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Quantized tensor-structured finite elements for second-order elliptic PDEs in two dimensions*

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Abstract

We analyze the approximation of the solutions of second-order elliptic problem, which have point singularities but belong to a countably normed space of analytic functions, with a first-order, h-version finite element (FE) method based on uniform tensor-product meshes. The FE solutions are well known to converge with algebraic rate at most $1/2$ in terms of the number of degrees of freedom, and even slower in the presence of singularities. We analyze the compression of the FE coefficient vectors represented in the so-called *quantized tensor train* format. We prove, in a reference square, that the resulting FE approximations converge exponentially in terms of the effective number N of degrees of freedom involved in the representation: $N = \mathcal{O}(\log^5 \varepsilon^{-1})$, where $\varepsilon \in (0,1)$ is the accuracy measured in the energy norm.

Numerically we show for solutions from the same class that the entire process of solving the tensor-structured Galerkin first-order FE discretization can achieve accuracy ε in the energy norm with $N = \mathcal{O}(\log^\kappa \varepsilon^{-1})$ parameters, where $\kappa < 3$.

Keywords: singular solution, analytic regularity, finite-element method, tensor decomposition, low rank, tensor rank, multilinear algebra, tensor train .
AMS Subject Classification (2000): 15A69, 35C99, 35J25, 65N12, 65N30, 65N35.

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1 Introduction

Linear second-order elliptic boundary-value problems with analytic data admit solutions analytic up to the singular support of the data and to the geometric singularities on the boundary or interfaces. Classical Lagrangian finite-element (FE for short) discretizations, based on uniformly refined meshes, can exploit only finite regularity of the solution. They thus realize asymptotic convergence rates that are at best *algebraic*, when measured in terms of a discretization parameter, such as the mesh width. In two dimensions, continuous piecewise-bilinear FE approximations converge, with respect to the number of degrees of freedom representing the approximations, with the rate $1/2$ at best. Adaptivity does not essentially improve the situation: adaptive mesh refinement can compensate for a local loss of regularity (e.g. due to corner singularities) but the maximal convergence rate achieved by adaptivity is still limited by the approximation order of the underlying FE method.

For singular solutions, the *exponential convergence* afforded by analytic regularity is realized by the so-called *hp*-FE method, as was proved in the 1980s in a series of papers, see [6] and references therein. The key ingredients in the analysis were a *geometric mesh refinement* towards the singularities of the solution (such as vertices of a polygonal domain Ω) and the use of high approximation order on large elements away from the singularities.

The idea of combining a tensor-structured representation with a low-order tensor-product discretization has been exploited computationally in a number of papers. A comprehensive overview is given in [28]; we provide further references in section 2. The reference most relevant to the present paper is [52, section 6.2.1], where the solution of a linear system corresponding to a boundary-value problem for the Poisson equation in $(-1,1)^3 \setminus [0,1)^3$ (partitioned into 7 cubic patches) is considered as a numerical example. In [52], the authors show that a tensor-structured solver of the discrete problem outperforms an algebraic multigrid method in solving the linear system. The algebraic treatment of that example in [52], however, bypasses the singular nature of the solution and the question of how well the FE space used to obtain the linear system approximates the solution.

In the present paper, we consider countably normed classes $\mathfrak{C}^2_\beta(\mathrm{Q}) \subset \mathbb{H}^1(\mathrm{Q})$ with $\beta \in [0,1)$. They consist of functions defined on the reference square $\mathrm{Q} = (0,1)^2$ that are analytic in its closure $\mathbf{cl}\,\mathrm{Q}$ except the origin, where the functions may exhibit algebraic singularities. Those countably-normed analyticity classes were introduced in [7, 5] following the regularity analysis of the solutions of linear elliptic boundary-value problems in *weighted* Sobolev spaces; see, e.g. [42, 43, 49, 9, 4, 44]. By the regularity and analyticity results of the aforementioned papers, the solutions of a broad, practically relevant class of boundary-value problems, namely for linear second-order elliptic operators with analytic coefficients in straight and curvilinear polygons, belong to $\mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ respectively, where Ω is, generally, a curvilinear polygon partitioned into quadrilaterals mapped analytically to Q and $\boldsymbol{\Theta}$ and $\boldsymbol{\beta}$ denote tuples of corners and singularity orders.

The purpose of the present paper is to establish, for the functions from $\mathfrak{C}^2_\beta(\mathrm{Q})$, the *exponential convergence* of tensor-structured approximations, which combine a naive quasi-uniform discretization with tensor decompositions. Namely, we consider *QTT-FE approximations*, by which we mean *continuous Lagrangian FE approximations* whose coefficient vectors are compressed in the *quantized-tensor-train* (QTT) format [56, 55, 54].

Instead of the function values at the nodes, which are associated to "hat functions", the *degrees of freedom* of QTT-FE approximations are the parameters of the corresponding QTT representations. We prove that, with respect to the number of such degrees of freedom, QTT-FE approximations of singular functions from the analyticity classes $\mathfrak{C}^2_\beta(\mathrm{Q})$, $\beta \in [0,1)$, do converge exponentially. This mathematical result paves the way for *exponentially-convergent low-order* FE approximations in elliptic boundary-value problems with analytic or, more generally, piecewise-analytic coefficients.

In the *QTT-FEM*, the method of solving such problems using QTT-FE approximations, the uniform mesh underlying the construction of the low-order FE space, whose refinement to high accuracies is computationally prohibitive, becomes *virtual*. Indeed, the entire mesh may never be explicitly accessed by a QTT-FEM solver.

We start with discussing, briefly, tensor decompositions and, in more detail, the tensor-train and quantized-tensor-train decompositions in section 2. We revisit the basic properties of these tensor formats, which make them suitable for the tensor-structured solution of PDEs.

In section 3, we give definitions of a curvilinear polygon Ω and of weighted Sobolev spaces and countably normed analyticity classes $\mathfrak{B}^2_\beta(\Omega)$ and $\mathfrak{C}^2_\beta(\Omega)$. By the analyticity shift result of [6, Theorem 3.1], the solutions of boundary-value problems in Ω for linear second-order elliptic operators with analytic, consistent data belong to $\mathfrak{B}^2_\beta(\Omega)$ with appropriate orders $\beta \in [0,1)$. In section 3.4, we consider a model boundary-value problem of that type.

In section 4, we define, in d dimensions, uniform tensor-product partitions and the corresponding FE spaces for functions with $\mathfrak{n}$ components defined on $\mathrm{Q} = [0,1]^d$. The components shall represent restrictions of functions defined in a domain of complex geometry to patches that are mapped to Q, similarly as it is done in composite-wavelet [22, 62, 35, 61, 20, 21] and composite-spectral [17, 18, 19, 63] methods. In section 4.3, we define the QTT-FE format for such multi-component functions.

In section 5, we return to the case of $d = 2$ dimensions and analyze an h-FE approximation based on hp approximation as *auxiliary*. We introduce hp spaces, prove the approximation and stability properties of corresponding projections and, finally, show the low-rank QTT-FE structure of the resulting approximations.

In section 6, we demonstrate the QTT-FE approximation in Q numerically and apply the QTT-FE method to solve a model problem in an L-shaped domain and in a domain with a cut.

Section 7 presents concluding remarks and the Appendix contains auxiliary results used in the paper.

2 Tensor Decompositions. TT and QTT Formats

2.1 Tensor-train (TT) representation

By *tensors* we mean multidimensional arrays, vectors and matrices being notable examples. A cornerstone of the present paper is the *tensor-train* (*TT* for short) decomposition, a non-linear low-parametric representation of multidimensional arrays based on the separation of variables, developed by Oseledets and Tyrtyshnikov [56, 55].

Let us consider a d-dimensional $n_1 \times \ldots \times n_d$-vector $\boldsymbol{u}$. If two- and three-dimensional arrays $U_1, U_2, \ldots, U_d$ satisfy the equation

$$\boldsymbol{u}_{i_1,\ldots,i_d} = \sum_{\alpha_1=1}^{r_1} \cdots \sum_{\alpha_{d-1}=1}^{r_{d-1}} U_1(i_1, \alpha_1) \cdot U_2(\alpha_1, i_2, \alpha_2) \cdots U_{d-1}(\alpha_{d-2}, i_{d-1}, \alpha_{d-1}) \cdot U_d(\alpha_{d-1}, i_d) \quad (2.1)$$

for $i_k = 0, \ldots, n_k - 1$ with $k = 1, \ldots, d$, then $\boldsymbol{u}$ is said to be represented in the TT decomposition in terms of the *core tensors* $U_1, U_2, \ldots, U_d$. The summation indices $\alpha_1, \ldots, \alpha_{d-1}$ and limits $r_1, \ldots, r_{d-1}$ on the right-hand side of (2.1) are called, respectively, *rank indices* and *ranks* of the representation.

A tensor-train decomposition with d cores, exact or approximate, can be constructed via the low-rank representation of each of $d-1$ matrices; for example, using the SVD. In particular, for every $k = 1, \ldots, d-1$ the representation (2.1) implies a rank-r_k factorization of an *unfolding matrix* $\boldsymbol{U}^k$ with the entries

$$\boldsymbol{U}^k_{\overline{i_1,\ldots,i_k},\, \overline{i_{k+1},\ldots,i_d}} = \boldsymbol{u}_{i_1,\ldots,i_k,i_{k+1},\ldots,i_d}. \quad (2.2)$$

Here, the overscore denotes the unfolding of a multi-index into a long scalar index:

$$\overline{i_1, \ldots, i_k} = \sum_{\kappa=1}^{k} i_\kappa \prod_{k'=\kappa+1}^{k} n_{k'} \quad (2.3)$$

for the row index, and similarly for the column index, so that $\boldsymbol{U}^k$ is a usual matrix with two "long" indices.

Remark 2.1. *Conversely, if a vector* $\boldsymbol{u}$ *is such that the unfolding matrices* $\boldsymbol{U}^1$, $\boldsymbol{U}^2, \ldots, \boldsymbol{U}^{d-1}$ *given by* (2.2) *are of ranks* $r_1, \ldots, r_{d-1}$ *respectively, then the cores* $U_1, \ldots, U_d$ *satisfying* (2.1) *do exist; see [55, theorem 2.1].*

Further, the TT format admits an efficient approximation algorithm, which is quasi-optimal with respect to the ℓ^2 norm. Specifically, if the unfolding matrices can be approximated with ranks $r_1, \ldots, r_{d-1}$ and accuracies $\varepsilon_1, \ldots, \varepsilon_{d-1}$ in the Frobenius norm, then the vector itself can be approximated in the TT format with ranks $r_1, \ldots, r_{d-1}$ and accuracy ε in the ℓ^2, where $\varepsilon^2 \leq \varepsilon_1^2 + \cdots + \varepsilon_{d-1}^2$. This opens the possibility of efficient, ℓ^2-stable low-rank TT-structured approximation of vectors given in the full format or in the TT format with larger ranks. For details, we refer the reader to theorem 2.2 with corollaries and algorithms 1 and 2 in [55].

So far, there has been mostly experimental evidence that many applications admit approximations in the TT or related formats with moderate ranks, e.g. $\mathcal{O}(d)$ or $\mathcal{O}(\log^\theta n)$ with a small $\theta \geq 1$ and $n = \max\{n_1, \ldots, n_d\}$. This property is crucial for the applicability of tensor-structured methods; we refer the reader to the papers [37, 58, 13, 46, 23, 41, 52, 24, 38, 39, 45], to the literature survey [28] and more recent works [10, 12, 1].

2.2 Quantized-tensor-train (QTT) representation

2.2.1 Quantization of a dimension.

The *quantization* of a dimension of a given tensor consists in folding it into a few modes representing different *levels*, or *scales*, of the former.

For the present paper, we assume that $n_k = 2^{l_k}$ with $l_k \in \mathbb{N}$. Then the index i_k running from 0 to $n_k - 1$ can be equivalently represented in the binary form, i.e. by l indices $i_{k1}, \ldots, i_{kl}$ taking values in $\{0, 1\}$:

$$(i_{k1}, \ldots, i_{kl}) \leftrightarrow i_k = \overline{i_{k1}, \ldots, i_{kl}} = \sum_{q=1}^{l} 2^{l-q}\, i_{kq}. \tag{2.4}$$

Here, i_{k1} and i_{kl} are the major and minor indices representing the coarsest and finest scales along the kth dimension. The value of i_{k1} selects between the “left” and “right” halves of $\{0, 1, \ldots, 2^l - 1\}$, and the value of i_{kl}, between odd and even elements of the same index set. Here, the overscore denotes such vectorizations of multi-indices, in which the scale of the indices refines from left to right.

We refer to original dimensions and indices representing them as “physical”, in contrast to the “virtual” dimensions produced by quantization. Transformations of this type are quite common: matrices are *unfolded* from representations with linear indexing, arrays are `reshape`d in MATLAB, and the positional notation for numerals relies on a bijection similar to (2.4). By quantizing every dimension, one recasts a

d-dimensional $2^l \times \cdots \times 2^l$-vector indexed by $i_1 = \overline{i_{11}, \ldots, i_{1l}}, \ldots, i_d = \overline{i_{d1}, \ldots, i_{dl}}$ as a dl-dimensional $2 \times \cdots\cdots \times 2$-vector indexed by $i_{11}, \ldots, i_{1l}, \ldots\ldots, i_{d1}, \ldots, i_{dl}$.

2.2.2 Ordering the indices. QTT representation

The idea of applying low-rank tensor decompositions to separate the "virtual" dimensions produced by what we call now quantization appeared in [64] in the context of the *canonical polyadic* decomposition of tensors. It has since been widely used with the *tensor-train (TT) decomposition*, which separates indices in a given ordering. Since the "virtual" indices can be grouped and ordered in many ways, quantization offers additional freedom in selecting the type of low-rank structure under consideration by arranging the indices in a particular way.

By applying the TT format to quantized vectors with the natural ordering of the "virtual" indices,

$$\underbrace{i_{11}, \ldots, i_{1l}}_{\text{1st dimension}}, \ldots\ldots, \underbrace{i_{d1}, \ldots, i_{dl}}_{d\text{th dimension}}, \tag{2.5}$$

one arrives at what is usually meant by the *quantized tensor-train (QTT)* format [50, 40, 54]. Then a *QTT decomposition* of a vector involves dl QTT cores and $dl - 1$ QTT ranks.

In the present paper, we use the *transposed indexing*: we merge the indices within each level to obtain

$$\underbrace{\overline{i_{11}, \ldots, i_{d1}}}_{\text{1st level}}, \underbrace{\overline{i_{12}, \ldots, i_{d2}}}_{\text{2nd level}}, \ldots\ldots, \underbrace{\overline{i_{1l}, \ldots, i_{dl}}}_{l\text{th level}}. \tag{2.6}$$

When $d = 2$, this indexing coincides with the indexing used by the standard QTT format for matrices, whose row and column dimensions correspond to the two spatial dimensions in our case, see [54, (1.3) and section 4.2].

For vectors, the transposition used in (2.6) was first suggested by Oseledets[1], who applied the TT format to separate all the dl virtual indices without merging them. In (2.6), the "virtual" indices corresponding to the same level of quantization of different physical dimensions are merged and shall reside in the same core. The resulting indices, each ranging from 0 to $2^d - 1$, shall then represent the l scales resolved by the quantization according to (2.4) in d-dimensional vectors of size $2^l \times \cdots \times 2^l$. The transposed indexing (2.6) was applied for the solution of the chemical master equation in [38]. We also refer to [14] for an adaptive algorithm selecting the ordering of virtual dimensions most suitable for given data.

We note that the *hierarchical tensor representation* [34, 26], a comprehensive exposition of which is given in [33], itself and in combination with *tensorization* [27] are

[1] I. V. Oseledets. QTT decomposition of the characteristic function of a simplex. September 2010, private communication.

closely related counterparts of the TT and QTT formats respectively. The TT and HT representations have been known in other fields for decades: as *matrix product states (MPS)*, see [59] and references therein, and as the *hierarchical* or *multi-layer MCTDH* method, see [65, 48].

3 Singularities. Model Problem

In this section, following [6, 7, 5], we consider a curvilinear polygon Ω and specify weighted Sobolev spaces denoted with $\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ (with $m, \ell \in \mathbb{N}_0$ such that $m \geq \ell$) and classes $\mathfrak{B}^{\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ (with $\ell \in \mathbb{N}_0$) and $\mathfrak{C}^{2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ of functions analytic in Ω. Then we consider a model boundary-value problem for a linear second-order elliptic differential operator of the divergence form in Ω.

3.1 Curvilinear polygons

We say that $\Omega \subset \mathbb{R}^2$ is a *curvilinear polygon* if it is a bounded open domain with a boundary

$$\boldsymbol{\partial}\,\Omega = \bigcup_{i=1}^{n} \gamma^i$$

consisting of $n \in \mathbb{N}$ disjoint curves γ^i, $1 \leq i \leq n$, which are piecewise-smooth, simple and closed (each curve can be parametrized on a closed interval by a piecewise-smooth function injective inside the interval and taking equal values at the endpoints).

Assume that, for each i, the ith curve is composed of $m_i \in \mathbb{N}$ distinct vertices ς^{ij}, $1 \leq j \leq m_i$, and of m_i smooth *edges* γ^{ij}, $1 \leq j \leq m_i$, not including the vertices:

$$\gamma^i = \bigcup_{j=1}^{m_i} \mathbf{cl}\,\gamma^{ij}.$$

We assume that every open edge $\gamma^{ij} = \varphi^{ij}(\mathrm{J})$ is parametrized on $\mathrm{J} = (0,1)$ by φ^{ij} whose both components are smooth on $\mathbf{cl}\,\mathrm{J}$. We assume that the numbering of the nodes and edges and the parametrizations of the edges satisfy

$$\gamma^{ij} = \varphi^{ij}(\mathrm{J}), \quad \varphi^{ij}(0) = \varsigma^{i,j-1}, \quad \varphi^{ij}(1) = \varsigma^{ij}$$

for $1 \leq i \leq n$ and $1 \leq j \leq m_i$, where, for notational convenience, we use an m_i-periodic indexing with $\varsigma^{i0} = \varsigma^{im_i}$ and $\gamma^{i0} = \gamma^{im_i}$.

We introduce $\Sigma = \left\{\varsigma^{ij} :\ 1 \leq i \leq n,\ 1 \leq j \leq m_i\right\}$, the set of all vertices of Ω. For each vertex ς^{ij}, by ω^{ij} we denote the angle between $\gamma^{i,j-1}$ and γ^{ij} at ς^{ij} that is internal for Ω and assume $0 < \omega^{ij} \leq 2\pi$.

3.2 Weighted Sobolev spaces

In this section, following [6], we recapitulate weighted Sobolev spaces of functions that may admit singularities at $s \in \mathbb{N}$ distinct vertices $\Theta_1, \dots, \Theta_s \in \Sigma$ of Ω. We collect those vertices in a tuple $\boldsymbol{\Theta} = (\Theta_j)_{j=1}^s$, with which we associate a tuple of singularity orders $\boldsymbol{\beta} = (\beta_j)_{j=1}^s$ with $\beta_j \in [0, 1)$ for every j.

Now we consider weighted spaces $\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$, with singularity orders $\boldsymbol{\beta}$ and smoothness indices $m, \ell \in \mathbb{N}_0$ such that $m \geq \ell$. For every $k \in \mathbb{N}_0$, we define a weight function $\Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+k} \colon \Omega \to \mathbb{R}$ as follows:

$$\Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+k}(x) = \prod_{j=1}^{s} \|x - \Theta_j\|_2^{\beta_j + k} \quad \text{for all} \quad x \in \Omega, \tag{3.1}$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^2$. These weight functions induce weighted Sobolev spaces $\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ with $\ell, m \in \mathbb{N}_0$ such that $m \geq \ell$:

$$\mathbb{H}^{m,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega) = \left\{ u \colon \Omega \to \mathbb{R} : \; \Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+|\boldsymbol{\alpha}|}\, \partial^{\boldsymbol{\alpha}} u \in \mathbb{L}^2(\Omega) \quad \text{for} \quad 0 \leq |\boldsymbol{\alpha}| \leq m \right\}$$

for all $\ell \geq 0$ and

$$\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega) = \left\{ u \in \mathbb{H}^{\ell-1}(\Omega) : \; \Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+|\boldsymbol{\alpha}|-\ell}\, \partial^{\boldsymbol{\alpha}} u \in \mathbb{L}^2(\Omega) \quad \text{for} \quad 0 \leq |\boldsymbol{\alpha}| \leq m \right\}$$

for all $m \geq \ell \geq 1$, where the differentiation is understood in the weak sense. By setting

$$|u|^2_{\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} = \sum_{|\boldsymbol{\alpha}|=m} \|\Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+m-\ell}\, \partial^{\boldsymbol{\alpha}} u\|^2_{\mathbb{L}^2(\Omega)} \quad \text{for all} \quad u \in \mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega), \tag{3.2}$$

we introduce $|\cdot|_{\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)}$, a seminorm on $\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$. Also, by setting

$$\begin{aligned}
\|u\|^2_{\mathbb{H}^{m,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} &= \sum_{k=0}^{m} |u|^2_{\mathbb{H}^{k,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)}, \quad u \in \mathbb{H}^{m,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega), \quad \ell \geq 0, \\
\|u\|^2_{\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} &= \|u\|^2_{\mathbb{H}^{\ell-1}(\Omega)} + \sum_{k=\ell}^{m} |u|^2_{\mathbb{H}^{k,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)}, \quad u \in \mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega), \quad m \geq \ell \geq 1,
\end{aligned}$$

we define $\|\cdot\|^2_{\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)}$, a norm on $\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ with $\ell, m \in \mathbb{N}_0$ such that $m \geq \ell$.

Let us note the following result.

Proposition 3.1. *There holds a continuous embedding* $\mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega) \subset \mathbf{C}(\mathbf{cl}\,\Omega)$.

Proof. The proof given for straight polygons in [9, section 2] is valid also for curvilinear polygons. ☐

In the particular case of only one ($s = 1$) singularity at the origin $\boldsymbol{\Theta} = \Theta = 0$ of order $\boldsymbol{\beta} = \beta$, we write $\Phi_{\beta+k}$ and $\mathbb{H}^{m,\ell}_{\beta}(\Omega)$ instead of $\Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+k}$ and $\mathbb{H}^{m,\ell}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$.

3.3 Analytic regularity

We recapitulate from [6, 7] analyticity classes $\mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and $\mathfrak{B}^\ell_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ with $\ell \in \mathbb{N}_0$, based on the weighted Sobolev spaces introduced in section 3.2.

Definition 3.2. *We say that $u \in \mathfrak{B}^\ell_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ with $\ell \in \mathbb{N}_0$ if $u \in \mathbb{H}^{m,\ell}_{\beta}(\Omega)$ for all integral $m \geq \ell$ and if there exist positive constants C_u and δ_u such that*

$$|u|_{\mathbb{H}^{m,\ell}_{\beta}(\Omega)} \leq C_u \delta_u^{m-\ell}(m-\ell)! \quad \textit{for all} \quad m \geq \ell.$$

The functions that belong to $\mathfrak{B}^\ell_\beta(\Omega)$ are analytic in an open domain containing $\mathbf{cl}\,\Omega\setminus\{0\}$ with possibly an algebraic singularity at the origin. The embedding $\mathfrak{B}^\ell_\beta(\Omega) \subset \mathbb{H}^{\ell-1}(\Omega)$ follows from the definition for all $\beta \in [0,1)$ and $\ell \in \mathbb{N}$. Furthermore, the space $\mathfrak{B}^\ell_\beta(\Omega)$ can be related to another space of analytic functions, with pointwise bounds on the derivatives. First, we define it following [7, 5].

Definition 3.3. *We say that $u \in \mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ if $u \in \mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ if there exist positive constants C_u and δ_u such that*

$$\Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+|\boldsymbol{\alpha}|-1}(x)\,|\partial^{\boldsymbol{\alpha}} u(x)| \leq C_u\,\delta_u^{|\boldsymbol{\alpha}|}\,|\boldsymbol{\alpha}|! \quad \textit{for all} \quad x \in \Omega \quad \textit{and} \quad \boldsymbol{\alpha} \in \mathbb{N}_0^2 \setminus \{0\}.$$

We shall use an equivalent definition, with $\boldsymbol{\alpha}! = \alpha_1!\alpha_2!$ instead of $|\boldsymbol{\alpha}|! = (\alpha_1+\alpha_2)!$ in the bound.

Definition 3.4. *We say that $u \in \mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ if $u \in \mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and if there exist positive constants C_u and δ_u such that*

$$\Phi_{\boldsymbol{\Theta},\boldsymbol{\beta}+|\boldsymbol{\alpha}|-1}(x)\,|\partial^{\boldsymbol{\alpha}} u(x)| \leq C_u\,\delta_u^{|\boldsymbol{\alpha}|}\,\boldsymbol{\alpha}! \quad \textit{for all} \quad x \in \Omega \quad \textit{and} \quad \boldsymbol{\alpha} \in \mathbb{N}_0^2 \setminus \{0\}.$$

Definitions 3.3 and 3.4 are equivalent. Obviously, $u \in \mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ in the sense of definition 3.4 with constants C_u and δ_u implies $u \in \mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ in the sense of definition 3.3 with the same constants. Conversely, definition 3.3 with constants C_u and δ_u implies definition 3.4 with the constants C_u and $2\delta_u$. We shall use definition 3.4 throughout the present paper.

The analyticity classes $\mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and $\mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ are related as follows.

Proposition 3.5 (theorems 2.2 and 2.3 in [5])**.** *For any $\varepsilon > 0$, there hold the inclusions $\mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega) \subset \mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega) \subset \mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}+\varepsilon}(\Omega)$.*

In the case of only one ($s = 1$) singularity at the origin $\boldsymbol{\Theta} = \Theta = 0$ of order $\boldsymbol{\beta} = \beta$, we write $\mathfrak{B}^2_\beta(\Omega)$ and $\mathfrak{C}^2_\beta(\Omega)$ instead of $\mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and $\mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$.

3.4 Model problem

To motivate the ensuing analysis, we consider a model second-order elliptic boundary-value problem in Ω, a curvilinear polygon.

