Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Seminar for Applied Mathematics SAM

Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients

D. Conus and A. Jentzen and R. Kurniawan

Research Report No. 2014-18
August 2014

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Weak convergence rates of spectral Galerkin approximations for SPDEs with nonlinear diffusion coefficients

Daniel Conus, Arnulf Jentzen and Ryan Kurniawan

Lehigh University, USA and ETH Zurich, Switzerland

August 13, 2014

Abstract

Strong convergence rates for (temporal, spatial, and noise) numerical approximations of semilinear stochastic evolution equations (SEEs) with smooth and regular nonlinearities are well understood in the scientific literature. Weak convergence rates for numerical approximations of such SEEs have been investigated since about 11 years and are far away from being well understood: roughly speaking, no essentially sharp weak convergence rates are known for parabolic SEEs with nonlinear diffusion coefficient functions; see Remark 2.3 in [A. Debussche, Weak approximation of stochastic partial differential equations: the nonlinear case, Math. Comp. 80 (2011), no. 273, 89–117] for details. In this article we solve the weak convergence problem emerged from Debussche's article in the case of spectral Galerkin approximations and establish esssentially sharp weak convergence rates for spatial spectral Galerkin approximations of semilinear SEEs with nonlinear diffusion coefficient functions. Our solution to the weak convergence problem does not use Malliavin calculus. Rather, key ingredients in our solution to the weak convergence problem emerged from Debussche's article are the use of appropriately modified versions of the spatial Galerkin approximation processes and applications of a mild Itô type formula for solutions and numerical approximations of semilinear SEEs. This article solves the weak convergence problem emerged from Debussche's article merely in the case of spatial spectral Galerkin approximations instead of other more complicated numerical approximations. Our method of proof extends, however, to a number of other kind of spatial, temporal, and noise numerical approximations for semilinear SEEs.

1 Introduction

Both strong and numerically weak convergence rates for numerical approximations of finite dimensional stochastic ordinary differential equations (SODEs) with smooth and regular nonlinearities are well understood in the literature; see, e.g., the monographs Kloeden & Platen [20] and Milstein [26]. The situation is different in the case of possibly infinite dimensional semilinear stochastic evoluation equations (SEEs). While strong convergence rates for (temporal, spatial, and noise) numerical approximations of semilinear SEEs with smooth and regular nonlinearities are well understood in the scientific literature, weak convergence rates for numerical approximations of such SEEs have been investigated since about 11 years and are far away from being well understood: roughly speaking, no essentially sharp weak convergence rates are known for parabolic SEEs with nonlinear diffusion coefficient functions (see Remark 2.3 in Debussche [12] for details). In this article we solve the weak convergence problem emerged from Debussche's article in the case of spectral Galerkin approximations and establish essentially sharp weak convergence rates for spatial spectral Galerkin approximations of semilinear SEEs with nonlinear diffusion coefficient functions. To illustrate the weak convergence problem emerged from Debussche's article and our solution to the problem we consider the following setting as a special case of our general setting in Section 5 below. Let $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$ and $(U, \langle \cdot, \cdot \rangle_U, \|\cdot\|_U)$ be separable $\mathbb{R}$-Hilbert spaces. Let $T \in (0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis and let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Let $(e_n)_{n\in\mathbb{N}} \subseteq H$ be an orthonormal basis of H and let $(\lambda_n)_{n\in\mathbb{N}} \subseteq (0, \infty)$ be an increasing sequence. Let $A \colon D(A) \subseteq H \to H$ be a closed linear operator such that $D(A) = \{v \in H \colon \sum_{n\in\mathbb{N}} |\lambda_n \langle e_n, v\rangle_H|^2 < \infty\}$ and such that for all $n \in \mathbb{N}$ it holds that $Ae_n = -\lambda_n e_n$. Let $(H_r, \langle \cdot, \cdot \rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $-A$ (see, e.g., Theorem and Definition 2.5.32 in [19]). Let $\iota \in [0, \frac{1}{4}]$ and let $\xi \in H_\iota$. Finally, let $\gamma \in [0, \frac{1}{2}]$, and let $F \in \cap_{r<\iota-\gamma} C_b^4(H_\iota, H_r)$, $B \in \cap_{r<\iota-\gamma/2} C_b^4(H_\iota, HS(U, H_r))$, where for two $\mathbb{R}$-Banach spaces $(V_1, \|\cdot\|_{V_1})$ and $(V_2, \|\cdot\|_{V_2})$ we denote by $C_b^4(V_1, V_2)$ the $\mathbb{R}$-vector space of all four times continuously Fréchet differentiable functions with globally bounded derivatives. The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) the existence of a continuous mild solution process $X \colon [0, T] \times \Omega \to H_\iota$ of the SEE

$$dX_t = [AX_t + F(X_t)]\, dt + B(X_t)\, dW_t, \qquad t \in [0, T], \qquad X_0 = \xi. \tag{1}$$

As an example for (1), we think of $H = U = L^2((0, 1); \mathbb{R})$ being the $\mathbb{R}$-Hilbert space of equivalence classes of Lebesgue square integrable functions from $(0, 1)$ to $\mathbb{R}$ and A being the Laplace operator with Dirichlet boundary conditions on H (cf., e.g., Da Prato & Zabczyk [9] and Debussche [12] for details). In the above setting the parameter γ controls the regularity of the operators F and B.

Strong convergence rates for (temporal, spatial, and noise) numerical approximations for SEEs of the form (1) are well understood. Weak convergence rates for numerical approximations of SEEs of the form (1) have been investigated since about 11 years; see, e.g., [32, 16, 11, 13, 15, 21, 14, 3, 22, 25, 23, 24, 4, 5, 2, 34, 36, 17, 12, 35]. Except for Debussche & De Bouard [11], Debussche [12] and Andersson & Larsson [3], all of the above mentioned references assume, beside further assumptions, that the considered SEE is driven by additive noise. In Debussche & De Bouard [11] weak convergence rates for the nonlinear Schrödinger equation, whose dominant linear operator generates a group (see Section 2 in [11]) instead of only a semigroup as in the general setting of the SEE (1), are analyzed. The method of proof in Debussche & De Bouard [11] strongly exploits this property of the nonlinear Schrödinger equation (see Section 5.2 in [11]). Therefore, the method of proof in [11] can, in general, not be used to establish weak convergence rates for the SEE (1). In Debussche's seminal article [12] (see also Andersson & Larsson [3]), essentially sharp weak convergence rates for SEEs of the form (1) are established under the hypothesis that the second derivative of the diffusion coefficient B satisfies the smoothing property that there exists a real number $L \in [0, \infty)$ such that for all $x, v, w \in H$ it holds that[1]

$$\|B''(x)(v, w)\|_{L(H)} \leq L \|v\|_{H_{-1/4}} \|w\|_{H_{-1/4}}. \tag{2}$$

As pointed out in Remark 2.3 in Debussche [12], assumption (2) is a serious restriction for SEEs of the form (1). Roughly speaking, assumption (2) imposes that the second derivative of the diffusion coefficient function vanishes and thus that the diffusion coefficient

[1]Assumption (2) above slighlty differs from the original assumption in [12] as we believe that there is a small typo in equation (2.5) in [12]; see inequality (4.3) in the proof of Lemma 4.5 in [12] for details.

function is affine linear. Remark 2.3 in Debussche [12] also asserts that assumption (2) is crucial in the weak convergence proof in [12], that assumption (2) is used in an essential way in Lemma 4.5 in [12] and that Lemma 4.5 in [12], in turn, is used at many points in the weak convergence proof in [12]. To the best of our knowledge, it remained an open problem to establish essentially sharp weak convergence rates for any type of temporal, spatial, or noise numerical approximation of the SEE (1) without imposing Debussche's assumption (2). In this article we solve this problem in the case of spatial spectral Galerkin approximations for the SEE (1). This is the subject of the following theorem (Theorem 1.1), which follows immediately from Corollary 5.2 (the $\mathbb{R}$-Hilbert space H in Corollary 5.2 corresponds to the $\mathbb{R}$-Hilbert space H_ι in Theorem 1.1).

Theorem 1.1. *Assume the setting in the first paragraph of Section 1, let $\varphi \in C_b^4(H_\iota, \mathbb{R})$, let $(P_N)_{N\in\mathbb{N}} \subseteq L(H_{-1})$ satisfy $P_N(v) = \sum_{n=1}^{N} \langle e_n, v\rangle_H \, e_n$ for all $v \in H$, $N \in \mathbb{N}$, and for every $N \in \mathbb{N}$ let $X^N \colon [0,T] \times \Omega \to P_N(H)$ be a continuous mild solution of the SEE*

$$dX_t^N = \left[P_N A X_t^N + P_N F(X_t^N)\right] dt + P_N B(X_t^N)\, dW_t, \quad t \in [0,T], \quad X_0^N = P_N(\xi). \tag{3}$$

Then for every $\varepsilon \in (0,\infty)$ there exists a real number $C_\varepsilon \in [0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\left|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(X_T^N)\big]\right| \leq C_\varepsilon \cdot (\lambda_N)^{-(1-\gamma-\varepsilon)} . \tag{4}$$

Let us add a few comments regarding Theorem 1.1. First, we would like to emphasize that in the general setting of Theorem 1.1, the weak convergence rate established in Theorem 1.1 can *essentially not be improved*. More specifically, in Corollary 6.5 in Section 6 below we give for every $\iota \in [0, \frac{1}{4}]$, $\gamma \in [0, \frac{1}{2}]$ examples of $A\colon D(A) \subseteq H \to H$, $F \in \cap_{r<\iota-\gamma} C_b^4(H_\iota, H_r)$, $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, $B \in \cap_{r<\iota-\gamma/2} C_b^4(H_\iota, HS(U, H_r))$, and $\varphi \in C_b^4(H_\iota, \mathbb{R})$ such that there exists a real number $C \in [0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\left|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(X_T^N)\big]\right| \geq C \cdot (\lambda_N)^{-(1-\gamma)} . \tag{5}$$

In addition, we emphasize that in the setting of Theorem 1.1 it is well known (cf., e.g., Corollary 6.1.11 in [19]) that for every $\varepsilon \in (0,\infty)$ there exists a real number $C_\varepsilon \in [0,\infty)$ such that for all $N \in \mathbb{N}$ it holds that

$$\left(\mathbb{E}\big[\|X_T - X_T^N\|_{H_\iota}^2\big]\right)^{1/2} \leq C_\varepsilon \cdot (\lambda_N)^{-\left(\frac{1-\gamma}{2}-\varepsilon\right)} . \tag{6}$$

The weak convergence rate $1-\gamma-\varepsilon$ established in Theorem 1.1 is thus *twice* the well known strong convergence rate $\frac{1-\gamma-\varepsilon}{2}$ in (6). Moreover, we add that Theorem 1.1 uses the assumption that the first four derivatives of φ, F, and B are globally bounded. While the proof of Theorem 1.1 can in a straightforward way be extended to the case where φ has at most polynomially growing derivatives, it is not clear to us how to treat the case where F and B are globally Lipschitz continuous with the first four derivatives having at most polynomial growth. Furthermore, we emphasize that Theorem 1.1 solves the weak convergence problem emerged from Debussche's article (see (2.5) and Remark 2.3 in Debussche [12]) merely in the case of spatial spectral Galerkin approximations instead of other more complicated numerical approximations for the SEE (1). The method of proof of our weak convergence results, however, *can be extended to a number of other kind of spatial, temporal, and noise numerical approximations* for SEEs of the form (1). This will be the subject of future research articles. Next we point out that the proof in Debussche's article [12] as well as many other proofs in the above mentioned weak convergence articles use *Malliavin calculus*. Our method of proof does not use Malliavin calculus but uses – in some sense – merely elementary arguments as well as the mild Itô formula in Da Prato et al. [8].

The paper is organized as follows. In Section 1.1 below we give a brief sketch of the proof without technical details. However, the main ideas needed in order to obtain an essentially sharp rate of convergence are highlighted. Sections 1.2 and 1.3 present the general notation and framework used in the paper. Section 2 addresses the weak convergence of the Galerkin projection $P_N(X_T)$ to the solution X_T to the SEE (1) as N goes to infinity. This result is then used in Section 3 to obtain the weak convergence of the Galerkin approximation (3) to the solution of (1) in the case where the drift and diffusion operators F and B as well as the initial condition are mollified in an appropriate sense. This provides a less general version of Theorem 1.1. Section 4 is devoted to the proof of a elementary strong convergence result which is then used in Section 5 to establish weak convergence (Proposition 5.2) for general drift and diffusion operators. Finally, in Section 6, we consider the case $F = 0$ and provide examples of constant (additive noise) functions B which show that the weak convergence rate established in Theorem 1.1 can essentially not be improved.

1.1 Sketch of the proof of Theorem 1.1.

In the following we give a brief sketch of our method of proof in the case where $\xi \in H_{\iota+1}$ (the case where $\xi \in H_\iota$ then follows from a standard mollification procedure; see (83) in the proof of Proposition 5.1 in Section 5 for details). In our weak convergence proof we intend to work (as it is often the case in the case of weak convergence for S(P)DEs; see, e.g., Rößler [31] and Debussche [12]) with the Kolmogorov backward equation associated to (1). In the case of an SEE with a general nonlinear diffusion coefficient it is, however, not clear whether there exists *any relation* between solutions of the Kolmogorov backward equation associated to (1) and solutions of the SEE (1); see Andersson et al. [1] and Da Prato [7]. We therefore work with suitable mollified versions of (1) and (3). More formally, for every $\kappa \in (0, \infty)$ let $F_\kappa \colon H_\iota \to H_1$ and $B_\kappa \colon H_\iota \to HS(U, H_1)$ be functions given by $F_\kappa(x) = e^{\kappa A} F(x)$ and $B_\kappa(x) = e^{\kappa A} B(x)$ for all $x \in H_\iota$. For every $\kappa \in (0, \infty)$, $x \in H_\iota$ let $\bar{X}^{x,\kappa} \colon [0, T] \times \Omega \to H_\iota$ be a continuous mild solution of the SEE

$$d\hat{X}_t^{x,\kappa} = \big[A\hat{X}_t^{x,\kappa} + F_\kappa(\hat{X}_t^{x,\kappa})\big]\,dt + B_\kappa(\hat{X}_t^{x,\kappa})\,dW_t, \qquad t \in [0, T], \qquad X_0^{x,\kappa} = x. \quad (7)$$

For every $\kappa \in (0, \infty)$, let $u_\kappa \colon [0, T] \times H_\iota \to \mathbb{R}$ be a function given by $u_\kappa(t, x) = \mathbb{E}\big[\varphi(\hat{X}_{T-t}^{x,\kappa})\big]$ for all $(t, x) \in [0, T] \times H_\iota$. In particular, notice that for all $\kappa \in (0, \infty)$ and all nonrandom $x \in H_\iota$ it holds that $u_\kappa(T, x) = \varphi(x)$. Then, for every $\kappa \in (0, \infty)$, $N \in \mathbb{N}$ let $X^{N,\kappa} \colon [0, T] \times \Omega \to H_\iota$ be a continuous mild solution of the SEE

$$dX_t^{N,\kappa} = \big[P_N A X_t^{N,\kappa} + P_N F_\kappa(X_t^{N,\kappa})\big]\,dt + P_N B_\kappa(X_t^{N,\kappa})\,dW_t, \quad t \in [0, T], \quad X_0^N = P_N(\xi). \quad (8)$$

The *first key idea* in our proof is then to bring *certain modified versions* of the SEEs (3) and (8) respectively into play to analyze the weak approximation errors $\big|\mathbb{E}\big[\varphi(\bar{X}_T^{\xi,\kappa})\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big|$ for $N \in \mathbb{N}$, $\kappa \in (0, \infty)$. More specifically, for every $\kappa \in (0, \infty)$, $N \in \mathbb{N}$ let $Y^{N,\kappa} \colon [0, T] \times \Omega \to H_{\iota+1}$ be a continuous mild solution of the SEE

$$dY_t^{N,\kappa} = \big[AY_t^{N,\kappa} + F_\kappa\big(P_N(Y_t^{N,\kappa})\big)\big]\,dt + B_\kappa\big(P_N(Y_t^{N,\kappa})\big)\,dW_t, \quad t \in [0, T], \quad Y_0^{N,\kappa} = \xi. \quad (9)$$

It is crucial in (9) that $P_N(\cdot)$ appears inside the arguments of F_κ and B_κ instead of in front of F_κ and B_κ as in (8) (and (3)). Moreover, notice the projection $P_N(Y_t^{N,\kappa}) = X_t^{N,\kappa}$ $\mathbb{P}$-a.s. for all $N \in \mathbb{N}$, $\kappa \in (0, \infty)$, $t \in [0, T]$. To estimate the weak approximation errors $\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\xi,\kappa})\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big|$ for $N \in \mathbb{N}$, $\kappa \in (0, \infty)$ we then apply the triangle inequality to obtain that for all $\kappa \in (0,\infty)$, $N \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\xi,\kappa})\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big| \\
&\le \big|\mathbb{E}\big[\varphi(\hat{X}_T^{\xi,\kappa})\big] - \mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big]\big| + \big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big| \\
&= \big|u_\kappa(0,\xi) - \mathbb{E}\big[u_\kappa(T,Y_T^{N,\kappa})\big]\big| + \big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(P_N(Y_T^{N,\kappa}))\big]\big| \\
&= \big|\mathbb{E}\big[u_\kappa(T,Y_T^{N,\kappa}) - u_\kappa(0,Y_0^{N,\kappa})\big]\big| + \big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(P_N(Y_T^{N,\kappa}))\big]\big|.
\end{aligned} \tag{10}$$

Roughly speaking, the processes $Y^{N,\kappa}$, $N \in \mathbb{N}$, $\kappa \in (0,\infty)$, are chosen in such a way so that it is not so difficult anymore to estimate $\big|\mathbb{E}\big[u_\kappa(T,Y_T^{N,\kappa}) - u_\kappa(0,Y_0^{N,\kappa})\big]\big|$ and $\big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(P_N(Y_T^{N,\kappa}))\big]\big|$ on the right hand side of (10). More formally, to estimate the term $\big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(P_N(Y_T^{N,\kappa}))\big]\big|$ on the right hand side of (10) (see Section 2 and Lemma 3.1 in Section 3) we apply the mild Itô formula in Corollary 2 in Da Prato et al. [8] to $\mathbb{E}\big[\varphi(Y_t^{N,\kappa})\big]$, $t \in [0,T]$, and to $\mathbb{E}\big[\varphi(P_N(Y_T^{N,\kappa}))\big]$, $t \in [0,T]$, and then estimate the difference of the resulting terms in a straightforward way (see the proof of Proposition 2.1 in Section 2 below for details). This allows us to prove (see Proposition 2.1 below) that there exist real numbers $C_\varepsilon^{(1)} \in [0,\infty)$, $\varepsilon \in (0,\infty)$, such that for all $\varepsilon, \kappa \in (0,\infty)$, $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big| = \big|\mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big] - \mathbb{E}\big[\varphi(P_N(Y_T^{N,\kappa}))\big]\big| \le C_\varepsilon^{(1)} \, (\lambda_N)^{-(1-\gamma-\varepsilon)}. \tag{11}$$

