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Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients

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Abstract

Many stochastic differential equations (SDEs) in the literature have a superlinearly growing nonlinearity in their drift or diffusion coefficient. Unfortunately, moments of the computationally efficient Euler-Maruyama approximation method diverge for these SDEs in finite time. This article develops a general theory based on rare events for studying integrability properties such as moment bounds for discrete-time stochastic processes. Using this approach, we establish moment bounds for fully and partially drift-implicit Euler methods and for a class of new explicit approximation methods which require only a few more arithmetical operations than the Euler-Maruyama method. These moment bounds are then used to prove strong convergence of the proposed schemes. Finally, we illustrate our results for several SDEs from finance, physics, biology and chemistry.

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CHAPTER 1

Introduction

This article investigates integrability and convergence properties of numerical approximation processes for stochastic differential equations (SDEs). In order to illustrate one of our main results, the following general setting is considered in this introductory chapter. Let $T \in (0,\infty)$, $d, m \in \mathbb{N} := \{1, 2, \ldots\}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $D \subset \mathbb{R}^d$ be an open set, let $\mu = (\mu_1, \ldots, \mu_d) \colon D \to \mathbb{R}^d$ and $\sigma = (\sigma_{i,j})_{i\in\{1,2,\ldots,d\}, j\in\{1,2,\ldots,m\}} \colon D \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions and let $X \colon [0,T] \times \Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths satisfying the SDE

$$X_t = X_0 + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s \tag{1.1}$$

$\mathbb{P}$-almost surely for all $t \in [0,T]$. Here μ is the infinitesimal mean and $\sigma \cdot \sigma^*$ is the infinitesimal covariance matrix of the solution process X of the SDE (1.1). To guarantee finiteness of some moments of the SDE (1.1), we assume existence of a Lyapunov-type function. More precisely, let $q \in (0,\infty)$, $\kappa \in \mathbb{R}$ be real numbers and let $V \colon D \to [1,\infty)$ be a twice continuously differentiable function with $\mathbb{E}[V(X_0)] < \infty$ and with $V(x) \geq \|x\|^q$ and

$$\sum_{i=1}^{d} \left(\frac{\partial V}{\partial x_i}\right)(x) \cdot \mu_i(x) + \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=1}^{m} \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)(x) \cdot \sigma_{i,k}(x) \cdot \sigma_{j,k}(x) \leq \kappa \cdot V(x) \tag{1.2}$$

for all $x \in D$. These assumptions ensure

$$\mathbb{E}\big[V(X_t)\big] \leq e^{\kappa t} \cdot \mathbb{E}\big[V(X_0)\big] \tag{1.3}$$

for all $t \in [0,T]$ and, therefore, finiteness of the q-th absolute moments of the solution process X_t, $t \in [0,T]$, of the SDE (1.1), i.e., $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^q\big] < \infty$. Note that in this setting both the drift coefficient μ and the diffusion coefficient σ of the SDE (1.1) may grow superlinearly and are, in particular, not assumed to be globally Lipschitz continuous. Our main goal in this introduction is to construct and to analyze numerical approximation processes that converge strongly to the exact solution of the SDE (1.1). The standard literature in computational stochastics (see, for instance, Kloeden & Platen [**47**] and Milstein [**61**]) concentrates on SDEs with globally Lipschitz continuous coefficients and can therefore not be applied here. Strong numerical approximations of the SDE (1.1) are of particular interest for the computation of statistical quantities of the solution process of the SDE (1.1) through computationally efficient multilevel Monte Carlo methods (see Giles [**20**], Heinrich [**28**] and, e.g., in Creutzig et al. [**13**], Hickernell et al. [**31**], Barth, Lang & Schwab [**5**] and in the references therein for further recent results on multilevel Monte Carlo methods).

Several SDEs from the literature satisfy the above setting (see Sections 4.2–4.11 below). For instance, the function $V(x) = \big(1 + \|x\|^2\big)^r$, $x \in D$, for an arbitrary $r \in (0,\infty)$ serves as a Lyapunov-type function for the stochastic van der Pol oscillator (4.4), for the stochastic Lorenz equation (4.20), for the Cox-Ingersoll-Ross process (4.74) and for the simplified Ait-Sahalia interest rate model (4.75) but not for the stochastic Duffing-van der Pol oscillator (4.13), not for the stochastic Brusselator (4.23), not for the stochastic SIR model (4.31), not for the Lotka-Volterra predator prey model (4.60) and, in general, also not for the Langevin equation (4.89). The function $V(x) = \big(1 + (x_1)^4 + 2(x_2)^2\big)^r$, $x = (x_1, x_2) \in D = \mathbb{R}^2$, for an arbitrary $r \in (0,\infty)$ is a Lyapunov-type function for the stochastic Duffing-van der Pol oscillator (4.13). For the stochastic SIR model (4.31), the function $V(x) = \big(1 + (x_1 + x_2)^2 + (x_3)^2\big)^r$, $x = (x_1, x_2, x_3) \in D = (0,\infty)^3$, for an arbitrary $r \in (0,\infty)$ serves as a Lyapunov-type function and for the stochastic Lotka-Volterra system (4.48), the function $V(x) = \big(1 + (v_1 x_1 + \dots v_d x_d)^2\big)^r$, $x \in D = (0,\infty)^d$, for an arbitrary $r \in (0,\infty)$ and an appropriate $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ is a Lyapunov-type function. More details on the examples can be found in Chapter 4.

The standard method for approximating SDEs with globally Lipschitz continuous coefficients is the Euler-Maruyama method. Unfortunately, the Euler-Maruyama method often fails to converge strongly to the exact solution of nonlinear SDEs of the form (1.1); see [**40**]. Indeed, if at least one of the coefficients of the SDE grows superlinearly, then the Euler-Maruyama scheme diverges in the strong sense. More precisely, let $Z^N \colon \{0, 1, \dots, N\} \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be Euler-Maruyama approximations for the SDE (1.1) defined recursively through $Z_0^N := X_0$ and

$$Z_{n+1}^N := Z_n^N + \bar{\mu}(Z_n^N)\tfrac{T}{N} + \bar{\sigma}(Z_n^N)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big) \tag{1.4}$$

for all $n \in \{0, 1, \dots, N-1\}$ and all $N \in \mathbb{N}$. Here $\bar{\mu} \colon \mathbb{R}^d \to \mathbb{R}^d$ and $\bar{\sigma} \colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$ are extensions of μ and σ given by $\bar{\mu}(x) = 0$, $\bar{\sigma}(x) = 0$ for all $x \in D^c$ and by $\bar{\mu}(x) = \mu(x)$, $\bar{\sigma}(x) = \sigma(x)$ for all $x \in D$ respectively (see also Sections 3.2 and 3.6 for more general extensions). Theorem 2.1 of [**39**] (which generalizes Theorem 2.1 of [**40**]) then implies in the case $d = m = 1$ that if there exists a real number $\varepsilon \in (0,\infty)$ such that $|\bar{\mu}(x)| + |\bar{\sigma}(x)| \geq \varepsilon |x|^{(1+\varepsilon)}$ for all $|x| \geq 1/\varepsilon$ and if $\mathbb{P}[\sigma(X_0) \neq 0] > 0$, then $\lim_{N\to\infty} \mathbb{E}\big[|Y_N^N|^r\big] = \infty$ for all $r \in (0,\infty)$ and therefore $\lim_{N\to\infty} \mathbb{E}\big[|X_T - Y_N^N|^r\big] = \infty$ for all $r \in (0, q]$ (see also Sections 4 and 5 in [**39**] for divergence results for the corresponding multilevel Monte Carlo Euler method). Due to these deficiencies of the Euler-Maruyama method, we look for numerical approximation methods whose computational cost is close to that of the Euler-Maruyama method and which converge strongly even in the case of SDEs with superlinearly growing coefficients.

There are a number of strong convergence results for temporal numerical approximations of SDEs of the form (1.1) with possibly superlinearly growing coefficients in the literature. Many of these results assume beside other assumptions that the drift coefficient μ of the SDE (1.1) is globally one-sided Lipschitz continuous and also prove rates of convergence in that case. In particular, if the drift coefficient is globally one-sided Lipschitz continuous and if the diffusion coefficient is globally Lipschitz continuous beside other assumptions, then strong convergence of the fully drift-implicit Euler method follows from Theorem 2.4 in Hu [**36**] and from Theorem 5.3 in Higham, Mao & Stuart [**34**], strong convergence of the split-step backward Euler method follows from Theorem 3.3 in Higham, Mao & Stuart [**34**], strong convergence of a drift-tamed Euler-Maruyama method follows from Theorem 1.1 in

[**38**] and strong convergence of a drift-tamed Milstein scheme follows from Theorem 3.2 in Gan & Wang [**19**]. Theorem 2 and Theorem 3 in Higham & Kloeden [**33**] generalize Theorem 3.3 and Theorem 5.3 in Higham, Mao & Stuart [**34**] to SDEs with Poisson-driven jumps. In addition, Theorem 6.2 in Szpruch et al. [**77**] establishes strong convergence of the fully drift-implicit Euler method of a one-dimensional Ait-Sahalia-type interest rate model having a superlinearly growing diffusion coefficient σ and a globally one-sided Lipschitz continuous drift coefficient μ which is unbounded near 0. Moreover, Theorem 4.4 in Mao & Szpruch [**56**] generalizes this result to a class of SDEs which have globally one-sided Lipschitz continuous drift coefficients and in which the function $V(x) = 1 + \|x\|^2$, $x \in D$, is a Lyapunov-type function (see also Mao & Szpruch [**57**] for related results but with rates of convergence). A similar method is used in Proposition 3.3 in Dereich, Neuenkirch & Szpruch [**17**] to obtain strong convergence of a drift-implicit Euler method for a class of Bessel type processes. Moreover, Gyöngy & Millet establish in Theorem 2.10 in [**25**] strong convergence of implicit numerical approximation processes for a class of possibly infinite dimensional SDEs whose drift μ and diffusion σ satisfy a suitable one-sided Lipschtz condition (see Assumption (C1) in [**25**] for details). Strong convergence of temporal numerical approximations for two-dimensional stochastic Navier-Stokes equations is obtained in Theorem 7.1 in Brzeźniak, Carelli & Prohl [**10**]. In all of the above mentioned results from the literature, the function $V(x) = 1 + \|x\|^2$, $x \in D$, is a Lyapunov-type function of the considered SDE. A result on more general Lyapunov-type functions is the framework in Schurz [**76**] which assumes general abstract conditions on the numerical approximations. The applicability of this framework is demonstrated in the case of SDEs which have globally one-sided Lipschitz continuous drift coefficients and in which the function $V(x) = 1 + \|x\|^2$, $x \in D$, is a Lyapunov-type function; see [**74, 75, 76**]. To the best of our knowledge, no strong numerical approximation results are known for the stochastic van der Pol oscillator (4.4), for the stochastic Duffing-van der Pol oscillator (4.13), for the stochastic Lorenz equation (4.20), for the stochastic Brusselator (4.23), for the stochastic SIR model (4.31), for the experimental psychology model (4.40) and for the Lotka-Volterra predator-prey model (4.48).

In this article, the following increment-tamed Euler-Maruyama scheme is proposed to approximate the solution process of the SDE (1.1) in the strong sense. Let $Y^N\colon \{0, 1 \ldots, N\} \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be numerical approximation processes defined through $Y_0^N := X_0$ and

$$Y_{n+1}^N := Y_n^N + \frac{\bar{\mu}\big(Y_n^N\big)\frac{T}{N} + \bar{\sigma}\big(Y_n^N\big)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)}{\max\big(1, \frac{T}{N}\|\bar{\mu}(Y_n^N)\frac{T}{N} + \bar{\sigma}(Y_n^N)\big(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big)\|\big)} \tag{1.5}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Note that the computation of (1.5) requires only a few additional arithmetical operations when compared to the computation of the Euler-Maruyama approximations (1.4). Moreover, we emphasize that the scheme (1.5) is a special case of a more general class of suitable tamed schemes proposed in Subsection 3.6.3 below. Next let $\bar{Y}^N\colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be linearly interpolated continuous-time versions of (1.5) defined through $\bar{Y}_t^N := \big(n+1-\frac{tN}{T}\big)Y_n^N + \big(\frac{tN}{T}-n\big)Y_{n+1}^N$ for all $t \in [nT/N, (n+1)T/N]$, $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. For proving strong convergence of the numerical approximation processes $\bar{Y}^N$, $N \in \mathbb{N}$, to the exact solution process X of the SDE (1.1), we additionally assume that $\mathbb{E}\big[\|X_0\|^r\big] < \infty$ for all $r \in [0, \infty)$ and that there exist real numbers $\gamma_0, \gamma_1, c \in [0, \infty)$, $p \in [3, \infty)$ and a three times continuously differentiable extension $\bar{V} \colon \mathbb{R}^d \to [1, \infty)$ of $V \colon D \to [1, \infty)$ such that $\bar{V}(x) \geq \|x\|^q$, $\|\bar{V}^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d, \mathbb{R})} \leq c\, |\bar{V}(x)|^{\left[1 - \frac{i}{p}\right]}$ and

$$\|\bar{\mu}(x)\| \leq c\, |\bar{V}(x)|^{\left[\frac{\gamma_0 + 1}{p}\right]} \qquad \text{and} \qquad \|\bar{\sigma}(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \leq c\, |\bar{V}(x)|^{\left[\frac{\gamma_1 + 2}{2p}\right]} \tag{1.6}$$

for all $x \in \mathbb{R}^d$ and all $i \in \{1, 2, 3\}$. These assumptions are satisfied in all of the example SDEs from Sections 4.2–4.9. In the case of the squared volatility process (4.63) in Section 4.10 and in case of the Langevin equation (4.89) in Section 4.11, these assumptions are also satisfied if the model parameters satisfy suitable regularity conditions (see Sections 4.10 and 4.11 for details). Under these assumptions, Theorem 3.15 below shows that

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\Big[\big\|X_t - \bar{Y}_t^N\big\|^r\Big] = 0 \tag{1.7}$$

for all $r \in (0, q)$ satisfying $r < \frac{p}{2\gamma_1 + 4 \max(\gamma_0, \gamma_1, 1/2)} - \frac{1}{2}$. Theorem 3.15 thereby proves strong convergence of the increment-tamed Euler-Maruyama method (1.5) for all example SDEs from Sections 4.2–4.9 and in parts also for the example SDEs from Sections 4.10–4.11. Moreover, using a whole family of Lyapunov-type functions, we will deduce from Theorem 3.15 for most of the examples of Chapter 4 that strong L^r-convergence (1.7) holds for all $r \in (0, \infty)$ (see Corollary 3.17 below for details). To the best of our knowledge, Theorem 3.15 is the first result in the literature that proves strong convergence of a numerical approximation method for the stochastic van der Pol oscillator (4.4), for the stochastic Duffing-van der Pol oscillator (4.13), for the stochastic Lorenz equation (4.20), for the stochastic Brusselator (4.23), for the stochastic SIR model (4.31), for the experimental psychology model (4.40) and for the stochastic Lotka-Volterra predator-prey model (4.60).

Theorem 3.15 proves the strong convergence (1.7) in a quite general setting. One may ask whether it is also possible to establish a strong convergence rate in this setting. There is a strong hint that this is not possible in this general setting. More precisely, Theorem 1.2 in Hairer et al. [**26**] shows that in this setting there exist SDEs with smooth and globally bounded coefficients whose solution processes are nowhere locally Hölder continuous in the strong mean square sense with respect to the initial values. This instability suggests that there exist SDEs with smooth and globally bounded coefficients for which there exist no one-step numerical approximation processes which converge in the strong sense with a convergence rate. In addition, Theorem 1.3 in Hairer et al. [**26**] proves that there exist SDEs with smooth and globally bounded coefficients to which the Euler-Maruyama scheme (see (1.4)) and other schemes such as the Milstein scheme converge in the strong mean square sense without any arbitrarily small positive rate of convergence. It remains an open question which conditions on the coefficients μ (more general than globally one-sided Lipschitz continuous) and σ of the SDE (1.1) are sufficient to ensure strong convergence of appropriate one-step numerical approximation processes to the exact solution of the SDE (1.1) with the standard strong convergence order 1/2 at least.

Finally, we summarize a few more results of this article. In Chapter 2, we establish uniform moment bounds of approximation processes for SDEs which are typically the first step in proving strong and numerically weak convergence results. In particular, Corollary 2.21 in Subsection 2.2.3 proves uniform moment bounds for the increment-tamed Euler-Maruyama scheme (1.5). Moreover, Corollary 2.27 in Subsection 2.3.1 yields uniform moment bounds for the fully drift-implicit Euler scheme and Lemma 2.28 in Subsection 2.3.2 establishes uniform moment bounds for partially drift-implicit approximation schemes. These results on uniform moment bounds are applications of a general theory which we develop in Section 2.1. In this theory (see Propositions 2.1 and 2.7 and Corollaries 2.2, 2.3 and 2.6) we assume a Lyapunov-type inequality to be satisfied by the approximation processes on large subevents of the probability space, i.e., on complements of *rare events*; see inequality (2.11) in Corollary 2.2. One of our main results (Theorem 2.13 in Subsection 2.2.1) establishes this Lyapunov-type condition for the Euler-Maruyama approximations (1.4). More precisely, whereas the Euler-Maruyama approximations often do not satisfy a Lyapunov-type inequality on events of probability one in the case of superlinearly growing coefficients according to Corollary 2.17 in Subsection 2.2.1, the Euler-Maruyama approximations do satisfy the Lyapunov-type inequality (2.11) on large subevents of the probability space according to Theorem 2.13 in Subsection 2.2.1. This integrability result on the Euler-Maruyama approximation processes can then be transfered to a large class of other one-step approximation processes. To be more precise, Lemma 2.18 in Subsection 2.2.2 proves that if two general one-step approximation schemes are close to each other in the sense of (2.102) (see Lemma 2.18 for the details) and if one approximation scheme satisfies the Lyapunov-type inequality (2.11) on large subevents, then the other approximation scheme satisfies the Lyapunov-type inequality (2.11) on large subevents of the probability space as well. After having established the Lyapunov-type inequality (2.11) on such complements of rare events, the general rare event based theory in Section 2.1 can be applied to derive moment bounds and further integrability properties of the approximation processes. In Chapter 3, we then proceed to study convergence in probability (see Section 3.3), strong convergence (see Section 3.4) and weak convergence (see Section 3.5) of approximation processes for SDEs. Definition 3.1 in Section 3.2 specifies a local consistency condition on approximation schemes which is, according to Theorem 3.3 in Section 3.3, sufficient for convergence in probability of the approximation processes to the exact solution of the SDE (1.1). This convergence in probability and the uniform moment bounds in Corollary 2.21 then result in the strong convergence (1.7) of the increment-tamed Euler-Maruyama approximations (1.5); see Theorem 3.15 in Subsection 3.4.3 for the details. Moreover, we obtain results for approximating moments and more general statistical quantities of solutions of SDEs of the form (1.1) in Section 3.5. In particular, Corollary 3.23 in Subsection 3.5.2 establishes convergence of the Monte Carlo Euler approximations for SDEs of the form (1.1).