Consider a second-order differential operator

$$\mathcal{A} = -\nabla^\top A\,\nabla + b^\top\nabla + c, \tag{3.3}$$

where the coefficients $x \mapsto A(x) \in \mathbb{R}^{2\times 2}$, $x \mapsto b(x) \in \mathbb{R}^{2\times 2}$ and $x \mapsto c(x) \in \mathbb{R}$ are analytic on $\mathbf{cl}\,\Omega$. For the diffusion term, we assume symmetricity, $A^\top = A$, and strong ellipticity with a constant $A_0 > 0$: $y^\top A\,y \geq A_0\,y^\top y$ in $\mathbf{cl}\,\Omega$ for all $y \in \mathbb{R}^2$.

Let us suppose that $\mathcal{D} \subset \{(i,j) :\ 1 \leq i \leq n,\ 1 \leq j \leq m_i\}$ is nonempty and denote

$$\gamma_{\mathrm{D}} = \bigcup_{(i,j)\in\mathcal{D}} \mathbf{cl}\,\gamma^{ij} \quad \text{and} \quad \gamma_{\mathrm{N}} = \boldsymbol{\partial}\,\Omega \setminus \gamma_{\mathrm{D}}. \tag{3.4}$$

The sets γ_{D} and γ_{N} are respectively closed and open within $\boldsymbol{\partial}\,\Omega$.

Let us assume that functions $f \in \mathbf{C}(\mathbf{cl}\,\Omega \setminus \Sigma)$ and $g_0 \in \mathbf{C}(\gamma_{\mathrm{D}})$, $g_1 \in \mathbf{C}^1(\gamma_{\mathrm{N}})$ are such that there exist $u_0 \in \mathbf{C}^1(\mathbf{cl}\,\Omega \setminus \Sigma)$, $u_1 \in \mathbf{C}^1(\mathbf{cl}\,\Omega \setminus \Sigma)$ satisfying $u_0|_{\gamma_{\mathrm{D}}} = g_0$, $u_1|_{\gamma_{\mathrm{D}}} = 0$, $u_1|_{\gamma_{\mathrm{N}}} = g_1$. Then we consider the following boundary-value problem.

$$\begin{aligned} &\text{Find } u \in \mathbf{C}^2(\Omega) \cap \mathbf{C}(\mathbf{cl}\,\Omega \setminus \Sigma) \text{ such that} \\ &\mathcal{A}u = f \quad \text{in} \quad \Omega, \quad u|_{\gamma_{\mathrm{D}}} = g_0, \quad (\mathbf{n}^\top A\,\nabla u)|_{\gamma_{\mathrm{N}}} = g_1. \end{aligned} \tag{3.5}$$

Here, $\mathbf{n}$ is the unitary normal to γ_{N} exterior with respect to Ω, which is defined except at the vertices.

The finite-element method is a Galerkin projection method based on the following weak formulation of (3.5). First, we define a bilinear form $\mathfrak{a} : \mathbb{H}^1(\Omega) \times \mathbb{H}^1(\Omega) \to \mathbb{R}$ and a linear form $\mathfrak{f} : \mathbb{H}^1(\Omega) \to \mathbb{R}$:

$$\begin{aligned} \mathfrak{a}(v,w) &= \int_\Omega \Big\{ (\nabla v)^\top A\,\nabla w + (b^\top \nabla v) w + c v w \Big\} \quad \text{for all} \quad v, w \in \mathbb{H}^1(\Omega), \\ \mathfrak{f}(w) &= \langle f, w\rangle_{V^*\times V} + \langle g_1, w\rangle_{W^*\times W} - \mathfrak{a}(u_0, w) \quad \text{for all} \quad w \in \mathbb{H}^1(\Omega), \end{aligned} \tag{3.6}$$

where $W = \mathbb{H}^{1/2}_{00}(\gamma_{\mathrm{N}})$ denotes the Lions–Magenes space [47, Chapter 1, section 11.5] of the elements of $\mathbb{H}^{1/2}(\gamma_{\mathrm{N}})$ whose zero extensions belong to $\mathbb{H}^{1/2}(\boldsymbol{\partial}\,\Omega)$. Then we consider the elliptic model problem (3.5) in the following weak formulation on the variational space $V = \{u \in \mathbb{H}^1(\Omega) :\ u|_{\gamma_{\mathrm{D}}} = 0\}$.

$$\begin{aligned} &\text{Find } u = u_0 + v \text{ with } v \in V \text{ such that} \\ &\mathfrak{a}(v,w) = \mathfrak{f}(w) \quad \text{for all} \quad w \in V. \end{aligned} \tag{3.7}$$

The bilinear form $\mathfrak{a}$ is continuous due to the boundedness of the coefficients: there exists a constant $\mathfrak{a}_1 > 0$ such that $|\mathfrak{a}(v,w)| \leq \mathfrak{a}_1 \|v\|_{\mathbb{H}^1(\Omega)} \|w\|_{\mathbb{H}^1(\Omega)}$ for all $v, w \in \mathbb{H}^1(\Omega)$.

Let us assume, for the weak formulation (3.7), that $f \in \mathbb{H}^{-1}(\Omega)$, that $g_0 \in \mathbb{H}^{1/2}(\gamma_{\mathrm{D}})$, i.e. it admits an extension to $u_0 \in \mathbb{H}^1(\Omega)$, and that $g_1 \in W^*$. Then the linear form $\mathfrak{f}$ is continuous on $\mathbb{H}^1(\Omega)$:

$$|\mathfrak{f}(w)| \leq \|f\|_{\mathbb{H}^{-1}(\Omega)} \|w\|_{\mathbb{H}^1(\Omega)} + \|g_1\|_{W^*} \|w\|_W + \mathfrak{a}_1 \|u_0\|_{\mathbb{H}^1(\Omega)} \|w\|_{\mathbb{H}^1(\Omega)},$$

then, by the continuous embedding $W = \mathbb{H}^{1/2}_{00}(\gamma_{\mathrm{N}}) \subset \mathbb{H}^{1/2}(\partial\Omega)$ [47, Chapter 1, section 11.5] and by the trace theorem, the assumptions on the data result in the existence of a positive constant $\mathfrak{f}_1$ such that $|\mathfrak{f}(w)| \leq \mathfrak{f}_1 \|w\|_{\mathbb{H}^1(\Omega)}$ for all $w \in \mathbb{H}^1(\Omega)$.

In addition, let us assume the *inf-sup* conditions with a constant $\mathfrak{a}_0 > 0$:

$$\inf_{\substack{v\in V\\ v\neq 0}} \sup_{\substack{w\in V\\ w\neq 0}} \frac{|\mathfrak{a}(v,w)|}{\|v\|_V \|w\|_V} \geq \mathfrak{a}_0, \quad \sup_{\substack{v\in V\\ v\neq 0}} \frac{|\mathfrak{a}(v,w)|}{\|v\|_V \|w\|_V} > 0 \quad \text{for every} \quad w \in V : w \neq 0. \tag{3.8}$$

By the Babuška–Lax–Milgram theorem, the problem (3.7) has a unique solution, see, e.g. [3, Theorem 2.1].

Let us assume that there exists a family of conforming partitions $\mathcal{T}^h$ of Ω into unions of quadrilaterals of diameter bounded from above by $h > 0$. Let $U^h = \mathrm{S}^1(\Omega, \mathcal{T}^h) \subset \mathbb{H}^1(\Omega)$ denote the corresponding spaces of continuous, piecewise-bilinear functions. Consider its subspace $V^h = U^h \cap V$. Then corresponding FE discretizations of (3.7) read as follows.

$$\begin{aligned} &\text{Find } u^h = u_0 + v^h \text{ with } v^h \in V^h \text{ such that} \\ &\mathfrak{a}(v^h, w^h) = \mathfrak{f}(w^h) \quad \text{for all} \quad w^h \in V^h. \end{aligned} \tag{3.9}$$

Assume that the bilinear form $\mathfrak{a}$ satisfies the discrete inf-sup condition with a constant $\tilde{\mathfrak{a}}_0 > 0$ uniformly in $h > 0$:

$$\inf_{\substack{v^h\in V^h\\ v^h\neq 0}} \sup_{\substack{w^h\in V\\ w^h\neq 0}} \frac{|\mathfrak{a}(v^h,w^h)|}{\|v^h\|_V \|w^h\|_V} \geq \tilde{\mathfrak{a}}_0, \quad \sup_{\substack{v^h\in V^h\\ v^h\neq 0}} \frac{|\mathfrak{a}(v^h,w^h)|}{\|v^h\|_V \|w^h\|_V} > 0 \quad \text{for every} \quad w^h \in V^h : w^h \neq 0. \tag{3.10}$$

Then, by [3, Theorem 2.2], for the discrete solutions there holds a bound

$$\|u - u^h\|_{\mathbb{H}^1(\Omega)} = \|v - v^h\|_{\mathbb{H}^1(\Omega)} \leq C_1 \inf_{w^h \in V^h} \|v - w^h\|_{\mathbb{H}^1(\Omega)} \tag{3.11}$$

with $C_1 = 1 + \mathfrak{a}_1/\tilde{\mathfrak{a}}_0$ independent of h.

For $u_0, u \in \mathbb{H}^2(\Omega)$, by classical approximation results, there exists a constant $C_2 > 0$ independent of the boundary data and of h such that

$$\|u - \hat{u}^h\|_{\mathbb{H}^1(\Omega)} \le C_2\, h\, \|u\|_{\mathbb{H}^2(\Omega)}, \qquad \|u_0 - \hat{u}_0^h\|_{\mathbb{H}^1(\Omega)} \le C_2\, h\, \|u_0\|_{\mathbb{H}^2(\Omega)}, \tag{3.12}$$

where $\hat{u}^h \in U^h$ and $\hat{u}_0^h \in U^h$ interpolate u and u_0 at the nodes of $\mathcal{T}^h$, see, e.g. [16, theorem 4.6.14] for tensor-product interpolation and [16, section 4.7] for isoparametric interpolation in domains of complex geometry. Considering $w^h = \hat{u}^h - \hat{u}_0^h \in V^h$, we obtain from (3.11)–(3.12) that

$$\|u - u^h\|_{\mathbb{H}^1(\Omega)} \le C_1\, C_2\, h \left\{ \|u\|_{\mathbb{H}^2(\Omega)} + \|u_0\|_{\mathbb{H}^2(\Omega)} \right\}. \tag{3.13}$$

In the present paper, we are interested in solutions $u \in \mathbb{H}^1(\Omega)$ of (3.7) which are, however, not in $\mathbb{H}^m(\Omega)$ for any $m \ge 2$, but do belong to the countably normed class $\mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ or $\mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ of analytic functions with some tuples $\boldsymbol{\Theta}$ and $\boldsymbol{\beta}$ of vertices and singularity orders, introduced in definitions 3.2 to 3.4. Then we have, in particular, $u \in \mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$. Let us assume also that the boundary-lifting term satisfies $u_0 \in \mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$. Then we have $u, u_0 \in \mathbf{C}(\mathbf{cl}\,\Omega)$ by proposition 3.1, and the nodal interpolation is still well defined. Instead of (3.12), similar bounds

$$\|u - \hat{u}^h\|_{\mathbb{H}^1(\Omega)} \le \tilde{C}_2\, h^{1-\beta_*}\, \|u\|_{\mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)}, \quad \|u_0 - \hat{u}_0^h\|_{\mathbb{H}^1(\Omega)} \le \tilde{C}_2\, h^{1-\beta_*}\, \|u_0\|_{\mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)}, \tag{3.14}$$

where $\beta_* = \max\{\beta_1, \dots, \beta_s\}$ and $\tilde{C}_2 > 0$ is independent of the boundary data and of h, combine with (3.11) to yield the quasi-optimality of the first-order Lagrangian FEM (3.9) for the problem (3.7).

Remark 3.6. *Let us consider the case when Ω is a polygonal domain, i.e. when the edges γ^{ij}, $1 \le i \le n$ and $1 \le j \le m_i$, are linear segments. Then, in the weighted Sobolev spaces defined in section 3.2, the weak formulation (3.7) of the model problem (3.5) satisfies the following regularity shift: for every $m \in \mathbb{N}$, there exists a constant $C_m > 0$ such that for every $f \in \mathbb{H}^{m-2,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and for all $u_1 \in \mathbb{H}^{m-1,1}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$, $u_0 \in \mathbb{H}^{m,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ the weak solution $u \in \mathbb{H}^1(\Omega)$ satisfies the following a-priori estimate:*

$$\begin{aligned} \|u\|_{\mathbb{H}^1(\Omega)} &\le C_1 \Big\{ \|f\|_{\mathbb{H}^{0,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} + \|u_1\|_{\mathbb{H}^{1,1}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} + \|u_0\|_{\mathbb{H}^{2,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} \Big\}, \\ \|u\|_{\mathbb{H}^{m,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} &\le C_m \Big\{ \|f\|_{\mathbb{H}^{m-2,0}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} + \|u_1\|_{\mathbb{H}^{m-1,1}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} + \|u_0\|_{\mathbb{H}^{m,2}_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)} \Big\}, \quad m \ge 2, \end{aligned} \tag{3.15}$$

see [6, lemma 3.1, remark 3 of section 2]. Here, the singular support and singularity orders indicated by $\boldsymbol{\Theta}$ and $\boldsymbol{\beta}$ are determined by the geometry of Ω, by the diffusion *tensor A and by the type of the boundary conditions imposed on every pair of adjacent edges.*

Moreover, if $f \in \mathfrak{B}^0_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and $u_0 \in \mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$, $u_1 \in \mathfrak{B}^1_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$, then $u \in \mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ [6, theorem 3.1]. Such regularity-shift results are available also for more evolved problems, see, e.g. [32, theorem 5.2] for linear elasticity models.

Remark 3.7. *For a general curvilinear polygon Ω described in section 3.1, there is a similar result: if $f \in \mathfrak{B}^0_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ and $u_0 \in \mathfrak{B}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$, $u_1 \in \mathfrak{B}^1_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$, then we have $u \in \mathfrak{C}^2_{\boldsymbol{\Theta},\boldsymbol{\beta}}(\Omega)$ [5, theorems 3.4–3.5].*

Again, the singular support and singularity orders indicated by $\boldsymbol{\Theta}$ and $\boldsymbol{\beta}$ are determined by the geometry of Ω, by the diffusion tensor A and by the type of the boundary conditions imposed on every pair of adjacent edges.

The regularity-shift results cited in remarks 3.6 and 3.7 prompts a question of how the infinite weighted regularity of the corresponding analyticity classes can be exploited for the efficient approximation of the functions from those classes. Building upon well-known results which address the question with hp approximations, the present paper provides an answer for the QTT-FE approximation.

Finally, when Ω has a cut, i.e. some edges γ^{ij} and $\gamma^{i'j'}$ share at least one (interior) point, the formulation of the problems (3.5), (3.7) becomes more technical. We refer to [29, section 1.7], where the Sobolev spaces $\mathbb{H}^m(\Omega)$, $m \in \mathbb{N}_0$, are defined over Ω via those for subdomains of Ω without cuts.

4 QTT-FEM in a Reference Domain in d Dimensions

In the present paper we are interested in the space dimension $d = 2$, but this section we develop, for future reference, in the case of a general space dimension $d \geq 2$. By the *reference domain* we mean $\mathrm{Q} = \mathrm{J}^d$, where $\mathrm{J} = (0,1)$. We consider the h-FE and QTT-FE approximation of functions from $\mathbf{C}(\mathbf{cl}\,\mathrm{Q})$, i.e. continuous in the closure of the reference domain Q.

We denote the $2d$ sides of dimension $d-1$ (faces when $d = 3$ and edges when $d = 2$) of Q as follows:

$$\Gamma_{k\jmath} = \mathrm{J}^{k-1} \times \{\jmath - 1\} \times \mathrm{J}^{d-k} \tag{4.1}$$

for every $k = 1, \dots, d$ and $\jmath = 1, 2$. In particular, Γ_{11}, Γ_{12}, Γ_{21} and Γ_{22} denote respectively the left, right, bottom and top edges of Q when $d = 2$.

We label each of the $2d$ sides of Q as *active* and *inactive*: we set $\mu_{k\jmath} = 1$ or $\mu_{k\jmath} = 0$ if the side $\Gamma_{k\jmath}$ is respectively active or inactive. Below, we shall use the binary matrix $\mu \in \{0,1\}^{d\times 2}$ resulting from this convention to encode the assignment of degrees of freedom and to construct suitable finite-element spaces.

4.1 h- and QTT-FE approximation of coupled functions in the reference domain

4.1.1 Uniform partitions of $\mathrm{J} = (0,1)$ and $\mathrm{Q} = (0,1)^d$

Next, we explicitly construct uniform tensor-product partitions of the reference domain and the corresponding finite-element space of functions that are continuous piecewise-d-linear on every element. We present the construction explicitly in order to ensure, for all $\mu \in \{0,1\}^{d\times 2}$, consistency with the corresponding auxiliary hp finite-element spaces introduced in section 5 below. Also, the proof of lemma 5.17 on the QTT structure of hp functions given below references the nodes explicitly.

Let us assume that $k \in \{1,2\}$ and $l \in \mathbb{N}$. First, we set

$$n_k^l = 2^l - \mu_{k1} - \mu_{k2} \quad \text{and} \quad h_k^l = (n_k^l + 1)^{-1}. \tag{4.2}$$

Then we introduce a uniform partition $\mathcal{T}_k^l$ of J with the nodes

$$t_{k,i_k}^l = (i_k + 1 - \mu_{k1})\, h_k^l, \quad i_k \in \mathcal{J}_k^l = \{\mu_{k1} - 1, \ldots, 2^l - \mu_{k2}\}, \tag{4.3a}$$

and the elements

$$\mathrm{I}_{k,i_k}^l = \left(t_{k,i_k}^l, t_{k,i_k+1}^l\right), \quad i_k \in \mathcal{E}_k^l = \{\mu_{k1} - 1, \ldots, 2^l - \mu_{k2} - 1\}. \tag{4.3b}$$

The number of interior nodes is n_k^l and the grid size is h_k^l.

For every $l \in \mathbb{N}$, consider $\mathcal{T}^l = \mathcal{T}_1^l \times \cdots \times \mathcal{T}_d^l$, a uniform tensor-product partition of Q, which consists of the Cartesian-product elements

$$\mathrm{Q}_i^l = \mathrm{I}_{1,i_1}^l \times \cdots \times \mathrm{I}_{d,i_d}^l \quad \text{with} \quad i = (i_1, \ldots, i_d) \in \mathcal{E}^l = \mathcal{E}_1^l \times \cdots \times \mathcal{E}_d^l, \tag{4.4}$$

where each interval I_{k,i_k}^l is given by (4.3b). The nodes of $\mathcal{T}^l$ are

$$t_i^l = (t_{1,i_1}^l, \ldots, t_{d,i_d}^l) \quad \text{with} \quad i = (i_1, \ldots, i_d) \in \mathcal{J}^l = \mathcal{J}_1^l \times \cdots \times \mathcal{J}_d^l.$$

4.1.2 h-FE space $\mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l)$. Active and inactive nodes

For all $k = 1, \ldots, d$ and $i_k \in \mathcal{J}_k^l$, we define $\phi_{k,i_k}^l \in \mathbf{C}(\mathbf{cl}\,\mathrm{J})$ by requiring linearity on each interval I_{k,i'_k}^l with $i'_k \in \mathcal{E}^l$ and by the interpolation condition $\phi_{k,i_k}^l(t_{k,i'_k}^l) = \delta_{i_k i'_k}$ for all $i'_k \in \mathcal{J}_k^l$. For all $i = (i_1, \ldots, i_d) \in \mathcal{J}^l$, we introduce $\phi_i^l = \phi_{1,i_1}^l \otimes \cdots \otimes \phi_{d,i_d}^l \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$. Then

$$\mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l) = \operatorname{span}\{\phi_i^l : \ i \in \mathcal{J}^l\} \subset \mathbf{C}(\mathbf{cl}\,\mathrm{Q}) \tag{4.5}$$

is the Lagrangian finite-element space of continuous, piecewise-d-linear functions induced by the Cartesian-product partition $\mathcal{T}^l$.

Let $\dot{\mathcal{J}}^l_k = \{0, \ldots, 2^l - 1\}$ for $k = 1, \ldots, d$ and

$$\dot{\mathcal{J}}^l = \dot{\mathcal{J}}^l_1 \times \cdots \times \dot{\mathcal{J}}^l_d. \tag{4.6}$$

The set $\mathcal{J}^l \setminus \dot{\mathcal{J}}^l$ indexes the nodes of $\mathcal{T}^l$ belonging to the closures of the inactive sides and $\dot{\mathcal{J}}^l$ indexes all the other nodes of $\mathcal{T}^l$. We refer to those nodes as *inactive* and *active* respectively. The purpose of the index shifts used in (4.2)–(4.3) was to ensure that, in each dimension, there are exactly 2^l active nodes and those are numbered starting from zero.

We define the corresponding *analysis operator* $\dot{\mathscr{A}}^l : \mathbf{C}(\mathbf{cl}\,\mathrm{Q}) \to \mathbb{R}^{2^l \times \cdots \times 2^l}$, which evaluates the components of a function at the active nodes:

$$(\dot{\mathscr{A}}^l u)_i = u(t^l_i) \quad \text{for all} \quad i \in \dot{\mathcal{J}}^l. \tag{4.7}$$

4.1.3 Boundary conditions

The approximations to be considered shall satisfy, in the sense specified below, Dirichlet boundary conditions on the closures of sides of Q. We assume that those sides are indexed by a set $\mathcal{B} \subset \{1, \ldots, d\} \times \{1, 2\}$.

Assumption 4.1 (sides with Dirichlet boundary conditions are inactive)**.** *The matrix $\mu \in \{0,1\}^{d\times 2}$ and set $\mathcal{B} \subset \{1, \ldots, d\} \times \{1, 2\}$ satisfy the following: for all $k = 1, \ldots, d$, and $\jmath = 1, 2$ such that $(k, \jmath) \in \mathcal{B}$, we have $\mu_{k\jmath} = 0$.*

Then we consider

$$\Gamma_0 = \bigcup_{(k,\jmath)\in\mathcal{B}} \mathbf{cl}\,\Gamma_{k\jmath} \subset \boldsymbol{\partial}\,\mathrm{Q}, \tag{4.8}$$

which may be empty. By assumption 4.1, the sides contained in Γ_0 are inactive. For all $l \in \mathbb{N}$, let us set

$$\mathcal{J}^l_0 = \{i \in \mathcal{J}^l : \ t^l_i \in \Gamma_0\} \tag{4.9}$$

and

$$\mathrm{S}^1_0(\mathrm{Q}, \mathcal{T}^l) = \left\{ u^l \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l) : \ u^l|_{\Gamma_0} = 0 \right\} = \mathrm{span}\{\phi^l_i : \ i \in \mathcal{J}^l \setminus \mathcal{J}^l_0\}, \tag{4.10}$$

We define a boundary-data interpolation operator $\mathfrak{I}^l_0 : \mathbf{C}(\mathbf{cl}\,\mathrm{Q}) \to \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l)$ by setting

$$\mathfrak{I}^l_0 u = \sum_{i\in\mathcal{J}^l_0} u(t^l_i)\, \phi^l_i \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l) \tag{4.11}$$

for all $u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$. By assumption 4.1, (4.9) and (4.6), we have $\mathcal{J}^l_0 \cap \dot{\mathcal{J}}^l = \varnothing$. For the analysis operator $\dot{\mathscr{A}}^l$ given by (4.7), we thus infer

$$\dot{\mathscr{A}}^l\, \mathfrak{I}^l_0\, u = 0 \quad \text{for all} \quad u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q}). \tag{4.12}$$

4.1.4 Admissible approximations

For every $u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$, we consider *admissible approximations*, i.e. those preserving the values at the nodes of $\mathcal{T}^l$ lying on Γ_0, which constitute the set

$$\mathscr{F}_u^l = \mathrm{S}_0^1(\mathrm{Q}, \mathcal{T}^l) + \mathfrak{I}_0^l u. \tag{4.13a}$$

Here, the two terms are given by (4.10) and (4.11) respectively. For every admissible approximation $u^l \in \mathscr{F}_u^l$, we shall consider the corresponding coefficient vector

$$\boldsymbol{u}^l = \dot{\mathscr{A}}^l u^l \in \mathbb{R}^{2^l \times \cdots \times 2^l}, \tag{4.13b}$$

which parametrizes—possibly partly—the approximation u^l and consists of

$$\aleph_l = \left|\dot{\mathcal{J}}^l\right| = 2^{2l} \tag{4.13c}$$

components.

4.2 Nodal approximation

One possible choice of an admissible approximation $u^l \in \mathscr{F}_u^l$ is the nodal interpolant $\mathfrak{I}^l u$ of $u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$, given by

$$\mathfrak{I}^l u = \sum_{i \in \mathcal{J}^l} u(t_i^l)\, \phi_i^l. \tag{4.14a}$$

For that approximation, we have

$$u^l(t_i^l) = \boldsymbol{u}_i^l = u(t_i^l) \quad \text{for all} \quad i \in \dot{\mathcal{J}}^l. \tag{4.14b}$$

Generally, the coefficient vector $\boldsymbol{u}^l = \dot{\mathscr{A}}^l u^l$ may not admit an *exact* low-rank tensor representation, and low-rank *approximations* need to be considered.

To prove the existence of such approximations, we consider an *auxiliary* approximation operator $\boldsymbol{\Pi}^l : \mathcal{D} \to \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$ with $\mathcal{D} \subset \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$. Then, instead of $u^l = \mathfrak{I}^l u$, we consider $u^l = \mathfrak{A}^l u$ with $\mathfrak{A}^l : \mathcal{D} \to \mathrm{S}_0^1(\mathrm{Q}, \mathcal{T}^l)$ given by

$$\mathfrak{A}^l = \mathfrak{I}^l\, \boldsymbol{\Pi}^l + \mathfrak{I}_0^l\,(\mathbf{id} - \boldsymbol{\Pi}^l). \tag{4.15a}$$

In the right-hand side, the first term corresponds to the reinterpolation of a projection obtained by $\boldsymbol{\Pi}^l$, and the second, to an appropriate boundary lifting. The corresponding approximation error reads

$$u - u^l = (\mathbf{id} - \boldsymbol{\Pi}^l)\, u + (\mathbf{id} - \mathfrak{I}^l)\, \boldsymbol{\Pi}^l u - \mathfrak{I}_0^l\,(\mathbf{id} - \boldsymbol{\Pi}^l)\, u. \tag{4.15b}$$

The corresponding coefficient vector $\boldsymbol{u}^l = \dot{\mathscr{A}}^l\, \boldsymbol{\Pi}^l u$ is determined entirely by the values of the auxiliary approximation $\boldsymbol{\Pi}^l u$ at the active nodes of $\mathcal{T}^l$. Thus, if $\boldsymbol{\Pi}^l u$ produces approximations which are both convergent and of *exact* low-rank tensor structure, the convergence of low-rank tensor-structured approximations follows immediately.