To estimate the term $\big|\mathbb{E}\big[u_\kappa(T,Y_T^{N,\kappa}) - u_\kappa(0,Y_0^{N,\kappa})\big]\big|$ on the right hand side of (10) we apply the standard Itô formula to the stochastic processes $\big(u_\kappa(t,Y_t^{N,\kappa})\big)_{t\in[0,T]}$, $\kappa \in (0,\infty)$, and use the fact that the functions u_κ, $\kappa \in (0,\infty)$, solve the Kolmogorov backward equation associated to (7) to obtain that for all $\kappa \in (0,\infty)$, $N \in \mathbb{N}$ it holds that

$$\begin{aligned}
&\big|\mathbb{E}\big[u_\kappa(T,Y_T^{N,\kappa}) - u_\kappa(0,Y_0^{N,\kappa})\big]\big| \le \int_0^T \big|\mathbb{E}\big[\big(\tfrac{\partial}{\partial x}u_\kappa\big)(s,Y_s^{N,\kappa})\big(F_\kappa(P_N(Y_s^{N,\kappa})) - F_\kappa(Y_s^{N,\kappa})\big)\big]\big| \, ds \\
&+ \sum_{j\in\mathcal{J}} \int_0^T \frac{\Big|\mathbb{E}\Big[\Big(\frac{\partial^2}{\partial x^2}u_\kappa\Big)(s,Y_s^{N,\kappa})\Big(\Big[B_\kappa(P_N(Y_s^{N,\kappa})) + B_\kappa(Y_s^{N,\kappa})\Big]g_j, \Big[B_\kappa(P_N(Y_s^{N,\kappa})) - B_\kappa(Y_s^{N,\kappa})\Big]g_j\Big)\Big]\Big|}{2} \, ds
\end{aligned} \tag{12}$$

where $(g_j)_{j\in\mathcal{J}} \subseteq U$ is an arbitrary orthonormal basis of U; cf. (39) in Section 3 below. The next key idea in our weak convergence proof is then to again apply the mild Itô formula (see Da Prato et al. [8]) to the terms appearing on the right hand side of (12). After applying the mild Itô formula, the resulting terms can be estimated in a straightforward way by using the estimates for the functions u_κ, $\kappa \in (0,\infty)$, from Andersson et al. [1]. This allows us (cf. (87) in the proof of Proposition 5.1 in Section 5) to prove that for all $\varepsilon \in (0,\infty)$ there exists a real number $C_\varepsilon^{(2)} \in [0,\infty)$ such that for all $\kappa \in (0,1]$, $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\xi,\kappa})\big] - \mathbb{E}\big[\varphi(Y_T^{N,\kappa})\big]\big| = \big|\mathbb{E}\big[u_\kappa(T,Y_T^{N,\kappa}) - u_\kappa(0,Y_0^{N,\kappa})\big]\big| \le \frac{C_\varepsilon^{(2)}}{\kappa^\varepsilon \, (\lambda_N)^{(1-\gamma-\varepsilon)}}. \tag{13}$$

Putting (13) and (11) into (10) then proves that for all $\kappa \in (0,1]$, $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\xi,\kappa})\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big| \le C_\varepsilon^{(2)} \, \kappa^{-\varepsilon} \, (\lambda_N)^{-(1-\gamma-\varepsilon)} + C_\varepsilon^{(1)} \, (\lambda_N)^{-(1-\gamma-\varepsilon)}. \tag{14}$$

Estimates (13) and (14) illustrate that we cannot simply let the mollifying parameter κ tend to 0 because the right hand side of (14) diverges as κ tends to 0. The last key idea in our proof is then to make use of the following – *somehow nonstandard – mollification procedure* to overcome this problem. For this mollification procedure we first use well-known *strong convergence analysis* to prove (see Proposition 4.1 in Section 4) that for all

$\varepsilon \in (0,\infty)$ there exists a real number $C_\varepsilon^{(3)} \in [0,\infty)$ such that for all $\kappa \in (0,1]$, $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{\xi,\kappa})\big]\big| + \big|\mathbb{E}\big[\varphi(X_T^N)\big] - \mathbb{E}\big[\varphi(X_T^{N,\kappa})\big]\big| \leq C_\varepsilon^{(3)}\, \kappa^{\left(\frac{1-\gamma}{2}-\varepsilon\right)}. \tag{15}$$

Combining (15) with (14) then shows that for all $\varepsilon \in (0,\infty)$, $\kappa \in (0,1]$, $N \in \mathbb{N}$ it holds that

$$\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(X_T^N)\big]\big| \leq \frac{C_\varepsilon^{(1)}}{(\lambda_N)^{(1-\gamma-\varepsilon)}} + \frac{C_\varepsilon^{(2)}}{\kappa^\varepsilon\,(\lambda_N)^{(1-\gamma-\varepsilon)}} + C_\varepsilon^{(3)}\, \kappa^{\left(\frac{1-\gamma}{2}-\varepsilon\right)}. \tag{16}$$

As the left hand side of (16) is independent of $\kappa \in (0,1]$, we can minimize the right hand side of (16) over $\kappa \in (0,1]$ (instead of letting κ tend to 0) and this will allow us to complete the proof of Theorem 1.1; see (88) and (90) in the proof of Proposition 5.1 in Section 5 below for details.

1.2 Notation

Throughout this article the following notation is used. For a set S we denote by $\mathrm{Id}_S\colon S \to S$ the identity mapping on S, that is, it holds for all $x \in S$ that $\mathrm{Id}_S(x) = x$. Moreover, for a set S we denote by $\mathcal{P}(S)$ the power set of S. Furthermore, let $\mathcal{E}_r\colon [0,\infty) \to [0,\infty)$, $r \in (0,\infty)$, be functions given by $\mathcal{E}_r(x) = \big[\sum_{n=0}^\infty \frac{x^{2n}\,\Gamma(r)^n}{\Gamma(nr+1)}\big]^{1/2}$ for all $x \in [0,\infty)$, $r \in (0,\infty)$ (cf. Chapter 7 in [18] and Chapter 3 in [19]). Moreover, for normed $\mathbb{R}$-vector spaces $(E_1, \|\cdot\|_{E_1})$ and $(E_2, \|\cdot\|_{E_2})$ and a nonnegative integer $k \in \mathbb{N}_0$, let $|\cdot|_{\mathrm{Lip}^k(E_1,E_2)}, \|\cdot\|_{\mathrm{Lip}^k(E_1,E_2)}\colon C^k(E_1, E_2) \to [0,\infty]$ be mappings given by

$$|f|_{\mathrm{Lip}^k(E_1,E_2)} = \begin{cases} \sup_{\substack{x,y\in E_1,\\ x\neq y}} \frac{\|f(x)-f(y)\|_{E_2}}{\|x-y\|_{E_1}} & : k = 0 \\ \sup_{\substack{x,y\in E_1,\\ x\neq y}} \frac{\|f^{(k)}(x)-f^{(k)}(y)\|_{L^{(k)}(E_1,E_2)}}{\|x-y\|_{E_1}} & : k \in \mathbb{N} \end{cases} \tag{17}$$

and $\|f\|_{\mathrm{Lip}^k(E_1,E_2)} = \|f(0)\|_{E_2} + \sum_{l=0}^k |f|_{\mathrm{Lip}^l(E_1,E_2)}$ for all $f \in C^k(E_1,E_2)$ and let $\mathrm{Lip}^k(E_1, E_2)$ be a set given by $\mathrm{Lip}^k(E_1,E_2) = \{f \in C^k(E_1,E_2)\colon \|f\|_{\mathrm{Lip}^k(E_1,E_2)} < \infty\}$. In addition, for a natural number $k \in \mathbb{N}$ and normed $\mathbb{R}$-vector spaces $(E_1, \|\cdot\|_{E_1})$ and $(E_2, \|\cdot\|_{E_2})$, let $|\cdot|_{C_b^k(E_1,E_2)}, \|\cdot\|_{C_b^k(E_1,E_2)}\colon C^k(E_1,E_2) \to [0,\infty]$ be mappings given by $|f|_{C_b^k(E_1,E_2)} = \sup_{x\in E_1} \|f^{(k)}(x)\|_{L^k(E_1,E_2)}$ and $\|f\|_{C_b^k(E_1,E_2)} = \|f(0)\|_{E_2} + \sum_{l=1}^k |f|_{C_b^l(E_1,E_2)}$ for all $f \in C^k(E_1,E_2)$ and let $C_b^k(E_1,E_2)$ be a set given by $C_b^k(E_1,E_2) = \{f \in C^k(E_1,E_2)\colon \|f\|_{C_b^k(E_1,E_2)} < \infty\}$.

1.3 Setting

Throughout this article the following setting is frequently used. Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ and $(U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$ be two separable $\mathbb{R}$-Hilbert spaces, let $T \in (0,\infty)$, $\eta \in [0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in[0,T]})$ be a stochastic basis, let $(W_t)_{t\in[0,T]}$ be a cylindrical Id_U-Wiener process with respect to $(\mathcal{F}_t)_{t\in[0,T]}$, let $\mathbb{H} \subseteq H$ be an orthonormal basis, let $\lambda\colon \mathbb{H} \to \mathbb{R}$ be a function satisfying $\sup_{b\in\mathbb{H}} \lambda_b < \eta$, let $A\colon D(A) \subseteq H \to H$ be a linear operator such that $D(A) = \{v \in H\colon \sum_{b\in\mathbb{H}} |\lambda_b \langle b, v\rangle_H|^2 < \infty\}$ and such that for all $v \in D(A)$ it holds that $Av = \sum_{b\in\mathbb{H}} \lambda_b \langle b, v\rangle_H\, b$, let $(H_r, \langle\cdot,\cdot\rangle_{H_r}, \|\cdot\|_{H_r})$, $r \in \mathbb{R}$, be a family of interpolation spaces associated to $\eta - A$ (see, e.g., Theorem and Definition 2.5.32 in [19]) and let $(P_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq L(H_{-1})$ fulfill that for all $v \in H$, $I \in \mathcal{P}(\mathbb{H})$ it holds that $P_I(v) = \sum_{b\in I} \langle b, v\rangle_H\, b$.

Throughout this article we also frequently use the well-known facts that for all $r \in (0,\infty)$, $I \in \mathcal{P}(\mathbb{H})$ it holds that $\|P_I\|_{L(H,H_{-r})} = [\inf_{b\in I}\,[\eta - \lambda_b]]^{-r} = \|P_I\|_{L(H,H_{-1})}^r$ and that for all $r \in [0,1]$ it holds that $\sup_{t\in(0,\infty)} \big\|(-tA)^r e^{At}\big\|_{L(H)} \leq \sup_{x\in(0,\infty)} \big[\frac{x^r}{e^x}\big] \leq \big[\frac{r}{e}\big]^r \leq 1$.

2 Weak convergence for Galerkin projections of SPDEs

In this section we establish weak convergence rates for *Galerkin projections* of SPDEs (see Proposition 2.1). Proposition 2.1, in particular, proves inequality (11) in Section 1.1. In Corollary 3.3 in Section 3 below we will use Proposition 2.1 to establish weak convergence rates for *Galerkin approximations* of SPDEs with mollified nonlinearities. Proposition 2.1 is a slightly modified version of Corollary 8 in Da Prato et al. [8].

2.1 Setting

Assume the setting in Section 1.3, assume that $\eta = 0$, and let $\vartheta \in [0,1)$, $F \in \operatorname{Lip}^0(H, H_{-\vartheta})$, $B \in \operatorname{Lip}^0(H, HS(U, H_{-\vartheta/2}))$, $\varphi \in \operatorname{Lip}^2(H,\mathbb{R})$, $\xi \in L^3(\mathbb{P}|_{\mathcal{F}_0}; H)$.

The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) that there exist an up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic process $X\colon [0,T]\times\Omega \to H$ which satisfies $\sup_{t\in[0,T]} \|X_t\|_{L^3(\mathbb{P};H)} < \infty$ and which satisfies that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$X_t = e^{At}\xi + \int_0^t e^{A(t-s)} F(X_s)\, ds + \int_0^t e^{A(t-s)} B(X_s)\, dW_s. \tag{18}$$

2.2 A weak convergence result

Proposition 2.1. *Assume the setting in Section 2.1 and let $\rho \in [0, 1-\vartheta)$, $I \in \mathcal{P}(\mathbb{H})$. Then*

$$\begin{aligned} &\big|\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi\big(P_I(X_T)\big)\big]\big| \leq \|\varphi\|_{\operatorname{Lip}^2(H,\mathbb{R})} \max\big\{1, \sup\nolimits_{t\in[0,T]} \mathbb{E}\big[\|X_t\|_H^3\big]\big\} \\ &\cdot \left[\frac{1}{T^\rho} + \frac{T^{(1-\rho-\vartheta)}\big[\|F\|_{\operatorname{Lip}^0(H,H_{-\vartheta})} + \|B\|_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\big]}{(1-\rho-\vartheta)}\right] \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}. \end{aligned} \tag{19}$$

Proof. Throughout this proof let $\mathbb{U} \subseteq U$ be an orthonormal basis of U and let $B^b \in C(H, H_{-\vartheta/2})$, $b \in \mathbb{U}$, be given by $B^b(v) = B(v)\, b$ for all $v \in H$, $b \in \mathbb{U}$. Next observe that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $P_I(X_t) = e^{At}P_I(\xi) + \int_0^t e^{A(t-s)} P_I F(X_s)\, ds + \int_0^t e^{A(t-s)} P_I B(X_s)\, dW_s$. The mild Itô formula in Corollary 2 in Da Prato et al. [8] hence yields that

$$\begin{aligned} &\mathbb{E}\big[\varphi(X_T)\big] - \mathbb{E}\big[\varphi(P_I(X_T))\big] = \mathbb{E}\big[\varphi(e^{AT}\xi)\big] - \mathbb{E}\big[\varphi(e^{AT}P_I(\xi))\big] \\ &+ \int_0^T \mathbb{E}\big[\varphi'(e^{A(T-t)}X_t)\, e^{A(T-t)}F(X_t)\big] - \mathbb{E}\big[\varphi'(e^{A(T-t)}P_I(X_t))\, e^{A(T-t)}P_I F(X_t)\big]\, dt \\ &+ \frac{1}{2}\sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[\varphi''(e^{A(T-t)}X_t)(e^{A(T-t)}B^b(X_t), e^{A(T-t)}B^b(X_t))\big]\, dt \\ &- \frac{1}{2}\sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[\varphi''(e^{A(T-t)}P_I(X_t))(e^{A(T-t)}P_I B^b(X_t), e^{A(T-t)}P_I B^b(X_t))\big]\, dt. \end{aligned} \tag{20}$$

Next observe that the fact that for all $r \in [0,1]$ it holds that $\sup_{t\in(0,\infty)} \big\|(-tA)^r e^{At}\big\|_{L(H)} \leq \sup_{x\in(0,\infty)} \big[\frac{x^r}{e^x}\big] \leq \big[\frac{r}{e}\big]^r \leq 1$ implies that

$$\big|\mathbb{E}\big[\varphi(e^{AT}\xi)\big] - \mathbb{E}\big[\varphi(e^{AT}P_I(\xi))\big]\big| \leq \frac{|\varphi|_{\operatorname{Lip}^0(H,\mathbb{R})}\, \|\xi\|_{L^1(\mathbb{P};H)}\, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{T^\rho}. \tag{21}$$

Inequality (21) provides us a bound for the first difference on the right hand side of (20). In the next step we bound the second difference on the right hand side of (20). For this observe that for all $x \in H$, $t \in [0,T)$ it holds that

$$\begin{aligned} &\left| \left[\varphi'(e^{A(T-t)}x) - \varphi'(e^{A(T-t)}P_I(x)) \right] e^{A(T-t)} F(x) \right| \\ &\le \frac{|\varphi|_{\mathrm{Lip}^1(H,\mathbb{R})} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, \|x\|_H \, \|F(x)\|_{H_{-\vartheta}}}{(T-t)^{(\rho+\vartheta)}} \end{aligned} \tag{22}$$

and that

$$\left| \varphi'(e^{A(T-t)}P_I(x)) \big([\mathrm{Id}_H - P_I] \, e^{A(T-t)} F(x) \big) \right| \le \frac{|\varphi|_{\mathrm{Lip}^0(H,\mathbb{R})} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, \|F(x)\|_{H_{-\vartheta}}}{(T-t)^{(\rho+\vartheta)}}. \tag{23}$$

Combining (22) and (23) proves that

$$\begin{aligned} &\left| \int_0^T \mathbb{E}\big[\varphi'(e^{A(T-t)}X_t) \, e^{A(T-t)} F(X_t) \big] \, dt - \int_0^T \mathbb{E}\big[\varphi'(e^{A(T-t)}P_I(X_t)) \, e^{A(T-t)} P_I F(X_t) \big] \, dt \right| \\ &\le \frac{T^{(1-\rho-\vartheta)} \, \sup_{t\in[0,T]} \mathbb{E}\Big[\|X_t\|_H \, \|F(X_t)\|_{H_{-\vartheta}} |\varphi|_{\mathrm{Lip}^1(H,\mathbb{R})} + \|F(X_t)\|_{H_{-\vartheta}} |\varphi|_{\mathrm{Lip}^0(H,\mathbb{R})} \Big] \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(1-\rho-\vartheta)} \\ &\le \frac{T^{(1-\rho-\vartheta)} \, \|\varphi\|_{\mathrm{Lip}^1(H,\mathbb{R})} \, \sup_{t\in[0,T]} \max\Big\{ \mathbb{E}\big[\|X_t\|_H \, \|F(X_t)\|_{H_{-\vartheta}} \big], \mathbb{E}\big[\|F(X_t)\|_{H_{-\vartheta}} \big] \Big\} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(1-\rho-\vartheta)} \\ &\le \frac{T^{(1-\rho-\vartheta)} \, \|\varphi\|_{\mathrm{Lip}^1(H,\mathbb{R})} \, \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \, \max\{1, \sup_{t\in[0,T]} \mathbb{E}[\|X_t\|_H^2]\} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(1-\rho-\vartheta)}. \end{aligned} \tag{24}$$

