A new embedding result for Kondratiev spaces and application to adaptive approximation of elliptic PDEs

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Abstract

In a continuation of recent work on Besov regularity of solutions to elliptic PDEs in Lipschitz domains with polyhedral structure, we prove an embedding between weighted Sobolev spaces (Kondratiev spaces) relevant for the regularity theory for such elliptic problems, and Triebel-Lizorkin spaces, which are known to be closely related to approximation spaces for nonlinear n-term wavelet approximation. Additionally, we also provide necessary conditions for such embeddings.

As a further application we discuss the relation of these embedding results with results by Gaspoz and Morin for approximation classes for adaptive Finite element approximation, and subsequently apply these result to parametric problems.

Keywords: Regularity for elliptic PDEs, Kondratiev spaces, Besov regularity, Triebel-Lizorkin spaces, wavelet decomposition, n-term approximation, adaptive Finite element approximation, parametric elliptic problems.

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1 Introduction

Ever since the emergence of (adaptive) wavelet algorithms for the numerical computation of solutions to (elliptic) partial differential equations there was also the interest in corresponding rates for n-term approximation rates, since these may be seen as the benchmark rates the optimal algorithm (which at each step would calculate an optimal n-term approximation) would converge with.

Later on, this question was seen to be closely related to the membership in a certain scale of Besov spaces. More precisely a famous result by DeVore, Jawerth and Popov [12] characterizes certain Approximation classes for approximation with respect to $L_p(D)$-norms as Besov spaces $B^r_{\tau,\tau}(D)$ with $\frac{1}{\tau} = \frac{r}{d} + \frac{1}{p}$, where r/d is the rate of the best n-term approximation.

In another famous article Dahlke and DeVore [4] later used this result to determine n-term approximation rates for the solution of Poisson's equation on general Lipschitz domains. This was done by proving that the solution of $-\Delta u = f$ belongs to Besov spaces $B^r_{\tau,\tau}(D)$ for parameters $r < r^*$, where r^* depends on the Lipschitz-character of the bounded domain $D \subset \mathbb{R}^d$, the dimension d and the regularity of the right-hand side f. In subsequent years this result was extended to more

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general elliptic operators [5], and for special domains more precise values for r^* were determined [6, 7, 10].

Recently, in [17] the point of view of these investigations was changed slightly: Instead of concentrating on the fixed scale of spaces $B_{\tau,\tau}^r(D)$ mentioned before, an embedding of weighted Sobolev spaces (a scale of function spaces adapted to elliptic problems on polygonal or polyhedral domains) into such Besov spaces was looked for, which allows the desired optimal approximation rate $n^{-m/d}$. This lead to conditions when solutions to elliptic problems belong to Besov spaces $B_{\tau,\infty}^{m+1}(D)$ with $\frac{1}{\tau} > \frac{m}{d} + \frac{1}{2}$, i.e. spaces which are compactly embedded into $H^1(D)$.

The original aim of this paper was to study certain limiting situations in the conditions obtained for such embeddings, which would have lead necessary and sufficient conditions for solutions to elliptic problems to belong to such regularity spaces which in turn would allow for the desired n-term approximation rate $n^{-m/d}$. While the method presented here does not work in these limiting cases, we were able to give a new proof for a very similar embedding result (into slightly smaller spaces), under the same condition as in [17]. Moreover, these conditions can be seen to be already optimal by considering some basic (typical) representatives in the respective function spaces.

Meanwhile, Gaspoz and Morin [15] obtained a counterpart of the result of DeVore, Jawerth and Popov for adaptive Finite element approximation. They proved a direct estimate for approximation of functions from Besov classes $B_{\tau,\tau}^s(D)$ with $\frac{1}{\tau} > \frac{s}{d} + \frac{1}{p}$, and supplemented this by corresponding Inverse Theorems. While not as sharp a characterization as for wavelet approximation, in this way a link between Besov regularity and Approximation classes for this type of Finite element approximation has been established. In a second part of this paper we will slightly extend their results using real interpolation, and combine these approximation results with our regularity results for elliptic problems.

As another application of our embedding results, we want to treat parametric elliptic problems. In recent years these problems got increasing attention as they particularly arise from rewriting problems with random inputs (coefficients) into deterministic problems with countably many parameters. As a particular aspect, these parametric problems were seen to admit very high (analytic) regularity w.r.t. the dependence on these parameters, which subsequently is exploited to derive and analyze approximation schemes either based on sampling (stochastic collocation) or on expanding the parametric solution into series of tensorized orthogonal polynomials (generalized polynomial chaos). Since generally neither the expansion coefficients nor the samples can be computed exactly, but only as solutions of PDEs, also the spatial regularity and its interplay with the parametric regularity needs to be known in the analysis of approximation schemes to predict possible convergence rates. Our embedding results for Besov and Triebel-Lizorkin spaces then bridge the gap between the known analyticity of the parametric solution as a mapping taking values in (weighted) Sobolev spaces on the one hand, and the results on adaptive approximation on the other hand.

The article is organized as follows. In Section 2, we will recall the necessary definitions of wavelet systems and related characterizations of Triebel-Lizorkin and Besov spaces as well as n-term approximation results, followed by the definition of weighted Sobolev spaces (Babuska-Kondratiev spaces) and regularity results for elliptic PDEs in that scale. In Section 3, particular functions and their membership to all these scales of function spaces are considered, which leads to necessary conditions on embeddings between Kondratiev spaces and Besov-/Triebel-Lizorkin spaces.

Section 4 is the main part: Here we present two new proofs for sufficient conditions for embedding into Triebel-Lizorkin spaces. The first one is based on wavelet decompositions and estimates for versions of the classical Hardy-Littlewood maximal operator. While the second one is restricted to the case of polygons in $d = 2$ or smooth cones in higher dimensions, its use of only localization

arguments is so far the most basic approach. Its main adavantage consists in the observation that no longer any splitting in interior and boundary terms is needed, and thus also no additional information about those boundary terms. In Section 5, we first extend the results of Gaspoz and Morin on approximation spaces of adaptive Finite element approximation by means of the real method of interpolation, and then combine these results with the regularity results for elliptic problems and the embeddings into Triebel-Lizorkin spaces. Finally, in Section 6 we transfer the embedding and approximation results for single elliptic problems to parametric ones.

2 Basic definitions and State of the art

In this section we will fix some notations corresponding to the used wavelet system, recall the definitions of the relevant function spaces, and formulate the regularity and n-term approximation results used later on.

2.1 Wavelets

We are not interested in utmost generality pertaining to the used wavelet system. Instead, for simplicity we will stick to Daubechies' Wavelets, the generalization to compactly supported biorthogonal wavelets constituting Riesz-bases being immediate.

Let ϕ be a univariate scaling function and η the associated wavelet corresponding to Daubechies' construction, where the smoothness of ϕ and η and the number of vanishing moments for η are assumed to be sufficiently large. Let E denote the nontrivial vertices of $[0,1]^d$, and put

$$\psi^e(x_1,\ldots,x_d) = \prod_{j=1}^{d} \psi^{e_j}(x_j), \qquad e \in E,$$

where $\psi^0 = \phi$ and $\psi^1 = \eta$. Then the set

$$\Psi' = \{\psi^e : e \in E\}$$

generates via shifts and dyadic dilates an orthonormal basis of $L_2(\mathbb{R}^d)$. More precisely, denoting by $\mathcal{D} = \{I \subset \mathbb{R}^d : I = 2^{-j}([0,1]^d + k), j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$ the set of all dyadic cubes in $\mathbb{R}^d$, then

$$\{\psi_I : I \in \mathcal{D}, \psi \in \Psi'\} = \{\psi_I = 2^{jd/2}\psi(2^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \psi \in \Psi'\}$$

forms an orthonormal basis in $L_2(\mathbb{R}^d)$. Denote by $Q(I)$ some dyadic cube (of minimal size) such that $\operatorname{supp}\psi_I \subset Q(I)$ for every $\psi \in \Psi'$. Then we clearly have $Q(I) = 2^{-j}k + 2^{-j}Q$ for some dyadic cube Q.

As usual $\mathcal{D}^+$ denotes the dyadic cubes with measure at most 1, and we put $\Lambda' = \mathcal{D}^+ \times \Psi'$. Additionally, we shall need the notation $\mathcal{D}_j = \{I \in \mathcal{D} : |I| = 2^{-jd}\}$. Then we can write every function $f \in L_2(\mathbb{R}^d)$ as

$$f = P_0 f + \sum_{(I,\psi)\in\Lambda'} \langle f, \psi_I \rangle \psi_I.$$

Therein $P_0 f$ denotes the orthogonal projector onto the closed subspace S_0, which is the closure in $L_2(\mathbb{R}^d)$ of the span of the function $\Phi(x) = \phi(x_1)\cdots\phi(x_d)$ and its integer shifts $\Phi(\cdot - k)$, $k \in \mathbb{Z}^d$.

Later on it will be convenient to include Φ into the set of generators Ψ' together with the notation $\Phi_I := 0$ for $|I| < 1$, and $\Phi_I = \Phi(\cdot - k)$ for $I = k + [0,1]^d$. Then we can simply write

$$f = \sum_{(I,\psi) \in \Lambda} \langle f, \psi_I \rangle \psi_I, \qquad \Lambda = \mathcal{D}^+ \times \Psi, \quad \Psi = \Psi' \cup \{\Phi\}.$$

Remark 1. *If not explicitly stated otherwise convergence of wavelet expansions is always understood in $S'(\mathbb{R}^d)$, the space of tempered distributions, or in some $L_p(\mathbb{R}^d)$, $1 < p < \infty$ (since all relevant spaces will be embedded into $L_p(\mathbb{R}^d)$).*

2.2 Triebel-Lizorkin spaces

Besov spaces are nowadays established as being closely related to many approximation schemes, starting with approximating periodic functions by trigonometric polynomials (where they actually first emerged in the works of Besov 1959/60), free-knot spline approximation (see [11]), n-term wavelet approximation and most recently adaptive Finite element schemes [15]. However, Triebel-Lizorkin spaces might (at least in some cases) be even closer to the respective approximation spaces. This will be made clear in the following subsections.

