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Research Report No. 2015-29
October 2015

Seminar für Angewandte Mathematik
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CONTINUOUS AND DISCRETE FRAMES GENERATED BY THE EVOLUTION FLOW OF THE SCHRÖDINGER EQUATION

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ABSTRACT. We study a family of coherent states, called Schrödingerlets, both in the continuous and discrete setting. They are defined in terms of the Schrödinger equation of a free quantum particle and some of its invariant transformations.

1. INTRODUCTION

In Quantum Mechanics, the time evolution of a d -dimensional free particle is described by the Schrödinger equation

$$(1.1) \quad \begin{cases} i \frac{\partial}{\partial t} f(x, t) = -\frac{1}{2\pi} \Delta f(x, t) \\ f(\cdot, 0) = f_0, \end{cases}$$

where Δ is the Laplace operator acting on the “space” variable $x \in \mathbb{R}^d$, and f_0 is a square-integrable function on \mathbb{R}^d describing the state of the quantum particle at time zero (for the sake of simplicity, the mass is normalised so that the Laplacian has the simple factor $1/2\pi$).

The aim of this paper is to introduce a new family of coherent states (*i.e.* a frame) generated by the time evolution unitary operator defined by the Schrödinger equation; following [1, 2, 15], its elements are called *Schrödingerlets*.

Clearly, the time evolution operator $e^{i\frac{t}{2\pi}\Delta}$ is not enough to generate a frame for $L^2(\mathbb{R}^d)$, hence we need to add other unitary transformations. Observe that equation (1.1) is invariant both with respect to the rotations $R \in SO(d)$, under the canonical action

$$f(x, t) \mapsto f(Rx, t),$$

and with respect to the dilations $a \in \mathbb{R}_+$, under the parabolic action

$$f(x, t) \mapsto a^{\frac{d}{4}} f(\sqrt{a}x, at),$$

where the factor $a^{\frac{d}{4}}$ ensures that the L^2 -norm of $f(\cdot, t)$ is preserved. Thus, it is natural to consider the group $G = (\mathbb{R} \times \mathbb{R}_+) \times SO(d)$, *i.e.* the direct product of

Date: October 15, 2015.

2010 *Mathematics Subject Classification.* 22D10, 42C40, 42C15.

G. S. Alberti was supported by the ERC Advanced Grant Project MULTIMOD-267184.

S. Dahlke was supported by Deutsche Forschungsgemeinschaft (DFG), Grant DA 360/19–1.

F. De Mari and E. De Vito were partially supported by Progetto PRIN 2010-2011 “Varietà reali e complesse: geometria, topologia e analisi armonica”. They are members of the Gruppo Nazionale per l’Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

the identity component of the one-dimensional affine group and $\mathrm{SO}(d)$, and the corresponding unitary representation π acting on $L^2(\mathbb{R}^d)$ as

$$(1.2) \quad \pi(t, a, R)f = a^{-\frac{d}{4}} e^{i\frac{t}{2\pi}\Delta} f_{a,R},$$

where $f_{a,R}(x) = f(a^{-\frac{1}{2}}R^{-1}x)$. It follows that the solution of (1.1) is given by

$$f(x, t) = \pi(t, 1, \mathbf{I})f_0(x),$$

and for any rotation $R \in \mathrm{SO}(d)$

$$f(Rx, t) = \pi(t, 1, R^{-1})f_0(x),$$

whereas for any dilation $a \in \mathbb{R}_+$

$$a^{\frac{d}{4}} f(\sqrt{a}x, at) = \pi(t, a^{-1}, \mathbf{I})f_0(x).$$

Our goal is to study the properties of the corresponding family of coherent states $\{\pi(x)\eta\}_{x \in G}$ where η is a suitable ‘‘ground state’’, *i.e.* an admissible vector. In the context of signal analysis, this amounts to analyze the *voice transform*

$$f \mapsto \langle f, \pi(\cdot)\eta \rangle$$

as a map from $L^2(\mathbb{R}^d)$ into a suitable Banach space of functions on G . We restrict ourselves to the L^2 -framework, both in the continuous and in the discrete setting. Our main contribution is twofold. First, we show that π is a reproducing representation of G and we characterize its admissible vectors. This result was already known for $d = 2$ [15], and here we extend the proof to arbitrary d . Furthermore, we construct a discrete Parseval frame of the form $\{\pi(x_i)\eta\}_{i \in I}$, where $\{x_i\}_{i \in I}$ is a suitable sampling of G .

In Section 2 we introduce the Schrödingerlets in two dimensions and we discuss the construction of a Parseval frame of two-dimensional Schrödingerlets. The purpose of this dimensionality restriction is twofold. Firstly, it allows to present the main ideas of this work in a simpler way, so that it may serve as a good introduction to the more involved general setting. Secondly, the two-dimensional case is somehow different from the higher dimensional cases, since when $d = 2$ the spherical harmonics on S^{d-1} correspond to the standard Fourier series; thus, a separate presentation allows to underline the peculiarities of the case $d = 2$.

Section 3 is devoted to studying the Schrödingerlets in any dimension. Proposition 3.3 shows that π is a reproducing representation and characterizes its admissible vectors. As a consequence, the Schrödingerlet voice transform permits to represent the quantum states as continuous functions on the parameter space $\mathbb{R} \times \mathbb{R}_+ \times \mathrm{SO}(d)$. Time evolution and rotations correspond to translations in the first and third variable, respectively, whereas dilations give rise to a multi-scale analysis of the original quantum state.

The main result of the paper is Theorem 3.4, which provides sufficient conditions in order to have a Parseval discrete frame.

We refer to [3, 19] for a general introduction to coherent states and reproducing formulæ associated with unitary representations. Schrödingerlets in dimension two were first introduced in [15] and further discussed in [1, 2], where G is regarded as a closed subgroup of the symplectic group and π is equivalent to the restriction to G of the metaplectic representation, whose role in signal analysis has been investigated in a series of papers [9, 10, 11, 12, 23]. We remark that the representation π is reducible and its reproducing kernel is not integrable. Hence, we cannot directly

apply the classical theory of square-integrable representations by Duflo and Moore [16], nor the coorbit space theory developed by Feichtinger and Gröchenig [17, 18].

Another construction based on the covariance properties of a free quantum particle is given by the coherent states associated to the isochronous Galilei group (see [3, Chapter 8.4.2] and references therein). However, in this case, the dilations are not present and the frame does not depend on the time parameter. Indeed, in order to make the representation square-integrable it is necessary to reduce the Galilei group by taking the quotient modulo a group that contains the time translations.

The proof that π is a reproducing representation is based on the general theory developed in [15]. However, since π is the direct sum of a countable family of square-integrable representations π_i . A general approach to obtain a discrete frame without assuming that the kernel is in $L^1(G)$ has been developed in [4, 5, 6, 20], but it requires the boundedness of a suitable convolution operator (see condition (R3) of [4]), which is hard to prove in our setting. We follow here a different approach. Taking into account that $\pi = \bigoplus_i \pi_i$, the discretization is achieved by a slight generalization of a well known result on discrete wavelet frames in $L^2(\mathbb{R})$ [22, Theorem 1.6, Chapter 7], by Schur's orthogonality relations for finite groups and a technical lemma about Parseval frames (Lemma 3.6). Comparing with the approach taken in [4], we are able to provide only Hilbert frames; we hope to extend our results in future work and succeed in describing Banach frames related to the function spaces introduced in [5, 6, 13].

2. THE MAIN RESULT IN TWO DIMENSIONS

We state here the main result of this paper particularized for two-dimensional signals. In the first part of the section we introduce the continuous Schrödingerlets following [13].

2.1. The continuous Schrödingerlets in 2D. For $d = 2$ by identifying the abelian group $\text{SO}(2)$ with the one dimensional torus $\mathcal{T} = \mathbb{R}/2\pi\mathbb{Z}$ as

$$\theta \longleftrightarrow R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

the group G is $(\mathbb{R} \times \mathbb{R}_+) \times \mathcal{T}$ and its elements are denote by (b, a, θ) , writing b instead of time variable t . In order to better visualize the action of π given by (1.2), it is worth rewriting it in an equivalent formulation by means of an intertwining operator S which we shall now define. We work in the Fourier domain with polar coordinates, and then perform a Fourier series with respect to the angular variable.

Below we write $\widehat{\mathbb{R}}^2$ for the dual space to \mathbb{R}^2 and dx and $d\xi$ denote the corresponding Lebesgue measures, whereas $d\theta$ is the Riemannian measure of \mathcal{T} (so that $\int_{\mathcal{T}} d\theta = 2\pi$). We let $\mathcal{F}: L^2(\mathbb{R}^2) \rightarrow L^2(\widehat{\mathbb{R}}^2)$ denote the Fourier transform given by

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^2} f(x)e^{-2\pi i x \cdot \xi} dx \quad \xi \in \widehat{\mathbb{R}}^2$$

whenever $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ and $x \cdot \xi$ is the Euclidean scalar product.