Inequality (24) provides us a bound for the second difference on the right hand side of (20). Next we bound the third difference on the right hand side of (20). To this end note that for all $x \in H$, $t \in [0,T)$ it holds that

$$\begin{aligned} &\left| \sum_{b\in\mathbb{U}} \left[\varphi''(e^{A(T-t)}x) - \varphi''(e^{A(T-t)}P_I(x)) \right] (e^{A(T-t)}B^b(x), e^{A(T-t)}B^b(x)) \right| \\ &\le \frac{|\varphi|_{\mathrm{Lip}^2(H,\mathbb{R})} \, \|B(x)\|^2_{HS(U,H_{-\vartheta/2})} \, \|x\|_H \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(T-t)^{(\rho+\vartheta)}} \end{aligned} \tag{25}$$

and that

$$\begin{aligned} &\left| \sum_{b\in\mathbb{U}} \varphi''(e^{A(T-t)}P_I(x)) ([\mathrm{Id}_H + P_I] e^{A(T-t)} B^b(x), [\mathrm{Id}_H - P_I] e^{A(T-t)} B^b(x)) \right| \\ &\le \frac{2 \, |\varphi|_{\mathrm{Lip}^1(H,\mathbb{R})} \, \|B(x)\|^2_{HS(U,H_{-\vartheta/2})} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(T-t)^{(\rho+\vartheta)}}. \end{aligned} \tag{26}$$

Combining (25) and (26) proves that

$$\begin{aligned} &\left| \frac{1}{2} \sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[\varphi''(e^{A(T-t)}X_t)(e^{A(T-t)}B^b(X_t), e^{A(T-t)}B^b(X_t)) \big] \, dt \right. \\ &\left. - \frac{1}{2} \sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[\varphi''(e^{A(T-t)}P_I(X_t))(e^{A(T-t)}P_I B^b(X_t), e^{A(T-t)}P_I B^b(X_t)) \big] \, dt \right| \\ &\le \frac{T^{(1-\rho-\vartheta)} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, \|\varphi\|_{\mathrm{Lip}^2(H,\mathbb{R})} \, \sup_{t\in[0,T]} \max\Big\{ \mathbb{E}\Big[\|X_t\|_H \|B(X_t)\|^2_{HS(U,H_{-\vartheta/2})} \Big], \mathbb{E}\Big[\|B(X_t)\|^2_{HS(U,H_{-\vartheta/2})} \Big] \Big\}}{(1-\rho-\vartheta)} \\ &\le \frac{T^{(1-\rho-\vartheta)} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, \|\varphi\|_{\mathrm{Lip}^2(H,\mathbb{R})} \, \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \, \max\{1, \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|_H^3 \big] \}}{(1-\rho-\vartheta)}. \end{aligned} \tag{27}$$

Combining (20), (21), (24), and (27) finally proves that

$$\begin{aligned}&\big|\mathbb{E}\big[\varphi(X_T)\big]-\mathbb{E}\big[\varphi(P_I(X_T))\big]\big|\leq\|\varphi\|_{\mathrm{Lip}^2(H,\mathbb{R})}\,\max\Big\{1,\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t\|_H^3\big]\Big\}\\&\cdot\left[\frac{1}{T^\rho}+\frac{T^{(1-\rho-\vartheta)}\big[\|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})}+\|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\big]}{(1-\rho-\vartheta)}\right]\|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}.\end{aligned}\tag{28}$$

This finishes the proof of Proposition 2.1. □

3 Weak convergence for Galerkin approximations of SPDEs with mollified nonlinearities

In this section we establish weak convergence rates for *Galerkin approximations of SPDEs with mollified nonlinearities*; see Corollary 3.3 and Corollary 3.4 below. Corollary 3.4, in particular, enables us to prove inequality (13) in the introduction. In Section 5 below we will use Corollary 3.4 to establish weak convergence rates for *Galerkin approximations of SPDEs with "non-mollified" nonlinearities.*

3.1 Setting

Assume the setting in Section 1.3, assume that $\eta = 0$, let $\mathbb{U} \subseteq U$ be an orthonormal basis of U, let $\vartheta \in [0, 1/2)$, $F \in C_b^4(H, H_1)$, $B \in C_b^4(H, HS(U, H_1))$, $\varphi \in C_b^4(H, \mathbb{R})$, $\xi \in L^4(\mathbb{P}|_{\mathcal{F}_0}; H_1)$, let $\varsigma_{F,B} \in \mathbb{R}$ be given by $\varsigma_{F,B} = \max\big\{1, \|F\|^2_{C_b^3(H,H_{-\vartheta})}, \|B\|^4_{C_b^3(H,HS(U,H_{-\vartheta/2}))}\big\}$, let $(F_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq C(H,H)$, $(B_I)_{I\in\mathcal{P}(\mathbb{H})} \subseteq C(H, HS(U,H))$, $(B^b)_{b\in\mathbb{U}} \subseteq C(H,H)$ and $(B_I^b)_{I\in\mathcal{P}(\mathbb{H}),b\in\mathbb{U}} \subseteq C(H,H)$ be given by

$$F_I(v) = F\big(P_I(v)\big), \quad B_I(v)\,u = B\big(P_I(v)\big)\,u, \quad B^b(v) = B(v)\,b, \quad B_I^b(v) = B\big(P_I(v)\big)\,b \tag{29}$$

for all $v \in H$, $u \in U$, $I \in \mathcal{P}(\mathbb{H})$, $b \in \mathbb{U}$, and let $(g_r)_{r\in[0,\infty)} \subseteq C(H,\mathbb{R})$ be given by $g_r(x) = \max\{1, \|x\|_H^r\}$ for all $r \in [0,\infty)$, $x \in H$.

The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^I\colon [0,T]\times\Omega \to P_I(H)$, $I \in \mathcal{P}(\mathbb{H})$, $Y^I\colon [0,T]\times\Omega \to H_1$, $I \in \mathcal{P}(\mathbb{H})$, and $X^{\mathbb{H},x}\colon [0,T]\times\Omega \to H$, $x \in H$, which satisfy that for all $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ it holds that $\sup_{t\in[0,T]}\big[\|X_t^I\|_{L^4(\mathbb{P};H)} + \|Y_t^I\|_{L^4(\mathbb{P};H_1)} + \|X_t^{\mathbb{H},x}\|_{L^4(\mathbb{P};H)}\big] < \infty$ and which satisfy that for all $t \in [0,T]$, $I \in \mathcal{P}(\mathbb{H})$, $x \in H$ it holds $\mathbb{P}$-a.s. that

$$X_t^I = e^{At}P_I(\xi) + \int_0^t e^{A(t-s)}P_I F(X_s^I)\,ds + \int_0^t e^{A(t-s)}P_I B(X_s^I)\,dW_s, \tag{30}$$

$$Y_t^I = e^{At}\xi + \int_0^t e^{A(t-s)}F_I(Y_s^I)\,ds + \int_0^t e^{A(t-s)}B_I(Y_s^I)\,dW_s, \tag{31}$$

$$X_t^{\mathbb{H},x} = e^{At}x + \int_0^t e^{A(t-s)}F(X_s^{\mathbb{H},x})\,ds + \int_0^t e^{A(t-s)}B(X_s^{\mathbb{H},x})\,dW_s. \tag{32}$$

Moreover, let $u\colon [0,T]\times H \to \mathbb{R}$ be a function given by $u(t,x) = \mathbb{E}\big[\varphi(X_{T-t}^{\mathbb{H},x})\big]$ for all $x \in H$, $t \in [0,T]$, let $c_{\delta_1,\ldots,\delta_k} \in [0,\infty]$, $\delta_1,\ldots,\delta_k \in \mathbb{R}$, $k \in \{1,2,3,4\}$, be extended real numbers given by

$$c_{\delta_1,\delta_2,\ldots,\delta_k} = \sup_{t\in(0,T]}\sup_{s\in[0,t]}\sup_{x\in H}\sup_{v_1,\ldots,v_k\in H\setminus\{0\}}\left[\frac{\big|(\frac{\partial^k}{\partial x^k}u)(t,e^{A(t-s)}x)(v_1,\ldots,v_k)\big|}{t^{(\delta_1+\ldots+\delta_k)}\,\|v_1\|_{H_{\delta_1}}\cdot\ldots\cdot\|v_k\|_{H_{\delta_k}}}\right] \tag{33}$$

for all $\delta_1, \ldots, \delta_k \in \mathbb{R}$, $k \in \{1,2,3,4\}$, and let $(K_r^I)_{r\in(0,\infty), I\in\mathcal{P}(\mathbb{H})} \subseteq [0,\infty)$ be given by $K_r^I = \sup_{t\in[0,T]} \mathbb{E}\big[g_r(Y_t^I)\big]$ for all $r \in (0,\infty)$, $I \in \mathcal{P}(\mathbb{H})$.

3.2 Weak convergence results

Lemma 3.1. *Assume the setting in Section 3.1 and let $\rho \in [0, 1-\vartheta)$, $I \in \mathcal{P}(\mathbb{H})$. Then*

$$\big|\mathbb{E}\big[\varphi(Y_T^I)\big] - \mathbb{E}\big[\varphi(X_T^I)\big]\big| \leq \left[\frac{1}{T^\rho} + \frac{2\,T^{(1-\rho-\vartheta)}}{(1-\rho-\vartheta)}\right] \|\varphi\|_{C_b^3(H,\mathbb{R})}\, \varsigma_{F,B}\, K_3^I\, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}. \quad (34)$$

Proof. First of all, note that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that

$$P_I(Y_t^I) = e^{At} P_I(\xi) + \int_0^t e^{A(t-s)} P_I F\big(P_I(Y_s^I)\big)\, ds + \int_0^t e^{A(t-s)} P_I B\big(P_I(Y_s^I)\big)\, dW_s. \quad (35)$$

The fact that mild solutions of (30) are within a suitable class of solutions unique up to modifications (see, e.g., Theorem 7.4 (i) in Da Prato & Zabczyk [9] for details) hence ensures that for all $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $P_I(Y_t^I) = X_t^I$. An application of Proposition 2.1 hence proves that

$$\begin{aligned}
&\big|\mathbb{E}\big[\varphi(Y_T^I)\big] - \mathbb{E}\big[\varphi\big(X_T^I\big)\big]\big| \leq \|\varphi\|_{C_b^3(H,\mathbb{R})}\, \max\big\{1, \sup\nolimits_{t\in[0,T]} \mathbb{E}\big[\|Y_t^I\|_H^3\big]\big\} \\
&\cdot \left[\frac{1}{T^\rho} + \frac{T^{(1-\rho-\vartheta)}\big[\|F\|_{C_b^1(H,H_{-\vartheta})} + \|B\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))}\big]}{(1-\rho-\vartheta)}\right] \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}.
\end{aligned} \quad (36)$$

This completes the proof of Lemma 3.1. ☐

Lemma 3.2. *Assume the setting in Section 3.1 and let $\rho \in [0, 1-\vartheta)$, $I \in \mathcal{P}(\mathbb{H})$. Then*

$$\begin{aligned}
\big|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(Y_T^I)\big]\big| &\leq \frac{T^{1-\vartheta-\rho}\, \varsigma_{F,B}\, K_4^I}{(1-\vartheta-\rho)} \left[1 + \frac{9\,T^{1-\vartheta}}{2\,(2-2\vartheta-\rho)}\right] \big\|P_{\mathbb{H}\setminus I}\big\|_{L(H,H_{-\rho})} \\
&\cdot \big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\big].
\end{aligned} \quad (37)$$

Proof. Throughout this proof let $u_t \colon [0,T]\times H \to \mathbb{R}$, $u_x \colon [0,T]\times H \to L(H,\mathbb{R})$, $u_{xx} \colon [0,T]\times H \to L^{(2)}(H,\mathbb{R})$, $u_{xxx} \colon [0,T]\times H \to L^{(3)}(H,\mathbb{R})$, $u_{xxxx} \colon [0,T]\times H \to L^{(4)}(H,\mathbb{R})$ be functions defined through $u_t(s,y) := \frac{\partial}{\partial s} u(s,y)$, $u_x(s,y) := \frac{\partial}{\partial y} u(s,y)$, $u_{xx}(s,y)(v_1,v_2) := \big(\frac{\partial^2}{\partial y^2} u(s,y)\big)(v_1,v_2)$, $u_{xxx}(s,y)(v_1,v_2,v_3) := \big(\frac{\partial^3}{\partial y^3} u(s,y)\big)(v_1,v_2,v_3)$, $u_{xxxx}(s,y)(v_1,\ldots,v_4) := \big(\frac{\partial^4}{\partial y^4} u(s,y)\big)(v_1,v_2,v_3,v_4)$ for all $s \in [0,T]$, $y, v_1, v_2, v_3, v_4 \in H$. Then observe that Itô's formula proves that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(Y_T^I)\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] = \mathbb{E}\big[u(T,Y_T^I) - u(0,Y_0^I)\big] \\
&= \int_0^T \mathbb{E}\big[u_t(t,Y_t^I) + u_x(t,Y_t^I)\big(AY_t^I + F_I(Y_t^I)\big)\big]\, dt \\
&+ \frac{1}{2} \sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[u_{xx}(t,Y_t^I)\big(B_I^b(Y_t^I), B_I^b(Y_t^I)\big)\big]\, dt.
\end{aligned} \quad (38)$$

Exploiting the fact that u is a solution of the Kolmogorov backward equation associated to $X^{\mathbb{H},x} \colon [0,T]\times\Omega \to H$, $x \in H$, hence shows that

$$\begin{aligned}
&\mathbb{E}\big[\varphi(Y_T^I)\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] \\
&= \int_0^T \mathbb{E}\big[u_x(t,Y_t^I)\big(F_I\big(Y_t^I\big) - F\big(Y_t^I\big)\big)\big]\, dt \\
&+ \frac{1}{2} \sum_{b\in\mathbb{U}} \int_0^T \mathbb{E}\big[u_{xx}(t,Y_t^I)\big(B_I^b\big(Y_t^I\big) + B^b\big(Y_t^I\big), B_I^b\big(Y_t^I\big) - B^b\big(Y_t^I\big)\big)\big]\, dt.
\end{aligned} \quad (39)$$

Below we will apply the mild Itô formula in Corollary 2 in Da Prato et al. [8] to the first summand on the right hand side of (39). To this end we define functions $\tilde{F}_{t,s} \colon H \to \mathbb{R}$, $t \in (s,T]$, $s \in [0,T)$, by

$$
\begin{aligned}
&\tilde{F}_{t,s}(x) \\
&:= u_{xx}(t, e^{A(t-s)}x)\big(F_I(e^{A(t-s)}x) - F(e^{A(t-s)}x), e^{A(t-s)}F_I(x)\big) \\
&+ u_x(t, e^{A(t-s)}x)\big(\big[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\big]\, e^{A(t-s)}F_I(x)\big) \\
&+ \tfrac{1}{2}\sum_{b\in\mathbb{U}} u_{xxx}(t, e^{A(t-s)}x)\big(F_I(e^{A(t-s)}x) - F(e^{A(t-s)}x), e^{A(t-s)}B_I^b(x), e^{A(t-s)}B_I^b(x)\big) \\
&+ \sum_{b\in\mathbb{U}} u_{xx}(t, e^{A(t-s)}x)\big(\big[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\big]\, e^{A(t-s)}B_I^b(x), e^{A(t-s)}B_I^b(x)\big) \\
&+ \tfrac{1}{2}\sum_{b\in\mathbb{U}} u_x(t, e^{A(t-s)}x)\big(\big[F_I''(e^{A(t-s)}x) - F''(e^{A(t-s)}x)\big]\big(e^{A(t-s)}B_I^b(x), e^{A(t-s)}B_I^b(x)\big)\big)
\end{aligned}
\tag{40}
$$

for all $x \in H$, $t, s \in [0,T]$ with $t > s$. An application of the mild Itô formula in Corollary 2 in Da Prato et al. [8] then proves that for all $t \in [0,T]$ it holds that

$$
\begin{aligned}
&\mathbb{E}\big[u_x(t, Y_t^I)\big(F_I\big(Y_t^I\big) - F\big(Y_t^I\big)\big)\big] \\
&= \mathbb{E}\big[u_x(t, e^{At}\xi)\big(F_I\big(e^{At}\xi\big) - F\big(e^{At}\xi\big)\big)\big] + \int_0^t \mathbb{E}\Big[\tilde{F}_{t,s}(Y_s^I)\Big]\, ds.
\end{aligned}
\tag{41}
$$