We shall only discuss the necessary facts and properties of these scales of function spaces. For more details on the history, definitions and properties we refer to the monographs by Triebel [25, 26, 27]. Instead of the usual (fourieranalytic) definition we will introduce them via their wavelet characterization.

Let $0 < p, q < \infty$ and $s > \max(0, d(\frac{1}{\min(p,q)} - 1))$. Then a function $v \in L_p(\mathbb{R}^d)$ belongs to the Triebel-Lizorkin space $F^s_{p,q}(\mathbb{R}^d)$ if, and only if

$$\|v|F^s_{p,q}(\mathbb{R}^d)\| := \|P_0 v|L_p(\mathbb{R}^d)\| + \left\|\left(\sum_{j=0}^{\infty} \sum_{(I,\psi) \in \mathcal{D}_j \times \Psi} 2^{j(s+\frac{d}{2})q} |\langle v, \psi_I \rangle|^q \chi_I(\cdot)\right)^{1/q}\middle|L_p(\mathbb{R}^d)\right\| < \infty.$$

Therein the function χ_I stands for the characteristic function of the cube I. Note that up to equivalent quasi-norms we can replace the cubes I by the cubes $Q(I)$; similarly we can replace cubes with vertices on the grid $2^{-j}\mathbb{Z}^d$ by cubes with centers in $2^{-j}k$, $k \in \mathbb{Z}^d$. For parameters $q = \infty$ we shall use the usual modification (replacing the sums by suprema), i.e.

$$\|v|F^s_{p,\infty}(\mathbb{R}^d)\| = \left\|\sup_{j \geq 0} \sup_{(I,\psi) \in \mathcal{D}_j \times \Psi} 2^{j(s+\frac{d}{2})} |\langle v, \psi_I \rangle| \chi_I(\cdot)\middle|L_p(\mathbb{R}^d)\right\| < \infty.$$

The corresponding characterization for Besov spaces is easier (which in turn is the reason why they are more commonly used): If $0 < p, q \leq \infty$ and $s > \max(0, d(\frac{1}{p} - 1))$, then a function $v \in L_p(\mathbb{R}^d)$ belongs to the Besov space $B^s_{p,q}(\mathbb{R}^d)$ if, and only if

$$\|v|B^s_{p,q}(\mathbb{R}^d)\| := \|P_0 v|L_p(\mathbb{R}^d)\| + \left(\sum_{j=0}^{\infty} 2^{j(s+d(\frac{1}{2} - \frac{1}{p}))q} \left(\sum_{(I,\psi) \in \mathcal{D}_j \times \Psi} |\langle v, \psi_I \rangle|^p\right)^{q/p}\right)^{1/q} < \infty,$$

with suprema instead of sums if p and/or q is infinite.

We shall use the notation $A^s_{p,q}(\mathbb{R}^d)$ if a statement refers to both Besov and Triebel-Lizorkin spaces alike.

Apart from these spaces on $\mathbb{R}^d$, for our main interest in boundary value problems for elliptic PDEs we also need to consider function spaces on domains. The easiest way to introduce these is via restriction, i.e.

$$A_{p,q}^s(D) := \left\{ f \in \mathcal{D}'(D) : \exists\, g \in A_{p,q}^s(\mathbb{R}^d)\,, g\big|_D = f \right\}\,, \qquad \|f|A_{p,q}^s(D)\| = \inf_{g|_D = f} \|g|A_{p,q}^s(\mathbb{R}^d)\|\,.$$

Alternative (different or equivalent) versions of this definition can be found, depending on possible additional properties for the distributions g (most often referring to their support). We refer to the monograph [28] for details and references.

A final important aspect of Triebel-Lizorkin spaces are their close relations to many classical function spaces. For our purposes, we particularly mention the identities $F_{p,2}^s(\mathbb{R}^d) = H_p^s(\mathbb{R}^d)$ and $H_p^m(\mathbb{R}^d) = W_p^m(\mathbb{R}^d)$, $1 < p < \infty$, $m \in \mathbb{N}$, $s \in \mathbb{R}$, which for Lipschitz domains D transfer to the respective scales of function spaces on D.

2.3 Babuska-Kondratiev spaces

As mentioned in the introduction our interest stems from elliptic boundary value problems such as (2.1) below. It is nowadays classical knowledge that the regularity of the solution depends not only on the one of the coefficient A and right-hand side f, but also on the regularity/roughness of the boundary of the considered domain. While for smooth coefficients A and smooth boundary we have $u \in H^{s+2}(D)$ for $f \in H^s(D)$, it is well-known that this becomes false for more general domains. In particular, if we only assume D to be a Lipschitz domain, then it was shown in [18] that in general we only have $u \in H^{3/2}$ for the solution of the Poisson equation, even for smooth right-hand side f. This behaviour is caused by singularities near the boundary.

To obtain similar shift theorems as for smooth domains, a possible approach is to adapt the function spaces. To compensate possible singularities one includes appropriate weights. For polyhedral domains, this idea has lead to the following definition of the Babuska-Kondratiev spaces $\mathcal{K}_{a,p}^m(D)$: If the function u admits m weak derivatives, we consider the norm

$$\|u|\mathcal{K}_{a,p}^m(D)\|^p = \sum_{|\alpha| \leq m} \int_D |\rho(x)^{|\alpha|-a} \partial^\alpha u(x)|^p\, dx\,,$$

where $a \in \mathbb{R}$ is an additional parameter, and the weight function $\rho : D \longrightarrow [0,1]$ is the smooth distance to the singular set of D. This means ρ is a smooth function, and in the vicinity of the singular set it is equal to the distance to that set. In 2D this singular set consists exactly of the vertices of the polygon, while in 3D it consists of the vertices and edges of the polyhedra. In case of mixed boundary conditions the singular set further includes points where the boundary conditions change (which can be interpreted as vertices with interior angle π), and for interface problems points where the interface touches the boundary, respectively in higher dimensions. Note that in general polygonal domains need not to be Lipschitz, i.e. the definition of the Kondratiev spaces and some related regularity results allow for cracks in the domain, which in turn corresponds to vertices with interior angle 2π. In case $p = 2$ we simply write $\mathcal{K}_a^m(D)$.

Within this scale of function spaces, a regularity result for boundary value problems for elliptic PDEs can be formulated as follows, see [1] and the references given there:

Proposition 1. *Let D be some bounded polyhedral domain without cracks in $\mathbb{R}^d$, $d = 2, 3$. Consider the problem*

$$-\nabla\big(A(x) \cdot \nabla u(x)\big) = f \quad in \quad D\,, \qquad u|_{\partial D} = 0\,, \tag{2.1}$$

where $A = (a_{i,j})_{i,j=1}^d$ is symmetric and

$$a_{i,j} \in \mathcal{W}_\infty^m = \left\{ v : D \longrightarrow \mathbb{C} : \rho^{|\alpha|} \partial^\alpha v \in L_\infty(D), |\alpha| \le m \right\}, \qquad 1 \le i, j \le d.$$

Let the bilinear form

$$B(v, w) = \int_D \sum_{i,j} a_{i,j}(x) \partial_i v(x) \partial_j w(x) dx$$

satisfy

$$|B(v, w)| \le R \|v | H^1(D)\| \cdot \|w | H^1(D)\| \qquad and \qquad r \|v | H^1(D)\|^2 \le B(v, v)$$

for some constants $0 < r \le R < \infty$. Then there exists some $\bar{a} > 0$ such that for any $m \in \mathbb{N}_0$, any $|a| < \bar{a}$ and any $f \in \mathcal{K}_{a-1}^{m-1}(D)$ the problem (2.1) admits a uniquely determined solution $u \in \mathcal{K}_{a+1}^{m+1}(D)$, and it holds

$$\|u | \mathcal{K}_{a+1}^{m+1}(D)\| \le C \|f | \mathcal{K}_{a-1}^{m-1}(D)\|$$

for some constant $C > 0$ independent of f.

We restrict ourselves in this presentation to this simplified situation. In the literature there are further results of this type, either treating different boundary conditions, or using slightly different scales of function spaces, see [19, 20, 21].

We finally shall add a comment on the possible domains D: While before and also in the sequel we will only refer to polyhedral domains, the analysis carries over without change to Lipschitz domains with polyhedral structure. Domains with polyhedral structure were seen to be a natural relaxation of polyhedra, for example replacing the flat faces of polyhedra by smooth surfaces. For precise definitions we refer to [8, 21]. As we shall see in the proofs, the only fact needed about the boundary ∂D are certain combinatorial aspects (counting the number of relevant wavelet coefficients), and these remain unchanged so long as the boundary remains Lipschitz; moreover, also Proposition 1 holds for this more general setting.