Define the unitary operator $J: L^2(\widehat{\mathbb{R}}^2) \rightarrow L^2(\widehat{\mathbb{R}}_+ \times \mathcal{T})$ by

$$J\hat{f}(\omega, \theta) = \hat{f}(\sqrt{\omega} \cos \theta, \sqrt{\omega} \sin \theta)/\sqrt{2} \quad \hat{f} \in L^2(\widehat{\mathbb{R}}^2), \omega \in \widehat{\mathbb{R}}_+, \theta \in \mathcal{T}.$$

The unitarily equivalent representation $(J\mathcal{F})\pi(J\mathcal{F})^{-1}$ acting on $L^2(\widehat{\mathbb{R}}_+ \times \mathcal{T})$ reads

$$(2.1) \quad (J\mathcal{F})\pi(J\mathcal{F})^{-1}(b, a, \phi)\hat{f}(\omega, \theta) = a^{1/2}e^{-2\pi i b\omega} \hat{f}(a\omega, \theta - \phi) \quad \omega \in \widehat{\mathbb{R}}_+, \theta \in \mathcal{T}$$

for all $(b, a, \phi) \in G$ and $\hat{f} \in L^2(\widehat{\mathbb{R}}_+ \times \mathcal{T})$. The action on the radial variable can be described by the representation of $\mathbb{R} \times \mathbb{R}_+$ on $L^2(\widehat{\mathbb{R}}_+)$ given by

$$(2.2) \quad \widehat{W}^+(b, a)g(\omega) = a^{1/2}e^{-2\pi i b \omega}g(a\omega) \quad \omega \in \widehat{\mathbb{R}}_+, (b, a) \in \mathbb{R} \times \mathbb{R}_+, g \in L^2(\widehat{\mathbb{R}}_+),$$

which is nothing else than the one-dimensional wavelet representation in the positive frequency domain. The action on the angular variable is simply given by a rotation $\rho(\phi)z(\theta) = z(\theta - \phi)$ for $z \in L^2(\mathcal{T})$. Therefore, the action of π on two-dimensional functions should be thought of as a classical one-dimensional wavelet representation on the radial component combined with rotations around the origin.

Consider now the Fourier series with respect to θ and define the unitary operator $S: L^2(\mathbb{R}^2) \rightarrow \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ by

$$(Sf)_n(\omega) = \int_0^{2\pi} (J\mathcal{F}f)(\omega, \theta)e^{-in\theta} \frac{d\theta}{\sqrt{2\pi}} \quad \omega \in \widehat{\mathbb{R}}_+, n \in \mathbb{Z}, f \in L^2(\mathbb{R}^2).$$

From now on, we shall consider the equivalent representation $\pi' = S\pi S^{-1}$ of G acting on $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$. In view of (2.1) and (2.2), the action of π' is given by

$$(\pi'(b, a, \phi)\hat{f})_n = e^{-in\phi} (\widehat{W}^+(b, a)\hat{f}_n) \quad n \in \mathbb{Z}, \hat{f} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+), (b, a, \phi) \in G.$$

Denoted by ρ_n the character $\phi \mapsto e^{-in\phi}$ of \mathcal{T} , the representation π' can be decomposed as

$$\pi' = \bigoplus_{n \in \mathbb{Z}} \rho_n \widehat{W}^+$$

where each component $\rho_n \widehat{W}^+$ acts irreducibly on $L^2(\widehat{\mathbb{R}}_+)$.

It was proven in [2, 13] that π' , and therefore π , is reproducing, namely

$$(2.3) \quad \|\hat{f}\|_{\bigoplus_n L^2(\widehat{\mathbb{R}}_+)}^2 = \int_G |\langle \pi'(b, a, \phi)\hat{\eta}, \hat{f} \rangle|^2 db \frac{da}{a^2} \frac{d\phi}{2\pi} \quad \hat{f} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$$

for some admissible vector $\hat{\eta} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$. A vector $\hat{\eta} = (\hat{\eta}_n)_n \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ is admissible for π' if and only if

$$(2.4) \quad \int_0^{+\infty} |\hat{\eta}_n(\omega)|^2 \frac{d\omega}{\omega} = 1 \quad n \in \mathbb{Z},$$

namely, if and only if each component $\hat{\eta}_n$ is a one-dimensional wavelet [14]. A simple way to construct admissible vectors in $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ satisfying (2.4) is to fix a one-dimensional wavelet $\hat{\eta}_0 \in L^2(\widehat{\mathbb{R}}_+)$ satisfying (2.4) and then construct all the other components $\hat{\eta}_n$ by dilating $\hat{\eta}_0$. Since (2.4) is invariant under positive dilations, it is immediately satisfied for all n . More precisely, set for all $n \in \mathbb{Z}$

$$(2.5) \quad \hat{\eta}_n(\omega) = \hat{\eta}_0(\alpha_n^{-1}\omega) \quad \omega \in \widehat{\mathbb{R}}_+,$$

for some weights $\alpha_n > 0$ that satisfy $\alpha_0 = 1$ and $\sum_n \alpha_n < \infty$. This last condition ensures that the resulting $\hat{\eta}$ has finite norm in $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$, because

$$\|\hat{\eta}\|^2 = \|\hat{\eta}_0\|_{L^2(\widehat{\mathbb{R}}_+)}^2 \sum_n \alpha_n.$$

2.2. The discrete Schrödingerlets in 2D. We now show how to construct a Parseval frame of $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ associated to π' . Then, by means of the intertwining operator S , this frame can be transformed into a Parseval frame of $L^2(\mathbb{R}^2)$ associated to π . Constructing a Parseval frame corresponds to a discretization of (2.3) of the form

$$\|\hat{f}\|_{\bigoplus_n L^2(\widehat{\mathbb{R}}_+)}^2 = \sum_{i \in \mathbb{N}} |\langle \pi'(x_i) \hat{\eta}, \hat{f} \rangle|^2 \quad \hat{f} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+),$$

for suitable choices of the admissible vector $\hat{\eta}$ and of a sampling $\{x_i\}_{i \in \mathbb{N}}$ of the group G .

Our approach is based on the fact that π' is the direct sum of one-dimensional wavelet representations. Thus, it is instructive to look first at the well known one-dimensional case, namely at the representation \widehat{W}^+ acting on $L^2(\widehat{\mathbb{R}}_+)$. Standard wavelet theory [22, Thm. 1.1, Chapter 7] gives that $\{\widehat{W}^+(2^j k, 2^j) \hat{\eta}_0 : k, j \in \mathbb{Z}\}$ is a Parseval frame for $L^2(\widehat{\mathbb{R}}_+)$, namely

$$\|\hat{f}\|_{L^2(\widehat{\mathbb{R}}_+)}^2 = \sum_{k, j \in \mathbb{Z}} |\langle \widehat{W}^+(2^j k, 2^j) \hat{\eta}_0, \hat{f} \rangle|^2 \quad \hat{f} \in L^2(\widehat{\mathbb{R}}_+),$$

provided that the conditions

$$(2.6a) \quad \sum_{j \in \mathbb{Z}} |\hat{\eta}_0(2^j \omega)|^2 = 1, \quad \text{for a.e. } \omega \in \widehat{\mathbb{R}}_+,$$

$$(2.6b) \quad \sum_{j \in \mathbb{N}} \hat{\eta}_0(2^j \omega) \overline{\hat{\eta}_0(2^j(\omega + 2\pi m))} = 0, \quad \text{for a.e. } \omega \in \widehat{\mathbb{R}}_+, m \in 2\mathbb{Z} + 1$$

hold true. Note that in this case the sampling of the group $\mathbb{R} \rtimes \mathbb{R}_+$ is the discrete set $\{(2^j k, 2^j) : k, j \in \mathbb{Z}\}$.

We now generalize this construction to the Schrödingerlets. In view of the above sampling of the affine group, it is natural to consider the discretization of G given by

$$\{x_{k,j,l} = (2^j k, 2^j, 2\pi l/L) : k, j \in \mathbb{Z}, l = 0, \dots, L-1\},$$

for some $L \in \mathbb{N}^*$, where $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Note that the angles $\phi_l = 2\pi l/L$ give a uniform sampling of \mathcal{T} and form a finite cyclic subgroup of order L . Let us now discuss suitable assumptions on the admissible vector $\hat{\eta}$, and therefore on $\hat{\eta}_0$ and the weights α_n in the case where $\hat{\eta}_n$ is given by (2.5), so that $\{\pi'(x_{k,j,l}) \hat{\eta} : k, j \in \mathbb{Z}, l = 0, \dots, L-1\}$ is a Parseval frame for $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$.

We first observe that for every $n \in \mathbb{Z}$ it is necessary that each $\hat{\eta}_n \in L^2(\widehat{\mathbb{R}}_+)$ give rise to a Parseval frame for the corresponding space $L^2(\widehat{\mathbb{R}}_+)$, *i.e.* that each $\hat{\eta}_n$ satisfies (2.6) (suitably normalized):

$$(2.7a) \quad \sum_{j \in \mathbb{Z}} |\hat{\eta}_n(2^j \omega)|^2 = 1/L, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+, n \in \mathbb{Z},$$

$$(2.7b) \quad \sum_{j \in \mathbb{N}} \hat{\eta}_n(2^j \omega) \overline{\hat{\eta}_n(2^j(\omega + 2\pi m))} = 0, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+, n \in \mathbb{Z}, m \in 2\mathbb{Z} + 1.$$

In the continuous setting, it is necessary and sufficient to assume that each $\hat{\eta}_n$ is a one-dimensional wavelet, *i.e.* that (2.4) holds true for every n , in order to have the

continuous reproducing formula (2.3). In the discrete case, however, assumptions (2.7) are not sufficient, and it is necessary to assume the following conditions:

(2.8a)

$$\sum_{j \in \mathbb{Z}} \widehat{\eta}_n(2^j \omega) \overline{\widehat{\eta}_{n+kL}(2^j \omega)} = 0, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+, n \in \mathbb{Z}, k \in \mathbb{Z}^*,$$

(2.8b)

$$\sum_{j \in \mathbb{N}} \widehat{\eta}_n(2^j \omega) \overline{\widehat{\eta}_{n+kL}(2^j(\omega + 2\pi m))} = 0, \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+, n \in \mathbb{Z}, k \in \mathbb{Z}^*, m \in 2\mathbb{Z} + 1,$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. When (2.5) holds true, the above expressions can be simplified into conditions involving $\widehat{\eta}_0$ and the weights α_n .