Below we will also apply the mild Itô formula in Corollary 2 in Da Prato et al. [8] to the second summand on the right hand side of (39). For this we define functions $\tilde{B}_{t,s} \colon H \to \mathbb{R}$, $t \in (s,T]$, $s \in [0,T)$, by

$$
\begin{aligned}
&\tilde{B}_{t,s}(x) \\
&:= \textstyle\sum_{b\in\mathbb{U}} u_{xx}(t, e^{A(t-s)}x)\Big(\big[(B_I^b)'(e^{A(t-s)}x) + (B^b)'(e^{A(t-s)}x)\big]\, e^{A(t-s)}F_I(x), \\
&\qquad\qquad B_I^b(e^{A(t-s)}x) - B^b(e^{A(t-s)}x)\Big) \\
&+ \sum_{b\in\mathbb{U}} u_{xx}(t, e^{A(t-s)}x)\Big(B_I^b(e^{A(t-s)}x) + B^b(e^{A(t-s)}x), \\
&\qquad\qquad \big[(B_I^b)'(e^{A(t-s)}x) - (B^b)'(e^{A(t-s)}x)\big]\, e^{A(t-s)}F_I(x)\Big) \\
&+ \sum_{b\in\mathbb{U}} u_{xxx}(t, e^{A(t-s)}x)\Big(B_I^b(e^{A(t-s)}x) + B^b(e^{A(t-s)}x), \\
&\qquad\qquad B_I^b(e^{A(t-s)}x) - B^b(e^{A(t-s)}x), e^{A(t-s)}F_I(x)\Big) \\
&+ \sum_{b_1,b_2\in\mathbb{U}} u_{xx}(t, e^{A(t-s)}x)\Big(\big[(B_I^{b_2})'(e^{A(t-s)}x) + (B^{b_2})'(e^{A(t-s)}x)\big]\, e^{A(t-s)}B_I^{b_1}(x), \\
&\qquad\qquad \big[(B_I^{b_2})'(e^{A(t-s)}x) - (B^{b_2})'(e^{A(t-s)}x)\big]\, e^{A(t-s)}B_I^{b_1}(x)\Big) \\
&+ \frac{1}{2}\sum_{b_1,b_2\in\mathbb{U}} u_{xx}(t, e^{A(t-s)}x)\Big(B_I^{b_2}(e^{A(t-s)}x) - B^{b_2}(e^{A(t-s)}x), \\
&\qquad\qquad \big[(B_I^{b_2})''(e^{A(t-s)}x) + (B^{b_2})''(e^{A(t-s)}x)\big]\big(e^{A(t-s)}B_I^{b_1}(x), e^{A(t-s)}B_I^{b_1}(x)\big)\Big) \\
&+ \frac{1}{2}\sum_{b_1,b_2\in\mathbb{U}} u_{xx}(t, e^{A(t-s)}x)\Big(B_I^{b_2}(e^{A(t-s)}x) + B^{b_2}(e^{A(t-s)}x), \\
&\qquad\qquad \big[(B_I^{b_2})''(e^{A(t-s)}x) - (B^{b_2})''(e^{A(t-s)}x)\big]\big(e^{A(t-s)}B_I^{b_1}(x), e^{A(t-s)}B_I^{b_1}(x)\big)\Big) \\
&+ \sum_{b_1,b_2\in\mathbb{U}} u_{xxx}(t, e^{A(t-s)}x)\Big(\big[(B_I^{b_2})'(e^{A(t-s)}x) + (B^{b_2})'(e^{A(t-s)}x)\big]\, e^{A(t-s)}B_I^{b_1}(x), \\
&\qquad\qquad B_I^{b_2}(e^{A(t-s)}x) - B^{b_2}(e^{A(t-s)}x), e^{A(t-s)}B_I^{b_1}(x)\Big)
\end{aligned}
\tag{42}
$$

$$+\sum_{b_1,b_2\in\mathbb{U}} u_{xxx}(t,e^{A(t-s)}x)\Big(B_I^{b_2}(e^{A(t-s)}x)+B^{b_2}(e^{A(t-s)}x),$$
$$\big[(B_I^{b_2})'(e^{A(t-s)}x)-(B^{b_2})'(e^{A(t-s)}x)\big]\,e^{A(t-s)}B_I^{b_1}(x),e^{A(t-s)}B_I^{b_1}(x)\Big)$$
$$+\frac{1}{2}\sum_{b_1,b_2\in\mathbb{U}} u_{xxxx}(t,e^{A(t-s)}x)\Big(B_I^{b_2}(e^{A(t-s)}x)+B^{b_2}(e^{A(t-s)}x),B_I^{b_2}(e^{A(t-s)}x)-B^{b_2}(e^{A(t-s)}x),$$
$$e^{A(t-s)}B_I^{b_1}(x),e^{A(t-s)}B_I^{b_1}(x)\Big)$$

for all $x\in H$, $t,s\in[0,T]$ with $t>s$. An application of the mild Itô formula in Corollary 2 in Da Prato et al. [8] then proves that for all $t\in[0,T]$ it holds that

$$\begin{aligned}&\sum_{b\in\mathbb{U}}\mathbb{E}\big[u_{xx}(t,Y_t^I)\big(B_I^b(Y_t^I)+B^b\big(Y_t^I\big),B_I^b\big(Y_t^I\big)-B^b\big(Y_t^I\big)\big)\big]\\&=\sum_{b\in\mathbb{U}}\mathbb{E}\big[u_{xx}(t,e^{At}\xi)\big(B_I^b\big(e^{At}\xi\big)+B^b\big(e^{At}\xi\big),B_I^b\big(e^{At}\xi\big)-B^b\big(e^{At}\xi\big)\big)\big]+\int_0^t\mathbb{E}\Big[\tilde{B}_{t,s}(Y_s^I)\Big]\,ds.\end{aligned}\tag{43}$$

Putting (41) and (43) into (39) proves that

$$\begin{aligned}&\mathbb{E}\big[\varphi(Y_T^I)\big]-\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]\\&=\int_0^T\mathbb{E}\big[u_x(t,e^{At}\xi)\big(F_I\big(e^{At}\xi\big)-F\big(e^{At}\xi\big)\big)\big]\,dt\\&+\frac{1}{2}\sum_{b\in\mathbb{U}}\int_0^T\mathbb{E}\big[u_{xx}(t,e^{At}\xi)\big(B_I^b\big(e^{At}\xi\big)+B^b\big(e^{At}\xi\big),B_I^b\big(e^{At}\xi\big)-B^b\big(e^{At}\xi\big)\big)\big]\,dt\\&+\int_0^T\int_0^t\mathbb{E}\Big[\tilde{F}_{t,s}(Y_s^I)\Big]+\frac{1}{2}\,\mathbb{E}\Big[\tilde{B}_{t,s}(Y_s^I)\Big]\,ds\,dt.\end{aligned}\tag{44}$$

In the following we estimate the absolute values of the summands on the right hand side of (44). To this end observe that the fact that for all $r\in[0,1]$ it holds that $\sup_{t\in(0,\infty)}\|(-tA)^r e^{At}\|_{L(H)}\leq\sup_{x\in(0,\infty)}\big[\frac{x^r}{e^x}\big]\leq\big[\frac{r}{e}\big]^r\leq 1$ ensures that for all $t\in(0,T]$ it holds that

$$\begin{aligned}&\big|u_x(t,e^{At}\xi)\big(F_I(e^{At}\xi)-F(e^{At}\xi)\big)\big|\leq\frac{c_{-\vartheta}}{t^\vartheta}\,\big\|F_I(e^{At}\xi)-F(e^{At}\xi)\big\|_{H_{-\vartheta}}\\&\leq\frac{c_{-\vartheta}\,|F|_{C_b^1(H,H_{-\vartheta})}\,\|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}\,\|\xi\|_H}{t^{(\rho+\vartheta)}}.\end{aligned}\tag{45}$$

This and the fact that $\mathbb{E}[\|\xi\|_H]\leq K_1^I$ imply that

$$\begin{aligned}&\left|\int_0^T\mathbb{E}\big[u_x(t,e^{At}\xi)\big(F_I\big(e^{At}\xi\big)-F\big(e^{At}\xi\big)\big)\big]\,dt\right|\\&\leq\frac{K_1^I\,c_{-\vartheta}\,T^{(1-\vartheta-\rho)}\,|F|_{C_b^1(H,H_{-\vartheta})}\,\|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(1-\vartheta-\rho)}.\end{aligned}\tag{46}$$

Inequality (46) provides us an estimate for the absolute value of the first summand on the right hand side of (44). In the next step we bound the absolute value of the second summand on the right hand side (44). For this we observe that the fact that $g_1(\xi)\|\xi\|_H\leq g_2(\xi)$ ensures that for all $t\in(0,T]$ it holds that

$$\begin{aligned}&\sum_{b\in\mathbb{U}}\big|u_{xx}(t,e^{At}\xi)\big(B_I^b\big(e^{At}\xi\big)+B^b\big(e^{At}\xi\big),B_I^b\big(e^{At}\xi\big)-B^b\big(e^{At}\xi\big)\big)\big|\\&\leq\frac{c_{-\vartheta/2,-\vartheta/2}}{t^\vartheta}\sum_{b\in\mathbb{U}}\big\|B_I^b\big(e^{At}\xi\big)+B^b\big(e^{At}\xi\big)\big\|_{H_{-\vartheta/2}}\big\|B_I^b\big(e^{At}\xi\big)-B^b\big(e^{At}\xi\big)\big\|_{H_{-\vartheta/2}}\\&\leq\frac{2\,c_{-\vartheta/2,-\vartheta/2}\,g_2(\xi)}{t^{(\rho+\vartheta)}}\,\|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))}\,\|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}.\end{aligned}\tag{47}$$

This and the fact that $\mathbb{E}[g_2(\xi)] \leq K_2^I$ imply that

$$\begin{aligned}
&\left| \sum_{b \in \mathbb{U}} \int_0^T \mathbb{E}\big[u_{xx}(t, e^{At}\xi)\big(B_I^b\big(e^{At}\xi\big) + B^b\big(e^{At}\xi\big), B_I^b\big(e^{At}\xi\big) - B^b\big(e^{At}\xi\big)\big)\big]\, dt \right| \\
&\leq \frac{2\, K_2^I\, c_{-\vartheta/2,-\vartheta/2}\, T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\| P_{\mathbb{H}\setminus I} \right\|_{L(H,H_{-\rho})}.
\end{aligned} \tag{48}$$

Inequality (48) provides us an estimate for the second term on the right hand side of (44). In the next step we bound the absolute value of the term $\int_0^T \int_0^t \mathbb{E}\big[\tilde{F}_{t,s}(Y_s^I)\big]\, ds\, dt$ on the right hand side of (44). For this we note that (40) shows that for all $s, t \in [0,T]$, $x \in H$ with $t > s$ it holds that

$$\begin{aligned}
&\big|\tilde{F}_{t,s}(x)\big| \\
&\leq \frac{c_{-\vartheta,0}}{t^\vartheta} \left\| F_I(e^{A(t-s)}x) - F(e^{A(t-s)}x) \right\|_{H_{-\vartheta}} \left\| e^{A(t-s)} F_I(x) \right\|_H \\
&+ \frac{c_{-\vartheta}}{t^\vartheta} \left\| \left[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\right] e^{A(t-s)} F_I(x) \right\|_{H_{-\vartheta}} \\
&+ \frac{c_{-\vartheta,0,0}}{2\, t^\vartheta} \left\| F_I(e^{A(t-s)}x) - F(e^{A(t-s)}x) \right\|_{H_{-\vartheta}} \left\| e^{A(t-s)} B_I(x) \right\|^2_{HS(U,H)} \\
&+ \frac{c_{-\vartheta,0}}{t^\vartheta} \sum_{b \in \mathbb{U}} \left\| \left[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\right] e^{A(t-s)} B_I^b(x) \right\|_{H_{-\vartheta}} \left\| e^{A(t-s)} B_I^b(x) \right\|_H \\
&+ \frac{c_{-\vartheta}}{2\, t^\vartheta} \sum_{b \in \mathbb{U}} \left\| \left[F_I''(e^{A(t-s)}x) - F''(e^{A(t-s)}x)\right] \big(e^{A(t-s)} B_I^b(x), e^{A(t-s)} B_I^b(x)\big) \right\|_{H_{-\vartheta}}.
\end{aligned} \tag{49}$$

Next observe that for all $x, v \in H$, $r \in [0, \vartheta]$, $s, t \in [0,T]$ with $s < t$ it holds that

$$\begin{aligned}
&\left\| \left[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\right] e^{A(t-s)}\, v \right\|_{H_{-\vartheta}} \\
&\leq \left\| \left[F'(e^{A(t-s)} P_I(x)) P_I - F'(e^{A(t-s)}x)\right] e^{A(t-s)}\, v \right\|_{H_{-\vartheta}} \\
&\leq \left\| \left[F'(e^{A(t-s)} P_I(x)) - F'(e^{A(t-s)}x)\right] P_I\, e^{A(t-s)}\, v \right\|_{H_{-\vartheta}} \\
&\quad + \left\| F'(e^{A(t-s)}x)\, P_{\mathbb{H}\setminus I}\, e^{A(t-s)}\, v \right\|_{H_{-\vartheta}} \\
&\leq |F|_{C_b^2(H,H_{-\vartheta})} \left\| e^{A(t-s)} P_{\mathbb{H}\setminus I} x \right\|_H \left\| P_I e^{A(t-s)} v \right\|_H + \|F\|_{C_b^1(H,H_{-\vartheta})} \left\| P_{\mathbb{H}\setminus I} e^{A(t-s)} v \right\|_H \\
&\leq \Big[g_1(x)\, |F|_{C_b^2(H,H_{-\vartheta})} + \|F\|_{C_b^1(H,H_{-\vartheta})}\Big] \frac{\|v\|_{H_{-r}} \left\| P_{\mathbb{H}\setminus I} \right\|_{L(H,H_{-\rho})}}{(t-s)^{\rho+r}} \\
&\leq \frac{g_1(x)\, \|F\|_{C_b^2(H,H_{-\vartheta})}\, \|v\|_{H_{-r}} \left\| P_{\mathbb{H}\setminus I} \right\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+r)}}.
\end{aligned} \tag{50}$$

This and the fact that for all $x \in H$ it holds that $g_1(x)\|x\|_H \leq g_2(x)$ imply that for all $x \in H$, $s, t \in [0,T]$ with $s < t$ it holds that

$$\left\| \left[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\right] e^{A(t-s)} F_I(x) \right\|_{H_{-\vartheta}} \leq \frac{g_2(x)\, \|F\|^2_{C_b^2(H,H_{-\vartheta})} \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(t-s)^{\rho+\vartheta}} \tag{51}$$

and

$$\begin{aligned}
&\left[\sum_{b \in \mathbb{U}} \left\| \left[F_I'(e^{A(t-s)}x) - F'(e^{A(t-s)}x)\right] e^{A(t-s)} B_I^b(x) \right\|^2_{H_{-\vartheta}} \right]^{\frac{1}{2}} \\
&\leq \frac{g_2(x)\, \|F\|_{C_b^2(H,H_{-\vartheta})}\, \|B\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))}\, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(t-s)^{\rho+\frac{\vartheta}{2}}}.
\end{aligned} \tag{52}$$

Moreover, note that for all $x \in H$, $s, t \in [0,T]$ with $s < t$ it holds that

$$
\begin{aligned}
&\sum_{b \in \mathbb{U}} \left\| \left[F_I''(e^{A(t-s)}x) - F''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^b(x), e^{A(t-s)} B_I^b(x) \right) \right\|_{H_{-\vartheta}} \\
&\leq \sum_{b \in \mathbb{U}} \left\| F''(e^{A(t-s)} P_I(x)) \left([\mathrm{Id}_H + P_I] \, e^{A(t-s)} B_I^b(x), [\mathrm{Id}_H - P_I] \, e^{A(t-s)} B_I^b(x) \right) \right\|_{H_{-\vartheta}} \\
&+ \sum_{b \in \mathbb{U}} \left\| \left[F''(e^{A(t-s)} P_I(x)) - F''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^b(x), e^{A(t-s)} B_I^b(x) \right) \right\|_{H_{-\vartheta}} \\
&\leq 2 \, \|F\|_{C_b^2(H,H_{-\vartheta})} \left\| e^{A(t-s)} B_I(x) \right\|_{HS(U,H)} \left\| P_{\mathbb{H}\setminus I} e^{A(t-s)} B_I(x) \right\|_{HS(U,H)} \\
&+ \frac{|F|_{C_b^3(H,H_{-\vartheta})} \, g_1(x) \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \left\| e^{A(t-s)} B_I(x) \right\|^2_{HS(U,H)}}{(t-s)^{\rho}} \\
&\leq \frac{2 \, \|F\|_{C_b^2(H,H_{-\vartheta})} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, g_2(x)}{(t-s)^{\rho+\vartheta}} \\
&+ \frac{|F|_{C_b^3(H,H_{-\vartheta})} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, g_3(x)}{(t-s)^{\rho+\vartheta}} \\
&\leq \frac{2 \, \|F\|_{C_b^3(H,H_{-\vartheta})} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, g_3(x)}{(t-s)^{\rho+\vartheta}}.
\end{aligned}
\tag{53}
$$

Putting (51), (52), and (53) into (49) proves that for all $x \in H$, $s, t \in [0,T]$ with $t > s$ it holds that

$$
\begin{aligned}
\left| \tilde{F}_{t,s}(x) \right| \leq \Big[& c_{-\vartheta,0} \, \|F\|^2_{C_b^1(H,H_{-\vartheta})} \, g_2(x) + c_{-\vartheta} \, \|F\|^2_{C_b^2(H,H_{-\vartheta})} \, g_2(x) \\
&+ c_{-\vartheta,0,0} \, \|F\|_{C_b^1(H,H_{-\vartheta})} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \, g_3(x) \\
&+ c_{-\vartheta,0} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \, g_3(x) \, \|F\|_{C_b^2(H,H_{-\vartheta})} \\
&+ c_{-\vartheta} \, \|B\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \, g_3(x) \, \|F\|_{C_b^3(H,H_{-\vartheta})} \Big] \frac{\|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{t^{\vartheta} \, (t-s)^{(\rho+\vartheta)}}.
\end{aligned}
\tag{54}
$$

This implies that for all $t \in (0,T]$, $s \in [0,t)$, $x \in H$ it holds that

$$
\left| \tilde{F}_{t,s}(x) \right| \leq \frac{2 \, [c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0}] \, \varsigma_{F,B} \, g_3(x) \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{t^{\vartheta} \, (t-s)^{\rho+\vartheta}}. \tag{55}
$$

This, in turn, proves that

$$
\left| \int_0^T \int_0^t \mathbb{E}\Big[\tilde{F}_{t,s}(Y_s^I) \Big] \, ds \, dt \right| \leq \frac{2 \, T^{(2-\rho-2\vartheta)} \, \varsigma_{F,B} \, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})} \, K_3^I \, \big[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} \big]}{(1-\rho-\vartheta) \, (2-\rho-2\vartheta)}. \tag{56}
$$

It thus remains to bound the term $\frac{1}{2} \int_0^T \int_0^t \mathbb{E}\big[\tilde{B}_{t,s}(Y_s^I) \big] \, ds \, dt$ on the right hand side of (44). To do so, we use a few auxiliary estimates. More formally, note that for all $r \in [0, \vartheta]$,