2.4 n-term approximation

The (error of the) best n-term approximation is defined as

$$\sigma_n(u; L_p(D)) = \inf_{\Gamma \subset \Lambda : \#\Gamma \le n} \inf_{c_\gamma} \left\| u - \sum_{\gamma = (I, \psi) \in \Gamma} c_\gamma \psi_I \Big| L_p(D) \right\|,$$

i.e. as the name suggests we consider the best approximation by linear combinations of the basis functions consisting of at most n terms. To study relations between decay properties of this quantity and certain function spaces, we shall further define related Approximation spaces. We define $\mathcal{A}_q^\alpha(L_p(D))$, $\alpha > 0$, $0 < q \le \infty$ to consist of all functions $f \in L_p(D)$ such that

$$\|u | \mathcal{A}_q^\alpha(L_p(D))\| = \left(\sum_{n=0}^\infty \left((n+1)^\alpha \sigma_n(u; L_p(D)) \right)^\tau \frac{1}{n+1} \right)^{1/q}, \quad 0 < q < \infty, \qquad (2.2)$$

or

$$\|u | \mathcal{A}_\infty^\alpha(L_p(D))\| = \sup_{n \ge 0} n^\alpha \sigma_n(u; L_p(D)),$$

respectively, are finite. Then a well-known result of DeVore, Jawerth and Popov [12] may be formulated as

$$\mathcal{A}_\tau^{s/d}(L_p(\mathbb{R}^d)) = B_{\tau,\tau}^s(\mathbb{R}^d)\,, \qquad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}\,. \tag{2.3}$$

However, when discussing the optimal convergence rate for adaptive algorithms this result is slightly stronger than required. We are rather interested in conditions on u that simply guarantee a certain decay rate, i.e. we are interested in the larger spaces $\mathcal{A}_\infty^{s/d}(L_p(\mathbb{R}^d))$.

We cite two further results. The first one [9, Theorem 7] incorporates the influence of the bounded domain D: If $s > d(\frac{1}{\tau} - \frac{1}{p})$ for $0 < \tau \le p$, $1 < p < \infty$, then

$$\sigma_n\big(u; L_p(D)\big) \lesssim n^{-s/d}\|u|B_{\tau,q}^s(D)\|\,, \qquad u \in B_{\tau,q}^s(D)\,, \tag{2.4}$$

independent of the microscopic parameter q. In view of the elementary embedding

$$B_{p,u}^s(D) \hookrightarrow F_{p,q}^s(D) \hookrightarrow B_{p,v}^s(D)\,, \quad 0 < p < \infty\,,$$

which holds if, and only if $u \le \min(p,q)$ and $\max(p,q) \le v$, this approximation result immediately transfers to spaces $F_{p,q}^s(D)$. Similar estimates are true for approximation in the energy norm, i.e. in the norm of the space $H^1(D)$, and more generally in the norm of $W_p^1(D)$,

$$\sigma_n\big(u; W_p^1(D)\big) \lesssim n^{-(s-1)/d}\|u|B_{\tau,q}^s(D)\|\,, \qquad u \in B_{\tau,q}^s(D)\,. \tag{2.5}$$

At last, the following result [16] demonstrates that the scale of Triebel-Lizorkin spaces is even closer to the approximation spaces than Besov spaces: For $s > 0$ and $1 < p < \infty$, we have

$$F_{\tau_*,\infty}^s(\mathbb{R}^d) \hookrightarrow \mathcal{A}_\infty^{s/d}(L_p(\mathbb{R}^d))\,, \qquad \frac{1}{\tau_*} = \frac{s}{d} + \frac{1}{p}\,.$$

Moreover, this space $F_{\tau_*,\infty}^s(\mathbb{R}^d)$ is maximal in the sense that if any other Besov or Triebel-Lizorkin space is embedded in $\mathcal{A}_\infty^{s/d}(L_p(\mathbb{R}^d))$, then it is already embedded in $F_{\tau_*,\infty}^s(\mathbb{R}^d)$.

In other words: To obtain optimal conditions for approximability of functions from Kondratiev spaces (and thus solutions of elliptic PDEs) we should turn our attention to conditions on embeddings of these spaces into Triebel-Lizorkin spaces $F_{\tau_*,\infty}^m(D)$.

3 Some basic representatives

In this section we shall consider some special functions and determine parameters a, m and p or s, τ and q, such that they belong to spaces $\mathcal{K}_{a,p}^m(D)$ or $A_{\tau,q}^s(D)$, respectively. The resulting conditions on the parameters will lead to necessary ones for embeddings $\mathcal{K}_{a,p}^m(D) \hookrightarrow A_{\tau,q}^s(D)$.

More precisely, we will study to which spaces the functions

$$f_{\beta,\gamma}(r,\theta) = g_{\beta,\gamma}(r)\Theta(\theta) = \eta(r)r^\beta(-\log r)^{-\gamma}\Theta(\theta)\,, \quad \beta \in \mathbb{R}, \gamma \ge 0\,,$$

belong, where η is a smooth cut-off function around $x = 0$, Θ is another sufficiently smooth function, and the origin is a vertex of some polygon $D \subset \mathbb{R}^2$. To determine whether $f_{\beta,\gamma}$ belongs to $\mathcal{K}_{a,p}^m(D)$ we only have to consider its behaviour sufficiently near the origin (if the support of

η is small enough), i.e. a subdomain where we can assume $\eta(x) \equiv 1$. Then, switching to polar coordinates, all partial derivatives $\partial^\alpha f_{\beta,\gamma}$ are of the form

$$\partial^\alpha f_{\beta,\gamma}(r,\theta) = \sum_{j=0}^{|\alpha|} r^{\beta-|\alpha|}(-\log r)^{-\gamma-j}\Theta_{j,\alpha,\beta,\gamma}(\theta)$$

with smooth functions $\Theta_{j,\alpha,\beta,\gamma}(\theta)$. Weighted integration then yields

$$\int_D \left|\rho(x)^{|\alpha|-a}\sum_{j=0}^{|\alpha|} r^{\beta-|\alpha|}(-\log r)^{-\gamma-j}\Theta_{j,\alpha,\beta,\gamma}(\theta)\right|^p dx \lesssim \int_0^{r_0} r^{p(\beta-a)}\sum_{j=0}^{|\alpha|}(-\log r)^{p(-\gamma-j)} r\, dr\,,$$

which is finite if either $p(\beta-a)+1 > -1$ or $p(\beta-a)+1 = -1$ and $p(-\gamma-j) < -1$ for all $0 \le j \le |\alpha|$. Hence we find

Lemma 3.1. $f_{\beta,\gamma} \in \mathcal{K}^m_{a,p}(D)$ *if, and only if either* $\beta > a - 2/p$ *or* $\beta = a - 2/p$ *and* $\gamma > 1/p$.

Remark 2.   *For us, requiring* $a \ge 0$ *will always be a quite natural restriction, since in this case finiteness of the natural norm in the Kondratiev spaces already implies that the function belongs to* $L_p(D)$.

This we shall now compare with the corresponding result for Besov and Triebel-Lizorkin spaces.

Lemma 3.2 (Runst/Sickel 96).   *Let D be a smooth cone in $\mathbb{R}^d$. Then the function $g_{\beta,\gamma}$ belongs to $B^s_{\tau,q}(D)$ if*

- $\gamma > 0$: *either* $s < \frac{d}{\tau} + \beta$ *or* $s = \frac{d}{\tau} + \beta$ *and* $q\gamma > 1$,

- $\gamma = 0$: *either* $s < \frac{d}{\tau} + \beta$ *or* $s = \frac{d}{\tau} + \beta$ *and* $q = \infty$.

The function $g_{\beta,\gamma}$ belongs to $F^s_{\tau,q}(D)$ if

- $\gamma > 0$: *either* $s < \frac{d}{\tau} + \beta$ *or* $s = \frac{d}{\tau} + \beta$ *and* $\tau\gamma > 1$,

- $\gamma = 0$ *and* $s < \frac{d}{\tau} + \beta$.

Comparing both lemmata leads to the following statement about necessary conditions for the existence of embeddings.

Theorem 1.   *Let $m \in \mathbb{N}$, $a \in \mathbb{R}$, $1 < p < \infty$ and $s \in \mathbb{R}$, $0 < \tau < \infty$, $0 < q \le \infty$, and let $D \subset \mathbb{R}^d$ be a polytope. Then in case $m > a$ the continuous embedding $\mathcal{K}^m_{a,p}(D) \hookrightarrow F^m_{\tau,q}(D)$ implies the condition*

$$0 < m - a < (d-\delta)\Big(\frac{1}{\tau} - \frac{1}{p}\Big), \tag{3.1}$$

and for $m \ge a$ we still find

$$m - a < (d-\delta)\Big(\frac{1}{\tau} - \frac{1}{p}\Big) \qquad or \qquad m - a = (d-\delta)\Big(\frac{1}{\tau} - \frac{1}{p}\Big) \quad and \quad q \ge p\,. \tag{3.2}$$

Moreover, an embedding $\mathcal{K}^m_{a,p}(D) \hookrightarrow B^m_{\tau,q}(D)$ also yields (3.2). Therein δ stands for the dimension of the singular set of D. These results extend to domains D which can be represented as diffeomorphic deformations of polytopes.

Proof. **Step 1:** Let D be a smooth cone. Starting with purely polynomially decaying functions, we first find that $g_{m-2/\tau,0} \notin A_{\tau,q}^m(D)$ gives an explicit counterexample for $\mathcal{K}_{a,p}^m(D) \not\hookrightarrow A_{\tau,q}^m(D)$ for arbitrary $0 < q \le \infty$ as long as $m - a > 2(\frac{1}{\tau} - \frac{1}{p})$.

The case of equality $m - a > 2(\frac{1}{\tau} - \frac{1}{p})$ is more delicate. Here the functions $g_{m-2/\tau,1/q}$ show $\mathcal{K}_{p,a}^m(D) \not\hookrightarrow B_{\tau,q}^m(D)$ for $0 < q < p$. Moreover, the functions $g_{m-2/\tau,1/\tau}$ finally yield $\mathcal{K}_{p,a}^m(D) \not\hookrightarrow F_{\tau,q}^m(D)$ for arbitrary $0 < q \le \infty$, as long as $\tau < p$, i.e. $m > a$.