These orthogonality relations do not contain all the cross terms between $\widehat{\eta}_n$ and $\widehat{\eta}_m$ for $n \neq m$, but only those corresponding to the cases when $m - n \in L\mathbb{Z}$. The reason for this simplification can be explained as follows. Two characters ρ_n and ρ_m restricted to the finite subgroup $\{2\pi l/L : l = 0, \dots, L-1\}$ are equivalent if and only if $m - n \in L\mathbb{Z}$. As a consequence, all the cross terms corresponding to m and n for which $m - n \notin L\mathbb{Z}$ are zero by Schur orthogonality relations for finite groups.

The following theorem shows that the above conditions are also sufficient.

Theorem 2.1. *Let $\widehat{\eta} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ be such that (2.7) and (2.8) hold true, and take $L \in \mathbb{N}^*$. Then $\{\pi'(x_{k,j,l})\widehat{\eta} : k, j \in \mathbb{Z}, l = 0, \dots, L-1\}$ is a Parseval frame for $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$, namely*

$$\|\widehat{f}\|_{\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)}^2 = \sum_{k,j,l} |\langle \pi'(x_{k,j,l})\widehat{\eta}, \widehat{f} \rangle|^2 \quad \widehat{f} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+).$$

The proof will be given below since the above result is a special instance of our main result, see Theorem 3.4. We just exhibit functions $\widehat{\eta}$ satisfying the assumptions. Take $\widehat{\eta}_0 \in L^2(\widehat{\mathbb{R}}_+)$ such that (2.7a) is satisfied for $n = 0$ and such that $\text{supp } \widehat{\eta}_0 \subseteq [0, 2\pi]$. Moreover, choose weights $\alpha_n \in (0, 1]$ such that $\alpha_0 = 1$, $\sum_n \alpha_n < \infty$ and

$$(2.9) \quad |\text{supp}(\widehat{\eta}_0) \cap \alpha_n^{-1} \alpha_{n+kL} \text{supp}(\widehat{\eta}_0)| = 0 \quad n \in \mathbb{Z}, k \in \mathbb{Z}^*,$$

where $|\cdot|$ denotes Lebesgue measure. It is easy to see that the admissible vector $\widehat{\eta} \in \bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$ defined by (2.5) satisfies (2.7) and (2.8). A simple choice valid for any L is $\widehat{\eta}_0 = L^{-1} \chi_{[1/2, 1]}$ and

$$\alpha_n = \begin{cases} 2^{-2n} & \text{if } n \geq 0 \\ 2^{2n+1} & \text{if } n < 0. \end{cases}$$

We now comment on the role of the number of rotations L . The conclusion of Theorem 2.1 still holds true when $L = 1$, namely when no rotations are considered. However, the rotations do play a role in the choice of the admissible vector $\widehat{\eta}$. Indeed, condition (2.8), or (2.9) in the case when (2.5) holds true, becomes weaker as L increases. More precisely, if L_2 is a multiple of L_1 and $\widehat{\eta}$ satisfies (2.8) with $L = L_1$, then the same equalities hold true with $L = L_2$. Note that this is equivalent to saying that the two corresponding discrete subgroups of \mathcal{T} are one contained into the other.

Note that for $L = 1$ a simple computation shows that $\|\widehat{\eta}\| = 1$, hence the frame obtained in Theorem 2.1 is in fact an orthonormal basis of $\bigoplus_{n \in \mathbb{Z}} L^2(\widehat{\mathbb{R}}_+)$. Indeed,

it is a standard general fact that a tight frame whose elements have norm (greater than or equal to) one is necessarily an orthonormal basis (see e.g. [22, Theorem 1.8, Ch. 7]).

3. THE d -DIMENSIONAL CASE

3.1. The continuous setting. We define $G = (\mathbb{R} \times \mathbb{R}_+) \times \text{SO}(d)$ as the direct product of the identity component of the one-dimensional affine group and $\text{SO}(d)$. Clearly, the set

$$H = \{(0, a, R) \mid a \in \mathbb{R}_+, R \in \text{SO}(d)\} \simeq \mathbb{R}_+ \times \text{SO}(d)$$

is a closed unimodular subgroup of G and its Haar measure is $dh = a^{-1}dadR$, and the set

$$\{(b, 1, \mathbf{I}) \mid b \in \mathbb{R}\} \simeq \mathbb{R}$$

is a normal abelian closed subgroup of G , whose Haar measure is the Lebesgue measure db . Moreover, G is the semi-direct product of \mathbb{R} and H with respect to the inner action of H on \mathbb{R} given by

$$h[b] = ab \quad b \in \mathbb{R}, h = (a, R) \in H.$$

We set

$$(3.1) \quad \gamma(h) = \det(b \mapsto h[b]) = a.$$

The Schrödinger representation π of G acts on $L^2(\mathbb{R}^d)$ as

$$(3.2a) \quad \pi(b, a, R) = U(b)V(a, R) \quad (b, a, R) \in G.$$

Here $V(a, R)$ is the unitary operator

$$V(a, R)f(x) = a^{-\frac{d}{4}}f(a^{-\frac{1}{2}}R^{-1}x) \quad f \in L^2(\mathbb{R}^d), x \in \mathbb{R}^d,$$

and $b \mapsto U(b)$ is the one-parameter group of unitary operators on $L^2(\mathbb{R}^d)$ associated with the Laplacian by the spectral calculus, namely

$$(3.2b) \quad U(b) = e^{i\frac{b}{2\pi}\Delta}.$$

Thus

$$(3.2c) \quad \mathcal{F}U(b)\mathcal{F}^{-1}\hat{f}(\xi) = e^{-2\pi ib\xi \cdot \xi}\hat{f}(\xi) \quad \xi \in \widehat{\mathbb{R}}^d.$$

Setting $\widehat{\pi} = \mathcal{F}\pi\mathcal{F}^{-1}$ we get

$$(3.2d) \quad \widehat{\pi}(b, a, R)\hat{f}(\xi) = a^{\frac{d}{4}}e^{-2\pi ib\xi \cdot \xi}\hat{f}(a^{\frac{1}{2}}R^{-1}\xi) \quad \hat{f} \in L^2(\widehat{\mathbb{R}}^d), \xi \in \widehat{\mathbb{R}}^d.$$

We now prove that π is a reproducing representation.

Proposition 3.1. *The Schrödinger representation π of G is a reproducing representation.*

Proof. It is enough to prove the result for $\widehat{\pi}$, which belongs to the family of mock-metaplectic representations introduced in [15], regarding G as semi-direct product of \mathbb{R} and H . Indeed, H acts on the dual group $\widehat{\mathbb{R}}$ of \mathbb{R} by the contra-gradient action

$${}^t h[\omega] = a^{-1}\omega \quad \omega \in \widehat{\mathbb{R}}, h = (a, R) \in H.$$

The group H acts on \mathbb{R}^d as well as on the dual space $\widehat{\mathbb{R}}^d$ by means of

$$\begin{aligned} h.x &= a^{\frac{1}{2}}Rx \\ {}^t h.\xi &= a^{-\frac{1}{2}}R\xi \end{aligned} \quad x \in \mathbb{R}^d, \xi \in \widehat{\mathbb{R}}^d, h = (a, R) \in H.$$

We set

$$\beta(h) = \det(\xi \mapsto {}^t h \cdot \xi) = a^{-\frac{d}{2}}.$$

The map

$$(3.3) \quad \Phi : \widehat{\mathbb{R}}^d \longrightarrow \widehat{\mathbb{R}}, \quad \Phi(\xi) = \xi \cdot \xi$$

is easily seen to satisfy the following properties:

- i) Φ is a smooth map whose gradient is $\nabla\Phi(\xi) = 2\xi$;
- ii) the set of critical points of Φ reduces to the origin, which is a Lebesgue negligible set, and $\Phi(\widehat{\mathbb{R}}^d \setminus \{0\}) = \widehat{\mathbb{R}}_+$;
- iii) $\Phi({}^t h \cdot \xi) = {}^t h[\Phi(\xi)]$ for all $\xi \in \widehat{\mathbb{R}}^d$ and $h \in H$;
- iv) the action of H on $\widehat{\mathbb{R}}_+$ is transitive, the stability subgroup at $1 \in \mathbb{R}_+$ is the compact group $\text{SO}(d)$, and $q : (0, +\infty) \rightarrow H$, $q(\omega) = \omega^{-1}$, is a smooth section, namely

$${}^t q(\omega)[1] = \omega \quad \omega \in \mathbb{R}_+;$$

- v) $\Phi^{-1}(1) = \text{S}^{d-1}$, where S^{d-1} is the unit sphere of $\widehat{\mathbb{R}}^d$ endowed with the Riemannian measure ds .

From (3.2d) it is clear that

$$(3.4a) \quad \widehat{\pi}(b, h)\hat{f}(\xi) = \beta(h)^{-\frac{1}{2}} e^{-2\pi i b \Phi(\xi)} \hat{f}({}^t h^{-1} \cdot \xi),$$

where $\xi \in \widehat{\mathbb{R}}^d$, $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$ and $(b, R) \in \mathbb{R} \times (\mathbb{R}_+ \times \text{SO}(d))$, which shows that $\widehat{\pi}$ is the mock-metaplectic representation associated with the map Φ . Theorem 9 of [15] then implies that $\widehat{\pi}$ is a reproducing representation. \square

We now study the admissible vectors of π . First, we need to recall some elementary facts.