1.1. Notation

Throughout this article, the following notation is used. For a set Ω, a measurable space $(E, \mathcal{E})$ and a mapping $Y\colon \Omega \to E$ we denote by $\sigma_\Omega(Y) := \{Y^{-1}(A) \subset \Omega\colon A \in \mathcal{E}\}$ the smallest sigma algebra with respect to which $Y\colon \Omega \to E$ is measurable. Furthermore, for a topological space $(E, \mathcal{E})$ we denote by $\mathcal{B}(E) := \sigma_E(\mathcal{E})$ the Borel sigma-algebra of $(E, \mathcal{E})$. Moreover, for a natural number $d \in \mathbb{N}$ and two sets

$A, B \subset \mathbb{R}^d$ we denote by

$$\operatorname{dist}(A,B) := \begin{cases} \inf\{\|a-b\| \in [0,\infty) \colon (a,b) \in A \times B\} & : A \neq \emptyset \text{ and } B \neq \emptyset \\ \infty & : \text{else} \end{cases} \tag{1.8}$$

the distance of A and B. In addition, for a natural number $d \in \mathbb{N}$, an element $x \in \mathbb{R}^d$ and a set $A \subset \mathbb{R}^d$ we denote by $\operatorname{dist}(x, A) := \operatorname{dist}(\{x\}, A)$ the distance of x and A. Throughout this article we also often calculate and formulate expressions in the extended positive real numbers $[0,\infty] = [0,\infty) \cup \{\infty\}$. For instance, we frequently use the conventions $\frac{a}{\infty} = 0$ for all $a \in [0,\infty)$, $\frac{a}{0} = \infty$ for all $a \in (0,\infty]$ and $0 \cdot \infty = 0$. Moreover, let $\chi_p \in [0,\infty)$, $p \in [1,\infty)$, be a family of real numbers such that for every $p \in [1,\infty)$, every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every stochastic processes $Z \colon \mathbb{N} \times \Omega \to \mathbb{R}$ with the property that $(\sum_{k=1}^n Z_k)_{n \in \mathbb{N}_0}$ is a martingale and every $N \in \mathbb{N}_0 := \{0, 1, \dots\}$ it holds that

$$\left\| \sup_{n \in \{0,1,\dots,N\}} \left| \sum\nolimits_{k=1}^n Z_k \right| \right\|^2_{L^p(\Omega;\mathbb{R})} \leq \chi_p \left(\sum\nolimits_{n=1}^N \|Z_n\|^2_{L^p(\Omega;\mathbb{R})} \right). \tag{1.9}$$

The Burkholder-Davis-Gundy inequality (see, e.g., Theorem 48 in Protter [**68**]) ensures that the real numbers $\chi_p \in [0,\infty)$, $p \in [1,\infty)$, in (1.9) do indeed exist. Next for two sets A and B we denote by $\mathcal{M}(A,B)$ the set of all mappings from A to B. Furthermore, for natural numbers $d, m \in \mathbb{N}$ and a $d \times m$-matrix $A \in \mathbb{R}^{d \times m}$ we denote by $A^* \in \mathbb{R}^{m \times d}$ the transpose of the matrix A. In addition, for $d, m \in \mathbb{N}$ and arbitrary functions $\mu = (\mu_1, \dots, \mu_d) \colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma = (\sigma_{i,j})_{i \in \{1,2,\dots,d\}, j \in \{1,2,\dots,m\}} = (\sigma_k)_{k \in \{1,2,\dots,m\}} \colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$ we denote by $\mathcal{G}_{\mu,\sigma} \colon C^2(\mathbb{R}^d, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^d, \mathbb{R})$ and $\tilde{\mathcal{G}}_{\mu,\sigma} \colon C^2(\mathbb{R}^d, \mathbb{R}) \to \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ linear operators defined through

$$\begin{aligned} &(\mathcal{G}_{\mu,\sigma}\varphi)(x) \\ &:= \varphi'(x)\,\mu(x) + \frac{1}{2}\sum_{k=1}^m \varphi''(x)\big(\sigma_k(x), \sigma_k(x)\big) \\ &= \langle \mu(x), (\nabla\varphi)(x) \rangle + \frac{1}{2}\operatorname{trace}\big(\sigma(x)\sigma(x)^* (\operatorname{Hess}\varphi)(x)\big) \\ &= \sum_{i=1}^d \left(\frac{\partial \varphi}{\partial x_i}\right)(x) \cdot \mu_i(x) + \frac{1}{2}\sum_{i,j=1}^d \sum_{k=1}^m \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)(x) \cdot \sigma_{i,k}(x) \cdot \sigma_{j,k}(x) \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &(\tilde{\mathcal{G}}_{\mu,\sigma}\varphi)(x,y) \\ &:= \varphi'(x)\,\mu(y) + \frac{1}{2}\sum_{k=1}^m \varphi''(x)\big(\sigma_k(y), \sigma_k(y)\big) \\ &= \langle \mu(y), (\nabla\varphi)(x) \rangle + \frac{1}{2}\operatorname{trace}\big(\sigma(y)\sigma(y)^* (\operatorname{Hess}\varphi)(x)\big) \\ &= \sum_{i=1}^d \left(\frac{\partial \varphi}{\partial x_i}\right)(x) \cdot \mu_i(y) + \frac{1}{2}\sum_{i,j=1}^d \sum_{k=1}^m \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)(x) \cdot \sigma_{i,k}(y) \cdot \sigma_{j,k}(y) \end{aligned} \tag{1.11}$$

for all $x, y \in \mathbb{R}^d$ and all $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$ where $\sigma_k \colon \mathbb{R}^d \to \mathbb{R}^d$, $k \in \{1, 2, \dots, m\}$, fulfill $\sigma_k(x) = (\sigma_{1,k}(x), \dots, \sigma_{d,k}(x))$ for all $x \in \mathbb{R}^d$ and all $k \in \{1, 2, \dots, m\}$. The linear operator in (1.10) is associated to the exact solution of the SDE (1.1) and the linear operator in (1.11) is associated to the Euler-Maruyama approximations of the SDE (1.1) (see, e.g., (2.53) in the proof of Lemma 2.10 below). Furthermore, for $d \in \mathbb{N}$ and a Borel measurable set $A \in \mathcal{B}(\mathbb{R}^d)$ we denote by $\lambda_A \colon \mathcal{B}(A) \to [0,\infty]$ the Lebesgue-Borel measure on $A \subset \mathbb{R}^d$. In addition, for $n, d \in \mathbb{N}$, $p \in (0,\infty]$ and a set $A \subset \mathbb{R}$ we denote by $C_p^n(\mathbb{R}^d, A)$ the set

(1.12)

$$\begin{aligned} &\mathcal{C}_p^n(\mathbb{R}^d, A) \\ &:= \left\{ f \in C^{n-1}(\mathbb{R}^d, A) \colon \begin{array}{c} f^{(n-1)} \text{ is locally Lipschitz continuous and there} \\ \text{exists a real number } c \in [0,\infty) \text{ such that for} \\ \lambda_{\mathbb{R}^d}\text{-almost all } x \in \mathbb{R}^d \text{ and all } i \in \{1,2,\dots,n\} \\ \text{we have } \|f^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \le c\,|f(x)|^{[1-i/p]} \end{array} \right\} \end{aligned}$$

throughout the rest of this article. Note that this definition is well-defined since Rademacher's theorem proves that a locally Lipschitz continuous function is almost everywhere differentiable.

CHAPTER 2

Integrability properties of approximation processes for SDEs

A central step in establishing strong and numerically weak convergence of approximation processes is to prove uniform moment bounds. For this analysis, we propose a Lyapunov-type condition on the one-step function of a one-step approximation scheme (see Definition 2.8 in Subsection 2.1.4). In Sections 2.2 and 2.3, we will show that many numerical approximation schemes including the Euler-Maruyama scheme, the increment-tamed Euler-Maruyama scheme (1.5) and some implicit approximation schemes satisfy this condition in the case of several nonlinear SDEs. Subject of Section 2.1 is to infer from this Lyapunov-type condition on the one-step function that the associated numerical approximations have certain uniform integrability properties.

2.1. General discrete-time stochastic processes

This section introduces a general approach for studying integrability and stability properties of discrete-time stochastic processes. We assume a Lyapunov-type estimate on a subevent of the probability space for each time step. From this, we derive Lyapunov-type estimates for the process uniformly in the time variable in Subsection 2.1.1. This approach is then applied to derive uniform moment bounds in finite time (see Subsection 2.1.3; see also Proposition 2.1 for infinite time) for a large class of possibly infinite dimensional approximation processes. Note that the state space $(E, \mathcal{E})$ appearing in Propositions 2.1 and 2.7 and in Corollaries 2.2, 2.3 and 2.6 is an arbitrary measurable space. In our examples in Sections 2.2 and 2.3 below we restrict ourself, however, to explicit (see Section 2.2) and implicit (see Section 2.3) approximation schemes for finite dimensional SDEs driven by standard Brownian motions. Our approach is mainly influenced by ideas in [**37, 9, 38**]; see the end of Subsection 2.1.4 for more details on these articles.

2.1.1. Lyapunov-type estimates on complements of rare events. The following result (Proposition 2.1) proves Lyapunov-type estimates for discrete-time stochastic processes which do, in general, not hold on the whole probability space but only on a family of typically large subevents of the probability space. These subevents are defined in terms of an appropriate *Lyapunov-type function* $V\colon E \to [0, \infty)$ on the measurable state space $(E, \mathcal{E})$ and in terms of a suitable *truncation function* $\zeta\colon [0, \infty) \to (0, \infty]$. In the case where the trunction function is infinity, i.e., $\zeta(t) = \infty$ for all $t \in [0, \infty)$, Proposition 2.1 and most of its consequences in Section 2.1 are well-known; see, e.g., Mattingly, Stuart & Higham [**59**] and Schurz [**75**].

Proposition 2.1. *Let $\rho \in \mathbb{R}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, be a non-decreasing sequence, let $\zeta \colon [0,\infty) \to (0,\infty]$ be a function, let $V \colon E \to [0,\infty)$ be an $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable function and let $Y \colon \mathbb{N}_0 \times \Omega \to E$, $Z \colon \mathbb{N} \times \Omega \to \mathbb{R}$ be stochastic processes with $\mathbb{E}[\mathbb{1}_{\Omega_n}|Z_n|] < \infty$ and*

$$\mathbb{1}_{\Omega_n} V(Y_n) \leq e^{\rho(t_n - t_{n-1})} V(Y_{n-1}) + \mathbb{1}_{\Omega_n} Z_n \tag{2.1}$$

for all $n \in \mathbb{N}$ where $\Omega_n := \cap_{k=0}^{n-1} \{V(Y_k) \leq \zeta(t_{k+1} - t_k)\} \in \mathcal{F}$ for all $n \in \mathbb{N}_0$. Then

$$\mathbb{1}_{\Omega_n} V(Y_n) \leq e^{\rho(t_n - t_0)} V(Y_0) + \sum_{k=1}^{n} e^{\rho(t_n - t_k)} \mathbb{1}_{\Omega_k} Z_k, \tag{2.2}$$

$$\mathbb{P}\big[(\Omega_n)^c\big] \leq \sum_{k=0}^{n-1} \left(\frac{e^{\rho(t_k - t_0)}\, \mathbb{E}[V(Y_0)] + \sum_{l=1}^{k} e^{\rho(t_k - t_l)}\, \mathbb{E}[\mathbb{1}_{\Omega_l} Z_l]}{\zeta(t_{k+1} - t_k)} \right) \tag{2.3}$$

for all $n \in \mathbb{N}_0$.

Proof of Proposition 2.1. First, observe that assumption (2.1) and the relation $\Omega_n \subset \Omega_{n-1}$ for all $n \in \mathbb{N}$ show that

$$\mathbb{1}_{\Omega_n} V(Y_n) \leq \mathbb{1}_{\Omega_{n-1}} e^{\rho(t_n - t_{n-1})} V(Y_{n-1}) + \mathbb{1}_{\Omega_n} Z_n \tag{2.4}$$

for all $n \in \mathbb{N}$. Estimate (2.4) is equivalent to the inequality

$$\mathbb{1}_{\Omega_n} e^{-\rho t_n} V(Y_n) - \mathbb{1}_{\Omega_{n-1}} e^{-\rho t_{n-1}} V(Y_{n-1}) \leq \mathbb{1}_{\Omega_n} e^{-\rho t_n} Z_n \tag{2.5}$$

for all $n \in \mathbb{N}$. Next note that (2.5) and the fact $\Omega_0 = \Omega$ imply

$$\begin{aligned} &\mathbb{1}_{\Omega_n} e^{-\rho t_n} V(Y_n) \\ &= \mathbb{1}_{\Omega_0} e^{-\rho t_0} V(Y_0) + \sum_{k=1}^{n} \big(\mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k) - \mathbb{1}_{\Omega_{k-1}} e^{-\rho t_{k-1}} V(Y_{k-1}) \big) \\ &\leq e^{-\rho t_0} V(Y_0) + \sum_{k=1}^{n} \mathbb{1}_{\Omega_k} e^{-\rho t_k} Z_k \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}_0$. This implies (2.2). For proving (2.3), note that the relation $\Omega_n \subset \Omega_{n-1}$ for all $n \in \mathbb{N}$ implies

$$(\Omega_n)^c = \big(\Omega_{n-1} \setminus \Omega_n\big) \uplus \big((\Omega_{n-1})^c \setminus \Omega_n \big) = \big(\Omega_{n-1} \setminus \Omega_n\big) \uplus \big((\Omega_{n-1})^c \big) \tag{2.7}$$

for all $n \in \mathbb{N}$. Iterating equation (2.7) and using again $\Omega_0 = \Omega$ shows

$$\begin{aligned} &(\Omega_n)^c \\ &= \left(\biguplus_{k=0}^{n-1} (\Omega_k \setminus \Omega_{k+1}) \right) \biguplus \big((\Omega_0)^c \big) = \biguplus_{k=0}^{n-1} (\Omega_k \setminus \Omega_{k+1}) \\ &= \biguplus_{k=0}^{n-1} (\Omega_k \cap \{V(Y_k) > \zeta(t_{k+1} - t_k)\}) = \biguplus_{k=0}^{n-1} \{\mathbb{1}_{\Omega_k} V(Y_k) > \zeta(t_{k+1} - t_k)\} \end{aligned} \tag{2.8}$$

for all $n \in \mathbb{N}_0$. Additivity of the probability measure $\mathbb{P}$, Markov's inequality and inequality (2.2) therefore imply

$$\begin{aligned}\mathbb{P}\big[(\Omega_n)^c\big] &= \sum_{k=0}^{n-1} \mathbb{P}\Big[\,\mathbb{1}_{\Omega_k} V(Y_k) > \zeta(t_{k+1}-t_k)\Big] \leq \sum_{k=0}^{n-1}\left[\frac{\mathbb{E}\big[\mathbb{1}_{\Omega_k} V(Y_k)\big]}{\zeta(t_{k+1}-t_k)}\right] \\ &\leq \sum_{k=0}^{n-1}\left[\frac{e^{\rho(t_k-t_0)}\,\mathbb{E}[V(Y_0)] + \sum_{l=1}^{k} e^{\rho(t_k-t_l)}\,\mathbb{E}[\mathbb{1}_{\Omega_l} Z_l]}{\zeta(t_{k+1}-t_k)}\right]\end{aligned} \tag{2.9}$$

for all $n \in \mathbb{N}_0$. This is inequality (2.3) and the proof of Proposition 2.1 is thus completed. □

Let us illustrate Proposition 2.1 with the following simple implication. If the assumptions of Proposition 2.1 are fulfilled, if $\rho \in (-\infty, 0)$, $\zeta \equiv \infty$, $\mathbb{E}[V(Y_0)] < \infty$ and if there exist real numbers $h \in (0,\infty)$ and $c \in [0,\infty)$ such that $\sup_{k\in\mathbb{N}} \mathbb{E}[Z_k] \leq ch$ and $t_n = nh$ for all $n \in \mathbb{N}_0$, then we infer from inequality (2.2) that

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{E}\big[V(Y_n)\big] \leq \limsup_{n\to\infty}\left(e^{\rho n h}\,\mathbb{E}[V(Y_0)] + \sum_{k=1}^{n} e^{\rho(n-k)h}\,\mathbb{E}[Z_k]\right) \\ &\leq \limsup_{n\to\infty}\left(e^{\rho n h}\,\mathbb{E}[V(Y_0)] + ch\left[\sum_{k=1}^{n} e^{\rho(n-k)h}\right]\right) \\ &= \limsup_{n\to\infty}\left(e^{\rho n h}\,\mathbb{E}[V(Y_0)] + \frac{ch\left(1-e^{\rho n h}\right)}{(1-e^{\rho h})}\right) = \frac{ch}{(1-\exp(\rho h))} \\ &= \frac{ch}{|\rho|\left(\int_0^h \exp(\rho s)\,ds\right)} \leq \frac{c}{|\rho|\,e^{\rho h}} = \frac{c\,e^{|\rho| h}}{|\rho|} < \infty.\end{aligned} \tag{2.10}$$

In many situations, the random variables $(Z_n)_{n\in\mathbb{N}}$ appearing in Proposition 2.1 are centered or even appropriate martingale differences. This is the subject of the next two corollaries (Corollary 2.2 and Corollary 2.3) of Proposition 2.1.

Corollary 2.2. *Let $\rho \in \mathbb{R}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, be a non-decreasing sequence, let $\zeta\colon [0,\infty) \to (0,\infty]$ be a function, let $V\colon E \to [0,\infty)$ be an $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable function and let $Y\colon \mathbb{N}_0 \times \Omega \to E$ be a stochastic process with $\mathbb{E}[V(Y_0)] < \infty$ and*

$$\mathbb{1}_{\cap_{k=0}^{n}\{V(Y_k)\leq\zeta(t_{k+1}-t_k)\}} \cdot \mathbb{E}\big[V(Y_{n+1})\,|\,(Y_k)_{k\in\{0,1,\dots,n\}}\big] \leq e^{\rho(t_{n+1}-t_n)} \cdot V(Y_n) \tag{2.11}$$

$\mathbb{P}$-a.s. for all $n \in \mathbb{N}_0$. Then the stochastic process $\mathbb{1}_{\Omega_n} e^{-\rho t_n} V(Y_n)$, $n \in \mathbb{N}_0$, is a non-negative supermartingale and

$$\mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Y_n)\big] \leq e^{\rho(t_n-t_0)}\,\mathbb{E}\big[V(Y_0)\big],\ \mathbb{P}\big[(\Omega_n)^c\big] \leq \left(\sum_{k=0}^{n-1}\frac{e^{\rho(t_k-t_0)}}{\zeta(t_{k+1}-t_k)}\right)\mathbb{E}\big[V(Y_0)\big], \tag{2.12}$$

$$\begin{aligned}\mathbb{E}\big[\bar{V}(Y_n)\big] &\leq e^{\rho(t_n-t_0)}\,\mathbb{E}\big[V(Y_0)\big] \\ &\quad + \|\bar{V}(Y_n)\|_{L^p(\Omega;\mathbb{R})}\left[\left(\sum_{k=0}^{n-1}\frac{e^{\rho(t_k-t_0)}}{\zeta(t_{k+1}-t_k)}\right)\mathbb{E}\big[V(Y_0)\big]\right]^{(1-1/p)}\end{aligned} \tag{2.13}$$

for all $n \in \mathbb{N}_0$, $p \in [1,\infty]$ and all $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable functions $\bar{V}\colon E \to [0,\infty)$ with $\bar{V}(x) \leq V(x)$ for all $x \in E$ where $\Omega_n := \cap_{k=0}^{n-1}\{V(Y_k) \leq \zeta(t_{k+1}-t_k)\} \in \mathcal{F}$ for all $n \in \mathbb{N}_0$.