Step 2: For the case $\delta > 0$, using suitable cut-off functions we can always restrict the considerations to a (sufficiently small) neighborhood of a single component of the singular set (an edge in polyhedra, or a $(d-2)$-face in a polytope). In this neighborhood, essentially the same examples can be used. In $d = 3$ and for some edge along the x_3-axis, simply multiply the $d = 2$-version with some smooth function in one variable x_3, accordingly in higher-dimensional situations. $\qquad \square$

Corollary 1. *For the adaptivity scale we find that an embedding $\mathcal{K}_{a,p}^m(D) \hookrightarrow B_{\tau,\tau}^m(D)$ with $\frac{1}{\tau} = \frac{m}{d} + \frac{1}{p}$ implies $a > \frac{\delta}{d}m$.*

The following (simple) reasoning shows that these conditions already seem to be optimal. We restrict ourselves here to two-dimensional polygons, and shall employ the splitting $u = u_{\text{reg}} + u_{\text{sing}}$ of solutions of elliptic PDEs into a regular part $u_{\text{reg}} \in H^{m+1}(D)$ (i.e. for the right-hand side of the equation we have $f \in H^{m-1}(D)$) and a singular part $u_{\text{sing}} \in H^s(D)$ for some $\frac{3}{2} < s \le 2$ depending on the interior angles of the domain D. Dahlke [6] showed that the singular part belongs to all spaces $B_{\tau,\tau}^\alpha(D)$, where $\frac{1}{\tau} = \frac{\alpha}{d} + \frac{1}{2}$ with arbitrary $\alpha > 0$. Now standard embeddings yield

$$u_{\text{reg}} \in H^{m+1}(D) \hookrightarrow F_{\tau,2}^{m+1}(D) \quad \text{as well as} \quad u_{\text{sing}} \in B_{\tau,\tau}^{m+1}(D) \hookrightarrow F_{\tau,2}^{m+1}(D) \,.$$

This implies $u \in F_{\tau,2}^{m+1}(D)$ for $\frac{1}{\tau} = \frac{m+1}{d} + \frac{1}{2}$. However, since we started with a right-hand side $f \in H^{m-1}(D)$ instead of $\mathcal{K}_{a-1}^{m-1}(D)$ this does not give any direct information on embeddings of Kondratiev spaces (and, as we have seen above, for the case of equality the embedding indeed does not hold).

Some results on sufficient conditions for embeddings into Besov spaces had already been presented in [17]. In the next section, we particularly consider embeddings into spaces $F_{\tau,\infty}^m(D)$.

4 Regularity in Triebel-Lizorkin spaces

In this section we will show that the necessary condition obtained above is also essentially sufficient, i.e. while our method requires a (relatively mild) additional assumption, that one will be fulfilled in the applications we have in mind.

We start with recalling a basic estimate for the wavelet coefficients.

Lemma 4.1. *Let $u \in \mathcal{K}_{a,p}^m(D)$. We put $\rho_I = \inf_{x \in Q(I)} \rho(x)$. Then it holds for all cubes I with $\rho_I > 0$ and $Q(I) \cap D \ne 0$*

$$|\langle \widetilde{u}, \psi_I \rangle| \le c_1 |I|^{\frac{m}{d} + \frac{1}{2} - \frac{1}{p}} \rho_I^{-m+a} \left(\sum_{|\alpha|=m} \int_{Q(I)} |\rho(x)^{m-a} \partial^\alpha \widetilde{u}(x)|^p dx \right)^{1/p} =: c_1 |I|^{\frac{m}{d} + \frac{1}{2} - \frac{1}{p}} \rho_I^{-m+a} m_I \,.$$

where $\widetilde{u}$ is the Stein extension of u to the whole of $\mathbb{R}^d$.

Proof. The proof is (essentially) already contained in the original work of Dahlke and DeVore [4]. We include it here for completeness' sake. Due to the vanishing moments of the wavelets we find

$$\langle \widetilde{u}, \psi_I \rangle = \langle \widetilde{u} - P, \psi_I \rangle$$

for all polynomials P of total degree less than m. Then a standard Whitney-estimate implies

$$\left| \langle \widetilde{u}, \psi_I \rangle \right| \leq \|\psi_I | L_{p'}(Q(I)) \| \cdot \|\widetilde{u} - P | L_p(Q(I)) \| \lesssim |I|^{m/d} |\widetilde{u}| H^m(Q(I))| \cdot |I|^{\frac{1}{2} - \frac{1}{p}}$$

Finally, the H^m-seminorm can be estimated by $\rho^{-m+a} m_I$. □

Remark 3. *(i) The boundedness of the Stein extension operator $\mathfrak{E}$ for Kondratiev spaces was discussed in detail in [17]. We shall work here with a slight modification: Let D_0 be some bounded open domain such that*

$$D_0 \supset \bigcup_{I:Q(I) \cap \overline{D} \neq \emptyset} Q(I) \,,$$

i.e. D_0 covers all supports of wavelets which have a nonempty intersection with D. Then let $\widetilde{u} = \varphi \mathfrak{E} u$, where φ is a smooth function with compact support and $\varphi(x) = 1$ for all $x \in D_0$. This choice of $\widetilde{u}$ clearly does not change the estimate in Lemma 4.1.
(ii) In this context, we always interpret ρ as being defined on the whole of $\mathbb{R}^d$, with otherwise the same properties: It is the regularized distance to the singular set S in the sense of Stein [23, Chapter VI] (there a construction for arbitrary closed subsets of $\mathbb{R}^d$ is given), but capped to take values in $[0,1]$.

To proceed, for $j \geq 0$ let $\Lambda_j \subset \Lambda$ be the set of all pairs (I, ψ) with $|I| = 2^{-jd}$, and for $k \geq 0$ let $\Lambda_{j,k} = \{(I, \psi) \in \Lambda_j : k2^{-j} \leq \rho_I < (k+1)2^{-j}\}$. Then we note that for every $(I, \psi) \in \Lambda_{j,k}$ with $k > 0$ we have $\rho(x) \sim \rho_I$ for every $x \in Q(I)$, since the diameter of $Q(I)$ is itself of the order 2^{-j}. With this observation we can re-interpret the estimate for the wavelet coefficients to obtain

$$|I|^{-\frac{m}{d} - \frac{1}{2}} |\langle \widetilde{u}, \psi_I \rangle| \chi_{Q(I)}(x) \lesssim |I|^{-\frac{1}{p}} \rho(x)^{-m+a} m_I \equiv \rho(x)^{-m+a} |I|^{-1/p} m_I \chi_{Q(I)}(x) \qquad (4.1)$$

for all $x \in Q(I))$. Thus we already arrived at terms which appear in the quasi-norm of $F^m_{\tau,\infty}(D)$. The precise result now can be formulated as follows.

Theorem 2. *Let $u \in \mathcal{K}^m_{a,p}(D)$, and let $\widetilde{u}$ be its Stein-extension. Moreover, define the operator P_{reg} by*

$$P_{reg} u = \sum_{(I,\psi): \rho_I > len(I)} \langle \widetilde{u}, \psi_I \rangle \psi_I \,,$$

where $len(I) = |I|^{1/d}$ denotes the side-length of the cube I. Then $P_{reg} : \mathcal{K}^m_{a,p}(D) \to F^m_{\tau,\infty}(D)$ is bounded whenever $m - a < (d - \delta)(\frac{1}{\tau} - \frac{1}{p})$.

Proof. We put $g_I(x) = |I|^{-1/p} m_I \chi_{Q(I)}(x)$, and note that the summation in the definition of P_{reg} ranges over $\bigcup_{j \geq 0} \bigcup_{k > 0} \Lambda_{j,k}$. Then taking the pointwise supremum in (4.1) and applying Hölder's inequality results in

$$\|P_{\text{reg}} u | F^m_{\tau,\infty}(D)\| \sim \left(\int_D \left(\sup_{(I,\psi) \in \Lambda^0} |I|^{-\frac{m}{d} - \frac{1}{2}} |\langle \widetilde{u}, \psi_I \rangle| \chi_{Q(I)}(x) \right)^\tau dx \right)^{1/\tau}$$

$$\lesssim \left(\int_{D_0} \left(\sup_{(I,\psi)\in\Lambda^0} \rho(x)^{-m+a} g_I(x) \right)^{\tau} dx \right)^{1/\tau}$$

$$\leq \left(\int_{D_0} \left(\rho(x)^{-m+a} \right)^{\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau}} dx \right)^{\frac{p(1-\varepsilon)-\tau}{p(1-\varepsilon)\tau}} \left(\int_{D_0} \sup_{(I,\psi)\in\Lambda^0} g_I(x)^{p(1-\varepsilon)} dx \right)^{\frac{1}{p(1-\varepsilon)}}.$$

$$(4.2)$$

Step 1: We shall show that the first integral in (4.2) is finite if, and only if $m - a < (d - \delta)(\frac{1}{\tau} - \frac{1}{p(1-\varepsilon)})$. To this end we note that we can find finitely many open neighbourhoods U_j covering D_0 such that on each U_j the distance function ρ is either bounded from below away from 0, or equivalent to the distance of one vertex, or equivalent to the distance to some edge. In other words: It is sufficient to consider this integral on cones, where the weight function is either the distance to the vertex or the distance to the axis (clearly, the third case for bounded ρ^{-1} is trivial). Consider first the case, where ρ is the distance to the vertex. Then we can switch to polar coordinates, and we find

$$\left(\int_{D_0} \left(\rho(x)^{-m+a} \right)^{\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau}} dx \right)^{\frac{p(1-\varepsilon)-\tau}{p(1-\varepsilon)\tau}} \sim \left(\int_{\Omega} \int_0^R \left(r^{-m+a} \right)^{\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau}} r^{d-1} dr\, dS \right)^{\frac{p(1-\varepsilon)-\tau}{p(1-\varepsilon)\tau}},$$

where $\Omega \subset S^{d-1}$, and dS is the surface measure in S^{d-1}. The integration over Ω simply amounts to a constant, and the r-integral is finite if, and only if $(-m + a)\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau} + d - 1 > -1$, which is equivalent to the condition stated in the beginning with $\delta = 0$.
Similarly we can argue in case of a cone and ρ being the distance to its axis. Switching to cylinder coordinates we have

$$\left(\int_{D_0} \left(\rho(x)^{-m+a} \right)^{\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau}} dx \right)^{\frac{p(1-\varepsilon)-\tau}{p(1-\varepsilon)\tau}} \sim \left(\int_{\Omega} \int_0^H \int_0^{R(z)} \left(r^{-m+a} \right)^{\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau}} r^{d-2} dr\, dz\, dS \right)^{\frac{p(1-\varepsilon)-\tau}{p(1-\varepsilon)\tau}},$$

