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Abstract

We present a finite difference-(Multi-level) Monte Carlo algorithm to efficiently compute statistical solutions of the two dimensional Navier-Stokes equations, with periodic boundary conditions and for arbitrarily high Reynolds number. We propose a reformulation of statistical solutions in the vorticity-stream function form. The vorticity-stream function formulation is discretized with a finite difference scheme. We obtain a convergence rate error estimate for this approximation. We also prove convergence and complexity estimates, for the (Multi-level) Monte Carlo finite-difference algorithm to compute statistical solutions. Numerical experiments illustrating the validity of our estimates are presented. They show that the Multi-level Monte Carlo algorithm significantly accelerates the computation of statistical solutions, even for very high Reynolds numbers.

Contents

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1 Introduction

The flow of a viscous, incompressible fluid is described by the (incompressible) Navier-Stokes equations,

$$\frac{\partial}{\partial t} u + \operatorname{div}(u \otimes u) + \nabla p = \nu \Delta u + f \ , \quad \nabla \cdot u = 0. \tag{1}$$

Here, u denotes the velocity field and p is the (scalar) pressure that serves as a Lagrange multiplier to enforce the divergence free condition on the velocity. The source function f represents the effects of a body force. The kinematic viscosity is denoted by ν. This system of PDEs needs to be supplemented with suitable initial and boundary conditions.

Given the fundamental role played by the Navier-Stokes equations (1) in fluid dynamics, these equations have been the subject of extensive analytical and numerical study over the past century. The existence of global weak solutions (in both two and three dimensions) goes back to the seminal work of Leray (and Hopf). Uniqueness results are only available in two space dimensions. The questions of uniqueness (and particularly regularity) in three space dimensions are open and constitute one of the Clay institute's millennium prize problems.

Moreover, it is well known that fluid motion, particularly in three space dimensions, is highly sensitive to initial conditions and consists of irregular and chaotic motions [10]. This is especially true when the viscosity ν is small, equivalently the Reynolds number is high. Given considerable experimental and numerical evidence, such high Reynolds number *turbulent* flows are difficult to describe in terms of a deterministic framework, for instance weak solutions. Hence, it has become customary in fluid dynamics to consider solutions of the Navier-Stokes equations within a suitable statistical (probabilistic) framework.

One such probabilistic framework is the concept of statistical solutions, as proposed by Foias and Prodi in [9], see also [21] and the text book [8] and references therein. Within this framework of statistical solutions, the object of interest is a one parameter (time) family of probability measures that are defined on the function space containing initial data (weak solutions) for the Navier-Stokes equations. Once an initial probability measure is specified, it is evolved in a manner that is consistent with the Navier-Stokes dynamics. Existence of such statistical solutions has been established, see [8]. In two space dimensions, it has been shown that these statistical solutions are defined as a push forward of the initial probability measure, under the action of the Navier-Stokes flow (solution operator). Moreover, it has been demonstrated (see [8] and references therein) that statistical solutions can be used to describe some of the well-known empirical (and analytical) laws of turbulence, particularly of statistically stationary homogeneous isotropic turbulence.

Given the above discussion, the numerical approximation of such statistical solutions is an important issue. However, the design of suitable numerical methods for computing statistical solutions is challenging as the object to be computed is (the statistics of) a measure on a (infinite dimensional) function space. Consequently, the sheer size (dimension) of any computation can be prohibitively large.

The computation of statistical solutions falls within the much wider class of uncertainty quantification (UQ) in computational fluid dynamics (CFD). A large number of methods have been proposed to efficiently quantify uncertain solutions of the PDEs that arise in fluid flows. A (highly incomplete) list includes the well known stochastic galerkin methods [22] and stochastic collocation methods [14]. Another class of methods are of the Monte-Carlo type, particularly multi-level Monte Carlo (MLMC) methods [19] and references therein. However, very few, if any, of these methods have been adapted to the specific task of computing statistical solutions of the Navier-Stokes equations.

The main objective of the current paper is to propose a class of statistical sampling methods, of the Monte-Carlo (MC) and Multi-level Monte Carlo (MLMC) type, based on a robust finite difference (volume) discretization of the Navier-Stokes equations , and to prove that the resulting finite difference (volume) - (ML)MC method can approximate statistical solutions of the Navier-Stokes equations efficiently. The basis of our method is an observation in a recent paper [3] that under the physically reasonable assumption of finite kinetic energy of the flow ensemble, there exist ν-independent bounds on the second-moments of underlying statistical solution. Consequently, these bounds suffice for computation (and convergence) of a Monte-Carlo approximation of flow statistics at arbitrarily high Reynolds numbers.

In the current paper, we carry out a detailed convergence analysis of the MC as well as the considerably faster MLMC algorithms to compute such statistical solutions. We restrict ourselves to the two-dimensional case as uniqueness and a clear characterization of statistical solutions is well established in this case. In two space dimensions, it is more natural [18] to consider an equivalent vorticity-stream function formulation of the incompressible Navier-Stokes equations as the vorticity is a scalar quantity. Hence, we provide a novel reformulation of the Foais-Prodi statistical solutions of [8] in terms of the vorticity-stream function formulation. Our finite difference discretization and the resulting (ML)MC method are defined for this vorticity-stream function formulation.

Since the design of a MLMC method crucially relies on the availability of a rigorous error estimate for the underlying spatio-temporal discretization, we propose a novel finite difference discretization of the two-dimensional Navier-Stokes equations and show that it converges to a weak solution as the mesh is refined. Moreover, we prove an error estimate for the approximation. It turns out that the resulting method (error estimate) is robust with respect to the Reynolds number. We can even approximate the incompressible Euler equations ($\nu = 0$) with our proposed finite difference approximation. Consequently, this method, combined with an MLMC algorithm, enables the computation of flow statistics at arbitrarily high Reynolds numbers (in two-space dimensions).

The remainder of this paper is organized as follows. In section 2, we recapitulate the notion of statistical solutions, in the sense of Foais-Prodi and introduce the reformulation of statistical solutions in terms of the vorticity. In section 3, we present a novel finite discretization of the Navier-Stokes equations, written in terms of the vorticity-streamfunction formulation and prove an error estimate. The finite difference scheme is presented in section 3 and the Monte Carlo and Multi-level Monte Carlo methods are described in section 4 and numerical experiments reported in section 5.

2 Statistical Solutions of the Navier-Stokes equations.

In this section, we will review the theory of statistical solutions for the incompressible Navier-Stokes equations and present a reformulation of statistical solutions in the vorticity-stream function formulation in two space dimensions.

We use standard notation. The set $\mathbb{T}^2$ will denote the torus in $\mathbb{R}^2$, identified with $[0,1)^2$.

For two Hilbert spaces H and K, we denote by $\mathcal{L}_{iso}(H,K)$ the linear space of all bounded isomorphisms from H to K. By $H^s_{per}(\mathbb{T}^2)$ and $W^{k,p}_{per}(\mathbb{T}^2)$, for $k,s \geq 0$ and for $1 \leq p \leq \infty$, we shall denote Sobolev spaces of real-valued, 1-periodic functions in $\mathbb{T}^2$. Space of complex-valued functions are denoted by $H^s_{per}(\mathbb{T}^2;\mathbb{C})$ and $W^{k,p}_{per}(\mathbb{T}^2;\mathbb{C})$, correspondingly.

We consider the incompressible Navier-Stokes equations (1) in the torus $\mathbb{T}^2$, and in the finite time interval $\bar{J} := [0,T]$, for $T < \infty$.

In order to describe statistical solutions of (1), we start with a concise (and self-contained) review of the theory of Leray-Hopf weak solutions.

2.1 Individual Leray-Hopf Solutions

2.1.1 Sobolev Spaces

For the formulation of the equations, as well as for various regularity statements for solutions and a-priori estimates, we require to define function spaces on $\mathbb{T}^2$ with "periodic boundary conditions". By this we mean that $\mathbb{T}^2$-periodic extensions of elements to all of $\mathbb{R}^2$ belong *locally* to the same function space. We denote such spaces by a subscript per.

We identify $\mathbb{T}^2 = [0,1)^2$, and consider the incompressible Navier–Stokes equations (1), with periodic boundary conditions.

We adopt the usual (as presented, eg., in [20, 8]) functional analytic setting, and denote by H a subspace of divergence-free, square integrable vector fields in $L^2(\mathbb{T}^2)^2$ and by V a subspace of the closure of H in $H^1_{per}(\mathbb{T}^2)^d$. Since we consider periodic boundary conditions, we choose

$$V = V_{per} := \{v \in H^1_{per}(\mathbb{T}^2)^2 : \nabla \cdot v = 0 \text{ in } L^2(\mathbb{T}^2), \int_{\mathbb{T}^2} v \, dx = 0\}, \tag{2}$$

$$H = \{v \in H_{per}(\mathrm{div};\mathbb{T}^2) : \nabla \cdot v = 0 \text{ in } L^2(\mathbb{T}^2), \int_{\mathbb{T}^2} v \, dx = 0\} \, . \tag{3}$$

Here, $H_{per}(\mathrm{div};\mathbb{T}^2)$ denotes the closure of the set $C^\infty_{per}(\mathbb{T}^2)^2$ with respect to the $H(\mathrm{div})$ norm in $\mathbb{T}^2$, given by $\|\varphi\|^2_{H(\mathrm{div};\mathbb{T}^2)} = \|\varphi\|^2 + \|\mathrm{div}\varphi\|^2$ with $\| \circ \|$ denoting the $L^2(\mathbb{T}^2)$ norm.

We observe that $H \subset L^2(\mathbb{T}^2)$ is a closed subspace, which "sees" the periodic BCs in a weak sense: since $H \subset H(\mathrm{div};\mathbb{T}^2)$, velocity fields $v \in H(\mathrm{div};\mathbb{T}^2)$ admit a normal trace $H \ni (v \mapsto n \cdot v)|_{\partial\mathbb{T}^2}$ which is extends to a continuous map from $H(\mathrm{div};\mathbb{T}^2)$ to $H^{-1/2}(\partial\mathbb{T}^2)$ (cp. eg. [13, Thm. 2.5]).

In H as defined in (3), the difference of normal traces $n \cdot v$ across opposite sides of $\mathbb{T}^2$ vanishes in $H^{-1/2}(\partial\mathbb{T}^2)$, so that $H \subset (L^2(\mathbb{T}^2)/\mathbb{R})^2$ with strict inclusion.

Evidently, we have compact and dense embeddings $V \subset H$ and, throughout the following, we identify the Hilbert space H with its own dual, i.e., $H \simeq H^*$.

The spaces H and V of L-periodic functions on the unit torus $\mathbb{T}^2 = [0,1)^2$ admit Fourier characterizations : any velocity field $u(x) \in L^2(\mathbb{T}^2)^2$ can be represented as

$$u(x) = \sum_{\boldsymbol{k} \in \mathbb{Z}^2} \hat{u}_{\boldsymbol{k}} \exp(2\pi \boldsymbol{i}\boldsymbol{k} \cdot x) \, , \tag{4}$$

with convergence in $L^2(\mathbb{T}^2)$. The characterization (4) will generally yield complex-valued functions, which we indicate by writing $L^2(\mathbb{T}^2;\mathbb{C})$. The subspace of square-integrable functions in $\mathbb{T}^2$ which are real-valued is denoted by $L^2(\mathbb{T}^2)$. By Parseval's identity, the Hilbert spaces $L^2(\mathbb{T}^2;\mathbb{C})$ and $L^2(\mathbb{T}^2)$ are isomorphic to sequence spaces of Fourier coefficients; specifically, with (4) hold isometric isomorphisms

$$L^2(\mathbb{T}^2;\mathbb{C}) \simeq \{(\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2)\}\ , \quad L^2(\mathbb{T}^2) \simeq \{(\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2) : \hat{u}_{\boldsymbol{k}} = -\overline{\hat{u}_{\boldsymbol{k}}}\}\ . \tag{5}$$

By $L^2(\mathbb{T}^2)/\mathbb{R}$, we denote the space of functions with vanishing mean over $\mathbb{T}^2$, ie, $\int_{\mathbb{T}^2} u_i(x)dx = 0$. There hold isometric isomorphisms

$$L^2(\mathbb{T}^2;\mathbb{C})/\mathbb{R} \simeq \{(\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2_0(\mathbb{Z}^2)\}\ , \quad L^2(\mathbb{T}^2)/\mathbb{R} \simeq \{(\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2_0(\mathbb{Z}^2) : \hat{u}_{\boldsymbol{k}} = -\overline{\hat{u}_{\boldsymbol{k}}}\}\ , \tag{6}$$

with $\ell^2_0(\mathbb{Z}^2) := \{(\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^2(\mathbb{Z}^2) : \hat{u}_0 = 0\}$.

The spaces H and V can be characterized in terms of the sequence $\{u_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}} \subset \mathbb{C}^2$ of Fourier coefficients of its elements: specifically, there holds

$$H = \left\{ u = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} \hat{u}_{\boldsymbol{k}} \exp(2\pi \boldsymbol{i}\boldsymbol{k}\cdot x) : (\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \hat{V}_0 \right\}\ , \tag{7}$$

and

$$V = \left\{ u \in H : (\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \hat{V}_1 \right\}\ . \tag{8}$$

Here, for any $s \in \mathbb{R}$, the sets $\hat{V}_s$ are defined by

$$\hat{V}_s := \left\{ (\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^{2,s}_0(\mathbb{Z}^2;\mathbb{C})^2 : \hat{u}_{-\boldsymbol{k}} = \overline{\hat{u}_{\boldsymbol{k}}},\ \boldsymbol{k}\cdot\hat{u}_{\boldsymbol{k}} = 0 \right\} \tag{9}$$

where, for *any* finite smoothness index $s \in \mathbb{R}$, we introduced the weighted sequence spaces

$$\ell^{2,s}(\mathbb{Z}^2) := \left\{ (\hat{c}_{\boldsymbol{k}})_{\boldsymbol{k}} \in \mathbb{C}^{\mathbb{Z}^2} : \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} |\boldsymbol{k}|^{2s} |\hat{c}_{\boldsymbol{k}}|^2 < \infty \right\}\ , \quad \ell^{2,s}_0(\mathbb{Z}^2) = \ell^{2,s}(\mathbb{Z}^2) \cap \{\hat{c}_0 = 0\}\ . \tag{10}$$

Via the (distributional) Fourier series (4) these sequence spaces are isometrically isomorphic to

$$V_s := \left\{ u = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} \hat{u}_{\boldsymbol{k}} \exp(2\pi \boldsymbol{i}\boldsymbol{k}\cdot x) : (\hat{u}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \hat{V}_s \right\}\ . \tag{11}$$

These spaces coincide with the spaces $H^s(\mathrm{div}0;\mathbb{T}^2)$ of $\mathbb{T}^2$-periodic, real-valued and divergence-free distributional velocity fields u whose components u_1 and u_2 belong to the set of generalized functions $H^s(\mathbb{T}^2)$. In particular, $V_0 = H$ as in (3) and $V_1 = V$ as in (2), and for any $-\infty < s_1 \leq s_2 < \infty$, $V_{s_1} \supseteq V_{s_2}$ with continuous embedding which is compact for any $s_1 < s_2$.

On the spaces H and V we have the (canonical) inner products

$$(v,w)_H = \int_{\mathbb{T}^2} v\cdot w\, d\boldsymbol{x} \quad \text{and} \quad (v,w)_V = \int_{\mathbb{T}^2} \sum_{i=1}^{2} \frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_i}\, d\boldsymbol{x},$$

where $\boldsymbol{x} = (x_1,x_2) \in \mathbb{T}^2$, with associated norms

$$\|v\|_H = ((v,v)_H)^{1/2},\ \text{for } v \in H, \quad \|v\|_V = ((v,v)_V)^{1/2},\ \text{for } v \in V\ .$$

2.1.2 De Rham Complex. Hodge Decompositions

The exact sequence for $\mathbb{T}^2$, with periodic boundary conditions, is

$$H^1_{per}(\mathbb{T}^2) \xrightarrow{\nabla} H_{per}(\mathrm{curl};\mathbb{T}^2) \xrightarrow{\mathrm{rot}} L^2(\mathbb{T}^2)/\mathbb{R} \ . \tag{12}$$

Here, rot $= \nabla\times$ and the related operator curl are defined, for $\phi \in C^\infty_{per}(\mathbb{T}^2)$ and for $\underline{v} \in C^\infty_{per}(\mathbb{T}^2)^2$, as

$$\mathrm{curl}\phi := \left(\frac{\partial\phi}{\partial x_2}\ ,\ -\frac{\partial\phi}{\partial x_1}\right)\ ,\quad \mathrm{rot}\underline{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\ . \tag{13}$$

These operators extend, in the sense of distributions (and defined in terms of their Fourier representations), to $H^s_{per}(\mathbb{T}^2;\mathbb{R})$ resp. to $H^s_{per}(\mathbb{T}^2;\mathbb{R})^2$ for any $s \in \mathbb{R}$. In two space dimensions, the space $H(\mathrm{div})$ is obtained from $H(\mathrm{curl})$ by "component flipping", i.e.

$$H_{per}(\mathrm{div};\mathbb{T}^2) = \left\{\underline{v}_{\underline{u}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\ :\ \underline{u} \in H_{per}(\mathrm{curl};\mathbb{T}^2)\right\}$$

and (12) implies the "flipped exact sequence"

$$H^1_{per}(\mathbb{T}^2) \xrightarrow{\mathrm{curl}} H_{per}(\mathrm{div};\mathbb{T}^2) \xrightarrow{\mathrm{div}} L^2(\mathbb{T}^2)/\mathbb{R} \ . \tag{14}$$

In order to reformulate the NSE (1) equivalently as a scalar vorticity equation, we shall require Hodge decompositions of vector fields. There hold the identities

$$\mathrm{rot}(\mathrm{curl}\phi) = -\Delta\phi,\quad \mathrm{curl}(\mathrm{rot}\underline{v}) = -\Delta\underline{v} + \underline{\nabla}(\mathrm{div}\underline{v})\ . \tag{15}$$

Divergence-free velocity fields in V_s admit representations as curls of a scalar potential, the so-called *stream function* (cp. eg. [13, Thms. I.3.1, I.3.2] with obvious modifications for the present, periodic setting).

Proposition 2.1. *For every $s \in \mathbb{R}$, the (distributional) operator* curl *is an isomorphism*

$$\mathrm{curl} \in \mathcal{L}_{iso}(H^{s+1}_{per}(\mathbb{T}^2;\mathbb{R}); V_s)\ . \tag{16}$$

Likewise,

$$\mathrm{rot} \in \mathcal{L}_{iso}(V_s; H^{s-1}_{per}(\mathbb{T}^2;\mathbb{R}))\ , \tag{17}$$

and, with (16), (17) *and* (15),

$$-\Delta = \mathrm{rot}\circ\mathrm{curl} \in \mathcal{L}_{iso}(H^{s+1}_{per}(\mathbb{T}^2;\mathbb{R}); H^{s-1}_{per}(\mathbb{T}^2;\mathbb{R})) \tag{18}$$

Proof. We prove (16). For any $s \in \mathbb{R}$, consider $\psi \in H^{s+1}_{per}(\mathbb{T}^2;\mathbb{R})$. Then

$$\psi = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} \hat{\psi}_{\boldsymbol{k}} \exp(2\pi\boldsymbol{ik}\cdot x),\quad (\text{ convergence in } H^{s+1}_{per}(\mathbb{T}^2;\mathbb{R})\ ) \tag{19}$$

where the Fourier coefficients $\hat{\psi}_{\boldsymbol{k}}$ satisfy $(\hat{\psi}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2} \in \ell^{2,s+1}_0(\mathbb{Z}^2)$ and $\hat{\psi}_{-\boldsymbol{k}} = \overline{\hat{\psi}_{\boldsymbol{k}}}$. Define $\underline{v} := \mathrm{curl}(\psi)$. Then, formally,

$$\underline{v} = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} \hat{v}_{\boldsymbol{k}} \exp(2\pi\boldsymbol{ik}\cdot x)\ ,\quad \hat{v}_{\boldsymbol{k}} = \begin{pmatrix} -\boldsymbol{i}k_2\hat{\psi}_{\boldsymbol{k}} \\ +\boldsymbol{i}k_1\hat{\psi}_{\boldsymbol{k}} \end{pmatrix}\ ,\quad \boldsymbol{k} \in \mathbb{Z}^2\ .$$

We verify convergence of the Fourier series (19). Evidently,

$$\hat{v}_{-\boldsymbol{k}} = \begin{pmatrix} +\boldsymbol{i}k_2\hat{\psi}_{-\boldsymbol{k}} \\ -\boldsymbol{i}k_1\hat{\psi}_{-\boldsymbol{k}} \end{pmatrix} = \overline{\begin{pmatrix} -\boldsymbol{i}k_2 \\ \boldsymbol{i}k_1 \end{pmatrix}}\overline{\hat{\psi}_{\boldsymbol{k}}} = \overline{\hat{v}_{\boldsymbol{k}}}$$

and

$$\|(\hat{v}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2}\|^2_{\ell_0^{2,s}(\mathbb{Z}^2)} = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} |\boldsymbol{k}|^{2s}|\hat{v}_{\boldsymbol{k}}|^2 = \sum_{0\neq\boldsymbol{k}\in\mathbb{Z}^2} |\boldsymbol{k}|^{2s}(k_1^2+k_2^2)|\hat{\psi}_{\boldsymbol{k}}|^2 = \left\|(\hat{\psi}_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{Z}^2}\right\|^2_{\ell_0^{2,s+1}(\mathbb{Z}^2)} .$$

As this implies that (19) converges in the norm of V_s, and as the mapping between (sequences of) Fourier coefficients is bijective from V_s to $H^{s+1}_{per}(\mathbb{T}^2;\mathbb{R})$, this proves (16).

