Combinig this with (219) implies that

$$\begin{aligned}
&1 - \left[\prod_{b\in\mathbb{H}} \frac{\big[1+2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}{\big[1+2\operatorname{Var}\big(\langle b, X\rangle_H\big)\big]}\right]^{1/2} \\
&\geq 1 - \left[1 + 2\sum_{b\in\mathbb{H}} \frac{\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)}{\big[1+2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]^{-1/2} \\
&\geq \left[\sum_{b\in\mathbb{H}} \frac{\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)}{\big[1+2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]\left[1 + 2\sum_{b\in\mathbb{H}} \frac{\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)}{\big[1+2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]^{-3/2}.
\end{aligned} \tag{220}$$

In the next step we combine (217), (218), and (220) with the fact that $\forall\, b \in \mathbb{H}\colon \operatorname{Var}\big(\langle b, Y\rangle_H\big) \leq \operatorname{Var}\big(\langle b, X\rangle_H\big) \leq \mathbb{E}\big[\|X\|_H^2\big]$ to obtain that

$$\begin{aligned}
\mathbb{E}\big[\varphi(Y)\big] - \mathbb{E}\big[\varphi(X)\big] &\geq \exp\!\big(-\mathbb{E}\big[\|X\|_H^2\big]\big)\left[\sum_{b\in\mathbb{H}} \frac{\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)}{\big[1+2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right] \\
&\quad \cdot \left[1 + 2\sum_{b\in\mathbb{H}} \frac{\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)}{\big[1+2\operatorname{Var}\big(\langle b, Y\rangle_H\big)\big]}\right]^{-3/2} \\
&\geq \exp\!\big(-\mathbb{E}\big[\|X\|_H^2\big]\big)\left[\sum_{b\in\mathbb{H}} \frac{\operatorname{Var}\big(\langle b, X\rangle_H\big) - \operatorname{Var}\big(\langle b, Y\rangle_H\big)}{\big[1+2\,\mathbb{E}\big[\|X\|_H^2\big]\big]}\right] \\
&\quad \cdot \left[1 + 2\sum_{b\in\mathbb{H}} \operatorname{Var}\big(\langle b, X\rangle_H\big)\right]^{-3/2} \\
&= \exp\!\big(-\mathbb{E}\big[\|X\|_H^2\big]\big)\,\big[1+2\,\mathbb{E}\big[\|X\|_H^2\big]\big]^{-5/2}\Big(\mathbb{E}\big[\|X\|_H^2\big] - \mathbb{E}\big[\|Y\|_H^2\big]\Big).
\end{aligned} \tag{221}$$

Combining (221) with the fact that $\forall\, x \in [0,\infty)\colon (1+x)^{-1} \geq e^{-x}$ implies (216). This completes the proof of Lemma 9.5. ☐

The next result, Corollary 9.6, is a direct consequence of Lemma 9.1, of Lemma 9.2, and of Lemma 9.5.

Corollary 9.6. *Assume the setting in Section 9.1 and let* $i \in \{1,2\}$, $\varphi \in \mathbb{M}(H,\mathbb{R})$ *satisfy that for all* $v \in H$ *it holds that* $\varphi(v) = \exp(-\|v\|_H^2)$. *Then*

$$\mathbb{E}\big[\varphi(Y_i)\big] - \mathbb{E}\big[\varphi(X)\big] \geq e^{-6\,\mathbb{E}[\|X\|_H^2]}\big(\mathbb{E}\big[\|X\|_H^2\big] - \mathbb{E}\big[\|Y_i\|_H^2\big]\big)\,. \tag{222}$$

The next result, Proposition 9.7, is an immediate consequence of Proposition 9.3 and of Corollary 9.6.

Proposition 9.7. *Assume the setting in Section 9.1, let* $b\colon \mathbb{N} \to \mathbb{H}$ *be a bijective function, let* $i \in \{1,2\}$, $c, \rho \in (0,\infty)$, $\delta \in \mathbb{R}$ *satisfy that for all* $n \in \mathbb{N}$ *it holds that* $\lambda_{b_n} = -c\,n^\rho$ *and* $\mu_{b_n} = |\lambda_{b_n}|^\delta$, *and let* $\varphi \in \mathbb{M}(H,\mathbb{R})$ *satisfy that for all* $v \in H$ *it holds that* $\varphi(v) = \exp(-\|v\|_H^2)$. *Then it holds that* $\varphi \in C_b^5(H,\mathbb{R})$, $B \in \cap_{r\in(-\infty,-\frac{1}{2}[1/\rho+2\delta])} HS(H,H_r)$ *and*

$$\mathbb{E}\big[\varphi(Y_i)\big] - \mathbb{E}\big[\varphi(X)\big] \geq \left[\frac{\big(1-e^{-2cT}\big)\big(1-e^{-1}\big)\,T^{(1/\rho+2\delta)^+}\,c^{2\delta}}{4^{(1+\rho\delta^-)}\,\exp\!\big(2^\rho ecT + 6\,\mathbb{E}[\|X\|_H^2]\big)\,(\rho + (1+2\rho\delta)^-)}\right] h^{(1-[1/\rho+2\delta]^+)} > 0. \tag{223}$$

In the next result, Corollary 9.8, we specialise Proposition 9.7 to the case where $c = \pi^2$ and $\rho = 2$ (Laplacian with Dirichlet boundary conditions on $(0,1)$) and slightly further estimates the right hand side of (223).

Corollary 9.8. *Assume the setting in Section 9.1, let $b\colon \mathbb{N} \to \mathbb{H}$ be a bijective function, let $\delta \in \mathbb{R}$, $i \in \{1,2\}$, assume that for all $n \in \mathbb{N}$ it holds that $\lambda_{b_n} = -\pi^2 n^2$ and $\mu_{b_n} = |\lambda_{b_n}|^\delta$, and let $\varphi \in \mathbb{M}(H,\mathbb{R})$ satisfy that for all $v \in H$ it holds that $\varphi(v) = \exp(-\|v\|_H^2)$. Then it holds that $\varphi \in C_b^5(H,\mathbb{R})$, $B \in \cap_{r\in(-\infty,-\delta-1/4)} HS(H,H_r)$ and*

$$\mathbb{E}\big[\varphi(Y_i)\big] - \mathbb{E}\big[\varphi(X)\big] \geq \left[\frac{\big(1-e^{-T}\big)\big(1-e^{-1}\big)\,T^{(1/2+2\delta)^+}\,\pi^{4\delta}}{4^{(1+2\delta^-)}\,(3+4\delta^-)\,\exp\!\big(12\pi^2 T + 6\,\mathbb{E}[\|X\|_H^2]\big)}\right] h^{\min\{1/2-2\delta,1\}} > 0. \tag{224}$$

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