$t \in (0,T]$, $s \in [0,t)$, $x, v \in H$ it holds that

$$\begin{aligned}
&\Big[\textstyle\sum_{b\in\mathbb{U}} \big\| \big[(B^b_I)'(e^{A(t-s)}x) - (B^b)'(e^{A(t-s)}x) \big]\, e^{A(t-s)}\, v \big\|^2_{H_{-\vartheta/2}} \Big]^{1/2} \\
&= \big\| \big[B'(e^{A(t-s)} P_I(x)) P_I - B'(e^{A(t-s)}x) \big]\, e^{A(t-s)}\, v \big\|_{HS(U,H_{-\vartheta/2})} \\
&\leq \big\| \big[B'(e^{A(t-s)} P_I(x)) - B'(e^{A(t-s)}x) \big]\, P_I\, e^{A(t-s)}\, v \big\|_{HS(U,H_{-\vartheta/2})} \\
&+ \big\| B'(e^{A(t-s)}x)\, P_{\mathbb{H}\setminus I}\, e^{A(t-s)}\, v \big\|_{HS(U,H_{-\vartheta/2})} \\
&\leq |B|_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \big\| e^{A(t-s)} P_{\mathbb{H}\setminus I} x \big\|_H \big\| P_I e^{A(t-s)} v \big\|_H \\
&+ \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \big\| P_{\mathbb{H}\setminus I} e^{A(t-s)} v \big\|_H \\
&\leq \Big[|B|_{C^2_b(H,HS(U,H_{-\vartheta/2}))}\, g_1(x) + \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \Big] \frac{\|v\|_{H_{-r}} \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+r)}} \\
&\leq \frac{g_1(x)\, \|B\|_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \|v\|_{H_{-r}} \big\| P_{\mathbb{H}\setminus I} \big\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+r)}}, \qquad \text{and}
\end{aligned} \tag{57}$$

$$\begin{aligned}
&\Big[\textstyle\sum_{b\in\mathbb{U}} \big\| \big[(B^b_I)'(e^{A(t-s)}x) + (B^b)'(e^{A(t-s)}x) \big]\, e^{A(t-s)}\, v \big\|^2_{H_{-\vartheta/2}} \Big]^{1/2} \\
&= \big\| \big[B'(e^{A(t-s)} P_I(x)) P_I + B'(e^{A(t-s)}x) \big]\, e^{A(t-s)}\, v \big\|_{HS(U,H_{-\vartheta/2})} \\
&\leq 2\, \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \big\| e^{A(t-s)} v \big\|_H \leq \frac{2\, \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \|v\|_{H_{-r}}}{(t-s)^r}.
\end{aligned} \tag{58}$$

Inequalities (57)–(58) imply that for all $t \in (0,T]$, $s \in [0,t)$, $x \in H$ it holds that

$$\begin{aligned}
&\Big[\textstyle\sum_{b\in\mathbb{U}} \big\| \big[(B^b_I)'(e^{A(t-s)}x) - (B^b)'(e^{A(t-s)}x) \big]\, e^{A(t-s)} F_I(x) \big\|^2_{H_{-\vartheta/2}} \Big]^{1/2} \\
&\leq \frac{g_1(x)\, \|B\|_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \|F_I(x)\|_{H_{-\vartheta}} \big\| P_{\mathbb{H}\setminus I} \big\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+\vartheta)}} \\
&\leq \frac{g_2(x)\, \|B\|_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \|F\|_{C^1_b(H,H_\vartheta)} \big\| P_{\mathbb{H}\setminus I} \big\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+\vartheta)}},
\end{aligned} \tag{59}$$

$$\begin{aligned}
&\Big[\textstyle\sum_{b\in\mathbb{U}} \big\| \big[(B^b_I)'(e^{A(t-s)}x) + (B^b)'(e^{A(t-s)}x) \big]\, e^{A(t-s)} F_I(x) \big\|^2_{H_{-\vartheta/2}} \Big]^{1/2} \\
&\leq \frac{2\, \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \|F_I(x)\|_{H_{-\vartheta}}}{(t-s)^\vartheta} \leq \frac{2\, g_1(x)\, \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \|F\|_{C^1_b(H,H_{-\vartheta})}}{(t-s)^\vartheta},
\end{aligned} \tag{60}$$

$$\begin{aligned}
&\Big[\textstyle\sum_{b_1,b_2\in\mathbb{U}} \big\| \big[(B^{b_2}_I)'(e^{A(t-s)}x) - (B^{b_2})'(e^{A(t-s)}x) \big]\, e^{A(t-s)} B^{b_1}_I(x) \big\|^2_{H_{-\vartheta/2}} \Big]^{1/2} \\
&\leq \frac{g_1(x)\, \|B\|_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \|B_I(x)\|_{HS(U,H_{-\vartheta/2})} \big\| P_{\mathbb{H}\setminus I} \big\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+\frac{\vartheta}{2})}} \\
&\leq \frac{g_2(x)\, \|B\|_{C^2_b(H,HS(U,H_{-\vartheta/2}))} \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \big\| P_{\mathbb{H}\setminus I} \big\|_{L(H,H_{-\rho})}}{(t-s)^{(\rho+\frac{\vartheta}{2})}},
\end{aligned} \tag{61}$$

$$\begin{aligned}
&\Big[\textstyle\sum_{b_1,b_2\in\mathbb{U}} \big\| \big[(B^{b_2}_I)'(e^{A(t-s)}x) + (B^{b_2})'(e^{A(t-s)}x) \big]\, e^{A(t-s)} B^{b_1}_I(x) \big\|^2_{H_{-\vartheta/2}} \Big]^{1/2} \\
&\leq \frac{2\, \|B\|_{C^1_b(H,HS(U,H_{-\vartheta/2}))} \|B_I(x)\|_{HS(U,H_{-\vartheta/2})}}{(t-s)^{\frac{\vartheta}{2}}} \leq \frac{2\, g_1(x)\, \|B\|^2_{C^1_b(H,HS(U,H_{-\vartheta/2}))}}{(t-s)^{\frac{\vartheta}{2}}}.
\end{aligned} \tag{62}$$

Moreover, observe that for all $x \in H$, $t \in (0,T]$, $s \in [0,t)$ it holds that

$$
\begin{aligned}
&\sum_{b_1 \in \mathbb{U}} \left[\sum_{b_2 \in \mathbb{U}} \left\| \left[(B_I^{b_2})''(e^{A(t-s)}x) + (B^{b_2})''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^{b_1}(x), e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{H_{-\vartheta/2}}^2 \right]^{1/2} \\
&= \sum_{b_1 \in \mathbb{U}} \left\| \left[(B_I)''(e^{A(t-s)}x) + (B)''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^{b_1}(x), e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{HS(U,H_{-\vartheta/2})} \\
&\leq \sum_{b_1 \in \mathbb{U}} \left\| B''(e^{A(t-s)} P_I(x)) \left(P_I e^{A(t-s)} B_I^{b_1}(x), P_I e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{HS(U,H_{-\vartheta/2})} \\
&+ \sum_{b_1 \in \mathbb{U}} \left\| B''(e^{A(t-s)}x) \left(e^{A(t-s)} B_I^{b_1}(x), e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{HS(U,H_{-\vartheta/2})} \\
&\leq \left\| B''(e^{A(t-s)} P_I(x)) \right\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \sum_{b_1 \in \mathbb{U}} \left\| P_I e^{A(t-s)} B_I^{b_1}(x) \right\|_H^2 \\
&+ \left\| B''(e^{A(t-s)}x) \right\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \sum_{b_1 \in \mathbb{U}} \left\| e^{A(t-s)} B_I^{b_1}(x) \right\|_H^2 \\
&\leq \|B\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} \left[\left\| P_I e^{A(t-s)} B_I(x) \right\|_{HS(U,H)}^2 + \left\| e^{A(t-s)} B_I(x) \right\|_{HS(U,H)}^2 \right] \\
&\leq \frac{2 \|B\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} \|B\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))}^2 \, g_2(x)}{(t-s)^{\vartheta}},
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
&\sum_{b_1 \in \mathbb{U}} \left[\sum_{b_2 \in \mathbb{U}} \left\| \left[(B_I^{b_2})''(e^{A(t-s)}x) - (B^{b_2})''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^{b_1}(x), e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{H_{-\vartheta/2}}^2 \right]^{1/2} \\
&= \sum_{b_1 \in \mathbb{U}} \left\| \left[(B_I)''(e^{A(t-s)}x) - (B)''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^{b_1}(x), e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{HS(U,H_{-\vartheta/2})} \\
&\leq \sum_{b_1 \in \mathbb{U}} \left\| \left[B''(e^{A(t-s)} P_I(x)) - B''(e^{A(t-s)}x) \right] \left(e^{A(t-s)} B_I^{b_1}(x), e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{HS(U,H_{-\vartheta/2})} \\
&+ \sum_{b_1 \in \mathbb{U}} \left\| B''(e^{A(t-s)} P_I(x)) \left((\mathrm{Id}_H - P_I) e^{A(t-s)} B_I^{b_1}(x), (\mathrm{Id}_H + P_I) e^{A(t-s)} B_I^{b_1}(x) \right) \right\|_{HS(U,H_{-\vartheta/2})} \\
&\leq \left\| B''(e^{A(t-s)} P_I(x)) - B''(e^{A(t-s)}x) \right\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \sum_{b_1 \in \mathbb{U}} \left\| e^{A(t-s)} B_I^{b_1}(x) \right\|_H^2 \\
&+ \left\| B''(e^{A(t-s)} P_I(x)) \right\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \\
&\cdot \sum_{b_1 \in \mathbb{U}} \left\| (\mathrm{Id}_H - P_I) \, e^{A(t-s)} B_I^{b_1}(x) \right\|_H \left\| (\mathrm{Id}_H + P_I) \, e^{A(t-s)} B_I^{b_1}(x) \right\|_H \\
&\leq \left\| B''(e^{A(t-s)} P_I(x)) - B''(e^{A(t-s)}x) \right\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \left\| e^{A(t-s)} B_I(x) \right\|_{HS(U,H)}^2 \\
&+ \left\| B''(e^{A(t-s)} P_I(x)) \right\|_{L^{(2)}(H,HS(U,H_{-\vartheta/2}))} \\
&\cdot \left\| (\mathrm{Id}_H - P_I) \, e^{A(t-s)} B_I(x) \right\|_{HS(U,H)} \left\| (\mathrm{Id}_H + P_I) \, e^{A(t-s)} B_I(x) \right\|_{HS(U,H)} \\
&\leq \frac{|B|_{C_b^3(H,HS(U,H_{-\vartheta/2}))} \|B\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))}^2 \left\| P_{\mathbb{H} \setminus I} \right\|_{L(H,H_{-\rho})} g_3(x)}{(t-s)^{(\rho+\vartheta)}} \\
&+ \frac{2 \|B\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} \|B\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))}^2 \left\| P_{\mathbb{H} \setminus I} \right\|_{L(H,H_{-\rho})} g_2(x)}{(t-s)^{(\rho+\vartheta)}} \\
&\leq \frac{2 \|B\|_{C_b^3(H,HS(U,H_{-\vartheta/2}))} \|B\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))}^2 \left\| P_{\mathbb{H} \setminus I} \right\|_{L(H,H_{-\rho})} g_3(x)}{(t-s)^{(\rho+\vartheta)}}.
\end{aligned}
\tag{64}
$$

Putting (59)–(64) into (42) shows that for all $t \in (0,T]$, $s \in [0,t)$, $x \in H$ it holds that

$$
\begin{aligned}
&\big|\tilde{B}_{t,s}(x)\big| \\
&\leq \Big[2c_{-\vartheta/2,-\vartheta/2} \left\|F\right\|_{C_b^1(H,H_{-\vartheta})} |B|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} g_2(x) \\
&+ 2c_{-\vartheta/2,-\vartheta/2} \left\|F\right\|_{C_b^1(H,H_{-\vartheta})} \left\|B\right\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} g_3(x) \\
&+ 2c_{-\vartheta/2,-\vartheta/2,0} \left\|F\right\|_{C_b^1(H,H_{-\vartheta})} |B|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} g_3(x) \\
&+ 2c_{-\vartheta/2,-\vartheta/2} \left\|B\right\|^3_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} g_3(x) \\
&+ c_{-\vartheta/2,-\vartheta/2} |B|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|^2_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} g_3(x) \\
&+ 2c_{-\vartheta/2,-\vartheta/2} \left\|B\right\|^3_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^3(H,HS(U,H_{-\vartheta/2}))} g_4(x) \\
&+ 2c_{-\vartheta/2,-\vartheta/2,0} |B|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|^3_{C_b^1(H,HS(U,H_{-\vartheta/2}))} g_3(x) \\
&+ 2c_{-\vartheta/2,-\vartheta/2,0} \left\|B\right\|^3_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|_{C_b^2(H,HS(U,H_{-\vartheta/2}))} g_4(x) \\
&+ c_{-\vartheta/2,-\vartheta/2,0,0} |B|_{C_b^1(H,HS(U,H_{-\vartheta/2}))} \left\|B\right\|^3_{C_b^1(H,HS(U,H_{-\vartheta/2}))} g_4(x)\Big] \\
&\cdot \frac{\left\|P_{\mathbb{H}\setminus I}\right\|_{L(H,H_{-\rho})}}{t^{\vartheta}\,(t-s)^{(\rho+\vartheta)}}.
\end{aligned}
\tag{65}
$$

This implies that for all $t \in (0,T]$, $s \in [0,t)$, $x \in H$ it holds that

$$
\big|\tilde{B}_{t,s}(x)\big| \leq \frac{9\left[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\right] \varsigma_{F,B}\, g_4(x) \left\|P_{\mathbb{H}\setminus I}\right\|_{L(H,H_{-\rho})}}{t^{\vartheta}\,(t-s)^{\rho+\vartheta}}. \tag{66}
$$

This proves that

$$
\begin{aligned}
&\frac{1}{2}\left|\int_0^T \int_0^t \mathbb{E}\Big[\tilde{B}_{t,s}(Y_s^I)\Big]\, ds\, dt\right| \\
&\leq \frac{9\,T^{(2-\rho-2\vartheta)}\, \varsigma_{F,B} \left\|P_{\mathbb{H}\setminus I}\right\|_{L(H,H_{-\rho})}}{2\,(1-\rho-\vartheta)\,(2-\rho-2\vartheta)} \left[c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\right] K_4^I.
\end{aligned}
\tag{67}
$$

Putting (46), (48), (56), and (67) into (44) finally yields

$$
\begin{aligned}
\big|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(Y_T^I)\big]\big| &\leq \left[c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\right] \\
&\quad \cdot \frac{T^{(1-\vartheta-\rho)}}{(1-\vartheta-\rho)} \left[1 + \frac{9\,T^{(1-\vartheta)}}{2\,(2-2\vartheta-\rho)}\right] \varsigma_{F,B}\, K_4^I \left\|P_{\mathbb{H}\setminus I}\right\|_{L(H,H_{-\rho})}.
\end{aligned}
\tag{68}
$$

This finishes the proof of Lemma 3.2. □

The next result, Corollary 3.3, is an immediate consequence of Lemma 3.1 and Lemma 3.2 above.

Corollary 3.3. *Assume the setting in Section 3.1 and let* $\rho \in [0, 1-\vartheta)$, $I \in \mathcal{P}(\mathbb{H})$. *Then*

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(X_T^I)\big]\big| \leq \tfrac{9}{2\,T^{\rho}} \left[1 + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\right]^2 \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-\rho})}\, \varsigma_{F,B}\, K_4^I \\
&\cdot \left[\|\varphi\|_{C_b^3(H,\mathbb{R})} + c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\right].
\end{aligned}
\tag{69}
$$

In the next result (Corollary 3.4 below) we use Proposition 5.1.11 in [19] to estimate the real numbers K_4^I, $I \in \mathcal{P}(\mathbb{H})$, on the right hand side of (69). For the formulation of Corollary 3.4 we recall that for all $x \in [0,\infty)$, $\theta \in [0,1)$ it holds that $\mathcal{E}_{1-\theta}(x) = \big[\sum_{n=0}^{\infty} \frac{x^{2n}\,\Gamma(1-\theta)^n}{\Gamma(n(1-\theta)+1)}\big]^{1/2}$ (see Section 1.2).

Corollary 3.4. *Assume the setting in Section 3.1. Then it holds for every $\theta \in [0,1)$, $\rho \in [0, 1-\vartheta)$, $I \in \mathcal{P}(\mathbb{H})$ that*

$$
\begin{aligned}
&\left|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(X_T^{I})\big]\right| \leq \frac{18}{T^\rho}\left[1 + \tfrac{T^{(1-\vartheta)}}{(1-\vartheta-\rho)}\right]^2 \mathbb{E}\big[\max\{1, \|\xi\|_H^4\}\big] \left\|P_{\mathbb{H}\setminus I}\right\|_{L(H,H_{-\rho})} \\
&\cdot \varsigma_{F,B} \left|\mathcal{E}_{(1-\theta)}\left[\tfrac{T^{1-\theta}\sqrt{2}\,\|F\|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\,T^{1-\theta}}\,\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^4 \\
&\cdot \left[\|\varphi\|_{C_b^3(H,\mathbb{R})} + c_{-\vartheta} + c_{-\vartheta,0} + c_{-\vartheta,0,0} + c_{-\vartheta/2,-\vartheta/2} + c_{-\vartheta/2,-\vartheta/2,0} + c_{-\vartheta/2,-\vartheta/2,0,0}\right].
\end{aligned}
\tag{70}
$$

4 Strong convergence of mollified solutions for SPDEs

In this section an elementary and essentially well-known strong convergence result, see Proposition 4.1 below, is established. Proposition 4.1, in particular, allows us to prove estimate (15) in the introduction. In Section 5 below we will use Proposition 4.1 in conjunction with Corollary 3.4 in Section 3 to establish weak convergence rates for Galerkin approximations of SPDEs.

4.1 Setting

Assume the setting in Section 1.3, assume that $\eta = 0$, and let $p \in [2,\infty)$, $\vartheta \in [0,1)$, $F \in \operatorname{Lip}^0(H, H_{-\vartheta})$, $B \in \operatorname{Lip}^0(H, HS(U, H_{-\vartheta/2}))$, $\xi \in L^p(\mathbb{P}|_{\mathcal{F}_0}; H)$.