Proof of Corollary 2.2. The relation $\Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}_0$ shows that assumption (2.11) is equivalent to the estimate

$$\mathbb{E}\big[\mathbb{1}_{\Omega_{n+1}} e^{-\rho t_{n+1}} V(Y_{n+1}) \,|\, (Y_k)_{k\in\{0,1,\dots,n\}}\big] \leq \mathbb{1}_{\Omega_n} e^{-\rho t_n} V(Y_n) \tag{2.14}$$

$\mathbb{P}$-a.s. for all $n \in \mathbb{N}_0$. Combining (2.14), the assumption $\mathbb{E}[V(Y_0)] < \infty$ and the relation

$$\sigma_\Omega\big((\mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k))_{k\in\{0,1,\dots,n\}}\big) \subset \sigma_\Omega\big((Y_k)_{k\in\{0,1,\dots,n\}}\big) \tag{2.15}$$

for all $n \in \mathbb{N}_0$ proves that the process $\mathbb{1}_{\Omega_n} e^{-\rho t_n} V(Y_n)$, $n \in \mathbb{N}_0$, is a non-negative supermartingale. This implies the first inequality in (2.12). In addition, this ensures that the stochastic process $Z\colon \mathbb{N}\times\Omega \to \mathbb{R}$ given by

$$Z_n = \mathbb{1}_{\Omega_n} V(Y_n) - \mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Y_n) \,|\, (Y_k)_{k\in\{0,1,\dots,n-1\}}\big] \tag{2.16}$$

$\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ satisfies $\mathbb{E}\big[\mathbb{1}_{\Omega_n}|Z_n|\big] < \infty$ and $\mathbb{E}[\mathbb{1}_{\Omega_n} Z_n] = 0$ for all $n \in \mathbb{N}$. Moreover, the definition of $Z\colon \mathbb{N}\times\Omega \to \mathbb{R}$ and assumption (2.11) ensure

$$\begin{aligned} \mathbb{1}_{\Omega_n} V(Y_n) &= \mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Y_n) \,|\, (Y_k)_{k\in\{0,1,\dots,n-1\}}\big] + Z_n \\ &= \mathbb{1}_{\Omega_n}\mathbb{E}\big[V(Y_n) \,|\, (Y_k)_{k\in\{0,1,\dots,n-1\}}\big] + \mathbb{1}_{\Omega_n} Z_n \\ &\leq e^{\rho(t_n - t_{n-1})} V(Y_{n-1}) + \mathbb{1}_{\Omega_n} Z_n \end{aligned} \tag{2.17}$$

$\mathbb{P}$-a.s. for all $n \in \mathbb{N}$. An application of Proposition 2.1 thus proves the second inequality in (2.12). Next observe that Hölder's inequality implies

$$\mathbb{E}[X] \leq \mathbb{E}[\mathbb{1}_{\tilde{\Omega}} X] + \big(\mathbb{P}\big[(\tilde{\Omega})^c\big]\big)^{(1-1/p)} \|X\|_{L^p(\Omega;\mathbb{R})} \tag{2.18}$$

for all $\tilde{\Omega} \in \mathcal{F}$, $p \in [1,\infty]$ and all $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mappings $X\colon \Omega \to [0,\infty)$. Combining (2.12) and (2.18) finally results in

$$\begin{aligned} \mathbb{E}\big[\bar{V}(Y_n)\big] &\leq \mathbb{E}\big[\mathbb{1}_{\Omega_n} V(Y_n)\big] + \|\bar{V}(Y_n)\|_{L^p(\Omega;\mathbb{R})} \big(\mathbb{P}\big[(\Omega_n)^c\big]\big)^{(1-1/p)} \\ &\leq e^{\rho(t_n - t_0)}\,\mathbb{E}\big[V(Y_0)\big] + \|\bar{V}(Y_n)\|_{L^p(\Omega;\mathbb{R})} \left[\left(\sum_{k=0}^{n-1} \frac{e^{\rho(t_k - t_0)}}{\zeta(t_{k+1}-t_k)}\right)\mathbb{E}\big[V(Y_0)\big]\right]^{(1-1/p)} \end{aligned} \tag{2.19}$$

for all $n \in \mathbb{N}_0$, $p \in [1,\infty]$ and all $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable functions $\bar{V}\colon E \to [0,\infty)$ with $\bar{V}(x) \leq V(x)$ for all $x \in E$. The proof of Corollary 2.2 is thus completed. □

Corollary 2.2, in particular, proves estimates on the quantities

$$\sup_{k\in\{0,1,\dots,n\}} \mathbb{E}\big[\mathbb{1}_{\Omega_k} V(Y_k)\big] \tag{2.20}$$

for $n \in \mathbb{N}$ (see the first inequality in (2.12)). Here $\Omega_n \subset \Omega$, $n \in \mathbb{N}_0$, are typically large subevents of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $V\colon E \to [0,\infty)$ is an appropriate Lyapunov-type function (see Corollary 2.2 for details). Under suitable additional assumptions, one can also obtain an estimate on the larger quantities

$$\mathbb{E}\left[\sup_{k\in\{0,1,\dots,n\}} \mathbb{1}_{\Omega_k} V(Y_k)\right] \tag{2.21}$$

for $n \in \mathbb{N}$. This is the subject of the next corollary.

Corollary 2.3. *Let $\rho \in \mathbb{R}$, $p \in [1,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, be a non-decreasing sequence, let $\zeta \colon [0,\infty) \to (0,\infty]$, $\nu \colon \mathbb{N} \to [0,\infty)$ be functions, let $V \colon E \to [0,\infty)$ be an $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable function and let $Y \colon \mathbb{N}_0 \times \Omega \to E$, $Z \colon \mathbb{N} \times \Omega \to \mathbb{R}$ be stochastic processes such that the process $\sum_{k=1}^{n} \mathbb{1}_{\Omega_k} Z_k$, $n \in \mathbb{N}$, is a martingale and such that*

$$\mathbb{1}_{\Omega_n} V(Y_n) \leq e^{\rho(t_n - t_{n-1})} V(Y_{n-1}) + \mathbb{1}_{\Omega_n} Z_n \qquad \mathbb{P}\text{-a.s.}, \tag{2.22}$$

$$\|\mathbb{1}_{\Omega_n} Z_n\|_{L^p(\Omega;\mathbb{R})} \leq \nu_n \left\| \sup_{k \in \{0,1,\dots,n-1\}} \mathbb{1}_{\Omega_k} e^{\rho(t_n - t_k)} V(Y_k) \right\|_{L^p(\Omega;\mathbb{R})} \tag{2.23}$$

for all $n \in \mathbb{N}$ where $\Omega_n := \cap_{k=0}^{n-1} \{V(Y_k) \leq \zeta(t_{k+1} - t_k)\} \in \mathcal{F}$ for all $n \in \mathbb{N}_0$. Then

$$\left\| \sup_{k \in \{0,1,\dots,n\}} \mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k) \right\|_{L^p(\Omega;\mathbb{R})} \leq \sqrt{2}\, \|V(Y_0)\|_{L^p(\Omega;\mathbb{R})} \exp\!\left(\chi_p \left[\sum_{k=1}^{n} |\nu_k|^2 \right] - \rho t_0 \right) \tag{2.24}$$

for all $n \in \mathbb{N}_0$.

Proof of Corollary 2.3. Inequality (2.2) in Proposition 2.1 implies

$$\sup_{k \in \{0,1,\dots,n\}} \mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k) \leq \sup_{k \in \{0,1,\dots,n\}} e^{-\rho t_0} V(Y_0) + \sup_{k \in \{0,1,\dots,n\}} \left[\sum_{l=1}^{k} \mathbb{1}_{\Omega_k} e^{-\rho t_k} Z_k \right] \tag{2.25}$$

$\mathbb{P}$-a.s. for all $n \in \mathbb{N}_0$. The triangle inequality and the estimate $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$ hence yield

$$\begin{aligned} &\left\| \sup_{k \in \{0,1,\dots,n\}} \mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k) \right\|_{L^p(\Omega;\mathbb{R})}^2 \\ &\leq 2 \left\| e^{-\rho t_0} V(Y_0) \right\|_{L^p(\Omega;\mathbb{R})}^2 + 2 \left\| \sup_{k \in \{0,1,\dots,n\}} \left| \sum_{l=1}^{k} \mathbb{1}_{\Omega_l} e^{-\rho t_l} Z_l \right| \right\|_{L^p(\Omega;\mathbb{R})}^2 \end{aligned} \tag{2.26}$$

for all $n \in \mathbb{N}_0$. The definition (1.9) of $\chi_p \in [0,\infty)$, $p \in [1,\infty)$, applied to the martingale $\sum_{l=1}^{k} \mathbb{1}_{\Omega_l} e^{-\rho t_l} Z_l$, $k \in \mathbb{N}_0$, therefore shows

$$\begin{aligned} &\left\| \sup_{k \in \{0,1,\dots,n\}} \mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k) \right\|_{L^p(\Omega;\mathbb{R})}^2 \\ &\leq 2 \left\| e^{-\rho t_0} V(Y_0) \right\|_{L^p(\Omega;\mathbb{R})}^2 + 2\chi_p \sum_{k=1}^{n} \left\| \mathbb{1}_{\Omega_k} e^{-\rho t_k} Z_k \right\|_{L^p(\Omega;\mathbb{R})}^2 \\ &\leq 2 \left\| e^{-\rho t_0} V(Y_0) \right\|_{L^p(\Omega;\mathbb{R})}^2 + 2\chi_p \sum_{k=0}^{n-1} |\nu_{k+1}|^2 \left\| \sup_{l \in \{0,1,\dots,k\}} \mathbb{1}_{\Omega_l} e^{-\rho t_l} V(Y_l) \right\|_{L^p(\Omega;\mathbb{R})}^2 \end{aligned} \tag{2.27}$$

for all $n \in \mathbb{N}_0$ where the last inequality follows from assumption (2.23). Consequently, Gronwall's lemma for discrete time yields

$$\begin{aligned}&\left\| \sup_{k\in\{0,1,\ldots,n\}} \mathbb{1}_{\Omega_k} e^{-\rho t_k} V(Y_k) \right\|^2_{L^p(\Omega;\mathbb{R})} \\ &\leq 2 \left\| e^{-\rho t_0} V(Y_0) \right\|^2_{L^p(\Omega;\mathbb{R})} \exp\left(2\chi_p \sum_{k=0}^{n-1} |\nu_{k+1}|^2 \right)\end{aligned} \tag{2.28}$$

for all $n \in \mathbb{N}_0$. This finishes the proof of Corollary 2.3. □

An application of Corollary 2.3 can be found in Lemma 2.28 below.

2.1.2. Moment bounds on complements of rare events. In the case of nonlinear SDEs, it has been shown in [**40**] that the Euler-Maruyama approximations often fail to satisfy moment bounds although the exact solution of the SDE does satisfy such moment bounds. Nonetheless, the Euler-Maruyama approximations often satisfy suitable moment bounds restricted to events whose probabilities converge to one sufficiently fast; see Corollary 4.4 and Lemma 4.5 in [**37**]. This is one motivation for the next definition.

Definition 2.4 (Semi boundedness). Let $\alpha \in (0,\infty]$, let $I \subset \mathbb{R}$ be a subset of $\mathbb{R}$, let $(E,\mathcal{E})$ be a measurable space, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $V\colon E \to [0,\infty)$ be an $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable mapping. A sequence $Y^N\colon I \times \Omega \to E$, $N \in \mathbb{N}$, of stochastic processes is then said to be α-semi V-bounded (with respect to $\mathbb{P}$) if there exists a sequence $\Omega_N \in \sigma_\Omega\big((Y^N_t)_{t\in I}\big) = \sigma_\Omega(Y^N) \subset \mathcal{F}$, $N \in \mathbb{N}$, of events such that

$$\limsup_{N\to\infty} \left(\sup_{t\in I} \mathbb{E}\big[\mathbb{1}_{\Omega_N} V(Y^N_t)\big] + N^\alpha \cdot \mathbb{P}\big[(\Omega_N)^c\big] \right) < \infty. \tag{2.29}$$

Moreover, a sequence $Y^N\colon I \times \Omega \to E$, $N \in \mathbb{N}$, of stochastic processes is said to be 0-semi V-bounded (with respect to $\mathbb{P}$) if there exists a sequence $\Omega_N \in \sigma_\Omega(Y^N)$, $N \in \mathbb{N}$, of events such that $\limsup_{N\to\infty} \sup_{t\in I} \mathbb{E}\big[\mathbb{1}_{\Omega_N} V(Y^N_t)\big] < \infty$ and $\lim_{N\to\infty} \mathbb{P}\big[(\Omega_N)^c\big] = 0$.

We now present some remarks concerning Definition 2.4. First, note that the concept of semi boundedness in the sense of Definition 2.4 is a property of the probability measures associated to the stochastic processes. More precisely, in the setting of Definition 2.4, a sequence of stochastic processes $Y^N\colon I \times \Omega \to E$, $N \in \mathbb{N}$, is α-semi V-bounded if and only if there exists a sequence $A_N \in \mathcal{E}^{\otimes I}$, $N \in \mathbb{N}$, of sets such that

$$\limsup_{N\to\infty} \left\{ \sup_{t\in[0,T]} \int_{E^{\times I}} \mathbb{1}_{A_N}(x)\, V(\pi_t(x))\, \mathbb{P}_{Y^N}(dx) + N^\alpha \cdot \mathbb{P}_{Y^N}\big[(A_N)^c\big] \right\} < \infty \tag{2.30}$$

where $\mathbb{P}_{Y^N}\colon \mathcal{E}^{\otimes I} \to [0,1]$, $N \in \mathbb{N}$, with $\mathbb{P}_{Y^N}[A] = \mathbb{P}[Y^N \in A]$ for all $A \in \mathcal{E}^{\otimes I}$ and all $N \in \mathbb{N}$ are the probability measures associated to $Y^N\colon I \times \Omega \to E$, $N \in \mathbb{N}$, and where $\pi_t\colon E^{\times I} \to E$, $t \in I$, with $\pi_t(x) = x(t)$ for all $x \in E^{\times I}$ and all $t \in I$ are projections from $E^{\times I}$ to E. Semi boundedness in the sense of Definition 2.4 is thus a property of the sequence $\mathbb{P}_{Y^N}\colon \mathcal{E}^{\otimes I} \to [0,1]$, $N \in \mathbb{N}$, of probability measures on the measurable path space $\big(E^{\times I}, \mathcal{E}^{\otimes I}\big)$. Moreover, observe that in the case where the index set I appearing in Definition 2.4 consists of only one real number $t_0 \in \mathbb{R}$, i.e., $I = \{t_0\}$, the sequence $Y^N\colon I \times \Omega \to E$, $N \in \mathbb{N}$, of stochastic processes reduces to a sequence $Z_N \colon \Omega \to E$, $N \in \mathbb{N}$, of random variables with $Z_N(\omega) = Y^N_{t_0}(\omega)$ for all $\omega \in \Omega$ and all $N \in \mathbb{N}$. Furthermore, observe that in the case $\alpha = \infty$ in Definition 2.4, condition (2.29) is equivalent to the condition $\limsup_{N\to\infty} \sup_{t\in I} \mathbb{E}\big[V(Y^N_t)\big] < \infty$. Next we would like to add some further comments to Definition 2.4. For this we first note in Lemma 2.5 a well-known characterization of convergence in probability (see, e.g., Exercise 6.2.1 (i) in Klenke [**44**] for a related excercise or, e.g., also Remark 9 in [**42**]). The proof of Lemma 2.5 is for completeness given below.