Inserting $R(z) = cz$ eventually leads to the condition $(-m + a)\frac{p(1-\varepsilon)\tau}{p(1-\varepsilon)-\tau} + d - 2 > -1$, and thus to the initial one with $\delta = 1$.
We finally note that the stated condition can always be fulfilled as a consequence of the assumption $m - a < (d - \delta)(\frac{1}{\tau} - \frac{1}{p})$, if $\varepsilon > 0$ is chosen sufficiently small.
Step 2: For the second term in (4.2) we find

$$\int_{D_0} \sup_{(I,\psi)\in\Lambda^0} g_I(x)^{p(1-\varepsilon)} dx = \int_D \left(\sup_{(I,\psi)\in\Lambda^0} \chi_{Q(I)}(x) \frac{1}{|I|} \int_{Q(I)} |\rho(y)^{m-a}\partial^\alpha \widetilde{u}(y)|^p dy \right)^{1-\varepsilon} dx$$

$$\sim \int_{D_0} \left(\sup_{(I,\psi)\in\Lambda^0: x\in Q(I)} \frac{1}{|Q(I)|} \int_{Q(I)} |\rho(y)^{m-a}\partial^\alpha \widetilde{u}(y)|^p dy \right)^{1-\varepsilon} dx$$

$$\leq \int_D \left(\sup_{\text{cube } Q\ni x} \frac{1}{|Q|} \int_Q |\rho(y)^{m-a}\partial^\alpha \widetilde{u}(y)|^p dy \right)^{1-\varepsilon} dx$$

$$\equiv \int_{D_0} \left(M_{D_0}(|\rho^{m-a}\partial^\alpha \widetilde{u}|^p)(x) \right)^{1-\varepsilon} dx \lesssim \left(\int_D |\rho(x)^{m-a}\partial^\alpha \widetilde{u}(x)|^p dx \right)^{1-\varepsilon}.$$

Therein M_{D_0} denotes a version of the classical Hardy-Littlewood maximal operator, where the supremum is restricted to cubes contained in the domain D_0. The last line then follows from Lemma 4.2 below. □

Lemma 4.2. *Let $D \subset \mathbb{R}^d$ be a bounded domain and $0 < r < 1$. Then the restricted Hardy-Littlewood Maximal operator M_D is bounded as a mapping from $L_1(D)$ into $L_r(D)$, i.e.*

$$\|M_D f\|_{L_r(D)} \leq C_r \|f\|_{L_1(D)}, \quad f \in L_1(D).$$

Recall that the maximal function Mf of some function $f \in L_1(\mathbb{R}^d)$ is never integrable on $\mathbb{R}^d$, and generally also fails to be integrable on bounded domains (it only holds $Mf \in L(\log L)(D)$). This lemma acts as a surrogate for this failure.

Proof. In view of the trivial pointwise estimate $M_D f(x) \leq Mf(x)$ the classical weak-type $(1,1)$ estimate yields

$$\left| \{ x \in D : M_D f(x) > t \} \right| \leq \left| \{ x \in \mathbb{R}^d : Mf(x) > t \} \right| \leq \frac{C_1}{t} \|f\|_{L_1(D)},$$

where we identify a function $f \in L_1(D)$ with its zero-extension to $\mathbb{R}^d$. On the other hand, we trivially have for the restricted Maximal operator

$$\left| \{ x \in D : M_D f(x) > t \} \right| \leq |D|,$$

i.e. the weak-type estimate is only relevant for $t \geq t_0 := \frac{C_1}{|D|} \|f\|_{L_1(D)}$. Then we obtain

$$\begin{aligned}
\|M_D f\|_{L_r(D)}^r &= r \int_0^\infty t^{r-1} \left| \{ x \in D : M_D f(x) > t \} \right| dt \\
&\leq r \int_{t_0}^\infty t^{r-1} \frac{C_1}{t} \|f\|_{L_1(D)} dt + r \int_0^{t_0} t^{r-1} |D| \, dt \\
&= C_1 \frac{r}{1-r} t_0^{r-1} \|f\|_{L_1(D)} + t_0^r |D| = \frac{C_1^r |D|^{1-r}}{1-r} \|f\|_{L_1(D)}^r.
\end{aligned}$$

This proves the claim. $\square$

Remark 4. *Comparing with Theorem 1 it is clear that the condition on the parameters in Theorem 2 is essentially optimal, at least as long as $0 \leq a < m$. For $m = a$ the method fails, and this can be easily demonstrated explicitly.*
It is well-known that the Maximal function Mf is never integrable for any nontrivial function $f \in L_1(\mathbb{R}^d)$. While one reason for this failure (the global support of the Maximal function) can be circumvented by considering the restricted Maximal operator M_D, the local problems for singularity functions which are only just integrable, still remain for functions $f = \rho^{m-a} \partial^\alpha u$, $|\alpha| = m$, $u \in \mathcal{K}_{a,p}^m(D)$.
For simplicity, we consider the situation of D being an unbounded cone in $\mathbb{R}^2$ (this can be transferred to bounded cones by multiplying with suitable cut-off functions). Moreover, we take $m = a = 1$ and $p = 2$, i.e. we consider the space $\mathcal{K}_1^1(D)$.
Consider the function $F_\gamma(x) = (-\log r)^\gamma \phi(\theta)$. We assume the function ϕ to be Lipschitz (i.e. it admits a bounded weak derivative). Then we find for the weak gradient

$$|f_\gamma(x)|^2 = |\nabla F_\gamma(x)|^2 = \gamma^2 r^{-2} (-\log r)^{2\gamma-2} |\phi(\theta)|^2 + r^{-2} (-\log r)^{2\gamma} |\phi'(\theta)|^2.$$

This is integrable on D if, and only if $\gamma < -1/2$. Moreover, we have $r^{-1} F_\gamma \in L_2(D)$ under exactly the same condition. Thus we have $F_\gamma \in \mathcal{K}_1^1(D)$ for suitable functions ϕ.

Now, consider the Maximal functions $M_D|f_\gamma|^2$. Integrating over an annulus $A_{r,R}$ with center in 0 and radii $0 < r < R$ yields $\int_{A_{r,R} \cap D} |f_\gamma|^2 dx \sim \int_r^R \rho^{-1}(-\log \rho)^{2\gamma} d\rho = (-\log \rho)^{2\gamma+1}|_r^R$. Hence the Maximal function can be estimated from below by $r^{-2}(-\log r)^{2\gamma+1}$ near the origin (choose $R = 2r$; while we integrated here over annuli instead of cubes, the estimate does not change when integrating over cubes contained in such annuli with $r \sim 2^{-j}$). This in turn is integrable if, and only if $\gamma < -1$. Hence for all values $-1 \le \gamma < -1/2$ we have $F_\gamma \in \mathcal{K}_1^1(D)$, but $M_D|\nabla F_\gamma|^2$ fails to be integrable.

To get a full embedding result, we still need to consider the boundary terms, i.e. the wavelets ψ_I with $(I, \psi) \in \Lambda_{j,0}$, for which we might have $\rho_I = 0$. These terms form the operator P_{sing},

$$P_{\text{sing}} u = \sum_{j=0}^\infty \sum_{(I,\psi) \in \Lambda_{j,0}} \langle \widetilde{u}, \psi_I \rangle \psi_I \,.$$

In [17] it was shown that this operator maps $B_{p,q}^s(D)$ into $B_{\tau,q}^{s+(d-\delta)(\frac{1}{\tau} - \frac{1}{p})}(\mathbb{R}^d)$, and we now want to consider its mapping properties in the Triebel-Lizorkin scale. The following result shows that there is no counterpart to this result for the Triebel-Lizorkin scale.

Lemma 4.3. *Let $s \ge 0$, $0 < \tau < p < \infty$ and $0 < q \le \infty$. Consider the operator*

$$\widetilde{P}_{sing} u = \sum_{j=0}^\infty \sum_{\lambda \in \Lambda_{j,0}} u_\lambda \psi_I \,,$$

where $u \in F_{p,q}^s(\mathbb{R}^d)$ is decomposed as $u = \sum_{\lambda=(I,\psi) \in \Lambda} u_\lambda \psi_I$ with convergence in $L_p(\mathbb{R}^d)$. Then there exists a function $u \in F_{p,q}^s(D)$ such that $\|\widetilde{P}_{sing} u | F_{\tau,\widetilde{q}}^{s+(d-\delta)(\frac{1}{\tau} - \frac{1}{p})}(\mathbb{R}^d)\| = \infty$ for all $0 < \widetilde{q} \le \infty$.