The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^\kappa \colon [0,T]\times\Omega \to H$, $\kappa \in [0,\infty)$, which satisfy that for all $\kappa \in [0,\infty)$ it holds that $\sup_{t\in[0,T]} \|X_t^\kappa\|_{L^p(\mathbb{P};H)} < \infty$ and which satisfy that for all $t \in [0,T]$, $\kappa \in [0,\infty)$ it holds $\mathbb{P}$-a.s. that

$$
X_t^\kappa = e^{At}\xi + \int_0^t e^{A(\kappa+t-s)} F(X_s^\kappa)\,ds + \int_0^t e^{A(\kappa+t-s)} B(X_s^\kappa)\,dW_s. \tag{71}
$$

4.2 A strong convergence result

Proposition 4.1. *Assume the setting in Section 4.1 and let $\kappa \in [0,\infty)$, $\rho \in [0, \frac{1-\vartheta}{2})$. Then*

$$
\begin{aligned}
&\left\|X_T^0 - X_T^\kappa\right\|_{L^p(\mathbb{P};H)} \leq \max\big\{1, \|\xi\|_{L^p(\mathbb{P};H)}\big\} \\
&\cdot 2\,\kappa^\rho \left|\mathcal{E}_{1-\vartheta}\left[\tfrac{T^{1-\vartheta}\sqrt{2}\,\|F\|_{\operatorname{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + \sqrt{T^{1-\vartheta}\,p\,(p-1)}\,\|B\|_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right]\right|^2 \\
&\cdot \left[\tfrac{T^{(1-\rho-\vartheta)}}{(1-\rho-\vartheta)}\|F\|_{\operatorname{Lip}^0(H,H_{-\vartheta})} + \tfrac{\sqrt{p\,(p-1)\,T^{(1-2\rho-\vartheta)}}}{\sqrt{2-4\rho-2\vartheta}}\|B\|_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right].
\end{aligned}
\tag{72}
$$

Proof. First of all, observe that Proposition 5.1.4 in [19] shows that

$$
\begin{aligned}
&\left\|X_T^0 - X_T^\kappa\right\|_{L^p(\mathbb{P};H)} \\
&\leq \sqrt{2}\cdot\mathcal{E}_{1-\vartheta}\left[\tfrac{T^{1-\vartheta}\sqrt{2}\,|F|_{\operatorname{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + \sqrt{T^{1-\vartheta}\,p\,(p-1)}\,|B|_{\operatorname{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right] \\
&\cdot \sup_{t\in[0,T]}\left\|\int_0^t e^{A(t-s)}\left(\operatorname{Id}_H - e^{A\kappa}\right)F(X_s^\kappa)\,ds + \int_0^t e^{A(t-s)}\left(\operatorname{Id}_H - e^{A\kappa}\right)B(X_s^\kappa)\,dW_s\right\|_{L^p(\mathbb{P};H)}.
\end{aligned}
\tag{73}
$$

In the next step we observe that the fact that for all $r \in [0,1]$, $u \in [0,1]$ it holds that $\sup_{t\in(0,\infty)} \left\|(-tA)^r e^{At}\right\|_{L(H)} \leq \left[\frac{r}{e}\right]^r \leq 1$ and $\sup_{t\in(0,\infty)} \|(-tA)^{-u}(\mathrm{Id}_H - e^{At})\|_{L(H)} \leq 1$ implies that for all $t \in [0,T]$, $r \in [0, 1-\vartheta)$ it holds that

$$\begin{aligned}
&\left\| \int_0^t e^{A(t-s)} \left(\mathrm{Id}_H - e^{A\kappa}\right) F(X_s^\kappa)\, ds \right\|_{L^p(\mathbb{P};H)} \\
&\leq \frac{T^{(1-r-\vartheta)}}{(1-r-\vartheta)} \left[\sup_{t\in[0,T]} \|F(X_t^\kappa)\|_{L^p(\mathbb{P};H_{-\vartheta})}\right] \kappa^r \\
&\leq \frac{T^{(1-r-\vartheta)}}{(1-r-\vartheta)} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} \max\left\{1, \sup_{t\in[0,T]} \|X_t^\kappa\|_{L^p(\mathbb{P};H)}\right\} \kappa^r
\end{aligned} \tag{74}$$

and that for all $t \in [0,T]$, $r \in [0, \frac{1-\vartheta}{2})$ it holds that

$$\begin{aligned}
&\left\| \int_0^t e^{A(t-s)} \left(\mathrm{Id}_H - e^{A\kappa}\right) B(X_s^\kappa)\, dW_s \right\|_{L^p(\mathbb{P};H)} \\
&\leq \sqrt{\frac{p\,(p-1)}{2}} \frac{\sqrt{T^{1-2r-\vartheta}}}{\sqrt{1-2r-\vartheta}} \left[\sup_{t\in[0,T]} \|B(X_t^\kappa)\|_{L^p(\mathbb{P};HS(U,H_{-\vartheta/2}))}\right] \kappa^r \\
&\leq \sqrt{\frac{p\,(p-1)}{2}} \frac{\sqrt{T^{1-2r-\vartheta}}}{\sqrt{1-2r-\vartheta}} \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))} \max\left\{1, \sup_{t\in[0,T]} \|X_t^\kappa\|_{L^p(\mathbb{P};H)}\right\} \kappa^r.
\end{aligned} \tag{75}$$

Putting (74) and (75) into (73) yields that for all $r \in [0, \frac{1-\vartheta}{2})$ it holds that

$$\begin{aligned}
&\left\|X_T^0 - X_T^\kappa\right\|_{L^p(\mathbb{P};H)} \leq \sqrt{2}\,\kappa^r \max\left\{1, \sup_{t\in[0,T]} \|X_t^\kappa\|_{L^p(\mathbb{P};H)}\right\} \\
&\cdot \mathcal{E}_{1-\vartheta}\left[\tfrac{T^{1-\vartheta}\sqrt{2}\,|F|_{\mathrm{Lip}^0(H,H_{-\vartheta})}}{\sqrt{1-\vartheta}} + \sqrt{T^{1-\vartheta}\,p\,(p-1)}\,|B|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right] \\
&\cdot \left[\tfrac{T^{(1-r-\vartheta)}}{(1-r-\vartheta)} \|F\|_{\mathrm{Lip}^0(H,H_{-\vartheta})} + \tfrac{\sqrt{p\,(p-1)\,T^{(1-2r-\vartheta)}}}{\sqrt{2-4r-2\vartheta}} \|B\|_{\mathrm{Lip}^0(H,HS(U,H_{-\vartheta/2}))}\right].
\end{aligned} \tag{76}$$

Proposition 5.1.9 in [19] combined with (76) finishes the proof of Proposition 4.1. □

5 Weak convergence for Galerkin approximations of SPDEs

In this section our main weak convergence results are established; see Proposition 5.1 and Corollary 5.2 below. The proof of Proposition 5.1 uses both Corollary 3.4 in Section 3 and Proposition 4.1 in Section 4. Theorem 1.1 in the introduction is an immediate consequence of Corollary 5.2 below.

5.1 Setting

Assume the setting in Section 1.3, let $\varphi \in C_b^4(H,\mathbb{R})$, $\theta \in [0,1)$, $F \in C_b^4(H, H_{-\theta})$, $B \in C_b^4(H, HS(U, H_{-\theta/2}))$, $\xi \in L^4(\mathbb{P}|_{\mathcal{F}_0}; H)$, and let $\varsigma_{F,B} \in \mathbb{R}$ be a real number given by $\varsigma_{F,B} = \max\left\{1, \|\eta\,\mathrm{Id}_H + F\|^2_{C_b^3(H,H_{-\theta})}, \|B\|^4_{C_b^3(H,HS(U,H_{-\theta/2}))}\right\}$.

The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^I\colon [0,T] \times \Omega \to P_I(H)$,

$I \in \mathcal{P}(\mathbb{H})$, and $X^{\mathbb{H},\kappa,x}\colon [0,T] \times \Omega \to H$, $\kappa \in [0,\infty)$, $x \in H$, which satisfy that for all $I \in \mathcal{P}(\mathbb{H})$, $\kappa \in [0,\infty)$, $x \in H$ it holds that $\sup_{t\in[0,T]} \big[\|X_t^I\|_{L^4(\mathbb{P};H)} + \|X_t^{\mathbb{H},\kappa,x}\|_{L^4(\mathbb{P};H)}\big] < \infty$ and which satisfy that for all $t \in [0,T]$, $I \in \mathcal{P}(\mathbb{H})$, $\kappa \in [0,\infty)$, $x \in H$ it holds $\mathbb{P}$-a.s. that

$$X_t^I = e^{At} P_I(\xi) + \int_0^t e^{A(t-s)} P_I F(X_s^I)\, ds + \int_0^t e^{A(t-s)} P_I B(X_s^I)\, dW_s, \tag{77}$$

$$X_t^{\mathbb{H},\kappa,x} = e^{At} x + \int_0^t e^{A(\kappa+t-s)} F(X_s^{\mathbb{H},\kappa,x})\, ds + \int_0^t e^{A(\kappa+t-s)} B(X_s^{\mathbb{H},\kappa,x})\, dW_s. \tag{78}$$

Moreover, let $u^{(\kappa)}\colon [0,T] \times H \to \mathbb{R}$, $\kappa \in [0,\infty)$, be functions given by $u^{(\kappa)}(t,x) = \mathbb{E}\big[\varphi(X_{T-t}^{\mathbb{H},\kappa,x})\big]$ for all $t \in [0,T]$, $x \in H$, $\kappa \in [0,\infty)$ and let $c^{(\kappa)}_{\delta_1,\dots,\delta_k} \in [0,\infty]$, $\delta_1,\dots,\delta_k \in \mathbb{R}$, $k \in \{1,2,3,4\}$, $\kappa \in [0,\infty)$, be extended real numbers given by

$$c^{(\kappa)}_{\delta_1,\delta_2,\dots,\delta_k} = \sup_{t\in(0,T]} \sup_{s\in[0,t)} \sup_{x\in H} \sup_{v_1,\dots,v_k \in H\setminus\{0\}} \left[\frac{\big|(\frac{\partial^k}{\partial x^k} u^{(\kappa)})(t, e^{A(t-s)}x)(v_1,\dots,v_k)\big|}{t^{(\delta_1+\ldots+\delta_k)}\, \|v_1\|_{H_{\delta_1}} \cdot \ldots \cdot \|v_k\|_{H_{\delta_k}}} \right] \tag{79}$$

for all $\kappa \in [0,\infty)$, $\delta_1,\dots,\delta_k \in \mathbb{R}$, $k \in \{1,2,3,4\}$.

5.2 A weak convergence result

Proposition 5.1. *Assume the setting in Section 5.1 and let $I \in \mathcal{P}(\mathbb{H})$, $\vartheta \in [0,\frac{1}{2}) \cap [0,\theta]$. Then it holds for all $r \in [0, 1-\vartheta)$, $\rho \in (0, 1-\theta)$ that*

$$\begin{aligned}
&\big|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(X_T^I)\big]\big| \le 22\,\mathbb{E}\big[|1 \vee \|\xi\|_H|^4\big]\big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-1})}\big]\, \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{\frac{r\rho}{\rho+4(\theta-\vartheta)}} \\
&\cdot \Bigg\{ \Bigg[\frac{T^{1-\rho/2-\theta}\, \|\eta \operatorname{Id}_H + F\|_{C_b^1(H,H_{-\theta})}}{1-\rho/2-\theta} + \frac{\sqrt{T^{1-\rho-\theta}}\, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\rho-\theta}} \Bigg] \|\varphi\|_{C_b^1(H,\mathbb{R})} + \tfrac{\varsigma_{F,B}}{T^r}\Big[1 + \tfrac{T^{1-\vartheta}}{1-\vartheta-r}\Big]^2 \\
&\cdot \Big[\|\varphi\|_{C_b^3(H,\mathbb{R})} + \sup_{\kappa\in(0,\infty)} \big[c^{(\kappa)}_{-\vartheta} + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0} \big] \Big] \Bigg\} \\
&\cdot \Bigg| \mathcal{E}_{(1-\theta)}\Bigg[\frac{T^{1-\theta}\sqrt{2}\, \|\eta \operatorname{Id}_H + F\|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\, T^{1-\theta}}\, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))} \Bigg] \Bigg|^4 < \infty.
\end{aligned} \tag{80}$$

Proof. First of all, let $\tilde{A}\colon D(A) \subseteq H \to H$ and $\tilde{F} \in C_b^4(H, H_{-\theta})$ be given by $\tilde{A}v = Av - \eta v$ and $\tilde{F}(v) = F(v) + \eta v$ for all $v \in D(A)$ and observe that Proposition 7.3.1 in [19] proves that for all $t \in [0,T]$, $I \in \mathcal{P}(\mathbb{H})$, $\kappa \in [0,\infty)$, $x \in H$ it holds $\mathbb{P}$-a.s. that

$$X_t^I = e^{\tilde{A}t} P_I(\xi) + \int_0^t e^{\tilde{A}(t-s)} P_I \tilde{F}(X_s^I)\, ds + \int_0^t e^{\tilde{A}(t-s)} P_I B(X_s^I)\, dW_s, \tag{81}$$

$$X_t^{\mathbb{H},\kappa,x} = e^{\tilde{A}t} x + \int_0^t e^{\tilde{A}(\kappa+t-s)} \tilde{F}(X_s^{\mathbb{H},\kappa,x})\, ds + \int_0^t e^{\tilde{A}(\kappa+t-s)} B(X_s^{\mathbb{H},\kappa,x})\, dW_s. \tag{82}$$

We intend to Proposition 5.1 through an application of Corollary 3.4. Corollary 3.4 assumes that the initial random variable of the considered SEE takes values in $H_1 \subseteq H$. In Section 5.1 above we, however, merely assume that the initial random variable ξ takes values in H. To overcome this difficulty, we mollify the initial random variable in an appropriate sense so that the assumptions of Corollary 3.4 are met and Corollary 3.4 can be applied. More formally, note that there exist (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $\hat{X}^{I,\kappa,\delta}\colon [0,T] \times \Omega \to H$,

$\kappa, \delta \in [0, \infty)$, such that for all $\kappa, \delta \in [0, \infty)$ it holds that $\sup_{t\in[0,T]} \|\hat{X}_t^{I,\kappa,\delta}\|_{L^4(\mathbb{P};H)} < \infty$ and such that for all $t \in [0, T]$, $\kappa, \delta \in [0, \infty)$ it holds $\mathbb{P}$-a.s. that

$$\hat{X}_t^{I,\kappa,\delta} = e^{\tilde{A}(\delta+t)} P_I(\xi) + \int_0^t e^{\tilde{A}(\kappa+t-s)} P_I \tilde{F}(\hat{X}_s^{I,\kappa,\delta})\, ds + \int_0^t e^{\tilde{A}(\kappa+t-s)} P_I B(\hat{X}_s^{I,\kappa,\delta})\, dW_s. \quad (83)$$

In the next step we observe that the triangle inequality ensures that for all $\kappa, \delta \in (0, \infty)$ it holds that

$$\begin{aligned} \big|\mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},0,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{I,0,\delta})\big]\big| &\leq \big|\mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},0,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},\kappa,\delta})\big]\big| \\ + \big|\mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},\kappa,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{I,\kappa,\delta})\big]\big| &+ \big|\mathbb{E}\big[\varphi(\hat{X}_T^{I,\kappa,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{I,0,\delta})\big]\big|. \end{aligned} \quad (84)$$

In the following we bound the three summands on the right hand side of (84). For the first and third summand on the right hand side of (84) we observe that Proposition 4.1 shows that for all $\kappa, \delta \in (0, \infty)$, $\rho \in [0, 1-\theta)$ it holds that

$$\begin{aligned} &\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},0,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},\kappa,\delta})\big]\big| + \big|\mathbb{E}\big[\varphi(\hat{X}_T^{I,\kappa,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{I,0,\delta})\big]\big| \\ &\leq 4\,\|\varphi\|_{C_b^1(H,\mathbb{R})} \left|\mathcal{E}_{(1-\theta)}\left[\frac{T^{1-\theta}\sqrt{2}\,\|\tilde{F}\|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{2\,T^{1-\theta}}\,\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^2 \\ &\cdot \left[\tfrac{T^{1-\rho/2-\theta}}{1-\rho/2-\theta}\|\tilde{F}\|_{C_b^1(H,H_{-\theta})} + \tfrac{\sqrt{T^{1-\rho-\theta}}}{\sqrt{1-\rho-\theta}}\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right] \max\big\{1, \|\xi\|_{L^2(\mathbb{P};H)}\big\}\, \kappa^{\frac{\rho}{2}}. \end{aligned} \quad (85)$$

Next we bound the second summand on the right hand side of (84). For this we note that for all $\kappa \in (0, \infty)$ it holds that

$$\max\big\{1, \|e^{\kappa\tilde{A}}\tilde{F}(\cdot)\|^2_{C_b^3(H,H_{-\vartheta})}, \|e^{\kappa\tilde{A}}B(\cdot)\|^4_{C_b^3(H,HS(U,H_{-\vartheta/2}))}\big\} \leq \varsigma_{F,B}\, \max\big\{1, \kappa^{-2(\theta-\vartheta)}\big\}. \quad (86)$$

This and Corollary 3.4 show that for all $\kappa, \delta \in (0, \infty)$, $r \in [0, 1-\vartheta)$ it holds that

$$\begin{aligned} &\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},\kappa,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{I,\kappa,\delta})\big]\big| \\ &\leq \left[1 + \tfrac{T^{1-\vartheta}}{1-\vartheta-r}\right]^2 \left|\mathcal{E}_{(1-\theta)}\left[\frac{T^{1-\theta}\sqrt{2}\,\|\tilde{F}\|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\,T^{1-\theta}}\,\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^4 \\ &\cdot \left[\|\varphi\|_{C_b^3(H,\mathbb{R})} + c^{(\kappa)}_{-\vartheta} + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\right] \\ &\cdot \frac{18\,\varsigma_{F,B}}{T^r}\,\mathbb{E}\big[\max\{1, \|\xi\|_H^4\}\big]\, \max\big\{1, \kappa^{-2(\theta-\vartheta)}\big\}\, \big\|P_{\mathbb{H}\setminus I}\big\|_{L(H,H_{-r})}. \end{aligned} \quad (87)$$

In the next step we plug (85) and (87) into (84) and we use the fact that for all $r \in (0, \infty)$ it holds that $\|P_I\|_{L(H,H_{-r})} = \|P_I\|^r_{L(H,H_{-1})}$ to obtain that for all $\kappa, \delta \in (0, \infty)$, $r \in [0, 1-\vartheta)$, $\rho \in [0, 1-\theta)$ it holds that

$$\begin{aligned} &\big|\mathbb{E}\big[\varphi(\hat{X}_T^{\mathbb{H},0,\delta})\big] - \mathbb{E}\big[\varphi(\hat{X}_T^{I,0,\delta})\big]\big| \leq \left[4\,\kappa^{\frac{\rho}{2}} + 18\max\big\{1, \kappa^{-2(\theta-\vartheta)}\big\}\|P_{\mathbb{H}\setminus I}\|^r_{L(H,H_{-1})}\right] \\ &\cdot \Bigg\{ \left[\tfrac{T^{1-\rho/2-\theta}}{1-\rho/2-\theta}\|\tilde{F}\|_{C_b^1(H,H_{-\theta})} + \tfrac{\sqrt{T^{1-\rho-\theta}}}{\sqrt{1-\rho-\theta}}\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right] \|\varphi\|_{C_b^1(H,\mathbb{R})} + \tfrac{\varsigma_{F,B}}{T^r}\left[1 + \tfrac{T^{1-\vartheta}}{1-\vartheta-r}\right]^2 \\ &\cdot \left[\|\varphi\|_{C_b^3(H,\mathbb{R})} + c^{(\kappa)}_{-\vartheta} + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\right] \Bigg\} \\ &\cdot \mathbb{E}\big[\max\{1, \|\xi\|_H^4\}\big] \left|\mathcal{E}_{(1-\theta)}\left[\frac{T^{1-\theta}\sqrt{2}\,\|\tilde{F}\|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\,T^{1-\theta}}\,\|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}\right]\right|^4. \end{aligned} \quad (88)$$