LEMMA 2.5 (Convergence in probability). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let (E, d_E) be a separable metric space and let $X \colon \Omega \to E$ and $Y_N \colon \Omega \to E$, $N \in \mathbb{N}$, be $\mathcal{F}/\mathcal{B}(E)$-measurable mappings. Then the following three assertions*

(i) *the sequence Y_N, $N \in \mathbb{N}$, converges to X in probability, i.e., it holds that $\lim_{N\to\infty} \mathbb{P}\big[d_E(X, Y_N) > \varepsilon\big] = 0$ for all $\varepsilon \in (0, \infty)$,*
(ii) *it holds that $\lim_{N\to\infty} \mathbb{P}\big[d_E(X, Y_N) \leq 1\big] = 1$ and it holds that $\lim_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{\{d_E(X,Y_N)\leq 1\}} d_E(X, Y_N)\big] = 0$,*
(iii) *there exists a sequence $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, with $\lim_{N\to\infty} \mathbb{P}\big[\Omega_N\big] = 1$ and $\lim_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{\Omega_N} d_E(X, Y_N)\big] = 0$*

are equivalent and each of these assertions implies for every $p \in (0, \infty)$ and every $x \in E$ with $\mathbb{E}\big[\,|d_E(x, X)|^p\,\big] < \infty$ that

$$\lim_{N\to\infty} \mathbb{P}\big[d_E(X, Y_N) > 1\big] = 0 \text{ and } \sup_{N\in\mathbb{N}} \mathbb{E}\big[\mathbb{1}_{\{d_E(X,Y_N)\leq 1\}}\,|d_E(x, Y_N)|^p\,\big] < \infty. \tag{2.31}$$

PROOF OF LEMMA 2.5. Note that if $\lim_{N\to\infty}\mathbb{P}\big[d_E(X, Y_N) > \varepsilon\big] = 0$ for all $\varepsilon \in (0, \infty)$, then

$$\lim_{N\to\infty} \mathbb{P}\big[d_E(X, Y_N) \leq 1\big] = 1 \quad \text{and} \quad \lim_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{\{d_E(X,Y_N)\leq 1\}} d_E(X, Y_N)\big] = 0 \tag{2.32}$$

due to Lebesgue's theorem of dominated convergence. This shows that (i) implies (ii). Next observe that (ii), clearly, implies (iii). Moreover, if there exists a sequence $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, with $\lim_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{\Omega_N} d_E(X, Y_N)\big] = 0$ and $\lim_{N\to 0} \mathbb{P}\big[\Omega_N\big] = 1$, then $\lim_{N\to\infty} \mathbb{P}\big[\mathbb{1}_{\Omega_N} d_E(X, Y_N) > \varepsilon\big] = 0$ for all $\varepsilon \in (0, \infty)$ and therefore

$$\begin{aligned}\lim_{N\to\infty} \mathbb{P}\big[d_E(X, Y_N) > \varepsilon\big] &\leq \lim_{N\to\infty} \mathbb{P}\big[\Omega_N \cap \{d_E(X, Y_N) > \varepsilon\}\big] + \lim_{N\to\infty} \mathbb{P}\big[(\Omega_N)^c\big] \\ &= \lim_{N\to\infty} \mathbb{P}\big[\mathbb{1}_{\Omega_N} d_E(X, Y_N) > \varepsilon\big] = 0\end{aligned} \tag{2.33}$$

for all $\varepsilon \in (0, \infty)$. This proves that (iii) implies (i). Finally, observe that

$$\begin{aligned}&\big\|\mathbb{1}_{\{d_E(X,Y_N)\leq 1\}} d_E(x, Y_N)\big\|_{L^p(\Omega;\mathbb{R})} \\ &\leq \big\|\mathbb{1}_{\{d_E(X,Y_N)\leq 1\}} d_E(x, X)\big\|_{L^p(\Omega;\mathbb{R})} + \big\|\mathbb{1}_{\{d_E(X,Y_N)\leq 1\}} d_E(X, Y_N)\big\|_{L^p(\Omega;\mathbb{R})} \\ &\leq \big\|d_E(x, X)\big\|_{L^p(\Omega;\mathbb{R})} + 1\end{aligned} \tag{2.34}$$

for all $x \in E$, $N \in \mathbb{N}$ and all $p \in (0, \infty)$. The proof of Lemma 2.5 is thus completed. □

Let us now study under which conditions Euler-Maruyama approximations are α-semi V-bounded with $\alpha \in [0, \infty]$ and $V \colon \mathbb{R}^d \to [0, \infty)$ appropriate and $d \in \mathbb{N}$. First, note that convergence in probability of the Euler-Maruyama approximations has been established in the literature for a large class of possibly highly nonlinear SDEs (see, e.g., Krylov [**49**], Gyöngy & Krylov [**24**], Gyöngy [**22**] and Jentzen, Kloeden & Neuenkirch [**43**]). For these SDEs, one can thus apply Lemma 2.5 to obtain the existence of a sequence of events whose probabilities converge to one and on which moments of the Euler approximations are bounded in the sense of (2.31). This is, however, not sufficient to establish α-semi $\|\cdot\|^p$-boundedness of the Euler-Maruyama approximations with $\alpha \in [0,\infty]$ and $p \in (0,\infty)$ since the events in (2.31) do, in general, not satisfy the required measurability condition in Definition 2.4. Moreover, we are mainly interested in α-semi $\|\cdot\|^p$-boundedness of the Euler-Maruyama approximations with $\alpha, p \in (0,\infty)$. Lemma 2.5 only shows that the complement of the events on the right side of (2.31) converge to zero but gives no information on the rate of convergence of the probabilities of these events. So, in general, α-semi $\|\cdot\|^p$-boundedness with $\alpha, p \in (0,\infty)$ cannot be inferred from convergence in probability. Here we employ the theory of Subsection 2.1.1 to obtain semi boundedness for the Euler-Maruyama approximations (see Theorem 2.13 and Corollary 2.9 below). In particular, Corollary 2.2 immediately implies the next corollary.

Corollary 2.6 (Semi moment bounds). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $\rho, T \in (0,\infty)$, $\alpha, q \in (1,\infty]$, $N_0 \in \mathbb{N}$, let $V\colon E \to [0,\infty)$ be an $\mathcal{E}/\mathcal{B}([0,\infty))$-measurable mapping and let $Y^N\colon \{0,1,\ldots,N\} \times \Omega \to E$, $N \in \mathbb{N}$, be a sequence of stochastic processes satisfying*

$$\mathbb{1}_{\{V(Y_n^N) \leq |\frac{N}{T}|^\alpha\}} \cdot \mathbb{E}\big[V(Y_{n+1}^N) \,|\, (Y_k^N)_{k \in \{0,1,\ldots,n\}}\big] \leq e^{\frac{\rho T}{N}} \cdot V(Y_n^N) \tag{2.35}$$

$\mathbb{P}$-a.s. for all $n \in \{0,1,\ldots,N-1\}$ and all $N \in \{N_0, N_0+1, \ldots\}$ and

$$\limsup_{N\to\infty} \mathbb{E}\big[V(Y_0^N)\big] < \infty. \tag{2.36}$$

Then the sequence $Y^N\colon \{0,1,\ldots,N\} \times \Omega \to E$, $N \in \mathbb{N}$, of stochastic processes is $(\alpha-1)$-semi V-bounded.

Proof of Corollary 2.6. Corollary 2.2 above with the truncation function $\zeta\colon [0,\infty) \to (0,\infty]$ given by

$$\zeta(0) = \infty \qquad \text{and} \qquad \zeta(t) = \frac{1}{t^\alpha} \tag{2.37}$$

for all $t \in (0,\infty)$ and with the sequence $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, given by $t_n = nT/N$ for all $n \in \mathbb{N}_0$ implies

$$\mathbb{E}\big[\mathbb{1}_{\Omega_N} V(Y_n^N)\big] \leq e^{\frac{\rho n T}{N}} \cdot \mathbb{E}\big[V(Y_0^N)\big] \leq e^{\rho T} \cdot \mathbb{E}\big[V(Y_0^N)\big] \tag{2.38}$$

and

$$\mathbb{P}\big[(\Omega_N)^c\big] \leq \left|\frac{T}{N}\right|^\alpha \cdot \left(\sum_{k=0}^{N-1} e^{\frac{\rho k T}{N}}\right) \cdot \mathbb{E}\big[V(Y_0^N)\big] \leq \left|\frac{T}{N}\right|^\alpha \cdot N \cdot e^{\rho T} \cdot \mathbb{E}\big[V(Y_0^N)\big] \tag{2.39}$$

for all $n \in \{0,1,\ldots,N\}$ and all $N \in \{N_0, N_0+1, \ldots\}$ where $\Omega_N := \cap_{k=0}^{N-1}\{V(Y_k^N) \leq (N/T)^\alpha\} \in \mathcal{F}$ for all $N \in \mathbb{N}$. Combining (2.38), (2.39) and assumption (2.36) implies

$$\limsup_{N\to\infty} \left(\sup_{n \in \{0,1,\ldots,N\}} \mathbb{E}\big[\mathbb{1}_{\Omega_N} V(Y_n^N)\big] + N^{(\alpha-1)} \cdot \mathbb{P}\big[(\Omega_N)^c\big] \right) < \infty. \tag{2.40}$$

The proof of Corollary 2.6 is thus completed. □

2.1.3. Moment bounds. Corollary 2.6 above, in particular, establishes moment bounds restricted to the complements of rare events for sequences of stochastic processes. In some situations, the sequence of stochastic processes fulfills an additional growth bound assumption (see (2.42) and (2.50) below for details) which can be used to prove moment bounds on the full probability space. This is the subject of the next proposition. The main idea of this proposition is a certain bootstrap argument which exploits inequality (2.13) in Corollary 2.2 (see also estimate (2.18)).

PROPOSITION 2.7 (Moment bounds). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $(E, \mathcal{E})$ be a measurable space, let $T \in (0, \infty)$, $\rho \in [0, \infty)$, $\alpha \in (1, \infty]$, $N_0 \in \mathbb{N}$, let $V, \bar{V} \colon E \to [0, \infty)$ be $\mathcal{E}/\mathcal{B}([0, \infty))$-measurable mappings with $\bar{V}(x) \leq V(x)$ for all $x \in E$ and let $Y^N \colon \{0, 1, \ldots, N\} \times \Omega \to E$, $N \in \mathbb{N}$, be a sequence of stochastic processes satisfying*

$$\mathbb{1}_{\left\{V(Y_n^N) \leq \left|\frac{N}{T}\right|^{\alpha}\right\}} \cdot \mathbb{E}\big[V(Y_{n+1}^N) \,|\, (Y_k^N)_{k \in \{0,1,\ldots,n\}}\big] \leq e^{\frac{\rho T}{N}} \cdot V(Y_n^N) \tag{2.41}$$

$\mathbb{P}$-a.s. for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \{N_0, N_0 + 1, \ldots\}$. Then

$$\begin{aligned} &\limsup_{N \to \infty} \sup_{n \in \{0,1,\ldots,N\}} \mathbb{E}\big[\bar{V}(Y_n^N)\big] \\ &\leq e^{\rho T} \left(1 + \limsup_{N \to \infty} \mathbb{E}\big[V(Y_0^N)\big]\right) \\ &\quad \cdot \left(1 + T^{\alpha(1-1/p)} \limsup_{N \to \infty} \left[N^{(1-\alpha)(1-1/p)} \sup_{0 \leq n \leq N} \|\bar{V}(Y_n^N)\|_{L^p(\Omega;\mathbb{R})}\right]\right) \end{aligned} \tag{2.42}$$

for all $p \in [1, \infty]$.

PROOF OF PROPOSITION 2.7. We proof Proposition 2.7 through an application of Corollary 2.2. More precisely, observe that with the truncation function $\zeta \colon [0, \infty) \to (0, \infty]$ given by

$$\zeta(0) = 0 \qquad \text{and} \qquad \zeta(t) = t^{-\alpha} \tag{2.43}$$

for all $t \in (0, \infty)$ and with the sequence $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, given by $t_n = nT/N$ for all $n \in \mathbb{N}_0$ we get from inequality (2.13) in Corollary 2.2 that

$$\begin{aligned} &\mathbb{E}\big[\bar{V}(Y_n^N)\big] \\ &\leq e^{\frac{\rho n T}{N}} \,\mathbb{E}\big[V(Y_0^N)\big] + \|\bar{V}(Y_n^N)\|_{L^p(\Omega;\mathbb{R})} \left[\left|\frac{T}{N}\right|^{\alpha} \left(\sum_{k=0}^{n-1} e^{\frac{\rho k T}{N}}\right) \mathbb{E}\big[V(Y_0^N)\big]\right]^{(1-1/p)} \\ &\leq e^{\rho T}\, \mathbb{E}\big[V(Y_0^N)\big] + e^{\rho T}\, \|\bar{V}(Y_n^N)\|_{L^p(\Omega;\mathbb{R})} \left[\left|\tfrac{T}{N}\right|^{\alpha} \cdot N \cdot \mathbb{E}\big[V(Y_0^N)\big]\right]^{(1-1/p)} \\ &\leq e^{\rho T} \left(1 + \mathbb{E}\big[V(Y_0^N)\big]\right) \left(1 + \left|\tfrac{T}{N}\right|^{\alpha(1-1/p)} N^{(1-1/p)} \,\|\bar{V}(Y_n^N)\|_{L^p(\Omega;\mathbb{R})}\right) \end{aligned} \tag{2.44}$$

for all $n \in \{0, 1, \ldots, N\}$, $N \in \{N_0, N_0 + 1, \ldots\}$ and all $p \in [1, \infty]$. This completes the proof of Proposition 2.7. □

2.1.4. One-step approximation schemes. Let $T \in (0, \infty)$ be a real number, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space and let $(E, \mathcal{E})$ be a measurable space. We are interested in general one-step approximation schemes for SDEs with state space E. The driving noise of the SDE could be, e.g., a standard Brownian motion, a fractional Brownian motion or a Levý process. In such a general situation, a sequence of approximation processes of the solution process with uniform time discretization is often given by a measurable mapping $\Psi\colon E\times[0,T]^2\times\Omega\to E$ as follows. Let $\xi\colon\Omega\to E$ be a measurable mapping. Define stochastic processes $Y^N\colon\{0,1,\ldots,N\}\times\Omega\to E$, $N\in\mathbb{N}$, through $Y_0^N=\xi$ and

$$Y_{n+1}^N=\Psi\big(Y_n^N,\tfrac{nT}{N},\tfrac{(n+1)T}{N}\big) \tag{2.45}$$

for all $n\in\{0,1,\ldots,N-1\}$ and all $N\in\mathbb{N}$. For example, in the setting of the introduction, the Euler-Maruyama scheme (1.4) is given by the one-step function

$$\Psi(x,s,t)=x+\bar{\mu}(x)(t-s)+\bar{\sigma}(x)\big(W_t-W_s\big) \tag{2.46}$$

for all $x\in\mathbb{R}^d$, $s,t\in[0,T]$. The general approach of Subsections 2.1.1–2.1.3 can be used to study moment bounds for stochastic processes defined as in (2.45).

In Sections 2.2 and 2.3 below, we focus on finite-dimensional SDEs driven by standard Brownian motions. Due to the Markov property of the Brownian motion, a one-step approximation scheme can in this case be specified by a function of the current position, of the time increment and of the increment of a Brownian motion. More precisely, let $d,m\in\mathbb{N}$, $\theta\in(0,T]$ and let $\Phi\colon\mathbb{R}^d\times[0,\theta]\times\mathbb{R}^m\to\mathbb{R}^d$ be a Borel measurable function. Using this function and a uniform time discretization, we define a family of stochastic processes $Y^N\colon\{0,1,\ldots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}\cap[\frac{T}{\theta},\infty)$, through $Y_0^N=\xi$ and

$$Y_{n+1}^N=\Phi\big(Y_n^N,\tfrac{T}{N},W_{\frac{(n+1)T}{N}}-W_{\frac{nT}{N}}\big) \tag{2.47}$$

for all $n\in\{0,1,\ldots,N-1\}$ and all $N\in\mathbb{N}\cap[\frac{T}{\theta},\infty)$. In this notation, the Euler-Maruyama scheme (1.4) is given by $\Phi(x,t,y)=x+\bar{\mu}(x)t+\bar{\sigma}(x)y$ for all $(x,t,y)\in\mathbb{R}^d\times[0,\theta]\times\mathbb{R}^m$. Moreover, in this setting, condition (2.11) in Corollary 2.2 on the approximation processes $Y^N\colon\{0,1,\ldots,N\}\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, follows from the following condition (see (2.48)) on the one-step function $\Phi\colon\mathbb{R}^d\times[0,\theta]\times\mathbb{R}^m\to\mathbb{R}^d$.

DEFINITION 2.8 (Semi stability with respect to Brownian motion). Let $\theta\in(0,\infty)$, $\alpha\in(0,\infty]$, $d,m\in\mathbb{N}$ and let $V\colon\mathbb{R}^d\to[0,\infty)$ be a Borel measurable function. A Borel measurable function $\Phi\colon\mathbb{R}^d\times[0,\theta]\times\mathbb{R}^m\to\mathbb{R}^d$ is then said to be α-semi V-stable with respect to Brownian motion if there exists a real number $\rho\in\mathbb{R}$ such that

$$\mathbb{E}\big[V(\Phi(x,t,W_t))\big]\le e^{\rho t}\cdot V(x) \tag{2.48}$$

for all $(x,t)\in\{(y,s)\in\mathbb{R}^d\times(0,\theta]\colon\alpha=\infty\text{ or }V(y)\le s^{-\alpha}\}$ where $W\colon[0,\theta]\times\Omega\to\mathbb{R}^m$ is an arbitrary standard Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. In addition, a Borel measurable function $\Phi\colon\mathbb{R}^d\times[0,\theta]\times\mathbb{R}^m\to\mathbb{R}^d$ is simply said to be V-stable with respect to Brownian motion if it is ∞-semi V-stable with respect to Brownian motion.

Let us add some remarks to Definition 2.8. Inequalities of the form (2.48) with $\alpha=\infty$ have been frequently used in the literature; see, e.g., Assumption 2.2 in Mattingly, Stuart & Higham [**59**] and Section 3.1 in Schurz [**75**]. In particular, inequality (16) in Schurz [**75**] (see also Schurz [**76**]) defines a numerical approximation of the form (2.45) to be (weakly) V-stable if inequality (2.48) holds with $\alpha=\infty$ where $V\colon\mathbb{R}^d\to[0,\infty)$ is Borel measurable.

In the case $\alpha\in(0,\infty)$, inequality (2.48) in Definition 2.8 is restricted to the subset $\{y\in\mathbb{R}^d\colon V(y)\le t^{-\alpha}\}\subset\mathbb{R}^d$ which increases to the full state space $\mathbb{R}^d$ as the time step-size $t\in(0,T]$ decreases to zero. Roughly speaking, the parameter $\alpha\in$

$(0,\infty)$ describes the speed how fast the subsets $\{y \in \mathbb{R}^d \colon V(y) \leq t^{-\alpha}\}$, $t \in (0,T]$, increase to the full state space $\mathbb{R}^d$.

Next note that if $\theta \in (0,\infty)$, $\beta \in (0,\infty]$, $d,m \in \mathbb{N}$, if $V \colon \mathbb{R}^d \to [1,\infty)$ is a Borel measurable function and if $\Phi \colon \mathbb{R}^d \times [0,\theta] \times \mathbb{R}^m \to \mathbb{R}^d$ is β-semi V-stable with respect to Brownian motion, then it holds for every $\alpha \in (0,\beta]$ that Φ is also α-semi V-stable with respect to Brownian motion. However, the converse is not true in general. In particular, Theorem 2.13 and Corollary 2.17 show in many situations that the Euler-Maruyama scheme is α-semi V-stable with respect to Brownian motion for some $\alpha \in (0,\infty)$ but not V-stable with respect to Brownian motion if the coefficients of the underlying SDE grow superlinearly.

It follows immediately from the Markov property of the Brownian motion that the approximation processes (2.47) associated to an α-semi V-stable one-step function satisfy condition (2.35) where $\alpha \in (0,\infty)$ and $V \colon \mathbb{R}^d \to [0,\infty)$ are appropriate and where $d \in \mathbb{N}$. The next corollary collects consequences of this observation.

Corollary 2.9 (Semi moment bounds and moment bounds based on semi stability with respect to Brownian motion). *Let $d,m \in \mathbb{N}$, $T \in (0,\infty)$, $\theta \in (0,T]$, $\alpha \in (1,\infty]$, $p \in [1,\infty]$, let $V,\bar{V} \colon \mathbb{R}^d \to [0,\infty)$ be Borel measurable mappings with $\bar{V}(x) \leq V(x)$ for all $x \in \mathbb{R}^d$, let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$ be a filtered probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\Phi \colon \mathbb{R}^d \times [0,\theta] \times \mathbb{R}^m \to \mathbb{R}^d$ be α-semi V-stable with respect to Brownian motion and let $Y^N \colon \{0,1,\ldots,N\} \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic processes satisfying $\limsup_{N\to\infty} \mathbb{E}[V(Y_0^N)] < \infty$ and*

$$Y_{n+1}^N = \Phi\big(Y_n^N, \tfrac{T}{N}, W_{(n+1)T/N} - W_{nT/N}\big) \tag{2.49}$$

for all $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N} \cap [T/\theta,\infty)$. Then the stochastic processes Y^N, $N \in \mathbb{N}$, are $(\alpha-1)$-semi V-bounded. Moreover, if

$$\limsup_{N\to\infty} \big(N^{(1-\alpha)(1-1/p)} \cdot \sup\nolimits_{n\in\{0,1,\ldots,N\}} \|\bar{V}(Y_n^N)\|_{L^p(\Omega;\mathbb{R})}\big) < \infty \tag{2.50}$$

in addition to the above assumptions, then the stochastic processes Y^N, $N \in \mathbb{N}$, are also ∞-semi $\bar{V}$-bounded, i.e., $\limsup_{N\to\infty} \sup_{n\in\{0,\ldots,N\}} \mathbb{E}\big[\bar{V}(Y_n^N)\big] < \infty$.