The proof of (17) is analogous. Finally, (18) follows from (15), (16) and (17). □

Remark 2.2. *For $s = 0$ there holds for $H = V_0$ in* (3)

$$H = \mathrm{curl}H^1_{per}(\mathbb{T}^2) , \quad \mathrm{rot}H = H^{-1}_{per}(\mathbb{T}^2) , \tag{20}$$

and, for the space $V = V_1$ in (2) *there holds*

$$V = \mathrm{curl}H^2_{per}(\mathbb{T}^2) , \quad \mathrm{rot}V = L^2_{per}(\mathbb{T}^2) . \tag{21}$$

2.1.3 Leray-Hopf weak solutions of the NSE

With the function spaces just defined, the *weak formulation* of Equation (1) reads: given $T > 0$, viscosity $\nu > 0$ and initial data $u_0 \in H$ and forcing $f \in L^2(J;H)$, find $u \in L^\infty(J;H) \cap L^2(J;V)$, such that, for all $v \in V$

$$\frac{d}{dt}(u,v)_H + \nu(u,v)_V + b(u,u,v) = (f,v)_H . \tag{22}$$

Since our formulation is divergence-free there is no pressure term entering in the weak formulation. In Equation (22), the trilinear form b is defined by

$$b(u,v,w) = \sum_{i,j=1}^{2} \int_D u_i \frac{\partial v_j}{\partial x_i} w_j \, d\boldsymbol{x} .$$

The trilinear form b is continuous on $V \times V \times V$ and

$$\forall u,v,w \in V : \quad b(u,v,v) = 0 \quad \text{and} \quad b(u,v,w) = -b(u,w,v) .$$

An equivalent formulation involves the Stokes operator

$$Av = -P\Delta v, \tag{23}$$

for all $v \in V_2$. Here, P denotes the Leray projection onto H in $L^2(D)^2$. The Stokes operator A in (23) is a positive, self-adjoint operator so that its fractional powers are well-defined. We denote the fractional powers by A^a, for $a \in \mathbb{R}$, and by $\mathcal{D}(A^a)$ the domain of A^a. We have $\mathcal{D}(A^{1/2}) = V$ and, more generally, $\mathcal{D}(A^{s/2}) = V_s$. Further, the trilinear form b induces, for fixed $u \in V$, a bilinear operator $B : \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \to \mathcal{D}(A^{-1/2})$ defined by

$${}_{\mathcal{D}(A^{-1/2})}\langle B(u,v), w\rangle_{\mathcal{D}(A^{1/2})} = b(u,v,w),$$

for all $u,v,w \in \mathcal{D}(A^{1/2})$. After these preliminaries, we introduce the operator formulation of the NSE (1): given $0 < T < \infty$, $u_0 \in H$ and $f \in L^2(J;H)$, in the time-interval $J = [0,T]$ find $u \in L^\infty(J;H) \cap L^2(J;V)$ with $u' \in L^1(J;\mathcal{D}(A^{-1/2}))$ such that

$$u' + \nu Au + B(u,u) = f . \tag{24}$$

Definition 2.3. *On a time interval $J \subset \mathbb{R}$, a function $u : J \mapsto H$ is called a* Leray–Hopf weak solution *of Equation* (1) *if*

1. $u \in L^\infty_{loc}(J; H) \cap L^2_{loc}(J; V)$,
2. $(\partial_t u)(\cdot) \in L^{4/3}_{loc}(J; V')$,
3. $t \mapsto u(t) \in \mathcal{C}_{loc}(J; H_w)$ *(i.e. for every $v \in H$, $t \mapsto (u(t), v)_H$ is continuous from J to $\mathbb{R}$),*
4. *u satisfies Equation* (24) *in the sense of distributions on J with values in V',*
5. *for almost all $t, t' \in J$, u satisfies the energy inequality*

$$\frac{1}{2}\|u(t)\|_H^2 + \nu \int_{t'}^t \|u(s)\|_V^2 \, ds \leq \frac{1}{2}\|u(t')\|_H^2 + \int_{t'}^t (f(s), u(s))_H \, ds. \tag{25}$$

If further $J = [t_0, t_1]$ is closed and left-bounded then

(vi) $u(t)$ is strongly right-continuous in H at t_0, i.e. $u_(t_0) = \lim_{t \downarrow t_0} u(t)$ in H.

As is well-known, in space dimension $d = 2$, for any $\nu > 0$ and for any initial data $\underline{u}_0 \in H$, individual Leray–Hopf solutions u of the NSE exist and are unique. For future reference we state the results (see, e.g., [8, Thms. II.7.3, II.7.4] and [20, Thm. 3.1] and the references there)).

Proposition 2.4. *Assume*

$$\underline{u}_0 \in H \ , \quad \underline{f} \in L^2(J; H) \ . \tag{26}$$

Then, for $\nu > 0$ exists a unique Leray–Hopf solution $\underline{u}$ of the NSE such that

$$\underline{u} \in L^2(J; V) \cap H^1(J; V_{-1}) \cap C(\overline{J}; H) \ .$$

If, moreover,

$$\underline{u}_0 \in V \ , \tag{27}$$

then the unique solution $\underline{u}$ of the NSE belongs to

$$L^2(J; V_2) \cap H^1(J; V) \cap C(\overline{J}; V) \ .$$

Under (26)*, if, in particular, the forcing $\underline{f}$ is independent of t, there exists a continuous solution operator $S^\nu : H \mapsto H$ such that $\underline{u}(t) = S^\nu(t)\underline{u}_0$.*

2.2 Statistical solutions

Contrary to the deterministic viewpoint of the Navier–Stokes equations, which associates a unique (in dimension $d = 2$) weak solution $u(t)$ to a given initial velocity $u_0 \in H$, statistical velocity solutions aim at describing the evolution of *ensembles of solutions* through their probability distribution. Throughout, we assume that a complete probability space, $(\Omega, \mathcal{A}, \mathbb{P})$, is given.

2.2.1 Statistical velocity solutions

In the definition of statistical velocity solutions we assume given a probability measure on H, which is equipped with the Borel-σ-algebra $\mathcal{B}(H)$. A statistical velocity solution is a (parametric family of) probability measure(s). Specifically, we consider statistical velocity solutions in the sense of Foiaş–Prodi: there, statistical velocity solutions of the Navier–Stokes equations are one-parameter families of probability measures which describe the evolution of velocity ensembles and their statistics.

The initial-boundary value problem for individual solutions in Equation (22) could be interpreted as a special case of a statistical solution: in this case, the measure μ_0 places unit mass at one initial velocity $u_0 \in H$. In general, the initial distribution is defined on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is assumed to be given as an image measure under an H-valued random variable with distribution μ_0. This means that

$$\mathbb{P}(\{u_0 \in E\}) = \mu_0(E) \ , \quad E \in \mathcal{B}(H) \ . \tag{28}$$

Rather than following an individual Leray-Hopf solution $\{\underline{u}(t)\}_{t\geq 0}$, *statistical velocity solutions* aim to describe *velocity ensembles* via a one-parameter family $\mu^\nu = \{\mu_t^\nu\}_{t\geq 0}$ of probability measures such that $\mu_0^\nu = \mu_0$ (up to null sets) for a given probability measure on all initial flow configurations and such that

$$\forall t > 0 : \qquad \mathbb{P}(\{u(t) \in E\}) = \mu_t^\nu(E) \ , \quad E \in \mathcal{B}(H) \ . \tag{29}$$

This random variable is defined as a mapping from the measurable space $(\Omega, \mathcal{F})$ into the measurable space $(H, \mathcal{B}(H))$ such that $\mu_0 = X \circ \mathbb{P}$. As a consequence of the global uniqueness of Leray-Hopf solutions in space dimension $d = 2$, there exists a unique, one-parameter family of probability measures $\mu^\nu = (\mu_t^\nu)_{t\in J}$ on the measurable space $(H, \mathcal{B}(H))$ of all flow configurations. At time $t \in [0, T]$, μ_t describes the statistics of the ensemble $\{\underline{u}(t)\}$ of "velocity snapshots" at time t. It is, in space dimension two, and in the absence of forcing, ie., for $f(t) = 0$, given by the initial measure μ_0 transported under the Leray-Hopf flow $S^\nu(t)$, ie.,

$$\mu_t^\nu(E) = \mu_0((S^\nu(t))^{-1}(E)) \quad \text{for all} \quad E \in \mathcal{B}(H) \ , \tag{30}$$

where $\{S^\nu(t)\}_{t\geq 0}$ denotes the Leray-Hopf solution flow of the NSE in space dimension $d = 2$ in Proposition 2.4. We shall refer to the above family in what follows as *statistical velocity solutions*. We will use the following existence result (see [7, Thm. 1 and Prop.1]).

Theorem 2.5. *Let μ_0 be a Borel probability measure on H with finite mean kinetic energy, ie.,*

$$\int_H \|v\|_H^2 \, d\mu_0(v) < +\infty \ .$$

Let, moreover, $f \in L^2(J; H)$ be a forcing term on the time interval $J = (0, T)$ for some $0 < T < \infty$. Then, for every $\nu > 0$ and for the periodic boundary conditions (with the spaces as in Equation (3)*, Equation* (2)*), there exists a unique statistical solution $(\mu_t^\nu, t \in J)$ of the Navier–Stokes equations* (1) *on H in the sense of* (30)*.*

If, moreover, μ_0 is supported in $B_H(R)$ for some $0 < R < \infty$, and if the forcing term $f \in H$ is time-independent, the statistical solution $\mu^\nu = (\mu_t^\nu)_{t\in J}$ is unique and explicitly given by (30) *i.e. by μ_0 transported under the LH flow $(S^\nu(t), t \in J)$ of Equation* (22)*.*

The proof of the existence (and uniqueness in dimension $d = 2$) result Theorem 2.5 can be found in [8, Thm. V.1.4] for the presently considered periodic boundary conditions in $\mathbb{T}^2$, in the spaces V_s of divergence-free velocity fields with vanishing average over $\mathbb{T}^2$.

2.2.2 Vorticity Reformulation of the NSE, Eqn. (1)

In two space dimensions, it is convenient to consider the *vorticity formulation* of the NSE (1); to present the reformulation, we assume for convenience that

$$f \equiv 0 \ . \tag{31}$$

The passage between (divergence-free) velocity fields $u(t)$ in the (periodic) function spaces V resp. in H defined in (2) and (3), respectively, and a (in the presently considered, two-dimensional case) scalar vorticity function is facilitated by the following results.

By Proposition 2.1, for any $s \in \mathbb{R}$ we may associate with velocity fields $\underline{u}(t) \in V_s$ unique, scalar in space dimension $d = 2$, *vorticities* $\eta(t) \in H^{s-1}_{per}(\mathbb{T}^2)$ via

$$\eta(t) = \mathrm{rot}\underline{u}(t) = \partial_2 u_1(t) - \partial_1 u_2(t) \ , \tag{32}$$

and a stream function $\psi(t) \in H^{s+1}_{per}(\mathbb{T}^2)$ which is scalar, in space dimension $d = 2$, via

$$\underline{u}(t) = \mathrm{curl}\psi(t) = (\partial_2\psi(t), -\partial_1\psi(t)) = \mathrm{curl} \circ (-\Delta)^{-1}\eta(t) \in V_s \ . \tag{33}$$

The relation $\underline{u}(t) = \mathrm{curl} \circ (-\Delta)^{-1}\eta(t)$ constitutes the *Biot-Savart Law.*

Proposition 2.6. *For any initial velocity* $\underline{u}_0 \in V \subset H$ *in the space-periodic case the vorticity* η *and the stream function* ψ *satisfy the a-priori estimates*

$$\|\eta\|^2_{L^2(J;H^1_{per}(\mathbb{T}^2;\mathbb{R}))} \leq \frac{1}{\nu}|u(0)|^2_{H^1} + \frac{T}{\nu^2}\|f\|^2_{L^\infty(J;H)} = \frac{1}{\nu}\|\eta_0\|^2_{L^2(\mathbb{T}^2)} + \frac{T}{\nu^2}\|f\|^2_{L^\infty(J;H)}$$

and

$$\sup_{0\leq t\leq T} \|\eta(t)\|^2_{L^2(\mathbb{T}^2)} \leq |u_0|^2_{H^1} + \frac{1}{\nu}\|f(t)\|^2_H = \|\eta_0\|^2_{L^2(\mathbb{T}^2)} + \frac{1}{\nu}\|f(t)\|^2_H \ .$$

Proof. The proposition is a straightforward consequence of [8, Eqn. (II.A.65-67)]:

$$\|\eta\|^2_{L^2_t H^1_x} \leq \|A\underline{u}\|^2_{L^2_t L^2_x} \overset{A.67}{\leq} \frac{1}{\nu}\,|\underline{u}(0)|^2_{H^1} + \frac{1}{\nu^2}\,\|f\|^2_{L^\infty_t H}\,|J| = \frac{1}{\nu}\,\|\eta(0)\|^2_{L^2_x} + \frac{1}{\nu^2}\,\|f\|^2_{L^\infty_t H}\,|J|$$

resp.

$$\begin{aligned}\|\eta\|^2_{L^\infty_t L^2_x} = |\underline{u}|^2_{L^\infty_t H^1_x} &\overset{A.66}{\leq} |\underline{u}(0)|^2_{H^1}\, C(t,\nu,\mathbb{T}^2) + \frac{1}{\nu^2\lambda_1}\,\|f\|^2_{L^\infty_t H}\,(1 - C(t,\nu,\mathbb{T}^2)) \\ &= \|\eta(0)\|^2_{L^2_x} + \frac{1}{\nu^2\lambda_1}\,\|f\|^2_{L^\infty_t H}\,(1 - C(t,\nu,\mathbb{T}^2))\end{aligned}$$

The constant $0 < C(t,\nu,\mathbb{T}^2) := e^{-\nu\lambda_1|J|} < 1$. In absence of forcing ($f \equiv 0$), the second reduces to an enstrophy stability statement. □

Likewise, by referring again to Proposition 2.1, for $f \equiv 0$, the vorticity $\eta(t) = \mathrm{rot}\underline{u}(t)$ satisfies *formally* in $J \times \mathbb{T}^2$ the *viscous vorticity equation*:

For $s \geq 0$, given $\nu > 0$, find $\eta \in X_s := L^2(J; H^{s+1}_{per}(\mathbb{T}^2)) \cap H^1(J; H^{s-1}_{per}(\mathbb{T}^2))$ such that

$$\partial_t\eta + \underline{u}\cdot\underline{\nabla}\eta = \nu\Delta\eta, \quad \text{in } L^2(J; H^{s-1}_{per}(\mathbb{T}^2)) \ , \tag{34a}$$

$$-\Delta\psi = \eta \quad \text{in } L^2(J; H^{s+1}_{per}(\mathbb{T}^2)) \ , \tag{34b}$$

$$\eta|_{t=0} = \eta_0 \quad \text{in } H^s_{per}(\mathbb{T}^2) \ . \tag{34c}$$

By (18), $H^{-1}_{per}(\mathbb{T}^2) = \Delta(H^1_{per}(\mathbb{T}^2))$.

2.2.3 Statistical Vorticity Solutions

In the context of statistical solutions, we assume in the following at least *finite variance enstrophy*, ie.

$$\eta_0 \in L^2(\Omega; L^2(\mathbb{T}^2)) \ . \tag{35}$$

Based on the isomorphism (20), the initial condition (35) is equivalent to the NSE (1) with a random initial velocity $u_0 \in L^2(\Omega; V)$ with law $\mu_0 = u_0 \circ \mathbb{P}$ on H. Under stronger regularity assumptions than (35) *equivalence of equations* (1) *and* (34) holds.

Proposition 2.7. *For $\nu > 0$, the NSE* (1) *with initial data $u_0 \in V_{1+s} = H^{1+s}_{per}(\mathrm{div}0; \mathbb{T}^2)$ for some $s > 0$ is equivalent to the problem* (34a) - (34c) *with initial vorticity $\eta_0 \in H^s_{per}(\mathbb{T}^2)$ with the same s.*

Proof. We verify that (34) is meaningful in the 'usual', parabolic Bochner spaces. To this end, observe that for any $s \geq 0$, we have the continuous embeddings

$$X_s = L^2(J; H^{s+1}_{per}(\mathbb{T}^2)) \cap H^1(J; H^{s-1}_{per}(\mathbb{T}^2)) \subset C^0(\overline{J}; H^s_{per}(\mathbb{T}^2)) \subset L^2(J; H^s_{per}(\mathbb{T}^2)) \ , \tag{36}$$

Therefore, the initial condition (34c) is meaningful in $H^s_{per}(\mathbb{T}^2)$.

Consider the multiplication $\underline{u} \cdot \underline{\nabla}\eta$ in (34a). By (33) and (34b), we have

$$\underline{u} = \mathrm{curl}\psi = \mathrm{curl} \circ (-\Delta)^{-1}\eta = \mathrm{rot}^{-1}\eta \ . \tag{37}$$

From the mapping properties (33) and (34b) of the operators curl and $-\Delta$, we also obtain

$$\underline{u} \in \left(\mathrm{curl} \circ (-\Delta)^{-1}\right) X_s = L^2(J; V_{s+2}) \cap H^1(J; V_s) \subset C^0(\overline{J}; V_{s+1}) \subset L^2(J; V_{s+1}) \ , \quad s \geq 0 \ .$$

In $\mathbb{T}^2$, we have for $s > 0$ from the Sobolev embedding

$$V_{s+1} = H^{s+1}_{per}(\mathrm{div}0; \mathbb{T}^2) \subset H^{s+1}_{per}(\mathbb{T}^2)^2 \subset C^{s'}_{per}(\mathbb{T}^2)^2$$

for any $0 \leq s' < s$. We may therefore estimate

$$\begin{aligned} \|\underline{u} \cdot \underline{\nabla}\eta\|_{L^2(J; L^2_{per}(\mathbb{T}^2))} &\leq C\|\underline{u}\|_{C^0(J; C^{s'}_{per}(\mathbb{T}^2))}\|\eta\|_{L^2(J; H^1_{per}(\mathbb{T}^2))} \\ &= C\|\mathrm{curl} \circ (-\Delta)^{-1}\eta\|_{C^0(J; C^{s'}_{per}(\mathbb{T}^2))}\|\eta\|_{L^2(J; H^1_{per}(\mathbb{T}^2))} \\ &\leq C\|(-\Delta)^{-1}\eta\|_{C^0(J; C^{s'+1}_{per}(\mathbb{T}^2))}\|\eta\|_{L^2(J; H^1_{per}(\mathbb{T}^2))} \ . \end{aligned}$$

This shows that for $s > 0$, the product in (34a) is defined as pointwise product, for a.e. $(t, x) \in J \times \mathbb{T}^2$. The uniqueness of the Leray-Hopf solutions $\{\underline{u}(t)\}_{t\in J}$ to (1) and the isomorphism in Proposition 2.1, imply that the corresponding vorticities $\{\eta(t)\}_{t\in J}$ given by (32) are weak solutions of the parabolic evolution equation (34a) in X_s.

The uniqueness of the Leray solution $\{\underline{u}(t)\}_{t\in J}$ and the isomorphism properties of the map rot in Proposition 2.1 imply that the solution family $\{\eta(t)\}_{t\in J}$ is unique in X_s. □

For $\nu > 0$, we use the isomorphism (20) and (34) to "transport" statistical velocity solutions $\{\mu^\nu_t\}_{t\geq 0}$ on ensembles $V_s \subset H$ of initial velocity fields to *statistical vorticity solutions* $\{\pi^\nu_t\}_{t\geq 0}$. We start by considering initial data. For any initial velocity ensemble of finite kinetic energy, ie., for $\underline{u}_0 \in L^2(\Omega; H)$ with respect to a probability measure $\mu_0 = \mathbb{P}_{\#u_0}$ on $(H, \mathcal{B}(H))$, and for $\nu > 0$, the isomorphism property $\mathrm{rot} \in \mathcal{L}_{iso}(H, H^{-1}_{per}(\mathbb{T}^2))$ in (17) yields an initial vorticity ensemble

$$\eta_0 = \mathrm{rot}\underline{u}_0 \in L^2(\Omega; H^{-1}_{per}(\mathbb{T}^2))$$

with respect to the transported initial measure $\pi_0 = \mathbb{P}_{\#\mathrm{rot}\underline{u}_0}$. By Proposition 2.1, for any $s \geq -1$, also $\underline{u}_0 \in L^2(\Omega; V_{1+s}) \subset L^2(\Omega; H)$ with law μ_0 on H is in 1-to-1 correspondence to $\eta_0 \in L^2(\Omega; H^s_{per}(\mathbb{T}^2; \mathbb{R}))$ with law π_0 on $H^{-1}_{per}(\mathbb{T}^2)$ via the isometry map rot.

Proposition 2.8. *For $\nu > 0$ and for any $s > 0$, for any initial probability measure π_0 with finite second moments on $H^s_{per}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2)$, there exists a unique vorticity statistical solution $\{\pi^\nu_t\}_{t\in J}$ such that*

$$\forall t \in J \; \forall F \in \mathcal{B}(H^s_{per}(\mathbb{T}^2)) : \quad \pi^\nu_t(F) = \pi_0((T^\nu)^{-1}(F)) \tag{38}$$

where T^ν denotes the "vorticity flow" from $H^s_{per}(\mathbb{T}^2)$ to X_s in (36), given by $\eta(t) = T^\nu(t)\eta_0$.

