Higher order Quasi Monte Carlo integration for holomorphic, parametric operator equations

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Abstract

We analyze the convergence of higher order Quasi-Monte Carlo (QMC) quadratures of solution-functionals to countably-parametric, nonlinear operator equations with distributed uncertain parameters taking values in a separable Banach space X. Such equations arise in numerical uncertainty quantification with random field inputs. Unconditional bases of X render the random inputs and the solutions of the forward problem countably parametric. We show that these parametric solutions belong to a class of weighted Bochner spaces of functions of countably many variables, with a particular structure of the QMC quadrature weights: up to a (problem-dependent, and possibly large) finite dimension, product weights can be used, and beyond this dimension, weighted spaces with so-called SPOD weights recently introduced in [F.Y. Kuo, Ch. Schwab, I.H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients. SIAM J. Numer. Anal. 50 (2012), 3351–3374] can be used to describe the solution regularity. The regularity results in the present paper extend those in [J. Dick, F.Y. Kuo, Q.T. Le Gia, D. Nuyens, Ch. Schwab, Higher order QMC (Petrov-)Galerkin discretization for parametric operator equations. To appear in SIAM J. Numer. Anal., 2015. Available at arXiv:1309.4624] established for affine parametric, linear operator families; they imply, in particular, efficient constructions of (sequences of) QMC quadrature methods there, which are applicable to these problem classes. We present a hybridized version of the fast component-by-component (CBC for short) construction of a certain type of higher order digital net. We prove that this construction exploits the product nature of the QMC weights with linear scaling with respect to the integration dimension up to a possibly large, problem dependent finite dimension, and the SPOD structure of the weights with quadratic scaling with respect to the weights beyond this dimension.

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1 Introduction

The numerical computation of statistical quantities for solutions of operator equations which depend on "uncertain input parameters" is a key task in uncertainty quantification in engineering and in the sciences. We consider here the case when the uncertain input quantities are random variables taking values in subsets of an infinite-dimensional, separable Banach space X. The system's *responses* to such random inputs are, in turn, random variables taking values in a state space $\mathcal{X}$. One is interested in *statistical moments* of these random responses, such as the mean response and (co)variance. These, and other *quantities of interest (QoI)* are then expressed as mathematical expectations over all realizations of the uncertain input $u \in X$.

The numerical approximation of such QoI's in these problems involves two basic steps: i) approximate (numerical) solution of the operator equation, and ii) approximate evaluation of the mathematical expectation by dimension-truncation and some form of dimension-robust numerical integration, i.e. an integration method that is free from the curse of dimensionality under certain assumptions on the integrand. In the present paper, we outline a strategy towards these two aims, which is based on i) a *(Petrov-)Galerkin discretization* of the parametric, nonlinear operator equation and on ii) higher order *QMC* integration. It is motivated in part by [21], where QMC integration using a family of *randomly shifted lattice rules* was combined with a Finite Element discretization for a model linear, parametric diffusion equation, and in part by [29], where the methodology of [21] was extended to problems described by an abstract family of *linear and affine-parametric* operator equations.

In contrast to [21, 29], we use *deterministic*, so-called *"interlaced polynomial lattice rules"*, which provide a convergence rate beyond order one for smooth integrands (cf. [8, 9]); convergence order one was the limitation in [21, 20, 29]. We establish sufficient conditions for higher order, dimension-independent convergence rates for the QMC evaluation of expectations of QoI's under the posterior, for a class of nonlinear, parametric operator equations.

The outline of this paper is as follows: in Section 2, we introduce a class of nonlinear, holomorphic-parametric operator equations with sufficient conditions on the nonlinear operators and on the uncertainty for the problems to be well-posed, as in [4, 14, 26]. We require that these conditions hold uniformly on the set $\tilde{X}$ of admissible uncertainties. We give a parametrization of the uncertain inputs which reduce the problem to a parametric, deterministic problem which depends on a possibly countable number of parameters $y_j \in [-1, 1]$. We review the theory of (Petrov-)Galerkin discretizations of these equations, and develop discretization error estimates. In Section 2.5 we review the notion of *holomorphy* of the integrand functions in these problems, from [4], whereas in Section 3, we present the first principal result of the present paper on analyticity and parametric regularity of the parametric integrand functions. Section 4 presents the convergence theory for higher order QMC quadratures, based on [12, 13], and, for the parametric integrands appearing here, on [10]. Based on the results in Section 3, the second principal result of this paper in Section 5 pertains to new variants of the fast component-by-component CBC constructions of generating vectors, which are developed based on [10, 16, 15] and which are tailored to the 'hybrid' nature of the QMC weights, with possibly more favorable complexity estimates for the CBC construction.

2 Holomorphic parametric operator equations

We present a class of operator equations which depend on an uncertain, "distributed parameter", being an element u in a real, separable Banach space X. For a given, known forcing term $f \in \mathcal{Y}'$, and any instance of u in (a subset of) X, the operator equation will admit a unique solution (also referred to as "response") $q \in \mathcal{X}$; here, $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be real, separable and reflexive Banach spaces and $\mathcal{Y}'$ is the dual space of $\mathcal{Y}$. In this section, we present a mathematical setting which accommodates this kind of problem and introduce conditions which ensure the (Lipschitz) continuous dependence of the response $q \in \mathcal{X}$ on the uncertain input $u \in X$. Assuming X to be separable and to admit an unconditional Schauder base $\Psi = \{\psi_j\}_{j \geq 1}$, with an eye towards QMC algorithms, we reformulate the operator equation with distributed uncertain input as infinite-dimensional, parametric operator equation where the uncertain input u is replaced by the sequence $\boldsymbol{y}$ of its coefficients y_j with respect to the basis Ψ. We then provide error bounds of the response subject to s-term truncations of the basis representation of u in terms of the Schauder basis Ψ. We also provide a general framework, from [26], for *Petrov-Galerkin approximation* of the responses $q \in \mathcal{X}$, and bound the combined error due to dimension-truncation and Petrov-Galerkin approximation. Finally, we investigate *analyticity* of the parameter dependence of the uncertainty-to-response map $X \ni u \to q \in \mathcal{X}$; to this end, we extend the Banach spaces X, $\mathcal{X}$ and $\mathcal{Y}$ to the coefficient field $\mathbb{C}$.

2.1 Nonlinear operator equations with uncertain input data

For a distributed, uncertain parameter $u \in X$, we consider a possibly nonlinear operator equation with input u which is defined by a "residual" operator $\mathcal{R} : X \times \mathcal{X} \to \mathcal{Y}'$, where $\mathcal{R}(u; q)$ acts, for given u, on $q \in \mathcal{X}$. We assume a known *"nominal parameter instance"* $\langle u \rangle \in X$ (such as, for example, the expectation of an X-valued random field u), and consider, for $u \in \mathscr{B}_X(\langle u \rangle; R)$, an open ball of radius $R > 0$ in X centered at $\langle u \rangle \in X$, the nonlinear operator equation

$$\text{given } u \in \mathscr{B}_X(\langle u \rangle; R) \text{ , find } q \in \mathcal{X} \quad \text{s.t.} \quad \mathcal{R}(u; q) = 0 \quad \text{in} \quad \mathcal{Y}' \text{ .}$$

Equivalently, with ${}_{\mathcal{Y}'}\langle \cdot, \cdot \rangle_{\mathcal{Y}}$ denoting the $\mathcal{Y}' \times \mathcal{Y}$-duality pairing,

$$\text{given } u \in \mathscr{B}_X(\langle u \rangle; R) \text{ , find } q \in \mathcal{X} \quad \text{s.t.} \quad {}_{\mathcal{Y}'}\langle \mathcal{R}(u; q), v \rangle_{\mathcal{Y}} = 0 \quad \forall v \in \mathcal{Y} \text{ .} \qquad (2.1)$$

Given $u \in \mathscr{B}_X(\langle u \rangle; R)$, we call a solution q_0 of (2.1) *regular at u* iff $\mathcal{R}(u; \cdot)$ is Fréchet differentiable with respect to q and the differential $D_q \mathcal{R}(u; q_0) \in \mathcal{L}(\mathcal{X}; \mathcal{Y}')$ is an isomorphism. Equivalently we can write: for every admissible $u \in \tilde{X} \subseteq X$, given a parametric forcing functional $F(u) \in \mathcal{Y}'$, find $q(u) \in \mathcal{X}$ which satisfies the residual equation

$$\mathcal{R}(u; q) = A(u; q) - F(u) = 0 \quad \text{in} \quad \mathcal{Y}' \text{ .} \qquad (2.2)$$

For the well-posedness of operator equations involving $\mathcal{R}(u; q)$ we assume the map $\mathcal{R}(u; \cdot) : \mathcal{X} \mapsto \mathcal{Y}'$ admits a family of regular solutions *locally, i.e. for each u in an open neighborhood of the nominal parameter instance $\langle u \rangle \in X$*. In particular, for all u in a sufficiently small, closed neighborhood $\tilde{X} \subseteq X$ of $\langle u \rangle \in X$ the problem (2.2) is well-posed. That is, for every fixed $u \in \tilde{X} \subset X$, and for every $F(u) \in \mathcal{Y}'$, there exists a unique solution $q(u)$ of (2.2)

which depends continuously on u. As in [3], we call the set $\{(u, q(u)) : u \in \tilde{X}\} \subset X \times \mathcal{X}$ a *regular branch of solutions* of (2.2) if

$$\tilde{X} \ni u \mapsto q(u) \text{ is continuous as mapping from } \tilde{X} \mapsto \mathcal{X} \text{ and}$$
$$\mathcal{R}(u; q(u)) = 0 \quad \text{in} \quad \mathcal{Y}' \text{ for all } u \in \tilde{X} . \tag{2.3}$$

We call (2.3) *nonsingular branch of solutions* if, in addition to (2.3), the differential

$$(D_q \mathcal{R})(u; q(u)) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}') \text{ is an isomorphism from } \mathcal{X} \text{ onto } \mathcal{Y}', \text{ for all } u \in \tilde{X} . \tag{2.4}$$

The following proposition collects well-known sufficient conditions for well-posedness of (2.2). For regular branches of nonsingular solutions given by (2.2) - (2.4), the differential $D_q \mathcal{R}$ satisfies the so-called *inf-sup conditions*.

Proposition 2.1. *Assume that $\mathcal{Y}$ is reflexive and that, for some nominal value $\langle u \rangle \in X$ of the uncertainty, the operator equation (2.2) admits a regular branch of nonsingular solutions (2.3), (2.4). Then the differential $D_q \mathcal{R}$ at $(\langle u \rangle, q_0)$, given by the bilinear map*

$$\mathcal{X} \times \mathcal{Y} \ni (\varphi, \psi) \mapsto {}_{\mathcal{Y}'}\langle D_q \mathcal{R}(\langle u \rangle; q_0)\varphi, \psi \rangle_{\mathcal{Y}} ,$$

is boundedly invertible, uniformly with respect to $u \in \tilde{X}$ where $\tilde{X} \subset X$ is an open neighborhood of the nominal instance $\langle u \rangle \in X$ of the uncertain parameter if and only if there exists a constant $0 < \mu \leq 1$ such that there holds

$$\forall u \in \tilde{X} : \quad
\begin{aligned}
&\inf_{0 \neq \varphi \in \mathcal{X}} \sup_{0 \neq \psi \in \mathcal{Y}} \frac{{}_{\mathcal{Y}'}\langle (D_q \mathcal{R})(u; q_0)\varphi, \psi \rangle_{\mathcal{Y}}}{\|\varphi\|_{\mathcal{X}} \|\psi\|_{\mathcal{Y}}} \geq \mu > 0 , \\
&\inf_{0 \neq \psi \in \mathcal{Y}} \sup_{0 \neq \varphi \in \mathcal{X}} \frac{{}_{\mathcal{Y}'}\langle (D_q \mathcal{R})(u; q_0)\varphi, \psi \rangle_{\mathcal{Y}}}{\|\varphi\|_{\mathcal{X}} \|\psi\|_{\mathcal{Y}}} \geq \mu > 0
\end{aligned} \tag{2.5}$$

and

$$\forall u \in \tilde{X} : \quad \|(D_q \mathcal{R})(u; q_0)\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} = \sup_{0 \neq \varphi \in \mathcal{X}} \sup_{0 \neq \psi \in \mathcal{Y}} \frac{{}_{\mathcal{Y}'}\langle (D_q \mathcal{R})(u; q_0)\varphi, \psi \rangle_{\mathcal{Y}}}{\|\varphi\|_{\mathcal{X}} \|\psi\|_{\mathcal{Y}}} \leq \mu^{-1} . \tag{2.6}$$

Under conditions (2.5) and (2.6), for every $u \in \tilde{X} \subseteq X$, there exists a unique, regular solution $q(u)$ of (2.2) which is uniformly bounded with respect to $u \in \tilde{X}$ in the sense that there exists a constant $C(F, \tilde{X}) > 0$, independent of q, such that

$$\sup_{u \in \tilde{X}} \|q(u)\|_{\mathcal{X}} \leq C(F, \tilde{X}) . \tag{2.7}$$

For (2.5) - (2.7) being valid, we shall say that the set $\{(u, q(u)) : u \in \tilde{X}\} \subset \tilde{X} \times \mathcal{X}$ forms a *regular branch of nonsingular solutions*.

If, in addition to Frechet differentiability of $\mathcal{R}$ with respect to q, for every $u \in \tilde{X} \subseteq X$, the nonlinear functional is also Frechet differentiable with respect to u at every point of the regular branch $\{(u, q(u)) : u \in \tilde{X}\} \subset \tilde{X} \times \mathcal{X}$, then the dependence of the mapping relating u to $q(u)$ with the branch of nonsingular solutions, is locally Lipschitz on $\tilde{X}$: i.e. there exists a Lipschitz constant $L(F, \tilde{X})$ such that

$$\forall u, v \in \tilde{X} : \quad \|q(u) - q(v)\|_{\mathcal{X}} \leq L(F, \tilde{X})\|u - v\|_X . \tag{2.8}$$

This follows from $(D_u q)(u) = -(D_q \mathcal{R})^{-1}(D_u \mathcal{R})$, from (2.5), and from the continuity (2.6) of the differential $D_q \mathcal{R}$ on the regular branch.

In what follows, we place ourselves in the abstract setting (2.2) with a uniformly continuously differentiable mapping $\mathcal{R}(u; q)$ in a product of neighborhoods $\mathscr{B}_X(\langle u \rangle; R) \times \mathscr{B}_{\mathcal{X}}(q(\langle u \rangle); R)$ of sufficiently small radius $R > 0$. The quantity $q(\langle u \rangle) \in \mathcal{X}$ is the corresponding regular solution of (2.2) at the nominal value $\langle u \rangle \in X$.

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2.2 Uncertainty parametrization

We shall be concerned with the particular case where $u \in X$ is a random variable taking values in a subset $\tilde{X}$ of the Banach space X. We assume that X is separable, infinite-dimensional, and admits an unconditional Schauder basis $\{\psi_j\}_{j \geq 1}$: $X = \text{span}\{\psi_j : j \geq 1\}$. Moreover, we assume the *summability condition*

$$\sum_{j \geq 1} \|\psi_j\|_X < \infty \; . \tag{2.9}$$

Let $U = [-1, 1]^{\mathbb{N}1}$ and assume that

$$\tilde{X} = \left\{ u = \langle u \rangle + \sum_{j \geq 1} y_j \psi_j : \boldsymbol{y} = (y_1, y_2, \ldots) \in U \right\} \; .$$

The properties of the set $\tilde{X}$ depend on the properties of the sequence $(\psi_j)_{j \geq 1}$. Uncertain data u with "higher regularity" (when measured in a smoothness scale $\{X_t\}_{t \geq 0}$ with $X = X_0 \supset X_1 \supset X_2 \supset \ldots$) corresponds to a stronger decay of the sequence $(\|\psi_j\|_X)_{j \geq 1}$: specifically, we shall assume in what follows that $\{\psi_j\}_{j \geq 1}$ is scaled such that the sequence $\boldsymbol{b} = (b_j)_{j \geq 1}$ given by

$$\boldsymbol{b} := \{\|\psi_j\|_X\}_{j \geq 1} \in \ell^p(\mathbb{N}) \text{ for some } 0 < p < 1 \; . \tag{2.10}$$

Once an unconditional Schauder basis $\Psi = \{\psi_j\}_{j \geq 1}$ of X has been selected, every realization $u \in \tilde{X}$ can be identified in a one-to-one fashion with the pair $(\langle u \rangle, \boldsymbol{y})$ via

$$u = \langle u \rangle + \sum_{j \geq 1} y_j \psi_j, \tag{2.11}$$

where $\langle u \rangle \in X$ denotes the *nominal instance* of the uncertain datum u and $\boldsymbol{y}$ is the coordinate vector of the basis representation (2.11).