Corollary 2.9 is an immediate consequence of Corollary 2.6 and Proposition 2.7. Below in Sections 2.2 and 2.3 we will study both explicit and implicit one-step numerical approximation processes of the form (2.47). For these approximation processes we will then give sufficient conditions which ensure that the function $\Phi \colon \mathbb{R}^d \times [0,\theta] \times \mathbb{R}^m \to \mathbb{R}^d$ appearing in (2.47) is α-semi V-stable with respect to Brownian motion with $\alpha \in (1,\infty]$ appropriate so that (2.41) is fulfilled and the abstract results developed in Subsections 2.1.1–2.1.4 can thus be applied.

To the best of our knowledge, the idea to restrict the Euler-Maruyama approximations to large subevents of the probability space which increase to the full probability space as the time discretization step-size decreases to 0 appeared first in [**37**] (see Section 4 in [**37**] and also [**38**] for details). The idea to restrict a Lyapunov-type condition of the form (2.48) on the one-step function of (Metropolis-adjusted) Euler-Maruyama schemes to subsets of the state space which increase to the full state space as the time discretization step-size decreases to 0 appeared first in Bou-Rabee & Hairer [**9**] in the setting of the Langevin equation (see Lemma 3.5 and Section 5 in [**9**] for details). In Section 5 in [**9**] it is also proved in the setting of the Langevin equation that the one-step function of (Metropolis-adjusted)

Euler-Maruyama schemes satisfy suitable Lyapunov-type conditions restricted to subsets of the state space which increase to the full state space as the time discretization step-size decreases to 0 (see Proposition 5.2 and Lemma 5.6 in [**9**]). In this article the Lyapunov-type condition (2.48) is proved in Subsection 2.2.1 in the case of the Euler-Maruyama scheme and in Subsection 2.2.2 in the case of suitable tamed methods. Finally, to the best of our knowledge, a bootstrap argument similar as in (2.13) and Proposition 2.7 appeared first in [**38**] for a class of drift-tamed Euler-Maruyama approximations (see the proof of Lemma 3.9 in [**38**] for details).

2.2. Explicit approximation schemes

This section investigates stability properties and moment bounds of explicit approximation schemes. We begin with the Euler-Maruyama scheme in Subsection 2.2.1 and then analyze further approximation methods in Subsections 2.2.2 and 2.2.3 below.

2.2.1. Semi stability for the Euler-Maruyama scheme. The main result of this subsection, Theorem 2.13 below, gives sufficient conditions for the Euler-Maruyama scheme to be α-semi V-stable with respect to Brownian motion with $\alpha \in (0,\infty)$, $V\colon \mathbb{R}^d \to [0,\infty)$ appropriate and $d \in \mathbb{N}$. For proving this result, we first present three auxiliary results (Lemmas 2.10–2.12). The first one (Lemma 2.10) establishes α-semi V-stability with respect to Brownian motion of the Euler-Maruyama scheme with $\alpha \in (0,\infty)$, $V\colon \mathbb{R}^d \to [0,\infty)$ appropriate and $d \in \mathbb{N}$ under a general abstract condition (see inequality (2.51) below for details).

LEMMA 2.10 (An abstract condition for semi V-stability with respect to Brownian motion of the Euler-Maruyama scheme). *Let $\alpha, T \in (0,\infty)$, $d, m \in \mathbb{N}$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions, $\rho, \tilde{\rho} \in \mathbb{R}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion and let $V\colon \mathbb{R}^d \to [0,\infty)$ be a twice continuously differentiable function with $(\mathcal{G}_{\mu,\sigma}V)(x) \leq \rho \cdot V(x)$ and*

$$\mathbb{E}\Big[\big(\tilde{\mathcal{G}}_{\mu,\sigma}V\big)\big(x+\mu(x)\cdot t+\sigma(x)W_t, x\big) - \big(\mathcal{G}_{\mu,\sigma}V\big)(x)\Big] \leq \tilde{\rho}\cdot V(x) \tag{2.51}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T]\colon V(y) \leq s^{-\alpha}\}$. Then

$$\mathbb{E}\big[V\big(x+\mu(x)\cdot t+\sigma(x)W_t\big)\big] \leq e^{(\rho+\tilde{\rho})t}\cdot V(x) \tag{2.52}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T]\colon V(y) \leq s^{-\alpha}\}$. In particular, the Euler-Maruyama scheme $\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto x + \mu(x)t + \sigma(x)y \in \mathbb{R}^d$ is α-semi V-stable with respect to Brownian motion.

PROOF OF LEMMA 2.10. Ito's formula shows

$$\begin{aligned}
&\mathbb{E}\Big[V\big(x+\mu(x)\cdot t+\sigma(x)W_t\big)\Big] \\
&= V(x) + \int_0^t \mathbb{E}\Big[\big(\tilde{\mathcal{G}}_{\mu,\sigma}V\big)\big(x+\mu(x)\cdot t+\sigma(x)W_t, x\big)\Big]\,ds \\
&= V(x) + \big(\mathcal{G}_{\mu,\sigma}V\big)(x)\cdot t \\
&\quad + \int_0^t \mathbb{E}\Big[\big(\tilde{\mathcal{G}}_{\mu,\sigma}V\big)\big(x+\mu(x)\cdot t+\sigma(x)W_t, x\big) - \big(\mathcal{G}_{\mu,\sigma}V\big)(x)\Big]\,ds
\end{aligned} \tag{2.53}$$

for all $x \in \mathbb{R}^d$ and all $t \in [0, T]$ and the assumption $(\mathcal{G}_{\mu,\sigma} V)(x) \leq \rho \cdot V(x)$ hence implies

$$\begin{aligned} &\mathbb{E}\Big[V\big(x + \mu(x) \cdot t + \sigma(x) W_t\big)\Big] \\ &\leq V(x) + \rho \cdot t \cdot V(x) \\ &\quad + \int_0^t \mathbb{E}\Big[\big(\tilde{\mathcal{G}}_{\mu,\sigma} V\big)\big(x + \mu(x) \cdot t + \sigma(x) W_t, x\big) - \big(\mathcal{G}_{\mu,\sigma} V\big)(x)\Big]\, ds \end{aligned} \tag{2.54}$$

for all $x \in \mathbb{R}^d$ and all $t \in [0, T]$. Inequality (2.51) therefore shows

$$\begin{aligned} \mathbb{E}\Big[V\big(x + \mu(x) \cdot t + \sigma(x) W_t\big)\Big] &\leq V(x) + \rho \cdot t \cdot V(x) + \tilde{\rho} \cdot t \cdot V(x) \\ &= V(x)\,(1 + \rho \cdot t + \tilde{\rho} \cdot t) \leq e^{(\rho + \tilde{\rho}) t} \cdot V(x) \end{aligned} \tag{2.55}$$

for all $(x, t) \in \{(y, s) \in \mathbb{R}^d \times (0, T] \colon V(y) \leq s^{-\alpha}\}$. The proof of Lemma 2.10 is thus completed. □

In many situations, it is tedious to verify (2.51). We therefore give more concrete conditions for α-semi V-stability with respect to Brownian motion of the Euler-Maruyama scheme with $\alpha \in (0, \infty)$, $V \colon \mathbb{R}^d \to [0, \infty)$ appropriate and $d \in \mathbb{N}$ in Theorem 2.13 below. For establishing this theorem, the next simple lemma is used.

LEMMA 2.11. *Let $T \in (0, \infty), c \in [0, \infty), p \in [1, \infty)$ be real numbers and let $y \colon [0, T] \to \mathbb{R}$ be an absolute continuous function with $y'(t) \leq c\,|y(t)|^{(1-1/p)}$ for $\lambda_{[0,T]}$-almost all $t \in [0, T]$. Then*

$$y(t) \leq \Big[|y(0)|^{1/p} + \tfrac{ct}{p}\Big]^p \leq 2^{(p-1)} \left[|y(0)| + \left|\tfrac{ct}{p}\right|^p\right] \tag{2.56}$$

and

$$\begin{aligned} y(t) &\leq |y(0)| + c\,t\,\big[\,|y(0)|^{1/p} + c\,t\,\big]^{(p-1)} \\ &\leq |y(0)| + 2^{(p-1)} \Big[c\,t\,|y(0)|^{(1-1/p)} + |c\,t|^p\Big] \end{aligned} \tag{2.57}$$

for all $t \in [0, T]$.

PROOF OF LEMMA 2.11. The assumption $y'(t) \leq c\,|y(t)|^{(1-1/p)}$ for $\lambda_{[0,T]}$-almost all $t \in [0, T]$ implies

$$\int_{t_0}^t y'(s)\,(y(s))^{(1/p-1)}\, ds \leq c\,(t - t_0) \tag{2.58}$$

for all $t_0 \in (\tau(t), t]$ and all $t \in [0, T]$ with $y(t) > 0$ where the function $\tau \colon [0, T] \to [0, T]$ is defined through

$$\tau(t) := \max\Big(\{0\} \cap \big\{s \in [0, t] \colon y(s) = 0\big\}\Big) \tag{2.59}$$

for all $t \in [0, T]$. Estimate (2.58) then gives

$$\big(y(t)\big)^{1/p} \leq \big(y(t_0)\big)^{1/p} + \frac{c\,(t - t_0)}{p} \tag{2.60}$$

for all $t_0 \in (\tau(t), t]$ and all $t \in [0, T]$ with $y(t) > 0$. This yields

$$\big(y(t)\big)^{1/p} \leq \big|y\big(\tau(t)\big)\big|^{1/p} + \frac{c\,(t - \tau(t))}{p} \leq \big|y(0)\big|^{1/p} + \frac{ct}{p} \tag{2.61}$$

for all $t \in [0,T]$ with $y(t) > 0$. Inequality (2.61) implies (2.56). In addition, note that combining (2.56) and the inequality

$$(x+y)^r = x^r + \int_0^1 r\,(x+sy)^{(r-1)}\,y\,ds \leq x^r + r\,y\,(x+y)^{(r-1)} \tag{2.62}$$

for all $x, y \in [0,\infty)$, $r \in [1,\infty)$ shows (2.57). The proof of Lemma 2.11 is thus completed. □

An immediate consequence of Lemma 2.11 are the following estimates.

LEMMA 2.12. *Let $c, p \in [1,\infty)$ be real numbers and let $V \in C^1_p(\mathbb{R}^d,\mathbb{R})$ with $\|(\nabla V)(x)\| \leq c\,|V(x)|^{(1-1/p)}$ for $\lambda_{\mathbb{R}^d}$-almost all $x \in \mathbb{R}^d$. Then*

$$V(x+y) \leq c^p\,2^{(p-1)}\Big(|V(x)| + \|y\|^p\Big), \tag{2.63}$$

$$V(x+y) \leq |V(x)| + c^p\,2^{(p-1)}\left(\|y\|\,|V(x)|^{(1-1/p)} + \|y\|^p\right) \tag{2.64}$$

for all $x, y \in \mathbb{R}^d$.

PROOF OF LEMMA 2.12. First of all, note that the assumption $\|(\nabla V)(x)\| \leq c\,|V(x)|^{(1-1/p)}$ for $\lambda_{\mathbb{R}^d}$-almost all $x \in \mathbb{R}^d$ implies

$$\begin{aligned}\frac{d}{dt}\Big(V(x+ty)\Big) &= V'(x+ty)\,y \leq \|(\nabla V)(x+ty)\|\,\|y\| \\ &\leq c\,|V(x+ty)|^{(1-1/p)}\,\|y\|\end{aligned} \tag{2.65}$$

for $\lambda_{\mathbb{R}}$-almost all $t \in \mathbb{R}$ and all $x, y \in \mathbb{R}^d$. Inequality (2.57) in Lemma 2.11 hence gives (2.64). Moreover, combining (2.65) and (2.56) implies

$$V(x+y) \leq 2^{(p-1)}\left(\,|V(x)| + \left|\tfrac{c\|y\|}{p}\right|^p\,\right) \leq c^p\,2^{(p-1)}\left(\,|V(x)| + \|y\|^p\,\right) \tag{2.66}$$

for all $x, y \in \mathbb{R}^d$. The proof of Lemma 2.12 is thus completed. □

We are now ready to present the promised theorem which shows α-semi V-stability with respect to Brownian motion of the Euler-Maruyama scheme with $\alpha \in (0,\infty)$, $V\colon \mathbb{R}^d \to [0,\infty)$ appropriate and $d \in \mathbb{N}$. It makes use of Lemma 2.10 and Lemma 2.12 above.

THEOREM 2.13 (Semi V-stability with respect to Brownian motion for the Euler-Maruyama scheme). *Let $T \in (0,\infty)$, $d, m \in \mathbb{N}$, $p \in [3,\infty)$, $c, \gamma_0, \gamma_1 \in [0,\infty)$ be real numbers with $\gamma_0 + \gamma_1 > 0$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions and let $V \in C^3_p(\mathbb{R}^d,[1,\infty))$ with $(\mathcal{G}_{\mu,\sigma}V)(x) \leq c \cdot V(x)$ and*

$$\|\mu(x)\| \leq c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]} \qquad \textit{and} \qquad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]} \tag{2.67}$$

for all $x \in \mathbb{R}^d$. Then the Euler-Maruyama scheme $\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto x + \mu(x)t + \sigma(x)y \in \mathbb{R}^d$ is $p/(\gamma_1 + 2(\gamma_0 \vee \gamma_1))$-semi V-stable with respect to Brownian motion.

PROOF OF THEOREM 2.13. Throughout this proof, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion. We will prove semi V-stability with respect to Brownian motion for the Euler-Maruyama scheme by applying Lemma 2.10. Our aim is thus to verify (2.51). For this note that

$$
\begin{aligned}
&\mathbb{E}\Big[\big|(\tilde{\mathcal{G}}_{\mu,\sigma}V)\big(x+\mu(x)\cdot t+\sigma(x)W_t, x\big)-(\mathcal{G}_{\mu,\sigma}V)(x)\big|\Big]\\
&\leq \mathbb{E}\Big[\big\|V'\big(x+\mu(x)\cdot t+\sigma(x)W_t\big)-V'(x)\big\|_{L(\mathbb{R}^d,\mathbb{R})}\Big]\,\|\mu(x)\|\\
&+\frac{1}{2}\cdot\mathbb{E}\Big[\big\|V''\big(x+\mu(x)\cdot t+\sigma(x)W_t\big)-V''(x)\big\|_{L^{(2)}(\mathbb{R}^d,\mathbb{R})}\Big]\left(\sum_{k=1}^{m}\|\sigma_k(x)\|^2\right)
\end{aligned}
\tag{2.68}
$$

for all $x\in\mathbb{R}^d$ and all $t\in[0,T]$. Assumption (2.67) hence shows

$$
\begin{aligned}
&\mathbb{E}\Big[\big|(\tilde{\mathcal{G}}_{\mu,\sigma}V)\big(x+\mu(x)\cdot t+\sigma(x)W_t, x\big)-(\mathcal{G}_{\mu,\sigma}V)(x)\big|\Big]\\
&\leq m\,(c+1)^2\\
&\cdot\left(\sum_{i=1}^{2}\mathbb{E}\Big[\big\|V^{(i)}\big(x+\mu(x)\cdot t+\sigma(x)W_t\big)-V^{(i)}(x)\big\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})}\Big]\,|V(x)|^{\left[\frac{\gamma_{(i-1)}+i}{p}\right]}\right)
\end{aligned}
\tag{2.69}
$$

for all $x\in\mathbb{R}^d$ and all $t\in[0,T]$. Next observe that the assumption

$$
V\in C_p^3(\mathbb{R}^d,[1,\infty))
\tag{2.70}
$$

and estimate (2.63) imply the existence of a real number $\hat{c}\in[c+1,\infty)$ such that

$$
\begin{aligned}
&\Big\|V^{(i)}(y)-V^{(i)}(x)\Big\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})}\\
&\leq\int_0^1\Big\|V^{(i+1)}\big(x+r(y-x)\big)\Big\|_{L^{(i+1)}(\mathbb{R}^d,\mathbb{R})}\|y-x\|\,dr\\
&\leq\hat{c}\left(\int_0^1\big|V\big(x+r(y-x)\big)\big|^{\frac{(p-i-1)}{p}}\,dr\right)\|y-x\|\\
&\leq|\hat{c}|^{(p+1)}\,2^p\left(|V(x)|^{\frac{(p-i-1)}{p}}+\|y-x\|^{(p-i-1)}\right)\|y-x\|\\
&=|\hat{c}|^{(p+1)}\,2^p\left(|V(x)|^{\frac{(p-i-1)}{p}}\|y-x\|+\|y-x\|^{(p-i)}\right)
\end{aligned}
\tag{2.71}
$$

for all $x,y\in\mathbb{R}^d$ and all $i\in\{1,2\}$. Putting (2.71) into (2.69) then results in

$$
\begin{aligned}
&\mathbb{E}\Big[\big|(\tilde{\mathcal{G}}_{\mu,\sigma}V)\big(x+\mu(x)\cdot t+\sigma(x)W_t, x\big)-(\mathcal{G}_{\mu,\sigma}V)(x)\big|\Big]\\
&\leq m\,|\hat{c}|^{(p+3)}\,2^p\left(\sum_{i=1}^{2}\mathbb{E}\Big[\|\mu(x)\cdot t+\sigma(x)W_t\|\Big]\,|V(x)|^{\left[\frac{\gamma_{(i-1)}+p-1}{p}\right]}\right)\\
&\quad+m\,|\hat{c}|^{(p+3)}\,2^p\left(\sum_{i=1}^{2}\mathbb{E}\Big[\|\mu(x)\cdot t+\sigma(x)W_t\|^{(p-i)}\Big]\,|V(x)|^{\left[\frac{\gamma_{(i-1)}+i}{p}\right]}\right)
\end{aligned}
\tag{2.72}
$$

for all $x \in \mathbb{R}^d$ and all $t \in [0, T]$. In the next step, we put the estimate