Proof. By Proposition 2.4, for $s > 0$ and for $u_0 \in V_{1+s} \subset H$ and for $\nu > 0$, there exists a unique individual Leray solution $\{\underline{u}(t)\}_{t\in J}$ of (1). By Proposition 2.7, for $s > 0$ the family of corresponding individual vorticities given by $\eta(t) = \mathrm{rot}\underline{u}(t)$ satisfies $(\eta(t))_{t\in J} \subset X_s$ and the individual vorticities are the unique, weak solutions of the vorticity equation (34). By the uniqueness of the Leray-Hopf velocity solutions in space dimension 2 and by the isomorphism property S^ν, the vorcity solutions are unique.

The Leray flow $\underline{u}(t) = S^\nu(t)\underline{u}_0$ induces a corresponding vorticity flow $(T^\nu(t))_{t\in J}$ from $H^s_{per}(\mathbb{T}^2)$ to X_s in (36) via

$$\eta(t) = \mathrm{rot}\underline{u}(t) = \mathrm{rot} \circ S^\nu(t)\underline{u}_0 = (\mathrm{rot} \circ S^\nu(t) \circ \mathrm{rot}^{-1})\eta_0 =: T^\nu(t)\eta_0 \; . \tag{39}$$

We use the isomorphism properties of curl and of rot in Proposition 2.1 to "transport" the unique statistical velocity solution $(\mu^\nu_t)_{t\in J}$ to the vorticity formulation (34) by the correspondence (37). Since $\mathrm{rot} \in \mathcal{L}_{iso}(V_{1+s}, H^s_{per}(\mathbb{T}^2))$, to each $F \in \mathcal{B}(H^s_{per}(\mathbb{T}^2))$ corresponds a unique $E = \mathrm{rot}^{-1}(F) \in \mathcal{B}(V_{1+s})$. We define the one-parameter of probability measures $(\pi^\nu_t)_{t\in J}$ by

$$\forall F \in \mathcal{B}(H^s_{per}(\mathbb{T}^2)) : \quad \pi^\nu_t(F) := \mu^\nu_t(\mathrm{rot}^{-1}(F)) \; .$$

Then

$$\forall F \in \mathcal{B}(H^s_{per}(\mathbb{T}^2)) : \quad \pi^\nu_t(F) = \pi_0((T^\nu(t))^{-1}(F)) \; . \tag{40}$$

To see this, we calculate with (39) for every $F \in \mathcal{B}(H^s_{per}(\mathbb{T}^2))$ such that $F = \mathrm{rot}E$ for $E \in \mathcal{B}(V_{1+s})$ that

$$\begin{aligned} \pi^\nu_t(F) &= \mu^\nu_t(\mathrm{rot}^{-1}(F)) \\ &= \mu_0((S^\nu(t))^{-1} \circ \mathrm{rot}^{-1}(F)) \\ &= \pi_0((\mathrm{rot} \circ S^\nu(t) \circ \mathrm{rot}^{-1})^{-1}(F)) \\ &= \pi_0((T^\nu(t))^{-1}(F)) \; . \end{aligned}$$

Therefore $(\pi^\nu_t)_{t\in J}$ is the family of images of the statistical velocity solution $(\mu^\nu_t)_{t\in J}$ under the isometry rot in (33) and therefore the unique vorticity statistical solution. □

3 A convergent finite difference scheme for the Navier-Stokes equations

Our main aim in this paper is to design an efficient numerical scheme to approximate the statistical vorticity solutions of the two-dimensional incompressible Navier-Stokes equations, defined in the previous section. As mentioned in the introduction, we will employ statistical sampling procedures such as Monte Carlo (MC) and Multi-level Monte Carlo (MLMC) in this endeavor. These schemes utilize independent identically distributed samples on the underlying space of initial data and weak solutions. Hence, we need an efficient discretization procedure for the deterministic two-dimensional Navier-Stokes equations. Moreover, a rigorous error estimate for such a scheme is also necessary in order to define the MLMC method.

A variety of efficient numerical methods have been designed to approximate the incompressible Navier-Stokes equations. These include finite element methods ([13] and references therein).

However, it is often problematic to approximate very high Reynolds number flows with finite element methods. Spectral (viscosity) methods ([6] and references therein) are an attractive alternative, particularly for periodic problems. Another class of methods are the so-called vortex methods ([18] and references therein), which are better suited for the zero viscosity limit i.e, the incompressible Euler equations.

A different class of numerical methods fall under the rubric of finite difference (finite volume) methods and include the well-known finite difference-Leray projection methods [16, 5]. More efficient variants are proposed in [4, 1]. These methods discretize the velocity-pressure form of the Navier-Stokes equations (1).

In this paper, we consider the statistical vorticity solutions of the Navier-Stokes equations. Therefore, we need a discretization of the vorticity-stream function form of these equations (34). Although many different finite difference discretizations of these methods have been proposed (see [17] and references therein), we were unable to find a finite difference discretization of the two-dimensional Navier-Stokes equations in the vorticity-stream function formulation, for which convergence with convergence rate bounds has been shown. Hence, we propose a *novel* finite difference discretization of the two-dimensional Navier-Stokes equations and prove a rigorous error estimate for it, in this section. Our scheme will be a variant of the finite difference projection method from [17].

3.1 Discrete derivatives and Preliminaries

We discretize the torus $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ (which we identify with $[0,1)^2$) with a Cartesian mesh $\mathbb{Z}/N_x\mathbb{Z} \times \mathbb{Z}/N_y\mathbb{Z} := M_x \times M_y \cong \mathcal{M} \ni \mathcal{C}_{i,j}, i = 0, \ldots, N_x, j = 0, \cdots, N_y$ (with $N_x, N_y \in \mathbb{N}$) of mesh sizes, Δx resp. Δy. Define $I := (N_x + 1) \times (N_y + 1)$. We fix a finite time $T > 0$ and consider the time interval $[0, T]$, discretized with $t^0, \cdots, t^N$ steps of difference Δt. The time step Δt is chosen such that $O(\Delta t) = O(\Delta x) = O(\Delta y) =: O(h)$. Hence, the discrete set of points are denoted as $\{x_i, y_j\}$. We denote $C_{i,j} := [x_{i-1/2}, x_{i+1/2}) \times [y_{j-1/2}, y_{i+1/2})$ as a *Cell* with cell center at (x_i, y_j)

As we are interested in designing a finite difference (volume) scheme, we define the following standard discrete derivatives (finite difference operators) for a generic (scalar or vector, depending on context) grid function $w_{i,j} \approx w(x_i, y_j)$,

$$D_+^x w_{i,j} = \frac{w_{i+1,j} - w_{i,j}}{\Delta x}, \quad D_-^x w_{i,j} = \frac{w_{i,j} - w_{i-1,j}}{\Delta x} \tag{41}$$

$$D_+^y w_{i,j} = \frac{w_{i,j+1} - w_{i,j}}{\Delta y}, \quad D_-^y w_{i,j} = \frac{w_{i,j} - w_{i,j-1}}{\Delta y} \tag{42}$$

$$\mathrm{grad}_\pm^h = \left(D_\pm^x, D_\pm^y\right), \quad \mathrm{curl}_\pm^h = \left(D_\pm^y, -D_\pm^x\right) \tag{43}$$

$$\Delta^h w_{i,j} = \left(D_+^x D_-^x + D_+^y D_-^y\right) w_{i,j}. \tag{44}$$

Similarly for a vector grid function $w = \left(w^1, w^2\right)$, we denote the following discrete derivatives,

$$\mathrm{rot}_\pm^h w_{i,j} = D_\pm^x w_{i,j}^2 - D_\pm^y w_{i,j}^1, \quad \mathrm{div}_\pm^h w_{i,j} = D_\pm^x w_{i,j}^1 + D_\pm^y w_{i,j}^2. \tag{45}$$

We are interested in approximating the following discrete versions of the vorticity, stream function and velocity fields,

$$\eta_{i,j}^{(n)} \approx \eta(x_i, y_j, t^n), \qquad \psi_{i,j}^{(n)} \approx \psi(x_i, y_j, t^n), \tag{46}$$

$$(u_1)_{i,j+1/2}^{(n)} \approx (u_1)(x_i, y_{j+1/2}, t^n), \qquad (u_2)_{i+1/2,j}^{(n)} \approx (u_2)(x_{i+1/2}, y_j, t^n). \tag{47}$$

Hence, we collocate the discrete vorticity and stream function at the cell centers of the cell $C_{i,j}$. On the other hand, the velocities are collocated on the mid-points of the tangential edges.

However, it is more reasonable to collocate the velocities on the midpoints of the normal edges. We do so by the following velocity repositioning,

Proposition 3.1. *Assume we have a velocity field, as defined in* (46) *that satisfies the discrete divergence relation,*

$$\frac{(u_1)_{i+1,j+1/2} - (u_1)_{i,j+1/2}}{\Delta x} + \frac{(u_2)_{i+1/2,j+1} - (u_2)_{i+1/2,j}}{\Delta y} = 0.$$

We define velocity components repositioned on the orthogonal edge:

$$(\tilde{u_1})_{i+1/2,j} := \frac{(u_1)_{i,j+1/2} + (u_1)_{i+1,j+1/2} + (u_1)_{i,j-1/2} + (u_1)_{i+1,j-1/2}}{4} \tag{48}$$

$$(\tilde{u_2})_{i,j+1/2} := \frac{(u_2)_{i+1/2,j+1} + (u_2)_{i+1/2,j} + (u_2)_{i-1/2,j+1} + (u_2)_{i-1/2,j}}{4}. \tag{49}$$

Then one has

$$\frac{(\tilde{u_1})_{i+1/2,j} - (\tilde{u_1})_{i-1/2,j}}{\Delta x} + \frac{(\tilde{u_2})_{i,j+1/2} - (\tilde{u_2})_{i,j-1/2}}{\Delta y} = 0. \tag{50}$$

Proof. The proof is a consequence of the following straightforward calculation,

$$\begin{aligned}
&\frac{(\tilde{u_1})_{i+1/2,j} - (\tilde{u_1})_{i-1/2,j}}{\Delta x} + \frac{(\tilde{u_2})_{i,j+1/2} - (\tilde{u_2})_{i,j-1/2}}{\Delta y} \\
= &\frac{(u_1)_{i,j+1/2} + (u_1)_{i+1,j+1/2} + (u_1)_{i,j-1/2} + (u_1)_{i+1,j-1/2}}{4\Delta x} \\
- &\frac{(u_1)_{i-1,j+1/2} + (u_1)_{i,j+1/2} + (u_1)_{i-1,j-1/2} + (u_1)_{i,j-1/2}}{4\Delta x} \\
+ &\frac{(u_2)_{i+1/2,j+1} + (u_2)_{i+1/2,j} + (u_2)_{i-1/2,j+1} + (u_2)_{i-1/2,j}}{4\Delta y} \\
- &\frac{(u_2)_{i+1/2,j} + (u_2)_{i+1/2,j-1} + (u_2)_{i-1/2,j} + (u_2)_{i-1/2,j-1}}{4\Delta y} = 0\ .
\end{aligned}$$

□

3.2 Discrete Poincaré inequalities

We will present a scheme for approximating the Navier-Stokes equations and prove an error estimate. Hence, we need suitable discrete norms. To this end, we define for the generic discrete spatial grid vector $w = \{w_{i,j}\}$,

$$\|w\|_p := \left(\sum_{(i,j)\in\mathcal{M}} |w_{i,j}^{(n)}|^p \Delta x \Delta y \right)^{1/p} \tag{51}$$

(the usual *discrete* L^p norm in space) and for the generic discrete space-time grid vector $w = \{w_{i,j}^{(n)}\}$

$$\|w\|_{p,q} := \left(\sum_{n=0}^{N} \left(\sum_{(i,j)\in\mathcal{M}} |x_{i,j}^{(n)}|^p \Delta x \Delta y \right)^{q/p} \Delta t \right)^{1/q} \tag{52}$$

the integration in space-time (with standard extensions to the ∞ norm).

Furthermore, we need suitable discrete version of the Poincaré inequality.

Theorem 3.2 (Discrete Poincaré inequality for periodic settings)**.** *Let the discrete vorticity and stream function be as defined in* (46) *and the finite difference operators defined in* (41), (45)*. Then, there exists constants* $C_{P_1}, C_{P_2}, C_{P_3} > 0$ *depending only on the domain (here, the domain is assumed to be* $[0,1)^2$*, so that the constants can be given explicitly in terms of smallest eigenvalue of Sturm-Liouville eigenvalue problem) and independent of the mesh, such that:*

$$C_{P_1}\|\mathrm{curl}_+^h\psi\|_2 = C_{P_1}\|\mathrm{grad}_+^h\psi\|_2 \geq \|\psi\|_2, \tag{53}$$

$$\eta = -\Delta^h\psi \Rightarrow \|\mathrm{curl}_+^h\psi\|_2 = \|\mathrm{grad}_+^h\psi\|_2 \leq C_{P_2}\|\eta\|_2, \tag{54}$$

$$\eta = -\Delta^h\psi \Rightarrow \|\psi\|_2 \leq C_{P_3}\|\eta\|_2. \tag{55}$$

and

$$\|\underline{\tilde{u}}\|_2 = \|\underline{u}\|_2 = \|\mathrm{curl}_+^h\psi\|_2 \leq C_{P_2}\|\eta\|_2.$$

The proof is analogous to the one in [15, Lemma 2.19 (pp. 120)].

Proof. Let ψ^h be a piecewise constant function on a uniform partition of $\mathbb{T}^2$ into congruent, axiparallel squares with edgelength h, with vanishing average over $\mathbb{T}^2$. Then $\psi^h \in L^2(\mathbb{T}^2)/\mathbb{R}$. First notice the following:

$$\|\mathrm{grad}_+^h\psi^h\|_2^2 = |\langle \mathrm{grad}_+^h\psi^h, \mathrm{grad}_+^h\psi^h\rangle| = |\langle \Delta^h\psi^h, \psi^h\rangle| \leq \|\Delta^h\psi^h\|_2\|\psi^h\|_2 = \|\eta\|_2\|\psi^h\|_2$$

On the torus $\mathbb{T}^2$ there exists a mesh-independent constant $C_{P_1} > 0$, such that

$$C_{P_1}\|\mathrm{grad}_+^h\psi^h\|_2 \geq \|\psi^h\|_2$$

The eigenvalues of the discrete negative Laplacian ($-\Delta^h$) are:

$$\lambda_{n_1,n_2}^{N_h} = 5\left(N_h^2\sin^2\left(\frac{\pi n_1}{2N_h}\right) + N_h^2\sin^2\left(\frac{\pi n_2}{2N_h}\right)\right)$$

Hence the smallest, non-trivial, one is $\lambda_{1,1}^{N_h}$:

$$\|\mathrm{grad}_+^h\psi\|_2^2 = -\langle \Delta^h\psi^h, \psi^h\rangle \geq +\lambda_{n-1}\langle \psi^h, \psi^h\rangle = +\lambda_{1,1}^{N_h}\|\psi\|_2^2\ .$$

With

$$C_{P_1}^2 := \lim_{h\to 0}((\lambda_{1,1}^{N_h})^{-1} + o(1)) = \frac{1}{2\pi^2}\quad ,$$

we obtain for any ψ^h as above,

$$C_{P_1}^2\|\psi^h\|_2\|\eta\|_2 \geq C_{P_1}^2\|\mathrm{grad}_+^h\psi^h\|_2^2 \geq \|\psi^h\|_2^2$$

and, simplifying, one gets the estimate:

$$C_{P_1}^2\|\eta\|_2 \geq \|\psi^h\|_2\ .$$

Moreover one has

$$C_{P_1}^2\|\eta\|_2^2 \geq \|\psi^h\|_2\|\eta\|_2 \geq \|\mathrm{grad}_+^h\psi^h\|_2^2\ .$$

This concludes the proof of the first bound. The proof of the last inequality in the statement follows from a straightforward adaptation of the above proof. □

3.3 The finite difference scheme

Armed with the above preliminaries, we will now describe a finite difference scheme to approximate the vorticity-stream function form of the two-dimensional Navier-Stokes equations, (34). We start with the following discretization of the vorticity evolution equation,

$$\frac{\eta_{i,j}^{(n+1)} - \eta_{i,j}^{(n)}}{\Delta t} + \frac{F_{i+1/2,j}^{n+1/2} - F_{i-1/2,j}^{n+1/2}}{\Delta x} + \frac{G_{i,j+1/2}^{n+1/2} - G_{i,j-1/2}^{n+1/2}}{\Delta y} = (\nu + \epsilon h)\left(\Delta^h \eta^{(n+1/2)}\right)_{i,j}. \tag{56}$$

Here, the diffusion is a sum of the physical diffusion ν and some numerical diffusion ϵh which acts on the vorticity at the time average,

$$\eta^{(n+1/2)} := \frac{\eta^{(n+1)} + \eta^{(n)}}{2}. \tag{57}$$

The numerical fluxes F, G are defined as,

$$F_{i+1/2,j}(\tilde{u}^{(n)}, \eta^{(n+1/2)}) := (\tilde{u_1})_{i+1/2,j}^{(n)}(\eta_{i+1,j}^{(n+1/2)} + \eta_{i,j}^{(n+1/2)}), \tag{58}$$

$$G_{i,j+1/2}(\tilde{v}^{(n)}, \eta^{(n+1/2)}) := (\tilde{u_2})_{i,j+1/2}^{(n)}(\eta_{i,j+1}^{(n+1/2)} + \eta_{i,j}^{(n+1/2)}), \tag{59}$$

with the averaged velocities $\tilde{u}, \tilde{v}$ being defined as in (48).

Moreover, the averaged velocities defined in (48) require the specification of velocities u_1^n, v_1^n at the normal edges. We define them as follows,

$$({u_1}^{(n)}, {u_2}^{(n)}) =: \underline{u}^{(n)} = \mathrm{curl}_+^h \psi^{(n)} := (D_+^y \psi^{(n)}, -D_+^x \psi^{(n)})^\top. \tag{60}$$

The discrete stream function satisfies the elliptic finite difference equation

$$-(\Delta^h \psi^{(n)})_{i,j} = \eta_{i,j}^{(n)}, \quad \text{periodic B.C.s}. \tag{61}$$

In the periodic setting, the matrix is rank deficient (due to the discrete solutions only being unique up to constants). To ensure uniqueness of discrete solutions, the additional condition $\sum_{(i,j)\in\mathcal{M}} \psi_{i,j} = 0$ has to be imposed (iterative solvers are able to efficiently take care of this). From now on, we will factor the stream functions, solutions of the discrete Laplace operator, by the equivalence relation $\psi_1 \equiv \psi_2 :\Leftrightarrow \exists c \in \mathbb{R}^n \; u_{i,j} + c = v_{i,j}$, and consider the quotient space with the induced norm. Notice that $\ker \Delta^h = \mathrm{span}\{\mathbf{1}\}$ and we are exclusively interested in derivatives of ψ. Therefore, the velocity at tangential edges as in (60) is well defined.

We use cell averages of the initial data,

$$\eta_{i,j}^0 = \frac{1}{\Delta x \Delta y} \int_{C_{i,j}} \eta_0(x) dx. \tag{62}$$

This completes the specification of the scheme.

3.4 Consistency and stability of the scheme

We claim that the above scheme is a consistent discretization of the incompressible Navier-Stokes equations in the velocity-stream function formulation (34). To show this, we will assert a few properties of the scheme below.

First, we check that the approximated velocity field satisfies a discrete divergence condition.

Proposition 3.3. *Let the discrete velocity be defined as in* (60)*. Then the following discrete divergence relation holds:*

$$\frac{(u_1)_{i+1,j+1/2} - (u_1)_{i,j+1/2}}{\Delta x} + \frac{(u_2)_{i+1/2,j+1} - (u_2)_{i+1/2,j}}{\Delta y} = 0. \tag{63}$$

Moreover, the discrete velocity and the discrete vorticity satisfy the relation (compare (13)*)*

$$-\frac{(u_1)_{i,j+1/2} - (u_1)_{i,j-1/2}}{\Delta y} + \frac{(u_2)_{i+1/2,j} - (u_2)_{i-1/2,j}}{\Delta x} = \eta_{i,j}\ . \tag{64}$$

Note that we have suppressed time dependence in the above formulas for the sake of notational brevity.