Let ρ be the regular representation of $\text{SO}(d)$ acting on $L^2(\text{S}^{d-1})$, namely

$$\rho(R)\varphi(s) = \varphi(R^{-1}s) \quad s \in \text{S}^{d-1}, \varphi \in L^2(\text{S}^{d-1}), R \in \text{SO}(d).$$

There holds that

$$(3.5) \quad L^2(\text{S}^{d-1}) = \bigoplus_{i \in \mathbb{N}} \mathcal{H}_i,$$

where each \mathcal{H}_i is the space of spherical harmonics, namely the complex polynomials in d variables, homogeneous of degree i and harmonic. Here each polynomial is regarded as a function on S^{d-1} , so that \mathcal{H}_i can be identified as a subspace of $L^2(\text{S}^{d-1})$. For an account of the role of spherical harmonics in the representation theory of the orthogonal groups see [8]. It is known that

$$(3.6) \quad \dim \mathcal{H}_0 = 1, \quad \dim \mathcal{H}_1 = d, \quad \dim \mathcal{H}_i = \binom{d+i-1}{d-1} - \binom{d+i-3}{d-1}, \quad i \geq 2.$$

Moreover,

$$(3.7) \quad \rho = \bigoplus_{i \in \mathbb{N}} \rho_i,$$

where ρ_i is the restriction of ρ to \mathcal{H}_i . We denote by P_i the projection from $L^2(\text{S}^{d-1})$ onto \mathcal{H}_i .

If $d > 2$, each representation ρ_i is irreducible, and two representations ρ_i and ρ_j are inequivalent whenever $i \neq j$ (the multiplicity of each ρ_i is one). For $d = 2$, every \mathcal{H}_i with $i \geq 1$ has dimension 2 and each ρ_i is the sum of two inequivalent

irreducible one-dimensional representations, namely $\rho_i^+(\theta) = e^{in\theta}$, $\rho_i^-(\theta) = e^{-in\theta}$, $\theta \in \text{SO}(2) \simeq \mathcal{T}$. Hence, we still obtain a decomposition into inequivalent irreducible representations if we just replace the index set \mathbb{N} with \mathbb{Z} . For ease of notation, we shall proceed assuming $d \geq 3$. The case $d = 2$ is described in Section 2.

Recall that the group $\mathbb{R} \rtimes \mathbb{R}_+$ has only two inequivalent infinite dimensional irreducible representations up to unitary equivalence, which we denote by \widehat{W}^\pm (see e.g. [24]). Each of them acts on $L^2(\widehat{\mathbb{R}}_\pm)$ as

$$(3.8) \quad \widehat{W}^\pm(b, a)\varphi(\omega) = a^{\frac{1}{2}}\varphi(a\omega)e^{-2\pi ib\omega} \quad \omega \in \widehat{\mathbb{R}}_\pm, (b, a) \in \mathbb{R} \rtimes \mathbb{R}_+,$$

where $\varphi \in L^2(\widehat{\mathbb{R}}_\pm)$.

Now, let $J : L^2(\widehat{\mathbb{R}}^d) \rightarrow L^2(\widehat{\mathbb{R}}_+ \times \mathbb{S}^{d-1})$ be the operator defined by

$$(3.9) \quad J\hat{f}(\omega, s) = \frac{\omega^{\frac{d-2}{4}}}{\sqrt{2}}\hat{f}(\sqrt{\omega}s) \quad \omega \in \widehat{\mathbb{R}}_+, s \in \mathbb{S}^{d-1}, \hat{f} \in L^2(\widehat{\mathbb{R}}^d).$$

We have the following simple lemma.

Lemma 3.2. *The operator J is unitary.*

Proof. If $\hat{f} \in L^2(\widehat{\mathbb{R}}^d)$, then the changes of variable $\omega = r^2$ and $\xi = rs$ yield

$$(3.10) \quad \int_{\widehat{\mathbb{R}}_+ \times \mathbb{S}^{d-1}} \omega^{\frac{d-2}{2}} |\hat{f}(\sqrt{\omega}s)|^2 \frac{d\omega ds}{2} = \int_{\widehat{\mathbb{R}}_+ \times \mathbb{S}^{d-1}} r^{d-1} |\hat{f}(rs)|^2 dr ds = \int_{\widehat{\mathbb{R}}^d} |\hat{f}(\xi)|^2 d\xi.$$

The inverse of J is given by

$$(J^{-1}g)(\xi) = \frac{\sqrt{2}}{(\xi \cdot \xi)^{\frac{d-2}{2}}} g(\xi \cdot \xi, \frac{\xi}{\sqrt{\xi \cdot \xi}}) \quad \xi \in \widehat{\mathbb{R}}^d, \xi \neq 0, g \in L^2(\widehat{\mathbb{R}}_+ \times \mathbb{S}^{d-1}),$$

which proves that J is unitary. \square

In what follows, we will freely identify

$$(3.11) \quad \begin{aligned} L^2(\widehat{\mathbb{R}}^d) &\simeq L^2(\widehat{\mathbb{R}}_+ \times \mathbb{S}^{d-1}) \\ &\simeq L^2(\widehat{\mathbb{R}}_+) \otimes L^2(\mathbb{S}^{d-1}) \\ &\simeq \bigoplus_{i \in \mathbb{N}} L^2(\widehat{\mathbb{R}}_+) \otimes \mathcal{H}_i \\ &\simeq \bigoplus_{i \in \mathbb{N}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i). \end{aligned}$$

We define the unitary operator $S : L^2(\mathbb{R}^d) \rightarrow \bigoplus_{i \in \mathbb{N}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$ by

$$(3.12) \quad (Sf)_i = (\text{Id} \otimes P_i)(J\mathcal{F}f) \quad f \in L^2(\mathbb{R}^d).$$

Proposition 3.3. *With the above notation,*

$$(3.13) \quad S\pi S^{-1} = \bigoplus_{i \in \mathbb{N}} \widehat{W}^+ \otimes \rho_i$$

where each component $\widehat{W}^+ \otimes \rho_i$ is irreducible and inequivalent to the others. A vector $\widehat{\eta} \in L^2(\mathbb{R}^d)$ is admissible for π if and only if

$$(3.14) \quad \int_0^{+\infty} \|(S\widehat{\eta})_i(\omega)\|_{\mathcal{H}_i}^2 \frac{d\omega}{\omega} = d_i \quad i \in \mathbb{N}.$$

Proof. The proof is based on the general theory developed in [15]. We sketch the main steps. For any $\omega \in \widehat{\mathbb{R}}_+$ we denote by ν_ω the measure on $\widehat{\mathbb{R}}^d$ which is the image measure of $\omega^{\frac{d-2}{2}} ds/2$ under the map

$$\mathbb{S}^{d-1} \ni s \mapsto \sqrt{\omega} s \in \widehat{\mathbb{R}}^d,$$

so that, for all compactly supported continuous functions φ , we have

$$\int_{\widehat{\mathbb{R}}^d} \varphi(\xi) d\nu_\omega(\xi) = \int_{\mathbb{S}^{d-1}} \varphi(\sqrt{\omega} s) \frac{\omega^{\frac{d-2}{2}}}{2} ds.$$

The change of variable in spherical coordinates (as in (3.10)) gives

$$\begin{aligned} \int_{\widehat{\mathbb{R}}^d} \varphi(\xi) d\xi &= \int_0^{+\infty} \left(\int_{\mathbb{S}^{d-1}} \varphi(rs) r^{d-1} ds \right) dr \\ &= \int_0^{+\infty} \left(\int_{\mathbb{S}^{d-1}} \varphi(\sqrt{\omega} s) \frac{\omega^{\frac{d-2}{2}}}{2} ds \right) d\omega \\ &= \int_0^{+\infty} \left(\int_{\widehat{\mathbb{R}}^d} \varphi(\xi) d\nu_\omega(\xi) \right) d\omega, \end{aligned}$$

where $r^2 = \omega$, so that the disintegration formula

$$(3.15) \quad d\xi = \int_0^{+\infty} \nu_\omega d\omega$$

holds true. Finally, Weil's formula for quasi-invariant measure on quotient spaces [19] reads

$$(3.16) \quad \int_H \varphi(a, R) \gamma(a)^{-1} \frac{da}{a} dR = C \int_0^{+\infty} \left(\int_{\text{SO}(d)} \varphi(q(\omega)R) dR \right) d\omega,$$

for some constant C , to be computed. Recalling (3.1) and $q(\omega) = \omega^{-1}$, we obtain $C = 1$ since

$$\int_0^{+\infty} \left(\int_{\text{SO}(d)} \varphi(\omega^{-1}R) dR \right) d\omega = \int_0^{+\infty} \left(\int_{\text{SO}(d)} \varphi(\omega, R) dR \right) \frac{d\omega}{\omega^2}.$$

Observe that

- i) $L^2(\mathbb{R}^d, 2\nu_1) \simeq L^2(\mathbb{S}^{d-1})$;
- ii) the “restriction” of the mock-metaplectic representation $\widehat{\pi}$ to the fiber $\Phi^{-1}(1)$ and to the stability subgroup $\text{SO}(d)$ is precisely ρ . Hence, (3.7) provides the decomposition of ρ into its irreducibles, all of them with multiplicity 1;
- iii) up to the normalization factor $1/\sqrt{2}$, the operator $S\mathcal{F}^{-1}$ coincides with the operator introduced in [15], whose main feature is that it decomposes $\widehat{\pi}$ into its irreducibles, each of which is the canonical representation obtained by inducing the irreducible representation of $\mathbb{R} \times \text{SO}(d)$ acting on \mathcal{H}_i as

$$(b, R) \mapsto e^{-2\pi i b} \rho_i(R)$$

from $\mathbb{R} \times \text{SO}(d)$ to G .