Remark 2.1. The operator $A(u; q)$ in (2.2) becomes, via the parametric dependence $u = u(\boldsymbol{y})$, a parametric operator family $A(u(\boldsymbol{y}); q)$ which we denote (with a slight abuse of notation) by $\{A(\boldsymbol{y}; q) : \boldsymbol{y} \in U\}$, with the parameter set $U = [-1, 1]^{\mathbb{N}}$. Similarly we write $\mathcal{R}(\boldsymbol{y}; q)$ instead of $\mathcal{R}(u, q)$ in the following. In the particular case that the parametric operator family is linear, we have $A(\boldsymbol{y}; q) = A(\boldsymbol{y})q$ with $A(\boldsymbol{y}) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$. We do not assume, however, that the maps $\mathcal{X} \ni q \mapsto A(\boldsymbol{y}; q) \in \mathcal{Y}'$ are linear in what follows, unless explicitly stated.

With these conventions and with (2.11), we may restate (2.2) as parametric operator equation: given $F : U \to \mathcal{Y}'$,

$$\text{find } q(\boldsymbol{y}; F) \in \mathcal{X} : \quad \forall \boldsymbol{y} \in U : \quad \mathcal{R}(\boldsymbol{y}; q) := A(\boldsymbol{y}; q) - F(\boldsymbol{y}) = 0 \quad \text{in} \quad \mathcal{Y}' \tag{2.12}$$

or, equivalently, with $\mathfrak{a}(\boldsymbol{y}; q, v) = {}_{\mathcal{Y}'}\langle A(\boldsymbol{y}; q), v \rangle_{\mathcal{Y}}$ and $\mathfrak{f}(\boldsymbol{y}; v) := {}_{\mathcal{Y}'}\langle F(\boldsymbol{y}), v \rangle_{\mathcal{Y}}$,

$${}_{\mathcal{Y}'}\langle \mathcal{R}(\boldsymbol{y}; q), v \rangle_{\mathcal{Y}} = \mathfrak{a}(\boldsymbol{y}; q, v) - \mathfrak{f}(\boldsymbol{y}; v) = 0 \quad \forall v \in \mathcal{Y} \; . \tag{2.13}$$

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[1]In the QMC part we rescale this set to $[-1/2, 1/2]^{\mathbb{N}}$, shift it to $[0, 1]^{\mathbb{N}}$ and integrate with respect to the product of the Lebesgue-measure in $[0, 1]^{\mathbb{N}}$.

With this understanding, and under the assumptions (2.7) and (2.8), the operator equation (2.2) will admit, for every $\boldsymbol{y} \in U$, a unique solution $q(\boldsymbol{y}; F)$ which is, due to (2.7) and (2.8), uniformly bounded and depends Lipschitz continuously on the parameter sequence $\boldsymbol{y} \in U$: there holds

$$\sup_{\boldsymbol{y} \in U} \|q(\boldsymbol{y}; F)\|_{\mathcal{X}} \leq C(F, U) \tag{2.14}$$

for some constant $C(F, U)$ which is independent of q, and, if the local Lipschitz condition (2.8) holds, there exists a Lipschitz constant $L > 0$ such that (denoting by $\boldsymbol{y}, \boldsymbol{y}' \in U$ the coefficient sequences associated with $u, v \in \tilde{X}$ via (2.11))

$$\begin{aligned}
\|q(\boldsymbol{y}; F) - q(\boldsymbol{y}'; F)\|_{\mathcal{X}} &\leq L(F, U)\|u - v\|_X \\
&\leq L(F, U)\|\boldsymbol{y} - \boldsymbol{y}'\|_{\ell^1(\mathbb{N})} \sup_{j \in \mathbb{N}} \|\psi_j\|_X .
\end{aligned} \tag{2.15}$$

We remark that the Lipschitz constant $L(F, U) > 0$ in (2.15) is not, in general, equal to $L(F, \tilde{X})$ in (2.8): it depends on $\langle u \rangle \in X$ and on the choice of $\{\psi_j\}_{j \geq 1}$.

2.3 (Petrov-)Galerkin discretization

In this section we present, based on the theory in [14, Chap. IV.3] and in [26], which goes back to [3] and to M. Crouzeix, an error analysis of (Petrov-)Galerkin discretizations of (2.12) for the approximation of regular branches of solutions of smooth, nonlinear problems (2.2). This will allow us, in the next section, to generalize the results [21, 20, 22] on Quasi-Monte Carlo (QMC) (Petrov-)Galerkin approximations for countably-parametric operator equations (2.12).

To this end, as in [29, 10], we assume that we are given two one-parameter sequences $\{\mathcal{X}^h\}_{h>0} \subset \mathcal{X}$ and $\{\mathcal{Y}^h\}_{h>0} \subset \mathcal{Y}$ of finite dimensional subspaces. We assume also that, as the discretization parameter $h \downarrow 0$, these sequences are dense in $\mathcal{X}$ and in $\mathcal{Y}$, respectively. For the computational complexity analysis, we further assume the following *approximation properties*: there is a scale $\{\mathcal{X}_t\}_{t \geq 0}$ of subspaces such that $\mathcal{X}_{t'} \subset \mathcal{X}_t \subset \mathcal{X}_0 = \mathcal{X}$ for any $0 < t < t' < \infty$ and such that, for $0 < t \leq \bar{t}$ and $0 < t' \leq \bar{t}'$, and for $0 < h \leq h_0$, there holds

$$\forall v \in \mathcal{X}_t : \quad \inf_{v^h \in \mathcal{X}^h} \|v - v^h\|_{\mathcal{X}} \leq C_t h^t \|v\|_{\mathcal{X}_t} . \tag{2.16}$$

Typical examples of smoothness scales $\{\mathcal{X}_t\}_{t \geq 0}$ are given by the Sobolev scale $\mathcal{X}_t = H^{1+t}(D)$ in smooth domains or by its weighted counterparts in polyhedra [24].

Proposition 2.2. *Assume that the subspace sequences $\{\mathcal{X}^h\}_{h>0} \subset \mathcal{X}$ and $\{\mathcal{Y}^h\}_{h>0} \subset \mathcal{Y}$ are stable, i.e., there exist $\bar{\mu} > 0$ and $h_0 > 0$ such that for every $0 < h \leq h_0$, there hold the uniform (with respect to $\boldsymbol{y} \in U$) discrete inf-sup conditions*

$$\forall \boldsymbol{y} \in U : \quad \inf_{0 \neq v^h \in \mathcal{X}^h} \sup_{0 \neq w^h \in \mathcal{Y}^h} \frac{_{\mathcal{Y}'}\langle (D_q\mathcal{R})(\boldsymbol{y}; q_0)v^h, w^h\rangle_{\mathcal{Y}}}{\|v^h\|_{\mathcal{X}}\|w^h\|_{\mathcal{Y}}} \geq \bar{\mu} > 0 , \tag{2.17}$$

$$\forall \boldsymbol{y} \in U : \quad \inf_{0 \neq w^h \in \mathcal{Y}^h} \sup_{0 \neq v^h \in \mathcal{X}^h} \frac{_{\mathcal{Y}'}\langle (D_q\mathcal{R})(\boldsymbol{y}; q_0)v^h, w^h\rangle_{\mathcal{Y}}}{\|v^h\|_{\mathcal{X}}\|w^h\|_{\mathcal{Y}}} \geq \bar{\mu} > 0 . \tag{2.18}$$

Assume in addition that the differential $(D_q\mathcal{R})(\boldsymbol{y}; q) \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$ is Lipschitz with respect to q, uniformly with respect to $\boldsymbol{y} \in U$, i.e.

$$\forall \boldsymbol{y} \in U : \quad \|(D_q\mathcal{R})(\boldsymbol{y}; q) - (D_q\mathcal{R})(\boldsymbol{y}; \tilde{q})\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \leq L\|q - \tilde{q}\|_{\mathcal{X}} , \tag{2.19}$$

where the Lipschitz constant is independent of $\boldsymbol{y}$.

Then, for every $0 < h \leq h_0$ the (Petrov-)Galerkin approximations: given $\boldsymbol{y} \in U$,

$$\text{find } q^h(\boldsymbol{y}) \in \mathcal{X}^h : \quad {}_{\mathcal{Y}'}\langle \mathcal{R}(\boldsymbol{y}; q^h(\boldsymbol{y})), w^h \rangle_{\mathcal{Y}} = 0 \quad \forall w^h \in \mathcal{Y}^h , \tag{2.20}$$

are uniquely defined and converge quasioptimally; i.e. there exists a constant $C > 0$ such that for all $\boldsymbol{y} \in U$

$$\|q(\boldsymbol{y}) - q^h(\boldsymbol{y})\|_{\mathcal{X}} \leq \frac{C}{\bar{\mu}} \inf_{0 \neq v^h \in \mathcal{X}^h} \|q(\boldsymbol{y}) - v^h\|_{\mathcal{X}} . \tag{2.21}$$

If the response $q(\boldsymbol{y}) \in \mathcal{X}_t$ is uniformly w.r.t. $\boldsymbol{y}$ and (2.16) holds, then

$$\|q(\boldsymbol{y}) - q^h(\boldsymbol{y})\|_{\mathcal{X}} \leq \frac{C}{\bar{\mu}} h^t \sup_{\boldsymbol{y} \in U} \|q(\boldsymbol{y})\|_{\mathcal{X}_t} . \tag{2.22}$$

This result follows, under the stated hypotheses, from [26, Thm. 4]. In the ensuing QMC convergence analysis we shall also require error bounds for the dimensionally truncated parameter sequences.

2.4   Dimension truncation

For a *truncation dimension* $s \in \mathbb{N}$, denote the s-term truncation of the series representation (2.11) of the uncertain datum u by $u^s \in X$. Then, dimension truncation is equivalent to setting $y_j = 0$ for $j > s$ in (2.11). We denote by $q^s(\boldsymbol{y})$ the solution of the corresponding parametric weak problem (2.12). Unique solvability of (2.12) for every $\boldsymbol{y} \in U$ implies also unique solvability for the dimension truncated problem with solution $q^s(\boldsymbol{y}) = q(\{y_1, y_2, ..., y_s, 0, ...\})$. For $\boldsymbol{y} \in U$, we define

$$\boldsymbol{y}_{\{1:s\}} := (y_1, y_2, ..., y_s, 0, 0, ...) \tag{2.23}$$

and introduce $u^s(\boldsymbol{y}) := u(\boldsymbol{y}_{\{1:s\}})$. We bound the *dimension truncation error* $q(\boldsymbol{y}) - q^s(\boldsymbol{y})$ based on

Assumption 1. *(i) $\boldsymbol{b} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$, i.e. (2.10) holds;*
(ii) the b_j are enumerated in non-increasing order, i.e.

$$b_1 \geq b_2 \geq \cdots \geq b_j \geq \cdots . \tag{2.24}$$

Under Assumption 1, we consider the *s-term truncated problem*: given $\boldsymbol{y}_{\{1:s\}} \in U$

$$\text{find } q^s \in \mathcal{X} : \quad {}_{\mathcal{Y}'}\langle \mathcal{R}(\boldsymbol{y}_{\{1:s\}}; q^s), w \rangle_{\mathcal{Y}} = 0 \quad \forall w \in \mathcal{Y} . \tag{2.25}$$

Under our assumption on well-posedness of the problem (2.1) uniformly for all $u \in \mathscr{B}_X(\langle u \rangle; R)$, the basis property (2.11) of the sequence $\{\psi_j\}$ implies that $u^s \in \tilde{X}$ and therefore (2.25) admits a unique solution.

Theorem 2.1. *Under the Assumptions in Section 2.1, and assuming (2.11) and (2.10), for every $f \in \mathcal{Y}'$, for every $\boldsymbol{y} \in U$ and for every $s \in \mathbb{N}$, the solution $q^s(\boldsymbol{y})$ of the parametric*

weak problem (2.12) *with s-term truncated parametric expansion* (2.11) *satisfies, with* b_j
as defined in (2.10),

$$\sup_{\boldsymbol{y} \in U} \|q(\boldsymbol{y}) - q^s(\boldsymbol{y})\|_{\mathcal{X}} \leq C(F, X) \sum_{j \geq s+1} b_j . \tag{2.26}$$

Moreover, for every observation functional $\mathscr{O}(\cdot) \in \mathcal{X}'$, *there holds the dimension-truncation error bound*

$$|I(\mathscr{O}(q)) - I(\mathscr{O}(q^s))| \leq \tilde{C} \left(\sum_{j \geq s+1} b_j \right) \tag{2.27}$$

for some constant $\tilde{C} > 0$ *independent of s. In addition, if conditions* (2.9), (2.10) *and* (2.24) *hold, then*

$$\sum_{j \geq s+1} b_j \leq \min\left(\frac{1}{1/p - 1}, 1 \right) \left(\sum_{j \geq 1} b_j^p \right)^{1/p} s^{-(1/p-1)} . \tag{2.28}$$

Proof. From the Lipschitz dependence (2.8), we obtain

$$\forall \boldsymbol{y} \in U : \quad \|q(\boldsymbol{y}) - q^s(\boldsymbol{y})\|_{\mathcal{X}} \leq L(F, \tilde{X}) \|u(\boldsymbol{y}) - u^s(\boldsymbol{y})\|_X .$$

From (2.11), the p-summability (2.10) of the sequence $\boldsymbol{b}$ and the monotonicity (2.24) we infer that the error of the s-term truncation $u^s(\boldsymbol{y})$, $\|u(\boldsymbol{y}) - u^s(\boldsymbol{y})\|_X$, can be bounded by a best s-term truncation error of $\boldsymbol{b}$ in the norm of $\ell^1(\mathbb{N})$ by

$$\sup_{\boldsymbol{y} \in U} \|u(\boldsymbol{y}) - u^s(\boldsymbol{y})\|_X \leq \sum_{j \geq s+1} b_j .$$

The p-summability $\boldsymbol{b} \in \ell^p(\mathbb{N})$ in Assumption 1(i) and the (assumed) ordering (2.24) imply (2.28). $\qquad\square$

As $\boldsymbol{y} \in U$ implies $\boldsymbol{y}_{\{1:s\}} \in U$ for all $s \in \mathbb{N}$, we obtain from Proposition 2.2 immediately

Corollary 2.1. *Under the assumptions of Proposition 2.2, for given* $\boldsymbol{y}_{\{1:s\}} \in U$, *the dimensionally truncated (Petrov-)Galerkin approximations*

$$\text{find } q^h(\boldsymbol{y}_{\{1:s\}}) \in \mathcal{X}^h : \quad {}_{\mathcal{Y}'}\langle \mathcal{R}(\boldsymbol{y}_{\{1:s\}}; q^h(\boldsymbol{y}_{\{1:s\}})), w^h \rangle_{\mathcal{Y}} = 0 \quad \forall w^h \in \mathcal{Y}^h , \tag{2.29}$$

admit unique solutions $q^h(\boldsymbol{y}_{\{1:s\}}) \in \mathcal{X}^h$ *which converge, as* $h \downarrow 0$, *quasioptimally to* $q(\boldsymbol{y}_{\{1:s\}}) \in \mathcal{X}$, *i.e.* (2.21) *and* (2.22) *hold with* $\boldsymbol{y}_{\{1:s\}}$ *in place of* $\boldsymbol{y}$, *with* $C > 0$ *and* $\bar{\mu} > 0$ *independent of s, of* $\boldsymbol{y} \in U$ *and of h.*

2.5 Holomorphic parameter dependence

In the error analysis for QMC integration methods as presented, e.g., in [21, 20, 22], derivative bounds for the integrand functions that are explicit with respect to the dimension s are essential. In [6, 4], such bounds were obtained via holomorphy of countably parametric families of operator equations and their parametric solutions. By this we mean that the parametric family of solutions permits, with respect to each parameter y_j, a holomorphic extension into the complex domain $\mathbb{C}$; for purposes of QMC integration, in addition, some uniform bounds on these holomorphic extensions must be satisfied in order to prove approximation rates and QMC quadrature error bounds which are independent of the number of parameters which are "activated" in the QMC quadrature process.

In the remainder of Section 2 and throughout the next Section 3, all spaces X, $\mathcal{X}$ and $\mathcal{Y}$ will be understood as Banach spaces over $\mathbb{C}$, without notationally indicating so.