$$\begin{aligned}
&\mathbb{E}\Big[\big\|\mu(x) \cdot t + \sigma(x) W_t\big\|^r\Big] \\
&\leq (m+1)^r \left(\|\mu(x)\|^r \cdot t^r + \sum_{k=1}^{m} \|\sigma_k(x)\|^r \cdot \mathbb{E}\Big[\big|W_t^{(k)}\big|^r\Big] \right) \\
&\leq c^r\,(m+1)^{(r+1)} \left(|V(x)|^{\left[\frac{r\gamma_0 + r}{p}\right]} t^r + |V(x)|^{\left[\frac{r\gamma_1 + 2r}{2p}\right]} \mathbb{E}\Big[\big|W_t^{(1)}\big|^r\Big] \right) \\
&\leq c^r\,(m+r+1)^{(2r+1)} \left(|V(x)|^{\left[\frac{r\gamma_0 + r}{p}\right]} t^r + |V(x)|^{\left[\frac{r\gamma_1 + 2r}{2p}\right]} t^{\frac{r}{2}} \right) \\
&= c^r\,(m+r+1)^{(2r+1)} \left(\sum_{j=1}^{2} |V(x)|^{\left[\frac{r\gamma_{(j-1)} + jr}{jp}\right]} t^{\frac{r}{j}} \right)
\end{aligned} \tag{2.73}$$

for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $r \in [0, \infty)$ into (2.72) to obtain

$$\begin{aligned}
&\mathbb{E}\Big[\big|(\tilde{\mathcal{G}}_{\mu,\sigma} V)\big(x + \mu(x) \cdot t + \sigma(x) W_t, x\big) - (\mathcal{G}_{\mu,\sigma} V)(x)\big|\Big] \\
&\leq (m+p)^{3p}\,|\hat{c}|^{(2p+3)}\,2^p \left(\sum_{i,j=1}^{2} |V(x)|^{\left[\frac{\gamma_{(i-1)} + \gamma_{(j-1)}/j}{p} + 1\right]} t^{\frac{1}{j}} \right) \\
&\quad + (m+p)^{3p}\,|\hat{c}|^{(2p+3)}\,2^p \left(\sum_{i,j=1}^{2} |V(x)|^{\left[\frac{\gamma_{(i-1)} + (p-i)\gamma_{(j-1)}/j}{p} + 1\right]} t^{\frac{(p-i)}{j}} \right)
\end{aligned} \tag{2.74}$$

and hence

$$\begin{aligned}
&\mathbb{E}\Big[\big|(\tilde{\mathcal{G}}_{\mu,\sigma} V)\big(x + \mu(x) \cdot t + \sigma(x) W_t, x\big) - (\mathcal{G}_{\mu,\sigma} V)(x)\big|\Big] \leq (m+p)^{3p}\,|\hat{c}|^{(2p+3)}\,2^p \\
&\cdot \left(\sum_{i,j=1}^{2} \left[|V(x)|^{\left[\frac{\gamma_{(i-1)} + \gamma_{(j-1)}/j}{p}\right]} t^{\frac{1}{j}} + |V(x)|^{\left[\frac{\gamma_{(i-1)} + (p-i)\gamma_{(j-1)}/j}{p}\right]} t^{\frac{(p-i)}{j}} \right] \right) V(x)
\end{aligned} \tag{2.75}$$

for all $x \in \mathbb{R}^d$ and all $t \in [0, T]$. This implies that there exists a real number $\tilde{\rho} \in [0, \infty)$ such that

$$\mathbb{E}\Big[\big|(\tilde{\mathcal{G}}_{\mu,\sigma} V)\big(x + \mu(x) \cdot t + \sigma(x) W_t, x\big) - (\mathcal{G}_{\mu,\sigma} V)(x)\big|\Big] \leq \tilde{\rho} \cdot V(x) \tag{2.76}$$

for all $(x, t) \in \{(y, s) \in \mathbb{R}^d \times (0, T] \colon V(y) \leq s^{-\alpha}\}$ where

$$\alpha := \min_{i,j \in \{1,2\}} \left(\min\left\{ \frac{p}{j \cdot \left(\gamma_{(i-1)} + \frac{\gamma_{(j-1)}}{j}\right)}, \frac{p \cdot (p-i)}{j \cdot \left(\gamma_{(i-1)} + \frac{(p-i)\gamma_{(j-1)}}{j}\right)} \right\} \right). \tag{2.77}$$

Lemma 2.10 therefore shows that the Euler-Maruyama scheme is α-semi V-stable with respect to Brownian motion, i.e., there exists a real number $\rho \in \mathbb{R}$ such that

$$\mathbb{E}\big[V(x + \mu(x) \cdot t + \sigma(x) W_t)\big] \leq e^{\rho t} \cdot V(x) \tag{2.78}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T] \colon V(y) \leq s^{-\alpha}\}$. Finally, note that

(2.79)
$$\begin{aligned}\alpha &= \min_{i,j\in\{1,2\}}\left(\min\left\{\frac{p}{j\cdot\Big(\gamma_{(i-1)}+\frac{\gamma_{(j-1)}}{j}\Big)},\frac{p}{j\cdot\Big(\frac{\gamma_{(i-1)}}{(p-i)}+\frac{\gamma_{(j-1)}}{j}\Big)}\right\}\right)\\ &= \min_{i,j\in\{1,2\}}\left(\frac{p}{\big(j\cdot\gamma_{(i-1)}+\gamma_{(j-1)}\big)}\right) = \frac{p}{\max_{j\in\{1,2\}}\Big(j\cdot\max(\gamma_0,\gamma_1)+\gamma_{(j-1)}\Big)}\\ &= \frac{p}{\gamma_1+2\max(\gamma_0,\gamma_1)}.\end{aligned}$$

This completes the proof of Theorem 2.13. □

If $V\colon \mathbb{R}^d \to [1,\infty)$ is a Lyapunov-type function in the sense of Theorem 2.13 with $d \in \mathbb{N}$, then the function $\mathbb{R}^d \ni x \mapsto |V(x)|^q \in [1,\infty)$ with $q \in (0,\infty)$ appropriate is in many situations a Lyapunov-type function too. This is the subject of Corollary 2.15 below. The next result is a simple lemma which will be used in the proof of Corollary 2.15.

LEMMA 2.14. *Let $q \in [1,\infty)$, $p \in [2,\infty)$, $d \in \mathbb{N}$ and let $V \in C^3_p(\mathbb{R}^d,[1,\infty))$. Then the function $\hat{V}\colon \mathbb{R}^d \to [1,\infty)$ given by $\hat{V}(x) = (V(x))^q$ for all $x \in \mathbb{R}^d$ satisfies $\hat{V} \in C^3_{pq}(\mathbb{R}^d,[1,\infty))$.*

PROOF OF LEMMA 2.14. Note that

$$\hat{V}'(x)(v_1) = q\,(V(x))^{(q-1)}\,V'(x)(v_1), \tag{2.80}$$

$$\begin{aligned}&\hat{V}''(x)(v_1,v_2)\\ &= q\,(q-1)\,(V(x))^{(q-2)}\,V'(x)(v_1)\,V'(x)(v_2) + q\,(V(x))^{(q-1)}\,V''(x)(v_1,v_2)\end{aligned} \tag{2.81}$$

for all $x, v_1, v_2 \in \mathbb{R}^d$ and

$$\begin{aligned}&\hat{V}^{(3)}(x)(v_1,v_2,v_3)\\ &= q\,(q-1)\,(q-2)\,(V(x))^{(q-3)}\,V'(x)(v_1)\,V'(x)(v_2)\,V'(x)(v_3)\\ &\quad + q\,(q-1)\,(V(x))^{(q-2)}\,V''(x)(v_1,v_2)\,V'(x)(v_3)\\ &\quad + q\,(q-1)\,(V(x))^{(q-2)}\,V''(x)(v_1,v_3)\,V'(x)(v_2)\\ &\quad + q\,(q-1)\,(V(x))^{(q-2)}\,V'(x)(v_1)\,V''(x)(v_2,v_3)\\ &\quad + q\,(V(x))^{(q-1)}\,V^{(3)}(x)(v_1,v_2,v_3)\end{aligned} \tag{2.82}$$

for all $v_1, v_2, v_3 \in \mathbb{R}^d$ and all $x \in \{y \in \mathbb{R}^d \colon V'' \text{ is differentiable in } y\}$. The assumption $V \in C^3_p(\mathbb{R}^d,[1,\infty))$ therefore shows that $\hat{V} \in C^3_{pq}(\mathbb{R}^d,[1,\infty))$. The proof of Lemma 2.14 is thus completed. □

COROLLARY 2.15 (Powers of the Lyapunov-type function). *Let $T \in (0,\infty)$, $d, m \in \mathbb{N}$, $p \in [3,\infty)$, $q \in [1,\infty)$, $c, \gamma_0, \gamma_1 \in [0,\infty)$ be real numbers with $\gamma_0 + \gamma_1 > 0$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions and let $V \in C^3_p(\mathbb{R}^d,[1,\infty))$ with $\|\mu(x)\| \leq c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]}$, $\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]}$ and*

$$(\mathcal{G}_{\mu,\sigma}V)(x) + \frac{(q-1)\,\|V'(x)\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R})}}{2\cdot V(x)} \leq c\cdot V(x) \tag{2.83}$$

for all $x \in \mathbb{R}^d$*. Then the Euler-Maruyama scheme*

$$\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto x + \mu(x)t + \sigma(x)y \in \mathbb{R}^d \tag{2.84}$$

is $pq/(\gamma_1 + 2(\gamma_0 \vee \gamma_1))$*-semi* $|V|^q$*-stable with respect to Brownian motion.*

Proof of Corollary 2.15. First, define the function $\hat{V} \colon \mathbb{R}^d \to [1,\infty)$ by $\hat{V}(x) = (V(x))^q$ for all $x \in \mathbb{R}^d$. Then note that this definition ensures

$$\|\mu(x)\| \leq c\,|\hat{V}(x)|^{\left[\frac{\gamma_0+1}{pq}\right]} \qquad \text{and} \qquad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,|\hat{V}(x)|^{\left[\frac{\gamma_1+2}{2pq}\right]} \tag{2.85}$$

for all $x \in \mathbb{R}^d$. In addition, observe that

$$\big(\mathcal{G}_{\mu,\sigma}(V^r)\big)(x) = r\,(V(x))^{(r-1)} \left((\mathcal{G}_{\mu,\sigma}V)(x) + \frac{(r-1)\,\|V'(x)\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R})}}{2 \cdot V(x)} \right) \tag{2.86}$$

for all $x \in \mathbb{R}^d$ and all $r \in (0,\infty)$. Therefore, we obtain

$$(\mathcal{G}_{\mu,\sigma}\hat{V})(x) = q\,(V(x))^{(q-1)} \left((\mathcal{G}_{\mu,\sigma}V)(x) + \frac{(q-1)\,\|V'(x)\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R})}}{2 \cdot V(x)} \right) \tag{2.87}$$

for all $x \in \mathbb{R}^d$ and (2.83) hence gives

$$(\mathcal{G}_{\mu,\sigma}\hat{V})(x) \leq q \cdot c \cdot \hat{V}(x) \tag{2.88}$$

for all $x \in \mathbb{R}^d$. Combining (2.85), (2.88), Lemma 2.14 and Theorem 2.13 then shows that the Euler-Maruyama scheme is $\frac{pq}{\gamma_1+2(\gamma_0\vee\gamma_1)}$-semi $\hat{V}$-stable with respect to Brownian motion. The proof of Corollary 2.15 is thus completed. □

Note that in (2.83) the norm in the Hilbert space of *Hilbert-Schmidt operators* from $\mathbb{R}^m$ to $\mathbb{R}$ is used where $m \in \mathbb{N}$. The definition of that norm and more details on Hilbert-Schmidt operators can, e.g., be found in Appendix B in Prévôt & Röckner [**66**]. In the next step, Theorem 2.13 is illustrated by a simple example. More precisely, the next corollary considers the special Lyapunov-type function $V \colon \mathbb{R}^d \to [1,\infty)$ given by $V(x) = 1 + \|x\|^p$ for all $x \in \mathbb{R}^d$ with $p \in [3,\infty)$ and $d \in \mathbb{N}$.

Corollary 2.16 (A special polynomial like Lyapunov-type function). *Let* $d, m \in \mathbb{N}$, $T \in (0,\infty)$, $c, \gamma_0, \gamma_1 \in [0,\infty)$, $p \in [3,\infty)$ *be real numbers with* $\gamma_0 + \gamma_1 > 0$*, let* $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ *be Borel measurable functions with*

$$\langle x, \mu(x)\rangle + \tfrac{(p-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^2\right), \tag{2.89}$$

$$\|\mu(x)\| \leq c\left(1 + \|x\|^{[\gamma_0+1]}\right) \qquad \textit{and} \qquad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^{\left[\frac{\gamma_1+2}{2}\right]}\right) \tag{2.90}$$

for all $x \in \mathbb{R}^d$*. Then the Euler-Maruyama scheme*

$$\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto x + \mu(x)t + \sigma(x)y \in \mathbb{R}^d \tag{2.91}$$

is $p/(\gamma_1 + 2(\gamma_0 \vee \gamma_1))$*-semi* $(1 + \|x\|^p)_{x\in\mathbb{R}^d}$*-stable with respect to Brownian motion.*

Proof of Corollary 2.16. Let $V \colon \mathbb{R}^d \to [1,\infty)$ be given by $V(x) = 1 + \|x\|^p$ for all $x \in \mathbb{R}^d$. Then note that inequality (2.89) implies

$$\begin{aligned} (\mathcal{G}_{\mu,\sigma}V)(x) &\leq p\,\|x\|^{(p-2)}\,\langle x, \mu(x)\rangle + \tfrac{p(p-1)}{2}\,\|x\|^{(p-2)}\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \\ &\leq 2 \cdot p \cdot c \cdot V(x) \end{aligned} \tag{2.92}$$

for all $x \in \mathbb{R}^d$. Next observe that

$$\|\mu(x)\| \leq c\left(1 + \|x\|^{[\gamma_0+1]}\right) \leq 2c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]}, \tag{2.93}$$

$$\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^{\left[\frac{\gamma_1+2}{2}\right]}\right) \leq 2c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]} \tag{2.94}$$

for all $x \in \mathbb{R}^d$. Combining (2.92)–(2.94), the fact $V \in C_p^3(\mathbb{R}^d, [1,\infty))$ and Theorem 2.13 hence shows that the Euler-Maruyama scheme is $p/(\gamma_1 + 2(\gamma_0 \vee \gamma_1))$-semi V-stable with respect to Brownian motion. The proof of Corollary 2.16 is thus completed. □

Theorem 2.13, Corollary 2.15 and Corollary 2.16 give sufficient conditions for the Euler-Maruyama scheme to be α-semi V-stable with respect to Brownian motion with $\alpha \in (0,\infty)$, $V\colon \mathbb{R}^d \to [0,\infty)$ appropriate and $d \in \mathbb{N}$. One may ask whether the Euler-Maruyama scheme is also V-stable with respect to Brownian motion. The next result, which is a corollary of Theorem 2.1 in [**39**] (which generalizes Theorem 2.1 in [**40**]), disproves this statement for one-dimensional SDEs in which at least one of the coefficients μ and σ grows more than linearly. There is thus a large class of SDEs in which the Euler-Maruyama scheme is α-semi V-stable but not V-stable with respect to Brownian motion.

Corollary 2.17 (Disproof of V-stability with respect to Brownian motion for the Euler-Maruyama scheme). *Let $T, \rho \in (0,\infty)$, $\alpha, c \in (1,\infty)$, let $\mu, \sigma\colon \mathbb{R} \to \mathbb{R}$ be Borel measurable functions with*

$$|\mu(x)| + |\sigma(x)| \geq \frac{|x|^\alpha}{c} \tag{2.95}$$

for all $x \in (-\infty, c] \cup [c,\infty)$ and with $\sigma \neq 0$ (i.e., with the property that there exists a real number $x_0 \in \mathbb{R}$ such that $\sigma(x_0) \neq 0$). Then there exists no Borel measurable function $V\colon \mathbb{R} \to [0,\infty)$ which fulfills

$$\limsup_{r\searrow 0} \sup_{x\in\mathbb{R}} \left(\frac{|x|^r}{(1+V(x))}\right) < \infty \tag{2.96}$$

and

$$\mathbb{E}\big[V\big(x + \mu(x)\cdot t + \sigma(x)W_t\big)\big] \leq e^{\rho t}\cdot V(x) \tag{2.97}$$

for all $(x,t) \in \mathbb{R}\times(0,T]$ where $W\colon [0,T]\times\Omega \to \mathbb{R}$ is an arbitrary standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof of Corollary 2.17. Suppose that $V\colon \mathbb{R} \to [0,\infty)$ is a Borel measurable function which fulfills (2.97) and

$$\limsup_{r\searrow 0} \sup_{x\in\mathbb{R}} \left(\frac{|x|^r}{(1+V(x))}\right) < \infty. \tag{2.98}$$

Then there exists a real number $r \in (0,\infty)$ such that

$$|x|^r \leq \frac{1}{r}\,(1 + V(x)) \tag{2.99}$$

for all $x \in \mathbb{R}$. Next observe that the assumption $\sigma \neq 0$ implies that there exists a real number $x_0 \in \mathbb{R}$ with $\sigma(x_0) \neq 0$. Now define a sequence $Y^N\colon \{0,1,\ldots,N\}\times\Omega \to \mathbb{R}$, $N \in \mathbb{N}$, of stochastic processes by $Y_0^N = x_0$ and

$$Y_{n+1}^N = Y_n^N + \mu(Y_n^N)\tfrac{T}{N} + \sigma(Y_n^N)\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right) \tag{2.100}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Corollary 2.2 then implies

$$\limsup_{N\to\infty} \mathbb{E}\big[\|Y_N^N\|^r\big] \leq \tfrac{1}{r}\left(1 + \limsup_{N\to\infty} \mathbb{E}\big[V(Y_N^N)\big]\right) \leq \tfrac{1}{r}\left(1 + e^{\rho T}\cdot V(x_0)\right) < \infty. \tag{2.101}$$

This contradicts to Theorem 2.1 in [**39**]. The proof of Corollary 2.17 is thus completed. □

2.2.2. Semi stability for tamed schemes. In the previous subsection, semi V-stability with respect to Brownian motion for the Euler-Maruyama scheme has been analyzed. In this subsection, semi V-stability with respect to Brownian motion for appropriately modified Euler-type methods is investigated. We begin with a general abstract result which shows that if two numerical schemes are close in some sense (see inequality (2.102) below for details) and if one of the two numerical schemes is α-semi V-stable with respect to Brownian motion with $\alpha \in (0,\infty)$, $V\colon \mathbb{R}^d \to [0,\infty)$ appropriate and $d \in \mathbb{N}$, then the other scheme is α-semi V-stable with respect to Brownian motion too.