We emphasize that, along these lines, that *is* the auxiliary approximation what realizes low-rank tensor approximation, even though its formulation may seem completely unrelated to tensor decompositions. Notable examples are trigonometric and polynomial approximations, global or piecewise. In section 5 below, we present and apply the piecewise-polynomial interpolation, known as *hp approximation*.

Note that the auxiliary approximation operator $\boldsymbol{\Pi}^l$ is not required for tensor-structured approximation. In the present paper, we use such an operator to prove the existence of low-rank approximations and, in the numerical experiments presented in section 6, to obtain reference low-rank solutions of certified accuracy.

4.3 QTT-FE representation of admissible approximations

In section 4.1.3, we defined an approximation space $\mathrm{S}_0^1(\mathrm{Q}, \mathcal{T}^l)$ of functions that are piecewise d-linear on the elements of a tensor-product partition of $\mathrm{Q} = (0,1)^d$ and vanish on the sides with Dirichlet boundary conditions. For every function $u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$, we consider approximations u^l from the set $\mathscr{F}_u^l$ given by (4.13a). By classical approximation results, those may achieve only algebraic convergence (of rate at most $1/d$ in the $\mathbb{H}^1$-norm) with respect to the number $\aleph_l$ (4.13c) of parameters.

In order to reduce the number of parameters of the discretization, we recast it in the QTT format presented in section 2.2. We require that the coefficient vector $\boldsymbol{u}^l = \dot{\mathscr{A}}^l u^l$ be represented in the QTT decomposition. Throughout the present paper, we refer to that combination as *QTT-FEM*, although QTT-structured finite-element methods can be envisaged more general.

The representation of the quantized coefficient vectors $\boldsymbol{u}^l$ in the TT format with the ordering and grouping of the "virtual" indices given by (2.6) corresponds to QTT decompositions given by

$$\boldsymbol{u}^l_{\overline{i_{11},\ldots,i_{1l}},\ldots\ldots,\overline{i_{d1},\ldots,i_{dl}}} = \sum_{\alpha_1=1}^{r_1}\cdots\sum_{\alpha_{l-1}=1}^{r_{l-1}} U_1(\overline{i_{11},\ldots,i_{d1}},\alpha_1)\cdot U_2(\alpha_1,\overline{i_{12},\ldots,i_{d2}},\alpha_2) \cdots U_{l-1}(\alpha_{l-2},\overline{i_{1,l-1},\ldots,i_{d,l-1}},\alpha_{l-1})\cdot U_l(\alpha_{l-1},\overline{i_{1l},\ldots,i_{dl}}) \quad (4.16)$$

for all values of the indices: for $i_{kq} = 0,1$ with $k = 1,\ldots,d$ and $q = 1,\ldots,l$. The ranks of the decomposition (4.16) are denoted by $r_1,\ldots,r_{l-1}$. The complexity of such a representation can be characterized by the *maximum rank* R_l and the *number of* *parameters* N_l,

$$R_l = \max_{1 \leq q < l} r_l \quad \text{and} \quad N_l = 2^d r_1 + \sum_{q=2}^{l-1} 2^d r_{q-1} r_q + 2^d r_{l-1} = \mathcal{O}(l R_l^2). \tag{4.17}$$

Analogously to the TT and standard QTT representations, the ranks of (4.16) are related to unfolding matrices. For $1 \leq q < l$, let us consider the unfolding matrix $\boldsymbol{U}^q$ whose entries are given by

$$\boldsymbol{U}^q_{\overline{\mathrm{i}, i_{11}, \ldots, i_{1q}, \ldots\ldots, i_{d1}, \ldots, i_{dq}}, \overline{i_{1,q+1}, \ldots, i_{1l}, \ldots\ldots, i_{d,q+1}, \ldots, i_{dl}}} = \boldsymbol{u}_{i, \overline{i_{11}, \ldots, i_{1l}}, \ldots, \overline{i_{d1}, \ldots, i_{dl}}} \tag{4.18}$$

for all values of the indices, cf. (2.2). Then (4.16) implies $\operatorname{rank} \boldsymbol{U}^q \leq r_q$ for $q = 1, \ldots, l-1$. A converse statement, similar to remark 2.1, holds as well.

5 Auxiliary *hp* Approximation of Singular Functions in the Reference Square

In this section, we consider the case of $d = 2$ dimensions. We construct and analyze auxiliary approximations in *hp* spaces consistent with the first-order *h*-FE spaces introduced in section 4.

We start with discussing piecewise-bilinear nodal approximation of an element of $\mathbb{H}^{2,2}_\beta(\mathrm{Q})$, in $\mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l)$, of a function proper and of its auxiliary approximation. Then we present, as auxiliary, *hp*-FE spaces $\tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ and $\mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$ analogous to those developed and analyzed in [2, 30, 31, 8, 60]. We recapitulate how approximations in such spaces can be obtained by polynomial quasi-interpolation. For the elements of $\mathfrak{C}^2_\beta(\mathrm{Q})$, we prove error and stability bounds of such approximations (theorems 5.13 and 5.14) and, then, analyze the error of first-order reinterpolation with boundary lifting (theorem 5.16).

We emphasize that the auxiliary *hp* spaces and approximations are introduced in this chapter exclusively for theoretical considerations. They will serve as a particular means of low-rank tensor approximation, and the corresponding QTT-FE structure of that approximation is revealed in section 5.6 below. The algorithmic realization of *hp* approximations, quite involved in practice, will *not* be required for computations with QTT-FE approximations.

The construction of this section is determined by a few parameters, of which we assume the following.

Assumption 5.1. *$\mu \in \{0,1\}^{2\times 2}$ and $\mathcal{B} \subset \{1,2\} \times \{1,2\}$ satisfy assumption 4.1. Additionally, $\delta > 0$ and $\beta \in [0,1)$.*

For each $l \in \mathbb{N}$, the auxiliary partitions $\mathcal{G}^l$ and corresponding hp spaces $\tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ and $\mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$ introduced below depend on μ. The operators $\tilde{\Pi}^{l,\mu}_{\delta,3/2,\beta}$ and $\Pi^{l,\mu}_{\delta,3/2,\beta}$ of hp interpolation depend also on δ, α and β. Additionally, the boundary lifting of hp approximations depends on $\mathcal{B}$. All bounds of this section are explicit in terms of $\delta > 0$ and β and uniform with respect to μ and $\mathcal{B}$.

5.1 Nodal approximation of singular functions

As in section 4.2, let us consider the nodal interpolation of $u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$ and of $\Pi^{l,\mu}_{\delta,\alpha,\beta}\, u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$, where $\Pi^{l,\mu}_{\delta,\alpha,\beta}$ is an auxiliary hp quasi-interpolation operator defined and analyzed below (in definition 5.9 and section 5.4). In the setting of this section, (4.14a) and (4.15a) reduce to

$$u^l = \mathfrak{I}^l u = \sum_{i \in \mathcal{I}^l \setminus \mathcal{I}^l_0} u(t^l_i)\, \phi^l_i + \sum_{i \in \mathcal{I}^l_0} u(t^l_i)\, \phi^l_i \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l), \tag{5.1}$$

$$u^l = \mathfrak{A}^l u = \sum_{i \in \mathcal{I}^l \setminus \mathcal{I}^l_0} \Pi^{l,\mu}_{\delta,\alpha,\beta}\, u(t^l_i)\, \phi^l_i + \sum_{i \in \mathcal{I}^l_0} u(t^l_i)\, \phi^l_i \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l). \tag{5.2}$$

For the exact continuous bilinear nodal interpolant $\mathfrak{I}^l u$, which is well defined by proposition 3.1, we note the following approximation result.

Lemma 5.2. *Let $\beta \in [0, 1)$. Then there exist a positive constant C such that the following error bound holds for all $u \in \mathbb{H}^{2,2}_\beta(\mathrm{Q})$ and $l \in \mathbb{N}$ such that $l \geq 2$:*

$$\|u - \mathfrak{I}^l u\|_{\mathbb{H}^1(\mathrm{Q})} \leq C\, 2^{-(1-\beta)l}\, |u|_{\mathbb{H}^{2,2}_\beta(\mathrm{Q})}.$$

Proof. Let $u \in \mathbb{H}^2_\beta(\mathrm{Q})$ and $l \geq 2$. Applying corollary A-8.7 in every Q^l_i except $\mathrm{Q}^l_{i_*} = (0, h^l_1) \times (0, h^l_2)$ with $i_* = (\mu_{11} - 1, \mu_{21} - 1)$ and proposition A-8.11 with a rescaling argument in $\mathrm{Q}^l_{i_*}$, we obtain

$$\begin{aligned}
\|u - \mathfrak{I}^l u\|^2_{\mathbb{H}^1(\mathrm{Q}^l_{i_*})} &\leq \left\{ D_0^2\, \frac{\Lambda^4_l}{\lambda^2_l}\, 2^{-2l} + D_1^2\, \frac{\Lambda^4_l}{\lambda^4_l} \right\} 2^{-2(1-\beta)l}\, |u|^2_{\mathbb{H}^{2,2}_\beta(\mathrm{Q}^l_{i_*})}, \\
\|u - \mathfrak{I}^l u\|^2_{\mathbb{H}^1(\mathrm{Q} \setminus \mathrm{Q}^l_{i_*})} &\leq \left\{ \frac{3}{64}\, \Lambda^4_l\, 2^{-2l} + \frac{1}{2}\, \frac{\Lambda^4_l}{\lambda^2_l} \right\} 2^{-2l} \sum_{i \in \mathcal{E}^l \setminus \{i_*\}} |u|^2_{\mathbb{H}^2(\mathrm{Q}^l_i)} \\
&\leq \left\{ \frac{3}{64}\, \Lambda^4_l\, 2^{-2l} + \frac{1}{2}\, \frac{\Lambda^4_l}{\lambda^2_l} \right\} \frac{1}{\lambda^2_l}\, 2^{-2(1-\beta)l}\, |u|^2_{\mathbb{H}^{2,2}_\beta(\mathrm{Q} \setminus \mathrm{Q}^l_{i_*})},
\end{aligned}$$

where D_0 and D_1 are positive constants depending only on β. The claim follows immediately. □

Lemma 5.2 shows that, as l increases, $\mathfrak{I}^l u$ converges to $u \in \mathbb{H}^{2,2}_\beta(\mathrm{Q})$ algebraically at the rate of $(1-\beta)/2$ with respect to the number $\aleph_l$ of active nodes (4.13c). In this section, we consider hp quasi-interpolation operators Π^l_p with $p \sim l$ such that, for every u from $\mathfrak{B}^2_\beta(\mathrm{Q})$ or $\mathfrak{C}^2_\beta(\mathrm{Q})$, the approximations u^l, $l \in \mathbb{N}$, given by (5.2) achieve the same convergence rate as $\mathfrak{I}^l u$, $l \in \mathbb{N}$.

5.2 Geometrically graded partitions of Q. hp spaces $\tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ and $\mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$

In this section, we introduce hp-FE spaces for the approximation of functions from $\mathfrak{B}^2_\beta(\mathrm{Q})$ and $\mathfrak{C}^2_\beta(\mathrm{Q})$ with $\beta \in [0,1)$. Both the spaces consist of functions that may have singularities at the origin, towards which the auxiliary hp spaces will refine with respect to l.

For all $l \in \mathbb{N}$ and $k \in \{1,2\}$, the construction of $\mathcal{T}^l_k$ ensures that the number n^l_k of interior nodes satisfies the inequality $2^l - 1 \le n^l_k + 1 \le 2^l + 1$. We introduce the nodes $x^{l,0}_k = 0$, $x^{l,l+1}_k = 1$ and

$$x^{l,j}_k = 2^{j-1}\, h^l_k \quad \text{with} \quad 1 \le j \le l. \tag{5.3}$$

Then, for $1 \le j \le l$, we define the intervals

$$\mathrm{J}^{l,j}_{k,0} = (0, x^{l,j}_k), \qquad \mathrm{J}^{l,j}_{k,1} = (x^{l,j}_k, x^{l,j+1}_k). \tag{5.4}$$

Further, we consider the partition of $(0,1)$ induced by the nodes $x^{l,j}_k$, $0 \le j \le l+1$, and consisting of the elements $\mathrm{J}^{l,1}_{k,0}$ and $\mathrm{J}^{l,j}_{k,1}$ with $1 \le j \le l$. That partition is graded geometrically towards 0: for $1 \le j \le l$, we have $x^{l,j}_k = t^l_{k,i}$ with $i = 2^{j-1} - 1 + \mu_{k1}$.

For every $k \in \{1,2\}$, the nodes satisfy $2^l/(2^l+1) \le x^{l,j}_k / 2^{j-1-l} \le 2^l/(2^l-1)$ for $1 \le j \le l$; for the lengths of the intervals, we have $x^{l,j+1}_k - x^{l,j}_k = x^{l,j}_k$ when $1 \le j \le l-1$ and $x^{l,l+1}_k - x^{l,l}_k = 1 - x^{l,l}_k$ for the last element. Then, for $1 \le j \le l$ and $k \in \{1,2\}$, the four inequalities

$$2^{j-1-l}\lambda_l \le x^{l,j}_k,\; x^{l,j+1}_k - x^{l,j}_k \le 2^{j-1-l}\Lambda_l \tag{5.5a}$$

hold with $\lambda_l = \min\{2^l/(2^l+1), 1-(2^l-1)^{-1}\}$ and $\Lambda_l = \max\{2^l/(2^l-1), 1+(2^l+1)^{-1}\}$, i.e. with

$$\lambda_l = 1 - (2^l-1)^{-1} \quad \text{and} \quad \Lambda_l = 1 + (2^l+1)^{-1}. \tag{5.5b}$$

Let us now define an auxiliary partition of Q. First, we introduce

$$\mathcal{N} = \{(1,1), (1,0), (0,1)\} = \{\nu \in \mathbb{N}^2_0 :\, \|\nu\|_\infty = 1\}. \tag{5.6}$$

Then, for every $l \in \mathbb{N}$, we consider $\mathcal{G}^l$, a partition of Q graded geometrically towards the origin, consisting of the elements $\mathrm{G}^{l,0} = \mathrm{J}^{l,1}_{1,0} \times \mathrm{J}^{l,1}_{2,0}$ and

$$\mathrm{G}^{l,j}_{\nu} = \mathrm{J}^{l,j}_{1,\nu_1} \times \mathrm{J}^{l,j}_{2,\nu_2} \quad \text{with} \quad 1 \le j \le l, \quad \nu = (\nu_1, \nu_2) \in \mathcal{N}. \tag{5.7}$$

For every $p \in \mathbb{N}_0$, we introduce auxiliary hp finite-element spaces:

$$\begin{aligned} \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l) &= \left\{ \begin{array}{ll} u \in \mathbb{L}^2(\mathrm{Q}) : & u|_{\mathrm{G}^{l,0}} \in \mathbb{Q}_{1,1}, \\ & u|_{\mathrm{G}^{l,j}_{\nu}} \in \mathbb{Q}_{p,p} \quad \text{for all} \quad j = 1, \dots, l, \quad \nu \in \mathcal{N} \end{array} \right\}, \\ \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l) &= \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l) \cap \mathbf{C}(\mathbf{cl}\, \mathrm{Q}). \end{aligned} \tag{5.8}$$

Below we see that the elements of $\mathfrak{B}^2_\beta(\mathrm{Q})$ and $\mathfrak{C}^2_\beta(\mathrm{Q})$ can be approximated by functions from $\tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$, generally discontinuous, at exponential rates with respect to $p \in \mathbb{N}$ (theorem 5.13). Furthermore, these approximations can be altered so as to belong to the corresponding spaces $\mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$ of continuous functions without compromising the convergence (theorem 5.14). The alteration consists in *trace lifting*, a standard procedure in hp approximation [60, section 4.6], which we present in lemma 5.8 below.

We shall use the following additional notation:

$$\Gamma^{l,j}_{\nu} = \boldsymbol{\partial}\, \mathrm{G}^{l,j}_{\nu} \cap \boldsymbol{\partial}\, \mathrm{Q} \quad \text{for all} \quad j = 1, \dots, l, \quad \nu \in \mathcal{N}, \tag{5.9a}$$

$$\gamma^l_i = \boldsymbol{\partial}\, \mathrm{Q}^l_i \cap \boldsymbol{\partial}\, \mathrm{Q} \quad \text{for all} \quad i \in \mathcal{E}^l. \tag{5.9b}$$

Also, for every $l \in \mathbb{N}$, we set

$$\mathrm{G}^l = \mathrm{Q} \setminus \mathrm{G}^{l,0} \tag{5.10a}$$

and

$$\Gamma^l = \boldsymbol{\partial}\, \mathrm{Q} \setminus \boldsymbol{\partial}\, \mathrm{G}^{l,0}. \tag{5.10b}$$

Finally, we shall refer to

$$\tilde{\mathrm{G}}^{l,0}_1 = \left(\frac{1}{2} x^{l,1}_1, x^{l,1}_1\right) \times (0, x^{l,1}_2), \quad \tilde{\mathrm{G}}^{l,0}_2 = (0, x^{l,1}_1) \times \left(\frac{1}{2} x^{l,1}_2, x^{l,1}_2\right), \tag{5.11}$$

which are the right and upper halves of $\mathrm{G}^{l,0}$.

5.3 Polynomial quasi-interpolation

In this section, we recapitulate univariate and bivariate polynomial quasi-interpolation in $\hat{\mathrm{J}} = (-1, 1)$ and $\hat{\mathrm{Q}} = \hat{\mathrm{J}} \times \hat{\mathrm{J}}$. Sections A-8.2 and A-8.3 consist of related approximation results, which we use in our analysis in sections 5.4 and 5.5.

5.3.1 Univariate quasi-interpolation

For $i \in \mathbb{N}_0$, by L_i we denote the ith Legendre polynomial with the standard normalization: $L_i(1) = 1$ and $\langle L_i, L_i \rangle_{\mathbb{L}^2(\hat{\mathrm{J}})} = (i + \frac{1}{2})^{-1}$.

Definition 5.3. *For every $p \in \mathbb{N}$, we define a quasi-interpolation operator $\hat{\pi}_p$: $\mathbb{H}^1(\hat{\mathrm{J}}) \to \mathbb{P}_p$ by setting*

$$\hat{u}(-1) = \hat{\pi}_p \hat{u}(-1) \quad \textit{and} \quad (\hat{\pi}_p \hat{u})' = \sum_{i=0}^{p-1} c_i L_i$$

for every $\hat{u} \in \mathbb{H}^1(\hat{\mathrm{J}})$, where $c_i = (i + \frac{1}{2}) \langle \hat{u}', L_i \rangle_{\mathbb{L}^2(\hat{\mathrm{J}})}$ for $i = 0, 1, \ldots, p-1$.

For every $p \in \mathbb{N}$, the quasi-interpolation operator $\hat{\pi}_p$ is continuous. Also, by [60, theorem 3.14] or [15, lemma 5], we have the following property

Proposition 5.4. *For every $p \in \mathbb{N}$, the operator $\hat{\pi}_p$ is nodally exact:*

$$\hat{u}(\pm 1) = \hat{\pi}_p \hat{u}(\pm 1)$$

holds for every $\hat{u} \in \mathbb{H}^1(\hat{\mathrm{J}})$.

5.3.2 Tensor-product bivariate quasi-interpolation

In the remainder of this section, we shall use tensor-product Sobolev spaces, which are isomorphic to Bochner spaces: for all $m_1 m_2 \in \mathbb{N}_0$, we consider

$$\mathbb{H}^{m_1 m_2}_{\mathrm{mix}}(\hat{\mathrm{Q}}) = \mathbb{H}^{m_1}(\hat{\mathrm{J}}) \otimes \mathbb{H}^{m_2}(\hat{\mathrm{J}}) \simeq \mathbb{H}^{m_2}(\hat{\mathrm{J}}, \mathbb{H}^{m_1}(\hat{\mathrm{J}}))$$

with the cross norm given by

$$\|u\|^2_{\mathbb{H}^{m_1 m_2}_{\mathrm{mix}}(\hat{\mathrm{Q}})} = \sum_{\alpha_1=0}^{m_1} \sum_{\alpha_2=0}^{m_2} \|\partial_1^{\alpha_1} \partial_2^{\alpha_2} u\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}$$

for all $u \in \mathbb{H}^{m_1 m_2}_{\mathrm{mix}}(\hat{\mathrm{Q}})$. For all $m_1, m_2 \in \mathbb{N}_0$, we have the following inclusion of a standard Sobolev space: $\mathbb{H}^{m_1+m_2}(\hat{\mathrm{Q}}) \subset \mathbb{H}^{m_1 m_2}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \subset \mathbb{H}^{\min\{m_1 m_2\}}(\hat{\mathrm{Q}})$. The subscript "mix" reflects that these spaces are often called *spaces with dominating mixed smoothness.* For all $m_1, m_2 \in \mathbb{N}_0$, the functions from $\mathbb{H}^{m_1 m_2}_{\mathrm{mix}}(\hat{\mathrm{Q}})$ admit continuous extensions to $\mathbf{cl}\,\hat{\mathrm{Q}}$.

For every $p \in \mathbb{N}$, we consider tensor-product operators $\hat{\pi}_p \otimes \mathbf{id}_2 : \mathbb{H}^{1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \to \mathbb{P}_p \otimes \mathbb{H}^1(\hat{\mathrm{J}})$ and $\mathbf{id}_1 \otimes \hat{\pi}_p : \mathbb{H}^{1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \to \mathbb{H}^1(\hat{\mathrm{J}}) \otimes \mathbb{P}_p$, where $\mathbf{id}_1$ and $\mathbf{id}_2$ denote the corresponding identity operators. For every $m \in \mathbb{N}$, these tensor-product operators are continuous mappings from $\mathbb{H}^{1,m}_{\mathrm{mix}}(\hat{\mathrm{Q}})$ to $\mathbb{P}_p \otimes \mathbb{H}^m(\hat{\mathrm{J}})$ and from $\mathbb{H}^{m,1}_{\mathrm{mix}}(\hat{\mathrm{Q}})$ to $\mathbb{H}^m(\hat{\mathrm{J}}) \otimes \mathbb{P}_p$ respectively, since the operator π_p is continuous from $\mathbb{H}^1(\hat{\mathrm{J}})$ to $\mathbb{P}_p$.

Definition 5.5. *For every* $p = (p_1, p_2) \in \mathbb{N}^2$*, we introduce a tensor-product projection operator:*

$$\hat{\Pi}_{p_1 p_2} = \hat{\pi}_{p_1} \otimes \hat{\pi}_{p_2} : \mathbb{H}^{1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \to \mathbb{Q}_{p_1 p_2}.$$

For every $p = (p_1, p_2) \in \mathbb{N}^2$, we consider the corresponding one-dimensional projections

$$\hat{\pi}_{1,p_1} = \hat{\pi}_{p_1} \otimes \mathbf{id}_2 \quad \text{and} \quad \hat{\pi}_{2,p_2} = \mathbf{id}_1 \otimes \hat{\pi}_{p_2},$$

where $\mathbf{id}_1$ and $\mathbf{id}_2$ denote the identity transformation of $\mathbb{H}^1(\hat{\mathrm{J}})$.

The operator $\hat{\Pi}_{p_1 p_2}$ is continuous and can be recast as a superposition of one-dimensional projections:

$$\hat{\Pi}_{p_1 p_2} = \hat{\pi}_{2,p_2} \circ \hat{\pi}_{1,p_1} = \hat{\pi}_{1,p_1} \circ \hat{\pi}_{2,p_2},$$

see [60, Lemma 4.67 (i)] or [15, section 4] for details. From proposition 5.4, we obtain the following proposition, $\hat{\Gamma}_{11}$, $\hat{\Gamma}_{12}$, $\hat{\Gamma}_{21}$ and $\hat{\Gamma}_{22}$ denoting the left, right, bottom and top edges of $\hat{\mathrm{Q}}$.

Proposition 5.6. *Assume* $p_1, p_2 \in \mathbb{N}$*. Then we have*

$$(\hat{\Pi}_{p_1 p_2} \hat{u})|_{\hat{\Gamma}_{kj}} = \hat{\pi}_{p_{k'}} \hat{u}|_{\hat{\Gamma}_{kj}}$$

for all $k, k', j = 1, 2$ *such that* $k' \neq k$ *and for every* $\hat{u} \in \mathbb{H}^{1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}})$*.*

5.4 *hp* quasi-interpolation

In this section, we recapitulate standard techniques of *hp* approximation based on the operators of polynomial quasi-interpolation described in section 5.3. Further details can be found, e.g. in [60, section 4.6].

The analysis of reinterpolation, which we give in section 5.5, requires the $\mathbb{H}^2$-stability of auxiliary approximations. In this section, to keep the presentation self-contained, we prove the convergence of *hp* approximations together with $\mathbb{H}^2$-stability bounds.

5.4.1 Definitions

Definition 5.7 (discontinuous *hp* quasi-interpolation)**.** *Let* $\mu \in \{0,1\}^{2\times 2}$*,* $l \in \mathbb{N}$ *and* $\mathcal{G}^l$ *be as described in section 5.2. Then, for all* $p \in \mathbb{N}$*, we define* $\tilde{\Pi}^l_p : \mathbb{H}^{1,1}_{\mathrm{mix}}(\mathrm{Q}) \to \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ *by setting*

$$(\tilde{\Pi}^l_p u) \circ \varphi^{l,0} = \hat{\Pi}_{1,1} (u \circ \varphi^{l,0}) \quad \text{in} \quad \hat{\mathrm{Q}},$$
$$(\tilde{\Pi}^l_p u) \circ \varphi^{l,j}_\nu = \hat{\Pi}_{p,p} (u \circ \varphi^{l,j}_\nu) \quad \text{in} \quad \hat{\mathrm{Q}} \quad \text{for all} \quad j = 1, \dots, l \quad \text{and} \quad \nu \in \mathcal{N},$$

where $\varphi^{l,0}$ *is an affine function mapping* $\hat{\mathrm{Q}}$ *onto* $\mathrm{G}^{l,0}$ *and every* $\varphi^{l,j}_\nu$ *is an affine function mapping* $\hat{\mathrm{Q}}$ *onto* $\mathrm{G}^{l,j}_\nu$*.*

The *hp* approximation $\tilde{\varPi}_p^l u$ to u may be discontinuous across edges of the elements of $\mathcal{G}^l$. These discontinuities can, however, be *lifted*, as the following lemma shows.