2.5.1 $(\boldsymbol{b}, p, \varepsilon)$-Holomorphy

In [17, 4], the notion of $(\boldsymbol{b}, p, \varepsilon)$-*holomorphy of parametric solutions* has been introduced. For $\kappa > 1$, we define the "κ-tube" about $[-1, 1]$, denoted by $\mathcal{T}_\kappa$, as the set

$$\mathcal{T}_\kappa = \{z \in \mathbb{C} \mid \operatorname{dist}(z, [-1, 1]) \leq \kappa - 1\} = \bigcup_{-1 \leq y \leq 1} \{z \in \mathbb{C} \mid |z - y| \leq \kappa - 1\} \subset \mathbb{C} . \quad (2.30)$$

Definition 2.1. $((\boldsymbol{b}, p, \varepsilon)$-holomorphy$)$ For $\varepsilon > 0$ and for a positive sequence $\boldsymbol{b} = (b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$, we say that a parametric solution family $q(\boldsymbol{y}) : U \mapsto \mathcal{X}$ of (2.2) satisfies the $(\boldsymbol{b}, p, \varepsilon)$-*holomorphy assumption* if and only if all of the following conditions hold:

1. For each $\boldsymbol{y} \in U$, the map $\boldsymbol{y} \mapsto q(\boldsymbol{y})$ from U to $\mathcal{X}$ is uniformly bounded w.r.t. the parameter sequence $\boldsymbol{y}$, i.e.

$$\sup_{\boldsymbol{y} \in U} \|q(\boldsymbol{y})\|_X \leq B_0 , \quad (2.31)$$

   for some finite constant $B_0 > 0$.

2. For any sequence $\rho := (\rho_j)_{j \geq 1}$ of numbers $\rho_j > 1$ that satisfies

$$\sum_{j \geq 1} (\rho_j - 1) b_j \leq \varepsilon, \quad (2.32)$$

   the parametric solution map $U \ni \boldsymbol{y} \mapsto q(\boldsymbol{y})$ admits an extension $\boldsymbol{z} \mapsto q(\boldsymbol{z})$ to the complex domain that is holomorphic with respect to each variable z_j in a cylindrical set of the form $\mathcal{O}_{\boldsymbol{\rho}} := \bigotimes_{j \geq 1} \mathcal{O}_{\rho_j}$, where, for every integer $j \geq 1$, $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing the closed tube $\mathcal{T}_{\rho_j}$. For a poly-radius $\boldsymbol{\rho}$ satisfying (2.32), we denote by $\mathcal{T}_{\boldsymbol{\rho}}$ the corresponding cylindrical set $\mathcal{T}_{\boldsymbol{\rho}} := \bigotimes_{j \geq 1} \mathcal{T}_{\rho_j} \subset \mathbb{C}^{\mathbb{N}}$.

3. For any poly-radius ρ satisfying (2.32), there is a second family $\tilde{\mathcal{O}}_{\boldsymbol{\rho}} := \bigotimes_{j \geq 1} \tilde{\mathcal{O}}_{\rho_j}$ of open, cylindrical sets

$$\mathcal{O}_{\rho_j} \subset \tilde{\mathcal{O}}_{\rho_j} \subset \mathbb{C}$$

   (strict inclusions), such that the extension is bounded on the closure $\overline{\tilde{\mathcal{O}}_{\boldsymbol{\rho}}}$ of $\tilde{\mathcal{O}}_{\boldsymbol{\rho}}$ according to

$$\sup_{\boldsymbol{z} \in \tilde{\mathcal{O}}_{\boldsymbol{\rho}}} \|q(\boldsymbol{z})\|_X \leq B_\varepsilon , \quad (2.33)$$

   where $B_\varepsilon > 0$ depends on ε, but is independent of $\boldsymbol{\rho}$.

The notion of $(\boldsymbol{b}, p, \varepsilon)$-holomorphy depends implicitly on the choice of sets $\mathcal{O}_{\boldsymbol{\rho}}$ and $\tilde{\mathcal{O}}_{\boldsymbol{\rho}}$. Depending on the approximation process in the parameter domain U under consideration, a particular choice of the sets $\tilde{\mathcal{O}}_{\rho_j}$ has to be made in order to obtain sharp convergence bounds under minimal holomorphy requirements.

In [4, 6], the sets $\mathcal{O}_{\boldsymbol{\rho}}$ were chosen to contain Bernstein ellipses $\mathcal{E}_{\boldsymbol{\rho}}$ which are natural in the context of Legendre polynomial chaos approximations. In the context of Taylor- or Tschebyscheff polynomial approximations, polydiscs $\mathcal{O}_{\boldsymbol{\rho}}$ are natural (cf. [17]). For the derivative bounds which arise in connection with higher order QMC error analysis (see, e.g., [10, 21]), we use the tubes $\mathcal{T}_{\boldsymbol{\rho}}$ (2.30) as continuation domains $\mathcal{O}_{\boldsymbol{\rho}}$ and $\tilde{\mathcal{O}} = \mathcal{T}_{\tilde{\boldsymbol{\rho}}}$ with $\tilde{\rho} > \rho > 1$.

2.5.2 Holomorphic parametric operator equations

We next consider parametric models (2.12) and the regularity of their (countably-) parametric solution families. The following result, established in [4, Thm. 4.1 and Lem. 4.4], ensures $(\boldsymbol{b}, p, \varepsilon)$-holomorphy of the parametric solution map $\boldsymbol{y} \mapsto q(\boldsymbol{y})$ with respect to $\mathcal{T}_{\boldsymbol{\rho}}$ under the assumption of $(\boldsymbol{b}, p, \varepsilon)$-holomorphy of the maps A and F in (2.2) and (2.12).

Theorem 2.2. *For $\varepsilon > 0$ and $0 < p < 1$, assume that there exist a positive sequence $\boldsymbol{b} = (b_j)_{j \geq 1} \in \ell^p(\mathbb{N})$, two constants $0 < \tilde{\mu} \leq \tilde{M} < \infty$ and a constant $M < \infty$ independent of $\boldsymbol{y} \in U$ such that the following holds:*

1. *For any sequence $\rho := (\rho_j)_{j \geq 1}$ of numbers strictly greater than 1 that satisfies (2.32), the parametric maps A and F in (2.12) admit extensions that are holomorphic with respect to every variable $\boldsymbol{z}$ in a cylindrical set $\widetilde{\mathcal{O}}_{\rho} = \bigotimes_{j \geq 1} \widetilde{\mathcal{O}}_{\rho_j}$, where $\widetilde{\mathcal{O}}_{\rho_j} \subset \mathbb{C}$ is an open set containing the closed tube $\mathcal{T}_{\rho_j}$.*

2. *These extensions satisfy, for all $\boldsymbol{z} \in \widetilde{\mathcal{O}}_{\rho}$, the uniform continuity conditions*

$$\sup_{w \in \mathcal{Y} \backslash \{0\}} \frac{|F(\boldsymbol{z}; w)|}{\|w\|_{\mathcal{Y}}} \leq M, \qquad \sup_{v \in \mathcal{X} \backslash \{0\}, w \in \mathcal{Y} \backslash \{0\}} \frac{|\mathfrak{a}(\boldsymbol{z}; v, w)|}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \leq \tilde{M}, \qquad (2.34)$$

and the uniform inf-sup conditions: there exists $\tilde{\mu} > 0$ such that for every $\boldsymbol{z} \in \widetilde{\mathcal{O}}_{\rho}$ hold the uniform inf-sup conditions

$$\inf_{v \in \mathcal{X} \backslash \{0\}} \sup_{w \in \mathcal{Y} \backslash \{0\}} \frac{|\mathfrak{a}(\boldsymbol{z}; v, w)|}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \tilde{\mu} \quad and \quad \inf_{w \in \mathcal{Y} \backslash \{0\}} \sup_{v \in \mathcal{X} \backslash \{0\}} \frac{|\mathfrak{a}(\boldsymbol{z}; v, w)|}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \tilde{\mu} . \qquad (2.35)$$

Then, the nonlinear, parametric residual operator $\mathcal{R}(\boldsymbol{z}; q) = A(\boldsymbol{z}; q) - F(\boldsymbol{z}; q)$ in (2.2), (2.13) satisfies the $(\boldsymbol{b}, p, \varepsilon)$-holomorphy assumptions for $\boldsymbol{z} \in \mathcal{T}_{\rho}$ with the same p and ε and with the same sequence $\boldsymbol{b}$.

3 Parametric regularity of solutions

In this section we study the dependence of the solution $q(\boldsymbol{y})$ of the parametric, variational problem (2.12) on the parameter vector $\boldsymbol{y}$, with precise bounds on the growth of the partial derivatives. These bounds imply, in conjunction with the results in [21], dimension independent convergence rates for QMC quadratures.

In the following, let $\mathbb{N}_0^{\mathbb{N}}$ denote the set of sequences $\boldsymbol{\nu} = (\nu_j)_{j \geq 1}$ of nonnegative integers ν_j, and let $|\boldsymbol{\nu}| := \sum_{j \geq 1} \nu_j$. For $|\boldsymbol{\nu}| < \infty$, we denote the partial derivative of order $\boldsymbol{\nu}$ of $q(\boldsymbol{y})$ with respect to $\boldsymbol{y}$ by

$$\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q(\boldsymbol{y}) := \frac{\partial^{|\boldsymbol{\nu}|}}{\partial_{y_1}^{\nu_1} \partial_{y_2}^{\nu_2} \cdots} q(\boldsymbol{y}) . \qquad (3.1)$$

In [5, 21, 19], bounds on the derivatives (3.1) were obtained by an induction argument which strongly relied on affine-parametric dependence of $A(\boldsymbol{y}; q)$ on $\boldsymbol{y}$.

Here, we derive alternative bounds on $\|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\boldsymbol{y})\|_{\mathcal{X}}$ based on complex variable methods from [6, 30, 27, 28, 4]. We shall see that in QMC integration these bounds give rise to product weights at least for a finite (possibly large, but in general operator-dependent)

"leading" dimension of the parameter space. The argument is based on *holomorphic extension of the parametric integrand functions into the complex domain* (we remark that not all PDE problems afford such extensions and refer to [18] for an example).

In certain cases, the possibility of covering the parameter intervals $[-1, 1]$ by a finite number of small balls (whose union is contained in a tube $\mathcal{T}_{\rho_j}$ (2.30) for a radius $\rho_j > 1$ sufficiently close to 1) is crucial to verify $(\boldsymbol{b}, p, \varepsilon)$-holomorphy for certain nonlinear operator equations, see for example [4, Lem. 4.4, Sec. 5.2].

Theorem 3.1. *For every mapping $q(\boldsymbol{y}) : U \mapsto \mathcal{X}$ which is $(\boldsymbol{b}, p, \varepsilon)$-holomorphic on a polytube $\mathcal{T}_{\boldsymbol{\rho}}$ of poly-radius $\boldsymbol{\rho} = (\rho_j)_{j \geq 1}$ with $\rho_j > 1$ satisfying (2.32), there exists a sequence $\boldsymbol{\beta} \in \ell^p(\mathbb{N})$ (depending on the sequence $\boldsymbol{b}$ in (2.32)) and a partition $\mathbb{N} = E \cup E^c$ such that the parametric solution $q(\boldsymbol{y})$ satisfies, for every $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$ with $|\boldsymbol{\nu}| < \infty$, the bound*

$$\sup_{\boldsymbol{y} \in U} \|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\boldsymbol{y})\|_{\mathcal{X}} \leq C \boldsymbol{\nu}_E! \prod_{j \in E} \beta_j^{\nu_j} \times |\boldsymbol{\nu}_{E^c}|! \prod_{j \in E^c} \beta_j^{\nu_j} . \tag{3.2}$$

Here, $E = \{1, 2, ..., J\}$ for some $J(\boldsymbol{b}) < \infty$ depending on the sequence $\boldsymbol{b}$ in (2.32), and for $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$, we set $\boldsymbol{\nu}_E := \{\nu_j : j \in E\}$. The sequence $\boldsymbol{\beta} = (\beta_j)_{j \geq 1}$ satisfies $\beta_j = 4\|\boldsymbol{b}\|_{\ell^1(\mathbb{N})}/\varepsilon$ for $1 \leq j \leq J$, i.e. it is in particular independent of j for $1 \leq j \leq J$. Moreover, $\beta_j \lesssim b_j$ for $j > J$ with the implied constant depending only on $J(\boldsymbol{b})$ and on $\|\boldsymbol{b}\|_{\ell^1(\mathbb{N})}$.

The proof of the derivative bound is divided into two steps. To simplify the notation, we give it for $|y_j| \leq 1$ and for a poly-radius $\boldsymbol{\rho}$ which satisfies $\rho_j > 1$. Later, in Section 4 it is natural to consider the parameter domain $[-1/2, 1/2]^{\mathbb{N}}$. The assertion for the parameter domain $U = [-1/2, 1/2]^{\mathbb{N}}$ then follows via scaling by a factor of $1/2$ (see Remark 4.1 for details).

In the first step, we infer from $(\boldsymbol{b}, p, \varepsilon)$-holomorphy of $q(\boldsymbol{y})$, via Cauchy's integral formula, bounds on $\sup_{\boldsymbol{y} \in [-1,1]^{\mathbb{N}}} \|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\boldsymbol{y})\|_{\mathcal{X}}$ in terms of the maximum of the analytic continuation of $q(\boldsymbol{y})$ to the domain $\mathcal{T}_{\boldsymbol{\rho}}$ of points in the "polytube" $\mathcal{T}_{\boldsymbol{\rho}}$. These derivative bounds are valid *for any poly-radius $\boldsymbol{\rho}$ which is $(\boldsymbol{b}, p, \varepsilon)$-admissible in the sense that (2.32) holds.* The result of the first step is recorded in Lemma 3.1.

In the second step of the proof, we use a $\boldsymbol{\nu}$-dependent choice of a $(\boldsymbol{b}, p, \varepsilon)$-admissible poly-radius $\boldsymbol{\rho}$ for which (2.32) holds to obtain the ($\boldsymbol{\nu}$-*independent*) weight sequence $\boldsymbol{\beta}$: for given $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$ such that $|\boldsymbol{\nu}| < \infty$, we then define a $(\boldsymbol{b}, p, \varepsilon)$-admissible poly-radius $\boldsymbol{\rho}(\boldsymbol{\nu})$ so that (3.2) is satisfied for this $\boldsymbol{\nu}$, with constants C_0 and the sequence $\boldsymbol{\beta}$ independent of $\boldsymbol{\nu}$.

In the following let for $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$ the support of $\boldsymbol{\nu}$ be denoted by $\operatorname{supp} \boldsymbol{\nu} := \{j \in \mathbb{N} : \nu_j \neq 0\} \subset \mathbb{N}$. For a subset $H \subseteq \mathbb{N}$, we denote its complement $H^c = \mathbb{N} \setminus H$ and for a vector $\boldsymbol{y} = (y_j)_{j \geq 1}$, $\boldsymbol{y}_H = (y_j)_{j \in H}$ denotes its "restriction" to H.

Lemma 3.1. *For every mapping $q(\boldsymbol{y}) : U \mapsto \mathcal{X}$ which is $(\boldsymbol{b}, p, \varepsilon)$-holomorphic on a polytube $\mathcal{T}_{\boldsymbol{\rho}}$ of poly-radius $\boldsymbol{\rho} = (\rho_j)_{j \geq 1}$ with $\rho_j > 1$ satisfying (2.32), there holds*

$$\sup_{\boldsymbol{y}_H \in \prod_{j \in H} [-1,1]} \|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\boldsymbol{y}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \leq \sup_{\boldsymbol{z}_H \in \prod_{j \in H} \mathcal{T}_{\rho_j}} \|q(\boldsymbol{z}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \boldsymbol{\nu}! \prod_{j \in H} \frac{\rho_j}{(\rho_j - 1)^{\nu_j + 1}} ,$$

for every $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$ with $|\boldsymbol{\nu}| < \infty$, where $H = \operatorname{supp} \boldsymbol{\nu}$ and for every $\boldsymbol{y}_{H^c} \in \prod_{j \in H^c} [-1, 1]$.

Proof. The condition (2.32) on the poly-radius $\boldsymbol{\rho}$ implies, with the assumption of $(\boldsymbol{b}, p, \varepsilon)$-holomorphy of the parametric map $q(\boldsymbol{y})$, the estimate

$$\|q(\boldsymbol{z}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \leq B \tag{3.3}$$

for some $B \geq 1$ (depending on ε) and for every $\boldsymbol{z}_H \in \prod_{j \in H} \mathcal{T}_{\rho_j}$ and every $\boldsymbol{y}_{H^c} \in \prod_{j \in H^c}[-1, 1]$. To simplify the notation in the following, w.l.o.g. we assume that $H = \{1, \ldots, K\}$ for some $K \in \mathbb{N}_0$ (this may always be achieved by re-indexing the variables). For $\boldsymbol{b} = (b_j)_{j \geq 1}$, we further define the sequence $\widetilde{\boldsymbol{\rho}}$ by

$$\widetilde{\rho}_j = \rho_j + \varepsilon, \quad j \in H, \quad \varepsilon = \frac{\delta}{\sum_{j \in H} b_j}, \qquad \widetilde{\rho}_j = \rho_j, \quad j \in H^c,$$

for some small real number $\delta > 0$. Then, for $\delta > 0$ sufficiently small, also $\widetilde{\rho}$ is an admissible poly-radius, in the sense that the parametric solution admits a holomorphic continuation to the set $\mathcal{T}_{\widetilde{\boldsymbol{\rho}}} \subset \mathbb{C}^{\mathbb{N}}$. In particular, q_H is analytic in an open neighborhood of $U_{\rho, H}$, where we are writing $q_H(z_1, \ldots, z_K) = q_H(\boldsymbol{z}_H) \equiv q(\boldsymbol{z}_H, 0)$.