Proof. By definition, we have

$$\begin{aligned}
&\frac{(u_1)_{i+1,j+1/2} - (u_1)_{i,j+1/2}}{\Delta x} + \frac{(u_2)_{i+1/2,j+1} - (u_2)_{i+1/2,j}}{\Delta y} \\
&= \frac{\psi_{i+1,j+1} - \psi_{i+1,j} - \psi_{i,j+1} + \psi_{i,j}}{\Delta x \Delta y} + \frac{-(\psi_{i+1,j+1} - \psi_{i,j+1}) + (\psi_{i+1,j} - \psi_{i,j})}{\Delta x \Delta y} \\
&= 0\ .
\end{aligned}$$

Moreover (cp. (34b))

$$\begin{aligned}
&-\frac{(u_1)_{i,j+1/2} - (u_1)_{i,j-1/2}}{\Delta y} + \frac{(u_2)_{i+1/2,j} - (u_2)_{i-1/2,j}}{\Delta x} \\
&= -\frac{\psi_{i,j+1} - \psi_{i,j} - \psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} - \frac{\psi_{i+1,j} - \psi_{i,j} - \psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} \\
&= -\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y^2} - \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x^2} = -(\Delta_h \psi)_{i,j} = \eta_{i,j}\ .
\end{aligned}$$

□

We observe that the scheme (56) is in conservative form whereas the vorticity transport equation (34a) is in the convective form. For the continuous problem, the divergence free condition automatically implies an equivalence between these two forms. A similar equivalence also holds in the discrete case as asserted below,

Proposition 3.4. *For any discrete* $\underline{\tilde{u}}$ *such that* $\mathrm{div}_+^h \underline{\tilde{u}} = \sigma$*, we have the following conservative/convective equivalence:*

$$\begin{aligned}
&\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} \\
&= (\tilde{u_1})_{i+1/2,j} \frac{\eta_{i+1,j} - \eta_{i,j}}{2\Delta x} + (\tilde{u_1})_{i-1/2,j} \frac{\eta_{i,j} - \eta_{i-1,j}}{2\Delta y} \\
&+ (\tilde{u_2})_{i,j+1/2} \frac{\eta_{i,j+1} - \eta_{i,j}}{2\Delta x} + (\tilde{u_2})_{i,j-1/2} \frac{\eta_{i,j} - \eta_{i,j-1}}{2\Delta y} \\
&+ \eta_{i,j}\sigma.
\end{aligned} \tag{65}$$

For $\sigma = 0$ *we obtain complete equivalence.*

Proof. We just add and subtract the same quantity:

$$\begin{aligned}
&\frac{F_{i+1/2,j} - F_{i-1/2,j}}{\Delta x} + \frac{G_{i,j+1/2} - G_{i,j-1/2}}{\Delta y} \\
&= \frac{(\tilde{u_1})_{i+1/2,j}(\eta_{i+1,j} + \eta_{i,j}) - (\tilde{u_1})_{i-1/2,j}(\eta_{i,j} + \eta_{i-1,j})}{2\Delta x} \\
&+ \frac{(\tilde{u_2})_{i,j+1/2}(\eta_{i,j+1} + \eta_{i,j}) - (\tilde{u_2})_{i,j-1/2}(\eta_{i,j} + \eta_{i,j-1})}{2\Delta y} \\
&= (\tilde{u_1})_{i+1/2,j}\frac{\eta_{i+1,j} - \eta_{i,j}}{2\Delta x} + (\tilde{u_1})_{i-1/2,j}\frac{\eta_{i,j} - \eta_{i-1,j}}{2\Delta y} \\
&+ (\tilde{u_2})_{i,j+1/2}\frac{\eta_{i,j+1} - \eta_{i,j}}{2\Delta x} + (\tilde{u_2})_{i,j-1/2}\frac{\eta_{i,j} - \eta_{i,j-1}}{2\Delta y} \\
&+ \eta_{i,j}\left(\frac{(\tilde{u_1})_{i+1/2,j} - (\tilde{u_1})_{i-1/2,j}}{\Delta x} + \frac{(\tilde{u_2})_{i,j+1/2} - (\tilde{u_2})_{i,j-1/2}}{\Delta y}\right) \\
&= (\tilde{u_1})_{i+1/2,j}\frac{\eta_{i+1,j} - \eta_{i,j}}{2\Delta x} + (\tilde{u_1})_{i-1/2,j}\frac{\eta_{i,j} - \eta_{i-1,j}}{2\Delta y} \\
&+ (\tilde{u_2})_{i,j+1/2}\frac{\eta_{i,j+1} - \eta_{i,j}}{2\Delta x} + (\tilde{u_2})_{i,j-1/2}\frac{\eta_{i,j} - \eta_{i,j-1}}{2\Delta y} \\
&+ \eta_{i,j}\sigma,
\end{aligned}$$

which is precisely (65). □

Note that following proposition 3.1 and the definition of our averaged velocity field, (48), the divergence error $\sigma = 0$ for our discrete averaged velocity field. Hence, the conservative form of our scheme in (56) is equivalent to the convective form and is consistent with (34a).

The natural form of stability for the vorticity-stream function formulation of incompressible Navier-Stokes is enstrophy (L^2) stability. We show that our scheme possesses a discrete version of this stability. The discrete analogue of enstrophy is defined by

$$E^n := \frac{1}{2}\Delta x\Delta y\sum_{i,j}\left(\eta_{i,j}^{(n)}\right)^2 . \tag{66}$$

We have the following estimate on the discrete enstrophy,

Lemma 3.5. *For the scheme defined in* (56)*, the discrete enstrophy decreases in time i.e,*

$$E^{n+1} \leq E^n, \quad \forall n . \tag{67}$$

Proof. For simplicity of exposition, we set $\Delta x = \Delta y = h$. The proof of (67) lies in multiplying $\eta_{i,j}^{(n+1/2)}$ to both sides of (56) and is a straightforward consequence of the following observations,

i. For the time derivative in (56), we calculate that

$$\sum_{(i,j)\in\mathcal{M}}(\eta^{(n+1)} - \eta^{(n)})\eta^{(n+1/2)}\Delta x\Delta y = E^{n+1} - E^n.$$

ii. We note that the fluxes are also evaluated at the time index $n + 1/2$. The calculation of the flux terms (suppressing the time dependence) is as follows,

$$
\begin{aligned}
2\sum_{i,j} F_{i+1/2,j}\eta_{i,j} &= \sum_{i,j} F_{i+1/2,j}(\eta_{i+1,j}+\eta_{i,j}) - \sum_{i,j} F_{i+1/2,j}(\eta_{i+1,j}-\eta_{i,j}) \\
&= \sum_{i,j} (\tilde{u_1})_{i+1/2,j}(\eta_{i+1,j}+\eta_{i,j})^2 - \sum_{i,j} (\tilde{u_1})_{i+1/2,j}(\eta^2_{i+1,j}-\eta^2_{i,j}), \\
2\sum_{i,j} F_{i-1/2,j}\eta_{i,j} &= \sum_{i,j} F_{i-1/2,j}(\eta_{i,j}+\eta_{i-1,j}) + \sum_{i,j} F_{i-1/2,j}(\eta_{i,j}-\eta_{i-1,j}) \\
&= \sum_{i,j} (\tilde{u_1})_{i-1/2,j}(\eta_{i,j}+\eta_{i-1,j})^2 + \sum_{i,j} (\tilde{u_1})_{i-1/2,j}(\eta^2_{i,j}-\eta^2_{i-1,j}).
\end{aligned}
$$

Most of the above terms are telescopic sums that vanish on account of the periodic boundary conditions. The remaining terms are of the form,

$$
\begin{aligned}
2\sum_{i,j} (\tilde{u_1})_{i+1/2,j}(\eta^2_{i+1,j}-\eta^2_{i,j}) &= \sum_{i,j} ((\tilde{u_1})_{i+3/2,j} + (\tilde{u_1})_{i+1/2,j})\eta^2_{i+1,j} \\
&\quad - \sum_{i,j} ((\tilde{u_1})_{i+3/2,j} - (\tilde{u_1})_{i+1/2,j})\eta^2_{i+1,j} \\
&= \sum_{i,j} ((\tilde{u_1})_{i+1/2,j} + (\tilde{u_1})_{i-1/2,j})\eta^2_{i,j} \\
&\quad - \sum_{i,j} ((\tilde{u_1})_{i+1/2,j} - (\tilde{u_1})_{i-1/2,j})\eta^2_{i,j}.
\end{aligned}
$$

Again, some terms vanish due to the telescopic sum. The remaining terms equal

$$
2\sum_{i,j} ((\tilde{u_1})_{i+1/2,j} - (\tilde{u_1})_{i-1/2,j})\eta^2_{i,j} \ .
$$

This, together with the term resulting from analogous manipulation of the flux G and from the discrete divergence relation result in

$$
\sum_{(i,j)\in\mathcal{M}} \frac{F_{i+1/2,j}-F_{i-1/2,j}}{\Delta x}\eta_{i,j}\Delta x\Delta y + \frac{G_{i,j+1/2}-G_{i,j-1/2}}{\Delta y}\eta_{i,j}\Delta x\Delta y = 0. \tag{68}
$$

iii. For the diffusion term in (56), we denote $\nu' := \nu + \epsilon h$ and observe that,

$$
\begin{aligned}
\nu'\left(\Delta^h \eta^{(n+1/2)}\right)\eta^{(n+1/2)} &= -(D^h_+\eta^{(n+1/2)}, D^h_+\eta^{(n+1/2)})^\top \qquad (69)\\
&= -\|\mathrm{grad}^h_+\eta^{(n+1/2)}\|^2_2 \le 0\ . \qquad (70)
\end{aligned}
$$

Hence, the viscous term has a negative contribution to the discrete enstrophy balance.

Combining the above observations yields the discrete enstrophy stability (67). □

Remark 3.6. *The fact that the discrete enstrophy is non-increasing in time implies that the approximate solution η^h is bounded in l^2, i.e, $\|\eta^h\|_{2,\infty}$ is bounded, thus proving that the approximation scheme is stable in this sense.*

3.5 Error estimate for the vorticity

The next step in the numerical analysis of the proposed finite difference scheme is to prove an error estimate for it. To this end, we will combine energy stability with consistency of the scheme by calculating the truncation error. To do so, we assume that a (smooth enough) solution η exists for the incompressible Navier-Stokes equations in the vorticity-stream function formulation (34). We denote the discrete point values as,

$$\bar{\eta}_{i,j}^{(n)} := \eta(x_i, y_j, t^n).$$

These discrete values satisfy the following discrete relation,

$$\frac{\bar{\eta}_{i,j}^{(n+1)} - \bar{\eta}_{i,j}^{(n)}}{\Delta t} + \frac{\bar{F}_{i+1/2,j}^{n+1/2} - \bar{F}_{i-1/2,j}^{n+1/2}}{\Delta x} + \frac{\bar{G}_{i,j+1/2}^{n+1/2} - \bar{G}_{i,j-1/2}^{n+1/2}}{\Delta y} = (\nu + \epsilon h)\left(\Delta^h \bar{\eta}^{n+1/2}\right)_{i,j} + \tau_1 \quad (71)$$

and the incompressibility condition

$$\operatorname{div}_+^h \underline{\tilde{u}} = \tau_2, \quad (72)$$

where $\underline{\bar{u}}$ denotes the exact velocity relative to the vorticity $\bar{\eta}$. In the above expressions, the fluxes are evaluated in terms of the discrete values of the velocity field and τ_1, τ_2 are the *truncation errors* for the vorticity transport and the incompressibility conditions, respectively.

We list a few bounds on the discrete exact solution $\bar{\eta}$ and the truncation errors $\tau_{1,2}$ that are useful in our subsequent analysis.

Theorem 3.7. *Let the initial vorticity $\eta_0 \in H^3_{per}$, then the discrete exact solution satisfies,*

$$\|\bar{\eta}\|_{\infty,1} < C_1(\eta_0, T, \mathbb{T}^2),\ \|\bar{\eta}\|_{\infty,\infty} < C_1(\eta_0, T, \mathbb{T}^2). \quad (73)$$

and,

$$\|\underline{\nabla}\bar{\eta}\|_{\infty,1} < C_2(T, \mathbb{T}^2),\ \|\underline{\nabla}\bar{\eta}\|_{\infty,\infty} < C_2(T, \mathbb{T}^2)\ . \quad (74)$$

Furthermore, the truncation errors satisfy,

$$\|\tau_1\|_{2,2} = C_3(T, \mathbb{T}^2, \|\eta\|_Z)h, \quad \|\tau_2\|_{2,\infty} = C_4(T, \mathbb{T}^2, \|\eta\|_Z)h\ .$$

Proof. Given the initial data in the above function space, the exact solution of the vorticity transport equation (34) satisfies (cf. [18, pag. 118]),

$$\|\eta\|_{H^m} \le exp(exp(C_m t) - 1)\, \|\eta_0\|_{H^m}^{exp(C_m t)}\ ,\ \forall m \ge 0\ , \quad (75)$$

in particular $\eta \in Z := L^2([0,T], H^3_{per}(\mathbb{T}^2)) \cap L^\infty([0,T], H^{2+\varepsilon}_{per}(\mathbb{T}^2))$. Hence, the bounds on the discrete exact solution are a straightforward consequence of Sobolev embeddings.

To prove the bounds on the truncation error, we notice that

$$\begin{aligned} \tau_1 &= \underbrace{\frac{\eta_{i,j}^{(n+1)} - \eta_{i,j}^{(n)}}{\Delta t} - \eta_t}_{\varphi_1 :=} \\ &+ \underbrace{\frac{F_{i+1/2,j}^{n+1/2} - F_{i-1/2,j}^{n+1/2}}{\Delta x} + \frac{G_{i,j+1/2}^{n+1/2} - G_{i,j-1/2}^{n+1/2}}{\Delta y} - \underline{u} \cdot \nabla \eta}_{\varphi_2 :=} \\ &- \underbrace{(\nu + \epsilon h)\left(\Delta^h \eta^{(n+1/2)}\right)_{i,j} - \nu \Delta \eta}_{\varphi_3 :=} + \underbrace{\epsilon h \left(\Delta^h \eta^{(n+1/2)}\right)_{i,j}}_{\varphi_4 :=} \\ &=: \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 \end{aligned}$$

As in [15], we average the truncation error over a cell and then take the discrete L^2-norm. Upon summation over indices n and integration over t, the term arising from φ_1 becomes zero.

For φ_3 one applies the same ideas of $\underline{\nabla}\eta$ estimates, requiring only first order (instead of second order, which with sufficient regularity of the data is possible). Here, we split φ_3 in its two time components (being a time average and everything being linear, upon integration the terms can be absorbed together). The viscosity term is bounded:

$$\|\varphi_3\|_2 \leq Ch \, \|\eta\|_{L^2([0,T],H^3(\mathbb{T}^2))} \, .$$

Notice that there is a small error term due to ϵh term, one has to also estimate:

$$\|\varphi_4\|_2 \leq C\epsilon h \left\|\Delta^h \eta\right\|_2 \leq Ch\epsilon \, \|\eta\|_{L^2([0,T],H^2(\mathbb{T}^2))} \; .$$

For the nonlinear term φ_2, let us split the term in the following way:

$$\varphi_2 = \underline{\tilde{u}}^n(\mathrm{grad}^h_+ - \underline{\nabla})\bar{\eta}^{n+1/2} + (\underline{\tilde{u}}^n - \underline{u}^{n+1/2})\underline{\nabla}\bar{\eta}^{n+1/2} + \tau_2\bar{\eta}^{n+1/2} \; .$$

Hence, if we take the norm:

$$\begin{aligned}
\|\varphi_2\|_2 &\leq \|\underline{\tilde{u}}^n\|_\infty \left\|(\mathrm{grad}^h_+ - \underline{\nabla})\bar{\eta}\right\|_2 + \left\|\underline{\tilde{u}}^n - \underline{u}^{n+1/2}\right\|_2 \|\underline{\nabla}\bar{\eta}\|_\infty + \|\tau_2\bar{\eta}\|_2 \\
&\leq Ch \, \|\underline{\tilde{\bar{u}}}\|_\infty \, \|\bar{\eta}\|_{L^2([0,T],H^2(\mathbb{T}^2))} + Ch(\|\underline{u}\|_{L^2([0,T],H^2(\mathbb{T}^2))} \\
&+ \|\underline{u}\|_{H^1([0,T],L^2(\mathbb{T}^2))}) \, \|\underline{\nabla}\bar{\eta}\|_\infty + \|\tau_2\|_2 \, \|\bar{\eta}\|_\infty \\
&\leq Ch \, \|\underline{\bar{u}}\|_\infty \, \|\bar{\eta}\|_{L^2([0,T],H^2(\mathbb{T}^2))} + Ch(\|\bar{\eta}\|_{L^2([0,T],H^1(\mathbb{T}^2))} \\
&+ \|\bar{\eta}\|_{H^1([0,T],L^2(\mathbb{T}^2))}) \, \|\underline{\nabla}\bar{\eta}\|_\infty + Ch \, \|\bar{\eta}\|_{L^\infty([0,T],H^1(\mathbb{T}^2))} \, \|\bar{\eta}\|_\infty \; .
\end{aligned}$$

Here we bound the $\|\underline{\bar{u}}\|_\infty$ and $\|\underline{\nabla}\bar{\eta}\|_\infty$ estimates in $L^\infty([0,T], H^{2+\varepsilon}(\mathbb{T}^2))$. We also have the divergence constraint truncation error:

$$\tau_2 = \mathrm{div}^h_+\underline{u} - \mathrm{div}\underline{u}$$

for which, analogously, we have:

$$\|\tau_2\|_{2,\infty} \leq Ch \, \|\underline{\bar{u}}\|_{L^\infty([0,T],H^2(\mathbb{T}^2))} \leq Ch \, \|\bar{\eta}\|_{L^\infty([0,T],H^1(\mathbb{T}^2))} \; .$$

Combining all estimates implies the asserted error bounds. □

Next, we assert the main error estimate for our numerical approximation.

Theorem 3.8. *We assume that the initial vorticity $\eta_0 \in H^3_{per}$ and $\bar{\eta}^{(n)}_{i,j}$ be the point value at (x_i, y_j, t^n) of the exact vorticity which solves the vorticity transport equation* (34a). *Let $\eta^{(n)}_{i,j}$ be the discrete vorticity, evolved by the numerical scheme* (56). *We denote the discrete error as*

$$\mathfrak{e}^n := \{\mathfrak{e}^n_{i,j}\}_{n,i,j}, \quad \mathfrak{e}^n_{i,j} := \eta^{(n)}_{i,j} - \bar{\eta}^{(n)}_{i,j} \; .$$

Then, the discrete error is bounded as,

$$\|\mathfrak{e}^{(N)}\|_2 \leq K_3(\eta_0, T, \mathbb{T}^2)(1 + K_4(T, \nu, \mathbb{T}^2)e^{K_4(\eta_0,T,\nu,\mathbb{T}^2)})h, \tag{76}$$

with $h = \Delta x = \Delta y \simeq \Delta t$ and with constants K_3 and K_4 depending only on the exact solution, the domain, the time interval and viscosity (for K_4). Sufficiently large viscosity allows to bound K_4 by 1.

Proof. We subtract the equations (71) and (56) and multiply this difference by the error $\mathfrak{e}^{(n+1/2)}$, to obtain

$$\begin{aligned}&\frac{\mathfrak{e}^{(n+1)}-\mathfrak{e}^{(n)}}{\Delta t}\frac{\mathfrak{e}^{(n+1)}+\mathfrak{e}^{(n)}}{2}+\mathbf{F}(\underline{u}^{(n)},\eta^{(n+1/2)})\mathfrak{e}^{(n+1/2)}-\mathbf{F}(\underline{\bar{u}}^{(n)},\bar{\eta}^{(n+1/2)})\mathfrak{e}^{(n+1/2)}\\ &=(\nu+\varepsilon h)\Delta^h\left(\mathfrak{e}^{(n+1/2)}\right)\left(\mathfrak{e}^{(n+1/2)}\right)-\tau_1\mathfrak{e}^{(n+1/2)}\ .\end{aligned}$$

We use that the flux is bilinear, i.e.

$$\mathbf{F}(\underline{u},\eta)-\mathbf{F}(\underline{\bar{u}},\bar{\eta})=\mathbf{F}(\underline{u},(\eta-\bar{\eta}))-\mathbf{F}(\underline{u}-\underline{\bar{u}},\bar{\eta})\ .$$

The first term drops out (upon integration) by the discrete divergence-free property and the discrete conservativeness of the fluxes (3.5). It remains estimate the second term, which is rewritten in discrete convective form (3.4):

$$\sum_{(i,j)\in\mathcal{M}}\mathbf{F}(\underline{u}-\underline{\bar{u}},\bar{\eta})(\eta-\bar{\eta})\Delta x\Delta y=\sum_{(i,j)\in\mathcal{M}}(\underline{\tilde{u}}-\underline{\tilde{\bar{u}}})\mathrm{grad}_+^h\bar{\eta}(\eta-\bar{\eta})-\bar{\eta}\tau_2(\eta-\bar{\eta})\Delta x\Delta y$$

Summing everything in space, one obtains:

$$\begin{aligned}&\left|\sum_{(i,j)\in\mathcal{M}}\mathbf{F}(\underline{u},\eta)\mathfrak{e}^{(n+1/2)}-\mathbf{F}(\underline{\bar{u}},\bar{\eta})\mathfrak{e}^{(n+1/2)}\Delta x\Delta y\right|\\ &\leq\left|\sum_{(i,j)\in\mathcal{M}}\left(\underline{\tilde{u}}-\underline{\tilde{\bar{u}}})\mathrm{grad}_+^h\bar{\eta}(\eta-\bar{\eta})-\bar{\eta}\tau_2(\eta-\bar{\eta})\right)\Delta x\Delta y\right|\\ &\leq\|\underline{\nabla}\bar{\eta}\|_\infty\|\underline{u}-\underline{\bar{u}}\|_2\|\eta-\bar{\eta}\|_2+\|\bar{\eta}\|_\infty\|\tau_2\|_2^2+\|\bar{\eta}\|_\infty\|\eta-\bar{\eta}\|_2^2\\ &\leq(C_{P_2}\|\underline{\nabla}\bar{\eta}\|_\infty+\|\bar{\eta}\|_\infty)\|\eta-\bar{\eta}\|_2^2+\|\bar{\eta}\|_\infty\|\tau_2\|_2^2\\ &\leq(C_{P_2}\|\underline{\nabla}\bar{\eta}\|_\infty+\|\bar{\eta}\|_\infty)\|\mathfrak{e}^{(n+1/2)}\|_2^2+\|\bar{\eta}\|_\infty\|\tau_2\|_2^2.\end{aligned}$$

The truncation residual becomes (using Young's inequality):

$$2\|\tau_1\mathfrak{e}^{(n+1/2)}\|_1\leq\|\mathfrak{e}^{(n+1/2)}\|_2^2+\|\tau_1\|_2^2=:\mathcal{E}_{n+1/2}+\|\tau_1\|_2^2.$$