Lemma 2.18 (A comparison principle for semi V-stability with respect to Brownian motion). *Let $\alpha, T \in (0,\infty)$, $c \in [0,\infty)$, $p \in [1,\infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $\Phi, \tilde{\Phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be Borel measurable functions and let $V \in C_p^1(\mathbb{R}^d, [0,\infty))$ be such that $\Phi\colon \mathbb{R}^d \times [0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ is α-semi V-stable with respect to Brownian motion and such that*

$$\left(\mathbb{E}\Big[\big\|\Phi(x,t,W_t) - \tilde{\Phi}(x,t,W_t)\big\|^p\Big]\right)^{1/p} \leq c\cdot t\cdot |V(x)|^{1/p} \tag{2.102}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T]\colon V(y) \leq s^{-\alpha}\}$. Then $\tilde{\Phi}\colon \mathbb{R}^d \times [0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ is also α-semi V-stable with respect to Brownian motion.

Proof of Lemma 2.18. Inequality (2.64) in Lemma 2.12 implies the existence of a real number $\hat{c} \in [c+1,\infty)$ such that

$$\begin{aligned}
&\mathbb{E}\Big[V\big(\tilde{\Phi}(x,t,W_t)\big)\Big]\\
&= \mathbb{E}\Big[V\Big(\Phi(x,t,W_t) + \big\{\tilde{\Phi}(x,t,W_t) - \Phi(x,t,W_t)\big\}\Big)\Big]\\
&\leq \mathbb{E}\Big[V\big(\Phi(x,t,W_t)\big)\Big]\\
&\quad + 2^p\,|\hat{c}|^p\,\mathbb{E}\Big[\big|V\big(\Phi(x,t,W_t)\big)\big|^{\frac{(p-1)}{p}}\,\big\|\Phi(x,t,W_t) - \tilde{\Phi}(x,t,W_t)\big\|\Big]\\
&\quad + 2^p\,|\hat{c}|^p\,\mathbb{E}\Big[\big\|\Phi(x,t,W_t) - \tilde{\Phi}(x,t,W_t)\big\|^p\Big]
\end{aligned} \tag{2.103}$$

and Hölder's inequality hence gives

$$\begin{aligned}
&\mathbb{E}\Big[V\big(\tilde{\Phi}(x,t,W_t)\big)\Big]\\
&\leq \mathbb{E}\Big[V\big(\Phi(x,t,W_t)\big)\Big]\\
&\quad + 2^p\,|\hat{c}|^p\left(\mathbb{E}\Big[V\big(\Phi(x,t,W_t)\big)\Big]\right)^{\frac{(p-1)}{p}}\left(\mathbb{E}\Big[\big\|\Phi(x,t,W_t) - \tilde{\Phi}(x,t,W_t)\big\|^p\Big]\right)^{\frac{1}{p}}\\
&\quad + 2^p\,|\hat{c}|^p\,\mathbb{E}\Big[\big\|\Phi(x,t,W_t) - \tilde{\Phi}(x,t,W_t)\big\|^p\Big]
\end{aligned} \tag{2.104}$$

for all $x \in \mathbb{R}^d$ and all $t \in [0,T]$. Moreover, by assumption there exists a real number $\rho \in [0,\infty)$ such that

$$\mathbb{E}\Big[V\big(\Phi(x,t,W_t)\big)\Big] \leq e^{\rho t} \cdot V(x) \tag{2.105}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T] \colon V(y) \leq s^{-\alpha}\}$. Putting (2.105) and (2.102) into (2.104) then results in

$$\begin{aligned}\mathbb{E}\Big[V\big(\tilde{\Phi}(x,t,W_t)\big)\Big] &\leq e^{\rho t} \cdot V(x) + 2^p\,|\hat{c}|^p\,\big(e^{\rho t} \cdot \hat{c} \cdot t \cdot V(x) + |\hat{c}|^p \cdot t^p \cdot V(x)\big)\\ &\leq \Big(e^{\rho t} + t\,(T+1)^p\,e^{\rho T}\,2^{(p+1)}\,|\hat{c}|^{2p}\Big) \cdot V(x)\\ &\leq \exp\!\Big(\Big[\rho + (T+1)^p\,e^{\rho T}\,2^{(p+1)}\,|\hat{c}|^{2p}\Big] \cdot t\Big) \cdot V(x)\end{aligned} \tag{2.106}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T] \colon V(y) \leq s^{-\alpha}\}$. The proof of Lemma 2.18 is thus completed. □

A direct consequence of Lemma 2.18 is the next corollary. It proves α-semi V-stability with respect to Brownian motion with $\alpha \in (0,\infty)$, $V \colon \mathbb{R}^d \to [0,\infty)$ appropriate and $d \in \mathbb{N}$ for a class of suitably "tamed" numerical methods.

Corollary 2.19 (Semi V-stability with respect to Brownian motion for an increment-taming principle). *Let $\alpha, T \in (0,\infty)$, $\beta, c \in [0,\infty)$, $p \in [1,\infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $V \in C^1_p(\mathbb{R}^d, [0,\infty))$, let $\Phi \colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be a Borel measurable function which is α-semi V-stable with respect to Brownian motion and assume that*

$$\mathbb{E}\Big[\|\Phi(x,t,W_t) - x\|^{2p}\Big] \leq c \cdot t^{p(1-\beta)} \cdot V(x) \tag{2.107}$$

for all $(x,t) \in \{(y,s) \in \mathbb{R}^d \times (0,T] \colon V(y) \leq s^{-\alpha}\}$. Then the function

$$\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto x + \frac{\Phi(x,t,y) - x}{\max(1, t^\beta\,\|\Phi(x,t,y) - x\|)} \in \mathbb{R}^d \tag{2.108}$$

is α-semi V-stable with respect to Brownian motion.

Proof of Corollary 2.19. We show Corollary 2.19 by an application of Lemma 2.18. We thus need to verify inequality (2.102). For this let $\tilde{\Phi} \colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be given by

$$\tilde{\Phi}(t,x,y) = x + \frac{\Phi(x,t,y) - x}{\max\,(1, t^\beta\,\|\Phi(x,t,y) - x\|)} \tag{2.109}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $y \in \mathbb{R}^m$ and note that

$$\begin{aligned}\big\|\Phi(x,t,W_t) - \tilde{\Phi}(x,t,W_t)\big\| &= \|\Phi(x,t,W_t) - x\| \cdot \frac{\max\big(1, t^\beta\,\|\Phi(x,t,y) - x\|\big) - 1}{\max(1, t^\beta\,\|\Phi(x,t,y) - x\|)}\\ &\leq t^\beta\,\|\Phi(x,t,W_t) - x\|^2\end{aligned} \tag{2.110}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$ and all $y \in \mathbb{R}^m$. Estimate (2.107) therefore shows (2.102) and Lemma 2.18 hence completes the proof of Corollary 2.19. □

The next result is an immediate consequence of Corollary 2.19 and Theorem 2.13. It shows α-semi V-stability with respect to Brownian motion with $\alpha \in (0, \infty)$, $V\colon \mathbb{R}^d \to [0, \infty)$ appropriate and $d \in \mathbb{N}$ for a suitable "tamed" Euler-Maruyama scheme.

Corollary 2.20 (Semi V-stability with respect to Brownian motion for an increment-tamed Euler-Maruyama scheme). *Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, $p \in [3, \infty)$, $c, \gamma_0, \gamma_1 \in [0, \infty)$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions and let $V \in C_p^3(\mathbb{R}^d, [1, \infty))$ with*

$$(\mathcal{G}_{\mu,\sigma}V)(x) \leq c \cdot V(x), \quad \|\mu(x)\| \leq c\, |V(x)|^{\left[\frac{\gamma_0+1}{p}\right]}, \quad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\, |V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]}$$

for all $x \in \mathbb{R}^d$. Then the function

$$\mathbb{R}^d \times [0, T] \times \mathbb{R}^m \ni (x, t, y) \mapsto x + \frac{\mu(x)t + \sigma(x)y}{\max(1, t\|\mu(x)t + \sigma(x)y\|)} \in \mathbb{R}^d \tag{2.111}$$

is $\frac{p}{\gamma_1+2\max(\gamma_0,\gamma_1,1/2)}$-semi V-stable with respect to Brownian motion.

Proof of Corollary 2.20. We prove Corollary 2.20 by using Theorem 2.13 and Corollary 2.19. We thus need to verify condition (2.107). For this let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W = (W^{(1)}, \ldots, W^{(m)})\colon [0, \infty) \times \Omega \to \mathbb{R}^m$ be an m-dimensional standard Brownian motion throughout this proof. Then note that the assumptions

$$\|\mu(x)\| \leq c\, |V(x)|^{(\gamma_0+1)/p} \quad \text{and} \quad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\, |V(x)|^{(\gamma_1+2)/(2p)} \tag{2.112}$$

for all $x \in \mathbb{R}^d$ imply

$$\begin{aligned}
&\mathbb{E}\Big[\|\mu(x) \cdot t + \sigma(x)W_t\|^{2p}\Big] \\
&\leq (m+1)^{(2p-1)} \left(\|\mu(x)\|^{2p}\, t^{2p} + \sum_{k=1}^{m} \|\sigma_k(x)\|^{2p}\, \mathbb{E}\big[|W_t^{(k)}|^{2p}\big]\right) \\
&\leq (m+1)^{(2p-1)}\, \mathbb{E}\big[|W_1^{(1)}|^{2p}\big] \left(\|\mu(x)\|^{2p}\, t^{2p} + \sum_{k=1}^{m} \|\sigma_k(x)\|^{2p}\, t^p\right) \\
&\leq c^{2p}\, (m+1)^{2p}\, \mathbb{E}\big[|W_1^{(1)}|^{2p}\big] \Big(|V(x)|^{(2\gamma_0+1)}\, t^{2p} + |V(x)|^{(\gamma_1+1)}\, t^p\Big)\, V(x)
\end{aligned} \tag{2.113}$$

for all $x \in \mathbb{R}^d$ and all $t \in [0, T]$. Applying Theorem 2.13 and Corollary 2.19 therefore shows that (2.111) is α-semi V-stable with respect to Brownian motion with

$$\alpha := \min\!\left(\frac{2p}{2\gamma_0+1}, \frac{p}{\gamma_1+1}, \frac{p}{\gamma_1+2(\gamma_0 \vee \gamma_1)}\right) \in (0, \infty). \tag{2.114}$$

Next note that

$$\begin{aligned}
\alpha &= \frac{p}{\max\!\big(\gamma_0+\frac{1}{2}, \gamma_1+1, \gamma_1+2(\gamma_0 \vee \gamma_1)\big)} \\
&= \frac{p}{\max\!\big(\gamma_0+\frac{1}{2}, \gamma_1+2\max(\gamma_0,\gamma_1,\frac{1}{2})\big)} = \frac{p}{\gamma_1+2\max(\gamma_0,\gamma_1,\frac{1}{2})}.
\end{aligned} \tag{2.115}$$

The proof of Corollary 2.20 is thus completed. □

Note that, under the assumptions of Theorem 2.13, the Euler-Maruyama scheme is $p/(\gamma_1 + 2\max(\gamma_0, \gamma_1))$-semi V-stable with respect to Brownian motion but the appropriate tamed numerical method in Corollary 2.20 is $p/(\gamma_1+2\max(\gamma_0, \gamma_1, 1/2))$-semi V-stable with respect to Brownian motion.

2.2.3. Moment bounds for an increment-tamed Euler-Maruyama scheme. In Subsections 2.2.1 and 2.2.2 above, we established under suitable assumptions that the Euler-Maruyama scheme (see Theorem 2.13 in Subsection 2.2.1) as well as appropriately tamed numerical methods (see Corollary 2.19 and Corollary 2.20 in Subsection 2.2.2) are α-semi V-stable with respect to Brownian motion with $\alpha \in (1, \infty)$, $V \colon \mathbb{R}^d \to [0, \infty)$ appropriate and $d \in \mathbb{N}$. Corollary 2.6 can then be applied to show that these approximations have uniformly bounded moments restricted to events whose probabilities converge to one with convergence order $(\alpha-1)$. In the case of appropriately tamed numerical methods (see Subsection 2.2.2 for a few simple examples and Section 3.6 below for more examples), it can even be shown that the approximations have uniformly bounded moments without restricting to a sequence of events whose probabilities converge to one sufficiently fast. In the next result this is illustrated in the case of the increment-tamed Euler-Maruyama scheme from Corollary 2.20. Its proof is based on an application of Corollary 2.9 above.

COROLLARY 2.21 (Moment bounds for an increment-tamed Euler-Maruyama scheme). *Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, $p \in [3, \infty)$, $c, \gamma_0, \gamma_1 \in [0, \infty)$, let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be Borel measurable functions, let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P}\big)$ be a filtered probability space, let $W \colon [0, T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t \in [0,T]}$-Brownian motion and let $\xi \colon \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}\big[\|\xi\|^q\big] < \infty$ for all $q \in [0, \infty)$. Moreover, let $V \in C_p^3(\mathbb{R}^d, [1, \infty))$ with*

$$(\mathcal{G}_{\mu,\sigma} V)(x) \leq c \cdot V(x), \ \|\mu(x)\| \leq c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]}, \ \|\sigma(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \leq c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]}$$

for all $x \in \mathbb{R}^d$. Furthermore, let $\bar{Y}^N \colon [0, T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes given by $\bar{Y}_0^N = \xi$ and

$$\bar{Y}_t^N = \bar{Y}_{\frac{nT}{N}}^N + \left(\tfrac{tN}{T} - n\right) \cdot \frac{\mu(\bar{Y}_{\frac{nT}{N}}^N)\frac{T}{N} + \sigma(\bar{Y}_{\frac{nT}{N}}^N)(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}})}{\max\!\big(1, \frac{T}{N}\|\mu(\bar{Y}_{\frac{nT}{N}}^N)\frac{T}{N} + \sigma(\bar{Y}_{\frac{nT}{N}}^N)(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}})\|\big)} \tag{2.116}$$

for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$. Then

$$\sup_{N \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\big[\|\bar{Y}_t^N\|^q\big] < \infty \tag{2.117}$$

for all $q \in [0, \infty)$ with $q < \frac{p}{2\gamma_1 + 4\max(\gamma_0, \gamma_1, 1/2)} - \frac{1}{2}$ and $\sup_{x \in \mathbb{R}^d} \|x\|^q / V(x) < \infty$.

PROOF OF COROLLARY 2.21. If there exists no real number $q \in (0, \infty)$ which satisfies

$$q < \frac{p}{2\gamma_1 + 4\max(\gamma_0, \gamma_1, 1/2)} - \frac{1}{2} \qquad \text{and} \qquad \sup_{x \in \mathbb{R}^d} \left(\frac{\|x\|^q}{V(x)}\right) < \infty, \tag{2.118}$$

then (2.117) follows immediately. We thus assume in the following that $q \in (0, \infty)$ is a real number which satisfies (2.118). Next observe that Corollary 2.20 shows that the function $\Phi \colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ given by

$$\Phi(x, t, y) = x + \frac{\mu(x)t + \sigma(x)y}{\max\left(1, t\,\|\mu(x)t + \sigma(x)y\|\right)} \tag{2.119}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $y \in \mathbb{R}^m$ is α-semi V-stable with respect to Brownian motion with

$$\alpha := \frac{p}{\gamma_1 + 2\max(\gamma_0, \gamma_1, 1/2)} \in (0,\infty). \tag{2.120}$$

Moreover, note that

$$\begin{aligned} &\big\|\bar{Y}_t^N\big\| \\ &\leq \|\xi\| + \sum_{n=0}^{N-1} \frac{\big\|\mu(\bar{Y}_{nT/N}^N)\frac{T}{N} + \sigma(\bar{Y}_{nT/N}^N)(W_{(k+1)T/N} - W_{kT/N})\big\|}{\max\big(1, \frac{T}{N}\|\mu(\bar{Y}_{nT/N}^N)\frac{T}{N} + \sigma(\bar{Y}_{nT/N}^N)(W_{(k+1)T/N} - W_{nT/N})\|\big)} \\ &\leq \|\xi\| + \frac{N^2}{T} \end{aligned} \tag{2.121}$$

for all $t \in [0,T]$ and all $N \in \mathbb{N}$ and therefore

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \Big(N^{-2q} \,\big\| \|\bar{Y}_t^N\|^q \big\|_{L^r(\Omega;\mathbb{R})} \Big) < \infty \tag{2.122}$$

for all $r \in (0,\infty)$. Next note that the inequality

$$1-\alpha = -2\left(\frac{\alpha}{2} - \frac{1}{2}\right) = -2\left(\frac{p}{2\gamma_1 + 4\max(\gamma_0,\gamma_1,\frac{1}{2})} - \frac{1}{2}\right) < -2q \tag{2.123}$$

proves that there exists a real number $r \in (1,\infty)$ such that

$$(1-\alpha)\left(1 - \tfrac{1}{r}\right) < -2q. \tag{2.124}$$

Combining this with (2.122) proves that

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \Big(N^{(1-\alpha)(1-1/r)} \,\big\| \|\bar{Y}_t^N\|^q \big\|_{L^r(\Omega;\mathbb{R})} \Big) < \infty. \tag{2.125}$$

Corollary 2.9 can thus be applied to give

$$\limsup_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\Big[\big\|\bar{Y}_{\frac{nT}{N}}^N\big\|^q\Big] < \infty. \tag{2.126}$$

Combining this and (2.122) finally implies

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\Big[\big\|\bar{Y}_t^N\big\|^q\Big] < \infty. \tag{2.127}$$

The proof of Corollary 2.21 is thus completed. $\square$

Corollary 2.22 (Powers of the Lyapunov-type function). *Let $T \in (0,\infty)$, $d, m \in \mathbb{N}$, $p \in [3,\infty)$, $q \in [1,\infty)$, $c, \gamma_0, \gamma_1 \in [0,\infty)$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions and let $V \in C_p^3(\mathbb{R}^d, [1,\infty))$ with $\|\mu(x)\| \leq c\,|V(x)|^{(\gamma_0+1)/p}$, $\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,|V(x)|^{(\gamma_1+2)/(2p)}$ and*

$$(\mathcal{G}_{\mu,\sigma}V)(x) + \frac{(q-1)\,\|V'(x)\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R})}}{2\cdot V(x)} \leq c\cdot V(x) \tag{2.128}$$

for all $x \in \mathbb{R}^d$. Moreover, let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$ be a filtered probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\xi\colon \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}\big[\|\xi\|^r\big] < \infty$ for all $r \in [0,\infty)$ and let

$\bar{Y}^N\colon [0,T]\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, *be a sequence of stochastic processes given by* $\bar{Y}_0^N=\xi$ *and* (2.116)*. Then*

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\|\bar{Y}_t^N\|^r\big]<\infty \tag{2.129}$$

for all $r\in[0,\infty)$ *which satisfy* $r<\frac{pq}{2\gamma_1+4\max(\gamma_0,\gamma_1,1/2)}-\frac{1}{2}$ *and* $\sup_{x\in\mathbb{R}^d}\|x\|^r/V(x)<\infty$.