Theorem 9 of [15] shows that $\widehat{\eta} \in L^2(\widehat{\mathbb{R}}^d)$ is admissible if and only if, for all $i \in \mathbb{N}$,

$$\int_0^{+\infty} \|(S\widehat{\eta})_i(\omega)\|_{\mathcal{H}_i}^2 \gamma(q(\omega)) d\omega = \frac{\dim \mathcal{H}_i}{C} = d_i.$$

Explicitly, this amounts to

$$\int_0^{+\infty} \|(S\hat{\eta})_i(\omega)\|_{\mathcal{H}_i}^2 \frac{d\omega}{\omega} = d_i. \quad \square$$

We remark that a second proof can be derived using Proposition 2.23 of [19], and it consists of three steps:

- i) a direct computation to show (3.13);
- ii) the observation that each component $\widehat{W}^+ \otimes \rho_i$ is a square-integrable representation, whose formal degree operator C_i is the unbounded multiplication operator

$$C_i \varphi(\omega) = d_i \omega \varphi(\omega),$$

where $\varphi \in L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$ and $\int_0^\infty \omega^2 |\varphi(\omega)|^2 d\omega < +\infty$;

- iii) a final application of Proposition 2.23 of [19], taking into account that $\widehat{W}^+ \otimes \rho_i$ and $\widehat{W}^+ \otimes \rho_j$ are inequivalent representations of G if $i \neq j$.

3.2. A family of admissible vectors. We now give an alternative description of the admissible vectors, which provides a direct strategy to construct them.

Let $\hat{\eta} \in L^2(\mathbb{R}^d)$ be an admissible vector. For any fixed $i \in \mathbb{N}$, we choose an orthonormal basis $\{e_k^i\}_{k=1}^{d_i}$ of \mathcal{H}_i , and define $\varphi_{i,k} : \widehat{\mathbb{R}}_+ \rightarrow \mathbb{C}$ by

$$\varphi_{i,k}(\omega) = \langle (S\hat{\eta})_i(\omega), e_k^i \rangle_{\mathcal{H}_i}.$$

By construction, $\varphi_{i,k} \in L^2(\widehat{\mathbb{R}}_+)$ and

$$\int_0^{+\infty} \frac{|\varphi_{i,k}(\omega)|^2}{\omega} d\omega < +\infty.$$

If $\varphi_{i,k} \neq 0$, up to a normalization we can always assume that

$$(3.17a) \quad \int_0^{+\infty} \frac{|\varphi_{i,k}(\omega)|^2}{\omega} d\omega = 1,$$

i.e. $\varphi_{i,k}$ is a 1D-wavelet for \widehat{W}^+ . Hence

$$(3.17b) \quad (S\hat{\eta})_i = \sum_{k=1}^{d_i} \varphi_{i,k} \otimes v_{i,k},$$

where $\{v_{i,k}\}_{k=1}^{d_i}$ is an orthogonal family in \mathcal{H}_i such that

$$(3.17c) \quad \sum_{k=1}^{d_i} \|v_{i,k}\|_{\mathcal{H}_i}^2 = d_i,$$

and all $\varphi_{i,k}$ satisfy (3.17a) (if for some k the function $\langle (S\hat{\eta})_i(\cdot), e_k^i \rangle_{\mathcal{H}_i}$ is zero, we set $v_{i,k} = 0$ and choose an arbitrary $\varphi_{i,k}$ satisfying (3.17a)).

The fact that $\hat{\eta} \in L^2(\mathbb{R}^d)$ implies

$$(3.17d) \quad \sum_{i=1}^{+\infty} \sum_{k=1}^{d_i} \|\varphi_{i,k}\|_2^2 \|v_{i,k}\|_{\mathcal{H}_i}^2 < +\infty.$$

Conversely, given a family $(\varphi_{i,k}, v_{i,k})_{i \in \mathbb{N}, k=1, \dots, d_i}$ such that

- a) each $\varphi_{i,k}$ is in $L^2(\widehat{\mathbb{R}}_+)$ and satisfies (3.17a),
- b) each family $\{v_{i,k}\}_{k=1}^{d_i}$ is orthogonal in \mathcal{H}_i and satisfies (3.17c) and (3.17d),

then $(\varphi_{i,k}, v_{i,k})_{i \in \mathbb{N}, k=1, \dots, d_i}$ defines an admissible vector via (3.17b). A simple solution is given as follows. Choose a 1D wavelet $\varphi \in L^2(\widehat{\mathbb{R}}_+)$. For all $i \in \mathbb{N}$, fix $\alpha_i > 0$ and $v_i \in \mathcal{H}_i$ with

$$\|v_i\|_{\mathcal{H}_i}^2 = d_i$$

and

$$\sum_{i \in \mathbb{N}} \alpha_i d_i < +\infty.$$

Define

$$\varphi_i(\omega) = \varphi(\alpha_i^{-1}\omega).$$

Then, the vector $\widehat{\eta} \in L^2(\mathbb{R}^d)$ such that

$$(S\widehat{\eta})_i = \varphi_i \otimes v_i$$

is admissible.

3.3. Discretization. The aim of this section is to construct a Parseval frame for $L^2(\mathbb{R}^d)$ based on a discretization of the reproducing representation π .

We fix a finite subgroup of $\text{SO}(d)$ of cardinality L

$$F = \{R_1, \dots, R_L\},$$

and we choose as grid points those in the family

$$x_{j,k,\ell} = (2^j k, 2^j, R_\ell) \quad j, k \in \mathbb{Z}, \ell = 1, \dots, L.$$

We denote by \widehat{F} the set of equivalence classes of irreducible (unitary) representations of F , and for each equivalence class in \widehat{F} we fix a representative $\chi : F \rightarrow \mathcal{U}(\mathcal{H}_\chi)$, where \mathcal{H}_χ is the Hilbert space on which χ acts and $\mathcal{U}(\mathcal{H}_\chi)$ is the corresponding set of unitary operators. The dimension of \mathcal{H}_χ , which is always finite, is denoted by d_χ .

For each $i \in \mathbb{N}$, the representation ρ_i restricted to F decomposes into its irreducibles

$$(3.18) \quad \mathcal{H}_i = \bigoplus_{\chi \in \widehat{F}} \mathcal{H}_\chi \otimes \mathbb{C}^{m_{i,\chi}} \quad \rho_i = \bigoplus_{\chi \in \widehat{F}} \chi \otimes \text{I}_{m_{i,\chi}},$$

where $m_{i,\chi} \in \mathbb{N}$ is the multiplicity of χ into ρ_i (with the convention that $\mathbb{C}^0 = \{0\}$ if $m_{i,\chi} = 0$, namely when the representation χ does not enter into the decomposition).

We remark that in the two-dimensional case the picture is clearer (see Section 2 and the remarks that follow (3.7)). Taking $F = \{2\pi l/L : l = 0, \dots, L-1\}$, the set \widehat{F} is given by L one-dimensional representations corresponding to the L -roots of unity, namely $\widehat{F} = \{\chi_l = e^{2\pi i l/L} : l = 0, \dots, L-1\}$. Writing $\mathcal{H}_k = \text{span}\{e^{ik\cdot}\}$ for $k \in \mathbb{Z}$ (as already observed, the natural index set in 2D is \mathbb{Z}), a simple calculation shows that ρ_k corresponds to $\chi_{\bar{k}}$, where $\bar{k} = k \bmod L$. Therefore, in the above decomposition one has

$$m_{k,\chi_l} = \begin{cases} 1 & \text{if } k - l \in L\mathbb{Z} \\ 0 & \text{otherwise,} \end{cases}$$

or, equivalently, $\mathcal{H}_k = \mathcal{H}_{\chi_{\bar{k}}}$.

From (3.5) and (3.18) we finally obtain the decomposition of ρ into its irreducibles

$$(3.19) \quad L^2(\mathbb{S}^{d-1}) = \bigoplus_{\chi \in \widehat{F}} \mathcal{H}_\chi \otimes \mathbb{C}^{m_\chi} \quad \rho = \bigoplus_{\chi \in \widehat{F}} \chi \otimes \text{I}_{m_\chi},$$

where $m_\chi = \sum_{i \in I} m_{i,\chi}$, the operator I_{m_χ} is the identity on \mathbb{C}^{m_χ} and $\mathbb{C}^\infty = \ell_2(\mathbb{N})$ if $\sum_{i \in I} m_{i,\chi} = \infty$. By (3.11) and (3.19), the following identifications hold true:

$$(3.20) \quad L^2(\widehat{\mathbb{R}}^d) = \bigoplus_{i \in \mathbb{N}, \chi \in \widehat{F}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi) \otimes \mathbb{C}^{m_{i,\chi}} = \bigoplus_{i \in \mathbb{N}, \chi \in \widehat{F}} \bigoplus_{\mu=1}^{m_{i,\chi}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi \otimes \mathbb{C}\{\epsilon_\mu\}),$$

where $(\epsilon_\mu)_{\mu \in \mathbb{N}}$ is the canonical basis of $\ell^2(\mathbb{N})$ and each $\mathbb{C}^{m_{i,\chi}}$ is regarded as a closed subspace of $\ell^2(\mathbb{N})$. According to this decomposition, we denote by $P_{i,\chi,\mu}$ the orthogonal projection from $L^2(\widehat{\mathbb{R}}^d)$ onto the closed subspace $L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi \otimes \mathbb{C}\{\epsilon_\mu\})$ of $L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$.