Cauchy's integral formula can be applied successively with respect to each coordinate $z_j \in \mathcal{T}_{\rho_j}$ with $j \in H$ to obtain for every $\boldsymbol{y} \in U$ the representation

$$q(y_1, \ldots, y_K, \boldsymbol{y}_{H^c}) = (2\pi i)^{-K} \oint_{\Gamma'_1(y_1)} \cdots \oint_{\Gamma'_K(y_K)} \frac{q(\boldsymbol{z}'_H, \boldsymbol{y}_{H^c})}{(z'_1 - y_1) \cdots (z'_K - y_K)} \, dz'_1 \cdots dz'_K,$$

where now $\Gamma'_j(y_j) \subset \mathbb{C}$ denotes the circle with radius $\rho_j - 1$ and center $y_j \in [-1, 1]$ for $j \in H$. Then, for all $\boldsymbol{y} \in U$, the integration domains are contained in $\prod_{j \in H} \mathcal{T}_{\rho_j}$. Changing the path of integration from $\Gamma'_j(y_j)$ to $\partial \mathcal{T}_{\rho_j}$, the boundary of $\mathcal{T}_{\rho_j}$, and differentiating under the integral sign in Cauchy's integral formula now yields for every $\boldsymbol{y} \in U$

$$(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\boldsymbol{y}) = \boldsymbol{\nu}!(2\pi i)^{-K} \oint_{\partial \mathcal{T}_{\rho_1}} \cdots \oint_{\partial \mathcal{T}_{\rho_K}} \frac{q(z'_1, \ldots, z'_K, \boldsymbol{y}_{H^c})}{(z'_1 - y_1)^{\nu_1 + 1} \cdots (z'_K - y_K)^{\nu_K + 1}} \, dz'_1 \cdots dz'_K$$

since $\Gamma'_j(y_j) \subset \mathcal{T}_{\rho_j}$. Then, (3.3), $|\partial \mathcal{T}_\kappa| = 2(2 + (\kappa - 1)\pi)$ and a standard estimate for the integral yields the derivative bound: for every fixed $\boldsymbol{y}_{H^c} \in U_{H^c}$,

$$
\begin{aligned}
\sup_{\boldsymbol{y}_H \in \prod_{j \in H}[-1,1]} & \|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} q)(\boldsymbol{y}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \\
&\leq \sup_{\boldsymbol{z}_H \in \prod_{j \in H} \partial \mathcal{T}_{\rho_j}} \|q(\boldsymbol{z}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \frac{\boldsymbol{\nu}!}{(2\pi)^K} \prod_{j \in H} |\partial \mathcal{T}_{\rho_j}|(\rho_j - 1)^{-\nu_j - 1} \\
&\leq \sup_{\boldsymbol{z}_H \in \prod_{j \in H} \mathcal{T}_{\rho_j}} \|q(\boldsymbol{z}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \frac{\boldsymbol{\nu}!}{(2\pi)^K} \prod_{j \in H} 2(2 + (\rho_j - 1)\pi)(\rho_j - 1)^{-\nu_j - 1} \\
&= \sup_{\boldsymbol{z}_H \in \prod_{j \in H} \mathcal{T}_{\rho_j}} \|q(\boldsymbol{z}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \boldsymbol{\nu}! \prod_{j \in H} (\frac{2}{\pi} + (\rho_j - 1))(\rho_j - 1)^{-(\nu_j + 1)} \\
&\leq \sup_{\boldsymbol{z}_H \in \prod_{j \in H} \mathcal{T}_{\rho_j}} \|q(\boldsymbol{z}_H, \boldsymbol{y}_{H^c})\|_{\mathcal{X}} \boldsymbol{\nu}! \prod_{j \in H} \frac{\rho_j}{(\rho_j - 1)^{\nu_j + 1}}.
\end{aligned}
\tag{3.4}
$$

Here we used

$$\inf_{y_j \in [-1,1], z'_j \in \partial \mathcal{T}_{\rho_j}} \{|z'_j - y_j|\} \geq \rho_j - 1 > 0.$$

$\square$

Proof of Theorem 3.1 In this proof, we will establish (3.2) using the result in Lemma 3.1. To obtain these derivative bounds, for given $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$ with $|\boldsymbol{\nu}| < \infty$ and for fixed $\varepsilon > 0$, we choose, with $B \geq 1$ as in (3.3), $J = J(\varepsilon, \boldsymbol{b}) \in \mathbb{N}$ as

$$J(\varepsilon, \boldsymbol{b}) := \min \left\{ s \in \mathbb{N} \mid \sum_{j > s} b_j \leq \frac{\varepsilon}{4B} \leq \frac{\varepsilon}{4} \right\} . \tag{3.5}$$

Since $\boldsymbol{b} \in \ell^1(\mathbb{N})$, (3.5) defines for every $\varepsilon > 0$ a unique $J(\varepsilon, \boldsymbol{b}) \in \mathbb{N}$ which, as *we emphasize, is independent of the particular multi-index $\boldsymbol{\nu}$.* With $J = J(\boldsymbol{b}, \varepsilon)$, we define the set $E := \{1, 2, ..., J\} \subset \mathbb{N}$ and define $E^c := \mathbb{N} \backslash E$. For any multi-index $\boldsymbol{\nu} \in \mathcal{F}$, we then introduce the partition $\boldsymbol{\nu} = (\boldsymbol{\nu}_E, \boldsymbol{\nu}_{E^c})$ where $\boldsymbol{\nu}_E := \{\nu_1, \nu_2, ..., \nu_J\}$ and $\boldsymbol{\nu}_{E^c} := \{\nu_{J+1}, \nu_{J+2}, ...\}$. Next, we define $\kappa := 1 + \varepsilon/(4\|\boldsymbol{b}\|_{\ell^1(\mathbb{N})}) > 1$ and introduce, for $\boldsymbol{\nu} \in \mathcal{F}$, the poly-radius $\boldsymbol{\rho}(\boldsymbol{\nu})$ by

$$\rho_j := \begin{cases} \kappa & \text{for } j \in E , \\ \kappa + \dfrac{\varepsilon}{2b_j} \dfrac{\nu_j}{1 + |\boldsymbol{\nu}_{E^c}|} & \text{for } j \in E^c . \end{cases} \tag{3.6}$$

With this choice of $\boldsymbol{\rho}(\boldsymbol{\nu})$ we verify that (2.32) holds. This follows since

$$\begin{aligned} \sum_{j \geq 1} (\rho_j - 1) b_j &\leq (\kappa - 1) \sum_{j=1}^{J} b_j + \sum_{j > J} b_j \left(\kappa - 1 + \frac{\varepsilon}{2b_j} \frac{\nu_j}{1 + |\boldsymbol{\nu}_{E^c}|} \right) \\ &= \frac{\varepsilon}{4\|\boldsymbol{b}\|_{\ell^1}} \sum_{j=1}^{J} b_j + \sum_{j > J} b_j \left(\frac{\varepsilon}{4\|\boldsymbol{b}\|_{\ell^1}} + \frac{\varepsilon}{2b_j} \frac{\nu_j}{1 + |\boldsymbol{\nu}_{E^c}|} \right) \\ &\leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \frac{|\boldsymbol{\nu}_{E^c}|}{1 + |\boldsymbol{\nu}_{E^c}|} \leq \varepsilon . \end{aligned}$$

We introduce the notation $\phi(\rho) := \frac{\pi}{2} \rho/(\rho - 1)$ for $\rho > 1$. The property $\phi'(\rho) < 0$ for $\rho > 1$ implies, for ρ_j as in (3.6), that $\phi(\rho_j) \leq \phi(\kappa)$ for all $j \in \mathbb{N}$. Further we have $\phi(\rho) \geq 1$ for all $\rho > 1$.

In the following we prove a bound on $\boldsymbol{\nu}! \prod_{j \in H} \frac{\rho_j}{(\rho_j - 1)^{\nu_j + 1}}$ (where $H = \operatorname{supp} \boldsymbol{\nu}$), which appears in Lemma 3.1. We obtain, assuming w.l.o.g. that $J \leq L := \max\{j : \nu_j > 0\}$, that there holds the bound

$$\begin{aligned} \boldsymbol{\nu}! \prod_{j \in H} \frac{\rho_j}{(\rho_j - 1)^{\nu_j + 1}} &= \boldsymbol{\nu}_E! \boldsymbol{\nu}_{E^c}! \prod_{1 \leq j \leq L} \frac{2}{\pi} \phi(\rho_j)(\rho_j - 1)^{-\nu_j} \\ &\leq \boldsymbol{\nu}_E! \left\{ \prod_{j \in E} \phi(\kappa) \left(\frac{4\|\boldsymbol{b}\|_{\ell^1}}{\varepsilon} \right)^{\nu_j} \right\} \times \boldsymbol{\nu}_{E^c}! \left\{ \prod_{\substack{j \in E^c \\ \nu_j > 0}} \phi(\rho_j) \left(\frac{2b_j}{\varepsilon \nu_j} (1 + |\boldsymbol{\nu}_{E^c}|) \right)^{\nu_j} \right\} \\ &=: \boldsymbol{\nu}_E! \boldsymbol{\nu}_{E^c}! \boldsymbol{\beta}_E(\boldsymbol{\nu}) \boldsymbol{\beta}_{E^c}(\boldsymbol{\nu}) . \end{aligned}$$

We estimate $\boldsymbol{\beta}_E(\boldsymbol{\nu})$ and $\boldsymbol{\beta}_{E^c}(\boldsymbol{\nu})$. We observe that in case that all $\nu_j \geq 1$ for $j \in E$

$$\boldsymbol{\beta}_E(\boldsymbol{\nu}) = \prod_{j=1}^{J} \phi(\kappa) \left(\frac{4\|\boldsymbol{b}\|_{\ell^1}}{\varepsilon} \right)^{\nu_j} \leq \left(\phi(\kappa) \frac{4\|\boldsymbol{b}\|_{\ell^1}}{\varepsilon} \right)^{|\boldsymbol{\nu}_E|} ,$$

which is of product weight form. In case some or all $\nu_j = 0$ for $j \in E$, we find the bound

$$\boldsymbol{\beta}_E(\boldsymbol{\nu}) \leq (\phi(\kappa))^J \left(\frac{4\|\boldsymbol{b}\|_{\ell^1}}{\varepsilon} \right)^{|\boldsymbol{\nu}_E|} ,$$

13

where we recall that $J = J(\boldsymbol{b}, \varepsilon)$ does not depend on $\boldsymbol{\nu}$.

Next we consider $\boldsymbol{\beta}_{E^c}(\boldsymbol{\nu})$. Using $\phi(\rho_j) \leq \phi(\kappa)$, we obtain that

$$
\begin{aligned}
\boldsymbol{\beta}_{E^c}(\boldsymbol{\nu}) &\leq \phi(\kappa)^{|\boldsymbol{\nu}_{E^c}|} \prod_{j \in E^c : \nu_j > 0} \left(\frac{2b_j}{\varepsilon}\right)^{\nu_j} \left(\frac{1 + |\boldsymbol{\nu}_{E^c}|}{\nu_j}\right)^{\nu_j} \\
&\leq \prod_{j \in E^c : \nu_j > 0} \left(\frac{2\phi(\kappa)b_j}{\varepsilon}\right)^{\nu_j} \left(\frac{1 + |\boldsymbol{\nu}_{E^c}|}{\nu_j}\right)^{\nu_j} .
\end{aligned}
$$

We set $d_j := 2\phi(\kappa)b_j/\varepsilon$, and $\bar{d}_j := ed_j$ for $j \in E^c$. Then

$$
\boldsymbol{\beta}_{E^c}(\boldsymbol{\nu}) \leq \prod_{j > J} \left(d_j \frac{1 + |\boldsymbol{\nu}_{E^c}|}{\nu_j}\right)^{\nu_j} = \frac{(1 + |\boldsymbol{\nu}_{E^c}|)^{|\boldsymbol{\nu}_{E^c}|}}{\boldsymbol{\nu}_{E^c}^{\boldsymbol{\nu}_{E^c}}} \prod_{j > J} d_j^{\nu_j} .
$$

Stirling's approximation implies that for all $n \in \mathbb{N}$ we have $\sqrt{2\pi}n^{n+1/2} \leq n!e^n \leq en^{n+1/2}$. This also implies that $(1+n)^n \leq n^n e^2/\sqrt{2\pi} \leq 3n^n$. Thus

$$
\frac{(1 + |\boldsymbol{\nu}_{E^c}|)^{|\boldsymbol{\nu}_{E^c}|}}{\boldsymbol{\nu}_{E^c}^{\boldsymbol{\nu}_{E^c}}} \leq 3\frac{|\boldsymbol{\nu}_{E^c}|^{|\boldsymbol{\nu}_{E^c}|}}{\boldsymbol{\nu}_{E^c}^{\boldsymbol{\nu}_{E^c}}} \leq 3\frac{|\boldsymbol{\nu}_{E^c}|!e^{|\boldsymbol{\nu}_{E^c}|}}{\sqrt{2\pi}\sqrt{|\boldsymbol{\nu}_{E^c}|}} \prod_{\substack{j \in E^c \\ \nu_j > 0}} \frac{e\sqrt{\nu_j}}{\nu_j!e^{\nu_j}} \leq \frac{3}{\sqrt{2\pi}} \frac{|\boldsymbol{\nu}_{E^c}|!}{\boldsymbol{\nu}_{E^c}!} \frac{\prod_{j \in E^c} e\sqrt{\nu_j}}{\sqrt{|\boldsymbol{\nu}_{E^c}|}} .
$$

Since $e\sqrt{\nu_j} \leq e^{\nu_j}$ for integers $\nu_j > 0$, we obtain

$$
\boldsymbol{\beta}_{E^c}(\boldsymbol{\nu}) \leq \frac{3}{\sqrt{2\pi}} \frac{|\boldsymbol{\nu}_{E^c}|!}{\boldsymbol{\nu}_{E^c}!} \bar{\boldsymbol{d}}^{\boldsymbol{\nu}_{E^c}} .
$$

Combining all bounds, we find there exists a constant $\hat{C} > 0$ (depending on p, ε, and on $\boldsymbol{b}$) such that there holds, for every $\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}}$ with $|\boldsymbol{\nu}| < \infty$, the bound

$$
\begin{aligned}
\boldsymbol{\nu}_E! \boldsymbol{\nu}_{E^c}! \boldsymbol{\beta}_E(\boldsymbol{\nu}) \boldsymbol{\beta}_{E^c}(\boldsymbol{\nu}) &\leq \frac{3}{\sqrt{2\pi}} \phi(\kappa)^J \left(\boldsymbol{\nu}_E! \prod_{j=1}^J \left(\frac{4\|\boldsymbol{b}\|_{\ell^1}}{\varepsilon}\right)^{\nu_j}\right) \times |\boldsymbol{\nu}_{E^c}|! \prod_{j > J} \bar{d}_j^{\nu_j} \\
&= \hat{C} \boldsymbol{\nu}_E! \prod_{j \in E} \beta_j^{\nu_j} \times |\boldsymbol{\nu}_{E^c}|! \prod_{j \in E^c} \beta_j^{\nu_j} .
\end{aligned}
$$

Here, $\beta_j = 4\|\boldsymbol{b}\|_{\ell^1}/\varepsilon$ for $1 \leq j \leq J$ is independent of $\boldsymbol{\nu}$ and we have $\beta_j = \bar{d}_j \sim b_j$ for $j > J$. By the choice of $J(\varepsilon, \boldsymbol{b})$, $\hat{C} = e\phi(\kappa)^{J(\varepsilon, \boldsymbol{b})}$ depends on $\boldsymbol{b}$ and ε, but not on $\boldsymbol{\nu}$. $\quad\square$

Remark 3.1. We see from the proof of Theorem 3.1 and, in particular, from (3.5), that the "crossover-dimension" $J(\boldsymbol{b}, \varepsilon)$ between product weights and the more general hybrids of product and of SPOD weights, depends on the precise structure of the decay of the sequence $\boldsymbol{b}$ (rather than only on the summability exponent). It is therefore of some interest to identify cases where J is large. This occurs for sequences $\boldsymbol{b}$ which exhibit a "plateau" up to dimension $J >> 1$, i.e.

$$
b_1 = b_2 = \dots = b_J > b_{J+1} \geq b_{J+2} \geq \dots \downarrow 0 . \tag{3.7}
$$

Such cases appear, for example, in Karhunen-Loève expansions of random fields $u(\boldsymbol{y})$, given by (2.11), with two-point correlation kernels which concentrate on a (non-dimensional) spatial correlation length scale $0 < \lambda << 1$, in $D \subset \mathbb{R}^d$ a bounded domain. In this case, typically $J \sim 1/\lambda^d$. E.g. for $\lambda \sim 0.01$ in three space dimensions, $J \sim 10^6$.