Lemma 5.8 (trace lifting)**.** *Let $\mu \in \{0,1\}^{2\times 2}$, $l \in \mathbb{N}$ and $\mathcal{G}^l$ be as described in section 5.2. Then, for all $p \in \mathbb{N}$, there exists a linear operator $\varPi_p^l : \mathbb{H}^{1,1}_{\mathrm{mix}}(\mathrm{Q}) \to \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$ such that, for every $u \in \mathbb{H}^{1,1}_{\mathrm{mix}}(\mathrm{Q})$, the function $(\varPi_p^l - \tilde{\varPi}_p^l)u \in \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ vanishes in $\mathrm{G}^{l,0}$ and on $\boldsymbol{\partial}\,\mathrm{Q}$.*

Proof. We refer the reader to the Appendix. □

Definition 5.9 (the selection of p and s)**.** *Under assumption 5.1 and for every $\alpha \geq 1$, we use definition 5.7 and lemma 5.8 for all $l \in \mathbb{N}$ to introduce the quasi-interpolation operators*

$$\tilde{\Pi}^{l,\mu}_{\delta,\alpha,\beta} = \tilde{\varPi}_p^l \quad \textit{and} \quad \Pi^{l,\mu}_{\delta,\alpha,\beta} = \varPi_p^l, \tag{5.12}$$

with

$$p = \lceil \varrho_\delta \chi_l \rceil, \tag{5.13a}$$

where

$$\varrho_\delta = 1 + \frac{\delta}{2} \quad \textit{and} \quad \chi_l = 1 + l(1+\alpha-\beta)\log 2. \tag{5.13b}$$

Together with the p chosen above, we shall also use

$$s = \lfloor p/\varrho_\delta \rfloor, \tag{5.13c}$$

which depends on δ, α, β and l.

Then we have $\varrho_\delta > 1$, $\chi_l > 1$, $\varrho_\delta\,\chi_l \leq p < \varrho_\delta\,\chi_l + 1$ and $\chi_l - 1 < s < \chi_l + \varrho_\delta^{-1} < \chi_l + 1$, so that $p \geq 2$ and $s \geq 1$.

5.4.2 Preliminary bounds

In this section, using the auxiliary results presented in sections A-8.2 and A-8.3 below, we prove preliminary approximation and stability results of *hp* approximation, which shall later be specified for the analyticity classes $\mathfrak{B}^2_\beta(\mathrm{Q})$ and $\mathfrak{C}^2_\beta(\mathrm{Q})$.

For all $r \in \mathbb{N}$, $\sigma \in \mathbb{R}$, we introduce $[\,\cdot\,]_{r+1,\sigma}$, a broken Sobolev seminorm on $\mathbb{H}^{r+1}(\mathrm{G}^l)$:

$$[u]^2_{r+1,\sigma} = \sum_{j=1}^{l} \sum_{\nu\in\mathcal{N}} (2^{j-2-l}\Lambda_l)^{2(r+1-\sigma)}\, |u|^2_{\mathbb{H}^{r+1}(\mathrm{G}^{l,j}_\nu)} \quad \text{for all} \quad u \in \mathbb{H}^{r+1}(\mathrm{G}^l). \tag{5.14}$$

We note that $[\cdot]_{r+1,\sigma}$ depends on μ and l, which define $\mathcal{G}^l$. For the sake of brevity, we do not indicate that dependence explicitly in the notation.

Lemma 5.10 (estimates for *hp* quasi-interpolation in terms of the broken Sobolev seminorms)**.** *Let $\beta \in [0,1)$. Then there exist positive constants D_0 and D_1 such that, for every $\mu \in \{0,1\}^{2\times 2}$, for all $l, p, s \in \mathbb{N}$ such that $l \geq 2$ and $s \leq p$ and for every $u \in \mathbb{H}^{2,2}_{\beta}(\mathrm{Q}) \cap \mathbb{H}^{s+2,2}_{\beta}(\mathrm{G}^l)$, the hp approximation $v^l = \tilde{\Pi}^l_p u \in \tilde{\mathbb{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ satisfies the following error and stability bounds:*

$$\|u - v^l\|^2_{\mathbb{L}^2(\mathrm{Q})} \leq 3\, \frac{\Upsilon_{ps}}{p(p+1)}\, [u]^2_{s+1,0} + D_0^2\, \frac{\Lambda_l^4}{\lambda_l^2}\, 2^{-2(2-\beta)l}\, |u|^2_{\mathbb{H}^{2,2}_{\beta}(\mathrm{G}^{l,0})}, \tag{5.15a}$$

$$|u - v^l|^2_{\mathbb{H}^1(\mathrm{Q})} \leq 4\, \frac{\Lambda_l^2}{\lambda_l^2}\, \Upsilon_{ps}\, [u]^2_{s+1,1} + D_1^2\, \frac{\Lambda_l^4}{\lambda_l^4}\, 2^{-2(1-\beta)l}\, |u|^2_{\mathbb{H}^{2,2}_{\beta}(\mathrm{G}^{l,0})}, \tag{5.15b}$$

$$\sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_{\nu})} \leq \frac{\Lambda_l^4}{\lambda_l^4}\, (p^2 - 1)\, [u]^2_{2,2}\,. \tag{5.15c}$$

Furthermore, on the boundary we have the inequalities

$$\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)} \leq \frac{3}{2}\, \frac{\Lambda_l}{\lambda_l}\, \frac{\Upsilon_{ps}}{p(p+1)} \left\{ [u]^2_{s+1,\frac{1}{2}} + [u]^2_{s+2,\frac{1}{2}} \right\}, \tag{5.16a}$$

$$|u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} \leq \frac{3}{2}\, \frac{\Lambda_l}{\lambda_l}\, \Upsilon_{ps} \left\{ [u]^2_{s+1,\frac{3}{2}} + [u]^2_{s+2,\frac{3}{2}} \right\}, \tag{5.16b}$$

$$\sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\Gamma^{l,j}_{\nu})} \leq \frac{3}{4}\, \frac{\Lambda_l}{\lambda_l} (p^2 - 1) \left\{ [u]^2_{2,\frac{5}{2}} + [u]^2_{3,\frac{5}{2}} \right\}, \tag{5.16c}$$

where $\Gamma^{l,j}_{\nu}$ with $j = 1, \ldots, l$ and $\nu \in \mathcal{N}$ are given by (5.9a).

Proof. We refer the reader to the Appendix. □

Lemma 5.11 (estimates for trace lifting in terms of the broken Sobolev seminorms)**.** *Let $\beta \in [0,1)$. Then, for every $\mu \in \{0,1\}^{2\times 2}$, for all $l, p, s \in \mathbb{N}$ such that $l \geq 2$ and $s \leq p$ and for every $u \in \mathbb{H}^{3,2}(\mathrm{Q}) \cap \mathbb{H}^{s+2,2}_{\beta}(\mathrm{G}^l)$, the trace-lifting term $w^l = \Pi^l_p u - \tilde{\Pi}^l_p u \in \tilde{\mathbb{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ of lemma 5.8 satisfies the bounds*

$$\|w^l\|^2_{\mathbb{L}^2(\mathrm{Q})} \leq \frac{1}{4}\, \frac{\Lambda_l^5}{\lambda_l}\, 2^{-4l}\, Z_0^2 + 3\, \frac{\Lambda_l}{\lambda_l}\, \frac{\Upsilon_{ps}}{p(p+1)} \left\{ [u]^2_{s+1,0} + [u]^2_{s+2,0} \right\}, \tag{5.17a}$$

$$\sum_{j\nu} |w^l|^2_{\mathbb{H}^1(\mathrm{G}^{l,j}_{\nu})} \leq \frac{11}{4}\, \frac{\Lambda_l^4}{\lambda_l^2}\, 2^{-2l}\, Z_0^2 + \frac{15}{2}\, \frac{\Lambda_l^2}{\lambda_l^2}\, \Upsilon_{ps} \left\{ [u]^2_{s+1,1} + [u]^2_{s+2,1} \right\}, \tag{5.17b}$$

$$\sum_{j\nu} |w^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_{\nu})} \leq 6\, \frac{\Lambda_l^2}{\lambda_l^2}\, Z_0^2 + 3p^2 \left\{ [u]^2_{2,2} + [u]^2_{3,2} \right\}, \tag{5.17c}$$

where

$$Z_0^2 = \sum_{k=1,2} \left\{ |u|^2_{\mathbb{H}^2(\tilde{\mathrm{G}}^{l,0}_k)} + (2^{-l-2}\Lambda_l)^2\, |u|^2_{\mathbb{H}^3(\tilde{\mathrm{G}}^{l,0}_k)} \right\}. \tag{5.18}$$

Proof. We refer the reader to the Appendix. □

5.4.3 *hp* quasi-interpolation of functions from $\mathfrak{C}^2_\beta(\mathrm{Q})$

Let $\mathrm{G} \subset \mathrm{Q}$ be rectangle. Assume that $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$ in the sense of definition 3.4 with positive constants C_u and δ_u. Then, using lemma A-8.1, we obtain for every $r \in \mathbb{N}$ that

$$\begin{aligned}
|u|^2_{\mathbb{H}^{r+1}(\mathrm{G})} &= \sum_{|\boldsymbol{\alpha}|=r+1} \|\partial^{\boldsymbol{\alpha}} u\|^2_{\mathbb{L}^2(\mathrm{G})} \\
&\le \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 \|\varPhi^{-1}_{\beta+r}\|^2_{\mathbb{L}^2(\mathrm{G})} \sum_{|\boldsymbol{\alpha}|=r+1} \left\{\frac{\boldsymbol{\alpha}!}{(r+1)!}\right\}^2 \\
&\le \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 \|\varPhi^{-1}_{\beta+r}\|^2_{\mathbb{L}^2(\mathrm{G})} \left\{\sum_{|\boldsymbol{\alpha}|=r+1} \frac{\boldsymbol{\alpha}!}{(r+1)!}\right\}^2 \\
&\le \frac{64}{9} \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 \|\varPhi^{-1}_{\beta+r}\|^2_{\mathbb{L}^2(\mathrm{G})}. \qquad (5.19\mathrm{a})
\end{aligned}$$

Let $l \in \mathbb{N}$, $j = 1, \dots, l$ and $\nu \in \mathcal{N}$. Using (5.5), we obtain the bound $\varPhi_{r+\beta}(x) \ge (2^{j-1-l}\lambda_l)^{r+\beta} \ge 2^{(r+\beta)(j-1-l)}\lambda_l^{r+1}$ for all $x \in \mathrm{G}^{l,j}_\nu$ and $r \in \mathbb{N}$. The corresponding Sobolev seminorm is thus bounded as follows:

$$\begin{aligned}
|u|^2_{\mathbb{H}^{r+1}(\mathrm{G}^{l,j}_\nu)} &\le \frac{64}{9} \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 \lambda_l^{-2(r+1)}\, 2^{-2(r+\beta)(j-1-l)}\, \Lambda_l^2\, 2^{2(j-1-l)} \\
&= \frac{64}{9} \frac{\Lambda_l^2}{\lambda_l^{2(r+1)}} \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 2^{-2(r+\beta-1)(j-1-l)} \qquad (5.19\mathrm{b})
\end{aligned}$$

for all $r \in \mathbb{N}$, $l \in \mathbb{N}$, $j = 1, \dots, l$ and $\nu \in \mathcal{N}$.

For $\tilde{\mathrm{G}}^{l,0}_k$ with $k = 1, 2$ given by (5.11), we obtain a similar bound: $\varPhi_{r+\beta}(x) \ge (2^{-1-l}\lambda_l)^{r+\beta} \ge 2^{-(r+\beta)(l+1)}\lambda_l^{r+1}$ for all $x \in \mathrm{G}^{l,j}_\nu$ and $r \in \mathbb{N}$, and, therefore,

$$\begin{aligned}
|u|^2_{\mathbb{H}^{r+1}(\tilde{\mathrm{G}}^{l,0}_k)} &\le \frac{64}{9} \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 \lambda_l^{-2(r+1)}\, 2^{2(r+\beta)(l+1)}\, \frac{1}{2}\, \Lambda_l^2\, 2^{-2l} \\
&= \frac{128}{9} \frac{\Lambda_l^2}{\lambda_l^{2(r+1)}} \left\{C_u\, \delta_u^{r+1}\, (r+1)!\right\}^2 2^{2(r+\beta-1)(l+1)} \qquad (5.19\mathrm{c})
\end{aligned}$$

holds for all $r \in \mathbb{N}$, $l \in \mathbb{N}$, and $k = 1, 2$.

Lemma 5.12 (estimates for the broken Sobolev seminorms). *Assume $\beta \in [0, 1)$, $\alpha \ge 1$, $\mu \in \{0, 1\}^{2\times 2}$ and $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$ in the sense of definition 3.4 with positive* *constants C_u and δ_u. Then, with p and s given by (5.13), the following bound is satisfied for all $l, r \in \mathbb{N}$ and $\sigma \in \mathbb{R}$:*

$$\Upsilon_{ps}\,[u]^2_{r+1,\sigma} \leq \frac{64}{3}\,\frac{e^3}{\sqrt{2\pi}}\,\frac{\Lambda_l^{2(r+2-\sigma)}}{\lambda_l^{2(r+1)}}\,\frac{\Sigma^2_{l\sigma}\,C_u^2\,\delta_u^{2(r+1-s)}}{2^{2(r+1-\sigma-s)}}\,s\left\{\frac{(r+1)!}{s!}\right\}^2 2^{-2(\alpha-\beta)l},$$

where

$$M^2_{l\sigma} = \max_{1\leq j\leq l} 2^{2(2-\beta-\sigma)(j-1-l)}.$$

Proof. Applying (5.19b) for all $j = 1,\dots,l$ and $\nu \in \mathcal{N}$, we deduce the following bound for $[u]^2_{r+1,\sigma}$ of (5.14) with arbitrary $r \in \mathbb{N}$ and $\sigma \in \mathbb{R}$:

$$\begin{aligned}
[u]^2_{r+1,\sigma} &\leq \frac{64}{9}\left\{C_u\,\delta_u^{r+1}\,(r+1)!\right\}^2 \sum_{j\nu} \frac{\Lambda_l^{2(r+2-\sigma)}\,2^{2(r+1-\sigma)(j-2-l)}}{\lambda_l^{2(r+1)}\,2^{2(r+\beta-1)(j-1-l)}} \\
&= \frac{64}{9}\,2^{-2(r+1-\sigma)}\,\frac{\Lambda_l^{2(r+2-\sigma)}}{\lambda_l^{2(r+1)}}\left\{C_u\,\delta_u^{r+1}\,(r+1)!\right\}^2 3\sum_{j=1}^{l} 2^{2(2-\beta-\sigma)(j-1-l)} \\
&= \frac{64}{3}\,\frac{\Lambda_l^{2(r+2-\sigma)}}{\lambda_l^{2(r+1)}}\,2^{-2(r+1-\sigma)}\,l\,M^2_{l\sigma}\left\{C_u\,\delta_u^{r+1}\,(r+1)!\right\}^2.
\end{aligned}$$

For p and s given by (5.13), using lemma A-8.2, we arrive at

$$\begin{aligned}
\Upsilon_{ps}\,[u]^2_{r+1,\sigma} &\leq (\varrho-1)^{2s}\,\Upsilon_{ps}\,(s!)^2\,\frac{64}{3}\,\frac{\Lambda_l^{2(r+2-\sigma)}}{\lambda_l^{2(r+1)}}\,\frac{l\,M^2_{l\sigma}\,C_u^2\,\delta_u^{2(r+1-s)}}{2^{2(r+1-\sigma-s)}}\left\{\frac{(r+1)!}{s!}\right\}^2 \\
&\leq \frac{64}{3}\,\frac{e^5}{\sqrt{2\pi}}\,s\,\exp\!\left(-\frac{2p}{\varrho}\right)\frac{\Lambda_l^{2(r+2-\sigma)}}{\lambda_l^{2(r+1)}}\,\frac{l\,M^2_{l\sigma}\,C_u^2\,\delta_u^{2(r+1-s)}}{2^{2(r+1-\sigma-s)}}\left\{\frac{(r+1)!}{s!}\right\}^2 \\
&\leq \frac{64}{3}\,\frac{e^3}{\sqrt{2\pi}}\,\frac{\Lambda_l^{2(r+2-\sigma)}}{\lambda_l^{2(r+1)}}\,\frac{l\,M^2_{l\sigma}\,C_u^2\,\delta_u^{2(r+1-s)}}{2^{2(r+1-\sigma-s)}}\,s\left\{\frac{(r+1)!}{s!}\right\}^2 2^{-2(\alpha-\beta)l}.
\end{aligned}$$

□

Theorem 5.13 (*hp* quasi-interpolation)**.** *Let assumption 5.1 hold. Assume that $\alpha \geq 1$ and $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$ in the sense of definition 3.4 with positive constants C_u and δ. Then there exist constants $C_1, C_2, c_0, c_1, c_2 > 0$ such that, for all $l \in \mathbb{N}$ such that $l \geq 2$, the approximation $v^l = \tilde{\Pi}^{l,\mu}_{\delta,\alpha,\beta} u$ satisfies the following error and stability bounds:*

$$\|u - v^l\|^2_{\mathbb{H}^1(\mathrm{Q})} \leq C_1^2\,l^3\,2^{-2(1-\beta)l}, \qquad \sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} \leq C_2^2\,l^2\,2^{2\beta l} \tag{5.20}$$

and

$$\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)} \le c_0^2\, l^4\, 2^{-2(\alpha-\beta)l}, \qquad |u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} \le c_1^2\, l^6\, 2^{-2(\alpha+\beta_*-2\beta)l},$$
$$\sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\Gamma^{l,j}_\nu)} \le c_2^2\, l^2\, 2^{(2\beta+1)l}, \tag{5.21}$$

where $\beta_* = \min\{\frac{1}{2}, \beta\}$.

Proof. We refer the reader to the Appendix. □

Theorem 5.14 (trace lifting)**.** *Let assumption 5.1 hold. Assume that* $\alpha \ge 1$ *and* $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$ *in the sense of definition 3.4 with positive constants* C_u *and* δ. *Then there exist constants* $\tilde{C}_1, \tilde{C}_2 > 0$ *such that, for all* $l \ge 2$, *the trace-lifting term* $w^l = \Pi^{l,\mu}_{\delta,\alpha,\beta} u - \tilde{\Pi}^{l,\mu}_{\delta,\alpha,\beta} u$ *satisfies the bounds*

$$\sum_{j\nu} \|w^l\|^2_{\mathbb{H}^1(\mathrm{G}^{l,j}_\nu)} \le \tilde{C}_1^2\, l^6\, 2^{-2(1-\beta)l}, \qquad \sum_{j\nu} |w^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} \le \tilde{C}_2^2\, l^2\, 2^{2\beta l}. \tag{5.22}$$

Proof. We refer the reader to the Appendix. □

5.5 h-FE reinterpolation

5.5.1 Preliminary bound

In this section, we prove the convergence of the approximations u^l given by (5.2) to u when $u \in \mathfrak{B}^2_\beta(\mathrm{Q})$ or $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$.

We note that the boundary lifting we use in this section is the same as in (4.15a). It is different from that discussed in the introductory section 3.4, which is independent of the discretization. In lemma 5.15, we introduce $\Delta^l \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l)$ to satisfy, in the sense of interpolation, the boundary conditions imposed on Γ_0 without affecting the values at active nodes and hence the tensor structure of the approximation. The term Δ^l corresponds to $\mathfrak{I}^l_0(u - \boldsymbol{\Pi}^l u)$ with $\mathfrak{n} = 1$, see (4.15a).

We set $i_* = (\mu_{11} - 1, \mu_{21} - 1)$, so that $\mathrm{G}^{l,0} = \mathrm{Q}^l_{i_*} = (0, h^l_1) \times (0, h^l_2)$. We also introduce

$$\mathcal{E}^l_0 = \Big\{ i \in \mathcal{E}^l \setminus \{i_*\} : \quad \boldsymbol{\partial}\, \mathrm{Q}^l_i \cap \Gamma_0 \neq \varnothing \Big\}, \tag{5.23}$$

where $\mathcal{E}^l$ is as in (4.4).

Lemma 5.15. *Consider* $l \in \mathbb{N}$ *such that* $l \ge 2$, $p \in \mathbb{N}$ *and* $\mu \in \{0,1\}^{2\times 2}$. *Assume that* $u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q}) \cap \mathbb{H}^2(\mathrm{G}^l)$. *Then the approximation* u^l *given by* (5.2) *satisfies the following error bound:*

$$
\begin{aligned}
\|u^l - u\|^2_{\mathbb{H}^1(\mathrm{Q})} \le 4\,\|v^l - u\|^2_{\mathbb{H}^1(\mathrm{Q})} + 4\,\|w^l\|^2_{\mathbb{H}^1(\mathrm{Q})} \\
+ 8\,\frac{\Lambda_l^4}{\lambda_l^2}\,2^{-2l}\left\{\sum_{j\nu}|v^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} + \sum_{j\nu}|w^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)}\right\} \\
+ 6\cdot 2^{-3l}\,|u|^2_{\mathbb{H}^2(\Gamma^l)} + 6\cdot 2^{-3l}\,\sum_{j\nu}|v^l|^2_{\mathbb{H}^2(\Gamma^{l,j}_\nu)} \\
+ \frac{16}{3}\,\Lambda_l\,2^{-l}\,|u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} + 32\cdot 2^l\,\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)}, \quad (5.24)
\end{aligned}
$$

where $v^l = \tilde{\Pi}^l_p u$ *and* $w^l = \Pi^l_p u - v^l$.

Proof. Let us decompose u^l into two terms: $u^l = \Xi^l + \Delta^l$, where

$$
\Xi^l = \sum_{i\in\mathcal{I}^l}\xi^l(t^l_i)\,\phi^l_i \;\in \mathrm{S}^1(\mathrm{Q},\mathcal{T}^l) \quad\text{and}\quad \Delta^l = \sum_{i\in\mathcal{I}^l_0}(u-\xi^l)(t^l_i)\,\phi^l_i \in \mathrm{S}^1(\mathrm{Q},\mathcal{T}^l) \quad (5.25)
$$

with $\xi^l = v^l + w^l \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$. To prove the error bound, we split the approximation error as follows:

$$
u^l - u = (v^l - u) + w^l + (\Xi^l - \xi^l) + \Delta^l, \quad (5.26)
$$

so that

$$
\begin{aligned}
\|u^l - u\|^2_{\mathbb{H}^1(\mathrm{Q})} \le 4\,\|v^l - u\|^2_{\mathbb{H}^1(\mathrm{Q})} + 4\,\|w^l\|^2_{\mathbb{H}^1(\mathrm{Q})} \\
+ 4\,\|\Xi^l - \xi^l\|^2_{\mathbb{H}^1(\mathrm{Q})} + 4\,\|\Delta^l\|^2_{\mathbb{H}^1(\mathrm{Q})}. \quad (5.27)
\end{aligned}
$$

In the corner element $\mathrm{G}^{l,0} = \mathrm{Q}^l_{i_*} = (0,h^l_1)\times(0,h^l_2)$ with $i_* = (\mu_{11}-1,\mu_{21}-1)$, the function ξ^l is bilinear; the interpolant Ξ^l thus coincides with ξ^l in that element. Then the third term of (5.27) may be estimated using corollary A-8.7 in every Q^l_i except $\mathrm{G}^{l,0} = \mathrm{Q}^l_{i_*}$:

$$
\begin{aligned}
\|\Xi^l - \xi^l\|^2_{\mathbb{H}^1(\mathrm{Q})} &\le \left\{\frac{3}{64}\,\Lambda_l^4\,2^{-2l} + \frac{1}{2}\,\frac{\Lambda_l^4}{\lambda_l^2}\right\}2^{-2l}\,\sum_{j\nu}|\xi^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} \\
&\le 2\,\frac{\Lambda_l^4}{\lambda_l^2}\,2^{-2l}\left\{\sum_{j\nu}|v^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} + \sum_{j\nu}|w^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)}\right\} \quad (5.28)
\end{aligned}
$$

Estimating the boundary term Δ^l is more technical. Due to the interpolation properties of $\tilde{\Pi}^l_1$ and $\tilde{\Pi}^l_p$ and because w^l vanishes in $\mathrm{G}^{l,0}$, the values of ξ^l and u at the vertices of $\mathrm{G}^{l,0}$ coincide, and Δ^l vanishes in $\mathrm{G}^{l,0}$. The boundary-lifting term Δ^l is thus nonzero only in those elements of $\mathcal{T}^l$ that are indexed by $\mathcal{E}^l_0$ given by (5.23).