Next, for each χ , we select an orthogonal family $w_1^\chi, \dots, w_{d_\chi}^\chi$ in \mathcal{H}_χ such that

$$(3.21) \quad \|w_\delta^\chi\|^2 = d_\chi \quad \delta = 1, \dots, d_\chi.$$

For each $i \in I$, we choose $m_{i,\chi}$ -vectors in this family and we denote by $\Delta_{i,\chi} = (\delta_1, \dots, \delta_{m_{i,\chi}})$ the corresponding family of indices, some of which might be repeated. We set

$$(3.22) \quad v_{i,\chi,\mu} = w_{\delta_\mu}^\chi \otimes \epsilon_\mu \quad \mu = 1, \dots, m_{i,\chi},$$

where each $v_{i,\chi,\mu}$ is a vector in \mathcal{H}_i by means of (3.18).

Finally, we select $m_{i,\chi}$ -functions $\varphi_{i,\chi,1}, \dots, \varphi_{i,\chi,m_{i,\chi}} \in L^2(\widehat{\mathbb{R}}_+)$ such that the following conditions hold true:

a) the series

$$(3.23) \quad \sum_{i \in \mathbb{N}} \sum_{\chi \in \widehat{F}} d_\chi \left(\sum_{\mu=1}^{m_{i,\chi}} \|\varphi_{i,\chi,\mu}\|_2^2 \right) < +\infty;$$

b) for each $i \in \mathbb{N}$, $\chi \in \widehat{F}$ and $\mu = 1, \dots, m_{i,\chi}$

$$(3.24a) \quad \sum_{j \in \mathbb{Z}} |\varphi_{i,\chi,\mu}(2^j \omega)|^2 = \frac{1}{L} \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+,$$

and for all odd integers m

$$(3.24b) \quad \sum_{j=0}^{+\infty} \varphi_{i,\chi,\mu}(2^j \omega) \overline{\varphi_{i,\chi,\mu}(2^j(\omega + 2\pi m))} = 0 \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+;$$

c) for all $\chi \in \widehat{F}$, if there exists $i, i' \in \mathbb{N}$ and $\mu = 1, \dots, m_{i,\chi}$, $\mu' = 1, \dots, m_{i',\chi}$ such that $(i, \mu) \neq (i', \mu')$, but $w_{\delta_\mu}^\chi = w_{\delta_{\mu'}}^\chi$ (where $\delta_\mu \in \Delta_{i,\chi}$ and $\delta_{\mu'} \in \Delta_{i',\chi}$), then

$$(3.25a) \quad \sum_{j \in \mathbb{Z}} \varphi_{i,\chi,\mu}(2^j \omega) \overline{\varphi_{i',\chi,\mu'}(2^j \omega)} = 0 \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+,$$

and for all odd integers m

$$(3.25b) \quad \sum_{j=0}^{+\infty} \varphi_{i,\chi,\mu}(2^j \omega) \overline{\varphi_{i',\chi,\mu'}(2^j(\omega + 2\pi m))} = 0 \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+.$$

Let us comment on the relation between these assumptions and the corresponding ones given in the two-dimensional case. Assumption (3.23) is simply a restatement of the fact that $\widehat{\eta}$ should have finite norm. Assumptions (3.24) and (3.25) correspond to assumptions (2.7) and (2.8), respectively. As we have already anticipated when

discussing the 2D case, the condition $(i, \mu) \neq (i', \mu')$ corresponds to $m \neq n$ and $w_{\delta_\mu}^\chi = w_{\delta_{\mu'}}^\chi$ corresponds to $m - n \in L\mathbb{Z}$.

We are now ready to state the main result of this paper.

Theorem 3.4. *Let $\widehat{\eta} \in L^2(\mathbb{R}^d)$ be defined by*

$$(3.26) \quad (S\widehat{\eta})_i = \sum_{\chi \in \widehat{F}} \sum_{\mu=1}^{m_{i,\chi}} \varphi_{i,\chi,\mu} \otimes v_{i,\chi,\mu}.$$

Then the family $\{\pi(2^j k, 2^j, R_\ell)\widehat{\eta}\}_{j,k \in \mathbb{Z}, \ell=1, \dots, L}$ is a Parseval frame for $L^2(\mathbb{R}^d)$.

The proof is in Section 3.4. We add a few comments. Since $\sum_{\chi \in \widehat{F}} m_{i,\chi} d_\chi = d_i$, we have

$$\sum_{\chi \in \widehat{F}} \sum_{\mu=1}^{m_{i,\chi}} \|v_{i,\chi,\mu}\|_{\mathcal{H}_i}^2 = d_i,$$

hence (3.23) ensures that (3.26) is well defined (compare with (3.17b)).

An important result in wavelet theory [22, Theorem 1.6, Chapter 7] shows that (3.24a) and (3.24b) are equivalent to the fact that for each $i \in \mathbb{N}$, $\chi \in \widehat{F}$ and $\mu = 1, \dots, m_{i,\chi}$ the family $\{\widehat{W}^+(2^j k, 2^j) \sqrt{L} \varphi_{i,\chi,\mu}\}_{j,k \in \mathbb{Z}}$ is a Parseval frame for $L^2(\widehat{\mathbb{R}}_+)$. Furthermore, (3.24a) implies that

$$(3.27) \quad \int_{\widehat{\mathbb{R}}_+} \frac{|\varphi_{i,\chi,\mu}(\omega)|^2}{\omega} d\omega = \frac{\ln 2}{L},$$

so that $\sqrt{L/\ln 2} \widehat{\eta}$ is an admissible vector for π by Proposition 3.3.

We now show that there exist families of $\{\varphi_{i,\chi,\mu}\}$, satisfying the above conditions. To this end, fix a function $\varphi \in L^2(\widehat{\mathbb{R}}_+)$ supported in $[0, 1]$ and such that

$$(3.28) \quad \sum_{j \in \mathbb{Z}} |\varphi(2^j \omega)|^2 = \frac{1}{L} \quad \text{a.e. } \omega \in \widehat{\mathbb{R}}_+.$$

Choose a sequence $\{\alpha_{i,\chi,\mu}\}$ such that $0 < \alpha_{i,\chi,\mu} < 1$ and

$$(3.29) \quad \sum_{i \in \mathbb{N}} \sum_{\chi \in \widehat{F}} d_\chi \sum_{\mu=1}^{m_{i,\chi}} \alpha_{i,\chi,\mu} < +\infty.$$

Suppose further that, for any $\chi \in \widehat{F}$, if there exists $i, i' \in \mathbb{N}$ and $\mu = 1, \dots, m_{i,\chi}$, $\mu' = 1, \dots, m_{i',\chi}$ such that $(i, \mu) \neq (i', \mu')$ but $w_{\delta_\mu}^\chi = w_{\delta_{\mu'}}^\chi$ (where $\delta_\mu \in \Delta_{i,\chi}$ and $\delta_{\mu'} \in \Delta_{i',\chi}$), then

$$(3.30) \quad |(\text{supp}(\varphi) \cap \alpha_{i,\chi,\mu}^{-1} \alpha_{i',\chi,\mu'} \text{supp}(\varphi))| = 0.$$

An explicit example is

$$\varphi = \chi_{(1/2, 1]},$$

$$\alpha_{i,\chi,\mu} = \frac{1}{2^{n_{i,\chi,\mu}}},$$

where $(i, \chi, \mu) \mapsto n_{i,\chi,\mu}$ is any bijection from the index set

$$\mathcal{N} = \{(i, \chi, \mu) \mid i \in \mathbb{N}, \chi \in \widehat{F}, m_{i,\chi} > 0, \mu = 1, \dots, m_{i,\chi}\}$$

onto \mathbb{N} .

With the above choices, define

$$\varphi_{i,\chi,\mu}(\omega) = \varphi(\alpha_{i,\chi,\mu}^{-1}\omega) \quad \omega \in \widehat{\mathbb{R}}_+.$$

Now, the sum in (3.24b) contains products of the form

$$\varphi\left(\frac{2^j\omega}{\alpha_{i,\chi,\mu}}\right)\varphi\left(\frac{2^j\omega}{\alpha_{i,\chi,\mu}} + \frac{2^j2\pi m}{\alpha_{i,\chi,\mu}}\right).$$

Since $|2^j2\pi m/\alpha_{i,\chi,\mu}| > |2^j2\pi m| > 1$ for every odd integer m and every non-negative integer j , one of the two factors must always vanish, so that (3.24b) holds true. Similarly, (3.30) implies (3.25a) and (3.25b).

3.4. Proof of Theorem 3.4. We first prove a technical lemma, which is a variant of a well known result (see Lemma 1.10 of [22]).

We recall that a family $(\psi_i)_{i \in \mathbb{N}}$ in a separable Hilbert space \mathcal{H} is a Parseval frame if one of the following two equivalent conditions is satisfied:

a) for all $f \in \mathcal{H}$

$$\sum_{i \in \mathbb{N}} \langle f, \psi_i \rangle \psi_i = f;$$

b) for all $f \in \mathcal{H}$

$$\sum_{i \in \mathbb{N}} |\langle \psi_i, f \rangle|^2 = \|f\|^2,$$

see Theorem 1.7 Chapter 7 of [22]. Both series convergence unconditionally. For a thorough discussion on frames see e.g. [7, 21].