To exploit the derivative bounds (3.2), it is of utmost importance to have a fast CBC construction of higher-order QMC rules which are able to exploit (3.7). We address a suitable CBC construction of corresponding QMC rules and estimates of the QMC errors incurred by these rules in the ensuing sections, thereby extending [1, 2, 10].

14

4 Quasi-Monte Carlo integration

In Theorem 3.1 we established bounds on the derivatives of $(\boldsymbol{b}, p, \varepsilon)$-analytic solution families of smooth, nonlinear parametric operator equations with $(\boldsymbol{b}, p, \varepsilon)$-analytic operators. Here, we establish error bounds for QMC quadratures for these integrand functions. The convergence estimates obtained here are uniform in the dimension s of the parameter domain. The application of the QMC quadratures to the formally countably-parametric problems must therefore be prepared by *dimension truncation*, i.e. we consider (2.25) and its (Petrov-)Galerkin discretization (2.29). As we explained in the introduction, in order to approximate the mathematical expectation of the random solutions by QMC methods, we truncate the infinite sum in (2.11) to a finite number of $s \geq 1$ terms.

4.1 Higher-order QMC quadrature based on digital nets

For an integrand $G \in C^0([0, 1]^s)$, we want to approximate the s-dimensional integral

$$I_s(G) := \int_{[0,1]^s} G(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \tag{4.1}$$

by an equal weight QMC quadrature rule of the form

$$Q_{N,s}(G) := \frac{1}{N} \sum_{n=0}^{N-1} G(\boldsymbol{y}_n), \tag{4.2}$$

with judiciously chosen points $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{N-1} \in [0, 1]^s$. For completeness we repeat the necessary definitions and results from [10] in the following.

Definition 4.1 (Norm and function space). Let $\alpha, s \in \mathbb{N}$, $1 \leq q \leq \infty$ and $1 \leq r \leq \infty$, and let $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$ be a collection of nonnegative real numbers, known as *weights*. Assume further that for every $s \in \mathbb{N}$, the integrand function $G : [0, 1]^s \to \mathbb{R}$ has partial derivatives of orders up to α with respect to each variable. Set $0/0 := 0$ and $a/0 := \infty$ for $a > 0$. Then we define the norm of G by the higher order unanchored Sobolev norm

$$\|G\|_{s,\alpha,\gamma,q,r}^r := \sum_{\mathfrak{u} \subseteq \{1:s\}} \left\| \gamma_{\mathfrak{u}}^{-1} \sum_{\mathfrak{v} \subseteq \mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u} \backslash \mathfrak{v}} \in \{1:\alpha\}^{|\mathfrak{u} \backslash \mathfrak{v}|}} \int_{[0,1]^{s-|\mathfrak{v}|}} (\partial_{\boldsymbol{y}}^{(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u} \backslash \mathfrak{v}}, \boldsymbol{0})} G)(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}_{\{1:s\} \backslash \mathfrak{v}} \right\|_{L_q}^r, \tag{4.3}$$

with the obvious modifications if q or r is infinite. Here $\{1 : s\}$ is a shorthand notation for the set $\{1, 2, \ldots, s\}$, and $(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u} \backslash \mathfrak{v}}, \boldsymbol{0})$ denotes a sequence $\boldsymbol{\nu}$ with $\nu_j = \alpha$ for $j \in \mathfrak{v}$, $\nu_j = \tau_j$ for $j \in \mathfrak{u} \backslash \mathfrak{v}$, and $\nu_j = 0$ for $j \notin \mathfrak{u}$. Let $\mathcal{W}_{s,\alpha,\gamma,q,r}$ denote the Banach space of all such functions F with finite norm.

By the definition of $0/0$ and $a/0$, if $\gamma_{\mathfrak{u}} = 0$ for some $\mathfrak{u}$ then the corresponding term $\sum_{\mathfrak{v} \subseteq \mathfrak{u}} \sum_{\boldsymbol{\tau}_{\mathfrak{u} \backslash \mathfrak{v}} \in \{1:\alpha\}^{|\mathfrak{u} \backslash \mathfrak{v}|}} \int_{[0,1]^{s-|\mathfrak{v}|}} (\partial_{\boldsymbol{y}}^{(\boldsymbol{\alpha}_{\mathfrak{v}}, \boldsymbol{\tau}_{\mathfrak{u} \backslash \mathfrak{v}}, \boldsymbol{0})} G)(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y}_{\{1:s\} \backslash \mathfrak{v}}$ has to be 0 for all $G \in \mathcal{W}_{s,\alpha,\gamma,q,r}$.

The following result is an upper bound on the worst-case integration error in $\mathcal{W}_{s,\alpha,\gamma,q,r}$ using a QMC rule based on a digital net, see [10, Thm. 3.5].

Theorem 4.1 (Worst case error bound). *Let $\alpha, s \in \mathbb{N}$ with $\alpha > 1$, $1 \leq q \leq \infty$ and $1 \leq r \leq \infty$, and let $\boldsymbol{\gamma} = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$ denote a collection of weights. Let $r' \geq 1$ satisfy*

$1/r + 1/r' = 1$. Let b be prime, $m \in \mathbb{N}$, and let $\mathcal{S} = \{\boldsymbol{y}_n\}_{n=0}^{b^m-1}$ denote a digital net with generating matrices $C_1, \ldots, C_s \in \mathbb{Z}_b^{\alpha m \times m}$. Then we have

$$\sup_{\|G\|_{s,\alpha,\boldsymbol{\gamma},q,r} \leq 1} \left| \frac{1}{b^m} \sum_{n=0}^{b^m-1} G(\boldsymbol{y}_n) - \int_{[0,1]^s} G(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \right| \leq e_{s,\alpha,\boldsymbol{\gamma},r'}(\mathcal{S}) \,,$$

with

$$e_{s,\alpha,\boldsymbol{\gamma},r'}(\mathcal{S}) := \left(\sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \left(C_{\alpha,b}^{|\mathfrak{u}|} \gamma_{\mathfrak{u}} \sum_{\boldsymbol{k}_\mathfrak{u} \in \mathfrak{D}_\mathfrak{u}^*} b^{-\mu_\alpha(\boldsymbol{k}_\mathfrak{u})} \right)^{r'} \right)^{1/r'} . \tag{4.4}$$

Here $\mathfrak{D}_\mathfrak{u}^*$ is the "dual net without 0 components" projected to the components in $\mathfrak{u}$, defined by

$$\mathfrak{D}_\mathfrak{u}^* := \left\{ \boldsymbol{k}_\mathfrak{u} \in \mathbb{N}^{|\mathfrak{u}|} : \sum_{j \in \mathfrak{u}} C_j^\top \mathrm{tr}_{\alpha m}(k_j) = \boldsymbol{0} \in \mathbb{Z}_b^m \right\} , \tag{4.5}$$

where $\mathrm{tr}_{\alpha m}(k) := (\varkappa_0, \varkappa_1, \ldots, \varkappa_{\alpha m-1})^\top$ if $k = \varkappa_0 + \varkappa_1 b + \varkappa_2 b^2 + \cdots$ with $\varkappa_i \in \{0, \ldots, b-1\}$. Moreover, we have $\mu_\alpha(\boldsymbol{k}_\mathfrak{u}) = \sum_{j \in \mathfrak{u}} \mu_\alpha(k_j)$ with

$$\mu_\alpha(k) := \begin{cases} 0 & \text{if } k = 0, \\ & \text{if } k = \kappa_1 b^{a_1-1} + \cdots + \kappa_\rho b^{a_\rho-1} \text{ with} \\ a_1 + \cdots + a_{\min(\alpha,\rho)} & \kappa_i \in \{1, \ldots, b-1\} \text{ and } a_1 > \cdots > a_\rho > 0, \end{cases} \tag{4.6}$$

and

$$C_{\alpha,b} := \max\left(\frac{2}{(2\sin\frac{\pi}{b})^\alpha}, \max_{1 \leq z \leq \alpha-1} \frac{1}{(2\sin\frac{\pi}{b})^z} \right)$$
$$\times \left(1 + \frac{1}{b} + \frac{1}{b(b+1)} \right)^{\alpha-2} \left(3 + \frac{2}{b} + \frac{2b+1}{b-1} \right) . \tag{4.7}$$

We recall the special case where the integrand $G(\boldsymbol{y})$ is a composition of a continuous, linear functional $\mathcal{O}(\cdot) \in \mathcal{X}'$ with the (Petrov-)Galerkin approximation $q_s^h(2\boldsymbol{y} - \boldsymbol{1})$ of the dimension-truncated, parametric and $(\boldsymbol{b}, p, \varepsilon)$-holomorphic, operator equation (2.1). In this case, for every $s \in \mathbb{N}$ and for every $h > 0$ sufficiently small, the integrand functions $G(\boldsymbol{y}) := (\mathcal{O} \circ q_s^h)(\boldsymbol{y}_{\{1:s\}})$ are, likewise, $(\boldsymbol{b}, p, \varepsilon)$-holomorphic *uniformly w.r.t.* $s \in \mathbb{N}$ *and to* $h > 0$. By Theorem 3.1, they satisfy the derivative estimates (3.2) uniformly w.r.t. $s \in \mathbb{N}$ and to $h > 0$. For integrand functions $G(\boldsymbol{y})$ which satisfy (3.2), we proved in [10] convergence rates of QMC quadratures which are based on higher order digital nets. Precisely, we showed in [10, Sec.3] the following result.

Proposition 4.1. *Let $s \geq 1$ and $N = b^m$ for $m \geq 1$ and prime b. Let $\boldsymbol{\beta} = (\beta_j)_{j \geq 1}$ be a sequence of positive numbers, and denote by $\boldsymbol{\beta}_s = (\beta_j)_{1 \leq j \leq s}$ its s-term truncation. Assume that*

$$\exists 0 < p \leq 1 : \quad \sum_{j=1}^\infty \beta_j^p < \infty . \tag{4.8}$$

Define, for $0 < p < 1$ as in (4.8),

$$\alpha := \lfloor 1/p \rfloor + 1 . \tag{4.9}$$

Consider integrand functions $G(\boldsymbol{y})$ whose mixed partial derivatives of order α satisfy

$$\forall\, \boldsymbol{y} \in U\ \forall s \in \mathbb{N}\ \forall \boldsymbol{\nu} \in \{0,1,\ldots,\alpha\}^s : \quad |(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} G)(\boldsymbol{y})| \leq c(G)\boldsymbol{\nu}_E!\prod_{j\in E}\beta_j^{\nu_j} \times |\boldsymbol{\nu}_{E^c}|!\prod_{j\in E^c}\beta_j^{\nu_j} \tag{4.10}$$

for some fixed integer $J \in \mathbb{N}$ where $E = \{1,2,\ldots,J\}$ and $E^c = \mathbb{N}\setminus E$, and where $c(G) > 0$ is independent of $\boldsymbol{y}$, s and of $\boldsymbol{\nu}$. Then, for every $N \in \mathbb{N}$, an interlaced polynomial lattice rule of order α with N points can be constructed using a fast component-by-component algorithm, using $\mathcal{O}(\alpha\,(\min\{s,J\} + \alpha(s-J)_+)\,N \log N)$ operations, plus $\mathcal{O}(\alpha^2(s-J)_+^2 N)$ update cost, plus $\mathcal{O}(N + \alpha(s-J)_+ N)$ memory cost, where $(w)_+ = \max\{0,w\}$, such that there holds the error bound

$$\forall s, N \in \mathbb{N} : \quad |I_s(G) - Q_{N,s}(G)| \leq C_{\alpha,\boldsymbol{\beta},b,p}\, N^{-1/p} ,$$

where $C_{\alpha,\boldsymbol{\beta},b,p} < \infty$ is a constant independent of s and N.

Remark 4.1. Notice that the bound (4.10) was shown in Theorem 3.1 for functions defined on $[-1,1]^{\mathbb{N}}$, whereas now we use (the dimension truncated version) $[0,1]^s$. The change from $[-1,1]$ to $[0,1]$ can be achieved by the simple linear transformation $y \mapsto (y+1)/2$. Using (3.2) together with this change of variable in Proposition 4.1 increases the constant in (4.7) by a factor of at most 2^α. Thus, in order for the theory to apply to the integrands from Sections 2 and 3, we need to multiply $C_{\alpha,b}$ in (4.7) by 2^α. In other words we need to replace $C_{\alpha,b}$ by $C'_{\alpha,b} = 2^\alpha C_{\alpha,b}$.

For a function G satisfying (4.10), its norm (4.3) with $r = \infty$ and for any q, can be bounded by

$$\|G\|_{s,\alpha,\boldsymbol{\gamma},q,\infty} \leq c \max_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \sum_{\boldsymbol{\nu}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \boldsymbol{\nu}_{\mathfrak{u}\cap E}! \prod_{j\in\mathfrak{u}\cap E}\left(2^{\delta(\nu_j,\alpha)}\beta_j^{\nu_j}\right) |\boldsymbol{\nu}_{\mathfrak{u}\cap E^c}|! \prod_{j\in\mathfrak{u}\cap E^c}\left(2^{\delta(\nu_j,\alpha)}\beta_j^{\nu_j}\right)$$

$$= c(G)\max_{\mathfrak{u}\subseteq\{1:s\}} \gamma_{\mathfrak{u}}^{-1} \sum_{\boldsymbol{\nu}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \boldsymbol{\nu}_{\mathfrak{u}\cap E}! |\boldsymbol{\nu}_{\mathfrak{u}\cap E^c}|! \prod_{j\in\mathfrak{u}}\left(2^{\delta(\nu_j,\alpha)}\beta_j^{\nu_j}\right) ,$$

where $\delta(\nu_j,\alpha)$ is 1 if $\nu_j = \alpha$ and is 0 otherwise. To make $\|G\|_{s,\alpha,\boldsymbol{\gamma},q,\infty} \leq c$, we choose

$$\gamma_{\mathfrak{u}} := \sum_{\boldsymbol{\nu}_{\mathfrak{u}}\in\{1:\alpha\}^{|\mathfrak{u}|}} \boldsymbol{\nu}_{\mathfrak{u}\cap E}! |\boldsymbol{\nu}_{\mathfrak{u}\cap E^c}|! \prod_{j\in\mathfrak{u}}\left(2^{\delta(\nu_j,\alpha)}\beta_j^{\nu_j}\right) . \tag{4.11}$$

4.2 Combined error bound

From the error bound in Theorem 2.1 on the impact of dimension truncation, the QMC integration error bound in Proposition 4.1, and from the properties (2.21) and (2.22) of the (Petrov-)Galerkin projection (2.29) we obtain

Theorem 4.2. *Consider the nonlinear, parametric operator equation (2.2) under the assumptions made in Section 2.1, and under Assumption 1 on p-summability (2.10) and the decreasing arrangement (2.24) of the sequence $\boldsymbol{b}$. If the approximation property (2.16) holds, and if the parametric solutions $q(\boldsymbol{y})$ of the problems (2.12) are uniformly $\mathcal{X}_t$-regular in the sense that there exists $C(F,t) < \infty$ such that*

$$\sup_{\boldsymbol{y}\in U} \|q(\boldsymbol{y})\|_{\mathcal{X}_t} \leq C(F,t) < \infty , \tag{4.12}$$

17

then for the QMC-integrated, (Petrov-)Galerkin-approximated responses $Q_{N,s}(\mathscr{O}(q_h^s))$ of the parametric (Petrov-)Galerkin approximations $q_h^s(\boldsymbol{y}) \in \mathcal{X}^h$ defined in (2.29), there holds the error bound

$$|I(\mathscr{O}(q(\cdot))) - Q_{N,s}(\mathscr{O}(q_h^s))| \leq C_1(N^{-1/p} + h^t + s^{-(1/p-1)}) \,.$$

Here, the constant $C_1 > 0$ is independent of N, h and of s.