Applying proposition A-8.10 and corollary A-8.5 to every Q_i^l with $i \in \mathcal{E}_0^l$ and using the fact that w^l vanishes on $\boldsymbol{\partial}\,\mathrm{Q}$, we obtain the following inequalities:

$$\begin{aligned}
\|\varDelta^l\|^2_{\mathbb{L}^2(\mathrm{Q})} &= \sum_{i\in\mathcal{E}_0^l} \|\varDelta^l\|^2_{\mathbb{L}^2(\mathrm{Q}_i^l)} \le 2 \sum_{i\in\mathcal{E}_0^l} \frac{1}{3}\, 2^{-l}\Lambda_l \, \|\varDelta^l\|^2_{\mathbb{L}^2(\gamma_i^l)} \\
&\le 4 \sum_{i\in\mathcal{E}_0^l} \frac{1}{3}\, 2^{-l}\Lambda_l \left\{ \|u - v^l - \varDelta^l\|^2_{\mathbb{L}^2(\gamma_i^l)} + \|u - v^l\|^2_{\mathbb{L}^2(\gamma_i^l)} \right\} \\
&\le \frac{\Lambda_l^5}{48}\, 2^{-5l} \sum_{i\in\mathcal{E}_0^l} |u - v^l|^2_{\mathbb{H}^2(\gamma_i^l)} + \frac{4}{3}\, 2^{-l}\Lambda_l \, \|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)} \\
&= \frac{\Lambda_l^5}{48}\, 2^{-5l} \sum_{j\nu} |u - v^l|^2_{\mathbb{H}^2(\Gamma_\nu^{l,j})} + \frac{4}{3}\, 2^{-l}\Lambda_l \, \|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)},
\end{aligned} \tag{5.29a}$$

$$\begin{aligned}
|\varDelta^l|^2_{\mathbb{H}^1(\mathrm{Q})} &= \sum_{i\in\mathcal{E}_0^l} |\varDelta^l|^2_{\mathbb{H}^1(\mathrm{Q}_i^l)} \le 2 \sum_{i\in\mathcal{E}_0^l} \left\{ \frac{1}{3}\, 2^{-l}\Lambda_l \, |\varDelta^l|^2_{\mathbb{H}^1(\gamma_i^l)} + \frac{2^l}{\lambda_l} \, \|\varDelta^l\|^2_{\mathbb{L}^2(\gamma_i^l)} \right\} \\
&\le \frac{4}{3}\, 2^{-l}\Lambda_l \sum_{i\in\mathcal{E}_0^l} \left\{ |u - v^l - \varDelta^l|^2_{\mathbb{H}^1(\gamma_i^l)} + |u - v^l|^2_{\mathbb{H}^1(\gamma_i^l)} \right\} \\
&\quad + 4\, \frac{2^l}{\lambda_l} \sum_{i\in\mathcal{E}_0^l} \left\{ \|u - v^l - \varDelta^l\|^2_{\mathbb{L}^2(\gamma_i^l)} + \|u - v^l\|^2_{\mathbb{L}^2(\gamma_i^l)} \right\} \\
&\le \frac{4}{3}\, 2^{-l}\Lambda_l \sum_{i\in\mathcal{E}_0^l} \left\{ \frac{\Lambda_l^2}{8}\, 2^{-2l} \, |u - v^l|^2_{\mathbb{H}^2(\gamma_i^l)} + |u - v^l|^2_{\mathbb{H}^1(\gamma_i^l)} \right\} \\
&\quad + 4\, \frac{2^l}{\lambda_l} \sum_{i\in\mathcal{E}_0^l} \left\{ \frac{\Lambda_l^4}{64}\, 2^{-4l} \, |u - v^l|^2_{\mathbb{H}^2(\gamma_i^l)} + \|u - v^l\|^2_{\mathbb{L}^2(\gamma_i^l)} \right\} \\
&\le \sum_{i\in\mathcal{E}_0^l} \left\{ \frac{11}{48}\, \frac{\Lambda_l^4}{\lambda_l}\, 2^{-3l} \, |u - v^l|^2_{\mathbb{H}^2(\gamma_i^l)} \right. \\
&\quad \left. + \frac{4}{3}\, 2^{-l}\Lambda_l \, |u - v^l|^2_{\mathbb{H}^1(\gamma_i^l)} + 4\, \frac{2^l}{\lambda_l} \, \|u - v^l\|^2_{\mathbb{L}^2(\gamma_i^l)} \right\} \\
&\le \frac{11}{48}\, \frac{\Lambda_l^4}{\lambda_l}\, 2^{-3l} \sum_{j\nu} |u - v^l|^2_{\mathbb{H}^2(\Gamma_\nu^{l,j})} + \frac{4}{3}\, 2^{-l}\Lambda_l \, |u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} + 4\, \frac{2^l}{\lambda_l} \, \|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)}.
\end{aligned} \tag{5.29b}$$

Combining (5.29), we proceed to

$$
\begin{aligned}
\|\Delta^l\|^2_{\mathbb{H}^1(\mathrm{Q})} &\le \left\{\frac{1}{48}\,2^{-2l}\Lambda_l^5 + \frac{11}{48}\,\frac{\Lambda_l^4}{\lambda_l}\right\} 2^{-3l} \sum_{j\nu} |u - v^l|^2_{\mathbb{H}^2(\Gamma_\nu^{l,j})} \\
&\quad + \frac{4}{3}\,\Lambda_l\,2^{-l}\,|u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} + 4\left\{\frac{1}{3}\,2^{-2l}\Lambda_l + \frac{1}{\lambda_l}\right\} 2^l\,\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)} \\
&\le \frac{3}{4}\,2^{-3l} \sum_{j\nu} |u - v^l|^2_{\mathbb{H}^2(\Gamma_\nu^{l,j})} + \frac{4}{3}\,\Lambda_l\,2^{-l}\,|u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} + 8\cdot 2^l\,\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)} \\
&\le \frac{3}{2}\,2^{-3l}\,|u|^2_{\mathbb{H}^2(\Gamma^l)} + \frac{3}{2}\,2^{-3l} \sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\Gamma_\nu^{l,j})} \\
&\quad + \frac{4}{3}\,\Lambda_l\,2^{-l}\,|u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} + 8\cdot 2^l\,\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)}. \qquad (5.30)
\end{aligned}
$$

The estimates (5.26), (5.28) and (5.30) yield the error bound claimed. □

5.5.2 Reinterpolation of functions from $\mathfrak{C}^2_\beta(\mathrm{Q})$

Theorem 5.16 (reinterpolation)**.** *Let assumption 5.1 hold. Assume that $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$ in the sense of definition 3.4 with positive constants C_u and δ. Then there exists a positive constant $\hat{C} > 0$ such that, for all $\mathcal{B} \subset \{1,2\}^{2\times 2}$ such that μ and $\mathcal{B}$ satisfy assumption 4.1 and for all $l \ge 2$, the function u^l defined in (5.2) with $\alpha = 3/2$ satisfies the bound*

$$
\|u - u^l\|_{\mathbb{H}^1(\mathrm{Q})} \le \hat{C}\,l^3\,2^{-(1-\beta)l}.
$$

Proof. The proof consists in combining the bounds of theorems 5.13 and 5.14 and lemma 5.15.

First, consider Q_i^l with $i \in \mathcal{E}_0^l$, where $\mathcal{E}_0^l$ is the index set given by (5.23). By the trace theorem (consider (A-8.11) with $\varkappa = 1/2$ and rescaled to Q_i^l with $i \in \mathcal{E}_0^l$), we have

$$
\begin{aligned}
|u|^2_{\mathbb{H}^2(\Gamma^l)} = \sum_{i\in\mathcal{E}_0^l} |u|^2_{\mathbb{H}^2(\gamma_i^l)} &\le 2\,\frac{\Lambda_l^4}{\lambda_l^4}\,\frac{1}{\lambda_l\,2^{-l-1}} \sum_{i\in\mathcal{E}_0^l} \left\{|u|^2_{\mathbb{H}^2(\mathrm{Q}_i^l)} + (\Lambda_l\,2^{-l-1})^2\,|u|^2_{\mathbb{H}^3(\mathrm{Q}_i^l)}\right\} \\
&\le \frac{2^9}{9}\,\frac{\Lambda_l^4}{\lambda_l^4}\,\frac{1}{\lambda_l\,2^{-l-1}}\,C_u^2\,\delta_u^4 \sum_{i\in\mathcal{E}_0^l} \left\{\|\varPhi^{-1}_{\beta+1}\|^2_{\mathbb{L}^2(\mathrm{Q}_i^l)} + 9\,\delta_u^2\,(\Lambda_l\,2^{-l-1})^2 \|\varPhi^{-1}_{\beta+2}\|^2_{\mathbb{L}^2(\mathrm{Q}_i^l)}\right\} \\
&\le \frac{2^{11}}{9}\,\frac{\Lambda_l^5}{\lambda_l^5}\,C_u^2\,\delta_u^4 \left\{\frac{(\lambda_l\,2^{-l-1})^{-(2\beta+1)}}{2\beta+1} + 9\,\delta_u^2\,(\Lambda_l\,2^{-l-1})^2\,\frac{(\lambda_l\,2^{-l-1})^{-(2\beta+3)}}{2\beta+3}\right\} \\
&\le \frac{2^{14}}{3}\,\frac{\Lambda_l^5}{\lambda_l^8}\,\frac{C_u^2\,\delta_u^4}{2\beta+3}\,2^{(2\beta+1)l} \left\{1 + 3\,\delta_u^2\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}, \qquad (5.31)
\end{aligned}
$$

where we bounded the standard Sobolev seminorms using (5.19a) similarly to as for (5.19). Summing with respect to $i \in \mathcal{E}_0^l$ and applying definition 3.4, we obtain By theorems 5.13 and 5.14, there exist positive constants C_1, C_2, c_0, c_1, c_2 and $\tilde{C}_1, \tilde{C}_2 > 0$ such that, for every $l \geq 2$ and for p given by (5.13), the discontinuous hp approximation $v^l = \tilde{\Pi}^{l,\mu}_{\delta,3/2,\beta} u$ and the lifting term $w^l = \tilde{\Pi}^{l,\mu}_{\delta,3/2,\beta} u - \tilde{\Pi}^{l,\mu}_{\delta,3/2,\beta} u$ sum to a continuous function $\xi^l = v^l + w^l \in \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$, where p is given by (5.13), and satisfy the error bounds (5.20), (5.21) and (5.22). Combining those bounds and (5.31) with (5.24) of lemma 5.15, we arrive at

$$\begin{aligned}\|u - u^l\|_{\mathbb{H}^1(\mathrm{Q})} &\leq 4\, C_1^2\, l^3\, 2^{-2(1-\beta)l} + 4\, \tilde{C}_1^2\, l^6\, 2^{-2(1-\beta)l} + 8\, \frac{\Lambda_l^4}{\lambda_l^2}\, l^2\, 2^{-2(1-\beta)l}\, (C_2^2 + \tilde{C}_2^2) \\ &\quad + 6\left[\frac{2^{14}}{3}\, \frac{\Lambda_l^5}{\lambda_l^8}\, \frac{C_u^2\, \delta_u^4}{2\beta+3}\left\{1 + 3\, \delta_u^2\, \frac{\Lambda_l^2}{\lambda_l^2}\right\} + c_2^2\, l^2\right] 2^{-2(1-\beta)l} \\ &\quad + \frac{16}{3}\, \Lambda_l\, c_1^2\, l^6\, 2^{-2(1-\beta+1+\beta_*-\beta)l} + 32\, c_0^2\, l^4\, 2^{-2(1-\beta)l} \leq \hat{C}^2\, l^6\, 2^{-(1-\beta)l}\end{aligned}$$

holds with any positive $\hat{C}$ such that

$$\begin{aligned}\hat{C}^2 &\geq 4\, C_1^2\, l^{-3} + 4\, \tilde{C}_1^2 + 8\, \frac{\Lambda_l^4}{\lambda_l^2}\, (C_2^2 + \tilde{C}_2^2)\, l^{-4} \\ &\quad + 6\left[\frac{2^{14}}{3}\, \frac{\Lambda_l^5}{\lambda_l^8}\, \frac{C_u^2\, \delta_u^4}{2\beta+3}\left\{1 + 3\, \delta_u^2\, \frac{\Lambda_l^2}{\lambda_l^2}\right\} l^{-6} + c_2^2\, l^{-4}\right] \\ &\quad + \frac{16}{3}\, \Lambda_l\, c_1^2\, 2^{-l} + 32\, c_0^2\, l^{-2},\end{aligned}$$

where the expression on the right-hand side is monotonically decreasing with respect to l. □

5.6 Rank structure of the unfolding matrices of the coefficient vector

Above in this section, we have studied the approximation of functions with a singularity at the origin. The approximation results generalize immediately to functions with singularities at $\theta \in \{0, 1\}^2$, which is an arbitrary vertex of Q. Indeed, we can map θ into the origin by the reflection of coordinates, apply the approximation results of this section and transform the hp approximation back. In this section, we consider the evaluation of an hp function under such a reflection of coordinates and analyze the QTT structure of the corresponding vector of values.

Let us consider polynomials $\hat{\psi}_0, \hat{\psi}_1 \in \mathbb{P}_1$: $\hat{\psi}_0(t) = t$ and $\hat{\psi}_1(t) = 1 - t$ for all $t \in [0, 1]$. We introduce a reflection function:

$$\psi = \psi_1 \otimes \psi_2 \quad \text{with} \quad \psi_k = \hat{\psi}_{\theta_k}, \quad k = 1, 2, \tag{5.32}$$

so that $\psi \circ \psi = \hat{\psi}_0 \circ \hat{\psi}_0$ and $\psi(\theta) = 0$, $\psi(0) = \theta$.

Lemma 5.17. *Consider $p, l \in \mathbb{N}$ and let $P_{\alpha_1\alpha_2}$, $\alpha_1, \alpha_2 = 0, \ldots, p$ form a basis of $\mathbb{Q}_{p,p}$. Assume that $\hat{u}^l \in \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$, $\theta \in \{0,1\}^2$ and $\boldsymbol{u}^l \in \mathbb{R}^{2^l \times 2^l}$ is given by*

$$\boldsymbol{u}^l_{i_1 i_2} = \hat{u}^l \circ \psi(t^l_{i_1 i_2})$$

for all $i_1, i_2 = 0, \ldots, 2^l - 1$. Then, for every $q = 1, \ldots, l-1$ and for all $\xi_1, \xi_2 = 0, \ldots, 2^q - 1$ except when $\xi_1 = \theta_1(2^q - 1)$ and $\xi_2 = \theta_2(2^q - 1)$, there exist coefficients $C_{\xi_1\xi_2\alpha_1\alpha_2}$, $\alpha_1, \alpha_2 = 0, \ldots, p$, such that

$$\boldsymbol{u}^l_{2^{l-q}\xi_1+\eta_1,\, 2^{l-q}\xi_2+\eta_2} = \sum_{\alpha_1,\alpha_2=0}^{p} C_{\xi_1\xi_2\alpha_1\alpha_2}\, P_{\alpha_1\alpha_2}(\eta_1, \eta_2) \tag{5.33}$$

for all $\eta_1, \eta_2 = 0, \ldots, 2^{l-q} - 1$.

Proof. First, we consider the case of $\theta = 0$. Let $q \in \{1, \ldots, l-1\}$ be arbitrary. For either $k = 1, 2$ and $\xi_k = 1, \ldots, 2^q - 1$, let us set $j_k = \lfloor \log_2 \xi_k \rfloor + 1 + l - q$. Then we have $1 \leq j_k \leq l$ and $2^{j_k - 1} \leq 2^{l-q}\xi_k$ and $2^{l-q}(\xi_k + 1) \leq 2^{j_k + q - l}$. That results in $x_k^{l,j_k} \leq h_k^l(2^{l-q}\xi_k + \eta_k + 1 - \mu_{k1}) \leq x_k^{l,j_k}$, i.e. $t^l_{k,2^{l-q}\xi_k+\eta_k} \in \mathrm{J}^{l,j_k}_{k,1}$, for all $\eta_k = 0, \ldots, 2^{l-q} - 1$. Therefore, if $(\xi_1, \xi_2) \neq 0$, there exist $j \in 1, \ldots, l$ and $\nu \in \mathcal{N}$ such that $t^l_{2^{l-q}\xi_1+\eta_1, 2^{l-q}\xi_2+\eta_2} \in \mathbf{cl}\, \mathrm{G}^{l,j}_\nu$ for all $\eta_k = 0, \ldots, 2^{l-q} - 1$, i.e. the nodes corresponding to fixed ξ_1 and ξ_2 belong to the same element of $\mathcal{G}^l$. Since $\hat{u}^l \in \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$, we have $\hat{u}^l|_{\mathbf{cl}\,\mathrm{G}^{l,j}_\nu} \in \mathbb{Q}_{p,p}$, and the coefficients $C_{\xi_1\xi_2\alpha_1\alpha_2}$, $\alpha_1, \alpha_2 = 0, \ldots, p$, satisfying (5.33) do exist.

Let us now consider the case of an arbitrary $\theta \in \{0,1\}^2$. Reflecting the kth coordinate of the function $u = \hat{u}^l \circ \psi$ corresponds to flipping the kth index i_k of the vector. We represent i_k by two indices: ξ_k and η_k. In (5.33), flipping the index ξ_k means reordering the coefficients $C_{\xi_1\xi_2\alpha_1\alpha_2}$, whereas flipping η_k results in composing the basis polynomials with an affine transformation. The transformed basis polynomials, however, can also be represented in the same polynomial basis. The structure given by (5.33) is therefore invariant under the reflection of coordinates, and the case of an arbitrary $\theta \in \{0,1\}^2$ reduces to that of $\theta = 0$. □

Lemma 5.17 describes the structure of the row space of each of the unfolding matrices $\boldsymbol{U}^q$, $q = 1, \ldots, l-1$, corresponding to the QTT format (4.16) with the transposed ordering (2.6). In the setting of this section, we have $\mathfrak{n} = 1$, and the index $\mathfrak{i} \equiv 0$ is void. A particular result of lemma 5.17 is that the ranks of the unfolding matrices are bounded by $1 + (p+1)^2$ for every $\theta \in \{0,1\}^2$ and for all $l \in \mathbb{N}$. Applying remark 2.1, we obtain the following statement.

Corollary 5.18. *Consider $p, l \in \mathbb{N}$. Assume that $\hat{u}^l \in \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$, $\theta \in \{0,1\}^2$. Then the vector $\boldsymbol{u}^l \in \mathbb{R}^{2^l \times 2^l}$ given by*

$$\boldsymbol{u}^l_{i_1 i_2} = \hat{u}^l \circ \psi(t^l_{i_1 i_2})$$

for all $i_1, i_2 = 0, \dots, 2^l - 1$ admits a QTT representation of the form (4.16) with ranks bounded from above by $1 + (p+1)^2$.

Lemma 5.17 is closely related to the following notable result: a vector of values of a univariate polynomial of degree p at 2^l uniformly distributed points admits a QTT representation with ranks bounded by $p+1$ independently of l [27, corollary 13]. The same result can also be obtained in a constructive way, see [51, theorem 6]. In [39, lemma 3.7], the result was generalized in the spirit of [27, theorem 18] to cover the case of piecewise-polynomial functions (still univariate). Lemma 5.17 and corollary 5.18 given above is a further extension of the result, to the case of bivariate functions. For hp-functions supported on similar partitions of $(0,1)^d$ graded geometrically towards a vertex, an analogous result, with the exponent d in the rank bound, follows in the same way.

5.7 Exponential convergence of QTT-FE approximations

Applying theorem 5.16 and corollary 5.18 to the approximations given by equation (5.2), we obtain the main result of the present work: for $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$, the set $\mathscr{F}^l_u$ given by (4.13a) contains an infinite sequence of approximations u^l, $l \in \mathbb{N}$, which converge exponentially with respect to the number of QTT parameters needed to *exactly* represent their coefficients $\boldsymbol{u}^l \in \mathbb{R}^{2^l \times 2^l}$, $l \in \mathbb{N}$.

Theorem 5.19 (exponential convergence of QTT-FE approximations)**.** *Let $\beta \in [0,1)$ and assume that μ and $\mathcal{B}$ satisfy assumption 4.1 and $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$. Then there exist positive constants C and c and $u^l \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l)$, $l \in \mathbb{N}$, such that, for every $l \in \mathbb{N}$, the error bound*

$$\|u - u^l\|_{\mathbb{H}^1(\mathrm{Q})} \le C\, l^3\, 2^{-(1-\beta)l}$$

holds and the coefficient vector $\boldsymbol{u}^l = \dot{\mathscr{A}}^l u^l$ admits a QTT representation of the form (4.16) with ranks $r_1, \dots, r_{l-1} \le c l^2$, i.e.

$$\|u - u^l\|_{\mathbb{H}^1(\mathrm{Q})} = \mathcal{O}\Big(l^3 \exp\big(-b N_l^{1/5}\big)\Big), \quad l \to \infty,$$

holds with $N_l^{1/5}$ given by (4.17) and a positive constant b independent of l.

Proof. The accuracy and rank bounds for $\boldsymbol{u}^l$, $l \in \mathbb{N}$, given by (4.14b) follow immediately from theorem 5.16 and corollary 5.18. Then, by (4.17), the number N_l of QTT parameters needed to represent $\boldsymbol{u}^l$ satisfies $N_l \le 4 l c^2 l^4$ for every $l \in \mathbb{N}$. That yields the exponential convergence with respect to $N_l^{1/5}$. □

6 Numerical Experiment

In the numerical experiments presented below, we study solutions $u \in \mathfrak{B}^2_\beta(\Omega) \subset \mathfrak{C}^2_\beta(\Omega)$ of the form

$$u(x) = r^\alpha(x)\,\sin(\alpha\varphi(x)) \quad \text{for all} \quad x \in \Omega, \tag{6.1}$$

where (r, φ) is the transformation to the standard polar coordinates, $\alpha > 0$ and Ω is a polygon such that $0 \in \boldsymbol{\partial}\Omega$. For every non-integral positive α, the function u given by (6.1) exhibits a singularity at the origin. For all $\alpha > 0$, we have $u \in \mathfrak{B}^2_\beta(\mathrm{Q})$ with $\beta \in (0,1)$ if and only if $\beta \in (1-\tilde{\alpha}, 1)$, where $\tilde{\alpha} = \min\{\alpha, 1\}$. In that sense, the analyticity class $\mathfrak{B}^\ell_\beta(\mathrm{Q})$ with $\ell = 2$ is suitable for quantifying the singularity strength for all $\alpha \in (0,1)$.

6.1 QTT-FE approximation

In this section, we illustrate the approximability of functions $u \in \mathfrak{B}^2_\beta(\Omega)$ given by (6.1) for $\Omega = \mathrm{Q} = (0,1)^2$ with QTT-FE functions, which are described in section 4. We consider $\alpha = \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}, \frac{7}{8}, \frac{3}{2}$. We set $\mu^0_{k\jmath} = 1$ for all $k, \jmath = 1,2$ and consider the corresponding uniform partition $\mathcal{T}^l$ of Q. We impose no boundary conditions: $\mathcal{B}^0 = \varnothing$.

For each α, we construct $u^l_{\mathrm{appr}} \in \mathrm{S}^1(\mathrm{Q}, \mathcal{T}^l)$, a QTT-FE approximation of u, whose coefficient vector $\boldsymbol{u}^l_{\mathrm{appr}} = \dot{\mathscr{A}}^l u^l_{\mathrm{appr}}$ has a low-rank QTT decomposition of the form (4.16). As a measure of approximation error, we use the relative $\mathbb{H}^1$-seminorm:

$$\varepsilon_l = |u^l_{\mathrm{appr}} - u|_{\mathbb{H}^1(\mathrm{Q})} \,/\, |u|_{\mathbb{H}^1(\mathrm{Q})}.$$

The results are presented in figure 1.

In each case, as figure 1a shows, the approximations u^l_{appr} achieve the convergence $\varepsilon_l \sim \aleph_l^{-\tilde{\alpha}/2} = 2^{-\tilde{\alpha}\, l}$, with the optimal rate $\tilde{\alpha} = \min\{\alpha, 1\}$. That rate is the same as suggested by lemma 5.2 and theorem 5.19. The convergence is algebraic with respect to $\aleph_l = 4^l$ (4.13c) and exponential with respect to l.

As figure 1c indicates, the QTT ranks of $\boldsymbol{u}^l_{\mathrm{appr}}$ grow sublinearly with respect to l. This is considerably better than the quadratic bound of corollary 5.18. A more refined observation, based on figure 1d and going in the same direction, is that for the number of QTT parameters N_l (4.17) we have $N_l \sim l^\kappa$ with $\kappa \approx 2.33$ instead of $\kappa = 5$ as in theorem 5.19. Those two dependencies amount to the exponential convergence of the QTT-FE approximations with respect to N_l, shown in figure 1b: $\log_2 \varepsilon_l^{-1} \sim N_l^{1/\kappa}$ with $\kappa < 3$. This exponent, observed numerically, is superior to $\kappa = 5$ of the theoretical error bound. It is even slightly better than $\kappa = 3$ of hp approximations in two dimensions [60]. We note that hp approximations in two dimensions may, in principle, achieve convergence with a better exponent than $\kappa = 3$, e.g. $\kappa = 2$ was proved [36] for approximation in hp spaces based on harmonic polynomials with respect to accuracy measured in a broken norm.

6.2 QTT-FEM

In this section, we demonstrate our approach as a *method* of solving the model problem (3.7) with $\mathcal{A} = -\Delta$ and $f = 0$. We consider solutions $u \in \mathfrak{B}^2_\beta(\Omega)$ given by (6.1) in the following three cases: $\alpha = \frac{2}{3}$ in an L-shaped domain and $\alpha = \frac{1}{2} \pm \frac{1}{4}$ in a domain with a cut.

- $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ with $\gamma_\mathrm{D} = \boldsymbol{\partial}\,\Omega$ and $\gamma_\mathrm{N} = \varnothing$, which we split Ω into $\mathfrak{n} = 3$ square patches;
- $\Omega = (-1,1)^2 \setminus [0,1) \times \{0\}$ with $\gamma_\mathrm{D} = \boldsymbol{\partial}(-1,1)^2 \cup [0,1] \times \{+0\}$ (including the top side of the cut) and $\gamma_\mathrm{N} = (0,1) \times \{-0\}$ (the bottom side of the cut), which we split Ω into $\mathfrak{n} = 4$ square patches.

6.2.1 Description of the experiment

In either domain, we consider the corresponding sets of admissible approximations $\mathscr{F}^l_u$ with $l \in \mathbb{N}$, see equation (4.13a).

First, similarly to section 6.1, we study the approximability of the solution u with QTT-FE functions, similarly to as described in section 4.1. The difference consists in the introduction of an additional index $\mathfrak{i}$ running from 0 to $\mathfrak{n}-1$ and indicating the patches. In each example, we have $u \in \mathfrak{B}^2_\beta(\Omega) \subset \mathfrak{C}^2_\beta(\Omega)$ for all $\beta \in (1-\alpha, 1)$. We construct $u^l_\mathrm{appr} \in \mathscr{F}^l_u$, a QTT-FE approximation of u, whose coefficient vector $\boldsymbol{u}^l_\mathrm{appr} \in \mathbb{R}^{\mathfrak{n} \times 2^l \times 2^l}$ has low QTT ranks in the sense of (4.16).