Next we use again the fact that for all $r \in (0,\infty)$ it holds that $\|P_I\|_{L(H,H_{-r})} = \|P_I\|^r_{L(H,H_{-1})}$ to obtain that for all $r \in (0,\infty)$, $\rho \in (0, 1-\theta)$ it holds that

$$
\begin{aligned}
&\inf_{\kappa\in(0,\infty)} \Big[4\,\kappa^{\frac{\rho}{2}} + 18\max\big\{1, \kappa^{-2(\theta-\vartheta)}\big\}\|P_{\mathbb{H}\setminus I}\|^r_{L(H,H_{-1})}\Big] \\
&\leq 4\left[\|P_{\mathbb{H}\setminus I}\|^{\frac{2r}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}\right]^{\frac{\rho}{2}} + 18\max\left\{1, \left[\|P_{\mathbb{H}\setminus I}\|^{\frac{2r}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}\right]^{-2(\theta-\vartheta)}\right\}\|P_{\mathbb{H}\setminus I}\|^r_{L(H,H_{-1})} \\
&\leq 22\max\left\{\|P_{\mathbb{H}\setminus I}\|^{\frac{r\rho}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}, \max\left\{1, \|P_{\mathbb{H}\setminus I}\|^{\frac{-4r(\theta-\vartheta)}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}\right\}\|P_{\mathbb{H}\setminus I}\|^{\frac{4r(\theta-\vartheta)}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}\|P_{\mathbb{H}\setminus I}\|^{r-\frac{4r(\theta-\vartheta)}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}\right\} \\
&\leq 22\,\|P_{\mathbb{H}\setminus I}\|^{\frac{r\rho}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}\max\left\{1, \max\left\{\|P_{\mathbb{H}\setminus I}\|^{\frac{4r(\theta-\vartheta)}{(\rho+4(\theta-\vartheta))}}_{L(H,H_{-1})}, 1\right\}\right\} \\
&\leq 22\,\max\big\{1, \|(-\tilde{A})^{\frac{-4r(\theta-\vartheta)}{\rho+4(\theta-\vartheta)}}\|_{L(H)}\big\}\,\|P_{\mathbb{H}\setminus I}\|^{\frac{r\rho}{\rho+4(\theta-\vartheta)}}_{L(H,H_{-1})}
\end{aligned} \tag{89}
$$

Putting (89) into (88) implies that for all $\delta \in (0,\infty)$, $r \in [0, 1-\vartheta)$, $\rho \in (0, 1-\theta)$ it holds that

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi(\hat{X}^{\mathbb{H},0,\delta}_T)\big] - \mathbb{E}\big[\varphi(\hat{X}^{I,0,\delta}_T)\big]\big| \leq 22\,\max\big\{1, \|(-\tilde{A})^{\frac{-4r(\theta-\vartheta)}{\rho+4(\theta-\vartheta)}}\|_{L(H)}\big\}\,\|P_{\mathbb{H}\setminus I}\|^{\frac{r\rho}{\rho+4(\theta-\vartheta)}}_{L(H,H_{-1})} \\
&\cdot\Bigg\{\Big[\tfrac{T^{1-\rho/2-\theta}}{1-\rho/2-\theta}\|\tilde{F}\|_{C^1_b(H,H_{-\theta})} + \tfrac{\sqrt{T^{1-\rho-\theta}}}{\sqrt{1-\rho-\theta}}\|B\|_{C^1_b(H,HS(U,H_{-\theta/2}))}\Big]\|\varphi\|_{C^1_b(H,\mathbb{R})} + \tfrac{\varsigma_{F,B}}{T^r}\Big[1 + \tfrac{T^{1-\vartheta}}{1-\vartheta-r}\Big]^2 \\
&\cdot\Big[\|\varphi\|_{C^3_b(H,\mathbb{R})} + \sup_{\kappa\in(0,\infty)}\big[c^{(\kappa)}_{-\vartheta} + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\big]\Big]\Bigg\} \\
&\cdot\mathbb{E}\big[\max\{1, \|\xi\|^4_H\}\big]\left|\mathcal{E}_{(1-\theta)}\left[\frac{T^{1-\theta}\sqrt{2}\,\|\tilde{F}\|_{C^1_b(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\,T^{1-\theta}}\,\|B\|_{C^1_b(H,HS(U,H_{-\theta/2}))}\right]\right|^4.
\end{aligned} \tag{90}
$$

Moreover, we note that Corollary 5.1.5 in [19] ensures that for all $J \in \mathcal{P}(\mathbb{H})$ it holds that $\lim_{\delta\to 0}\mathbb{E}\big[\varphi(\hat{X}^{J,0,\delta}_T)\big] = \mathbb{E}\big[\varphi(X^J_T)\big]$. Combining this with inequality (90) proves the first inequality in (80). The second inequality in (80) follows from Andersson et al. [1]. The proof of Proposition 5.1 is thus completed. □

In a number of cases the difference $\theta - \vartheta \geq 0$ can be chosen to be an arbitrarily small positive real number (cf. Theorem 1.1 above). In the next result, Corollary 5.2, we further estimate the right hand side of (80).

Corollary 5.2. *Assume the setting in Section 5.1 and let $I \in \mathcal{P}(\mathbb{H})$. Then it holds for all $\rho \in (0, 1-\theta) \cap \big(4(\theta-\vartheta), \infty\big)$ that*

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi(X^{\mathbb{H}}_T)\big] - \mathbb{E}\big[\varphi(X^I_T)\big]\big| \leq 22\,\big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-2})}\big]\left[\inf_{b\in\mathbb{H}\setminus I}[\eta - \lambda_b]\right]^{-(\rho-4(\theta-\vartheta))} \\
&\cdot\Bigg\{\Bigg[\frac{T^{1-\rho/2-\theta}\,\|\eta\operatorname{Id}_H + F\|_{C^1_b(H,H_{-\theta})}}{1-\rho/2-\theta} + \frac{\sqrt{T^{1-\rho-\theta}}\,\|B\|_{C^1_b(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\rho-\theta}}\Bigg]\|\varphi\|_{C^1_b(H,\mathbb{R})} + \tfrac{\varsigma_{F,B}}{T^\rho}\Big[1 + \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho}\Big]^2 \\
&\cdot\Big[\|\varphi\|_{C^3_b(H,\mathbb{R})} + \sup_{\kappa\in(0,\infty)}\big[c^{(\kappa)}_{-\vartheta} + c^{(\kappa)}_{-\vartheta,0} + c^{(\kappa)}_{-\vartheta,0,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0} + c^{(\kappa)}_{-\vartheta/2,-\vartheta/2,0,0}\big]\Big]\Bigg\} \\
&\cdot\mathbb{E}\big[|1 \vee \|\xi\|_H|^4\big]\left|\mathcal{E}_{(1-\theta)}\left[\frac{T^{1-\theta}\sqrt{2}\,\|\eta\operatorname{Id}_H + F\|_{C^1_b(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\,T^{1-\theta}}\,\|B\|_{C^1_b(H,HS(U,H_{-\theta/2}))}\right]\right|^4 < \infty.
\end{aligned} \tag{91}
$$

Proof of Corollary 5.2. First of all, we choose $r = \rho$ in (80) in Proposition 5.1 above to obtain that for all $\rho \in (0, 1-\theta)$ that

$$
\begin{aligned}
&\big|\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(X_T^I)\big]\big| \leq 22\, \mathbb{E}\big[|1 \vee \|\xi\|_H|^4\big] \big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-1})}\big] \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{\frac{\rho^2}{\rho+4(\theta-\vartheta)}} \qquad (92) \\
&\cdot \left\{ \left[\frac{T^{1-\rho/2-\theta} \|\eta \operatorname{Id}_H + F\|_{C_b^1(H,H_{-\theta})}}{1-\rho/2-\theta} + \frac{\sqrt{T^{1-\rho-\theta}}\, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))}}{\sqrt{1-\rho-\theta}} \right] \|\varphi\|_{C_b^1(H,\mathbb{R})} + \tfrac{\varsigma_{F,B}}{T^\rho} \left[1 + \tfrac{T^{1-\vartheta}}{1-\vartheta-\rho}\right]^2 \right. \\
&\cdot \left. \left[\|\varphi\|_{C_b^3(H,\mathbb{R})} + \sup_{\kappa\in(0,\infty)} \left[c_{-\vartheta}^{(\kappa)} + c_{-\vartheta,0}^{(\kappa)} + c_{-\vartheta,0,0}^{(\kappa)} + c_{-\vartheta/2,-\vartheta/2}^{(\kappa)} + c_{-\vartheta/2,-\vartheta/2,0}^{(\kappa)} + c_{-\vartheta/2,-\vartheta/2,0,0}^{(\kappa)} \right] \right] \right\} \\
&\cdot \left| \mathcal{E}_{(1-\theta)} \left[\frac{T^{1-\theta}\sqrt{2}\, \|\eta \operatorname{Id}_H + F\|_{C_b^1(H,H_{-\theta})}}{\sqrt{1-\theta}} + \sqrt{12\, T^{1-\theta}}\, \|B\|_{C_b^1(H,HS(U,H_{-\theta/2}))} \right] \right|^4 < \infty.
\end{aligned}
$$

Next we note that for all $\rho \in \big(0, 1-\theta\big) \cap \big(4(\theta-\vartheta), \infty\big)$ it holds that

$$
\begin{aligned}
&\big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-1})}\big] \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{\frac{\rho^2}{\rho+4(\theta-\vartheta)}} \\
&= \big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-1})}\big] \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{\rho\left[\frac{1}{1+4(\theta-\vartheta)/\rho} - 1 + \frac{4(\theta-\vartheta)}{\rho}\right]} \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{\rho\left[1-\frac{4(\theta-\vartheta)}{\rho}\right]} \\
&\leq \big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-1})}\big] \big[1 \vee \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}\big]^{\rho\left[\frac{1}{1+4(\theta-\vartheta)/\rho} - 1 + \frac{4(\theta-\vartheta)}{\rho}\right]} \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{(\rho-4(\theta-\vartheta))} \qquad (93) \\
&\leq \big[1 \vee \|\operatorname{Id}_H\|_{L(H,H_{-1})}\big] \big[1 \vee \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}\big]^{\rho} \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{(\rho-4(\theta-\vartheta))} \\
&\leq \max\big\{1, \|\operatorname{Id}_H\|_{L(H,H_{-1})}^2\big\} \|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{(\rho-4(\theta-\vartheta))}
\end{aligned}
$$

Combining this with (92), the fact that $\|\operatorname{Id}_H\|_{L(H,H_{-1})}^2 = \|\operatorname{Id}_H\|_{L(H,H_{-2})}$ and the fact that $\|P_{\mathbb{H}\setminus I}\|_{L(H,H_{-1})}^{(\rho-4(\theta-\vartheta))} = \big[\inf_{b\in\mathbb{H}\setminus I}\, [\eta - \lambda_b]\big]^{-(\rho-4(\theta-\vartheta))}$ completes the proof of Corollary 5.2. ☐

6 Lower bounds for the weak error of Galerkin approximations for SPDEs

In this section a few specific lower bounds for weak approximation errors are established in the case of concrete example SEEs. Lower bounds for strong approximation errors for examples SEEs and whole classes of SEEs can be found in [10, 27, 29, 30]. In the case of finite dimensional stochastic ordinary differential equations lower bounds for both strong and weak approximation errors have been established; for details see, e.g., the references in the overview article Müller-Gronbach & Ritter [28].

6.1 Setting

Assume the setting in Section 1.3, assume that $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H) = (U, \langle\cdot,\cdot\rangle_U, \|\cdot\|_U)$, assume that $\eta = 0$, let $\beta \in [0, \frac{1}{2})$, let $\mu\colon \mathbb{H} \to \mathbb{R}$ be a function such that $\sum_{b\in\mathbb{H}} |\mu_b|^2 |\lambda_b|^{-2\beta} < \infty$, and let $B \in HS(H, H_{-\beta})$ satisfy that for all $v \in H$ it holds that $Bv = \sum_{b\in\mathbb{H}} \mu_b \langle b, v\rangle_H\, b$.

The above assumptions ensure (cf., e.g., Proposition 3 in Da Prato et al. [8], Theorem 4.3 in Brzeźniak [6], Theorem 6.2 in Van Neerven et al. [33]) that there exist up-to-modifications unique $(\mathcal{F}_t)_{t\in[0,T]}$-predictable stochastic processes $X^I\colon [0,T]\times\Omega \to H$, $I \in \mathcal{P}(\mathbb{H})$, such that for all $p \in (0,\infty)$, $I \in \mathcal{P}(\mathbb{H})$ it holds that $\sup_{t\in[0,T]} \|X_t^I\|_{L^p(\mathbb{P};H)} < \infty$ and such that for all $I \in \mathcal{P}(\mathbb{H})$, $t \in [0,T]$ it holds $\mathbb{P}$-a.s. that $X_t^I = \int_0^t e^{A(t-s)} P_I B\, dW_s$.

6.2 Lower bounds for the weak error

Lemma 6.1. *Assume the setting in Section 6.1 and let $I \in \mathcal{P}(\mathbb{H})$, $b \in \mathbb{H}$, $t \in [0,T]$. Then* $\operatorname{Var}\big(\langle b, X_t^I\rangle_H\big) = \frac{\mathbb{1}_I(b)\,|\mu_b|^2\,(e^{2\lambda_b t}-1)}{2\lambda_b}$.

Proof. Observe that it holds $\mathbb{P}$-a.s. that

$$\begin{aligned}\langle b, X_t^I\rangle_H &= \left\langle b, \int_0^t e^{A(t-s)} P_I B\, dW_s \right\rangle_H = \int_0^t \left\langle P_I\, e^{A(t-s)}\, b, B\, dW_s\right\rangle_H \\ &= \mathbb{1}_I(b) \int_0^t e^{\lambda_b(t-s)}\, \langle b, B\, dW_s\rangle_H = \mathbb{1}_I(b)\, \mu_b \int_0^t e^{\lambda_b(t-s)}\, \langle b, dW_s\rangle_H .\end{aligned} \tag{94}$$

This and Itô's isometry yield that

$$\operatorname{Var}\big(\langle b, X_t^I\rangle_H\big) = \mathbb{1}_I(b)\,|\mu_b|^2 \int_0^t e^{2\lambda_b(t-s)}\, ds = \frac{\mathbb{1}_I(b)\,|\mu_b|^2\,\big(e^{2\lambda_b t}-1\big)}{2\lambda_b}. \tag{95}$$

The proof of Lemma 6.1 is thus completed. ☐

The next elementary result, Lemma 6.2, is an immediate consequence of Lemma 6.1 above.

Lemma 6.2. *Assume the setting in Section 6.1, let $I \in \mathcal{P}(\mathbb{H})$, and let $\varphi\colon H \to [0,\infty)$ fulfill that for all $x \in H$ it holds that $\varphi(x) = \|x\|_H^2$. Then $\varphi \in C^\infty(H,[0,\infty))$ and*

$$\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] - \mathbb{E}\big[\varphi(X_T^I)\big] = \mathbb{E}\big[\|X_T^{\mathbb{H}\setminus I}\|_H^2\big] \geq \left[\tfrac{1-e^{-2T\inf_{b\in\mathbb{H}}|\lambda_b|}}{2}\right]\left[\textstyle\sum_{b\in\mathbb{H}\setminus I} \frac{|\mu_b|^2}{|\lambda_b|}\right]. \tag{96}$$

Lemma 6.2 establishes a lower bound in the case of the squared norm as the test function. The next result, Lemma 6.3, establishes a similar lower bound for a test function in $C_b^4(H,\mathbb{R})$.