As a consequence if we start with functions $u \in F_{p,q}^s(D)$, the best possible result is

$$P_{\text{sing}} : F_{p,q}^s(D) \to B_{\tau,\max(p,q)}^{s+(d-\delta)(\frac{1}{\tau} - \frac{1}{p})}(\mathbb{R}^d) \,, \tag{4.3}$$

where we simply combined the elementary embedding $F_{p,q}^s(D) \hookrightarrow B_{p,\max(p,q)}^s(D)$ with the result for Besov spaces from [17].

Proof. For simplicity, we consider only the case $\delta = 0$, the general case follows by an easy modification. Moreover, let the vertex be the origin. Then we put $u_j \equiv u_{(\psi,I_j)} = 2^{-j(s+d/2)} j^{-\alpha} 2^{jd/p}$ for some fixed $\psi \in \Psi$ and $I_j = 2^{-j}[0,1]^d$, $j \ge 1$. Furthermore, we put $E_j = I_j \setminus I_{j+1}$. Then we can calculate directly, using the disjointness of the sets E_k,

$$\|\widetilde{P}_{\text{sing}} u | F_{p,q}^s(\mathbb{R}^d)\|^p = \int_{\mathbb{R}^d} \left(\sum_{j \ge 1} |2^{j(s+d/2)} u_j \chi_{I_j}(x)|^q \right)^{p/q} dx$$

$$= \int_{\mathbb{R}^d} \left(\sum_{k \ge 1} \sum_{1 \le j \le k} |j^{-\alpha} 2^{jd/p}|^q \chi_{E_k}(x) \right)^{p/q} dx$$

$$= \int_{\mathbb{R}^d} \sum_{k \ge 1} \left(\sum_{1 \le j \le k} |j^{-\alpha} 2^{jd/p}|^q \right)^{p/q} \chi_{E_k}(x) dx \sim \sum_{k \ge 1} k^{-\alpha p} 2^{jd} 2^{-jd} \,,$$

which is finite if, and only if $\alpha > 1/p$. Note that this result is independent of the microscopic parameter q. A completely analogous calculation shows

$$\|\widetilde{P}_{\mathrm{sing}}u|F_{p,\widetilde{q}}^{s+(d-\delta)(\frac{1}{\tau}-\frac{1}{p})}(\mathbb{R}^d)\|^{\tau} \sim \sum_{k\geq 1} k^{-\alpha\tau}\,.$$

Thus choosing $\frac{1}{p} < \alpha \leq \frac{1}{\tau}$ gives the desired example. $\qquad\square$

Both results on P_{reg} and P_{sing} are summarized in the following theorem.

Theorem 3. *Let $1 < p < \infty$, $m \in \mathbb{N}$ and $a,s \geq 0$. Moreover, let $D \subset \mathbb{R}^d$ be a Lipschitz domain with polyhedral structure. Assume $\min(a,s) > \frac{\delta}{d}m$. Then there exists $\tau_* < \tau_0 \leq p$ such that for all $\tau < \tau_0$ we have an embedding*

$$\mathcal{K}_{a,p}^m(D) \cap B_{p,\infty}^s \hookrightarrow F_{\tau,\infty}^m(D) \hookrightarrow \mathcal{A}_\infty^{m/d}(L_p(D))\,.$$

Proof. If $a > \frac{\delta}{d}m$ then we have $m - a < \frac{d-\delta}{d}m = (d-\delta)(\frac{1}{\tau_*} - \frac{1}{p})$. Since this inequality is strict, it remains true for τ sufficiently close to τ_*. In view of Theorem 2 this takes care of the regular part of $u \in \mathcal{K}_{a,p}^m(D) \cap B_{p,\infty}^s(D)$. For the singular part we can argue similarly as $s > \frac{\delta}{d}m$ ensures $m < s + (d-\delta)(\frac{1}{\tau} - \frac{1}{p})$ for τ sufficiently close to τ_*. Hence we can use (4.3) together with the embedding $B_{\tau,\infty}^{s+(d-\delta)(\frac{1}{\tau}-\frac{1}{p})}(D) \hookrightarrow F_{\tau,\infty}^m(D)$. $\qquad\square$

Remark 5. *We want to emphasize that the above condition $\min(a,s) > \frac{\delta}{d}m$ generally is optimal: In view of Lemma 4.3 $s = \frac{\delta}{d}m$ is impossible, and similarly, for $a = \frac{\delta}{d}m$ by Theorem 1 the embedding $\mathcal{K}_{a,p}^m(D) \hookrightarrow F_{\tau_*,\infty}^m(D)$ becomes false.*

In case of solutions to elliptic PDEs the additional condition needed for the boundary terms is naturally satisfied, at least on polygons in $d = 2$.

Corollary 2. *Let D be a polygon, and let the assumptions of Proposition 1 be satisfied. If $0 \leq a < \bar{a}$, then for every $f \in \mathcal{K}_{a-1}^{m-1}(D)$ the solution $u \in \mathcal{K}_{a+1}^{m+1}(D)$ also belongs to $F_{\tau,\infty}^{m+1}(D)$.*

Proof. In view of the relation $\mathcal{K}_1^1(D) \cap \{u : u|_{\partial D} = 0\} = H_0^1(D)$, for $a \geq 0$ we thus always have $u \in H^1(D) = B_{2,2}^1(D) \hookrightarrow B_{2,\infty}^1(D)$. Then simply apply the embedding from Theorem 3 for $\mathcal{K}_{a+1}^{m+1}(D)$. $\qquad\square$

In the case $d \geq 3$ we obtain a relation between a and m, and also assertions on the Sobolev regularity of the solution come into play. Both can easily obtained from Theorem 3.

5   Adaptive Finite element approximation

5.1   Approximation classes for Finite element methods

In a recent article Gaspoz and Morin [15] established a connection between Besov classes and certain adaptive Finite element methods strikingly similar to the well-known one for n-term wavelet approximation by DeVore, Jawerth and Popov. We shall briefly describe these results.

The starting point is an initial triangulation $\mathcal{T}_0$ of some polyhedral domain D, and $\mathbb{T}$ denotes the family of all conforming, shape-regular partitions of D obtained from $\mathcal{T}_0$ by refinement using bisection rules (these in turn correspond to the newest-vertex bisection in two dimensions). Moreover, $V_\mathcal{T}$ denotes the finite element space of continuous piecewise polynomials of degree at most r, i.e.

$$V_\mathcal{T} = \left\{ v \in C(\overline{D}) : v|_T \in \mathcal{P}_r \text{ for all } T \in \mathcal{T} \right\}.$$

A benchmark for adaptive Finite element algorithms, choosing the optimal triangulation after a given number of refinements, is the quantity

$$\sigma_N^{FE}\big(u; L_p(D)\big) = \min_{\substack{\mathcal{T} \in \mathbb{T}: \\ \#\mathcal{T} - \#\mathcal{T}_0 \le N}} \inf_{v \in V_\mathcal{T}} \|u - v\|_{L_p(D)}, \quad 0 < p < \infty,$$

which may be interpreted as a counterpart of the error of the best n-term wavelet approximation. One of the main results of [15] then consists in a direct estimate,

$$\sigma_N^{FE}\big(u; L_p(D)\big) \le C\, N^{-s/d} \|u|\widetilde{B}_{\tau,\tau}^s(D)\|, \tag{5.1}$$

where $0 < p < \infty$, $0 < s \le r + \frac{1}{\tau_*}$, $\tau_* = \min(1, \tau)$, $\frac{1}{\tau} < \frac{s}{d} + \frac{1}{p}$. Therein the Besov classes $\widetilde{B}_{p,p}^\alpha$ are introduced as subspaces of $L_p(D)$ via finite differences, hence they differ from the previously defined spaces $B_{p,p}^\alpha$ for $p \le 1$ and $\alpha \le d(\frac{1}{p} - 1)$.

We shall use a reformulation of this result in terms of Approximation classes. Similar to (2.2) we define spaces $\mathcal{A}_{q,FE}^\alpha(L_p(D))$, $\alpha > 0$, $0 < q \le \infty$ by requiring

$$\|u|\mathcal{A}_{q,FE}^\alpha(L_p(D))\| = \left(\sum_{n=0}^\infty \Big((n+1)^\alpha \sigma_n^{FE}\big(u; L_p(D)\big)\Big)^\tau \frac{1}{n+1} \right)^{1/q} < \infty, \quad 0 < q < \infty \tag{5.2}$$

as well as

$$\|u|\mathcal{A}_{\infty,FE}^\alpha(L_p(D))\| = \sup_{n \ge 0}(n+1)^\alpha \sigma_n^{FE}\big(u; L_p(D)\big) < \infty.$$

Using these approximation spaces, the estimate (5.1) is equivalent to an embedding $\widetilde{B}_{\tau,\tau}^s(D) \hookrightarrow \mathcal{A}_{\infty,FE}^{s/d}(L_p(D))$.

Furthermore, there is also an inverse theorem, [15, Theorem 2.5], which can be stated as

$$\mathcal{A}_{\tau,FE}^{s/d}\big(L_p(D)\big) \hookrightarrow \widetilde{B}_{\tau,\tau}^s(D), \quad \frac{1}{\tau} = \frac{s}{d} + \frac{1}{p}.$$

Corresponding results are also available when the error is measured in some space $\widetilde{B}_{p,p}^\alpha(D)$ instead of $L_p(D)$, in particular for $\widetilde{B}_{2,2}^1(D) = H^1(D)$.

5.2 Application to elliptic PDEs

We shall now use interpolation arguments to extend the above direct estimate to spaces $\widetilde{B}_{\tau,\infty}^s(D)$, and afterwards combine this with our embedding result for weighted Sobolev spaces and regularity results for elliptic PDEs to obtain error bounds for Finite element algorithms for this class of problems.