Second, we approximately solve the model problem in the weak formulation (3.7) with the variational space is and the FEM space

$$V = \mathbb{H}^1_0(\Omega) = \left\{u \in \mathbb{H}^1_0(\Omega) : u|_{\gamma_\mathrm{D}} = 0\right\}.$$

For each l, we choose the offset term $\mathfrak{I}^l_0$ in place of u_0 of (3.6) and denote the FEM space with V^l. We solve the discretization (3.9) as a linear system of the form

$$\boldsymbol{A}^l \boldsymbol{u}^l_\mathrm{sol} = \boldsymbol{f}^l \tag{6.2}$$

using the solver `amen_solve2` from the TT Toolbox[2] [53]. The solver implements the AMEn method for the TT-structured solution of linear systems, developed in [25]. Among the parameters of the solver are the maximum rank of the solution sought, the maximum number of iterations ("sweeps") and the target relative ℓ^2 norm of the corresponding residual, which we indiscriminately set equal to 50, 500 and 10^{-10} for all

[2]We use the master branch of the GitHub version 2.2+ of July 24, 2014 (git tag http://github.com/oseledets/TT-Toolbox/tree/v2.3-4-ge1a3f2c) .

α and l. Finally, we truncate every QTT-FE solution u^l_{sol} to obtain its approximation $u^l_{\mathrm{tr}} \in \mathscr{F}^l_u$ with smaller QTT ranks and such that

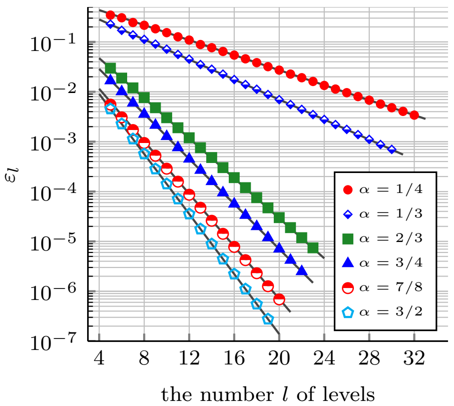
$$\|u^l_{\mathrm{tr}} - u^l_{\mathrm{sol}}\|_{\mathbb{L}^2(\Omega)} \leq 0.05 \cdot 2^{-(1+\alpha)l} \, \|w - u\|_{\mathbb{L}^2(\Omega)}, \tag{6.3}$$

where w, in every patch, is a bilinear function interpolating u at the vertices of the patch. This allows to adapt the low-rank structure of the FE solution with respect to the $\mathbb{L}^2$-error (and, due to the Markov brothers' inequality, to the error measured in the $\mathbb{H}^1$-norm), whereas the AMEn solver minimizes the ℓ^2 norm of the residual.

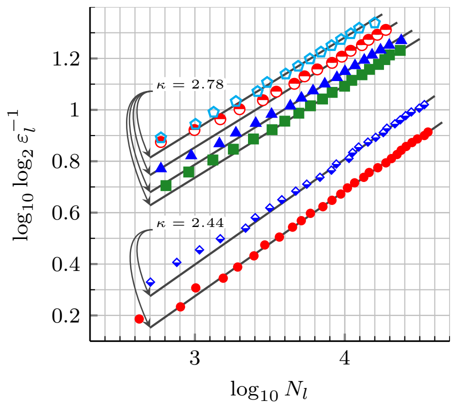
6.2.2 Discussion of the results

The results are presented in figure 2. For all u^l_{appr}, u^l_{sol} and u^l_{tr}, we measure the error with respect to the exact solution u by the relative $\mathbb{H}^1$-seminorm, $\varepsilon_l = |\cdot - u|_{\mathbb{H}^1(\mathrm{Q})} \, / \, |u|_{\mathbb{H}^1(\mathrm{Q})}$.

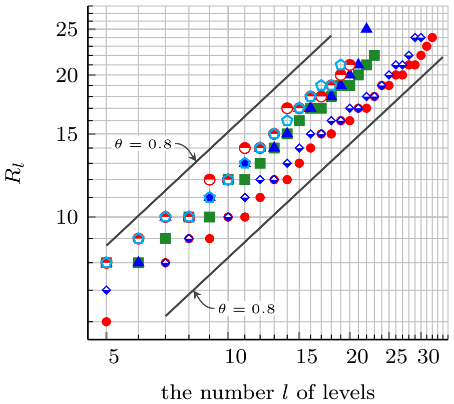
First, we note that the observations made regarding u^l_{appr} in section 6.1 remain true in this case, where Ω consists of multiple patches. Second, the QTT-FE solutions u^l_{sol} exhibit approximately the same convergence with respect to l (figure 2a) and N_l (figure 2b). For small l, as figures 2c and 2d show, the AMEn solver significantly overestimates the ranks of the QTT-FE solution u^l_{sol}. This is due to that the target residual norm is inadequately small for those runs. For larger l, the behavior observed for u^l_{appr} reappears. The rank truncation subject to (6.3) optimizes the QTT structure of the FE solution, so that the maximum rank R_l, the number of QTT parameters N_l and the convergence with respect to N_l for u^l_{tr} are much closer to those of u^l_{appr}. As in section 6.1 for u^l_{appr}, we observe the exponential convergence of u^l_{appr}, u^l_{sol} and u^l_{tr} to u with respect to $N_l^{1/\kappa}$ with $\kappa < 3$, see figure 2b: $\log_2^\kappa \varepsilon_l^{-1} \sim N_l$. Again, the exponent is slightly better than for general hp approximations in two dimensions, with $\kappa = 3$ [60].



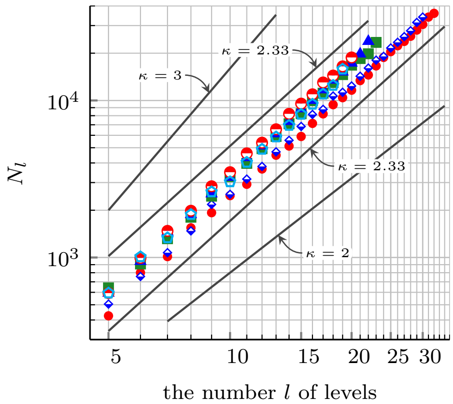
(a) Convergence with respect to l. The reference lines correspond to the exponential convergence $\varepsilon_l = C_\alpha\, 2^{-\tilde{\alpha}\, l}$ with C_α independent of l and with $\tilde{\alpha} = \min\{\alpha, 1\}$.



(b) Convergence with respect to N_l (4.17). The reference lines correspond to the exponential convergence $\log_2^\kappa \varepsilon_l = -b_\alpha\, N_l$ with κ and b_α independent of l.

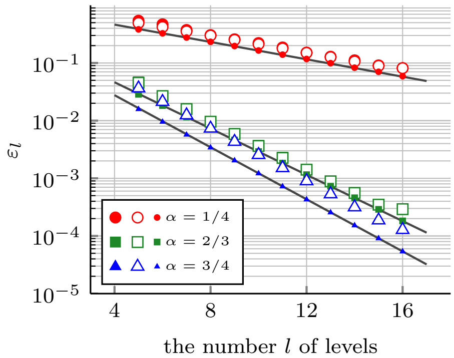


(c) The maximum QTT rank R_l (4.17) of $\boldsymbol{u}^l_{\mathrm{appr}}$ vs. l. The reference lines correspond to the algebraic growth $R_l = c_\alpha\, l^\theta$ with θ and c_α independent of l.

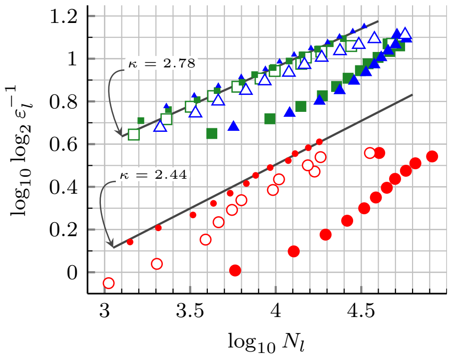


(d) The number N_l (4.17) of QTT parameters vs. l. The reference lines correspond to the algebraic growth $N_l = C_\alpha\, l^\kappa$ with κ and C_α independent of l.

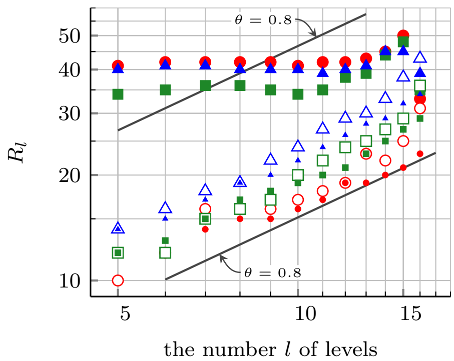
Figure 1: QTT-FE approximation with u^l_{appr} in $\mathrm{Q} = (0,1)^2$ for $\alpha = \frac{1}{4}$ ●, $\frac{1}{3}$ ◇, $\frac{2}{3}$ ■, $\frac{3}{4}$ ▲, $\frac{7}{8}$ ◓, $\frac{3}{2}$ ⬠. **Top**. Convergence to u with respect to the number l of levels and to the number N_l (4.17) of QTT parameters. The error is $\varepsilon_l = |u^l_{\mathrm{appr}} - u|_{\mathbb{H}^1(\mathrm{Q})} \,/\, |u|_{\mathbb{H}^1(\mathrm{Q})}$. **Bottom**. The QTT structure of u^l_{appr} vs. l.



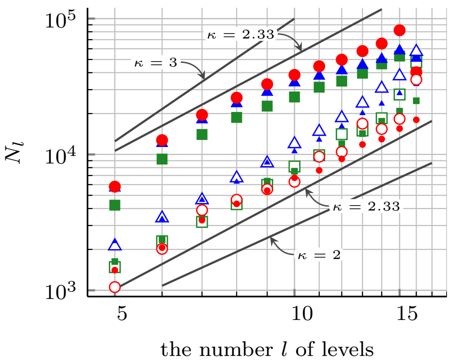
(a) Convergence with respect to l. The reference lines correspond to the exponential convergence $\varepsilon_l = C_\alpha\, 2^{-\alpha l}$ with C_α independent of l. The markers for u^l_{sol} and u^l_{tr} mostly coincide.



(b) Convergence with respect to N_l (4.17). The reference lines correspond to the exponential convergence $\log_2^\kappa \varepsilon_l = -b_\alpha\, N_l$ with κ and b_α independent of l.



(c) The maximum QTT rank R_l (4.17) vs. l. The reference lines correspond to the algebraic growth $R_l = c_\alpha\, l^\theta$ with θ and c_α independent of l.



(d) The number N_l (4.17) of QTT parameters vs. l. The reference lines correspond to the algebraic growth $N_l = C_\alpha\, l^\kappa$ with κ and C_α independent of l.

Figure 2: QTT-FEM for $\alpha = \frac{1}{4}$ ● ○ • (the domain with a cut), $\alpha = \frac{2}{3}$ ■ □ ▪ (the L-shaped domain) and $\alpha = \frac{3}{4}$ ▲ △ ▴ (the domain with a cut): QTT-FE solutions u^l_{sol} ● ■ ▲, lower-rank truncations u^l_{tr} ○ □ △ of u^l_{sol}, QTT-FE approximations u^l_{appr} • ▪ ▴ of u. **Top**. Convergence to u with respect to the number l of levels and to the number N_l (4.17) of QTT parameters. The error is $\varepsilon_l = |\cdot - u|_{\mathbb{H}^1(\Omega)} \,/\, |u|_{\mathbb{H}^1(\Omega)}$. **Bottom**. The QTT structure of the solutions vs. l.

7 Conclusion

We summarize the main findings of the present paper. For a function $u \in \mathfrak{C}^2_\beta(\mathrm{Q})$, where $\mathrm{Q} = (0,1)^2$, there exists QTT-FE approximations u^l, $l \in \mathbb{N}$, that satisfy two properties. First, as elements of the corresponding FE spaces, they are quasi-optimal in the sense that, in the energy norm, they realize the optimal algebraic convergence rate $1-\beta$ with respect to the mesh size. Second, the coefficient vectors can be represented in the QTT format with the number of parameters N_l that is polylogarithmic with respect to the mesh size. This means that the analytic regularity exhibited by the functions from $\mathfrak{C}^2_\beta(\mathrm{Q})$, although irrelevant for the convergence of low-order Courant FE approximations with respect to the number of degrees of freedom, is implicitly encoded in the low-rank tensor structure of the coefficient vectors. Furthermore, that regularity may be recovered by the QTT representation and results in exponential convergence with respect to $N^{1/\kappa}$, where $\kappa > 1$.

These results are established theoretically with $\kappa = 5$ in theorem 5.19 and are observed experimentally with $\kappa < 3$ in section 6. An analogous exponent for *hp*-FE approximations is $\kappa = 3$, and the QTT-FE approach turns out to be comparable.

Storing the coefficients implicitly, in a low-parametric form of the QTT decomposition, allows for both accurate approximation and efficient computation. The mesh underlying the FE space becomes *virtual* in the sense that the solution is never represented or processed locally, i.e. with respect to single finite elements. The number of levels of the virtual mesh, however, remains an important, “real” parameter. First, it limits the approximation properties of the QTT-FE approximations. Second, it is equal to the number of cores in the QTT representations of the approximations.

For very fine discretizations (large l), the nodal approximation considered in the present paper suffers *in practice* from the ill-conditioning of the nodal bases, which prohibited the solution of the Galerkin system with larger l in the experiments presented in section 6.2. The approximation results obtained in the present paper, however, carry over to $\mathbb{H}^1$-stable tensor-product wavelet bases, which promise to be more efficient for the solution of boundary-value problems using the QTT-FEM, see, e.g. the recent works [10, 11].

A-8 Appendix

A-8.1 Auxiliary lemmas

Lemma A-8.1. *Let $n \in \mathbb{N}$. Then for the sum of the inverses of the corresponding $n+1$ binomial coefficients*

$$\sum_{k=0}^{n} \binom{n}{k}^{-1} \leq \frac{8}{3}\,. \tag{A-8.1}$$

Proof. For $n \in \mathbb{N}$, let us denote the left-hand side of (A-8.1) by I_n. For every $n \in \mathbb{N}$, we have the recurrence relation $I_{n+1} = (n+2)/(2n+2)\, I_n + 1$, see [57, theorem 1]. By induction, starting from $n = 3$, it follows that $I_n \geq 2(1 + n^{-1})$ for all $n \geq 3$. Then $I_{n+1} \leq I_n + (2n+2)^{-1} I_n - 1$ for all $n \geq 3$. Again by induction, we obtain $I_n \leq I_3 = 8/3$ for all $n > 3$. Since $I_1, I_2 \leq I_3$, that proves the claim. $\square$

As above, for all $p \in \mathbb{N}$ and $s = 0, 1, \ldots, p$, we use the notation

$$\Upsilon_{ps} = \frac{(p-s)!}{(p+s)!}.$$

Lemma A-8.2. *For all $\varrho > 1$ and $p \in \mathbb{N}$ such that $p \geq \varrho$, let us set $s = \lfloor p/\varrho \rfloor$, so that $1 \leq s \leq p$. Then the bound*

$$(\varrho - 1)^{2s} (s!)^2\, \Upsilon_{ps} \leq c^2\, s\, \exp\left(-\frac{2p}{\varrho}\right)$$

holds with $c^2 = e^5/\sqrt{2\pi}$.

Proof. Using Stirling's bound for the Euler's Gamma function, we obtain the bounds

$$\Upsilon_{ps} \leq \frac{e\,(p-s)^{p-s+\frac{1}{2}}\, e^{-(p-s)}}{\sqrt{2\pi}\,(p+s)^{p+s+\frac{1}{2}}\, e^{-(p+s)}} = \frac{e}{\sqrt{2\pi}} \left(\frac{p-s}{p+s}\right)^{p-s+\frac{1}{2}} \frac{e^{2s}}{(p+s)^{2s}},$$
$$(s!)^2 \leq e^2\, s^{2s+1}\, e^{-2s},$$

which yield together

$$(\varrho - 1)^{2s} (s!)^2\, \Upsilon_{ps} \leq \frac{e^3}{\sqrt{2\pi}}\, s \left(\frac{p-s}{p+s}\right)^{p-s+\frac{1}{2}} \left[(\varrho - 1)\frac{s}{p+s}\right]^{2s} \leq .$$

For $s = \lfloor p/\varrho \rfloor$, we have $p/\varrho - 1 < s \leq p/\varrho$, so that $(\varrho - 1)s \leq p - s$. We denote $t = 2s/(p+s)$ and, using that $(1-t)^{1/t} < e^{-1}\sqrt{1-t}$ holds for $0 < t \leq 1$, obtain

$$\begin{aligned}(\varrho - 1)^{2s} (s!)^2\, \Upsilon_{ps} \leq \frac{e^3}{\sqrt{2\pi}}\, s \left(\frac{p-s}{p+s}\right)^{p+s+\frac{1}{2}} &\leq \frac{e^3}{\sqrt{2\pi}} \sqrt{\frac{p-s}{p+s}}\, s\, (1-t)^{2s/t} \\ &\leq \frac{e^3}{\sqrt{2\pi}} \sqrt{\frac{p-s}{p+s}}\, s\, \exp(-2s) \leq \frac{e^5}{\sqrt{2\pi}}\, s\, \exp\left(-\frac{2p}{\varrho}\right).\end{aligned}$$

$\square$

A-8.2 Bounds for univariate quasi-interpolation

We shall use the following bound, for which one may refer to either of [60, Corollary 3.15] or [15, Corollary 2]

Proposition A-8.3. *Assume that $p \in \mathbb{N}$ and $s \in \mathbb{N}_0$ are such that $s \le p$. Then, for any function $\hat{u} \in \mathbb{H}^{s+1}(\hat{\mathrm{J}})$, the interpolant $\hat{\pi}_p \hat{u}$ satisfies the error bounds*

$$\begin{aligned} |\hat{u} - \hat{\pi}_p \hat{u}|^2_{\mathbb{H}^1(\hat{\mathrm{J}})} &\le \Upsilon_{ps}\, |\hat{u}|^2_{\mathbb{H}^{s+1}(\hat{\mathrm{J}})}, \\ \|\hat{u} - \hat{\pi}_p \hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} &\le \frac{1}{p\,(p+1)} \Upsilon_{ps}\, |\hat{u}|^2_{\mathbb{H}^{s+1}(\hat{\mathrm{J}})}. \end{aligned} \tag{A-8.2}$$

We shall also use the following stability bound on the second derivative of the interpolant.

Lemma A-8.4. *For every $p \in \mathbb{N}$ and for every $u \in \mathbb{H}^2(\hat{\mathrm{J}})$, the following bounds hold:*

$$|\hat{\pi}_p \hat{u}|^2_{\mathbb{H}^2(\hat{\mathrm{J}})} \le \frac{1}{4}\, p^2(p^2-1)|\hat{u}|^2_{\mathbb{H}^1(\hat{\mathrm{J}})}, \quad |\hat{\pi}_p \hat{u}|^2_{\mathbb{H}^2(\hat{\mathrm{J}})} \le \frac{1}{2}(p^2-1)|\hat{u}|^2_{\mathbb{H}^2(\hat{\mathrm{J}})}. \tag{A-8.3}$$

Proof. Consider $\mathbb{L}^2_w(\hat{\mathrm{J}})$, a weighted space of square-integrable functions defined on $\hat{\mathrm{J}}$, with the weight w given by $w(x) = 1 - x^2$ for all $x \in \hat{\mathrm{J}}$. For this weight, we have $\hat{u}'' \in \mathbb{L}^2_w(\hat{\mathrm{J}})$ with $\|\hat{u}''\|_{\mathbb{L}^2_w(\hat{\mathrm{J}})} \le \|\hat{u}''\|_{\mathbb{L}^2(\hat{\mathrm{J}})}$. Note that L_i', $i \in \mathbb{N}$, are orthogonal in $\mathbb{L}^2_w(\hat{\mathrm{J}})$, namely

$$\langle L_i', L_{i'}' \rangle_{\mathbb{L}^2_w(\hat{\mathrm{J}})} = \frac{\delta_{ii'}}{i+\frac{1}{2}} \frac{(i+1)!}{(i-1)!} \quad \text{for all} \quad i, i' \in \mathbb{N}.$$

Below, we shall also use that $\|L_i'\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} = i(i+1)$ for every $i \in \mathbb{N}_0$. That follows readily from integration by parts and the orthogonality of the Legendre polynomials, as in [60, theorem 3.91]:

$$\int_{-1}^{1} L_i' L_i' = L_i L_i'|_{-1}^{1} - \int_{-1}^{1} L_i L_i'' = L_i L_i'|_{-1}^{1} = 2\, L_i(1)\, L_i'(1) = i(i+1).$$

Since $\hat{u}' \in \mathbb{L}^2(\hat{\mathrm{J}})$, we have a Legendre representation $\hat{u}' = \sum_{i=0}^{\infty} c_i L_i$ in $\mathbb{L}^2(\hat{\mathrm{J}})$ with coefficients c_i, $i \in \mathbb{N}_0$. Then $\hat{u}'' = \sum_{i=1}^{\infty} c_i L_i'$ holds in $\mathbb{L}^2(\hat{\mathrm{J}})$ and, thus, also in $\mathbb{L}^2_w(\hat{\mathrm{J}})$. This results in the bound

$$\sum_{i=1}^{\infty} \frac{1}{i+\frac{1}{2}} \frac{(i+1)!}{(i-1)!} |c_i|^2 = \|\hat{u}''\|^2_{\mathbb{L}^2_w(\hat{\mathrm{J}})} \le \|\hat{u}''\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}. \tag{A-8.4}$$

By Definition 5.3, we have $(\hat{\pi}_p \hat{u})' = \sum_{i=0}^{p-1} c_i L_i$, and by the triangle inequality we obtain

$$\|(\hat{\pi}_p \hat{u})''\|_{\mathbb{L}^2(\hat{\mathrm{J}})} = \Big\|\sum_{i=1}^{p-1} c_i L_i'\Big\|_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq \sum_{i=1}^{p-1} \sqrt{i\Big(i+\frac{1}{2}\Big)(i+1)} \cdot \left[\frac{c_i^2}{i+\frac{1}{2}}\right]^{\frac{1}{2}} \tag{A-8.5}$$

$$= \sum_{i=1}^{p-1} \sqrt{i+\frac{1}{2}} \cdot \left[\frac{c_i^2}{i+\frac{1}{2}} \frac{(i+1)!}{(i-1)!}\right]^{\frac{1}{2}}. \tag{A-8.6}$$

Finally, we apply the Cauchy–Bunyakovsky–Schwarz inequality to (A-8.5) and (A-8.6) to arrive at the bounds

$$\|(\hat{\pi}_p \hat{u})''\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq \|\hat{u}'\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \sum_{i=1}^{p-1} i\Big(i+\frac{1}{2}\Big)(i+1) = \frac{1}{4}\, p^2(p^2-1)\|\hat{u}'\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})},$$

$$\|(\hat{\pi}_p \hat{u})''\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq \|\hat{u}''\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \sum_{i=1}^{p-1} \Big(i+\frac{1}{2}\Big) = \frac{1}{2}(p^2-1)\|\hat{u}''\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}.$$

□

By rescaling from an interval I to $\hat{\mathrm{J}} = (-1, 1)$, from proposition A-8.3 and lemma A-8.4 we obtain the following statement.

Corollary A-8.5. *Consider an interval* $\mathrm{I} = (2a, 2a+2h)$ *with* $0 < h < \infty$. *Let* φ *be an affine map from* $\hat{\mathrm{J}}$ *onto* I. *Then, for all* $p, s \in \mathbb{N}$ *such that* $s \leq p$, *for every* $u \in \mathbb{H}^{s+1}(\mathrm{G})$ *and for* $v \in \mathbb{Q}_{p,p}$ *given by* $v \circ \varphi = \hat{\pi}_{k,p}\,(u \circ \varphi)$, *the following inequalities hold:*

$$\|u - v\|^2_{\mathbb{L}^2(\mathrm{I})} \leq \frac{1}{p\,(p+1)}\, h^{2(s+1)}\, \Upsilon_{ps}\, |u|^2_{\mathbb{H}^{s+1}(\mathrm{I})},$$

$$|u - v|^2_{\mathbb{H}^1(\mathrm{I})} \leq h^{2s}\, \Upsilon_{ps}\, |u|^2_{\mathbb{H}^{s+1}(\mathrm{I})},$$

$$|v|^2_{\mathbb{H}^2(\mathrm{I})} \leq \frac{1}{4}\, \frac{p^2(p^2-1)}{2h}\, |u|^2_{\mathbb{H}^1(\mathrm{I})}, \quad |v|^2_{\mathbb{H}^2(\mathrm{I})} \leq \frac{1}{2}\,(p^2-1)\, |u|^2_{\mathbb{H}^2(\mathrm{I})}.$$

A-8.3 Bounds for tensor-product bivariate quasi-interpolation

We shall use the following error bound for the projection $\hat{\Pi}_{p_1,p_2}$. For similar bounds for tensor-product interpolation operators, we also refer to [60, lemma 4.67] and [15, theorem 5].

Lemma A-8.6. *Let $p = (p_1, p_2) \in \mathbb{N}^2$ and $q_1, q_2, r_1, r_2, s_1, s_2 \in \mathbb{N}_0$ be such that $q_1, r_1, s_1 \le p_1$ and $q_2, r_2, s_2 \le p_2$. Then the following bounds hold true for every $\hat{u} \in \mathbb{H}^{q_1+1,2}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \cap \mathbb{H}^{2,q_2+1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \cap \mathbb{H}^{r_1+1,r_2+1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \cap \mathbb{H}^{s_1+1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \cap \mathbb{H}^{1,s_2+1}_{\mathrm{mix}}(\hat{\mathrm{Q}})$:*

$$\begin{aligned}
\|\hat{u} - \hat{\mathit{\Pi}}_{p_1p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} &\le 3\sum_{k=1}^{2} \frac{1}{p_k(p_k+1)} \Upsilon_{p_k s_k} \|\partial_k^{s_k+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \\
&\quad + 3\left[\prod_{k=1}^{2} \frac{1}{p_k(p_k+1)} \Upsilon_{p_k r_k}\right] \|\partial_1^{r_1+1}\partial_2^{r_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}, \\
|\hat{u} - \hat{\mathit{\Pi}}_{p_1p_2}\hat{u}|^2_{\mathbb{H}^1(\hat{\mathrm{Q}})} &\le 2\sum_{k=1}^{2} \Upsilon_{p_k s_k} \|\partial_k^{s_k+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \\
&\quad + \frac{2}{p_1(p_1+1)} \Upsilon_{p_1 q_1} \|\partial_1^{q_1+1}\partial_2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \\
&\quad + \frac{2}{p_2(p_2+1)} \Upsilon_{p_2 q_2} \|\partial_1\partial_2^{q_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}, \\
|\hat{\mathit{\Pi}}_{p_1p_2}\hat{u}|^2_{\mathbb{H}^2(\hat{\mathrm{Q}})} &\le (p_1^2-1)\|\partial_1^2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + (p_2^2-1)\|\partial_2^2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \\
&\quad + \left\{1 + \frac{1}{2}\frac{p_1^2}{p_2}\frac{p_1^2-1}{p_2+1} + \frac{1}{2}\frac{p_2^2}{p_1}\frac{p_2^2-1}{p_1+1}\right\}\|\partial_1\partial_2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}.
\end{aligned}$$

Proof. Note that the inequality

$$\|\partial_k \hat{\pi}_{k,p_k}\hat{v}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \le \|\partial_k \hat{v}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \quad \text{for every} \quad \hat{v} \in \mathbb{H}^{1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \tag{A-8.7}$$

follows from Definition 5.3.