Lemma 6.3. *Assume the setting in Section 6.1, let $I \in \mathcal{P}(\mathbb{H})$ and let $\varphi\colon H \to \mathbb{R}$ be given by $\varphi(v) = \exp(-\|v\|_H^2)$ for all $v \in H$. Then $\varphi \in C_b^4(H,\mathbb{R})$ and*

$$\mathbb{E}\big[\varphi(X_T^I)\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] \geq \tfrac{\mathbb{E}[\varphi(X_T^{\mathbb{H}})]\,\mathbb{E}[\|X_T^{\mathbb{H}\setminus I}\|_H^2]}{2\,(1+\mathbb{E}[\|X_T^{\mathbb{H}\setminus I}\|_H^2])^{3/2}} \geq \tfrac{\mathbb{E}[\varphi(X_T^{\mathbb{H}})]\,\left(1-e^{-2T\inf_{b\in\mathbb{H}}|\lambda_b|}\right)}{4\,(1+\mathbb{E}[\|X_T^{\mathbb{H}\setminus I}\|_H^2])^{3/2}} \left[\sum_{b\in\mathbb{H}\setminus I} \tfrac{|\mu_b|^2}{|\lambda_b|}\right].$$

Proof. First of all, observe that for all $x, u_1, u_2, u_3, u_4 \in H$ it holds that

$$\varphi^{(1)}(x)(u_1) = -2\,\varphi(x)\,\langle x, u_1\rangle_H , \tag{97}$$

$$\begin{aligned}\varphi^{(2)}(x)(u_1,u_2) &= -2\left[\varphi^{(1)}(x)(u_2)\,\langle x, u_1\rangle_H + \varphi(x)\,\langle u_2, u_1\rangle_H\right] \\ &= -2\,\varphi(x)\left[\langle u_1, u_2\rangle_H - 2\,\langle x, u_1\rangle_H\,\langle x, u_2\rangle_H\right],\end{aligned} \tag{98}$$

$$\begin{aligned}\varphi^{(3)}(x)(u_1,u_2,u_3) &= -2\left[\varphi^{(1)}(x)\big(\langle u_3, u_1\rangle_H\, u_2 + \langle u_2, u_1\rangle_H\, u_3\big)\right. \\ &\qquad \left. + \varphi^{(2)}(x)(u_2,u_3)\,\langle x, u_1\rangle_H\right],\end{aligned} \tag{99}$$

$$\begin{aligned}\varphi^{(4)}(x)(u_1,u_2,u_3,u_4) &= -2\big[\varphi^{(3)}(x)(u_2,u_3,u_4)\,\langle x, u_1\rangle_H + \varphi^{(2)}(x)(u_2,u_3)\,\langle u_4, u_1\rangle_H \\ &\qquad + \varphi^{(2)}(x)\big(\langle u_3, u_1\rangle_H\, u_2 + \langle u_2, u_1\rangle_H\, u_3, u_4\big)\big].\end{aligned} \tag{100}$$

Identity (97) and the fact that for all $r \in [0,\infty)$ it holds that $\sup_{x\in H}\left([1+\|x\|_H^r]\,\varphi(x)\right) < \infty$ show that for all $r \in [0,\infty)$ it holds that $\sup_{x\in H}\left([1+\|x\|_H^r]\left[\varphi(x) + \|\varphi^{(1)}(x)\|_{L(H,\mathbb{R})}\right]\right) < \infty$. This and identity (98) imply that for all $r \in [0,\infty)$ it holds that

$$\sup_{x\in H}\left([1+\|x\|_H^r]\left[\textstyle\sum_{k=1}^2 \|\varphi^{(k)}(x)\|_{L^{(k)}(H,\mathbb{R})}\right]\right) < \infty. \tag{101}$$

This and (99) yield that for all $r \in [0,\infty)$ it holds that

$$\sup_{x\in H}\left(\left[1+\|x\|_H^r\right]\left[\textstyle\sum_{k=1}^3 \|\varphi^{(k)}(x)\|_{L^{(k)}(H,\mathbb{R})}\right]\right) < \infty. \tag{102}$$

This and (100) prove that $\varphi \in C_b^4(H,\mathbb{R})$. Next observe that for all $\sigma \in \mathbb{R}$ it holds that

$$\begin{aligned}&\int_{\mathbb{R}} \exp\!\left(-[\sigma x]^2\right)\cdot \frac{1}{\sqrt{2\pi}}\exp\!\left(-\frac{x^2}{2}\right)dx = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}\exp\!\left(-\frac{x^2}{2}\left[1+2\sigma^2\right]\right)dx\\ &= \frac{1}{[1+2\sigma^2]^{1/2}}\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi}}\exp\!\left(-\frac{x^2}{2}\right)dx = \frac{1}{\sqrt{1+2\sigma^2}}.\end{aligned} \tag{103}$$

This and Lemma 6.1 imply that

$$\begin{aligned}&\mathbb{E}\big[\varphi(X_T^I)\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]\\ &= \prod_{b\in I}\left[1+\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-1/2} - \prod_{b\in \mathbb{H}}\left[1+\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-1/2}\\ &= \prod_{b\in I}\left[1+\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-1/2}\left[1-\prod_{b\in \mathbb{H}\setminus I}\left[1+\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-1/2}\right]\\ &\geq \prod_{b\in \mathbb{H}}\left[1+\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-1/2}\left[1-\left[\prod_{b\in \mathbb{H}\setminus I}\left[1+\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]\right]^{-1/2}\right]\\ &\geq \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]\left[1-\left[1+\sum_{b\in\mathbb{H}\setminus I}\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-1/2}\right].\end{aligned} \tag{104}$$

In the next step we note that the fundamental theorem of calculus ensures that for all $x \in [0,\infty)$ it holds that $1-[1+x]^{-1/2} = \frac{1}{2}\int_0^x [1+y]^{-3/2}\,dy \geq \frac{1}{2}x\,[1+x]^{-3/2}$. Combining this with (104) and Lemma 6.1 proves that

$$\begin{aligned}&\mathbb{E}\big[\varphi(X_T^I)\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]\\ &\geq \frac{\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]}{2}\left[\sum_{b\in\mathbb{H}\setminus I}\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]\left[1+\sum_{b\in\mathbb{H}\setminus I}\tfrac{|\mu_b|^2}{\lambda_b}\left(e^{2\lambda_b T}-1\right)\right]^{-3/2}\\ &\geq \frac{\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]\,\mathbb{E}\big[\|X_T^{\mathbb{H}\setminus I}\|_H^2\big]}{2\left(1+\mathbb{E}\big[\|X_T^{\mathbb{H}\setminus I}\|_H^2\big]\right)^{3/2}}.\end{aligned} \tag{105}$$

This and Lemma 6.2 complete the proof of Lemma 6.3. □

Proposition 6.4 (A more concrete lower bound for the weak error)**.** *Assume the setting in Section 6.1, let $b\colon \mathbb{N}\to\mathbb{H}$ be a bijective function, let $I\in\mathcal{P}(\mathbb{H})$, $N\in\mathbb{N}$, $c,\rho\in(0,\infty)$, $\delta\in(-\infty,\frac{1}{2}-\frac{1}{2\rho})$ satisfy that for all $n\in\mathbb{N}$ it holds that $\lambda_{b_n} = -c\,n^\rho$ and $\mu_{b_n} = |\lambda_{b_n}|^\delta$, and let $\varphi\colon H\to\mathbb{R}$ be given by $\varphi(v) = \exp(-\|v\|_H^2)$ for all $v\in H$. Then $\varphi\in C_b^5(H,\mathbb{R})$, $B\in\cap_{r\in(-\infty,-\frac{1}{2}[1/\rho+2\delta])}HS(H,H_r)$ and*

$$\mathbb{E}\big[\varphi(X_T^{\{b_1,\dots,b_N\}})\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] \geq \frac{\mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big]\left(1-e^{-2Tc}\right)|\lambda_{b_N}|^{-(1-[1/\rho+2\delta])}}{2^{(1-2\delta\rho+\rho)}\,c^{1/\rho}\left(\rho-2\delta\rho+c^{(2\delta-1)}\,(\rho-2\delta\rho-1)^{-1/3}\right)^{3/2}}. \tag{106}$$

Proof. First of all, observe that Lemma 6.3 ensures that $\varphi \in C_b^5(H,\mathbb{R})$ and that

$$\mathbb{E}\big[\varphi(X_T^{\{b_1,\dots,b_N\}})\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] \geq \tfrac{\mathbb{E}\left[\varphi(X_T^{\mathbb{H}})\right]\left(1-e^{-2Tc}\right)c^{(2\delta-1)}}{4\left(1+c^{(2\delta-1)}\sum_{n=N+1}^{\infty}n^{\rho(2\delta-1)}\right)^{3/2}}\left[\sum_{n=N+1}^{\infty}n^{\rho(2\delta-1)}\right]. \tag{107}$$

Next note that the assumption that $\delta < \frac{1}{2} - \frac{1}{2\rho}$ ensures that $\rho\,(2\delta-1) < -1$. This, in turn, implies that

$$\begin{aligned}
&\sum_{n=N+1}^{\infty} n^{\rho\,(2\delta-1)} = \sum_{n=N+1}^{\infty}\int_n^{n+1}\frac{1}{n^{\rho\,(1-2\delta)}}\,dx \geq \sum_{n=N+1}^{\infty}\int_n^{n+1}\frac{1}{x^{\rho\,(1-2\delta)}}\,dx \\
&= \int_{N+1}^{\infty} x^{\rho\,(2\delta-1)}\,dx = \frac{-(N+1)^{[1+\rho(2\delta-1)]}}{[1+\rho\,(2\delta-1)]} \geq \frac{(2N)^{\rho\,(1/\rho+2\delta-1)}}{[\rho\,(1-2\delta)-1]} \\
&= \frac{\left[2^{\rho}/c\right]^{(1/\rho+2\delta-1)}|\lambda_{b_N}|^{(1/\rho+2\delta-1)}}{[\rho\,(1-2\delta)-1]}.
\end{aligned} \tag{108}$$

Putting this into (107) proves that

$$\mathbb{E}\big[\varphi(X_T^{\{b_1,\dots,b_N\}})\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] \geq \tfrac{\mathbb{E}\left[\varphi(X_T^{\mathbb{H}})\right]\left(1-e^{-2Tc}\right)2^{(1+2\delta\rho-\rho)}\,|\lambda_{b_N}|^{(1/\rho+2\delta-1)}}{4\,c^{1/\rho}\,(\rho-2\delta\rho-1)\left(1+c^{(2\delta-1)}\sum_{n=N+1}^{\infty}n^{\rho(2\delta-1)}\right)^{3/2}}. \tag{109}$$

This and the fact that

$$\begin{aligned}
&\sum_{n=N+1}^{\infty} n^{\rho\,(2\delta-1)} = \sum_{n=N+1}^{\infty}\int_{n-1}^{n}\frac{1}{n^{\rho\,(1-2\delta)}}\,dx \leq \sum_{n=N+1}^{\infty}\int_{n-1}^{n}\frac{1}{x^{\rho\,(1-2\delta)}}\,dx \\
&= \int_{N}^{\infty} x^{\rho\,(2\delta-1)}\,dx = \frac{-N^{[1+\rho(2\delta-1)]}}{[1+\rho\,(2\delta-1)]} = \frac{N^{(1+2\delta\rho-\rho)}}{(\rho-2\delta\rho-1)} \leq \frac{1}{(\rho-2\delta\rho-1)}
\end{aligned} \tag{110}$$

complete the proof of Proposition 6.4. ☐

In the next result, Corollary 6.4, we specialise Proposition 6.4 to the case where $\rho = 2$, $c = \pi^2$ (we think of A being, e.g., the Laplacian with Dirichlet boundary conditions on $H = L^2((0,1);\mathbb{R})$) and $\delta \in (-\infty, 1/4)$.

Corollary 6.5. *Assume the setting in Section 6.1, let $b\colon \mathbb{N} \to \mathbb{H}$ be a bijective function, let $I \in \mathcal{P}(\mathbb{H})$, $N \in \mathbb{N}$, $\delta \in (-\infty, 1/4)$ satisfy that for all $n \in \mathbb{N}$ it holds that $\lambda_{b_n} = -\pi^2 n^2$ and $\mu_{b_n} = |\lambda_{b_n}|^{\delta}$, and let $\varphi\colon H \to \mathbb{R}$ be given by $\varphi(v) = \exp(-\|v\|_H^2)$ for all $v \in H$. Then $\varphi \in C_b^5(H,\mathbb{R})$, $B \in \cap_{r\in(-\infty,-\frac{1}{2}[1/2+2\delta])} HS(H,H_r)$ and*

$$\mathbb{E}\big[\varphi(X_T^{\{b_1,\dots,b_N\}})\big] - \mathbb{E}\big[\varphi(X_T^{\mathbb{H}})\big] \geq \left[\tfrac{\mathbb{E}\left[\varphi(X_T^{\mathbb{H}})\right]2^{(4\delta-5)}\,(1-e^{-T})}{\left(2-4\delta+2^{(7\delta-7)}\,(1-4\delta)^{-1/3}\right)^{3/2}}\right]|\lambda_{b_N}|^{-(1-[1/2+2\delta])}. \tag{111}$$

Acknowledgements

Jarred Foster is gratefully acknowledged for a few helpful suggestions regarding English wording. Xiaojie Wang is gratefully acknowledged for bringing a few typos into our notice.

References

[1] Andersson, A., Jentzen, A., and Larsson, S. Existence, uniqueness and regularity for stochastic evolution equations with rough initial values and Kolmogorov equations infinite dimensions. *working paper* (2014).

[2] Andersson, A., Kruse, R., and Larsson, S. Duality in refined Watanabe-Sobolev spaces and weak approximations of SPDE. *arXiv:1312.5893* (2013), 28 pages.

[3] Andersson, A., and Larsson, S. Weak convergence for a spatial approximation of the nonlinear stochastic heat equation. *arXiv:1212.5564* (2013), 19 pages.

[4] Bréhier, C.-E. Approximation of the invariant measure with an Euler scheme for stochastic PDEs driven by space-time white noise. *Potential Analysis 40*, 1 (2014), 1–40.

[5] Bréhier, C.-E., and Kopec, M. Approximation of the invariant law of SPDEs: error analysis using a poisson equation for a full-discretization scheme. *arXiv:1311.7030* (2013), 50 pages.

[6] Brzeźniak, Z. On stochastic convolution in Banach spaces and applications. *Stochastics Stochastics Rep. 61*, 3-4 (1997), 245–295.

[7] Da Prato, G. Kolmogorov equations for stochastic PDEs with multiplicative noise. *Stochastic Anal. Appl. 2* (2007), 235–263.

[8] Da Prato, G., Jentzen, A., and Röckner, M. A mild Itô formula for SPDEs. *arXiv:1009.3526* (2012).

[9] Da Prato, G., and Zabczyk, J. *Stochastic equations in infinite dimensions*, vol. 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[10] Davie, A. M., and Gaines, J. G. Convergence of numerical schemes for the solution of parabolic stochastic partial differential equations. *Math. Comp. 70*, 233 (2001), 121–134 (electronic).

[11] de Bouard, A., and Debussche, A. Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation. *Appl. Math. Optim. 54*, 3 (2006), 369–399.

[12] Debussche, A. Weak approximation of stochastic partial differential equations: the nonlinear case. *Math. Comp. 80*, 273 (2011), 89–117.

[13] Debussche, A., and Printems, J. Weak order for the discretization of the stochastic heat equation. *Math. Comp. 78*, 266 (2009), 845–863.

[14] Dörsek, P. Semigroup splitting and cubature approximations for the stochastic Navier-Stokes equations. *SIAM J. Numer. Anal. 50*, 2 (2012), 729–746.

[15] Geissert, M., Kovács, M., and Larsson, S. Rate of weak convergence of the finite element method for the stochastic heat equation with additive noise. *BIT 49*, 2 (2009), 343–356.

[16] Hausenblas, E. Weak approximation for semilinear stochastic evolution equations. In *Stochastic analysis and related topics VIII*, vol. 53 of *Progr. Probab.* Birkhäuser, Basel, 2003, pp. 111–128.

[17] Hausenblas, E. Weak approximation of the stochastic wave equation. *J. Comput. Appl. Math. 235*, 2 (2010), 33–58.

[18] Henry, D. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[19] Jentzen, A. Numerical analysis of stochastic partial differential equations. Lecture notes, ETH Zurich, summer semester 2014, available online at http://www.math.ethz.ch/education/bachelor/lectures/fs2014/math/numsol, 2014.

[20] Kloeden, P. E., and Platen, E. *Numerical solution of stochastic differential equations*, vol. 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992.

[21] Kovács, M., Larsson, S., and Lindgren, F. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise. *BIT Numerical Mathematics* (2011), 24 pages.

[22] Kovács, M., Larsson, S., and Lindgren, F. Weak convergence of finite element approximations of linear stochastic evolution equations with additive noise II. Fully discrete schemes. *BIT 53*, 2 (2013), 497–525.

[23] Kruse, R. *Strong and weak approximation of semilinear stochastic evolution equations*, vol. 2093 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[24] Lindgren, F. On weak and strong convergence of numerical approximations of stochastic partial differential equations. *PhD thesis, Chalmers University of Technology and University of Gothenburg* (2012), 60 pages.

[25] Lindner, F., and Schilling, R. L. Weak order for the discretization of the stochastic heat equation driven by impulsive noise. *Potential Anal. 38*, 2 (2013), 345–379.

[26] Milstein, G. N. *Numerical integration of stochastic differential equations*, vol. 313 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1995. Translated and revised from the 1988 Russian original.

[27] Müller-Gronbach, T., and Ritter, K. Lower bounds and nonuniform time discretization for approximation of stochastic heat equations. *Found. Comput. Math. 7*, 2 (2007), 135–181.

[28] Müller-Gronbach, T., and Ritter, K. Minimal errors for strong and weak approximation of stochastic differential equations. In *Monte Carlo and quasi-Monte Carlo methods 2006*. Springer, Berlin, 2008, pp. 53–82.

[29] Müller-Gronbach, T., Ritter, K., and Wagner, T. Optimal pointwise approximation of a linear stochastic heat equation with additive space-time white noise. In *Monte Carlo and quasi-Monte Carlo methods 2006*. Springer, Berlin, 2007, pp. 577–589.

[30] Müller-Gronbach, T., Ritter, K., and Wagner, T. Optimal pointwise approximation of infinite-dimensional Ornstein-Uhlenbeck processes. *Stoch. Dyn. 8*, 3 (2008), 519–541.

[31] Rössler, A. *Runge-Kutta Methods for the Numerical Solution of Stochastic Differential Equations*. Shaker Verlag, Aachen, 2003. Dissertation, Technische Universität Darmstadt, Darmstadt, Germany.

[32] Shardlow, T. Weak convergence of a numerical method for a stochastic heat equation. *BIT 43*, 1 (2003), 179–193.

[33] van Neerven, J. M. A. M., Veraar, M. C., and Weis, L. Stochastic evolution equations in UMD Banach spaces. *J. Funct. Anal. 255*, 4 (2008), 940–993.

[34] Wang, X. An exponential integrator scheme for time discretization of nonlinear stochastic wave equation. *arXiv:1103.4504v1* (2013), 27 pages.

[35] Wang, X. Weak error estimates of the exponential Euler scheme for semi-linear SPDEs without Malliavin calculus. *arXiv:1408.0713* (2014), 16 pages.

[36] Wang, X., and Gan, S. Weak convergence analysis of the linear implicit Euler method for semilinear stochastic partial differential equations with additive noise. *J. Math. Anal. Appl. 398*, 1 (2013), 151–169.

Recent Research Reports

Nr.	Authors/Title
2014-08	P. Grohs and M. Sprecher and T. Yu Scattered Manifold-Valued Data Approximation
2014-09	P. Grohs and U. Wiesmann and Z. Kereta A Shearlet-Based Fast Thresholded Landweber Algorithm for Deconvolution
2014-10	P. Grohs and S. Vigogna Intrinsic Localization of Anisotropic Frames II: α-Molecules
2014-11	S. Etter and P. Grohs and A. Obermeier FFRT - A Fast Finite Ridgelet Transform for Radiative Transport
2014-12	A. Paganini Approximate Shape Gradients for Interface Problems
2014-13	E. Fonn and P. Grohs and R. Hiptmair Polar Spectral Scheme for the Spatially Homogeneous Boltzmann Equation
2014-14	J. Dick and F.Y. Kuo and Q.T. Le Gia and Ch. Schwab Multi-level higher order QMC Galerkin discretization for affine parametric operator equations
2014-15	Ch. Schwab Exponential convergence of simplicial <i>hp</i>-FEM for <i>H</i> ¹-functions with isotropic singularities
2014-16	P. Grohs and S. Keiper and G. Kutyniok and M. Schaefer α-Molecules
2014-17	A. Hiltebrand and S. Mishra Efficient computation of all speed flows using an entropy stable shock-capturing space-time discontinuous Galerkin method