Proof. We write

$$
\begin{aligned}
|I(\mathscr{O}(q)) - Q_{N,s}(\mathscr{O}(q_h^s))| &\leq |I(\mathscr{O}(q)) - I(\mathscr{O}(q^s))| + |I(\mathscr{O}(q^s)) - Q_{N,s}(\mathscr{O}(q^s))| \\
&\quad + |Q_{N,s}(\mathscr{O}(q^s - q_h^s))| \\
&=: E_I + E_{II} + E_{III} \,.
\end{aligned}
$$

The dimension truncation error E_I is bounded by (2.27) and (2.28) in Theorem 2.1. Term E_{II} is a QMC error which is bounded by Proposition 4.1; this Proposition is applicable based on Theorem 3.1, upon noting (2.23), i.e., that for finite truncation dimension s the dimensionally truncated, parametric solution $q^s(\boldsymbol{y})$ can be interpreted as evaluation of $q(\boldsymbol{y})$ (to which Theorem 3.1 applies) at the particular parameter value $\boldsymbol{y} := (\boldsymbol{y}_{\{1:s\}}, \boldsymbol{0})$. The last term E_{III} is bounded using the equal weight property (4.2) of $Q_{N,s}$ to infer

$$E_{III} \leq \|\mathscr{O}\|_{\mathcal{X}'} \sup_{\boldsymbol{y}_{\{1:s\}} \in [-1/2, 1/2)^s} \|q^s(\boldsymbol{y}_{\{1:s\}}) - q_h^s(\boldsymbol{y}_{\{1:s\}})\|_{\mathcal{X}} \leq \|\mathscr{O}\|_{\mathcal{X}'} \sup_{\boldsymbol{y} \in U} \|q(\boldsymbol{y}) - q_h(\boldsymbol{y})\|_{\mathcal{X}}$$

and the (Petrov-)Galerkin error $\sup_{\boldsymbol{y} \in U} \|q(\boldsymbol{y}) - q_h(\boldsymbol{y})\|_{\mathcal{X}} \leq Ch^t$. □

5   Fast component-by-component construction

Here, we outline, based on [25, 16, 10], a modification of the fast CBC construction of the generating vector for the QMC rule; while asymptotically, as $s \to \infty$, the complexity of this construction equals that of the CBC construction for the SPOD weights in [10], for finite, large values of the index J in the proof of Theorem 3.1 (which do occur in practical situations as outlined in Remark 3.1), we obtain quantitative advantages for the construction based on " hybrid QMC-weights ", as outlined in what follows. We follow [10] closely in our exposition below.

As quadrature rule we use (interlaced) polynomial lattice rules which are a special class of (higher order) digital nets, and which were introduced by Niederreiter, see [23]. We state the definition of these rules in the following. Let b be a prime number, $\mathbb{Z}_b$ be the finite field with b elements, $\mathbb{Z}_b[x]$ be the set of all polynomials with coefficients in $\mathbb{Z}_b$ and $\mathbb{Z}_b((x^{-1}))$ be the set of all formal Laurent series $\sum_{\ell=w}^{\infty} t_\ell x^{-\ell}$, where w is an arbitrary integer and $t_\ell \in \mathbb{Z}_b$ for all ℓ.

Definition 5.1 (Polynomial lattice rules)**.** For a prime b and any $m \in \mathbb{N}$, let $P \in \mathbb{Z}_b[x]$ be an irreducible polynomial with $\deg(P) = m$. For a given dimension $s \geq 1$, select s polynomials $q_1(x), \ldots, q_s(x)$ from the set

$$\mathscr{P}_{b,m} := \{q(x) \in \mathbb{Z}_b[x] \setminus \{0\} : \deg(q) < m\} \,, \tag{5.1}$$

and write collectively

$$\boldsymbol{q} = \boldsymbol{q}(x) = (q_1(x), \ldots, q_s(x)) \in \mathscr{P}_{b,m}^s \,. \tag{5.2}$$

For each integer $0 \le n < b^m$, let $n = \eta_0 + \eta_1 b + \cdots + \eta_{m-1}b^{m-1}$ be the b-adic expansion of n, and associate with n the polynomial

$$n(x) = \sum_{r=0}^{m-1} \eta_r \, x^r \in \mathbb{Z}_b[x] \ .$$

Furthermore, we denote by v_m the map from $\mathbb{Z}_b((x^{-1}))$ to the interval $[0,1)$ defined for any integer w by

$$v_m\left(\sum_{\ell=w}^{\infty} t_\ell \, x^{-\ell}\right) = \sum_{\ell=\max(1,w)}^{m} t_\ell \, b^{-\ell} \ .$$

Then, the QMC point set $\mathcal{S}_{P,b,m,s}(\boldsymbol{q})$ of a (classical) *polynomial lattice rule* comprises the points

$$\boldsymbol{y}_n = \left(v_m\left(\frac{n(x)q_1(x)}{P(x)}\right), \ldots, v_m\left(\frac{n(x)q_s(x)}{P(x)}\right)\right) \in [0,1)^s, \quad n = 0, \ldots, b^m - 1 \ .$$

Interlaced polynomial lattice rules are special families of higher order digital nets [7, 8]. These quadrature rules were first studied in [15, 16] since they yield faster CBC constructions.

Definition 5.2 (Interlaced polynomial lattice rules)**.** Define the *digit interlacing function* with interlacing factor $\alpha \in \mathbb{N}$ by

$$\begin{array}{rcl}
\mathscr{D}_\alpha : [0,1)^\alpha & \to & [0,1) \\
(x_1, \ldots, x_\alpha) & \mapsto & \sum_{a=1}^{\infty}\sum_{j=1}^{\alpha} \xi_{j,a} b^{-j-(a-1)\alpha} \ ,
\end{array} \tag{5.3}$$

where $x_j = \xi_{j,1}b^{-1} + \xi_{j,2}b^{-2} + \cdots$ for $1 \le j \le \alpha$. We also define such a function for vectors by setting

$$\begin{array}{rcl}
\mathscr{D}_\alpha : [0,1)^{\alpha s} & \to & [0,1)^s \\
(x_1, \ldots, x_{\alpha s}) & \mapsto & (\mathscr{D}_\alpha(x_1, \ldots, x_\alpha), \ldots, \mathscr{D}_\alpha(x_{(s-1)\alpha+1}, \ldots, x_{s\alpha})) \ .
\end{array} \tag{5.4}$$

Then, an *interlaced polynomial lattice rule of order α with b^m points in s dimensions* is a QMC rule using $\mathscr{D}_\alpha(\mathcal{S}_{P,b,m,\alpha s}(\boldsymbol{q})) = \{\mathscr{D}_\alpha(\boldsymbol{y}_n) : n = 0, \ldots, b^m - 1\}$ as quadrature points, for some given modulus P and generating vector $\boldsymbol{q} \in \mathscr{P}_{b,m}^{\alpha s}$.

We have the following upper bound for the worst-case error of interlaced polynomial lattice rules [10, Section 3.2]

$$e_{s,\alpha,\boldsymbol{\gamma},1}(\mathcal{S}) \le \sum_{\emptyset \neq \mathfrak{v} \subseteq \{1:\alpha s\}} (C'_{\alpha,b})^{|\mathfrak{u}(\mathfrak{v})|} \, \gamma_{\mathfrak{u}(\mathfrak{v})} \, b^{\alpha(\alpha-1)|\mathfrak{u}(\mathfrak{v})|/2} \sum_{\boldsymbol{\ell}_\mathfrak{v} \in \mathcal{D}_\mathfrak{v}^*} b^{-\alpha\mu_1(\boldsymbol{\ell}_\mathfrak{v})} \ , \tag{5.5}$$

where $\mathcal{D}_\mathfrak{v}^*$ is the "dual net without 0 components" defined in terms of the generating polynomials, see [10, Eq (3.28)] and where we replaced $C_{\alpha,b}$ by $C'_{\alpha,b}$. Eq. (5.5) is derived from (4.4) by setting $r' = 1$ and using interlaced polynomial lattice rules, see [10] for details. Here, for a given set $\emptyset \neq \mathfrak{v} \subseteq \{1:\alpha s\}$, we define

$$\mathfrak{u}(\mathfrak{v}) := \{\lceil j/\alpha \rceil : j \in \mathfrak{v}\} \subseteq \{1:s\} \ , \tag{5.6}$$

19

where each element appears only once. The set $\mathfrak{u}(\mathfrak{v})$ can be viewed as an indicator on whether the set $\mathfrak{v}$ includes any element from each block of α components from $\{1:\alpha s\}$.

Since we do not have a suitable expression for the worst-case error $e_{s,\alpha,\gamma,1}$ we use the right-hand side of (5.5) as our search criterion in the CBC construction instead. To simplify our notation, we define

$$\mathcal{E}_d(\boldsymbol{q}) := \sum_{\emptyset \neq \mathfrak{v} \subseteq \{1:d\}} \widetilde{\gamma}_{\mathfrak{v}} \sum_{\boldsymbol{\ell}_{\mathfrak{v}} \in \mathcal{D}_{\mathfrak{v}}^*} b^{-\alpha \mu_1(\boldsymbol{\ell}_{\mathfrak{v}})} . \tag{5.7}$$

The case $d = \alpha s$ and the weights

$$\widetilde{\gamma}_{\mathfrak{v}} := (C'_{\alpha,b})^{|\mathfrak{u}(\mathfrak{v})|} \gamma_{\mathfrak{u}(\mathfrak{v})} b^{\alpha(\alpha-1)|\mathfrak{u}(\mathfrak{v})|/2} \tag{5.8}$$

are of particular interest for our purposes here. However, as shown in [10], the theorem below holds for any d and also for general weights $\widetilde{\gamma}_{\mathfrak{v}}$.

Theorem 5.1 (CBC error bound). *Let $b \geq 2$ be prime, and $\alpha \geq 2$ and $m, d \geq 1$ be integers, and let $P \in \mathbb{Z}_b[x]$ be an irreducible polynomial with $\deg(P) = m$. Let $(\widetilde{\gamma}_{\mathfrak{v}})_{\mathfrak{v} \subseteq \{1:d\}}$ be positive real numbers. Then a generating vector $\boldsymbol{q}^* = (1, q_2^*, \ldots, q_d^*) \in G_{b,m}^d$ can be constructed using a component-by-component approach, minimizing $\mathcal{E}_d(\boldsymbol{q})$ in each step, such that*

$$\mathcal{E}_d(\boldsymbol{q}^*) \leq \left(\frac{2}{b^m - 1} \sum_{\emptyset \neq \mathfrak{v} \subseteq \{1:d\}} \widetilde{\gamma}_{\mathfrak{v}}^{\lambda} \left(\frac{b-1}{b^{\alpha\lambda} - b} \right)^{|\mathfrak{v}|} \right)^{1/\lambda} \quad for \ all \quad \lambda \in (1/\alpha, 1] . \tag{5.9}$$

It follows from Theorem 5.1 that an interlaced polynomial lattice rule with interlacing factor α in s dimensions can be constructed using a CBC algorithm with weights (5.8), such that

$$e_{\alpha,\gamma,s,1}(\mathcal{S}) \leq \mathcal{E}_{\alpha s}(\boldsymbol{q}^*)$$
$$\leq \left(\frac{2}{b^m - 1} \sum_{\emptyset \neq \mathfrak{v} \subseteq \{1:\alpha s\}} \left((C'_{\alpha,b})^{|\mathfrak{u}(\mathfrak{v})|} \gamma_{\mathfrak{u}(\mathfrak{v})} b^{\alpha(\alpha-1)|\mathfrak{u}(\mathfrak{v})|/2} \right)^{\lambda} \left(\frac{b-1}{b^{\alpha\lambda} - b} \right)^{|\mathfrak{v}|} \right)^{1/\lambda}$$
$$= \left(\frac{2}{b^m - 1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \left((C'_{\alpha,b})^{|\mathfrak{u}|} \gamma_{\mathfrak{u}} b^{\alpha(\alpha-1)|\mathfrak{u}|/2} \right)^{\lambda} \left(\left(1 + \frac{b-1}{b^{\alpha\lambda} - b}\right)^{\alpha} - 1 \right)^{|\mathfrak{u}|} \right)^{1/\lambda} .$$

By substituting in $\gamma_{\mathfrak{u}}$ from (4.11) and using Jensen's inequality, we get

$$e_{\alpha,\gamma,s,1}(\mathcal{S}) \leq \left(\frac{2}{b^m - 1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \sum_{\boldsymbol{\nu}_{\mathfrak{u}} \in \{1:\alpha\}^{|\mathfrak{u}|}} (\boldsymbol{\nu}_{\mathfrak{u} \cap E}! |\boldsymbol{\nu}_{\mathfrak{u} \cap E^c}|!)^{\lambda} \prod_{j \in \mathfrak{u}} \left(B \, 2^{\delta(\nu_j,\alpha)} \beta_j^{\nu_j} \right)^{\lambda} \right)^{1/\lambda}$$
$$= \left(\frac{2}{b^m - 1} \sum_{\boldsymbol{0} \neq \boldsymbol{\nu} \in \{0:\alpha\}^s} (\boldsymbol{\nu}_{E \cap \{1:s\}}! |\boldsymbol{\nu}_{E^c \cap \{1:s\}}|!)^{\lambda} \prod_{\substack{j=1 \\ \nu_j > 0}}^{s} \left(B \, 2^{\delta(\nu_j,\alpha)} \beta_j^{\nu_j} \right)^{\lambda} \right)^{1/\lambda} , \tag{5.10}$$

where

$$B := C'_{\alpha,b} \, b^{\alpha(\alpha-1)/2} \left(\left(1 + \frac{b-1}{b^{\alpha\lambda} - b}\right)^{\alpha} - 1 \right)^{1/\lambda} . \tag{5.11}$$

We now show how we can choose λ such that the sum in (5.10) is bounded independently of s. Let $\widetilde{\beta}_j := 2\max(B,1)\beta_j$. Using the same argument as in [10], the sum in (5.10) is bounded by

$$\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\{0:\alpha\}^s}\left(\boldsymbol{\nu}_{E\cap\{1:s\}}!|\boldsymbol{\nu}_{E^c\cap\{1:s\}}|!\prod_{j=1}^s\widetilde{\beta}_j^{\nu_j}\right)^\lambda,$$

where each term in the sum to be raised to the power of λ is of the form

$$\nu_1!\nu_2!\cdots\nu_J!(\nu_{J+1}+\nu_{J+2}+\cdots+\nu_s)!\underbrace{\widetilde{\beta}_1\cdots\widetilde{\beta}_1}_{\nu_1}\underbrace{\widetilde{\beta}_2\cdots\widetilde{\beta}_2}_{\nu_2}\cdots\underbrace{\widetilde{\beta}_s\cdots\widetilde{\beta}_s}_{\nu_s}, \tag{5.12}$$

where for $s\leq J$ we set $(\nu_{J+1}+\nu_{J+2}+\ldots+\nu_s)!=0!=1$ and $\nu_{s+1}!=\cdots=\nu_J!=0!=1$.

We now define a sequence $d_j := \widetilde{\beta}_{\lceil j/\alpha\rceil}$ so that $d_1=\cdots=d_\alpha=\widetilde{\beta}_1$ and $d_{\alpha+1}=\cdots=d_{2\alpha}=\widetilde{\beta}_2$, and so on. Then any term of the form (5.12) is bounded by a term of the form

$$\left(\prod_{j\in\mathfrak{v}\cap\alpha E}\alpha!d_j\right)|\mathfrak{v}\cap\alpha E^c|!\prod_{j\in\mathfrak{v}\cap\alpha E^c}d_j$$

for some finite subset of indices $\mathfrak{v}\subset\mathbb{N}$. As before, $E=\{1:J\}$ and we write

$$\alpha E=\{1,2,\ldots,\alpha J\}\text{ and }\alpha E^c=\{\alpha J+1,\alpha J+2,\ldots,\}\,.$$

Thus we conclude that

$$\sum_{\mathbf{0}\neq\boldsymbol{\nu}\in\{0:\alpha\}^s}\left(\boldsymbol{\nu}_{E\cap\{1:s\}}!\,|\boldsymbol{\nu}_{E^c\cap\{1:s\}}|!\prod_{\substack{j=1\\\nu_j>0}}^s\widetilde{\beta}_j^{\nu_j}\right)^\lambda\leq\sum_{\substack{\mathfrak{v}\subset\mathbb{N}\\|\mathfrak{v}|<\infty}}\left(\left(\prod_{j\in\mathfrak{v}\cap\alpha E}\alpha!d_j\right)|\mathfrak{v}\cap\alpha E^c|!\prod_{j\in\mathfrak{v}\cap\alpha E^c}d_j\right)^\lambda$$

$$=\sum_{\mathfrak{w}\subset\alpha E}\sum_{\substack{\mathfrak{u}\subset\alpha E^c\\|\mathfrak{u}|<\infty}}\left(\left(\prod_{j\in\mathfrak{w}}\alpha!d_j\right)|\mathfrak{u}|!\prod_{j\in\mathfrak{u}}d_j\right)^\lambda$$

$$=\sum_{\mathfrak{w}\subset\alpha E}\left(\prod_{j\in\mathfrak{w}}\alpha!d_j\right)^\lambda\sum_{\substack{\mathfrak{u}\subset\alpha E^c\\|\mathfrak{u}|<\infty}}\left(|\mathfrak{u}|!\prod_{j\in\mathfrak{u}}d_j\right)^\lambda$$

$$\leq\prod_{j=1}^{\alpha J}(1+(\alpha!d_j)^\lambda)\sum_{\ell=0}^\infty(\ell!)^\lambda\sum_{\substack{\mathfrak{u}\subset\alpha E^c\\|\mathfrak{u}|=\ell}}\prod_{j\in\mathfrak{u}}d_j^\lambda$$

$$\leq\exp\left((\alpha!)^\lambda\sum_{j=1}^\infty d_j^\lambda\right)\sum_{\ell=0}^\infty(\ell!)^{\lambda-1}\left(\sum_{j=1}^\infty d_j^\lambda\right)^\ell, \tag{5.13}$$

where in the last step we used the estimation $1+x=\exp(\log(1+x))\leq\exp(x)$.