Let $\hat{u} \in \mathbb{H}^{s_1+1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \cap \mathbb{H}^{1,s_2+1}_{\mathrm{mix}}(\hat{\mathrm{Q}})$. Since $\hat{\mathit{\Pi}}_{p_1p_2} = \hat{\pi}_{1,p_1} \circ \hat{\pi}_{2,p_2}$, we have the decomposition

$$\begin{aligned}
\hat{u} - \hat{\mathit{\Pi}}_{p_1p_2}\hat{u} &= (\mathbf{id} - \hat{\pi}_{1,p_1})\hat{u} + \hat{\pi}_{1,p_1}(\mathbf{id} - \hat{\pi}_{2,p_2})\hat{u} \\
&= (\mathbf{id} - \hat{\pi}_{1,p_1})\hat{u} + (\mathbf{id} - \hat{\pi}_{2,p_2})\hat{u} - (\mathbf{id} - \hat{\pi}_{1,p_1})(\mathbf{id} - \hat{\pi}_{2,p_2})\hat{u}.
\end{aligned} \tag{A-8.8}$$

$\mathbb{L}^2$ error bound. By the triangle inequality, from (A-8.8) we obtain

$$\begin{aligned}
\|\hat{u} - \hat{\mathit{\Pi}}_{p_1p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \le 3\|\hat{u} - \hat{\pi}_{1,p_1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + 3\|\hat{u} - \hat{\pi}_{2,p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \\
+ 3\|(\mathbf{id} - \hat{\pi}_{1,p_1})(\hat{u} - \hat{\pi}_{2,p_2}\hat{u})\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}.
\end{aligned}$$

Using proposition A-8.3, we bound the first two terms as follows: for every $k \in \{1, 2\}$, we have

$$\|\hat{u} - \hat{\pi}_{k,p_k}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \le \frac{1}{p_k(p_k+1)} \Upsilon_{p_k s_k} \|\partial_k^{s_k+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \tag{A-8.9}$$

for every $s_k \in \mathbb{N}_0$ such that $s_k \leq p_k$. Similarly, for the third term we obtain

$$\|(\mathbf{id} - \hat{\pi}_{1,p_1})(\hat{u} - \hat{\pi}_{2,p_2}\hat{u})\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \leq \frac{1}{p_1(p_1+1)}\Upsilon_{p_1 r_1}\,\|\partial_1^{r_1+1}(\hat{u} - \hat{\pi}_{2,p_2}\hat{u})\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \leq \left[\prod_{k=1}^{2}\frac{1}{p_k(p_k+1)}\Upsilon_{p_k r_k}\right]\|\partial_1^{r_1+1}\partial_2^{r_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}$$

for all $r_1, r_2 \in \mathbb{N}_0$ such that $r_1 \leq p_1$ and $r_2 \leq p_2$. By combining the bounds for the three terms, we obtain the $\mathbb{L}^2$-norm estimate claimed.

$\mathbb{H}^1$ error bound. We use (A-8.8), the triangle inequality, (A-8.7) and proposition A-8.3 to arrive at

$$\begin{aligned}\|\partial_1(\hat{u} - \hat{\Pi}_{p_1p_2}\hat{u})\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} &\leq 2\,\|\partial_1(\mathbf{id} - \hat{\pi}_{1,p_1})\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + 2\,\|\partial_1\hat{\pi}_{1,p_1}(\mathbf{id} - \hat{\pi}_{2,p_2})\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}\\ &\leq 2\,\|\partial_1(\mathbf{id} - \hat{\pi}_{1,p_1})\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + 2\,\|(\mathbf{id} - \hat{\pi}_{2,p_2})\partial_1\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}\\ &\leq 2\,\Upsilon_{p_1 s_1}\,\|\partial_1^{s_1+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + \frac{2}{p_2(p_2+1)}\,\Upsilon_{p_2 q_2}\,\|\partial_2^{q_2+1}\partial_1\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}.\end{aligned}$$

An analogous bound holds for $\|\partial_2(\hat{u} - \hat{\Pi}_{p_1p_2}\hat{u})\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}$. Together, the two bounds prove the $\mathbb{H}^1$-norm error estimate claimed.

$\mathbb{H}^2$-stability estimate. Let us decompose the interpolant as follows:

$$\hat{\Pi}_{p_1p_2}\hat{u} = \hat{\pi}_{1,p_1}\hat{u} - (\mathbf{id} - \hat{\pi}_{2,p_2})\hat{\pi}_{1,p_1}\hat{u}. \tag{A-8.10}$$

Using the triangle inequality and the properties of the interpolation operators, we obtain

$$\begin{aligned}\|\partial_1^2\hat{\Pi}_{p_1p_2}\hat{u}\|_{\mathbb{L}^2(\hat{\mathrm{J}})} &\leq \|\partial_1^2(\mathbf{id} - \hat{\pi}_{2,p_2})\hat{\pi}_{1,p_1}\hat{u}\|_{\mathbb{L}^2(\hat{\mathrm{J}})} + \|\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|_{\mathbb{L}^2(\hat{\mathrm{J}})}\\ &= \|(\mathbf{id} - \hat{\pi}_{2,p_2})\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|_{\mathbb{L}^2(\hat{\mathrm{J}})} + \|\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|_{\mathbb{L}^2(\hat{\mathrm{J}})}\end{aligned}$$

For these two terms, let us use the corresponding bounds of lemma A-8.4. For the second, we obtain $\|\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq \frac{1}{2}(p_1^2 - 1)\|\partial_1^2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}$. For the first term, we use the error bound of proposition A-8.3: $\|(\mathbf{id} - \hat{\pi}_{2,p_2})\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq p_2^{-1}(p_2 + 1)^{-1}\|\partial_2\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} = p_2^{-1}(p_2+1)^{-1}\|\partial_1^2\hat{\pi}_{1,p_1}\partial_2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}$. Then, by the first bound of lemma A-8.4, the inequality $\|(\mathbf{id} - \hat{\pi}_{2,p_2})\partial_1^2\hat{\pi}_{1,p_1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq p_2^{-1}(p_2+1)^{-1}p_1^2/4\,(p_1^2 -$

$1)\|\partial_1\partial_2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}$ holds true. By combining the bounds for the two terms, we arrive at

$$\|\partial_1^2 \hat{\mathit{\Pi}}_{p_1 p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq (p_1^2-1)\|\partial_1^2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} + \frac{1}{2}\frac{p_1^2}{p_2}\frac{p_1^2-1}{p_2+1}\|\partial_1\partial_2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}.$$

An analogous estimate follows for $\|\partial_2^2\hat{\mathit{\Pi}}_{p_1p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}$. For the mixed derivative, (A-8.7) allows to obtain

$$\begin{aligned}\|\partial_1\partial_2\hat{\mathit{\Pi}}_{p_1p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} = \|\partial_1\hat{\pi}_{1,p_1}\partial_2\hat{\pi}_{2,p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} &\leq \|\partial_1\partial_2\hat{\pi}_{2,p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \\ &= \|\partial_2\hat{\pi}_{2,p_2}\partial_1\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \leq \|\partial_2\partial_1\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})}.\end{aligned}$$

Together, the bounds on the second-order derivatives yield the stability estimate. □

Corollary A-8.7. *Consider a rectangle* $\mathrm{G} = (2a_1, 2a_1+2h_1)\times(2a_2, 2a_2+2h_2)$ *with* $\lambda h \leq h_1, h_2 \leq \Lambda h$*, where* $h > 0$*,* $0 < \lambda \leq 1$ *and* $\Lambda \geq 1$*. Let* φ *be an affine map from* $\hat{\mathrm{Q}}$ *onto* G*. Then, for all* $p, s \in \mathbb{N}$ *such that* $s \leq p$*, for every* $u \in \mathbb{H}^{s+2}(\mathrm{G})$ *and for* $v \in \mathbb{Q}_{p,p}$ *given by* $v\circ\varphi = \hat{\mathit{\Pi}}_{p,p}\,(u\circ\varphi)$*, the following inequalities hold:*

$$\begin{aligned}\|u-v\|^2_{\mathbb{L}^2(\mathrm{G})} &\leq 3\,(\Lambda h)^{2(s+1)}\,\frac{\Upsilon_{ps}}{p\,(p+1)}\,|u|^2_{\mathbb{H}^{s+1}(\mathrm{G})}, \\ |u-v|^2_{\mathbb{H}^1(\mathrm{G})} &\leq 4\,\frac{\Lambda^2}{\lambda^2}\,(\Lambda h)^{2s}\,\Upsilon_{ps}\,|u|^2_{\mathbb{H}^{s+1}(\mathrm{G})}, \\ |v|^2_{\mathbb{H}^2(\mathrm{G})} &\leq \frac{\Lambda^4}{\lambda^4}\,(p^2-1)\,|u|^2_{\mathbb{H}^2(\mathrm{G})}.\end{aligned}$$

Proof. The statement follows by a rescaling argument from lemma A-8.6 with $s_1 = s_2 = s$ and $r_1 = s-1$, $r_2 = 0$. □

Lemma A-8.8. *Consider* $p_1, p_2 \in \mathbb{N}$ *and* $s_2 \in \mathbb{N}_0$ *such that* $s_2 \leq p_2$*. Then the following bounds on* $\hat{\Gamma}$*, the left edge of* $\hat{\mathrm{Q}}$*, hold for every* $\hat{u} \in \mathbb{H}^{1,s_2+1}_{\mathrm{mix}}(\hat{\mathrm{Q}})$*:*

$$\begin{aligned}\|\hat{u}-\hat{\mathit{\Pi}}_{p_1p_2}\hat{u}\|^2_{\mathbb{L}^2(\hat{\Gamma})} &\leq \frac{3}{2}\frac{\Upsilon_{p_2s_2}}{p_2(p_2+1)}\left\{\|\partial_2^{s_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + \|\partial_1\partial_2^{s_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}\right\}, \\ |\hat{u}-\hat{\mathit{\Pi}}_{p_1p_2}\hat{u}|^2_{\mathbb{H}^1(\hat{\Gamma})} &\leq \frac{3}{2}\,\Upsilon_{p_2s_2}\left\{\|\partial_2^{s_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + \|\partial_1\partial_2^{s_2+1}\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}\right\}, \\ |\hat{\mathit{\Pi}}_{p_1p_2}\hat{u}|^2_{\mathbb{H}^2(\hat{\Gamma})} &\leq \frac{3}{4}\,(p_2^2-1)\left\{\|\partial_2^2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + \|\partial_1\partial_2^2\hat{u}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})}\right\}.\end{aligned}$$

Proof. First, we note that

$$\|\hat{v}\|^2_{\mathbb{L}^2(\hat{\Gamma})} \leq \frac{1}{2}(1+\varkappa^{-1})\|\hat{v}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} + (1+\varkappa)\|\partial_1\hat{v}\|^2_{\mathbb{L}^2(\hat{\mathrm{Q}})} \quad \text{for all} \quad \hat{v} \in \mathbb{H}^{1,1}_{\mathrm{mix}}(\hat{\mathrm{Q}}) \qquad \text{(A-8.11)}$$

holds with every $\varkappa > 0$. This explicit form of the trace theorem follows immediately from the formula $\hat{v}(-1,y) = \hat{v}(x,y) - \int_{-1}^{x} \partial_1 \hat{v}(t,y)\,\mathrm{d}t$, valid for all $(x,y) \in \hat{\mathrm{Q}}$, and from the Cauchy–Bunyakovsky–Schwarz inequality for $\mathbb{R}^2$ and $\mathbb{L}^2(\hat{\mathrm{J}})$.

The interpolation property of $\hat{\pi}_{1,p_1}$ and $\hat{\pi}_{2,p_2}$ implies the relations

$$\partial_2^{\alpha_2} \hat{\mathit{\Pi}}_{p_1 p_2} \hat{u}(-1,y) = \hat{\pi}_{1,p_1} \partial_2^{\alpha_2} \hat{\pi}_{2,p_2} \hat{u}(-1,y) = \partial_2^{\alpha_2} \hat{\pi}_{2,p_2} \hat{u}(-1,y) = (\hat{\pi}_{p_2}[\hat{u}(-1,\cdot)])^{(\alpha_2)}(y)$$

for all $y \in \hat{\mathrm{J}}$ and $\alpha_2 \in \{0,1,2\}$. Therefore, by proposition A-8.3 and lemma A-8.4, the following inequalities hold:

$$\begin{aligned}
\|\hat{u} - \hat{\mathit{\Pi}}_{p_1 p_2} \hat{u}\|^2_{\mathbb{L}^2(\hat{\Gamma})} &= \|\hat{u}(-1,\cdot) - \hat{\pi}_{p_2} \hat{u}(-1,\cdot)\|^2_{\mathbb{L}^2(\hat{\mathrm{J}})} \\
&\le \frac{\Upsilon_{p_2 s_2}}{p_2(p_2+1)} \|\partial_2^{s_2+1} \hat{u}\|^2_{\mathbb{L}^2(\hat{\Gamma})}, \\
|\hat{u} - \hat{\mathit{\Pi}}_{p_1 p_2} \hat{u}|^2_{\mathbb{H}^1(\hat{\Gamma})} &= |\hat{u}(-1,\cdot) - \hat{\pi}_{p_2} \hat{u}(-1,\cdot)|^2_{\mathbb{H}^1(\hat{\mathrm{J}})} \le \Upsilon_{p_2 s_2} \|\partial_2^{s_2+1} \hat{u}\|^2_{\mathbb{L}^2(\hat{\Gamma})}, \\
|\hat{\mathit{\Pi}}_{p_1 p_2} \hat{u}|^2_{\mathbb{H}^2(\hat{\Gamma})} &= |\hat{\pi}_{p_2} \hat{u}(-1,\cdot)|^2_{\mathbb{H}^2(\hat{\mathrm{J}})} \le \frac{1}{2}\,(p_2^2 - 1) \|\partial_2^2 \hat{u}\|^2_{\mathbb{L}^2(\hat{\Gamma})}.
\end{aligned}$$

Then, applying inequality (A-8.11) with $\varkappa = \frac{1}{2}$, we obtain the claim. □

Corollary A-8.9. *Consider a rectangle* $\mathrm{G} = (2a_1, 2a_1 + 2h_1) \times (2a_2, 2a_2 + 2h_2) \subset \mathrm{Q}$. *Let* γ *denote the left edge of* G *and* φ *be an affine map from* $\hat{\mathrm{Q}}$ *onto* G.

Then, for all $p \in \mathbb{N}$ *and* $s \in \mathbb{N}_0$ *such that* $s \le p$, *for every* $u \in \mathbb{H}^{s+3}(\mathrm{G})$ *and for* $v \in \mathbb{Q}_{p,p}$ *given by* $v \circ \varphi = \hat{\mathit{\Pi}}_{p,p}\,(u \circ \varphi)$, *the following inequalities hold:*

$$\begin{aligned}
\|u - v\|^2_{\mathbb{L}^2(\gamma)} &\le \frac{3}{2} \frac{h_2^{2(s+1)}}{h_1} \frac{\Upsilon_{ps}}{p(p+1)} \Big\{ \|\partial_2^{s+1} u\|^2_{\mathbb{L}^2(\mathrm{G})} + h_1^2 \,\|\partial_1 \partial_2^{s+1} u\|^2_{\mathbb{L}^2(\mathrm{G})} \Big\}, \\
|u - v|^2_{\mathbb{H}^1(\gamma)} &\le \frac{3}{2} \frac{h_2^{2s}}{h_1} \Upsilon_{ps} \Big\{ \|\partial_2^{s+1} u\|^2_{\mathbb{L}^2(\mathrm{G})} + h_1^2 \,\|\partial_1 \partial_2^{s+1} u\|^2_{\mathbb{L}^2(\mathrm{G})} \Big\}, \\
|v|^2_{\mathbb{H}^2(\gamma)} &\le \frac{3}{4} \frac{p^2 - 1}{h_1} \Big\{ \|\partial_2^2 u\|^2_{\mathbb{L}^2(\mathrm{G})} + h_1^2 \,\|\partial_1 \partial_2^2 u\|^2_{\mathbb{L}^2(\mathrm{G})} \Big\}.
\end{aligned}$$

Proof. Follows from lemma A-8.8 by a rescaling argument. □

Proposition A-8.10. *Consider a rectangle* $\mathrm{G} = (2a_1, 2a_1 + 2h_1) \times (2a_2, 2a_2 + 2h_2) \subset \mathrm{Q}$. *Let* $w \in \mathbb{P}_1$ *be the polynomial satisfying* $w(2a_1) = 1$ *and* $w(2a_1 + 2h_1) = 0$.

Then, for every $v \in \mathbb{P}_p$ *with* $p \in \mathbb{N}$, *the polynomial* $\xi = w \otimes v \in \mathbb{Q}_{1,p}$ *satisfies*

$$\|\xi\|^2_{\mathbb{L}^2(\mathrm{G})} = \frac{2h_1}{3} \|v\|^2_{\mathbb{L}^2(\gamma)} \quad \text{and} \quad |\xi|^2_{\mathbb{H}^{m+1}(\mathrm{G})} = \frac{2h_1}{3} |v|^2_{\mathbb{H}^{m+1}(\gamma)} + \frac{1}{2h_1} |v|^2_{\mathbb{H}^m(\gamma)},$$

with $m = 0, 1$.

For a square with 0 as a vertex, we shall use the following approximation result.

Proposition A-8.11. *Consider a rectangle* $\mathrm{Q} = (0,1)^2$ *and* $\beta \in [0,1)$. *Then there exist positive constants* D_0 *and* D_1 *such that, for every* $u \in \mathbb{H}^{2,2}_{\beta}(\mathrm{Q})$, *the polynomial* $v \in \mathbb{Q}_{1,1}$ *interpolating* u *at the vertices of* Q *is well-defined and satisfies the error bounds*

$$\|u - v\|^2_{\mathbb{L}^2(\mathrm{Q})} \leq D_0^2 \, |u|^2_{\mathbb{H}^{2,2}_{\beta}(\mathrm{Q})} \quad \textit{and} \quad |u - v|^2_{\mathbb{H}^1(\mathrm{Q})} \leq D_1^2 \, |u|^2_{\mathbb{H}^{2,2}_{\beta}(\mathrm{Q})}.$$

Proof. By proposition 3.1, the function u admits a continuous extension to $\mathbf{cl}\,\mathrm{Q}$. That ensures that the interpolant is well defined. The error estimates follow from [30, lemma 3.6], see also [60, lemma 4.25]. □

A-8.4 Proofs of theorems for *hp* quasi-interpolation

A-8.4.1 Lemma 5.8

Proof. For an arbitrary function $u \in \mathbb{H}^{1,1}_{\mathrm{mix}}(\mathrm{Q})$, we shall define a lifting term $w^l \in \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ so that $\tilde{\Pi}^l_p u + w^l \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$ and the mapping $u \mapsto w^l$ is linear.

For $1 \leq j \leq l$, let γ_1^j and $\tilde{\gamma}_1^j$ denote the right edges of $\mathrm{G}^{l,j}_{11}$ and $\mathrm{G}^{l,j}_{10}$ respectively, γ_2^j and $\tilde{\gamma}_2^j$, the top edges of $\mathrm{G}^{l,j}_{11}$ and $\mathrm{G}^{l,j}_{01}$ respectively, $\tilde{\Gamma}_1^j$ and $\tilde{\Gamma}_2^j$, the left edge of $\mathrm{G}^{l,j}_{10}$ and the bottom edge of $\mathrm{G}^{l,j}_{0,1}$ respectively.

Let us set consider the polynomial interpolants

$$v^j_\nu = (\tilde{\Pi}^l_p u)|_{\mathrm{G}^{l,j}_\kappa} \in \mathbb{Q}_{p,p} \quad \text{with} \quad j = 1, \ldots, l \quad \text{and} \quad \nu \in \mathcal{N},$$
$$v^0 = (\tilde{\Pi}^l_p u)|_{\mathrm{G}^{l,0}} \in \mathbb{Q}_{1,1}$$

given in definition 5.7. For $1 \leq j < l$ and $k \in \{1,2\}$, we define linear univariate polynomials $\psi_k^j \in \mathbb{P}_1$ by requiring

$$\psi_k^j(x_k^{l,j}) = 0 \quad \text{and} \quad \psi_k^j(x_k^{l,j+1}) = 1.$$

Using these as factors, we introduce bivariate lifting polynomials $\eta_1^j, \tilde{\eta}_1^j \in \mathbb{P}_{1,p}$ and $\eta_2^j, \tilde{\eta}_2^j \in \mathbb{P}_{p,1}$ with $1 \leq j < l$ by setting

$$\begin{aligned} \eta_1^j &= \psi_1^j \otimes (v^{j+1}_{1,0} - v^j_{1,1})|_{\gamma_1^j}, & \tilde{\eta}_1^j &= \psi_1^j \otimes (v^{j+1}_{1,0} - v^j_{1,0})|_{\tilde{\gamma}_1^j} \\ \eta_2^j &= (v^{j+1}_{0,1} - v^j_{1,1})|_{\gamma_2^j} \otimes \psi_2^j, & \tilde{\eta}_2^j &= (v^{j+1}_{0,1} - v^j_{0,1})|_{\tilde{\gamma}_2^j} \otimes \psi_2^j. \end{aligned} \tag{A-8.12a}$$

Additionally, we introduce lifting polynomials $\zeta_1 \in \mathbb{P}_{1,p}$ and $\zeta_2 \in \mathbb{P}_{p,1}$:

$$\zeta_1 = (1 - \psi_1^1) \otimes (v^0 - v^1_{1,0})|_{\tilde{\Gamma}_1^1}, \quad \zeta_2 = (v^0 - v^1_{0,1})|_{\tilde{\Gamma}_2^1} \otimes (1 - \psi_2^1). \tag{A-8.12b}$$

From the lifting polynomials defined in (A-8.12), we construct the lifting term $w^l \in \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$:

$$w^l = \mathbb{1}_{\mathrm{G}_{10}^{l,1}} \zeta_1 + \mathbb{1}_{\mathrm{G}_{01}^{l,1}} \zeta_2 + \sum_{j=1}^{l-1} \Big\{ \mathbb{1}_{\mathrm{G}_{11}^{l,j}} \eta_1^j + \mathbb{1}_{\mathrm{G}_{11}^{l,j}} \eta_2^j + \mathbb{1}_{\mathrm{G}_{10}^{l,j}} \tilde{\eta}_1^j + \mathbb{1}_{\mathrm{G}_{01}^{l,j}} \tilde{\eta}_2^j \Big\}. \quad \text{(A-8.13)}$$

By Definition 5.5, the interpolants of u in any two elements sharing an entire edge coincide on that edge. Then, due to the nodal exactness of interpolation in each element, the construction of (A-8.12) ensures that, first, w^l vanishes in $\mathrm{G}^{l,0}$ and on $\boldsymbol{\partial}\,\mathrm{Q}$ and, second, the lifted interpolant extends to a function continuous across the edges of the elements of $\mathcal{G}^l$: $w^l + \tilde{\Pi}_p^l u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$.

Since the operator $\tilde{\Pi}_p^l$ is linear, so is the mapping $u \mapsto w^l$. This allows to define a linear operator $\Pi_p^l \colon \mathbb{H}^{1,1}_{\mathrm{mix}}(\mathrm{Q}) \to \mathrm{S}^p(\mathrm{Q}, \mathcal{G}^l)$ on u by setting $\Pi_p^l u = \tilde{\Pi}_p^l u + w^l$. □

A-8.4.2 Lemma 5.10

Proof. By proposition A-8.11 and a rescaling argument, the interpolant v^l satisfies the error bounds

$$\|u - v^l\|^2_{\mathbb{L}^2(\mathrm{G}^{l,0})} \le D_0^2 \, \frac{\Lambda_l^4}{\lambda_l^2} \, 2^{-2(2-\beta)l} \, |u|^2_{\mathbb{H}^{2,2}_\beta(\mathrm{G}^{l,0})}, \quad \text{(A-8.14a)}$$

$$|u - v^l|^2_{\mathbb{H}^1(\mathrm{G}^{l,0})} \le D_1^2 \, \frac{\Lambda_l^4}{\lambda_l^4} \, 2^{-2(1-\beta)l} \, |u|^2_{\mathbb{H}^{2,2}_\beta(\mathrm{G}^{l,0})} \quad \text{(A-8.14b)}$$

with positive constants D_0 and D_1 depending only on β.

For all $j = 1, \ldots, l$, $\nu \in \mathcal{N}$ and $s \in \mathbb{N}$ such that $s \le p$, corollary A-8.7 yields the following:

$$\begin{aligned} \|u - v^l\|^2_{\mathbb{L}^2(\mathrm{G}_\nu^{l,j})} &\le 3\,(2^{j-2-l}\Lambda_l)^{2s+2} \, \frac{\Upsilon_{ps}}{p\,(p+1)} \, |u|^2_{\mathbb{H}^{s+1}(\mathrm{G}_\nu^{l,j})}, \\ |u - v^l|^2_{\mathbb{H}^1(\mathrm{G}_\nu^{l,j})} &\le 4\, \frac{\Lambda_l^2}{\lambda_l^2} \,(2^{j-2-l}\Lambda_l)^{2s} \, \Upsilon_{ps} \, |u|^2_{\mathbb{H}^{s+1}(\mathrm{G}_\nu^{l,j})}, \\ |v^l|^2_{\mathbb{H}^2(\mathrm{G}_\nu^{l,j})} &\le \frac{\Lambda_l^4}{\lambda_l^4} \,(p^2-1)\, |u|^2_{\mathbb{H}^2(\mathrm{G}_\nu^{l,j})}. \end{aligned} \quad \text{(A-8.15)}$$

By combining the inequalities of (A-8.14) and (A-8.15), we obtain the bounds of (5.15).