It is a basic property of Approximation spaces that this scale of spaces is closed under the real method of interpolation. More precisely, for arbitrary $0 < s_0 \neq s_1 < \infty$, $0 < q, q_0, q \leq \infty$ and $0 < \Theta < 1$ we have

$$\left(\mathcal{A}_{q_0}^{s_0}(L_p), \mathcal{A}_{q_1}^{s_1}(L_p)\right)_{\Theta,q} = \mathcal{A}_q^s(L_p)\,, \quad s = (1-\Theta)s_0 + \Theta s_1\,.$$

This result is independent of the particular approximation scheme (i.e. it holds for n-term approximation, adaptive Finite element approximation and beyond). For details we refer, e.g., to the monograph [13], and more information on interpolation theory can be found in [24, 2]. On the other hand, interpolation results for Besov spaces are known as well. For our purposes, we shall need the following relation, see [14]: If $s_0 \neq s_1 > 0$, $0 < \tau < \infty$ and arbitrary $0 < q_0, q_1 \leq \infty$ it holds

$$\left(\widetilde{B}_{\tau,q_0}^{s_0}(D), \widetilde{B}_{\tau,q_1}^{s_1}(D)\right)_{\Theta,q} = \widetilde{B}_{\tau,q}^s(D)\,, \quad s = (1-\Theta)s_0 + \Theta s_1\,.$$

Both results now shall be applied for $q = \infty$. Then the interpolation property yields the embedding

$$\widetilde{B}_{\tau,\infty}^s(D) = \left(\widetilde{B}_{\tau,q_0}^{s_0}(D), \widetilde{B}_{\tau,q_1}^{s_1}(D)\right)_{\Theta,\infty} \hookrightarrow \left(\mathcal{A}_{q_0,FE}^{s_0/d}(L_p), \mathcal{A}_{q_1,FE}^{s_1/d}(L_p)\right)_{\Theta,\infty} = \mathcal{A}_{q,FE}^{s/d}(L_p)\,,$$

which proves the desired estimate

$$\sigma_N^{FE}\left(u; L_p(D)\right) \leq C\, N^{-s/d}\|u|\widetilde{B}_{\tau,\infty}^s(D)\|\,, \tag{5.3}$$

under the same restrictions on p, s and τ as before.

Theorem 4. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with polyhedral structure. Further, let $1 < p < \infty$, $m \in \mathbb{N}$ and $\min(a,s) > \frac{\delta}{d}m$. Then, given an initial triangulation $\mathcal{T}_0$, for every function $u \in \mathcal{K}_{a,p}^m(D) \cap H_p^s(D)$ and for every $N \in \mathbb{N}$ there exists a triangulation $\mathcal{T}_N$ with $\#\mathcal{T}_N - \#\mathcal{T}_0 \leq N$ such that the corresponding Finite element space $V_N = V_{\mathcal{T}_N}$ (continuous piecewise polynomials of degree $r \geq m$) satisfies*

$$\inf_{v \in V_N} \|u - v \,|\, L_p(D)\| \leq C\, N^{-m/d}\|u \,|\, \mathcal{K}_{a,p}^m(D) \cap H_p^s(D)\|\,,$$

where the constant C is independent of u and N.

This is simply a combination of the above approximation result (Equation (5.3)) and the embedding from Theorem 3, together with the elementary embedding $F_{\tau,\infty}^m(D) \hookrightarrow B_{\tau,\infty}^m(D)$ and (due to $1 < p < \infty$) the identity $\widetilde{B}_{\tau,\infty}^m(D) = B_{\tau,\infty}^m(D) \hookrightarrow L_p(D)$.

Finally, we can combine this approximation result with regularity results for elliptic PDEs. For simplicity, we restrict ourselves here to the case $d = 2$. Then the following theorem results immediately from combining Proposition 1, Theorem 3 and (5.3).

Theorem 5. *Let $D \subset \mathbb{R}^2$ be a polygon. Moreover, let the assumptions of Proposition 1 be satisfied. Then for all $|a| < \overline{a}$ and all functions $f \in \mathcal{K}_{a-1}^{m-1}(D)$ the unique solution to the problem (2.1) belongs to the space $F_{\tau,2}^m(D)$. Furthermore, given an initial triangulation $\mathcal{T}_0$, for every $N \in \mathbb{N}$ there exists a triangulation $\mathcal{T}_N$ with $\#\mathcal{T}_N - \#\mathcal{T}_0 \leq N$ such that the corresponding Finite element space $V_N = V_{\mathcal{T}_N}$ satisfies*

$$\inf_{v \in V_N} \|u - v \,|\, L_p(D)\| \leq C\, N^{-m/d}\|f \,|\, \mathcal{K}_{a-1}^{m-1}(D)\|\,,$$

where the constant C is independent of u and N.

6 Application to parametric problems

In this final section we shall apply our results to parametric problems. More precisely, we shall consider problems

$$
\begin{aligned}
-\nabla\big(A(y)\nabla u(y)\big) &= f(y) &&\text{in}\quad D\,,\\
u(y) &= 0 &&\text{on}\quad \partial_D D\,,\\
\nabla_\nu^A u(y) &= g(y) &&\text{on}\quad \partial_N D\,,
\end{aligned}
\tag{6.1}
$$

where $A = (a_{i,j})_{i,j=1}^d$ and $a_{i,j} : U \longrightarrow L^\infty(D)$ are given fixed mappings with D being a bounded Lipschitz domain in $\mathbb{R}^d$ and parameter domain $U = [-1,1]^{\mathbb{N}}$, i.e. the countable cartesian product of intervals $[-1,1]$, either interpreted as the unit ball of $\ell_\infty(\mathbb{N})$ or as the compact subset of the Frechet space $\mathbb{R}^{\mathbb{N}}$ (i.e. equipped with the product topology). Moreover, the boundary ∂D is partitioned into the Dirichlet part $\partial_D D$ and a Neumann part $\partial_N D$. With $\nabla_\nu^A u$ we denote the co-normal derivative of u,

$$
\nabla_\nu^A u(x,y) = \sum_{i,j=1}^d \nu_i a_{i,j}(x,y)\partial_j u(x,y)\,, \quad x \in \partial_N D\,, y \in U\,,
$$

where ν is the outward unit normal in $x \in \partial_N D$. In other words: For every fixed parameter $y \in U$ we have an elliptic problem similar to the one discussed in Proposition 1, for which an analogous shift-theorem is valid. In addition to the spatial regularity, i.e. regularity w.r.t. the variable $x \in D$, we now can also discuss regularity w.r.t. $y \in U$. The following result has been proved in [22].

Proposition 2. *Let $D \subset \mathbb{R}^d$ be a domain with polyhedral structure. Assume $f : U \to \mathcal{K}_{a-1}^{m-1}(D)$, $g : U \to \mathcal{K}_{a-1/2}^{m-1/2}(D)$ and $A : U \to \mathcal{W}_\infty^m(D)^{d\times d}$ to be analytic. Suppose*

$$
0 < r|\xi|^2 \leq \operatorname*{ess\,inf}_{x\in D} \xi^\top A(x,y)\xi \leq \operatorname*{ess\,sup}_{x\in D} \xi^\top A(x,y)\xi \leq R|\xi|^2\,, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\}\ \forall y \in U\,.
\tag{6.2}
$$

Then for every $y \in U$ the solution of the problem (6.1) exists, and it is analytic as a mapping from U into $\mathcal{K}_{a+1}^{m+1}(D)$.

Note that the precise meaning of "analyticity" and its consequences depend on the topology chosen on U, but it always includes the following: For every finite set $E \subset \mathbb{N}$, $J = |E|$, $E = \{j_1,\ldots,j_J\}$, we put $y_E = (y_{j_1},\ldots,y_{j_J})$, $F = \mathbb{N} \setminus E$, and split $y = (y_E, y_F)$. Then for every choice of y_F, the function $u_E(z) = u(z,y_F)$ is analytic on $[-1,1]^J$ (in the usual sense of analytic functions in several variables). We will refer to this as "analytic in every finite choice of variables."

Subsequently, this property implies estimates on derivatives, and thus also on Taylor coefficients, and coefficients in expansions w.r.t. several systems of orthogonal polynomials on U. Below we will concentrate on Taylor coefficients and power series, but all results can immediately be transferred to Legendre and Chebyshev series. We refer again to [22] for related notions and further references. By $\mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\operatorname{supp}\nu| < \infty\}$ we denote the set of multiindices with finite support, $\operatorname{supp}\nu = \{j \in \mathbb{N} : \nu_j \neq 0\}$. Moreover, we put with $E = \operatorname{supp}\nu$, $J = |E|$,

$$
\partial^\nu u(y) = \frac{\partial^J u_E}{\partial y_{j_1}^{\nu_{j_1}} \cdots \partial y_{j_J}^{\nu_{j_J}}}(y_E)\,,
$$

$y^\nu = \prod_{j\in\operatorname{supp}\nu} y_j^{\nu_j}$, and $|\nu| = \sum_{j\geq 1} \nu_j$, $\nu! = \prod_{j\in\mathbb{N}} \nu_j!$.