Lemma 3.5. *Let $(\psi_i)_{i \in \mathbb{N}}$ be a family of vectors in \mathcal{H} . If there exists a total subset \mathcal{S} of \mathcal{H} such that*

- a) for all $f \in \mathcal{S}$ the sequence $(\langle f, \psi_i \rangle)_{i \in \mathbb{N}}$ is in $\ell^2(\mathbb{N})$;
b) for all $f, g \in \mathcal{S}$

$$(3.31) \quad \sum_{i \in \mathbb{N}} \langle f, \psi_i \rangle \langle \psi_i, g \rangle = \langle f, g \rangle,$$

then the family $(\psi_i)_{i \in \mathbb{N}}$ is a Parseval frame.

Proof. Define

$$\mathcal{D} = \{f \in \mathcal{H} \mid \sum_{i \in \mathbb{N}} |\langle f, \psi_i \rangle|^2 < +\infty\}$$

and $V : \mathcal{D} \rightarrow \ell^2(\mathbb{N})$

$$Vf = (\langle f, \psi_i \rangle)_{i \in \mathbb{N}}.$$

By construction, \mathcal{D} is a linear subspace containing \mathcal{S} , so that \mathcal{D} is dense and V is a linear operator. It is known that V is a closed operator, see Proposition 2.8 of [19]. By (3.31), the restriction of V to \mathcal{S} preserves the scalar product. By linearity, the same property holds on the linear subspace spanned by \mathcal{S} , which is contained in \mathcal{D} and dense in \mathcal{H} since \mathcal{S} is total in \mathcal{H} . Then V extends to a unique isometry W from \mathcal{H} into $\ell^2(\mathbb{N})$. Since V is closed, then $\mathcal{D} = \mathcal{H}$ and $V = W$. By definition of V , the family $(\psi_i)_{i \in \mathbb{N}}$ is a Parseval frame. \square

The following lemma is a variant of a result given in [19] in the context of admissible representations, see Proposition 2.23.

Lemma 3.6. *Take two countable families $(\mathcal{H}_j)_{j \in \mathbb{N}}$ and $(\mathcal{H}'_j)_{j \in \mathbb{N}}$ of separable Hilbert spaces, set $\mathcal{H} = \bigoplus_{j \in \mathbb{N}} \mathcal{H}_j \otimes \mathcal{H}'_j$ and, for all $j \in \mathbb{N}$, denote the canonical projection by $P_j : \mathcal{H} \rightarrow \mathcal{H}_j \otimes \mathcal{H}'_j$. A family $(\psi_i)_{i \in \mathbb{N}}$ is a Parseval frame for \mathcal{H} if and only if the following two conditions hold true:*

a) *for all $j \in \mathbb{N}$ and all $f \in \mathcal{H}_j$, $f' \in \mathcal{H}'_j$*

$$\sum_{i \in \mathbb{N}} |\langle f \otimes f', P_j \psi_i \rangle|^2 = \|f\|_{\mathcal{H}_j}^2 \|f'\|_{\mathcal{H}'_j}^2;$$

b) *for all $j, k \in \mathbb{N}$, $j \neq k$ and for all $f \in \mathcal{H}_j$, $f' \in \mathcal{H}'_j$, $g \in \mathcal{H}_k$, $g' \in \mathcal{H}'_k$*

$$\sum_{i \in \mathbb{N}} \langle f \otimes f', P_j \psi_i \rangle \langle P_k \psi_i, g \otimes g' \rangle = 0.$$

Proof. Assume that $(\psi_i)_{i \in \mathbb{N}}$ is a Parseval frame for \mathcal{H} and fix $j \in \mathbb{N}$. Given $f \in \mathcal{H}_j$ and $f' \in \mathcal{H}'_j$, we have

$$P_j^*(f \otimes f'_j) = \sum_{i \in \mathbb{N}} \langle P_j^*(f \otimes f'_j), \psi_i \rangle \psi_i.$$

For all $k \in \mathbb{N}$, P_k is a bounded linear operator, and $P_k P_j^* = \delta_{jk} P_j P_j^* = \delta_{jk} \text{Id}_{\mathcal{H}_j \otimes \mathcal{H}'_j}$. Then

$$\begin{cases} \sum_{i \in \mathbb{N}} \langle f \otimes f'_j, P_j \psi_i \rangle P_j \psi_i = f \otimes f'_j & k = j \\ \sum_{i \in \mathbb{N}} \langle f \otimes f'_j, P_j \psi_i \rangle P_k \psi_i = 0 & k \neq j, \end{cases}$$

whence a) and b) easily follow.

Conversely, set

$$\mathcal{S} = \bigcup_{j \in \mathbb{N}} \{P_j^*(f \otimes f') \mid f \in \mathcal{H}_j, f' \in \mathcal{H}'_j\},$$

which is total in \mathcal{H} by construction. Conditions a) and b) imply that (3.31) of Lemma 3.5 is satisfied, hence, $(\psi_i)_{i \in \mathbb{N}}$ is a Parseval frame. \square

The following result is a restatement of the well known characterization of wavelet Parseval frames. For the sake of clarity, we set $\lambda = (j, k) \in \Lambda = \mathbb{Z}^2$ and $x_\lambda = (2^j k, 2^j) \in \mathbb{R} \times \mathbb{R}_+$.

Lemma 3.7. *If the family $\{\varphi_{i, \chi, \mu}\}$ in $L^2(\widehat{\mathbb{R}}_+)$ satisfies (3.24a), (3.24b), (3.25a) and (3.25b), then*

a) *for each $i \in \mathbb{N}$, $\chi \in \widehat{F}$ and $\mu = 1, \dots, m_{i, \chi}$*

$$(3.32) \quad \sum_{\lambda \in \Lambda} |\langle \varphi, \widehat{W}^+(x_\lambda) \varphi_{i, \chi, \mu} \rangle_2|^2 = \frac{1}{L} \|\varphi\|_2^2$$

for all $\varphi \in L^2(\widehat{\mathbb{R}}_+)$;

b) *for all $\chi \in \widehat{F}$, if there exists $i, i' \in \mathbb{N}$ and $\mu = 1, \dots, m_{i, \chi}$, $\mu' = 1, \dots, m_{i', \chi}$ such that $(i, \mu) \neq (i', \mu')$ but $w_{\delta_\mu}^\chi = w_{\delta_{\mu'}}^\chi$ (where $\delta_\mu \in \Delta_{i, \chi}$ and $\delta_{\mu'} \in \Delta_{i', \chi}$), then*

$$(3.33) \quad \sum_{\lambda \in \Lambda} \langle \varphi, \widehat{W}^+(x_\lambda) \varphi_{i, \chi, \mu} \rangle_2 \langle \widehat{W}^+(x_\lambda) \varphi_{i', \chi, \mu'}, \varphi' \rangle_2 = 0$$

for all $\varphi, \varphi' \in L^2(\widehat{\mathbb{R}}_+)$.

Proof. The fact that (3.32) is equivalent to (3.24a) and (3.24b) is one of the fundamental results at the root of wavelet frames, see Theorem 1.6 of [22]. The fact that (3.25a) and (3.25b) imply (3.33) follows by Lemma 1.18 of [22], which, by polarization, can be rewritten as

$$\begin{aligned} & 2\pi \sum_{\lambda \in \Lambda} \langle \varphi, \widehat{W}^+(x_\lambda) \varphi_{i,\chi,\mu} \rangle_2 \langle \widehat{W}^+(x_\lambda) \varphi_{i',\chi,\mu'}, \varphi' \rangle_2 \\ &= \int_{\widehat{\mathbb{R}}} \varphi(\omega) \overline{\varphi'(\omega)} \sum_{j \in \mathbb{Z}} \varphi_{i',\chi,\mu'}(2^j \omega) \overline{\varphi_{i,\chi,\mu}(2^j \omega)} d\omega \\ &+ \int_{\widehat{\mathbb{R}}} \overline{\varphi'(\omega)} \sum_{j \in \mathbb{Z}} \sum_{m \in 2\mathbb{Z}+1} \varphi_{i',\chi,\mu'}(\omega + 2^j 2\pi m) h_m(2^j \omega) d\omega, \end{aligned}$$

where

$$h_m(\omega) = \sum_{n=0}^{+\infty} \varphi_{i',\chi,\mu'}(2^n \omega) \overline{\varphi_{i,\chi,\mu}(2^n(\omega + 2\pi m))}.$$

Indeed, (3.25a) implies that the first summand vanishes, whereas (3.25b) implies that h_m vanish for all odd integers, hence the second summand is zero. \square

Proof of Theorem 3.4. By means of the unitary operator S , we can prove the result for the family of vectors

$$S\pi(x_{\lambda,\ell})\widehat{\eta} \quad \lambda \in \Lambda, \quad \ell = 1, \dots, L$$

in the space $\bigoplus_{i \in \mathbb{N}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_i)$, which by (3.11) and (3.20) can be identified with

$$L^2(\widehat{\mathbb{R}}^d) = \bigoplus_{i \in \mathbb{N}, \chi \in \widehat{F}} \bigoplus_{\mu=1}^{m_{i,\chi}} L^2(\widehat{\mathbb{R}}_+, \mathcal{H}_\chi \otimes \mathbb{C}\{\epsilon_\mu\}).$$