Finally, for all $j = 1, \ldots, l$ and $\nu \in \mathcal{N}$ such that $\Gamma_\nu^{l,j}$ of (5.9a) is nonempty, we apply corollary A-8.9 to $\mathrm{G}_\nu^{l,j}$ and $\Gamma_\nu^{l,j}$:

$$\|u - v^l\|^2_{\mathbb{L}^2(\Gamma_\nu^{l,j})} \leq \frac{3}{2}\,\frac{\Lambda_l}{\lambda_l}\,(2^{j-2-l}\Lambda_l)^{2s+1}\,\frac{\Upsilon_{ps}}{p(p+1)}\Big\{|u|^2_{\mathbb{H}^{s+1}(\mathrm{G}_\nu^{l,j})} + (2^{j-2-l}\Lambda_l)^2\,|u|^2_{\mathbb{H}^{s+2}(\mathrm{G}_\nu^{l,j})}\Big\},$$

$$|u - v^l|^2_{\mathbb{H}^1(\Gamma_\nu^{l,j})} \leq \frac{3}{2}\,\frac{\Lambda_l}{\lambda_l}\,(2^{j-2-l}\Lambda_l)^{2s-1}\,\Upsilon_{ps}\Big\{|u|^2_{\mathbb{H}^{s+1}(\mathrm{G}_\nu^{l,j})} + (2^{j-2-l}\Lambda_l)^2\,|u|^2_{\mathbb{H}^{s+2}(\mathrm{G}_\nu^{l,j})}\Big\},$$

$$|v^l|^2_{\mathbb{H}^2(\Gamma_\nu^{l,j})} \leq \frac{3}{4}\,\frac{p^2-1}{2^{j-2-l}\lambda_l}\Big\{|u|^2_{\mathbb{H}^2(\mathrm{G}_\nu^{l,j})} + (2^{j-2-l}\Lambda_l)^2\,|u|^2_{\mathbb{H}^3(\mathrm{G}_\nu^{l,j})}\Big\}.$$

By summing over all $j = 1, \ldots, l$ and $\nu \in \mathcal{N}$, we obtain (5.16). □

A-8.4.3 Lemma 5.11

Proof. Using the Cauchy–Bunyakovsky–Schwarz inequality in each $\mathrm{G}_\nu^{l,j}$, we may bound the lifting term w^l of lemma 5.8 as follows:

$$\sum_{j=1}^{l-1}\sum_{\nu\in\mathcal{N}}|w^l|^2_{\mathbb{H}^m(\mathrm{G}_\nu^{l,j})} \leq 2\Big\{|\zeta_1|^2_{\mathbb{H}^m(\mathrm{G}_{10}^{l,1})} + |\zeta_2|^2_{\mathbb{H}^m(\mathrm{G}_{01}^{l,1})}\Big\} + 2\sum_{j=1}^{l-1}\sum_{\nu\in\mathcal{N}}\Big\{|\tilde\eta_1^j|^2_{\mathbb{H}^m(\mathrm{G}_{10}^{l,j})} + |\tilde\eta_2^j|^2_{\mathbb{H}^m(\mathrm{G}_{01}^{l,j})} + |\eta_1^j|^2_{\mathbb{H}^m(\mathrm{G}_{11}^{l,j})} + |\eta_2^j|^2_{\mathbb{H}^m(\mathrm{G}_{11}^{l,j})}\Big\} \tag{A-8.16}$$

for $m = 0, 1, 2$.

Applying proposition A-8.10 and the Cauchy–Bunyakovsky–Schwarz inequality on edges of the elements, we obtain from (A-8.13) and (A-8.16) that

$$\|w^l\|^2_{\mathbb{L}^2(\mathrm{Q})} \leq \frac{4}{3}\,2^{-l}\Lambda_l\Big\{\|u - v^0\|^2_{\mathbb{L}^2(\tilde\Gamma_1^1)} + \|u - v^1_{1,0}\|^2_{\mathbb{L}^2(\tilde\Gamma_1^1)} + \|u - v^0\|^2_{\mathbb{L}^2(\tilde\Gamma_2^1)} + \|u - v^1_{0,1}\|^2_{\mathbb{L}^2(\tilde\Gamma_2^1)}\Big\}$$
$$+ \sum_{j=1}^{l-1}\frac{4}{3}\,2^{j-1-l}\Lambda_l\Big\{\|u - v^j_{1,0}\|^2_{\mathbb{L}^2(\tilde\gamma_1^j)} + \|u - v^j_{1,1}\|^2_{\mathbb{L}^2(\gamma_1^j)} + \|u - v^{j+1}_{1,0}\|^2_{\mathbb{L}^2(\tilde\Gamma_1^{j+1})} + \|u - v^j_{0,1}\|^2_{\mathbb{L}^2(\tilde\gamma_2^j)} + \|u - v^j_{1,1}\|^2_{\mathbb{L}^2(\gamma_2^j)} + \|u - v^{j+1}_{0,1}\|^2_{\mathbb{L}^2(\tilde\Gamma_2^{j+1})}\Big\}.$$

Rearranging the terms and using corollary A-8.9, we arrive at

$$
\begin{aligned}
\|w^l\|^2_{\mathbb{L}^2(\mathrm{Q})} \leq \frac{4}{3}\, 2^{-l}\Lambda_l \,\|u - v^0\|^2_{\mathbb{L}^2(\tilde{\Gamma}^1_1 \cup \tilde{\Gamma}^1_2)} \\
+ \frac{4}{3} \sum_{j=1}^{l} 2^{j-1-l}\Lambda_l \Big\{ \frac{1}{2}\,\|u - v^j_{1,0}\|^2_{\mathbb{L}^2(\tilde{\Gamma}^j_1)} + \|u - v^j_{1,0}\|^2_{\mathbb{L}^2(\tilde{\gamma}^j_1)} \\
+ \frac{1}{2}\,\|u - v^j_{0,1}\|^2_{\mathbb{L}^2(\tilde{\Gamma}^j_2)} + \|u - v^j_{0,1}\|^2_{\mathbb{L}^2(\tilde{\gamma}^j_2)} + \|u - v^j_{1,1}\|^2_{\mathbb{L}^2(\gamma^j_1 \cup \gamma^j_2)} \Big\} \\
\leq \frac{1}{4}\,\frac{\Lambda_l}{\lambda_l}\,(2^{-l}\Lambda_l)^4 \sum_{k=1}^{2} \Big\{ |u|^2_{\mathbb{H}^2(\tilde{\mathrm{G}}^{l,0}_k)} + (2^{-l-2}\Lambda_l)^2\, |u|^2_{\mathbb{H}^3(\tilde{\mathrm{G}}^{l,0}_k)} \Big\} \\
+ 3\,\frac{\Lambda_l}{\lambda_l}\,\frac{\Upsilon_{ps}}{p(p+1)} \Big\{ [u]^2_{s+1,0} + [u]^2_{s+2,0} \Big\}. \quad \text{(A-8.17a)}
\end{aligned}
$$

Analogously to (A-8.17a), we obtain the bounds

$$
\begin{aligned}
\sum_{j=1}^{l-1} \sum_{\nu\in\mathcal{N}} |w^l|^2_{\mathbb{H}^1(\mathrm{G}^{l,j}_\nu)} \leq \frac{4}{3}\, 2^{-l}\Lambda_l \,|u - v^0|^2_{\mathbb{H}^1(\tilde{\Gamma}^1_1 \cup \tilde{\Gamma}^1_2)} + \frac{4}{2^{-l}\lambda_l}\,\|u - v^0\|_{\mathbb{L}^2(\tilde{\Gamma}^1_1 \cup \tilde{\Gamma}^1_2)} \\
+ \frac{4}{3} \sum_{j=1}^{l} 2^{j-1-l}\Lambda_l \Big\{ \frac{1}{2}\,|u - v^j_{1,0}|^2_{\mathbb{H}^1(\tilde{\Gamma}^j_1)} + |u - v^j_{1,0}|^2_{\mathbb{H}^1(\tilde{\gamma}^j_1)} \\
+ \frac{1}{2}\,|u - v^j_{0,1}|^2_{\mathbb{H}^1(\tilde{\Gamma}^j_2)} + |u - v^j_{0,1}|^2_{\mathbb{H}^1(\tilde{\gamma}^j_2)} + |u - v^j_{1,1}|^2_{\mathbb{H}^1(\gamma^j_1 \cup \gamma^j_2)} \Big\} \\
+ 4 \sum_{j=1}^{l} \frac{1}{2^{j-1-l}\lambda_l} \Big\{ 2\,\|u - v^j_{1,0}\|^2_{\mathbb{L}^2(\tilde{\Gamma}^j_1)} + \|u - v^j_{1,0}\|^2_{\mathbb{L}^2(\tilde{\gamma}^j_1)} \\
+ 2\,\|u - v^j_{0,1}\|^2_{\mathbb{L}^2(\tilde{\Gamma}^j_2)} + \|u - v^j_{0,1}\|^2_{\mathbb{L}^2(\tilde{\gamma}^j_2)} + \|u - v^j_{1,1}\|^2_{\mathbb{L}^2(\gamma^j_1 \cup \gamma^j_2)} \Big\} \\
\leq \frac{\Lambda_l}{\lambda_l}\left[2 + \frac{3}{4}\,\frac{\Lambda_l}{\lambda_l}\right] (2^{-l}\Lambda_l)^2 \sum_{k=1}^{2} \Big\{ |u|^2_{\mathbb{H}^2(\tilde{\mathrm{G}}^{l,0}_k)} + (2^{-l-2}\Lambda_l)^2\, |u|^2_{\mathbb{H}^3(\tilde{\mathrm{G}}^{l,0}_k)} \Big\} \\
+ 6\,\frac{\Lambda_l}{\lambda_l}\left[1 + \frac{3}{2}\,\frac{\Lambda_l}{\lambda_l}\,\frac{1}{p(p+1)}\right] \Upsilon_{ps} \Big\{ [u]^2_{s+1,1} + [u]^2_{s+2,1} \Big\} \quad \text{(A-8.17b)}
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=1}^{l-1} \sum_{\nu\in\mathcal{N}} |w^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} \leq \frac{4}{2^{-l}\lambda_l}\,|u - v^0|^2_{\mathbb{H}^1(\tilde{\Gamma}^1_1 \cup \tilde{\Gamma}^1_2)} \\
+ \frac{4}{3} \sum_{j=1}^{l} 2^{j-1-l}\Lambda_l \Big\{ \frac{1}{2}\,|v^j_{1,0}|^2_{\mathbb{H}^2(\tilde{\Gamma}^j_1)} + |v^j_{1,0}|^2_{\mathbb{H}^2(\tilde{\gamma}^j_1)}
\end{aligned}
$$

$$+\frac{1}{2}\,|v_{0,1}^j|^2_{\mathbb{H}^2(\tilde{\Gamma}_2^j)} + |v_{0,1}^j|^2_{\mathbb{H}^2(\tilde{\gamma}_2^j)} + |v_{1,1}^j|^2_{\mathbb{H}^2(\gamma_1^j\cup\gamma_2^j)}\Big\}$$

$$+\,4\sum_{j=1}^{l}\frac{1}{2^{j-1-l}\lambda_l}\Big\{2\,|u-v_{1,0}^j|^2_{\mathbb{H}^1(\tilde{\Gamma}_1^j)} + |u-v_{1,0}^j|^2_{\mathbb{H}^1(\tilde{\gamma}_1^j)}$$

$$+\,2\,|u-v_{0,1}^j|^2_{\mathbb{H}^1(\tilde{\Gamma}_2^j)} + |u-v_{0,1}^j|^2_{\mathbb{H}^1(\tilde{\gamma}_2^j)} + |u-v_{1,1}^j|^2_{\mathbb{H}^1(\gamma_1^j\cup\gamma_2^j)}\Big\}$$

$$\le 6\,\frac{\Lambda_l^2}{\lambda_l^2}\sum_{k=1}^{2}\Big\{|u|^2_{\mathbb{H}^2(\tilde{\mathbb{G}}_k^{l,0})} + (2^{-l-2}\Lambda_l)^2\,|u|^2_{\mathbb{H}^3(\tilde{\mathbb{G}}_k^{l,0})}\Big\}$$

$$+\,3\,\frac{\Lambda_l}{\lambda_l}\left[p^2-1+3\,\frac{\Lambda_l}{\lambda_l}\,\frac{1}{p(p+1)}\right]\Big\{[u]^2_{2,2}+[u]^2_{3,2}\Big\}. \qquad \text{(A-8.17c)}$$

The final inequalities of (A-8.17) prove the bounds of (5.17). □

A-8.4.4 Theorem 5.13

Proof. For p and s given by (5.13), lemma 5.12 yields the following bounds:

$$\Upsilon_{ps}\,[u]^2_{s+1,1} \le \frac{64}{3}\,\frac{e^3}{\sqrt{2\pi}}\,l\,C_u^2\,\delta_u^2\left[\frac{\Lambda_l}{\lambda_l}\right]^{2(s+1)} s\,(s+1)^2\,2^{-2(\alpha-\beta)l}$$

$$\le \frac{64}{3}\,\frac{e^3}{\sqrt{2\pi}}\,C_u^2\,\delta_u^2\left[\frac{\Lambda_l}{\lambda_l}\right]^{2(\chi_l+2)}\frac{\chi_l+1}{\chi_l-1}\,(\chi_l+2)^2\,l\,2^{-2(\alpha-\beta)l}, \qquad \text{(A-8.18a)}$$

$$\Upsilon_{ps}\Big\{[u]^2_{s+1,\frac{1}{2}}+[u]^2_{s+2,\frac{1}{2}}\Big\} \le \frac{64}{3}\left\{\frac{1}{s^2}+\frac{\delta_u^2}{4}\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}\frac{e^3}{\sqrt{2\pi}}\,\frac{\Lambda_l^{2s+3}}{\lambda_l^{2s+2}}\,\frac{C_u^2}{2}\,\delta_u^4\,l$$

$$\cdot\, s\,(s+1)^2\,(s+2)^2\,2^{-2(\alpha-\beta)l}$$

$$\le \frac{32}{3}\,\frac{e^3}{\sqrt{2\pi}}\,C_u^2\,\delta_u^4\left\{\frac{1}{(\chi_l-1)^2}+\frac{\delta_u^2}{4}\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}\Lambda_l\left[\frac{\Lambda_l}{\lambda_l}\right]^{2(\chi_l+3)}$$

$$\cdot\,(\chi_l+1)(\chi_l+2)^2(\chi_l+3)^2\,l\,2^{-2(\alpha-\beta)l}, \qquad \text{(A-8.18b)}$$

$$\Upsilon_{ps}\Big\{[u]^2_{s+1,\frac{3}{2}}+[u]^2_{s+2,\frac{3}{2}}\Big\} \le \frac{64}{3}\left\{\frac{1}{s^2}+\frac{\delta_u^2}{4}\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}\frac{e^3}{\sqrt{2\pi}}\,\frac{\Lambda_l^{2s+1}}{\lambda_l^{2s}}\,2^{-2(\beta_*-\beta)l}\,2\,C_u^2\,\delta_u^4\,l$$

$$\cdot\, s\,(s+1)^2\,(s+2)^2\,2^{-2(\alpha-\beta)l}$$

$$\le \frac{128}{3}\,\frac{e^3}{\sqrt{2\pi}}\,C_u^2\,\delta_u^4\left\{\frac{1}{(\chi_l-1)^2}+\frac{\delta_u^2}{4}\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}\Lambda_l\left[\frac{\Lambda_l}{\lambda_l}\right]^{2\chi_l+2}$$

$$\cdot\,(\chi_l+1)(\chi_l+2)^2(\chi_l+3)^2\, l\, 2^{-2(\alpha+\beta_*-2\beta)l}. \quad \text{(A-8.18c)}$$

Also, using (5.14) and (5.19b), we obtain the estimates

$$[u]^2_{2,2} \le \frac{256}{9}\, C_u^2\, \delta_u^4\, \frac{\Lambda_l^2}{\lambda_l^4}\, 2^{2\beta(l+1)}, \tag{A-8.19a}$$

$$[u]^2_{2,\frac{5}{2}} + [u]^2_{3,\frac{5}{2}} \le \frac{256}{9}\, C_u^2\, \delta_u^4\, \frac{\Lambda_l^4}{\lambda_l^4}\, \Lambda_l^{-1}\lambda_l^{-2}\left\{1 + \frac{\delta_u^2}{4}\,\frac{\Lambda_l^2}{\lambda_l^2}\right\} 2^{(2\beta+1)(l+1)}. \tag{A-8.19b}$$

Combining the bounds of lemma 5.10 with (A-8.18)–(A-8.19) and applying definition 3.3 again, we obtain

$$\begin{aligned}
\|u - v^l\|^2_{\mathbb{H}^1(\mathrm{Q})} &\le \left\{2^{-2l}\,\lambda_l^2\, D_0^2 + D_1^2\right\}\frac{\Lambda_l^4}{\lambda_l^4}\, 2^{-2(1-\beta)l}\, |u|^2_{\mathbb{H}^{2,2}_\beta(\mathrm{G}^{l,0})} \\
&\qquad + \left\{3\,\frac{2^{-4}\Lambda_l^2}{p(p+1)} + 4\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}\Upsilon_{ps}\,[u]^2_{s+1,1} \le C_1^2\, l^3\, 2^{-2(1-\beta)l}, \\
\sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\mathrm{G}^{l,j}_\nu)} &\le \frac{\Lambda_l^4}{\lambda_l^4}\,(p^2-1)\,[u]^2_{2,2} \le C_2^2\, l^2\, 2^{2\beta l}, \\
\|u - v^l\|^2_{\mathbb{L}^2(\Gamma^l)} &\le \frac{3}{2}\,\frac{\Lambda_l}{\lambda_l}\,\frac{\Upsilon_{ps}}{p(p+1)}\left\{[u]^2_{s+1,\frac{1}{2}} + [u]^2_{s+2,\frac{1}{2}}\right\} \le c_0^2\, l^4\, 2^{-2(\alpha-\beta)l}, \\
|u - v^l|^2_{\mathbb{H}^1(\Gamma^l)} &\le \frac{3}{2}\,\frac{\Lambda_l}{\lambda_l}\,\Upsilon_{ps}\left\{[u]^2_{s+1,\frac{3}{2}} + [u]^2_{s+2,\frac{3}{2}}\right\} \le c_1^2\, l^6\, 2^{-2(\alpha+\beta_*-2\beta)l}, \\
\sum_{j\nu} |v^l|^2_{\mathbb{H}^2(\Gamma^{l,j}_\nu)} &\le \frac{3}{4}\,\frac{\Lambda_l}{\lambda_l}(p^2-1)\left\{[u]^2_{2,\frac{5}{2}} + [u]^2_{3,\frac{5}{2}}\right\} \le c_2^2\, l^2\, 2^{(2\beta+1)l}
\end{aligned}$$

with any positive C_1, C_2, c_0, c_1, c_2 such that

$$\begin{aligned}
C_1^2 &\ge \frac{64}{3}\,\frac{e^3}{\sqrt{2\pi}}\, C_u^2\, \delta_u^2\left\{\frac{1}{32}\,\frac{1}{\varrho_\delta^2\chi_l^2} + 4\right\}\left[\frac{\Lambda_l}{\lambda_l}\right]^{2\chi_l+6}\frac{\chi_l+1}{\chi_l-1}\,\frac{(\chi_l+2)^2}{l^2}\, 2^{-2(\alpha-1)l} \\
&\qquad + \frac{1}{l^3}\left\{2^{-2l}\,\lambda_l^2\, D_0^2 + D_1^2\right\}\frac{\Lambda_l^4}{\lambda_l^4}\, |u|^2_{\mathbb{H}^{2,2}_\beta(\mathrm{Q})}, \\
C_2^2 &\ge \frac{256}{9}\, 2^{2\beta}\, C_u^2\, \delta_u^4\, \frac{\Lambda_l^6}{\lambda_l^8}\,\frac{(\varrho_\delta\chi_l+1)^2}{l^2}, \\
c_0^2 &\ge 16\,\frac{e^3}{\sqrt{2\pi}}\,\frac{C_u^2\,\delta_u^4}{\varrho_\delta^2}\,\frac{(\chi_l+1)(\chi_l+2)^2(\chi_l+3)^2}{\chi_l^2\, l^3} \\
&\qquad \cdot\left\{\frac{1}{(\chi_l-1)^2} + \frac{\delta_u^2}{4}\,\frac{\Lambda_l^2}{\lambda_l^2}\right\}\Lambda_l\left[\frac{\Lambda_l}{\lambda_l}\right]^{2\chi_l+5},
\end{aligned}$$

$$c_1^2 \geq 64\, \frac{e^3}{\sqrt{2\pi}}\, C_u^2\, \delta_u^4\, \frac{(\chi_l+1)(\chi_l+2)^2(\chi_l+3)^2}{l^5} \cdot \left\{ \frac{1}{(\chi_l-1)^2} + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2} \right\} \Lambda_l \left[\frac{\Lambda_l}{\lambda_l}\right]^{2\chi_l+3},$$

$$c_2^2 \geq \frac{128}{3}\, 2^{2\beta}\, C_u^2\, \delta_u^4\, \frac{\Lambda_l^4}{\lambda_l^7}\, \frac{(\varrho_\delta \chi_l+1)^2}{l^2} \left\{1 + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2}\right\},$$

where the expressions on the right-hand side are monotonically decreasing with respect to l. □

A-8.4.5 Theorem 5.14

Proof. Let $l \in \mathbb{N}$ be greater than one. Consider p and s given by (5.13). According to lemma 5.11, there exists $w^l \in \tilde{\mathrm{S}}^p(\mathrm{Q}, \mathcal{G}^l)$ vanishing in $\mathrm{G}^{l,0}$ and on $\boldsymbol{\partial}\,\mathrm{Q}$ and such that $w^l + \tilde{\Pi}_p^l u \in \mathbf{C}(\mathbf{cl}\,\mathrm{Q})$ and the bounds of (5.17) are satisfied. From those bounds we obtain

$$\sum_{j\nu} \|w^l\|^2_{\mathrm{H}^1(\mathrm{G}_\nu^{l,j})} \leq 3\, \frac{\Lambda_l^4}{\lambda_l^2}\, 2^{-2l}\, Z_0^2 + 9\, \frac{\Lambda_l^2}{\lambda_l^2}\, \Upsilon_{ps} \left\{[u]^2_{s+1,1} + [u]^2_{s+2,1}\right\}, \tag{A-8.20a}$$

$$\sum_{j\nu} |w^l|^2_{\mathrm{H}^2(\mathrm{G}_\nu^{l,j})} \leq 6\, \frac{\Lambda_l^2}{\lambda_l^2}\, Z_0^2 + 3p^2 \left\{[u]^2_{2,2} + [u]^2_{3,2}\right\}, \tag{A-8.20b}$$

where Z_0^2 is given by (5.18) and can be estimated using (5.19c) as follows:

$$Z_0^2 \leq \frac{512}{9}\, \frac{\Lambda_l^2}{\lambda_l^4}\, C_u^2\, \delta_u^4 \left\{1 + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2}\right\} 2^{2\beta(l+1)}. \tag{A-8.21}$$

By lemma 5.12, there holds a bound

$$\begin{aligned} \Upsilon_{ps} \left\{[u]^2_{s+1,1} + [u]^2_{s+2,1}\right\} &\leq \frac{64}{3}\, \frac{e^3}{\sqrt{2\pi}}\, \frac{\Lambda_l^{2s+2}}{\lambda_l^{2s+2}}\, C_u^2\, \delta_u^4 \left\{\frac{1}{s^2(s+1)^2} + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2}\right\} l \\ &\qquad \cdot s\,(s+1)^2\,(s+2)^2\, 2^{-2(\alpha-\beta)l} \\ &\leq \frac{64}{3}\, \frac{e^3}{\sqrt{2\pi}} \left[\frac{\Lambda_l}{\lambda_l}\right]^{2\chi_l+4} C_u^2\, \delta_u^4 \left\{1 + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2}\right\} (\chi_l+1)(\chi_l+2)^2(\chi_l+3)^2\, l\, 2^{-2(\alpha-\beta)l}. \end{aligned} \tag{A-8.22a}$$

From (5.14), (5.19c) and definition 3.2, we obtain also that

$$[u]^2_{2,2} + [u]^2_{3,2} \leq \frac{256}{9}\, \frac{\Lambda_l^2}{\lambda_l^4}\, C_u^2\, \delta_u^4 \left\{1 + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2}\right\} 2^{2\beta l}. \tag{A-8.22b}$$

Combining the bounds of (A-8.21)–(A-8.22) with (A-8.20), we obtain the inequalities (5.22) with any positive $\tilde{C}_1, \tilde{C}_2$ such that

$$C_1^2 \geq \frac{64}{9}\, C_u^2\, \delta_u^4 \left\{ 1 + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2} \right\} \left\{ 48 \cdot 2^{2\beta}\, \frac{1}{l^6}\, \frac{\Lambda_l^6}{\lambda_l^6} \right.$$
$$\left. + 27\, \frac{e^3}{\sqrt{2\pi}} \left[\frac{\Lambda_l}{\lambda_l} \right]^{2\chi_l+6} \frac{(\chi_l+1)(\chi_l+2)^2(\chi_l+3)^2}{l^5}\, 2^{-2(\alpha-1)l} \right\},$$
$$C_2^2 \geq \frac{64}{9}\, C_u^2\, \delta_u^4\, \frac{\Lambda_l^2}{\lambda_l^4} \left\{ 1 + \frac{\delta_u^2}{4}\, \frac{\Lambda_l^2}{\lambda_l^2} \right\} \left\{ 96 \cdot 2^{2\beta}\, \frac{1}{l^2}\, \frac{\Lambda_l^2}{\lambda_l^2} + 9\, \frac{(\varrho_\delta \chi_l + 2)^2}{l^2} \right\},$$

where the expressions on the right-hand side are monotonically decreasing with respect to l. □

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