Note that $\sum_{j=1}^\infty\beta_j^p<\infty$ holds if and only if $\sum_{j=1}^\infty d_j^p<\infty$. The last expression in (5.13) is finite for $p\leq\lambda\leq1$. The last expression in (5.13) is also finite if $\lambda=1$ and $\sum_{j=1}^\infty d_j<1$. Since λ also needs to satisfy $1/\alpha<\lambda\leq1$, we choose

$$\lambda=p\qquad\text{and}\qquad\alpha=\lfloor1/p\rfloor+1\,,$$

21

and for $p = 1$ we assume additionally that $\sum_{j=1}^{\infty} d_j < 1$, which is equivalent to

$$\sum_{j=1}^{\infty} \beta_j < \frac{1}{2\alpha \max(B, 1)} . \tag{5.14}$$

Thus we obtain a convergence of $\mathcal{O}(N^{-1/p})$ with the implied constant independent of the dimension s.

5.1 Component-by-component algorithm

We first derive a closed form expression for $\mathcal{E}_d(\boldsymbol{q})$ in (5.7) which can be used for computation. Recall from Definition 5.1 that the j-th coordinate of the n-th point of the interlaced polynomial lattice point set is

$$y_j^{(n)} = v_m \left(\frac{n(x)\, q_j(x)}{P(x)} \right) .$$

Note that $y_j^{(n)}$ depends on the j-th component q_j of the generating vector. We have

$$\sum_{\boldsymbol{\ell}_{\mathfrak{v}} \in \mathcal{D}_{\mathfrak{v}}^*} b^{-\alpha\mu_1(\boldsymbol{\ell}_{\mathfrak{v}})} = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\boldsymbol{\ell} \in \mathbb{N}^{|\mathfrak{v}|}} b^{-\alpha\mu_1(\boldsymbol{l})} \operatorname{wal}_{\boldsymbol{\ell}}(\boldsymbol{y}_{\mathfrak{v}}^{(n)}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \prod_{j \in \mathfrak{v}} \omega(y_j^{(n)}) ,$$

where $\boldsymbol{y}_{\mathfrak{v}}^{(n)} = (y_j^{(n)})_{j \in \mathfrak{v}}$ is the projection of the n-th point $\boldsymbol{y}^{(n)}$ onto the coordinates in $\mathfrak{v}$,

$$\omega(y) = \sum_{\ell=1}^{\infty} b^{-\alpha\mu_1(\ell)} \operatorname{wal}_{\ell}(y) = \frac{b-1}{b^{\alpha} - b} - b^{\lfloor \log_b y \rfloor(\alpha-1)} \frac{b^{\alpha} - 1}{b^{\alpha} - b} ,$$

and where for $y = 0$ we set $b^{\lfloor \log_b 0 \rfloor(\alpha-1)} := 0$. The last equality can be obtained by multiplying [11, Eq. (2)] by $b^{-\alpha}$. Thus we have

$$\mathcal{E}_d(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\emptyset \neq \mathfrak{v} \subseteq \{1:d\}} \widetilde{\gamma}_{\mathfrak{v}} \prod_{j \in \mathfrak{v}} \omega(y_j^{(n)}) . \tag{5.15}$$

The CBC construction proceeds inductively on the dimension, keeping the components already calculated fixed and searching for the polynomial q_d which minimizes $\mathcal{E}_d$. To do so, we separate the terms in $\mathcal{E}_d$ which depend on q_d from those which do not depend on q_d. This depends on the particular form of the weights.

From (4.11) and (5.8) we obtain hybrid weights

$$\widetilde{\gamma}_{\mathfrak{v}} = \sum_{\boldsymbol{\nu}_{\mathfrak{u}(\mathfrak{v})} \in \{1:\alpha\}^{|\mathfrak{u}(\mathfrak{v})|}} \boldsymbol{\nu}_{\mathfrak{u}(\mathfrak{v}) \cap E}!\, |\boldsymbol{\nu}_{\mathfrak{u}(\mathfrak{v}) \cap E^c}|!\, \prod_{j \in \mathfrak{u}(\mathfrak{v})} \gamma_j(\nu_j), \quad \text{with} \quad \gamma_j(\nu_j) := C_{\alpha,b}'\, b^{\alpha(\alpha-1)/2}\, 2^{\delta(\nu_j,\alpha)} \beta_j^{\nu_j} .$$

Substituting this into (5.15) yields

$$\mathcal{E}_d(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\emptyset \neq \mathfrak{v} \subseteq \{1:d\}} \sum_{\boldsymbol{\nu}_{\mathfrak{u}(\mathfrak{v})} \in \{1:\alpha\}^{|\mathfrak{u}(\mathfrak{v})|}} \boldsymbol{\nu}_{\mathfrak{u}(\mathfrak{v}) \cap E}!\, |\boldsymbol{\nu}_{\mathfrak{u}(\mathfrak{v}) \cap E^c}|! \left(\prod_{j \in \mathfrak{u}(\mathfrak{v})} \gamma_j(\nu_j) \right) \left(\prod_{j \in \mathfrak{v}} \omega(y_j^{(n)}) \right) .$$

Every block of α components in the generating vector $\boldsymbol{q}$ yields one component for the interlaced polynomial lattice rule. In order to keep track of the block and position within each block, we replace the index d by a double index (s,t) such that s is the index for the block and t is the index within the block, that is, we set

$$s = \lceil d/\alpha \rceil \quad \text{and} \quad t = (d-1) \bmod \alpha + 1 \quad \text{such that} \quad d = \alpha(s-1) + t \,.$$

We now reorder the sums in $\mathcal{E}_d(\boldsymbol{q})$ according to $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_s) \in \{0:\alpha\}^s$ and $\mathfrak{v} \subseteq \{1:d\}$ so that the set $\mathfrak{u}(\mathfrak{v})$ consists of the indices j for which $\nu_j > 0$. This yields

$$\mathcal{E}_{s,t}(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^s \\ |\boldsymbol{\nu}| \neq 0}} \sum_{\substack{\mathfrak{v} \subseteq \{1:d\} \text{ s.t.} \\ \mathfrak{u}(\mathfrak{v}) = \{1 \leq j \leq s : \nu_j > 0\}}} \boldsymbol{\nu}_E! \, |\boldsymbol{\nu}_{E^c}|! \left(\prod_{j \in \mathfrak{u}(\mathfrak{v})} \gamma_j(\nu_j) \right) \left(\prod_{j \in \mathfrak{v}} \omega(y_j^{(n)}) \right)$$

$$= \frac{1}{b^m} \sum_{n=0}^{b^m-1} (S_1(n,s,t) + S_2(n,s,t) + S_3(n,s,t)), \tag{5.16}$$

where

$$S_1(n,s,t) := \sum_{\substack{\boldsymbol{\nu}_{E_s} \in \{0:\alpha\}^{|E_s|} \\ |\boldsymbol{\nu}_{E_s}| \neq 0}} \sum_{\substack{\mathfrak{v}_1 \subseteq \{1:\min(d,\alpha J)\} \text{ s.t.} \\ \mathfrak{u}(\mathfrak{v}_1) = \{1 \leq j \leq \min(s,J) : \nu_j > 0\}}} \boldsymbol{\nu}_{E_s}! \left(\prod_{j \in \mathfrak{u}(\mathfrak{v}_1)} \gamma_j(\nu_j) \right) \left(\prod_{j \in \mathfrak{v}_1} \omega(y_j^{(n)}) \right),$$
$$\tag{5.17}$$

$$S_2(n,s,t) := \sum_{\substack{\boldsymbol{\nu}_{E_s^c} \in \{0:\alpha\}^{|E_s^c|} \\ |\boldsymbol{\nu}_{E_s^c}| \neq 0}} \sum_{\substack{\mathfrak{v}_2 \subseteq \{1+\min(d,\alpha J):d\} \text{ s.t.} \\ \mathfrak{u}(\mathfrak{v}_2) = \{\min(s,J) < j \leq s : \nu_j > 0\}}} |\boldsymbol{\nu}_{E_s^c}|! \left(\prod_{j \in \mathfrak{u}(\mathfrak{v}_2)} \gamma_j(\nu_j) \right) \left(\prod_{j \in \mathfrak{v}_2} \omega(y_j^{(n)}) \right),$$
$$\tag{5.18}$$

$$S_3(n,s,t) = S_1(n,s,t) \cdot S_2(n,s,t),$$

with $E_s := E \cap \{1:s\}$, $E_s^c := E^c \cap \{1:s\}$, $\nu_{E_s} = (\nu_j)_{j \in E_s}$ and $\nu_{E_s^c} = (\nu_j)_{j \in E_s^c}$. For $s \leq J$ we set $S_2(n,s,t) = 0$.

We note that $S_1(n)$ has a product weight structure while $S_2(n)$ has an SPOD weight structure. If $d > \alpha J$ then $S_1(n)$ is fixed and we need to compute $S_2(n)$ only.

For $d = \alpha(s-1) + t \leq \alpha J$ we have

$$\mathcal{E}_{s,t}(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^s \\ |\boldsymbol{\nu}| \neq 0}} \sum_{\substack{\emptyset \neq \mathfrak{v}_1 \subseteq \{1:\alpha(s-1)+t\} \text{ s.t.} \\ \mathfrak{u}(\mathfrak{v}_1) = \{1 \leq j \leq \min(s,J) : \nu_j > 0\}}} \left(\prod_{j \in \mathfrak{u}(\mathfrak{v}_1)} \nu_j! \gamma_j(\nu_j) \right) \left(\prod_{j \in \mathfrak{v}_1} \omega(y_j^{(n)}) \right)$$

$$= \frac{1}{b^m} \sum_{n=0}^{b^m-1} \sum_{\emptyset \neq \mathfrak{u} \subseteq \{1:s\}} \left(\prod_{j \in \mathfrak{u}} \sum_{\nu_j=1}^{\alpha} \nu_j! \gamma_j(\nu_j) \right) \sum_{\substack{\mathfrak{v}_1 \subseteq \{1:\alpha(s-1)+t\} \\ \mathfrak{u}(\mathfrak{v}_1) = \mathfrak{u}}} \left(\prod_{j \in \mathfrak{v}_1} \omega(y_j^{(n)}) \right).$$

Replacing d by the double index (s,t) as before, we obtain for $t = \alpha$ that

$$\mathcal{E}_{s,\alpha}(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \underbrace{\prod_{j=1}^{s} \left[1 + \sum_{\nu_j=1}^{\alpha} \nu_j! \gamma_j(\nu_j) \left(\prod_{i=1}^{\alpha} (1 + \omega(y_{j,i}^{(n)})) - 1 \right) \right]}_{=: Y_s(n)} - 1 \,,$$

where we defined the quantity $Y_s(n)$, with $Y_0(n) := 1$. For $t < \alpha$ we have

$$\mathcal{E}_{s,t}(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} \left[1 + \sum_{\nu_s=1}^{\alpha} \nu_s! \gamma_s(\nu_s) \underbrace{\left(\prod_{i=1}^{t} (1 + \omega(y_{s,i}^{(n)})) - 1 \right)}_{=: V_{s,t}(n)} \right] Y_{s-1}(n) - 1,$$

where $V_{s,t}(n)$ is defined above. The part of $\mathcal{E}_{s,t}(\boldsymbol{q})$ that is affected by $q_{s,t}$ is

$$\sum_{n=1}^{b^m-1} \omega(y_{s,t}^{(n)}) V_{s,t-1}(n) Y_{s-1}(n).$$

In order to compute this quantity for every $q_{s,t} \in G_{b,m}$ we need to perform the matrix-vector multiplication using the matrix

$$\boldsymbol{\Omega} := \left[\omega \left(v_m \left(\frac{n(x)q(x)}{P(x)} \right) \right) \right]_{\substack{1 \le n \le b^m-1 \\ q \in \overline{G}_{b,m}}}$$

and the vector $[V_{s,t-1}(n) Y_{s-1}(n)]_{1 \le n \le b^m-1}$. A permutation can be applied to $\boldsymbol{\Omega}$ using the so-called Rader transform (see, e.g., [25])) such that the fast Fourier transform can be used to carry out the matrix-vector multiplication. As shown in [25], this reduces the cost of the matrix-vector multiplication to $\mathcal{O}(M \log M) = \mathcal{O}(N \log N)$ operations, where $M = b^m - 1$ and $N = b^m$.

Once $q_{s,t}$ has been computed for a given dimension, one has to update the products $V_{s,t}(n)$. This can be done in $\mathcal{O}(N)$ operations. After an entire block of α dimensions has been computed, the products $Y_s(n)$ need to be updated, which can be done in $\mathcal{O}(N)$ operations. The total computational cost is then $\mathcal{O}(\alpha s N \log N)$ operations, with a memory requirement of $\mathcal{O}(N)$.

When $d > \alpha J$ we have

$$\mathcal{E}_{s,t}(\boldsymbol{q}) = \frac{1}{b^m} \sum_{n=0}^{b^m-1} S_1(n, J, \alpha) + S_2(n, s, t) \cdot (1 + S_1(n, J, \alpha)),$$

where $S_1(n, J, \alpha) = Y_J(n) - 1$. Thus $S_1(n, J, \alpha)$ has been computed in the first part of the algorithm and is therefore now fixed. When the final block is complete and therefore $t = \alpha$, we have

$$S_2(n, s, \alpha) = \sum_{\ell=1}^{\alpha(s-J)} \ell! \underbrace{\sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^{s-J} \\ |\boldsymbol{\nu}|=\ell}} \prod_{\substack{j=J+1 \\ \nu_j>0}}^{s} \left[\gamma_j(\nu_j) \left(\prod_{i=1}^{\alpha} (1 + \omega(y_{j,i}^{(n)})) - 1 \right) \right]}_{=: U_{s,\ell}(n)}, \qquad (5.19)$$

where $\boldsymbol{\nu} \in \{0 : \alpha\}^{s-J}$ is given by $\boldsymbol{\nu} = (\nu_j)_{j \in \{J+1, J+2, \dots, s\}}$ and where we defined the quantity $U_{s,\ell}(n)$, with $U_{J,\ell}(n) := 1$, $U_{s,0}(n) := 0$, and $U_{s,\ell}(n) := 0$ for $\ell > \alpha(s-J)$. When the final

block is incomplete, that is, $t < \alpha$, by separating out the case $\nu_s = 0$ in (5.16), we get

$$
S_2(n, s, t) = \sum_{\ell=1}^{\alpha(s-1-J)} \ell! \sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^{s-1-J} \\ |\boldsymbol{\nu}|=\ell}} \prod_{\substack{j=J+1 \\ \nu_j>0}}^{s-1} \left[\gamma_j(\nu_j) \left(\prod_{i=1}^{\alpha} (1 + \omega(y_{j,i}^{(n)})) - 1 \right) \right]
$$

$$
+ \sum_{\ell=1}^{\alpha(s-J)} \sum_{\nu_s=1}^{\min(\alpha,\ell)} \ell! \sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^{s-1-J} \\ |\boldsymbol{\nu}|=\ell-\nu_s}} \left(\prod_{\substack{j=J+1 \\ \nu_j>0}}^{s-1} \left[\gamma_j(\nu_j) \left(\prod_{i=1}^{\alpha} (1 + \omega(y_{j,i}^{(n)})) - 1 \right) \right] \right.
$$

$$
\left. \times \gamma_s(\nu_s) \left(\prod_{i=1}^{t} (1 + \omega(y_{s,i}^{(n)})) - 1 \right) \right) ,
$$

and thus

$$
S_2(n, s, t) = S_2(n, s-1, \alpha) \tag{5.20}
$$

$$
+ \underbrace{\left(\prod_{i=1}^{t} (1 + \omega(y_{s,i}^{(n)})) - 1 \right)}_{=:V_{s,t}(n)} \underbrace{\left(\sum_{\ell=1}^{\alpha(s-J)} \underbrace{\sum_{\nu_s=1}^{\min(\alpha,\ell)} \gamma_s(\nu_s) \frac{\ell!}{(\ell-\nu_s)!} U_{s-1,\ell-\nu_s}(n)}_{=:X_{s,\ell}(n)} \right)}_{=:W_s(n)} ,
$$

where we defined $V_{s,t}(n)$, $W_s(n)$, and $X_{s,\ell}(n)$ as indicated, with $V_{s,0}(n) := 1$.