The complete estimate for the evolution of the errors is of the form,

$$\begin{aligned}&\sum_{(i,j)\in\mathcal{M}}\frac{(\mathfrak{e}^{(n+1)})^2-(\mathfrak{e}^{(n)})^2}{\Delta t}\Delta x\Delta y\\ &=\sum_{(i,j)\in\mathcal{M}}\left((\underline{u}-\underline{\bar{u}})\mathrm{grad}_+^h\bar{\eta}(\eta-\bar{\eta})-\bar{\eta}\tau_2(\eta-\bar{\eta})+\nu'(\Delta^h\mathfrak{e}^{(n+1/2)})\mathfrak{e}^{(n+1/2)}+\tau_1\mathfrak{e}^{(n+1/2)}\right)\Delta x\Delta y\\ &\leq\|(\underline{u}-\underline{\bar{u}})\underline{\nabla}\bar{\eta}(\eta-\bar{\eta})\|_1+\|\bar{\eta}\tau_2(\eta-\bar{\eta})\|_1-\nu' C_{P_1}^{-1}\|\mathfrak{e}^{(n+1/2)}\|_2^2+\mathcal{E}_{n+1/2}+\|\tau_1\|_2^2.\end{aligned}\qquad(77)$$

Summing in time and using the previous estimates, gives:

$$\begin{aligned}
\|\mathfrak{e}^{(N+1)}\|_2^2 &\leq \|\mathfrak{e}^{(0)}\|_2^2 + \sum_{n=0}^{N} \Big((C_{P_2}\|\underline{\nabla}\bar{\eta}\|_\infty + \|\bar{\eta}\|_\infty)\|\mathfrak{e}^{(n+1/2)}\|_2^2 + \|\bar{\eta}\|_\infty \|\tau_2\|_2^2 \\
&\quad - \nu' C_{P_1}^{-1}\|\mathfrak{e}^{(n+1/2)}\|_2^2 + \frac{1}{2}\|\mathfrak{e}^{(n+1/2)}\|_2^2 + \frac{1}{2}\|\tau_1\|_2^2 \Big) \Delta t \\
&\leq \|\mathfrak{e}_0\|_2^2 + \sum_{n=0}^{N+1} \Big(\max\{C_{P_2}\|\underline{\nabla}\bar{\eta}\|_\infty + \|\bar{\eta}\|_\infty - \nu' C_{P_1}^{-1} + 1, 0\}\|\mathfrak{e}^{(n)}\|_2^2 \\
&\quad + \|\bar{\eta}\|_\infty \|\tau_2\|_2^2 + \frac{1}{2}\|\tau_1\|_2^2 \Big) \Delta t \\
&=: \mathcal{E}_0 + \sum_{n=0}^{N+1} \Big(K^{n+1/2}\mathcal{E}_n + D^{n+1/2} \Big) \Delta t,
\end{aligned}$$

with

$$\min\{C_{P_2}\|\underline{\nabla}\bar{\eta}\|_\infty + \|\bar{\eta}\|_\infty - \nu C_{P_1}^{-1} + 1, 0\} =: c_n \ , \quad \|\bar{\eta}\|_\infty \|\tau_2\|_2^2 + \frac{1}{2}\|\tau_1\|_2^2 =: d_n$$

finite under previous assumptions. Before applying Gronwall, we bring the "temporal boundary term" to the correct side. In the ensuing derivations, we denote by $C^{-1} := (1 - c_{N+1}\Delta t)$ and assume Δt sufficiently small, so that $c_{N+1}\Delta t < 1$:

$$\begin{aligned}
\mathcal{E}_{N+1}(1 - c_{N+1}\Delta t) &\leq \mathcal{E}_0 + \sum_{n=0}^{N} R_n \mathcal{E}_n \Delta t + \|d_n\|_1 \\
\mathcal{E}_{N+1} &\leq \frac{\mathcal{E}_0 + \|d_n\|_1}{1 - c_{N+1}\Delta t} + \sum_{n=0}^{N} \frac{c_n}{1 - c_{N+1}\Delta t}\mathcal{E}_n \Delta t \ .
\end{aligned}$$

Applying a discrete Gronwall Lemma yields

$$\begin{aligned}
\mathcal{E}_{N+1} &\leq \frac{\mathcal{E}_0 + \|d_n\|_1}{1 - K^{N+1}\Delta t} + \sum_{n=0}^{N} \frac{c_n}{1 - K^{N+1}\Delta t}\frac{\mathcal{E}_0 + \|d_n\|_1}{1 - K^{N+1}\Delta t} e^{\sum_{m=n}^{N} \frac{K^m}{1-K^{N+1}\Delta t}\Delta t} \Delta t \\
&\leq (\mathcal{E}_0 + \|d_n\|_1 C) \left(1 + C \sum_{n=0}^{N} c_n e^{C\sum_{m=n}^{N} K^m \Delta t} \Delta t \right) \\
&\leq (\mathcal{E}_0 + \|d_n\|_1 C) \left(1 + C \sum_{n=0}^{N} c_n e^{C\sum_{n=0}^{N} c_n \Delta t} \Delta t \right) \\
&\leq (\mathcal{E}_0 + \|d_n\|_1 C) \left(1 + C\|c_n\|_1 e^{C\|c_n\|_1} \right) .
\end{aligned}$$

Applying the bounds in theorem 3.7, we obtain,

$$\begin{aligned}
\|c_n\|_1 &= \max\{C_{P_2}\|\underline{\nabla}\bar{\eta}\|_{\infty,1} + \|\bar{\eta}\|_{\infty,1} - \nu C_{P_1}^{-1} + 1, 0\} \\
&\leq \max\{C_{P_2}C_2(T,\mathbb{T}^2) + C_1(T,\mathbb{T}^2) - \nu C_{P_1}^{-1} + 1, 0\} =: K_1(T,\nu,\mathbb{T}^2) \\
\|d^n\|_1 &= \|\bar{\eta}\|_{\infty,1}\|\tau_2\|_{2,\infty}^2 + \frac{1}{2}\|\tau_1\|_{2,2}^2 \\
&\leq C_1(T,\mathbb{T}^2)C_4(T,\mathbb{T}^2)^2 h^2 + \frac{1}{2}C_3(T,\mathbb{T}^2)^2 h^2 \leq K_2(T,\mathbb{T}^2)h^2 \\
C &= (1 - K^{N+1}\Delta t)^{-1} \\
&\leq (1 - \max\{C_{P_2}\|\underline{\nabla}\bar{\eta}\|_{\infty,\infty} + \|\bar{\eta}\|_{\infty,\infty} - \nu C_{P_1}^{-1}, 0\}\Delta t)^{-1} \leq 1 + \delta(T,\nu,\mathbb{T}^2) \\
\mathcal{E}_0 &= C_5(T,\mathbb{T}^2)h^2 \ .
\end{aligned}$$

Then

$$\|\mathfrak{e}^{(N)}\|_2 \leq K_3(T,\mathbb{T}^2)h(1+K_4(T,\nu,\mathbb{T}^2)e^{K_4(T,\nu,\mathbb{T}^2)}), \tag{78}$$

with

$$K_3(T,\mathbb{T}^2) := (\mathcal{E}_0 + \|d_n\|_1 C),\ K_4(T,\nu,\mathbb{T}^2) := C\|c_n\|_1.$$

Notice that, as ν increases, the constant K_4 can be chosen arbitrary small. □

A straightforward corollary of the above estimate on the discrete error is its continuous counterpart, defined in terms of the piecewise constant approximate solution,

$$\eta^h(x,y,t) := \eta_{i,j}^{(n)}, \quad \text{if} \quad (x,y,t) \in C_{i,j} \times [t^n \times t^{(n+1)}] \ ,$$

and estimated in the following,

Theorem 3.9. *Given an initial vorticity $\eta_0 \in H^3(\mathbb{T}^n)$, let us denote the unique global solution to the NSE equations* (34) *by $\bar{\eta} \in L^1([0,T],H^3(\mathbb{T}^n))$. Then the sequence of piecewise constant approximations η^h obtained through the numerical scheme* (56) *satisfies the L^2 error bound:*

$$\left\|\eta(t,\cdot) - \eta^h(t,\cdot)\right\|_{L^2(\mathbb{T}^2)} \leq C(\eta_0,T,\mathbb{T}^2)h, \tag{79}$$

where the constant C depends on only on the initial data η_0, the domain and the final time.

We end this section with a few remarks on the numerical scheme and the error estimate.

Remark 3.10. *The constant in the error estimate* (79) *only depends on the final time and on higher derivative H^3 norm on the initial data. This constant can be rather large for some initial data, see* (75)*. However, it is finite for a finite final time and is independent of the Reynolds number. To see the fact that the constant is Reynolds number independent, we readily observe that in the proof of* (78)*, we can easily drop the ν dependent terms as they all have negative signs. However, we chose to retain the ν-dependence in order to observe the effect of viscosity on the constant. As argued in the proof of* (79)*, the constant in the error estimate is reduced when the viscosity is large. We reemphasize that the constant in the error bound is uniformly bounded with respect to $\nu \to 0$. Thus, our scheme* (56) *can be employed to approximate two-dimensional incompressible flows with periodic boundary conditions at arbitrary large Reynolds numbers. In particular,* (56) *is also a robust discretization (with error estimate* (79) *for the incompressible Euler equations, by setting $\nu = 0$.*

Remark 3.11. *We can also set numerical diffusion to zero in* (56) *by letting $\nu' = \nu$. We do not require any numerical diffusion to stabilize our scheme, even in the inviscid limit, as we employ an implicit time stepping.*

Remark 3.12. *The $H^3_{per}(\mathbb{T}^2)$ regularity assumption on the initial data is necessary as this smoothness is used to define and to bound the truncation error. It can be relaxed slightly. However, a convergence rate estimate (which is the basis for the design of the Multi-Level Monte-Carlo sampling) for our scheme that uses less regular data than H^2 seems elusive, at this stage. It might be possible to prove ν dependent bounds with H^1 regularity. However, these bounds would blow up when $\nu \to 0$. Energy inequalities (e.g. [8, Eqn. (V.1.9)]) imply that second moments of statistical velocity solutions for initial data with finite variance enstrophy* (35) *are (in the presently adopted setting of vanishing volume forcing, ie.* (31)*), are bounded independent of ν. We are therefore interested in computing statistics of individual flows with very large Reynolds numbers, and continue to work under strong regularity assumptions on the initial data.*

Remark 3.13. *Our scheme is of the finite difference type as we approximate point values. As our grid is Cartesian, we can readily interpret our scheme as a finite volume scheme by requiring approximation of cell averages. Extension to (block)-structured meshes is straightforward. However, extending the scheme to unstructured grids is more challenging.*

3.6 $H_{per}(\mathrm{div};\mathbb{T}^2)$ convergence rate bounds for the velocity fields

We have shown in Theorem 3.9 that the approximate solution $\eta^h(t,\cdot)$ of piecewise constant vorticity approximations produced by the finite difference scheme in $\mathbb{T}^2$ on a uniform mesh of axiparallel quadrilaterals of width h converges, as $h \to 0$, as $O(h)$ in the norm $L^2(\mathbb{T}^2)$. As we show in the next sections, this will allow for a convergence analysis of MLMCFVM approximations of statistical vorticity solutions π_t^ν in Section 2.2.3.

The finite difference scheme being a stable and consistent discretization of the vorticity formulation of the NSE (34), consistent velocity approximations are produced as well by the scheme. This allows for a convergence analysis of MLMC first order FVM statistical velocity solutions in the sense of Foias and Prodi, as introduced in Section 2.2.1. We indicate the (short) derivation of the relevant estimates. For every η^h, there exists a unique stream function

$$\psi^h \in H^2_{per}(\mathbb{T}^2): \quad -\Delta\psi^h = \eta^h \quad \text{in} \quad L^2(\mathbb{T}^2)\,. \tag{80}$$

The unique solution ψ^h satisfies, due to the L^2-stability of the FV scheme,

$$\|\psi^h\|_{H^2(\mathbb{T}^2)} \le C\|\Delta\psi^h\|_0 = C\|\eta^h(t)\|_0 \le CC_{stab}\,, \quad 0 < h \le 1\,. \tag{81}$$

where the constants are independent of ν, and where C_{stab} denotes a $L^2(\mathbb{T}^2)$-stability bound obtained from the bounds in the mesh-dependent norms in Theorem 3.7. We note that the exact stream function ψ^h corresponding to the FD stepfunction vorticity approximation η^h is not a piecewise polynomial function. We now show that we can recover a continuous, piecewise polynomial stream function at cost $O(N)$ by a Galerkin projection. To this end, let $S^1_H(\mathbb{T}^2)$ denote the space of continuous, piecewise *bilinear* functions on a uniform mesh of axiparallel quadrilaterals of meshwidth $H \le h$, where $h/H \in \mathbb{N}$, which belong to $H^1_{per}(\mathbb{T}^2)$. We define the Galerkin-projected stream function by

$$\text{find } \psi^h_H \in S^1_H(\mathbb{T}^2): \quad (\nabla\psi^h_H, \varphi_H) = (\eta^h(t), \varphi_H) \quad \forall \varphi_H \in S^1_H(\mathbb{T}^2)\,. \tag{82}$$

As $(\nabla\cdot, \nabla\cdot)$ is coercive on $H^1_{per}(\mathbb{T}^2)$, there exists a unique solution $\psi^h_H \in S^1_H(\mathbb{T}^2)$ and

$$\|\nabla(\psi^h - \psi^h_H)\|_0 = \inf_{\varphi_H \in S^1_H(\mathbb{T}^2)} \|\nabla(\psi^h - \varphi_H)\|_0 \le CH\|\eta^h(t)\|_0 \tag{83}$$

where we used (81) and the standard approximation property of the space $S^1_H(\mathbb{T}^2)$, and where the constants are independent of h and of H. We estimate

$$\begin{aligned} \|\nabla(\psi - \psi^h_H)\|_0 &\le \|\nabla(\psi - \psi^h)\|_0 + \|\nabla(\psi^h - \psi^h_H)\|_0 \\ &\le \|\nabla \circ (-\Delta)^{-1}(\eta - \eta^h)(t)\|_0 + CH\|\eta^h(t)\|_0 \\ &\le \|(\eta - \eta^h)(t)\|_0 + CH\|\eta^h(t)\|_0 \\ &\le C(h + H)\,. \end{aligned} \tag{84}$$

We choose $H = h$ and note that Galerkin equations (82) can be solved to first order accuracy $O(h)$ in $H^1(\mathbb{T}^2)$ in work and memory of $O(h^{-2}) = O(N)$ using, e.g. a multigrid iteration.

With $\psi^h_h \in S^1_h(\mathbb{T}^2)$ available, we define an approximate velocity by

$$u^h := \mathrm{curl}\psi^h_h = (\partial_2\psi^h_h, -\partial_1\psi^h_h)^\top\,. \tag{85}$$

Due to the low regularity of $S_h^1(\mathbb{T}^2)$, $u^h \notin H_{per}^1(\mathbb{T}^2)$. However, due to the tensor product structure $S_h^1(\mathbb{T}^2) = S_h^1(I^x) \otimes S_h^1(I^y)$ with $I^x, I^y := (0,1)$, we note that e.g. $\partial_2 \psi_h^h \in S_h^1(I^x) \otimes S_h^0(I^y)$, where $S_h^0(I^y)$ denotes the space of piecewise constant functions on a uniform partition of the interval $I^y = (0,1)$ into subintervals of length h. Analogous reasoning for ∂_1 implies

$$u^h = (\partial_2 \psi_h^h, -\partial_1 \psi_h^h) \in S_h^1(I^x) \otimes S_h^0(I^y) \times S_h^0(I^x) \otimes S_h^1(I^y) \subset H(\mathrm{div}; \mathbb{T}^2)$$

and there holds discrete conservation: $\mathrm{div} u^h = \mathrm{div}\mathrm{curl}\psi_h^h = 0$. This implies with (84)

$$\begin{aligned} \|u - u^h\|_{H(\mathrm{div};\mathbb{T}^2)}^2 &= \|u - u^h\|_{L^2(\mathbb{T}^2)}^2 + \|\mathrm{div}(u - u^h)\|_{L^2(\mathbb{T}^2)}^2 \\ &= \|\mathrm{curl}(\psi - \psi_h^h)\|^2 = \|\nabla(\psi - \psi_h^h)\|^2 \leq Ch^2 \ . \end{aligned} \tag{86}$$

We remark that an $O(h)$ error bound in $L^2(\mathbb{T}^2)^2$ for the discrete velocities also follows from error bounds for the finite difference Laplacian on uniform grids in [15, Sec. 2.3.2].

4 (Multi-level) Monte Carlo methods

Our aim is to compute statistics properties of the two-dimensional Navier-Stokes flow (34), with respect to the statistical vorticity solution π_t^ν defined in (38). To this end, we will utilize the equivalence between statistical solutions and random fields. We assume that the initial data for the statistical solution π_0 is given as the law of a random field $\eta_0 \in L^2(\Omega, H_{per}^s(\mathbb{T}^2))$, defined on an underlying complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Computing a typical observable with respect to the statistical solution π_t^ν amounts to calculating, for some admissible test function $g \in C_b(H_{per}^s)$,

$$\int_{H_{per}^s} g(\eta) d\pi_t^\nu(\eta) \ . \tag{87}$$

We recast the problem of computing the statistics (87) in terms of random fields in the following calculation,

$$\begin{aligned} \int_{H_{per}^s} g(\eta) d\pi_t^\nu &= \int_{H_{per}^s} g(T^\nu(t)\eta_0) d\pi_0(\eta_0) \quad \text{(by (40))} \\ &= \int_\Omega g(T^\nu(t)\eta_0(\omega)) d\mathbb{P}(\omega), \\ &= \int_\Omega g(\eta(t, \cdot; \omega)) d\mathbb{P}(\omega) \ . \end{aligned}$$

Here, $\eta = T^\nu(t)\eta_0$ denotes the solution for the two-dimensional incompressible Navier-Stokes equation (34), corresponding to initial data η_0. Computing statistics of the flow (with respect to the statistical vorticity solution) amounts to approximating integrals of the form,

$$\int_\Omega g(\eta(t, \cdot; \omega)) d\mathbb{P}(\omega) \ , \tag{88}$$

with $\eta(\omega)$ being the (deterministic) solution of the incompressible Navier-Stokes equations, corresponding to an initial vorticity realization $\eta_0(\omega)$, $\omega \in \Omega$. The choice $g(\eta) = \eta$ yields the mean vorticity field (or ensemble average). We will approximate the above integral by sampling, using Monte Carlo type methods.

To obtain a computational realization, the random field $\eta(t, x; \omega)$ also needs to be approximated in physical space-time. As mentioned earlier, we will approximate individual vorticity solutions of (34) with the finite difference approximation (56) introduced in the last section. We assume that any numerical approximation of the (deterministic) Navier-Stokes equations (34) satisfies,

Assumption 4.1. *The numerical scheme for approximating the evolution of individual vorticity solutions* $\eta(\cdot;\omega)$ *of* (34) *is of order* s, *i.e. there exists* $s > 0$ *and a constant* $C(T, \|\eta_0\|_{H^m_{per}(\mathbb{T}^2)}) > 0$ *depending continuously on* $\|\eta_0\|_{H^m_{per}(\mathbb{T}^2)}$, *but possibly exponentially on* T *such that, for each realization* $\omega \in \Omega$ *the following error bound is satisfied:*

$$\left\|\eta(\cdot, T; \omega) - \eta^{\Delta x, \Delta t}(\cdot, T; \omega)\right\|_{L^2(\mathbb{T}^2)} \leq C(T, \|\eta_0(\cdot;\omega)\|_{H^m_{per}(\mathbb{T}^2)})\Delta x^s \tag{89}$$

Here, we have implicitly assumed that $\Delta t \approx \Delta x \approx \Delta y$. As shown in the previous section, the scheme (56) satisfies (89) with $s = 1, m = 3$. Moreover, the constants in (89) can depend (non-linearly) on the initial data, as in the worst-case exponential dependence of the constants on the higher-order derivatives of the solution in (79) for the scheme (56). Hence, one needs to assume that the expectation and variance of these high-order norms be finite.

4.1 Singlelevel Monte Carlo

For $g(\eta) = \eta$, we approximate (88) with the "sample average" (singlelevel) Monte Carlo (SLMC) estimator E_M. For $x \in \mathcal{T}_{\Delta x} \subseteq \mathbb{T}^2$ it is defined by

$$\mathbb{E}\left[\eta(x, T; \cdot)\right] \approx \frac{1}{M}\sum_{m=1}^{M} \eta^{\Delta x, \Delta t}(x, T; \omega_m) =: E_M\left[\eta^{\Delta x \Delta t}(x, T; \cdot)\right] \,. \tag{90}$$

Lemma 4.2. *Let* $\eta = \eta(x;\omega) \in L^2(\Omega, L^2(\mathbb{T}^2))$, *then:*

$$\|\mathbb{E}\left[\eta(\cdot;\omega)\right]\|^2_{L^2(\mathbb{T}^2)} \leq \mathbb{E}\left[\|\eta(\cdot;\omega)\|^2_{L^2(\mathbb{T}^2)}\right]. \tag{91}$$

Proof. The lemma follows as special case of Jensen's inequality on the space $(\Omega, \mathcal{A}, \mathbb{P})$:

$$\|\mathbb{E}\left[\eta(\cdot;\omega)\right]\|^2_{L^2(\mathbb{T}^2)} \leq \int_{\omega\in\Omega}\int_{x\in\mathbb{T}^2} |\eta(\cdot;\omega)|^2\, dx d\mathbb{P}(\omega).$$

□

For simplicity, we drop the dependency on x, T and ω and write, by slight abuse of notation, $\eta(x, T; \omega) =: \eta$ and $\eta^{\Delta x \Delta t}(x, T; \omega) =: \eta^{\Delta x \Delta t}$. Next, we prove the error estimates for the Monte Carlo-finite difference approximation,

Theorem 4.3 (Singlelevel Monte Carlo error estimate)**.** *Under assumption* (89) *, the following holds:*

$$\mathbb{E}\left[\|\mathbb{E}[\eta] - E_M[\eta^{\Delta x, \Delta t}]\|^2_{L^2(\mathbb{T}^2)}\right]^{1/2} \leq \frac{1}{M^{1/2}}\mathbb{E}\left[\|\eta - \eta^{\Delta x, \Delta t}\|^2_{L^2(\mathbb{T}^2)}\right]^{1/2} \tag{92}$$

$$\leq \left(\frac{1}{M^{1/2}} + \Delta x^s\right)\mathbb{E}\left[C(T, \|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right]^{1/2}. \tag{93}$$

The proof of the MLMC estimate in the next section with $L = 0$ amounts to proving the above estimate.