We mean to apply Lemma 3.6. So, let us fix $i, i' \in \mathbb{N}$, $\chi, \chi' \in \widehat{F}$ and $\mu \in \{1, \dots, m_{i,\chi}\}$, $\mu' \in \{1, \dots, m_{i',\chi'}\}$. Given $\varphi, \varphi' \in L^2(\widehat{\mathbb{R}}_+)$ and $w \in \mathcal{H}_\chi$, $w' \in \mathcal{H}_{\chi'}$, we look at the quantity

$$\begin{aligned} & A(i, \chi, \mu, i', \chi', \mu') \\ &= \sum_{\lambda \in \Lambda} \sum_{\ell=1}^L \langle \varphi \otimes w \otimes \epsilon_\mu, P_{i,\chi,\mu} S\pi(x_{\lambda,\ell})\widehat{\eta} \rangle_2 \langle P_{i',\chi',\mu'} S\pi(x_{\lambda,\ell})\widehat{\eta}, \varphi' \otimes w' \otimes \epsilon_{\mu'} \rangle_2. \end{aligned}$$

Recall that, since $x_{\lambda,\ell} = (x_\lambda, R_\ell)$,

$$P_{i,\chi,\mu} S\pi(x_{\lambda,\ell})\widehat{\eta} = \widehat{W}^+(x_\lambda) \varphi_{i,\chi,\mu} \otimes \chi(R_\ell) w_{\delta_\mu}^\chi \otimes \epsilon_\mu,$$

hence we have

$$\begin{aligned} A(i, \chi, \mu, i', \chi', \mu') &= \left(\sum_{\lambda \in \Lambda} \langle \varphi, \widehat{W}^+(x_\lambda) \varphi_{i,\chi,\mu} \rangle_2 \langle \widehat{W}^+(x_\lambda) \varphi_{i',\chi',\mu'}, \varphi' \rangle_2 \right) \\ &\quad \times \left(\sum_{\ell=1}^L \langle w, \chi(R_\ell) w_{\delta_\mu}^\chi \rangle_{\mathcal{H}_\chi} \langle \chi'(R_\ell) w_{\delta_{\mu'}}^{\chi'}, w' \rangle_{\mathcal{H}_{\chi'}} \right), \end{aligned}$$

where the series are absolutely summable because of (3.32) and the Cauchy-Schwarz inequality.

From the Schur orthogonality relations applied to the pair of irreducible representations χ, χ' of F , we know that

$$\frac{1}{L} \sum_{\ell=1}^L \langle w, \chi(R_\ell) w_{\delta_\mu}^\chi \rangle_{\mathcal{H}_\chi} \langle \chi'(R_\ell) w_{\delta_{\mu'}}^{\chi'}, w' \rangle_{\mathcal{H}_{\chi'}} = \begin{cases} 0 & \chi \neq \chi' \\ \frac{1}{d_\chi} \langle w_{\delta_{\mu'}}^\chi, w_{\delta_\mu}^\chi \rangle_{\mathcal{H}_\chi} \langle w, w' \rangle_{\mathcal{H}_\chi} & \chi = \chi'. \end{cases}$$

Thus, if $\chi \neq \chi'$, we get $A(i, \chi, \mu, i', \chi', \mu') = 0$. From now on assume $\chi = \chi'$, for which

$$\begin{aligned} A(i, \chi, \mu, i', \chi, \mu') &= L \left(\sum_{\lambda \in \Lambda} \langle \varphi, \widehat{W}^+(x_\lambda) \varphi_{i, \chi, \mu} \rangle_2 \langle \widehat{W}^+(x_\lambda) \varphi_{i', \chi, \mu'}, \varphi' \rangle_2 \right) \\ &\quad \times \frac{1}{d_\chi} \langle w_{\delta_{\mu'}}^\chi, w_{\delta_\mu}^\chi \rangle_{\mathcal{H}_\chi} \langle w, w' \rangle_{\mathcal{H}_\chi}. \end{aligned}$$

If $(i, \mu) \neq (i', \mu')$ and $\delta_\mu \neq \delta_{\mu'}$, then $A(i, \chi, \mu, i', \chi, \mu') = 0$ since the family $w_1^\chi, \dots, w_{d_\chi}^\chi$ is orthonormal. If $(i, \mu) \neq (i', \mu')$ but $\delta_\mu = \delta_{\mu'}$, then by (3.33) it follows that $A(i, \chi, \mu, i', \chi, \mu') = 0$. Finally, if $(i, \mu) = (i', \mu')$, then (3.21) yields

$$\begin{aligned} A(i, \chi, \mu, i, \chi, \mu) &= L \left(\sum_{\lambda \in \Lambda} \langle \varphi, \widehat{W}^+(x_\lambda) \varphi_{i, \chi, \mu} \rangle_2 \langle \widehat{W}^+(x_\lambda) \varphi_{i, \chi, \mu}, \varphi' \rangle_2 \right) \langle w, w' \rangle_{\mathcal{H}_\chi} \\ &= \langle \varphi, \varphi' \rangle_2 \langle w, w' \rangle_{\mathcal{H}_\chi} \\ &= \langle \varphi \otimes w \otimes \epsilon_\mu, \varphi' \otimes w' \otimes \epsilon_\mu \rangle_2, \end{aligned}$$

where the second equality is a consequence of (3.32).

Summarizing the above results in a single equation, we obtain

$$A(i, \chi, \mu, i', \chi', \mu') = \begin{cases} \langle \varphi \otimes w \otimes \epsilon_\mu, \varphi' \otimes w' \otimes \epsilon_\mu \rangle_2 & \text{if } \chi = \chi' \text{ and } (i, \mu) = (i', \mu'), \\ 0 & \text{if } \chi \neq \chi' \text{ or } (i, \mu) \neq (i', \mu'). \end{cases}$$

The conclusion follows from Lemma 3.6. \square

Acknowledgement. The authors would like to thank Fulvio Ricci for suggesting the extension from the 2D case to the higher dimensional case.

REFERENCES

- [1] G. S. Alberti, L. Balletti, F. De Mari, and E. De Vito. Reproducing subgroups of $Sp(2, \mathbb{R})$. Part I: Algebraic classification. *J. Fourier Anal. Appl.*, 19(4):651–682, 2013.
- [2] G. S. Alberti, F. De Mari, E. De Vito, and L. Mantovani. Reproducing subgroups of $Sp(2, \mathbb{R})$. Part II: admissible vectors. *Monatsh. Math.*, 173(3):261–307, 2014.
- [3] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. *Coherent states, wavelets, and their generalizations*. Theoretical and Mathematical Physics. Springer, New York, second edition, 2014.
- [4] J. G. Christensen. Sampling in reproducing kernel Banach spaces on Lie groups. *J. Approx. Theory*, 164(1):179–203, 2012.
- [5] J. G. Christensen and G. Ólafsson. Examples of coorbit spaces for dual pairs. *Acta Appl. Math.*, 107(1-3):25–48, 2009.
- [6] J. G. Christensen and G. Ólafsson. Coorbit spaces for dual pairs. *Appl. Comput. Harmon. Anal.*, 31(2):303–324, 2011.
- [7] O. Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2003.
- [8] J. L. Clerc. *Analyse Harmonique*, chapter “Les représentations des groupes compacts”. C.I.M.P.A., 1981. Ecole “Analyse harmonique”, 1980 Nancy (France).
- [9] E. Cordero, F. De Mari, K. Nowak, and A. Tabacco. Analytic features of reproducing groups for the metaplectic representation. *J. Fourier Anal. Appl.*, 12(2):157–180, 2006.

- [10] E. Cordero, F. De Mari, K. Nowak, and A. Tabacco. Reproducing groups for the metaplectic representation. In *Pseudo-differential operators and related topics*, volume 164 of *Oper. Theory Adv. Appl.*, pages 227–244. Birkhäuser, Basel, 2006.
- [11] E. Cordero, F. De Mari, K. Nowak, and A. Tabacco. Dimensional upper bounds for admissible subgroups for the metaplectic representation. *Math. Nachr.*, 283(7):982–993, 2010.
- [12] E. Cordero and A. Tabacco. Triangular subgroups of $Sp(d, \mathbb{R})$ and reproducing formulae. *J. Funct. Anal.*, 264(9):2034–2058, 2013.
- [13] S. Dahlke, F. De Mari, E. De Vito, D. Labate, G. Steidl, G. Teschke, and S. Vigogna. Coorbit spaces with voice in a Fréchet space. *to appear on J. Fourier. Anal. Appl.*
- [14] I. Daubechies. *Ten lectures on wavelets*, volume 61 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.
- [15] F. De Mari and E. De Vito. Admissible vectors for mock metaplectic representations. *Appl. Comput. Harmon. Anal.*, 34(2):163–200, 2013.
- [16] M. Duflo and C. C. Moore. On the regular representation of a nonunimodular locally compact group. *J. Functional Analysis*, 21(2):209–243, 1976.
- [17] H. G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. I. *J. Funct. Anal.*, 86(2):307–340, 1989.
- [18] H. G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. II. *Monatsh. Math.*, 108(2-3):129–148, 1989.
- [19] H. Führ. *Abstract harmonic analysis of continuous wavelet transforms*, volume 1863 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.
- [20] H. Führ and K. Gröchenig. Sampling theorems on locally compact groups from oscillation estimates. *Math. Z.*, 255(1):177–194, 2007.
- [21] C. Heil. *A basis theory primer*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, expanded edition, 2011.
- [22] E. Hernández and G. Weiss. *A first course on wavelets*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1996. With a foreword by Yves Meyer.
- [23] E. J. King. *Wavelet and frame theory: frame bound gaps, generalized shearlets, Grassmannian fusion frames, and p-adic wavelets*. PhD thesis, University of Maryland, College Park, 2009.
- [24] M. E. Taylor. *Noncommutative harmonic analysis*, volume 22 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1986.

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