Since the polynomial $q_{s,t}$ only appears in the final factor of the products $V_{s,t}(n)$, the only part of $\mathcal{E}_{s,t}(\boldsymbol{q})$ that is affected by $q_{s,t}$ is

$$
\sum_{n=1}^{b^m-1} \omega(y_{s,t}^{(n)}) \, V_{s,t-1}(n) \, W_s(n) \, (1 + S_1(n, J, \alpha)) .
$$

Computing this quantity for every $q_{s,t} \in G_{b,m}$ requires the matrix-vector multiplication with the matrix $\boldsymbol{\Omega}$ and the vector $[V_{s,t-1}(n) \, W_s(n) \, (1 + S_1(n, J, \alpha)]_{1 \le n \le b^m - 1}$. Again, one can apply a permutation to the matrix $\boldsymbol{\Omega}$ such that the fast Fourier transform can be used [25]. The cost is then $\mathcal{O}(M \log M) = \mathcal{O}(N \log N)$ operations, where $M = b^m - 1$ and $N = b^m$.

Once $q_{s,t}$ is chosen for dimension $\alpha(s-1) + t$, we update the products $V_{s,t}(n)$ using

$$
V_{s,t}(n) = (1 + \omega(y_{s,t}^{(n)})) \, V_{s,t-1}(n) .
$$

This requires $\mathcal{O}(N)$ operations. After completing an entire block of α dimensions, also

the values $U_{s,\ell}(n)$ need to be updated. This can be done using the equation

$$
U_{s,\ell}(n) = \ell! \sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^{s-J-1} \\ |\boldsymbol{\nu}|=\ell}} \prod_{\substack{j=J+1 \\ \nu_j>0}}^{s-1} \left[\gamma_j(\nu_j) \left(\prod_{i=1}^{\alpha}(1 + \omega(y_{j,i}^{(n)})) - 1 \right) \right]
$$

$$
+ \ell! \sum_{\nu_s=1}^{\min(\alpha,\ell)} \sum_{\substack{\boldsymbol{\nu} \in \{0:\alpha\}^{s-J-1} \\ |\boldsymbol{\nu}|=\ell-\nu_s}} \left(\prod_{\substack{j=J+1 \\ \nu_j>0}}^{s-1} \left[\gamma_j(\nu_j) \left(\prod_{i=1}^{\alpha}(1 + \omega(y_{j,i}^{(n)})) - 1 \right) \right] \right.
$$

$$
\left. \times \gamma_s(\nu_s) \left(\prod_{i=1}^{\alpha}(1 + \omega(y_{s,i}^{(n)})) - 1 \right) \right)
$$

$$
= U_{s-1,\ell}(n) + (V_{s,\alpha}(n) - 1) X_{s,\ell}(n) .
$$

Since the quantities $V_{s,\alpha}(n)$ and $X_{s,\ell}(n)$ can be pre-computed and stored, this update requires $\mathcal{O}(\alpha (s - J)_+ N)$ operations, where $(x)_+ = \max\{0, x\}$. In the next step, the products $V_{s+1,0}(n)$ need to be initialized by 1 with $\mathcal{O}(N)$ operations, and the quantities $W_{s+1}(n)$ and $X_{s+1,\ell}(n)$ need to be computed, which can be done in $\mathcal{O}(\alpha^2 (s - J)_+ N)$ operations. The algorithm then continuous the search in the new block.

We need to store the quantities $U_{s,\ell}(n)$, $V_{s,t}(n)$, $W_s(n)$, and $X_{s,\ell}(n)$, which can be overwritten as we increase s and t. Hence, the total memory requirement is $\mathcal{O}(\alpha\, s\, N)$.

The *total computational cost for the CBC construction up to dimension* αs is therefore bounded by

$$
\begin{aligned}
\mathcal{O}\left(\alpha \min\{s, J\} N \log N + \alpha^2(s - J)_+ N \log N \right) &\quad \text{search cost, plus} \\
\mathcal{O}(\alpha^2(s - J)_+^2 N) &\quad \text{update cost, plus} \\
\mathcal{O}(N + \alpha(s - J)_+ N) &\quad \text{memory cost .}
\end{aligned}
$$

Hence, for large values of J (as may occur in practice, cf. Remark 3.1), and for higher orders α the product structure of the QMC weights up to dimension J, implied by (3.2), imply quantitative advantages in the CBC construction.

We summarize the algorithm in Pseudocode 1 below; there, $.*$ means element-wise multiplication. Note that $\boldsymbol{U}(\ell)$ for $\ell = 0, \ldots, \alpha(s_{\max} - J)$, and $\boldsymbol{V}$, $\boldsymbol{W}$, $\boldsymbol{X}(\ell)$ for $\ell = 1, \ldots, \alpha(s_{\max} - J)$, and $\boldsymbol{\mathcal{E}}$ are all vectors of length $N - 1$, while $\boldsymbol{\Omega}^{\mathrm{perm}}$ denotes the permuted version of the matrix $\boldsymbol{\Omega}$. The vector $\boldsymbol{\mathcal{E}}$ stores the values of $\mathcal{E}_d$.

6   Conclusion

We have analyzed the convergence of a class of higher order Quasi Monte-Carlo (HO-QMC) quadrature methods for the approximate evaluation of response-statistics of a class of nonlinear operator equations subject to distributed uncertainty, corresponding (via a Schauder basis) to infinite-dimensional, iterated integrals. We showed that for operators with analytic dependence on the uncertain input, the HOMQC quadratures achieve convergence rates which are independent of the parameter dimension and which are, in a sense, best possible for a given sparsity measure of the parameter dependence. The argument in the proof of Theorem 3.1 involved analytic continuation. It allows us to control parametric derivatives of arbitrary order and is, therefore, also applicable to

other settings; we mention in particular the problem of Bayesian estimation. Here, the posterior densities admit an infinite-dimensional, parametric deterministic representation which "inherits" analytic properties from the forward map (cf. [30, 27, 28] and the references there). Details on the extension of the present analysis to this problem class, as well as numerical experiments, will be presented elsewhere.

In the present paper, we have confined the analysis to the so-called single-level version of the HOQMC-PG discretization, and assumed minimal regularity $G(\cdot) \in \mathcal{X}'$. Based on the present results, *multilevel discretizations* can be designed which are more complicated but which are expected to exhibit, in certain cases, superior performance (we refer to [10] for the analysis of a higher order, multilevel QMC-PG algorithm in the particular case of affine-parametric, linear operators). The analysis of such multilevel algorithms in the present general context, will likewise be presented elsewhere.

References

[1] J. Baldeaux, J. Dick, J. Greslehner and F. Pillichshammer, Construction algorithms for higher order polynomial lattice rules. J. Complexity, 27, 281–299, 2011.

[2] J. Baldeaux, J. Dick, G. Leobacher, D. Nuyens and F. Pillichshammer, Efficient calculation of the worst-case error and (fast) component-by-component construction of higher order polynomial lattice rules. Numer. Algorithms, 59, 403–431, 2012.

[3] F. Brezzi, J. Rappaz and P.A. Raviart, Finite dimensional approximation of nonlinear problems I: branches of nonsingular solutions. Numer. Math., 36, 1–25, 1980.

[4] A. Chkifa, A. Cohen and Ch. Schwab, Breaking the curse of dimensionality in sparse polynomial approximation of parametric PDEs. To appear in Journ. Math. Pures et Appliquees, 2014. Available at: Report 2013-25, Seminar for Applied Mathematics, ETH Zürich 2013.

[5] A. Cohen, R. DeVore and Ch. Schwab, Convergence rates of best N-term Galerkin approximation for a class of elliptic sPDEs. Found. Comput. Math., 10, 615–646, 2010.

[6] A. Cohen, R. DeVore and Ch. Schwab, Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDEs. Analysis and Applications, 9, 1–37, 2011.

[7] J. Dick, Explicit constructions of Quasi-Monte Carlo rules for the numerical integration of high-dimensional periodic functions. SIAM J. Numer. Anal., 45, 2141–2176, 2007.

[8] J. Dick, Walsh spaces containing smooth functions and Quasi-Monte Carlo rules of arbitrary high order. SIAM J. Numer. Anal., 46, 1519–1553, 2008.

[9] J. Dick, The decay of the Walsh coefficients of smooth functions. Bull. Aust. Math. Soc., 80, 430–453, 2009.

[10] J. Dick, F.Y. Kuo, Q. T. Le Gia, D. Nuyens and Ch. Schwab, Higher order QMC Galerkin discretization for parametric operator equations. To appear in SIAM J. Numer. Anal., 2015. [Available at arXiv:1309.4624]

[11] J. Dick and F. Pillichshammer, Multivariate integration in weighted Hilbert spaces based on Walsh functions and weighted Sobolev spaces. J. Complexity, 21,149–195, 2005.

[12] J. Dick and F. Pillichshammer, Strong tractability of multivariate integration of arbitrary high order using digitally shifted polynomial lattice rules. J. Complexity, 23, 436–453, 2007.

[13] J. Dick and F. Pillichshammer, *Digital Nets and Sequences*, Cambridge University Press, 2010.

[14] V. Girault and P.A. Raviart, *Finite Element Methods for Navier-Stokes Equations.* Springer Verlag, Berlin, 1986.

[15] T. Goda, Good interlaced polynomial lattice rules for numerical integration in weighted Walsh spaces. Preprint (2013) (in review) [Available at arXiv:1306.4573v1]

[16] T. Goda and J. Dick, Construction of interlaced scrambled polynomial lattice rules of arbitrary high order. To appear in Found. Comput. Math., 2015. [Available at arXiv:1301.6441v2]

[17] M. Hansen and Ch. Schwab, Analytic regularity and best N-term approximation of high dimensional, parametric initial value problems. Vietnam Journal of Mathematics, 41, 181–215, 2013.

[18] V. H. Hoang and Ch. Schwab, Regularity and Generalized Polynomial Chaos Approximation of Parametric and Random Second-Order Hyperbolic Partial Differential Equations. Analysis and Applications (Singapore) 10(3), 2012.

[19] A. Kunoth and Ch. Schwab, Analytic Regularity and GPC Approximation for Stochastic Control Problems Constrained by Linear Parametric Elliptic and Parabolic PDEs. SIAM Journ. Control and Optimization, 51, 2442 – 2471, 2013.

[20] F. Y. Kuo, Ch. Schwab, and I. H. Sloan, Quasi-Monte Carlo methods for very high dimensional integration: the standard weighted-space setting and beyond. ANZIAM Journal, 53, 1–37, 2011.

[21] F. Y. Kuo, Ch. Schwab, and I. H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficient. SIAM J. Numer. Anal., 50, 3351–3374, 2012.

[22] F. Y. Kuo, Ch. Schwab, and I. H. Sloan, Multi-Level Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficient. (In review.)

[23] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, Philadelphia, 1992.

[24] V. Nistor and Ch. Schwab, High order Galerkin approximations for parametric second order elliptic partial differential equations. Math. Mod. Meth. Appl. Sci., 23, 1729–1760, 2013.

[25] D. Nuyens and R. Cools, Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernel Hilbert spaces. Math. Comp., 75, 903–920, 2006.

[26] J. Pousin and J. Rappaz, Consistency, Stability, apriori an aposteriori errors for Petrov-Galerkin methods applied to nonlinear problems. Numer. Math., 69, 213–231, 1994.

[27] Cl. Schillings and Ch. Schwab, Sparse, adaptive Smolyak quadratures for Bayesian inverse problems. Inverse Problems, 29, no. 6, 065011, 28 pp., 2013.

[28] Cl. Schillings and Ch. Schwab, Sparsity in Bayesian Inversion of Parametric Operator Equations. Inverse Problems, 30, 065007, 30 pp., 2014.

[29] Ch. Schwab, QMC Galerkin discretizations of parametric operator equations. In J. Dick, F. Y. Kuo, G. W. Peters and I. H. Sloan (eds.), Monte Carlo and Quasi-Monte Carlo methods 2012, Springer Verlag, Berlin, 2013, pp. 613–630.

[30] Ch. Schwab and A.M. Stuart, Sparse deterministic approximation of Bayesian inverse problems. Inverse Problems, 28, 045003, 2012.

Pseudocode 1 (Fast CBC implementation for hybrid weights)

$\boldsymbol{Y} := \mathbf{1}$
for s **from** 1 **to** $\min(J, s_{\max})$ **do**
    $\boldsymbol{V} := \mathbf{1}$
    for t **from** 1 **to** α **do**
        $\boldsymbol{\mathcal{E}} := \boldsymbol{\Omega}^{\mathrm{perm}}\left(\boldsymbol{V}.{*}\,\boldsymbol{Y}\right)$ $\triangleright$ **compute** $-$ **use FFT**
        $q_{s,t} := \mathrm{argmin}_{q \in G_{b,m}}\mathcal{E}(q)$ $\triangleright$ **select** $-$ **pick the correct index**
        $\boldsymbol{V} := \left(\mathbf{1} + \boldsymbol{\Omega}^{\mathrm{perm}}(q_{s,t}, :)\right).{*}\,\boldsymbol{V}$ $\triangleright$ **update products**
    end for
    $\boldsymbol{Y} := (\mathbf{1} + \sum_{\nu_j=1}^{\alpha} \nu_s!\gamma_s(\boldsymbol{V} - \mathbf{1})).{*}\,\boldsymbol{Y}$ $\triangleright$ **update products**
end for
$\boldsymbol{S}_1 := \boldsymbol{Y} - \mathbf{1}$
if $s_{\max} \leq J$ **then return**
end if
$\boldsymbol{U}(0) := \mathbf{1}$
$\boldsymbol{U}(1 : \alpha\,(s_{\max} - J)) := \mathbf{0}$
for s **from** $J+1$ **to** $s_{\max}$ **do**
    $\boldsymbol{V} := \mathbf{1}$ $\triangleright$ **initialize products and sums**
    $\boldsymbol{W} := \mathbf{0}$
    for ℓ **from** 1 **to** $\alpha(s - J)$ **do**
        $\boldsymbol{X}(\ell) := \mathbf{0}$
        for ν **from** 1 **to** $\min(\alpha, \ell)$ **do**
            $\boldsymbol{X}(\ell) := \boldsymbol{X}(\ell) + \gamma_s(\nu)\dfrac{\ell!}{(\ell - \nu)!}\,\boldsymbol{U}(\ell - \nu)$
        end for
        $\boldsymbol{W} := \boldsymbol{W} + \boldsymbol{X}(\ell)$
    end for
    for t **from** 1 **to** α **do**
        $\boldsymbol{\mathcal{E}} := \boldsymbol{\Omega}^{\mathrm{perm}}\left(\boldsymbol{S}_1 + (\mathbf{1} + \boldsymbol{S}_1).{*}\,\boldsymbol{V}.{*}\,\boldsymbol{W}\right)$ $\triangleright$ **compute** $-$ **use FFT**
        $q_{s,t} := \mathrm{argmin}_{q \in G_{b,m}}\mathcal{E}(q)$ $\triangleright$ **select** $-$ **pick the correct index**
        $\boldsymbol{V} := \left(\mathbf{1} + \boldsymbol{\Omega}^{\mathrm{perm}}(q_{s,t}, :)\right).{*}\,\boldsymbol{V}$ $\triangleright$ **update products**
    end for
    for ℓ **from** 1 **to** $\alpha(s - J)$ **do** $\triangleright$ **update sums**
        $\boldsymbol{U}(\ell) := \boldsymbol{U}(\ell) + (\boldsymbol{V} - \mathbf{1}).{*}\,\boldsymbol{X}(\ell)$
    end for
end for

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