4.2 Multilevel Monte Carlo

Even with a first order deterministic numerical scheme, the overall error is dominated by the SLMC error. Multilevel Monte Carlo methods were developed for path simulation of stochastic ODEs in [11] to "accelerate" the convergence of SLMC by performing many inexpensive computations and a few expensive computations. In the context of PDE, MLMC was successfully applied to a wide range of equations (e.g. [19, 3]). In this context, a nested mesh hierarchy is used in multilevel algorithms to reduced computational work, analogous to multigrid methods used in the numerical solution of elliptic PDE.

In $\mathbb{T}^2 = [0,1)^2$, we consider nested, uniform grids of congruent, axiparallel rectangles of size $(\Delta x_l)_{l=0}^L$, $\Delta x_l = \Delta y_l = 2^{-l}$. For each mesh in the family we denote, for simplicity, by $\eta_l := \eta^{\Delta x_l \Delta t_l}$ the discrete (piecewise constant) solution computed at the level $\mathbb{N} \ni l < L$. Also, denote $\eta := \eta(x,T;\omega)$. As convention we denote $\eta_{-1} := 0$. Then, we define the MLMC estimator:

$$\mathbb{E}\left[\eta(x,T;\omega)\right] \approx \mathbb{E}\left[\eta_L\right] = \mathbb{E}\left[\sum_{l=0}^{L}(\eta_l - \eta_{l-1})\right] = \sum_{l=0}^{L}\mathbb{E}\left[\eta_l - \eta_{l-1}\right] \approx \sum_{l=0}^{L} E_{M_l}\left[\eta_l - \eta_{l-1}\right]. \tag{94}$$

The following lemma will facilitate estimating the MLMC error:

Lemma 4.4. *Under assumption* (89)*, for some* $m \geq 0$*, there exists a constant, depending only on the initial data and the time, s.t.*

$$\begin{aligned}\mathbb{E}\left[\left\|(\mathbb{E}-E_M)[\eta - \eta^{\Delta x,\Delta t}]\right\|^2_{L^2(\mathbb{T}^2)}\right]^{1/2} &\leq \frac{1}{M^{1/2}}\mathbb{E}\left[\left\|\eta - \eta^{\Delta x,\Delta t}\right\|^2_{L^2(\mathbb{T}^2)}\right]^{1/2} &(95)\\ &\leq \frac{1}{M^{1/2}}\Delta x^s \mathbb{E}\left[C(T, \|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right]^{1/2}. &(96)\end{aligned}$$

Proof. Let us consider the expectation of the Monte Carlo error and denote by $\mathfrak{e}_m = \eta(x,T;\omega_m) - \eta^{\Delta x,\Delta t}(x,T;\omega_m)$ the error and by $\mathfrak{e} := \mathbb{E}\left[\eta(x,T;\omega) - \eta^{\Delta x,\Delta t}(x,T;\omega_m)\right]$:

$$\begin{aligned}\mathbb{E}\left[\left\|(\mathbb{E}-E_M)[\eta - \eta^{\Delta x,\Delta t}]\right\|^2_{L^2(\mathbb{T}^2)}\right] &\leq \mathbb{E}\left[\frac{1}{M^2}\left\|\sum_{m=0}^{M}(\mathfrak{e} - \mathfrak{e}_m)\right\|^2_{L^2(\mathbb{T}^2)}\right]\\ &\leq \mathbb{E}\left[\frac{1}{M^2}\left(\sum_{m=0}^{M}\|\mathfrak{e} - \mathfrak{e}_m\|_{L^2(\mathbb{T}^2)}\right)^2\right]\\ &\leq \mathbb{E}\left[\frac{1}{M^2}\sum_{m=0}^{M}\|\mathfrak{e} - \mathfrak{e}_m\|^2_{L^2(\mathbb{T}^2)}\right].\end{aligned}$$

Using the fact that the samples $\eta^{\Delta x,\Delta t}(x,T;\omega_m)$ are i.i.d:

$$\begin{aligned}\mathbb{E}\left[\frac{1}{M^2}\sum_{m=0}^{M}\|\mathfrak{e} - \mathfrak{e}_m\|^2_{L^2(\mathbb{T}^2)}\right] &\leq \frac{2}{M^2}\sum_{m=0}^{M}\left(\mathbb{E}\left[\|\mathfrak{e}\|^2_{L^2(\mathbb{T}^2)}\right] + \mathbb{E}\left[\|\mathfrak{e}_i\|^2_{L^2(\mathbb{T}^2)}\right]\right)\\ &\leq \frac{4}{M}\mathbb{E}\left[\left\|\eta - \eta^{\Delta x,\Delta t}\right\|^2_{L^2(\mathbb{T}^2)}\right]\\ &\leq \frac{4}{M}\Delta x^{2s}\mathbb{E}\left[C(T, \|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right].\end{aligned}$$

Taking the square root and absorbing the factor 4 in the constant concludes the proof. □

The preceding lemma allows to provide an a-priori error bound for the combined FV - MLMC approximation of statistical solutions.

Theorem 4.5. *Under assumption* (89)*, and the requirement that*

$$\mathbb{E}\left[C(T, \|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right] < \infty \tag{97}$$

$$\mathbb{E}\left[\|\eta(x,T;\cdot)\|^2_{L^2(\mathbb{T}^2)}\right] < \infty \tag{98}$$

the error of the MLMC approximation can be bounded in the following way:

$$\left\| \mathbb{E}\left[\eta(x,T;\cdot)\right] - \sum_{l=0}^{L} E_{M_l}\left[\eta_l - \eta_{l-1}\right] \right\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \leq \mathbb{E}\left[C(T,\|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right] \left(\sum_{l=0}^{L} \frac{1}{\sqrt{M_l}}\Delta x_l^s + \frac{1}{\sqrt{M_0}}\right), \tag{99}$$

for a constant $0 < C$ *depending on* T *and* $\|\eta_0\|_{H^m_{per}(\mathbb{T}^2)}$.

Proof. We write the error of the MC approximations combined with the space-time discretizations in the usual, telescopic sum as

$$\begin{aligned}
\mathbb{E}\left[\eta\right] - \sum_{l=0}^{L} E_{M_l}\left[\eta_l - \eta_{l-1}\right] = &\quad \mathbb{E}\left[\eta - \eta_L\right] + \mathbb{E}\left[\eta_L\right] - \sum_{l=0}^{L} E_{M_l}\left[\eta_l - \eta_{l-1}\right] \\
= &\quad \underbrace{\mathbb{E}\left[\eta - \eta_L\right]}_{I} + \underbrace{\sum_{l=0}^{L} \mathbb{E}\left[\eta_l - \eta_{l-1}\right] - \sum_{l=0}^{L} E_{M_l}\left[\eta_l - \eta_{l-1}\right]}_{II} \, .
\end{aligned}$$

The second term can be rewritten as:

$$II = \sum_{l=0}^{L} (\mathbb{E} - E_{M_l})[\eta_l - \eta_{l-1}] = \sum_{l=0}^{L} (\mathbb{E} - E_{M_l})[\eta_l - \eta + \eta - \eta_{l-1}]$$

then, using triangle inequality and Lemma 4.2:

$$\begin{aligned}
\|II\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \leq &\sum_{l=0}^{L} 2\,\|(\mathbb{E} - E_{M_l})[\eta_l - \eta]\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \\
&+ \sum_{l=0}^{L} \|(\mathbb{E} - E_{M_l})[\eta - \eta_{l-1}]\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \\
\leq &\sum_{l=0}^{L} 2\mathbb{E}\left[(\mathbb{E} - E_{M_l})[\|\eta_l - \eta\|^2_{L^2(\mathbb{T}^2)}]\right]^{1/2} \\
&+ \sum_{l=0}^{L} \mathbb{E}\left[(\mathbb{E} - E_{M_l})[\|\eta - \eta_{l-1}\|^2_{L^2(\mathbb{T}^2)}]\right]^{1/2} \\
=: &\sum_{l=0}^{L} 2II_{a,l} + II_{b,l} \, .
\end{aligned}$$

Each individual term can be estimated using the SLMC error bounds from theorem 4.3:

$$\begin{aligned}
II^2_{a,l} \leq \frac{1}{M_l}\mathbb{E}\left[\|\eta - \eta_l\|^2_{L^2(\mathbb{T}^2)}\right] \leq \frac{1}{M_l}\Delta x_l^{2s}\mathbb{E}\left[C(T,\|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right] \\
II^2_{b,l} \leq \frac{1}{M_l}\mathbb{E}\left[\|\eta - \eta_{l-1}\|^2_{L^2(\mathbb{T}^2)}\right] \leq \frac{1}{M_l}\Delta x_{l-1}^{2s}\mathbb{E}\left[C(T,\|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right] .
\end{aligned}$$

Substituting in the previous expression, and defining $\Delta x_{-1} := 1$, we obtain:

$$\|II\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \leq 3\mathbb{E}\left[C(T,\|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right]^{1/2} \sum_{l=0}^{L} \frac{1}{\sqrt{M_l}}\Delta x_{l-1}^s$$

and with the bound

$$\|I\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \leq \mathbb{E}\left[C(T, \|\eta_0\|^2_{H^m_{per}(\mathbb{T}^2)})\right]^{1/2} \Delta x^s_L,$$

we obtain the bound in the proposition, including all the constants in C. □

The best possible approximation, assuming a sharp order of convergence (89), is Δx^s_L. The MC sample numbers are to be selected as small as possible, subject to ensuring an overall error which is consistent with this bound, i.e:

$$\sum_{l=0}^{L} \frac{1}{\sqrt{M_l}} \Delta x^s_l \overset{!}{\lesssim} \Delta x^s_L.$$

With the particular (nonadaptive) choice of sample numbers on levels $0 \leq l \leq L$,

$$\frac{1}{\sqrt{M_l}} \Delta x^s_l = \Delta x^s_L \Rightarrow M_l = 2^{2s(L-l)}, \tag{100}$$

we obtain the requirements

$$M_L \sim 1, \quad M_1 \sim 2^{2sL} .$$

Then the combined statistical and discretization error is of the form,

$$\left\| \mathbb{E}\left[\eta(x,T;\cdot)\right] - \sum_{l=0}^{L} E_{M_l}\left[\eta_l - \eta_{l-1}\right] \right\|_{L^2(\Omega;L^2(\mathbb{T}^2))} \leq C(L+1)\Delta x^s_L . \tag{101}$$

Remark 4.6. *The expectations* (97)*,* (98) *are taken with respect to the probability measure* π_0 *on the ensemble of initial vorticities.*

A necessary condition for (97) *to be finite is that the subspace* $H^m_{per}(\mathbb{T}^2)$ *for* $m \geq 3$ *of initial vorticities is charged by* π_0*, i.e,* $\pi_0(H^m_{per}(\mathbb{T}^2)) = 1$*. Inspecting the error analysis for the spatio-temporal discretization, we observe that the expression* $C(T,\cdot)$ *is a smooth, but at least exponentially growing function of the second argument. Therefore, finiteness of the expectations* (97)*,* (98) *stipulates that* π_0 *is supported on bounded subsets of* $H^m_{per}(\mathbb{T}^2)$*, which we assume from now on.*

4.3 Complexity of MLMC

Given the above error estimates for both SLMC and MLMC algorithms (with the same space-time discretization of (56)), we will compare the complexity of both algorithms in terms of the *error vs. computational work* criterion. To this end, we consider the SLMC error estimate in theorem 4.3 and choose the number of samples in order to balance the space-time and sampling errors. As $s = 1$, the MC error behaves like

$$\mathcal{E}_{SLMC} = O(M^{-1/2} + \Delta x),$$

a sufficient choice of the number of samples, for a first order numerical scheme on individual solutions $M = \Delta x^{-2}$. This choice is due to the (non improvable) convergence order 1/2 of the MC sampling and is particularly problematic for the finer meshes (which are computationally most costly), for which with the single-level Monte-Carlo Method also a prohibitively large number of samples is required.

Furthermore, the computational cost of a single deterministic simulation behaves like

$$\mathcal{W}_{DET} \sim \Delta x^{-3}$$

(in two spatial dimensions[1] and one temporal dimension). Consequently, we arrive at the following asymptotic work versus Δx relation

$$\mathcal{W}_{SLMC} \sim \Delta x^{-5}.$$

Hence, the complexity (error vs. work) scales as,

$$\mathcal{E}_{SLMC} \sim \mathcal{W}_{SLMC}^{-1/5}.$$

On the other hand, for MLMC, we have shown in (101), with the choice $M_l = 2^{2s(L-l)}$ we obtain the asymptotic error bound (with L being the number of levels):

$$\mathcal{E}_{MLMC} = O(\Delta x(L+1)) \ .$$

Hence, the total work of MLMC can be calculated as follows,

$$\mathcal{W}_{MLMC} \sim \sum_{l=0}^{L} \Delta x_l^{-3} \Delta x_l^{2s} / \Delta x_L^{2s} = \Delta x_L^{-2} \sum_{l=0}^{L} \Delta x_l^{-1} = \Delta x_L^{-3}(2 - 2^{-L}).$$

Hence, the MLMC complexity estimate (with the choices (100)) scales $\forall \delta > 0$ and for $L \to \infty$, as

$$\mathcal{W}_{MLMC}^{-1/3+\delta} \sim \mathcal{E}_{MLMC} \ .$$

Comparing the SLMC and MLMC estimates, we observe that MLMC promises to be significantly more efficient (at least asymptotically) as far as computational work is concerned. We will test this assertion in the next section on a suite of numerical experiments.

Remark 4.7. *The term $L+1$ arising in the error estimate* (101) *(or, equivalently, on the complexity vs. work) is difficult to eliminate. In fact, the space-time discretization algorithm is of computational complexity $O(\Delta x^{-3})$ (for smooth data and with $\Delta t = O(\Delta x)$, we obtain* 2 *orders of complexity in space and one in time) and of convergence order $O(\Delta x)$, implying that we are unable to apply the MLMC theorem in [12] (which assumes $1 = \alpha \geq 1/2\gamma = 1.5$, α being the order of convergence and γ being the complexity order).*

5 Numerical experiments

In this section, we will present a set of numerical experiments to illustrate our SLMC and MLMC-finite difference approximations of the statistical solutions of the incompressible Navier-Stokes equations (34). We start with a brief overview of the implementation of our algorithms.

5.1 Implementation

Both the deterministic (finite difference) "evolver" and the statistical (SLMC / MLMC) integrators were implemented in C++ using the PETSc library. Both sets of algorithms were parallelized, for implementation on HPC architectures.

The deterministic scheme was parallelized exploiting domain decomposition via MPI communication. The linear algebra routines required by the Hodge decomposition and implicit time stepping exploit the efficiency of multigrid (whenever possible) or other algorithms (LU-decomposition and Krylov methods).

[1]Multigrid ensures that the treatment of the viscosity and of the velocity reconstruction behaves as $O(\Delta x^{-2})$.

The parallel SLMC/MLMC component of the numerical scheme is (almost) trivially parallel (i.e. communication free), except for pre- and post-processing stages. Load balancing is the major issue in this case. Our software is able to perform MC estimates online (i.e. in a single, uninterrupted run) and offline (with the use of postprocessing routines after the completion of every simulation). The tradeoff online vs offline depends on discretization parameters (meshwidth, processor availability, numerical schemes, storage/memory requirements). However, the online MLMC implementation requires particular care in load balancing and in the communication of subdomain data between processors for the estimation of statistical quantities.

Simulations where performed on the Cray XC40 “Piz Dora” at the Swiss National Supercomputing Centre (CSCS). Each node is composed of two 12-core Intel Haswell CPUs. Numerical experiments demonstrate that the simulation runtime is limited by memory bandwidth.

5.2 Numerical experiment 1: Deterministic simulations

We start by presenting a numerical experiment that illustrates our deterministic finite difference scheme (56). To this end, we consider (34) in the torus $\mathbb{T}^2$ (identified with $[0,1)^2$ and periodic boundary conditions) with initial data

$$\eta_0(x_1, x_2) := x_1(1 - x_1)x_2(1 - x_2) \ .$$

There is no analytical formula for the exact solution in this case. Hence, we compute a reference solution η_{ref} at a very fine resolution of 4096^2 cells, by a Gauss-Legendre quadrature with 10 points for the statistical integration. We calculate the relative error

$$\|\eta - \eta_{ref}\|_2 \, / \, \|\eta_{ref}\|_2$$

of the vorticity on a sequence of grids, ranging from 32^2 to 1024^2, at time $T = 1$. We use a range of values of the viscosity ν ranging from $\nu = 10^{-1}$ to $\nu = 10^{-7}$ and include the incompressible Euler case, i.e $\nu = 0$. The corresponding results are plotted in figure 1. These results show that,

- The finite difference scheme (56) converges to the (reference) solution at the rate of 1, as predicted by the error estimate (79), as the mesh is refined.

- The convergence order is completely independent of the viscosity (Reynolds number). As shown in figure 1, the same convergence order is obtained for a six orders of magnitude variation in the Reynolds number. Furthemore, as discussed in the proof of (78), we also obtain a rate of convergence of order 1, even for inviscid Euler case. Hence, our finite difference scheme (56) is *robust* with respect to arbitrarily high Reynolds numbers.

- The role of the viscosity ν is also evidenced from figure 1. As shown in the error estimate (78), higher values of ν lead to a smaller value of the constant in the error estimate. Thus, reducing the physical viscosity only affects the constants (mildly) but does not affect the rate of convergence of the scheme (56).

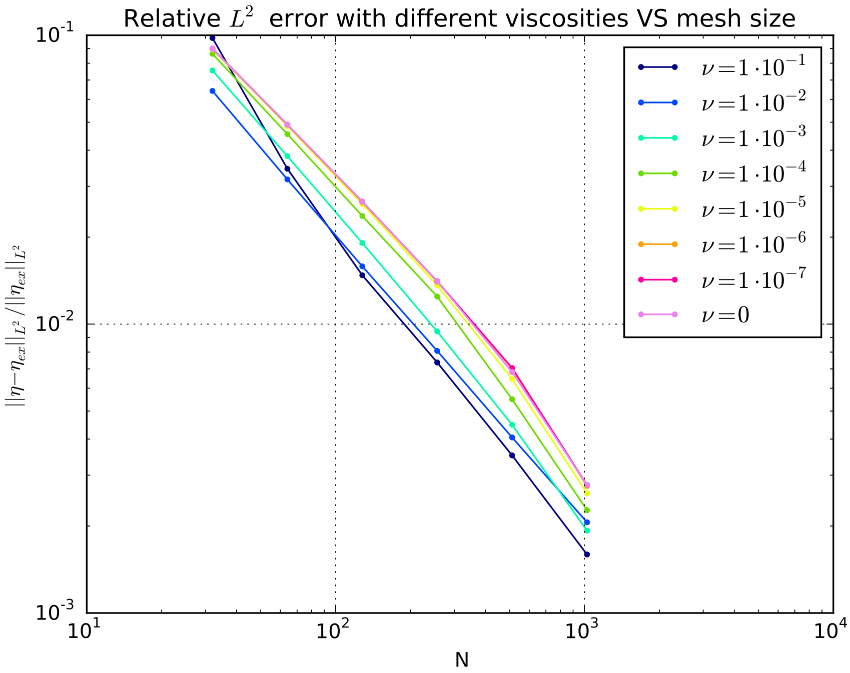


Figure 1: Relative error of the NS solver for individual solutions with respect to the mesh size Δx for various values of ν.

5.3 Numerical experiment 2: Single mode stochastic perturbation

Equipped with a robust finite difference scheme for the incompressible Navier-Stokes equations (34), we now proceed to approximate statistical solutions. We start with an initial probability measure π_0 concentrated on vorticities of the form:

$$\eta_0(x;\omega) = \bar{\eta_0}(x) + Y_1(\omega)\eta_1(x)$$

with $Y_1 \sim \mathcal{U}(-1,1)$ where $\bar{\eta_0}(x) \in H^s_{per}(\mathbb{T}^2)$ with $s < 3/2$, denotes the mean initial vorticity corresponding to the *vortex blob*

$$\bar{\eta_0}(x) := x_1(1-x_1)x_2(1-x_2) \in H^s_{per}(\mathbb{T}^2) \quad \text{for} \quad s < 3/2 \,, \tag{102}$$

and the single-mode stochastic fluctuation at point $x = (x_1, x_2) \in \mathbb{T}^2$ is given by

$$\eta_1(x) = \eta_1(x_1,x_2) := \sin(2\pi x_1)\sin(2\pi x_2) \in C^\infty_{per}(\mathbb{T}^2) \,.$$

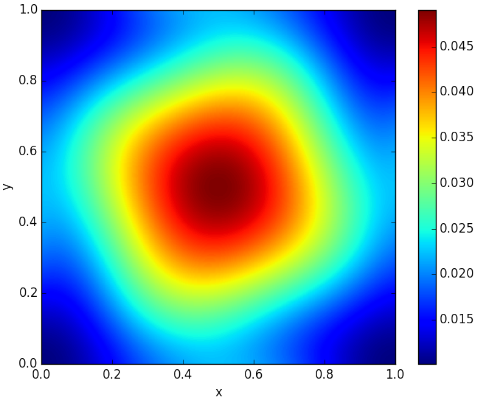
Our aim is to test both the SLMC and MLMC methods in order to check the validity of the error estimates, for both these algorithms, proved in the previous section. To this end, we calculate a reference solution by Gauss-Legendre quadrature with respect to y_1 with 10 nodes. The mean and variance of the solution, for various values of the viscosity ν, are shown in figure 2.