Proposition 3. *Let f, g and D fulfill the assumption of the previous proposition. Assume A is of the special form*

$$A(y) = \overline{\psi} + \sum_{j \geq 1} y_j \psi_j \,, \quad \overline{\psi} \in \mathcal{W}_\infty^m(D) \,, \{\psi_j\}_{j \in \mathbb{N}} \subset \mathcal{W}_\infty^m(D) \,.$$

Further assume

$$\left(\|\psi_j | \mathcal{W}_\infty^m(D) \| \right)_{j \in \mathbb{N}} \in \ell_p(\mathbb{N})$$

for some $0 < p \leq 1$. Then $A : U \subset \ell_\infty(\mathbb{N}) \to \mathcal{W}_\infty^m(D)$ is well-defined and analytic.
Now further suppose A fulfills condition (6.2). Then $u : U \to \mathcal{K}_{a+1}^{m+1}(D)$ is analytic for all $|a| < \overline{a}$. Moreover, for its Taylor coefficients $t_\nu = \frac{1}{\nu!} \partial^\nu u(0)$, $\nu \in \mathcal{F}$, we find

$$\left(\|t_\nu | \mathcal{K}_{a+1}^{m+1}(D) \| \right)_{\nu \in \mathcal{F}} \in \ell_p(\mathcal{F}) \,.$$

Finally, it holds

$$u(y) = \sum_{\nu \in \mathcal{F}} t_\nu y^\nu \,, \quad y \in U \,, \tag{6.3}$$

with absolute and uniform on U convergence in the norm of $\mathcal{K}_{a+1}^{m+1}(D)$.

In view of our embedding results, the natural question now becomes whether we can replace $\mathcal{K}_{a+1}^{m+1}(D)$ in these results by appropriate Besov or Triebel-Lizorkin spaces. Here we will give an answer to this question, but only in case $\delta = 0$: While Theorem 3 can be applied also for $\delta > 0$, this would involve either restrictions on m, or we would have to apply results on the H^s-regularity of elliptic problems for $s > 1$, which is beyond the scope of the present work. Recall that due to $\mathcal{K}_1^1(D) \cap \{U|_{\partial D} = 0\} = H_0^1(D)$ the above proposition for $m = a = 0$ particularly includes an assertion about analyticity w.r.t. $H_0^1(D)$.
• Applying Theorem 3 and Corollary 2 we now immediately obtain conditions on a such that $u(y)$ belongs to $F_{\tau,\infty}^{m+1}(D)$, and particularly for $a \geq 0$ we can choose $\tau > \tau_*$, $\frac{1}{\tau_*} = \frac{m+1}{d} + \frac{1}{2}$.
To conclude analyticity of $u : U \to F_{\tau,\infty}^{m+1}(D)$, we recall that the mapping $(T, u) \mapsto Tu$ itself is analytic as a mapping from $L(X, Y) \times X$ into Y, where $L(X, Y)$ is the space of bounded linear operators from the Banach space X into another Banach space Y. Hence applying a fixed bounded linear operator to some analytic mapping preserves analyticity. This will be used with the operators P_{reg} and P_{sing}. Unfortunately, in our case the target space of these operators is no longer a Banach space. However, the notion of analyticity in every finite choice of variables is still preserved: This notion is equivalent to the existence of Frechèt derivatives (for functions in a finite number of complex variables), which clearly remains true after applying a bounded operator.
• The second aspect is the summability of the Taylor coefficients: The direct method as used in previous works, in particular applying Cauchy's integral formula, does no longer work. While the formula still holds in the sense of a Pettis integral (which is rather immediate, due to the fact that the Besov and Triebel-Lizorkin spaces considered here have a separating dual) and also as a Bochner-type integral in the sense of Vogt [29], both variants do not yield suitable estimates for the integral. However, a far more simpler approach already does the job: Once more applying Theorem 3, now to $t_\nu \in \mathcal{K}_{a+1}^{m+1}(D)$, yields

$$\|t_\nu | F_{\tau,\infty}^{m+1}(D)\| \lesssim \|t_\nu | \mathcal{K}_{a+1}^{m+1}(D)\| \,,$$

and accordingly for the $\ell_p(\mathcal{F})$-quasi-norm.

● Finally, we can also consider the respective Taylor series. Again, an application of Theorem 3 suffices to transfer the (uniform) convergence from $\mathcal{K}_{a+1}^{m+1}(D)$ to $F_{\tau,2}^{m+1}(D)$. However, note that the notion of absolute convergence changes in quasi-Banach spaces. It is well-known that for every quasi-norm on some quasi-normed space X there is an equivalent r-norm, i.e. a quasi-norm which satisfies $\|f + g\|^r \leq \|f\|^r + \|g\|^r$. Choosing this $r \leq 1$ as large as possible, a series $\sum_{j \geq 1} f_j$ converges absolutely w.r.t. this r-norm if, and only if $\sum_{j \geq 1} \|f_j\|^r < \infty$. Since the usual quasi-norm on $F_{\tau,2}^{m+1}(D)$ is a τ-norm (and $r = \tau$ is already the optimal choice), the power series (6.3) converges absolutely in $F_{\tau,2}^{m+1}(D)$ only if $p \leq \tau$.

Theorem 6. *Let the assumptions of Proposition 3 be fulfilled. Further assume $m - a < d(\frac{1}{\tau} - \frac{1}{2})$. Then for every finite choice of variables the mapping $u : U \to F_{\tau,2}^{m+1}(D)$ is analytic. Moreover, for its Taylor coefficients we find $\left(\|t_\nu|F_{\tau,2}^{m+1}(D)\| \right)_{\nu \in \mathcal{F}} \in \ell_p(\mathcal{F})$. The power series (6.3) converges uniformly on U in the quasi-norm of $F_{\tau,2}^{m+1}(D)$, and in case $p \leq \tau$ it converges absolutely for every $y \in U$.*

After discussing the regularity of the parametric mapping, we shall consider the implications to approximating the parametric solutions by partial sums of the power series, and particularly also with only approximately known expansion coefficients. A similar discussion can be found in [3, Section 8], and we will only give the necessary modifications of these arguments.

We want to approximate $u(y) = \sum_{\nu \in \mathcal{F}} t_\nu y^\nu$ by $\sum_{\nu \in \Lambda} \widetilde{t}_\nu y^\nu$, where $|\Lambda| \leq N$, and $\widetilde{t}_\nu$ is a Finite element or wavelet approximation of t_ν. When directly (naively) approximating the power series and estimating the error, we only obtain

$$\left\| u(y) - \sum_{\nu \in \Lambda} t_\nu y^\nu \Big| F_{\tau,2}^{m+1}(D) \right\|^\tau \leq \sum_{\nu \in \mathcal{F} \backslash \Lambda} \|t_\nu|F_{\tau,2}^{m+1}(D)\|^\tau \leq N^{-\tau/p+1} \left(\sum_{\nu \in \mathcal{F} \backslash \Lambda} \|t_\nu|F_{\tau,2}^{m+1}(D)\|^p \right)^{\tau/p},$$

where the last step is due to Stechkin, if we choose the index set Λ to correspond to the N largest coefficients. However, the situation is much more favorable, if we measure the error in the $H_0^1(D)$-norm: This time we obtain

$$\left\| u(y) - \sum_{\nu \in \Lambda} \widetilde{t}_\nu y^\nu \Big| H_0^1(D) \right\| \leq \left\| u(y) - \sum_{\nu \in \Lambda} t_\nu y^\nu \Big| H_0^1(D) \right\| + \left\| \sum_{\nu \in \Lambda} t_\nu y^\nu - \sum_{\nu \in \Lambda} \widetilde{t}_\nu y^\nu \Big| H_0^1(D) \right\|$$

$$\leq N^{-\frac{1}{p}+1} \left\| \left(\|t_\nu\|_{H_0^1(D)} \right)_{\nu \in \mathcal{F}} \right\|_{\ell_p(\mathcal{F})} + \sum_{\nu \in \Lambda^N} \left\| t_\nu - \widetilde{t}_\nu \right\|_{H_0^1(D)}$$

$$\leq N^{-\frac{1}{p}+1} \left\| \left(\|t_\nu\|_{H_0^1(D)} \right)_{\nu \in \mathcal{F}} \right\|_{\ell_p(\mathcal{F})} + c \sum_{\nu \in \Lambda} (N_\nu)^{-m/d} \|t_\nu|F_{\tau,2}^{m+1}(D)\|.$$

Therein N_ν stands for the number of degrees of freedom used to calculate the approximation of t_ν; i.e. either t_ν is an N_ν-term approximation w.r.t. some suitable wavelet system on D, or it is an adaptive Finite element approximation obtained from a triangulation with at most N_ν refinements as compared to the initial one. Balancing N_ν against the size of the coefficient, and further balancing both error contributions (i.e. approximation by partial sums, and approximating the coefficients within the partial sum) against each other, the final outcome can be formulated as follows:

Theorem 7. *Under the assumptions of Proposition 3 and Theorem 6 there exists an adaptive wavelet or Finite element approximation $\widetilde{u}(y) \in \left\{ \sum_{\nu \in \Lambda} c_\nu y^\nu : c_\nu \in H_0^1(D), |\Lambda| \leq N \right\}$ of the*

parametric solution $u(y)$ of (6.1) such that

$$\sup_{y \in U} \|u(y) - \widetilde{u}(y)|H_0^1(D)\| \lesssim N_{dof}^{-\min(\frac{1}{p}-1, \frac{m}{d})} \sup_{y \in U} \|f(y)|\mathcal{K}_{a-1}^{m-1}(D)\|.$$

7 Conclusions

We have given a new approache for proofs of embeddings between weighted Sobolev spaces (Kondratiev spaces), which appear in the regularity theory for solutions of (elliptic) PDEs on piecewise smooth domains, and Triebel-Lizorkin spaces, which together with Besov spaces are closely related to adaptive approximation schemes. This was based on estimates of wavelet coefficients and Maximal inequalities.

Moreover, we were able to show that the conditions for this embedding, as well as related regularity results for elliptic PDEs, were optimal by considering suitable representative functions. Ultimately, the embeddings were applied to obtain conditions for the optimal approximation rate $N^{-m/d}$ for adaptive approximation of solutions to elliptic PDEs by either wavelet or Finite element methods, and subsequently also to parametric problems.

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