We perform both SLMC and MLMC computations, using the same deterministic finite difference scheme (56), on a sequence of grids ranging from 32^2 to 1024^2. The number of samples for the both the SLMC and MLMC calculations as well as the number of levels for the MLMC method, are chosen based on the discussion of the previous section. For MLMC, we choose $M = 20$ samples on the finest level and choose the coarsest level to be at 16^2. For SLMC, we choose $M = 20$ samples on the coarsest 16^2 mesh. We repeat the experiments $S = 10$ times in order to estimate the relative error (which is a random variable itself). The relative error of the expectation (at time $T = 1$) is approximated with

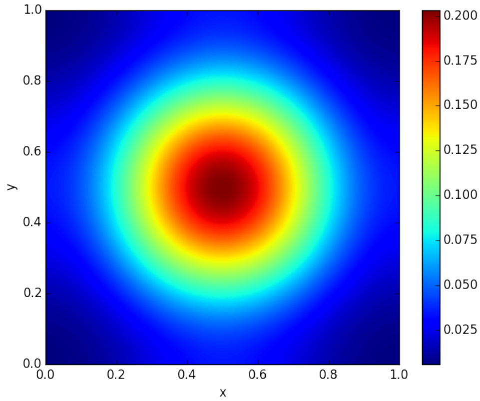
$$err = \frac{1}{S}\sum_{s=0}^{S}(\|E[\eta_s] - E[\eta_{ref}]\|_2 \,/\, \|E[\eta_{ref}]\|_2), \tag{103}$$

where E represents the sample expectation, and is plotted in figure 3. From this figure, we observe that

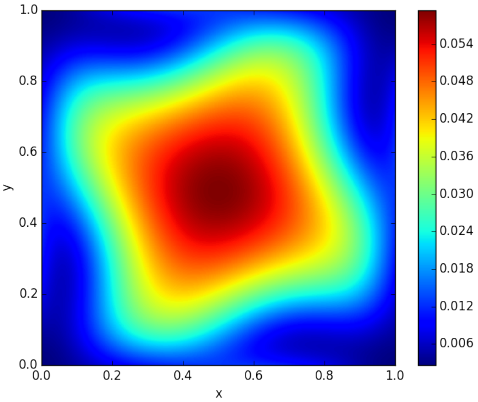
- Both the SLMC and MLMC algorithms approximate the mean of the underlying statistical solution accurately. In fact, for a fixed mesh resolution (consequently sample numbers), the SLMC method is slightly more accurate than the MLMC method.
- However, the big difference between the methods appears when one considers the computational complexity in terms of the error vs. computational work (right column of figure 3). Here, as predicted by the discussion in the previous section, the MLMC method is considerably faster (more efficient) than the SLMC method. In fact, in the range of engineering accuracy (5% - 1% error) the MLMC is consistently faster by approximately two orders of magnitude, when compared to the SLMC method, for a range of viscosity parameters ν.
- The approximation of statistical solutions is robust with respect to the increase in Reynolds numbers. In fact, the MLMC method continues to be at least two orders of magnitude faster even for the inviscid $\nu = 0$ case. However, there is a slight improvement in speed up when one considers a higher value of the viscosity parameter.



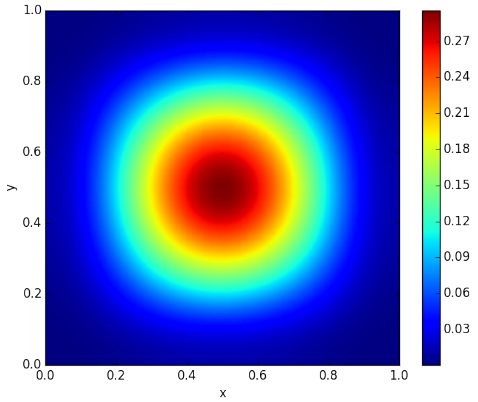
(a) Mean, $\nu = 10^{-2}/4$



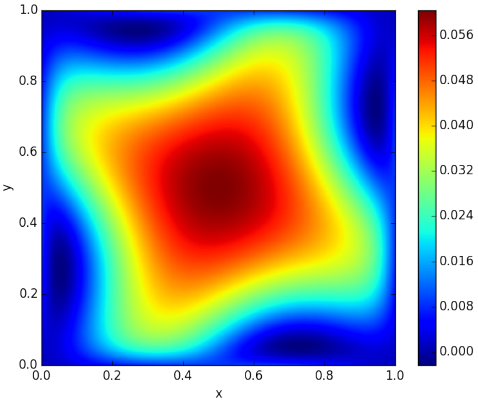
(b) Variance, $\nu = 10^{-2}/4$



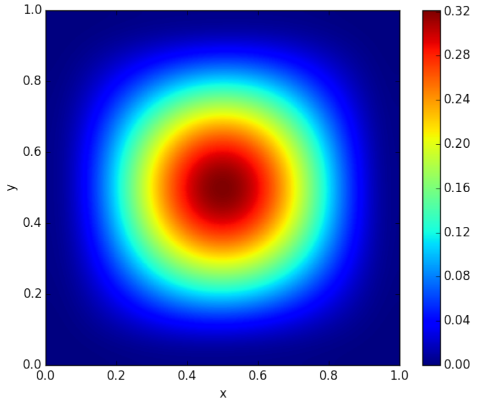
(c) Mean, $\nu = 10^{-2}/16$



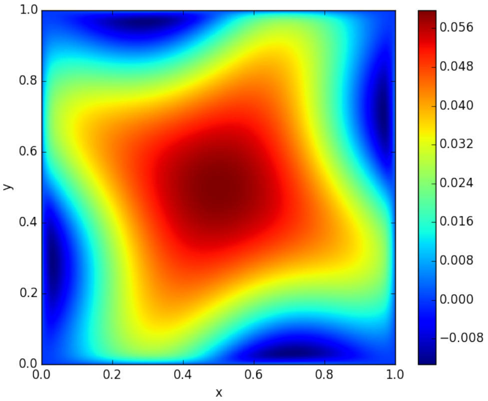
(d) Variance, $\nu = 10^{-2}/16$



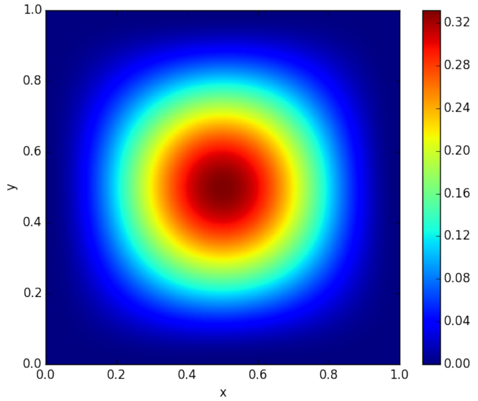
(e) Mean, $\nu = 10^{-2}/64$



(f) Variance, $\nu = 10^{-2}/64$

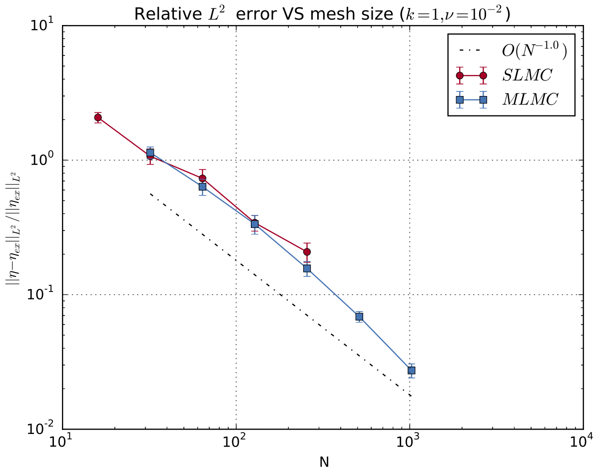


(g) Mean, $\nu = 0$

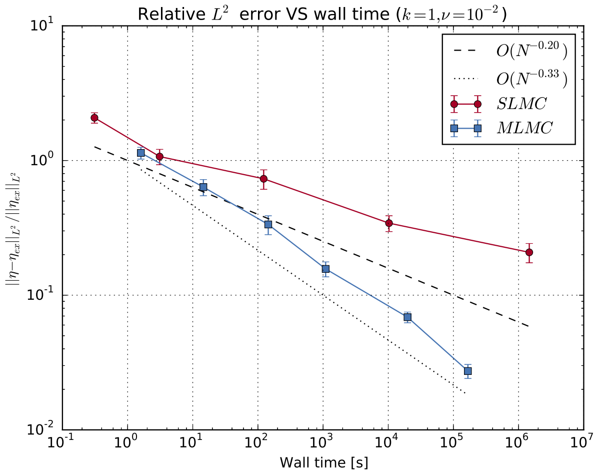


(h) Variance, $\nu = 0$

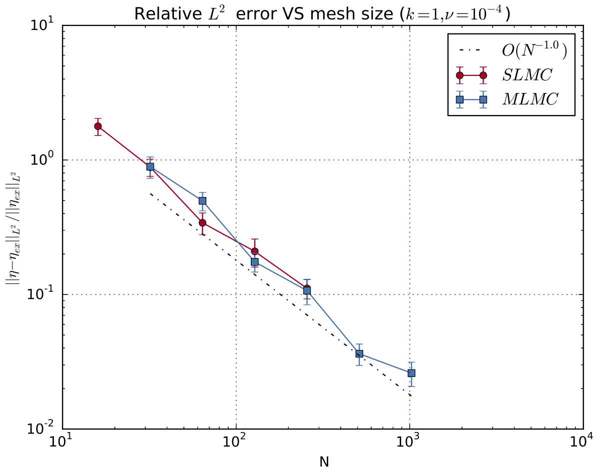
Figure 2: Mean and variance of a 1-term KL expansion for different viscosities.



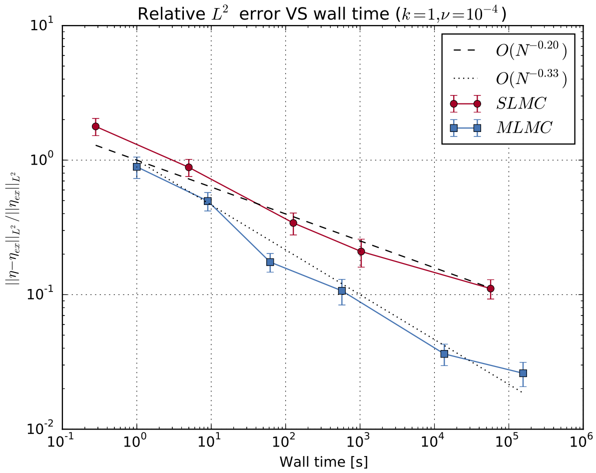
(a) Error VS mesh width



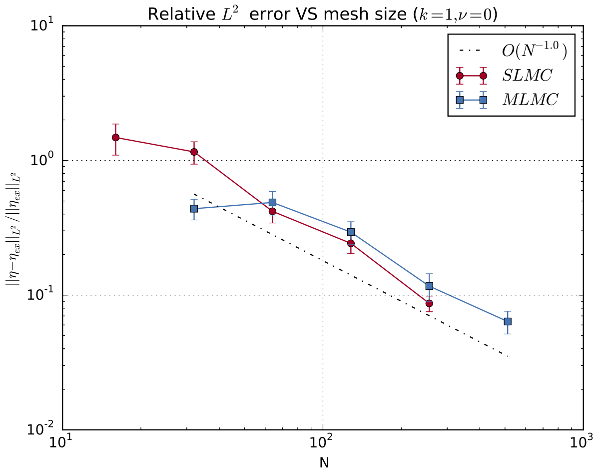
(b) Error VS CPU Time



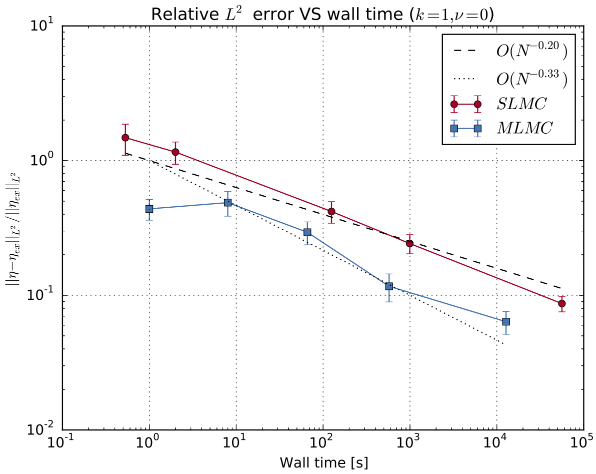
(c) Error VS mesh width



(d) Error VS CPU Time



(e) Error VS mesh width



(f) Error VS CPU Time

Figure 3: relative L_2 error of the mean for different viscosities with SLMC and MLMC, with respect to the mesh width h (left) and w.r.t. wall clock time (right) of a 1-parametric statistical initial data. The reference solution is computed using Gaussian quadrature. There is no qualitative difference in error vs. meshwidth Δx between SLMC and MLMC.

5.4 Numerical experiment 3: Karhunen-Loève expansion

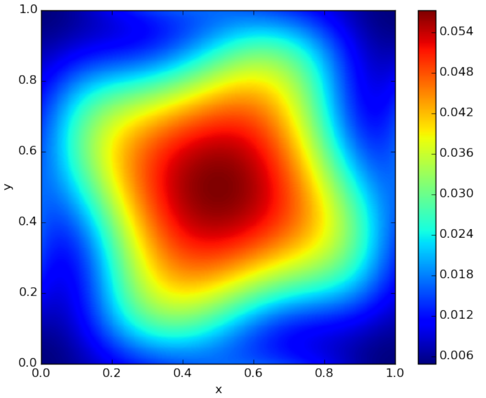
We consider the initial statistical solution as a probability measure, realized as the law of a random field, given in terms of a Karhunen-Loéve expansion of the form,

$$\eta_0(x;\omega) = \bar{\eta_0}(x) + \sum_{k=1}^{K} Y_k(\omega)\eta_k(x)$$

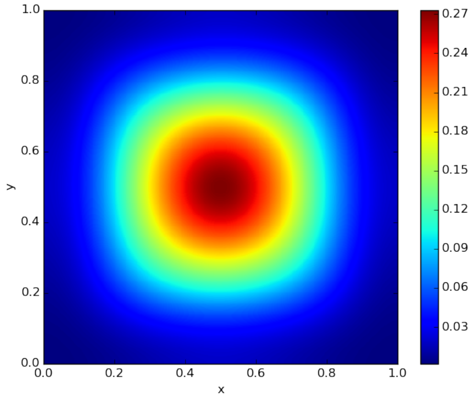
with $Y_k \in \mathcal{U}(-1,1)$ random variables, $\bar{\eta_0}(x) \in H^1_{per}(\mathbb{T}^2)$ the mean vorticity, and

$$\eta_k(x) = \eta_k(x_1, x_2) := \lambda_k \sin(2\pi k x_1)\sin(2\pi k x_2) \in H^1_{per}(\mathbb{T}^2),$$

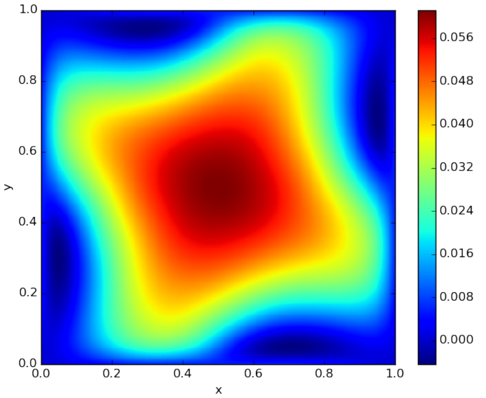
where $(\lambda_k)_{k=0}^K$ is given by $\lambda_k = k^{-2}$. The mean initial vorticity is the vortex blob (102). In this case, we compute a reference solution using a MLMC simulation on a suitably refined mesh (using the fine mesh of 2048^2 with 20 samples on the finest level and 16^2 as the coarsest level). The mean and the variance of the resulting random field is plotted in figure 4. We show the estimated relative error, with respect to the computed reference solution, in figure 5. The results are very similar to those of the previous numerical experiment. In particular, although both the SLMC and MLMC methods provide a robust approximation of the mean, the MLMC method is considerably faster than the SLMC method. The speed up is at least two orders of magnitude in terms of the run time. Furthermore, this speed up is consistent across a range of increasing Reynolds numbers and also holds in the inviscid case.



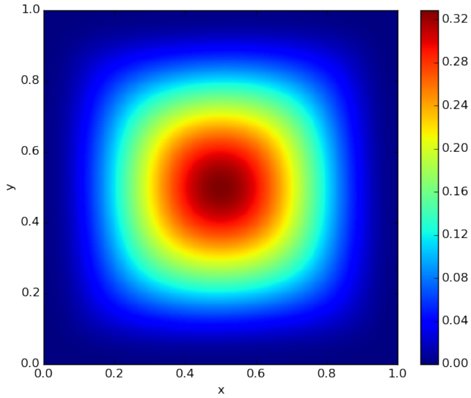
(a) Mean, $\nu = 10^{-2}$



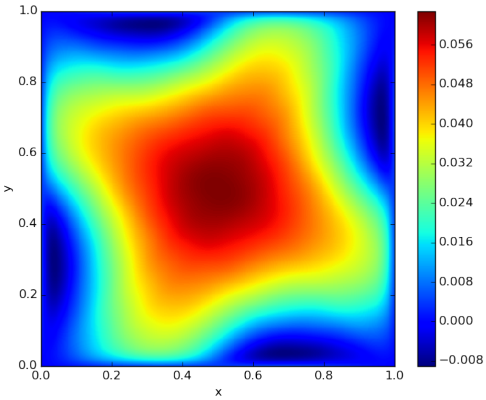
(b) Variance, $\nu = 10^{-2}$



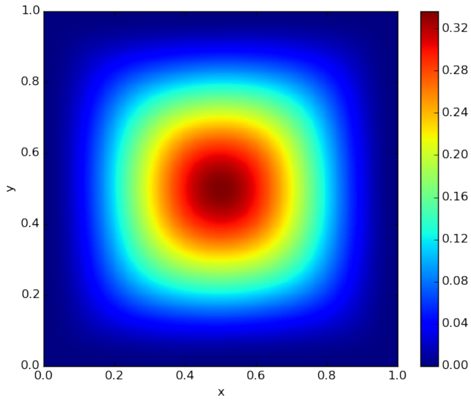
(c) Mean, $\nu = 10^{-3}$



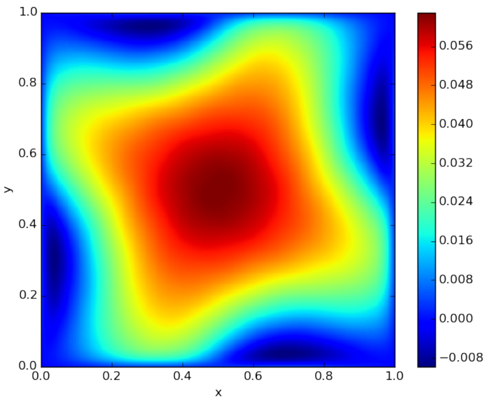
(d) Variance, $\nu = 10^{-3}$



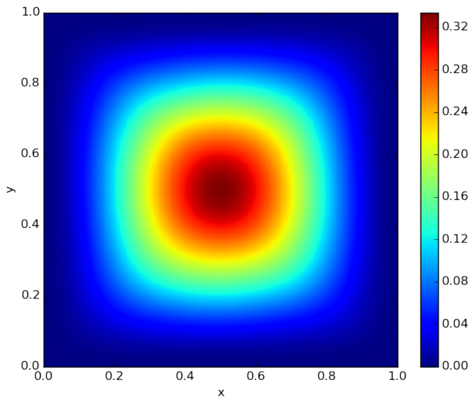
(e) Mean, $\nu = 10^{-7}$



(f) Variance, $\nu = 10^{-7}$

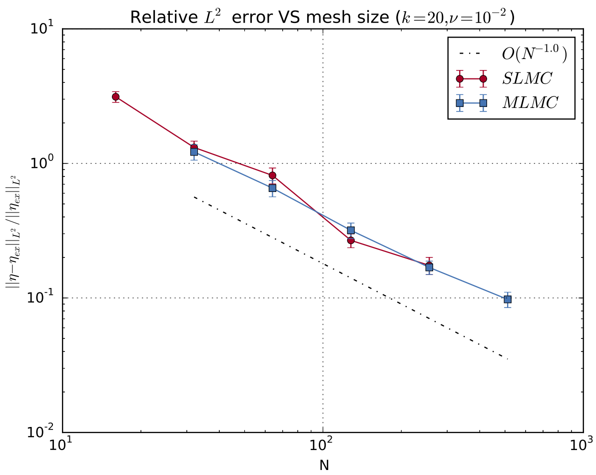


(g) Mean, $\nu = 0$

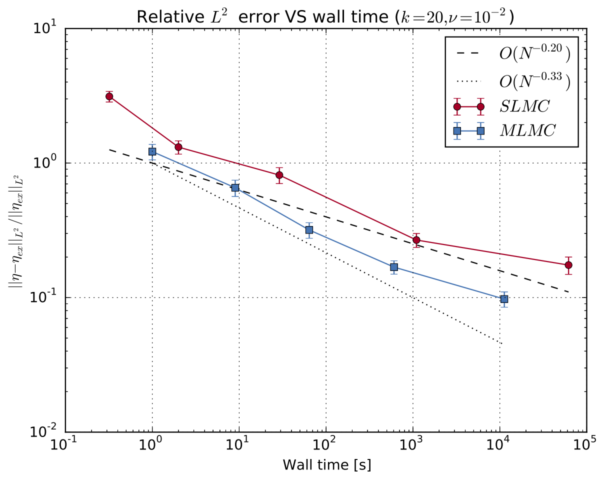


(h) Variance, $\nu = 0$

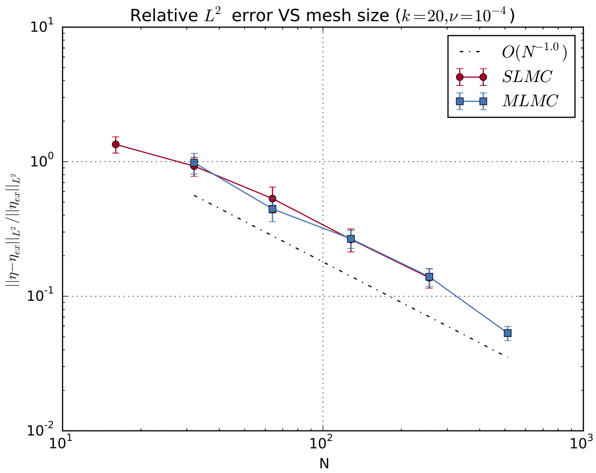
Figure 4: Mean and variance of a 20-term KL expansion for different viscosities.



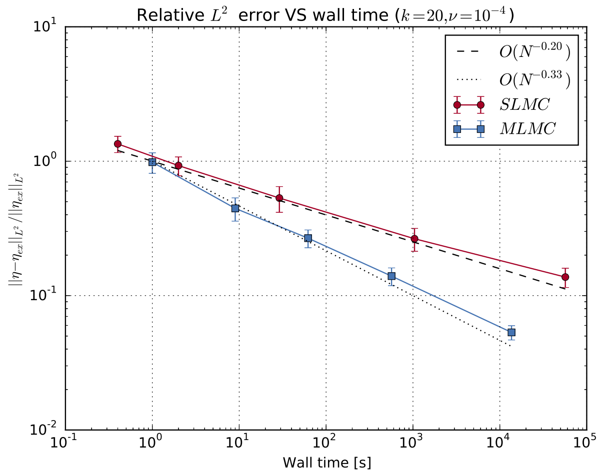
(a) Error VS mesh width



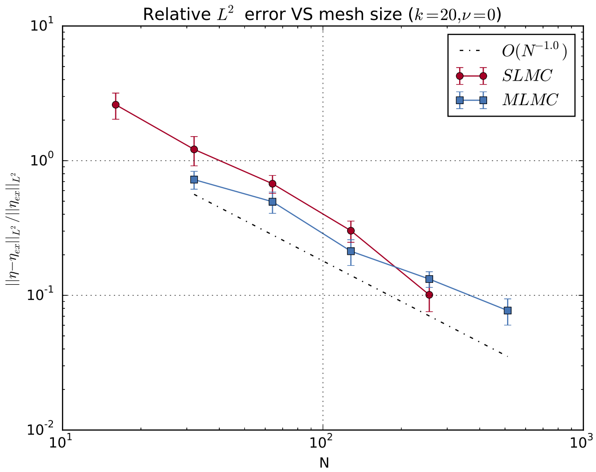
(b) Error VS CPU Time



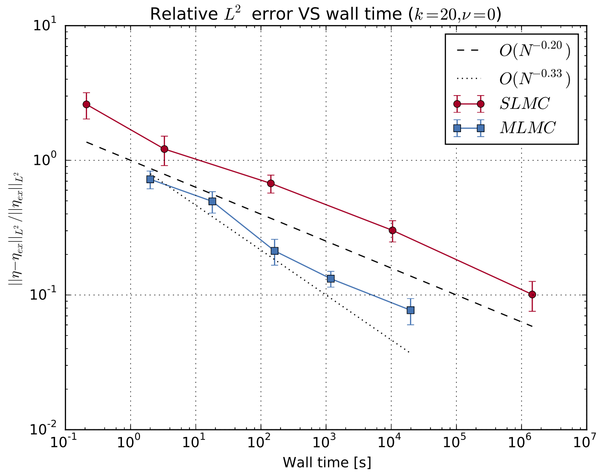
(c) Error VS mesh width



(d) Error VS CPU Time



(e) Error VS mesh width



(f) Error VS CPU Time

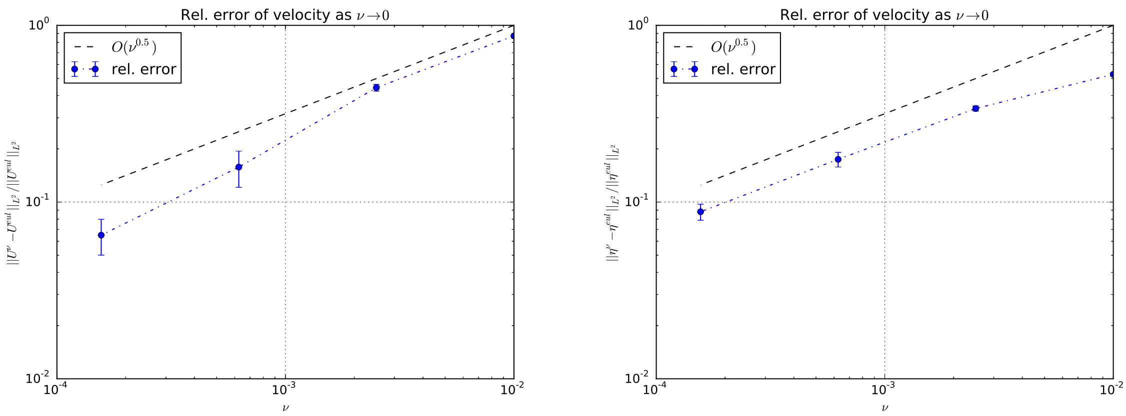
Figure 5: relative L_2 error of the mean of a statistical simulation with 20 random parameters for different viscosities with SLMC and MLMC, with respect to the mesh width h (left) and wall time (right). The reference solution is computed using a spatially refined MLMC simulation.

5.5 Numerical experiment 4: ν-stability of KL expansion

We perform the same experiment as number 3 investigating the behaviour as $\nu \to 0$. We compute the relative error on a sufficiently fine mesh of 1024^2 of a simulation with $\nu > 0$ with respect to the "reference simulation" obtained with $\nu = 0$:

$$err = \frac{1}{S}\sum_{s=0}^{S} \left\| E_{MLMC}[\eta_s^{\nu}] - E_{MLMC}[\eta_s^{eul}] \right\|_{L_x^2} / \left\| E_{MLMC}[\eta_s^{eul}] \right\|_{L_x^2}$$

In order to observe convergence we must to limit to sufficiently small Reynolds number, such that it can be resolved by the mesh. If the viscosity is too small we observe a flat plot due to the viscous cut-off. We observe in figure 6, a convergence order of $O(\sqrt{\nu})$, which is consistent with the convergence rate estimates towards the inviscid limit of the Cauchy problem in [2, Chap. 7].



(a) Velocity (b) Vorticity

Figure 6: Relative error and standard deviation of the mean versus different values of viscosity.

5.6 Numerical experiment 5: Uncertain smooth shear layer.

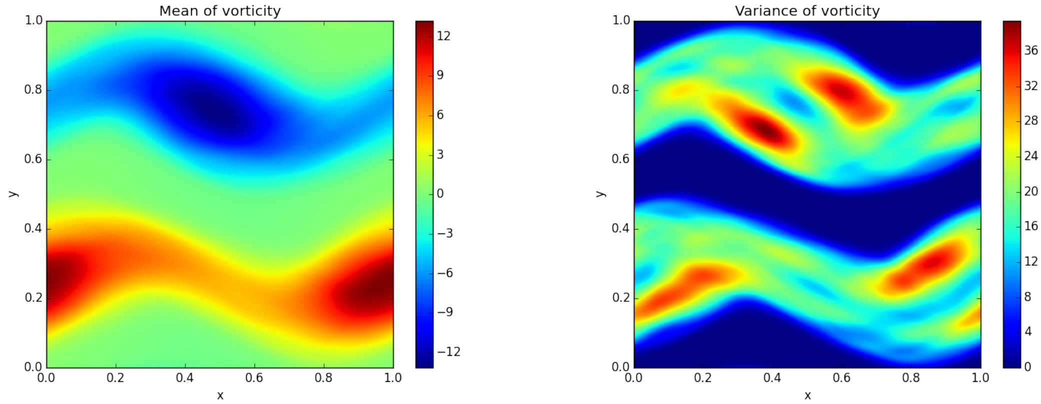
We consider an uncertain version of the well known benchmark test case of smooth shear layer by considering the initial data, given $\rho > 0$ we start from the initial datum in $[0,1)^2$

$$u_0(x,y) := \begin{cases} -\tanh((y+0.75)/\rho) & y > 0 \\ -\tanh((-y+0.25)/\rho) & y < 0 \end{cases}$$
$$v_0(x,y) := \delta \sin(2\pi x)$$

and add a random, small amplitude perturbation using $\omega \sim \mathcal{U}([0,1)^K)$ to the interface, by defining $f(\omega; x, y) := (x, y + \gamma \sum_{k=0}^{K/2} Y_{2k} \sin(2\pi(x + Y_{2k+1}))$ and using as initial data

$$\eta_0(\omega; x, y) := \mathrm{rot}(\underline{u} \circ f)(\omega; x, y).$$

The initial measure, on the ensemble of vorticities, is supported on $C_{per}^{\infty}(\mathbb{T}^2)$ and satisfies all the assumptions for the convergence rate (99), see remark 4.6. We compute a reference statistical solution with the MLMC-finite difference algorithm and present the mean and the variance with respect to the underlying statistical solution, for $\nu = 10^{-4}$ and at time $T = 0.4$, in figure 7. The plot shows that both the mean and the variance are well resolved. The mean vorticity field



(a) Mean

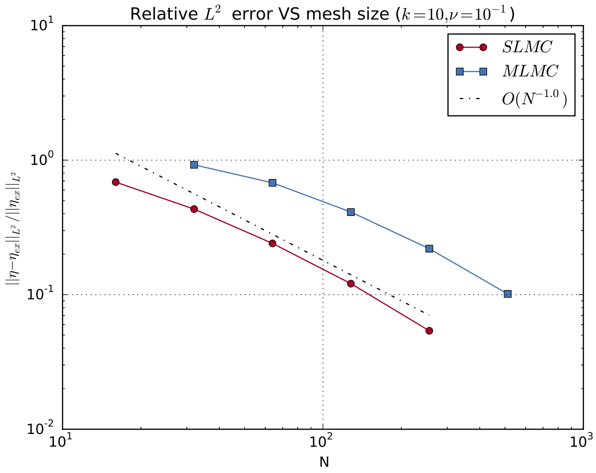
(b) Variance

Figure 7: Mean and variance of a smooth Vortex-Sheet ($\rho, \delta, \gamma = 0.05$) with uncertain vortex location and magnitude.

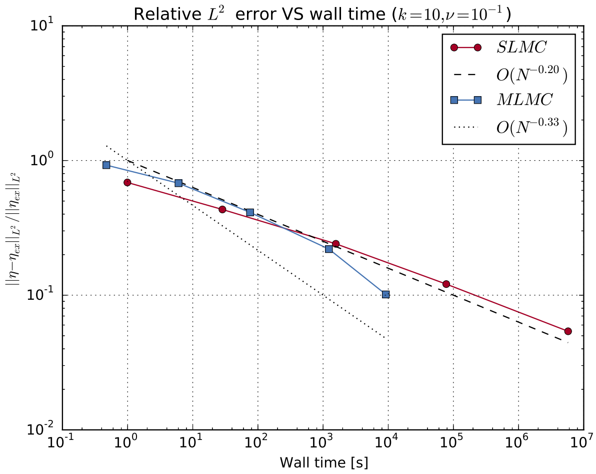
consists of strong vortices (of both negative and positive signs) separated by a smooth field. The variance seems to be concentrated near the edge of the vortices. In particular, it appears that the vortex location is quite sensitive to the initial perturbation.

We compute relative errors with respect to this reference solution for different grid resolutions and different values of ν and plot the results in figure 8. The results are completely consistent with the previous numerical experiments as the MLMC method is consistently two orders of magnitude faster than the corresponding SLMC method, across a large range of Reynolds number, upto and including the Euler case, i.e, infinite Reynolds number.

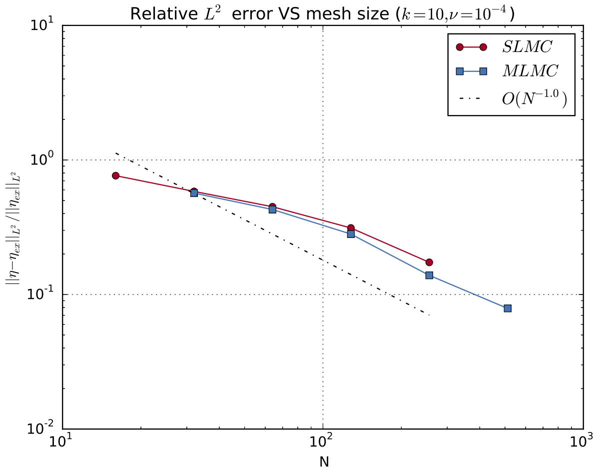
Remark 5.1. *The MLMC error estimate* (99) *required us to assume that the initial measure* π_0*, on the ensemble of initial vorticities, was supported on a subspace* $H^m_{per}(\mathbb{T}^2)$ *(with* $m \geq 3$*). However, we see from numerical experiments* 2 *and* 3 *where the initial measure was supported on a less regular space, namely* $H^m_{per}(\mathbb{T}^2)$ *with* $m < \frac{3}{2}$*, that a convergence rate for the MLMC method, similar to that predicted in* (99)*, also holds in practice for less regular initial data.*



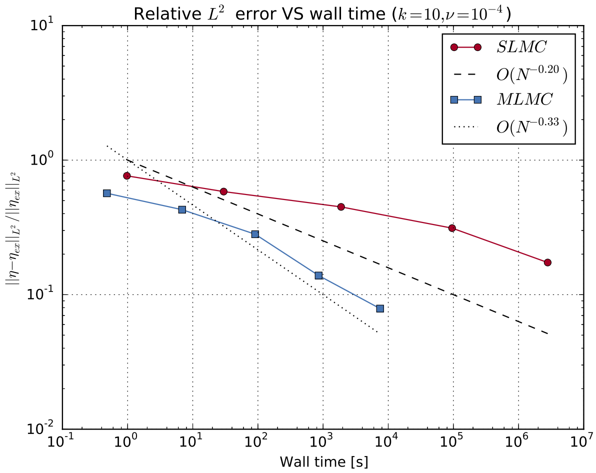
(a) Error VS mesh width



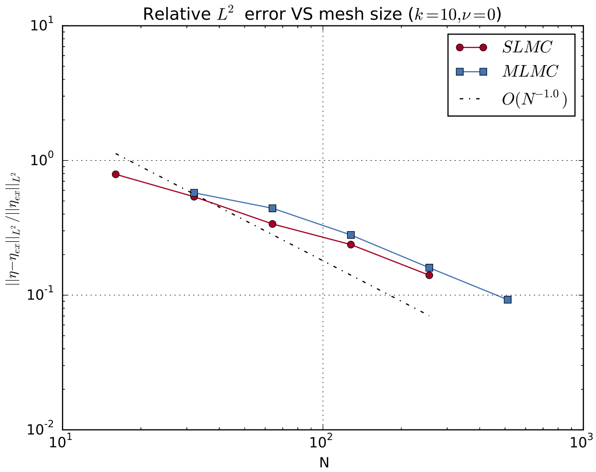
(b) Error VS CPU Time



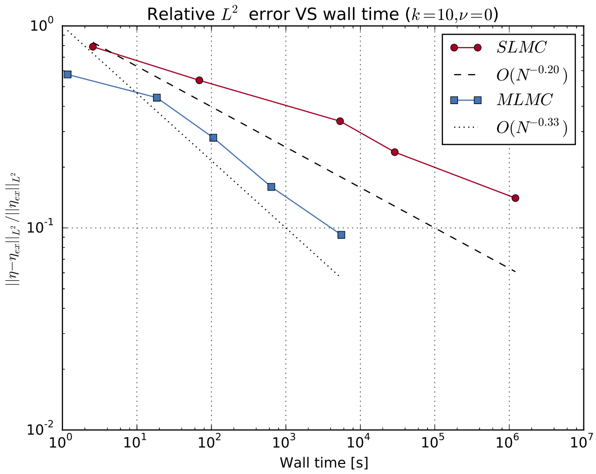
(c) Error VS mesh width



(d) Error VS CPU Time



(e) Error VS mesh width



(f) Error VS CPU Time

Figure 8: L_2 error of the mean for different viscosities with SLMC and MLMC, with respect to the mesh width h and wall clock time of a smooth Shear-Layer ($\rho = 0.05$) with uncertain vortex location.

6 Conclusion

The incompressible Navier-Stokes equations are the fundamental governing equations for fluids. They are supposed to describe very complex phenomena, that includes highly irregular and chaotic motions that constitute the core of very high Reynolds number unstable and turbulent flows. Given this observed highly sensitive dependence on initial data, the deterministic framework of weak solutions does not suffice in describing the complexity, that solutions of the Navier-Stokes equations entail. Furthermore, the process of determining initial conditions, body forces and other input data, via measurement or observation, is prone to errors on account of uncertainty. Given these factors, it is natural to seek a probabilistic framework of (otherwise deterministic) solutions for the Navier-Stokes equations. The concept of statistical solutions, introduced by Foias and Prodi, is one such notion.

In two space dimensions, statistical solutions of the NSE in the sense of Foias and Prodi are (time) parametrized probability measures on the function space $H = H(\mathrm{div}; \mathbb{T}^2)$ of divergence-free, square integrable velocity fields (in which the deterministic solutions take values in). The evolution of the statistical solutions is constrained in terms of the Liouville equation. Although existence results for statistical solutions are available in both two and three dimensions, uniqueness has been established in two space dimensions only. Furthermore, in two dimensions, statistical solutions are completely identified as the push forward of the initial probability measure under the Navier-Stokes flow (solution operator).

Although statistical solutions have been widely studied and their numerous links to turbulence are by now well established, very few (if any) papers have considered the efficient numerical approximation of statistical solutions. The main aim of the present paper has been to provide an efficient numerical method for approximating statistical solutions for the Navier-Stokes equations in space dimension two.

In two dimensions, it is natural to work with a vorticity-stream function formulation of the Navier-Stokes equations (34). Hence, we recast the standard concept of statistical velocity solutions in terms of the vorticity-stream function formulation. To this end, we have introduced a novel reformulation of the statistical vorticity solutions for (34).

We realize the statistical vorticity solutions in terms of laws of random fields. The random fields are computed by discretizing individual solutions in space-time using a finite difference (volume) scheme and in the probability space by Monte-Carlo sampling. In particular, we propose a novel fully implicit finite difference scheme (56) and prove that it satisfies a rigorous error estimate, with a convergence of order one, as the mesh is refined. For discretizing the probability space, we use both the standard Monte Carlo method and the more recent Multilevel Monte Carlo (MLMC) method. We prove error and complexity estimates for both methods. In particular, we show that the MLMC method is (upto logarithmic term) optimally complex, i.e, it has same complexity (asymptotically) as a single deterministic run.

We present several numerical experiments to illustrate our methods. From the numerical experiments, we observe that

- Both the MC and MLMC methods converge to an underlying statistical solution, as predicted by our numerical analysis. At the same grid resolution, the MC method is slightly more accurate than the MLMC method.
- However, the MLMC method is considerably more efficient. In particular, to achieve relative accuracy of one percent, it requires at least two orders of magnitude less run time than the corresponding MC method.
- The convergence rate, as well as the observed (and predicted) speed up, are totally robust with respect to the Reynolds number. We also observe a two orders of magnitude speed

up even for the inviscid (infinite Reynolds number) case i.e, for the incompressible Euler equations.

Given the above considerations, it is fair to claim that the MLMC-finite difference scheme provides a robust and efficient approximation of the statistical solutions of the two-dimensional Navier-Stokes equations, for arbitrarily high Reynolds numbers.

The work presented in this paper, forms the basis of various extensions. Some of them are outlined below,

- The current paper is restricted to the two dimensional case. The extension of the algorithms to three space dimensions is straightforward. As there are no uniqueness results for the three dimensional case, it will not be possible to extend the analysis, given the state of the art. However, the MLMC-finite difference algorithm might still provide an attractive numerical framework for the three dimensional case, particularly for the computation of turbulent flows.

- The finite difference (volume) scheme (56) is defined on Cartesian grids. The extension to unstructured grids needs to be performed, in order to compute statistical solutions in the case of Dirichlet boundary conditions, for instance in exterior flows.

- We have defined the statistical solutions only for the Navier-Stokes case i.e, $\nu > 0$. However, the numerical scheme (56) and the MC (MLMC) algorithms also cover the inviscid $\nu = 0$ case i.e, the incompressible Euler equations. The numerical results present in section 5 also indicated that the MLMC-finite difference algorithm is able to robustly compute the statistical solutions even for the Euler equations.

These questions will be addressed in forthcoming papers.

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