Proof of Corollary 2.22. Define the function $\hat{V}\colon\mathbb{R}^d\to[1,\infty)$ by $\hat{V}(x)=(V(x))^q$ for all $x\in\mathbb{R}^d$ and observe that

$$\|\mu(x)\|\leq c\,|\hat{V}(x)|^{\left[\frac{\gamma_0+1}{pq}\right]}\qquad\text{and}\qquad\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}\leq c\,|\hat{V}(x)|^{\left[\frac{\gamma_1+2}{2pq}\right]} \tag{2.130}$$

for all $x\in\mathbb{R}^d$. Lemma 2.14 implies that $\hat{V}\in C^3_{pq}(\mathbb{R}^d,[1,\infty))$ and inequality (2.128) yields that $(\mathcal{G}_{\mu,\sigma}\hat{V})(x)\leq q\cdot c\cdot\hat{V}(x)$ for all $x\in\mathbb{R}^d$. Therefore, Corollary 2.21 shows that

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\|\bar{Y}_t^N\|^r\big]<\infty \tag{2.131}$$

for all $r\in[0,\infty)$ with $r<\frac{pq}{2\gamma_1+4\max(\gamma_0,\gamma_1,1/2)}-\frac{1}{2}$ and $\sup_{x\in\mathbb{R}^d}\|x\|^r/\hat{V}(x)<\infty$. The proof of Corollary 2.22 is thus completed. □

We now illustrate the moment bounds of Corollary 2.21 and of Corollary 2.22 by two simple corollaries.

Corollary 2.23. *Let* $c,T\in(0,\infty)$, $d,m\in\mathbb{N}$, *let* $\mu\colon\mathbb{R}^d\to\mathbb{R}^d$, $\sigma\colon\mathbb{R}^d\to\mathbb{R}^{d\times m}$ *be Borel measurable functions and let* $V\in\cup_{p\in(0,\infty)}C^3_p(\mathbb{R}^d,[1,\infty))$ *be a function with* $\limsup_{q\searrow 0}\sup_{x\in\mathbb{R}^d}\frac{\|x\|^q}{V(x)}<\infty$ *and with*

$$\sup_{x\in\mathbb{R}^d}\left[\frac{(\mathcal{G}_{\mu,\sigma}V)(x)}{V(x)}+\frac{r\,\|V'(x)\sigma(x)\|^2_{L(\mathbb{R}^m,\mathbb{R})}}{|V(x)|^2}\right]<\infty, \tag{2.132}$$

$$\sup_{x\in\mathbb{R}^d}\left[\frac{\|\mu(x)\|+\|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}}{(1+\|x\|^c)}\right]<\infty \tag{2.133}$$

for all $r\in[0,\infty)$*. Moreover, let* $\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P}\big)$ *be a filtered probability space, let* $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ *be a standard* $(\mathcal{F}_t)_{t\in[0,T]}$*-Brownian motion, let* $\xi\colon\Omega\to\mathbb{R}^d$ *be an* $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$*-measurable mapping with* $\mathbb{E}\big[\|\xi\|^q\big]<\infty$ *for all* $q\in[0,\infty)$ *and let* $\bar{Y}^N\colon[0,T]\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, *be a sequence of stochastic processes given by* $\bar{Y}_0^N=\xi$ *and* (2.116)*. Then*

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\mathbb{E}\big[\|\bar{Y}_t^N\|^q\big]<\infty \tag{2.134}$$

for all $q\in[0,\infty)$.

Corollary 2.23 is an immediate consequence of Corollary 2.22. Note also for every $d\in\mathbb{N}$ that a function $V\colon\mathbb{R}^d\to[1,\infty)$ is in $\cup_{p\in(0,\infty)}C^3_p(\mathbb{R}^d,[1,\infty))$ if and only if it is twice differentiable with a locally Lipschitz continuous second derivative and if there exists a real number $c\in(0,\infty)$ such that $\sum_{i=1}^3\|V^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})}\leq c\,|V(x)|^{[1-1/c]}$ for $\lambda_{\mathbb{R}^d}$-almost all $x\in\mathbb{R}^d$. The next corollary of Corollary 2.21 is the counterpart to Corollary 2.16

Corollary 2.24 (A special polynomial like Lyapunov-type function). *Let $T \in (0,\infty)$, $c, \gamma_0, \gamma_1 \in [0,\infty)$, $p \in [3,\infty)$, $d, m \in \mathbb{N}$, let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions with*

$$\langle x, \mu(x)\rangle + \tfrac{(p-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^2\right), \tag{2.135}$$

$$\|\mu(x)\| \leq c\left(1+\|x\|^{[\gamma_0+1]}\right) \qquad \text{and} \qquad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1+\|x\|^{\left[\frac{\gamma_1+2}{2}\right]}\right) \tag{2.136}$$

for all $x \in \mathbb{R}^d$. Moreover, let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\xi\colon \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable mapping with $\mathbb{E}[\|\xi\|^q] < \infty$ for all $q \in [0,\infty)$ and let $\bar{Y}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes given by $\bar{Y}^N_0 = \xi$ and (2.116). Then

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big[\|\bar{Y}^N_t\|^q\big] < \infty \tag{2.137}$$

for all $q \in [0,\infty)$ which satisfy $q < \frac{p}{2\gamma_1 + 4\max(\gamma_0,\gamma_1,1/2)} - \frac{1}{2}$.

Proof of Corollary 2.24. Let $V\colon \mathbb{R}^d \to [1,\infty)$ be given by $V(x) = 1 + \|x\|^p$ for all $x \in \mathbb{R}^d$. Then note, as in the proof of Corollary 2.16, that

$$(\mathcal{G}_{\mu,\sigma}V)(x) \leq 2\cdot p\cdot c\cdot V(x), \tag{2.138}$$

$$\|\mu(x)\| \leq 2c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]} \qquad \text{and} \qquad \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq 2c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]} \tag{2.139}$$

for all $x \in \mathbb{R}^d$. Combining this, the fact $V \in C^3_p(\mathbb{R}^d, [1,\infty))$ and Corollary 2.21 then shows that

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}\big[\|\bar{Y}^N_t\|^q\big] < \infty \tag{2.140}$$

for all $q \in [0,\infty)$ which satisfy $q < \frac{p}{2\gamma_1+4\max(\gamma_0,\gamma_1,\frac{1}{2})} - \frac{1}{2}$ and $\sup_{x\in\mathbb{R}^d} \|x\|^q/V(x) < \infty$. The estimate $\frac{p}{2\gamma_1+4\max(\gamma_0,\gamma_1,\frac{1}{2})} - \frac{1}{2} < p$ hence implies (2.137) and this completes the proof of Corollary 2.24. □

2.3. Implicit approximation schemes

In this section, stability properties and moment bounds for implicit approximation schemes are analyzed. The main results of this section are Corollary 2.27 for the fully drift-implicit Euler scheme and Lemma 2.28 for partially drift-implicit approximation schemes.

2.3.1. Fully drift-implicit approximation schemes. Corollary 2.27 below proves uniform bounds on the q-th moments of fully drift-implicit Euler approximations for a class of SDEs with globally one-sided Lipschitz continuous drift coefficients where $q \in [0,\infty)$. This result generalizes, in the case of the fully drift-implicit Euler scheme, Theorem 3.6 of Mao & Szpurch [**56**] which establishes uniform bounds on the second moments of the numerical approximation processes. First we prove two auxiliary lemmas (Lemma 2.25 and Lemma 2.26). The first lemma is a slight generalization of Lemma 3.2 of Mao & Szpruch [**56**].

LEMMA 2.25. *Let $d \in \mathbb{N}$, $c \in (0,\infty)$ and let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ be a function with $\langle x, \mu(x)\rangle \leq c\left(1 + \|x\|^2\right)$ for all $x \in \mathbb{R}^d$. Then*

$$1 + \|x - \mu(x)s\|^2 \leq e^{4c(t-s)}\left(1 + \|x - \mu(x)t\|^2\right) \tag{2.141}$$

for all $x \in \mathbb{R}^d$ and all $s, t \in [0, \frac{1}{4c}]$ with $s \leq t$.

PROOF OF LEMMA 2.25. Throughout this proof, let $F\colon [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be a function defined through

$$F_t(x) := x - \mu(x)t \tag{2.142}$$

for all $(t, x) \in [0,\infty) \times \mathbb{R}^d$. The assumption $\langle x, \mu(x)\rangle \leq c\left(1 + \|x\|^2\right)$ for all $x \in \mathbb{R}^d$ then implies

$$\begin{aligned} 1 + \|F_s(x)\|^2 &= 1 + \|F_t(x) + (t-s)\mu(x)\|^2 \\ &= 1 + \|F_t(x)\|^2 + 2\langle x - t\mu(x), (t-s)\mu(x)\rangle + (t-s)^2\|\mu(x)\|^2 \\ &= 1 + \|F_t(x)\|^2 + 2(t-s)\langle x, \mu(x)\rangle - (t+s)(t-s)\|\mu(x)\|^2 \\ &\leq 1 + \|F_t(x)\|^2 + 2c\,(t-s)\left(1 + \|x\|^2\right) \end{aligned} \tag{2.143}$$

for all $x \in \mathbb{R}^d$ and all $s, t \in [0,\infty)$ with $s \leq t$. The special case $s = 0$ in (2.143) shows

$$1 + \|x\|^2 \leq \frac{\left(1 + \|F_t(x)\|^2\right)}{(1 - 2tc)} \tag{2.144}$$

for all $x \in \mathbb{R}^d$ and all $t \in [0, \frac{1}{2c})$. Next we apply (2.144) to (2.143) and arrive at

$$\begin{aligned} 1 + \|F_s(x)\|^2 &\leq 1 + \|F_t(x)\|^2 + 2c\,(t-s)\,\tfrac{(1+\|F_t(x)\|^2)}{(1-2ct)} \\ &= \left(1 + \|F_t(x)\|^2\right)\left(1 + \tfrac{2c(t-s)}{(1-2ct)}\right) \leq \left(1 + \|F_t(x)\|^2\right)e^{\frac{2c(t-s)}{(1-2ct)}} \end{aligned} \tag{2.145}$$

for all $x \in \mathbb{R}^d$ and all $s, t \in [0, \frac{1}{2c})$ with $s \leq t$. Combining this with the estimate

$$\frac{2c\,(t-s)}{(1-2ct)} \leq 4c\,(t-s) \tag{2.146}$$

for all $s, t \in [0, \frac{1}{4c}]$ with $s \leq t$ completes the proof of Lemma 2.25. □

LEMMA 2.26 (Stability of the fully drift-implicit Euler scheme). *Let $d, m \in \mathbb{N}$, $c \in (0,\infty)$, $p \in [2,\infty)$ and let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be functions with*

$$\langle x, \mu(x)\rangle + \tfrac{(p-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^2\right) \tag{2.147}$$

for all $x \in \mathbb{R}^d$. Then there exists a real number $\rho \in \mathbb{R}$ such that

$$\mathbb{E}\Big[\left(1 + \|x + \sigma(x)W_t\|^2\right)^q\Big] \leq e^{\rho t}\left(1 + \|x - \mu(x)t\|^2\right)^q \tag{2.148}$$

for all $x \in \mathbb{R}^d$, $t \in [0, \frac{1}{4c}]$ and all $q \in [0, \frac{p}{2}]$ where $W\colon [0,\infty) \times \Omega \to \mathbb{R}^m$ is an arbitrary standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Proof of Lemma 2.26. Throughout this proof, let $F\colon [0,\infty)\times\mathbb{R}^d\to\mathbb{R}^d$ be a function defined through $F_t(x) := x-\mu(x)t$ for all $(t,x)\in[0,\infty)\times\mathbb{R}^d$ and let $e_1^m := (1,0,\dots,0)$, …, $e_m^m := (0,\dots,0,1)\in\mathbb{R}^m$ be the canonical basis of $\mathbb{R}^m$. Itô's lemma yields

$$\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]\\
&= \Big(1+\|x\|^2\Big)^q + q\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\int_0^t \mathbb{E}\Big[\big(1+\|x+\sigma(x)W_s\|^2\big)^{(q-1)}\Big]\,ds\\
&+ q\,(2q-2)\sum_{k=1}^m\int_0^t \mathbb{E}\Big[\big(1+\|x+\sigma(x)W_s\|^2\big)^{(q-2)}\,|\langle x+\sigma(x)W_s,\sigma(x)e_k^m\rangle|^2\Big]\,ds\\
&\le \Big(1+\|x\|^2\Big)^q\\
&+ q\,(2q-1)\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\int_0^t \mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_s\|^2\Big)^{(q-1)}\Big]\,ds
\end{aligned}\tag{2.149}$$

for all $t\in[0,\infty)$, $x\in\mathbb{R}^d$ and all $q\in[1,\infty)$. In addition, the fundamental theorem of calculus implies

$$\begin{aligned}
&\Big(1+\|F_t(x)\|^2\Big)^q\\
&= \Big(1+\|F_0(x)\|^2\Big)^q + 2q\int_0^t\Big(1+\|F_s(x)\|^2\Big)^{(q-1)}\,\langle F_s(x),\tfrac{\partial}{\partial s}F_s(x)\rangle\,ds
\end{aligned}\tag{2.150}$$

and therefore

$$\begin{aligned}
&\Big(1+\|x\|^2\Big)^q\\
&= \Big(1+\|F_t(x)\|^2\Big)^q + 2q\int_0^t\Big(1+\|F_s(x)\|^2\Big)^{(q-1)}\,\langle F_s(x),\mu(x)\rangle\,ds
\end{aligned}\tag{2.151}$$

for all $t\in[0,\infty)$, $x\in\mathbb{R}^d$ and all $q\in[1,\infty)$. Putting (2.151) into (2.149) gives

$$\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]\\
&\le \Big(1+\|F_t(x)\|^2\Big)^q + 2q\int_0^t\Big(1+\|F_s(x)\|^2\Big)^{(q-1)}\,\langle F_s(x),\mu(x)\rangle\,ds\\
&\quad + q\,(2q-1)\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\int_0^t \mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_s\|^2\Big)^{(q-1)}\Big]\,ds
\end{aligned}\tag{2.152}$$

for all $t\in[0,\infty)$, $x\in\mathbb{R}^d$ and all $q\in[1,\infty)$. Roughly speaking, we now use (2.152) to prove (2.148) by induction on $q\in[0,\frac{p}{2}]$. More precisely, let $\kappa\colon[0,\frac{p}{2}]\to[0,\infty)$ be a function defined recursively through $\kappa(q) := 6cq$ for all $q\in[0,1]$ and through

$$\kappa(q) := 2p^3c\exp\!\left(\frac{\kappa(q-1)}{2c}+p\right)\tag{2.153}$$

for all $q\in(n,n+1]\cap[0,\frac{p}{2}]$ and all $n\in\mathbb{N}$. We then prove

$$\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big] \le e^{\kappa(q)t}\Big(1+\|F_t(x)\|^2\Big)^q\tag{2.154}$$

for all $t\in[0,\frac{1}{4c}]$, $x\in\mathbb{R}^d$ and all $q\in(n,n+1]\cap[0,\frac{p}{2}]$ by induction on $n\in\mathbb{N}_0$. For the case $n=0$ and $q=1$, we apply assumption (2.147) and Lemma 2.25 to the right-hand side of (2.152) and get

$$
\begin{aligned}
&\mathbb{E}\Big[1+\|x+\sigma(x)W_t\|^2\Big] \\
&\leq 1+\|F_t(x)\|^2+2\int_0^t \langle F_s(x),\mu(x)\rangle\,ds+\int_0^t \|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\,ds \\
&= 1+\|F_t(x)\|^2+2\int_0^t \Big(\langle x,\mu(x)\rangle+\tfrac{1}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}-s\|\mu(x)\|^2\Big)\,ds \\
&\leq 1+\|F_t(x)\|^2+2\int_0^t \Big(\langle x,\mu(x)\rangle+\tfrac{(p-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\Big)\,ds \\
&\leq 1+\|F_t(x)\|^2+2c\int_0^t \big(1+\|x\|^2\big)\,ds \leq 1+\|F_t(x)\|^2+2tc\,e^{4tc}\Big(1+\|F_t(x)\|^2\Big) \\
&\leq (1+2ect)\Big(1+\|F_t(x)\|^2\Big)\leq e^{2ect}\Big(1+\|F_t(x)\|^2\Big)\leq e^{\kappa(1)t}\Big(1+\|F_t(x)\|^2\Big)
\end{aligned}
\tag{2.155}
$$

for all $t\in[0,\frac{1}{4c}]$ and all $x\in\mathbb{R}^d$. Next observe for every $q\in(0,1)$ that the function $[0,\infty)\ni z\mapsto z^q\in[0,\infty)$ is concave. Hence, Jensen's inequality and (2.155) imply

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]\leq\Big(\mathbb{E}\Big[1+\|x+\sigma(x)W_t\|^2\Big]\Big)^q \\
&\leq\Big(e^{\kappa(1)t}\Big(1+\|F_t(x)\|^2\Big)\Big)^q=e^{\kappa(1)qt}\Big(1+\|F_t(x)\|^2\Big)^q=e^{\kappa(q)t}\Big(1+\|F_t(x)\|^2\Big)^q
\end{aligned}
\tag{2.156}
$$

for all $t\in[0,\frac{1}{4c}]$, $x\in\mathbb{R}^d$ and all $q\in[0,1]$. This proves (2.154) in the base case $n=0$. For the induction step $n\to n+1$, apply the induction hypothesis on the right-hand side of (2.152) to obtain

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]-\Big(1+\|F_t(x)\|^2\Big)^q \\
&\leq\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \\
&\cdot\Big[2q\,\langle F_s(x),\mu(x)\rangle+q\,(2q-1)\,e^{\kappa(q-1)s}\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\Big]\,ds \\
&=2q\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)}\Big[\langle x,\mu(x)\rangle+\tfrac{(2q-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\Big]\,ds \\
&+2q\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \\
&\cdot\Big[\tfrac{(2q-1)}{2}\Big(e^{\kappa(q-1)s}-1\Big)\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}-s\|\mu(x)\|^2\Big]\,ds
\end{aligned}
\tag{2.157}
$$

and assumption (2.147) and the fact that the function $(0,\infty) \ni x \mapsto \frac{(e^x-1)}{x} \in (0,\infty)$ is increasing hence show

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big] - \Big(1+\|F_t(x)\|^2\Big)^q \\
&\le 2qc\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \left(1+\|x\|^2\right) ds \\
&+ 2q\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \\
&\cdot s\left[\left(q-\tfrac{1}{2}\right)\tfrac{(e^{\kappa(q-1)t}-1)}{t}\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} - \|\mu(x)\|^2\right] ds \\
&\le 2qc\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \left(1+\|x\|^2\right) ds \\
&+ 2q\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \\
&\cdot s\left[\left(q-\tfrac{1}{2}\right)4c\left(e^{\frac{\kappa(q-1)}{4c}}-1\right)\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} - \|\mu(x)\|^2\right] ds \\
&\le 2qc\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \left(1+\|x\|^2\right) ds \\
&+ 2q\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \\
&\cdot s\left[2cp\exp\!\Big(\tfrac{\kappa(q-1)}{4c}\Big)\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} - \|\mu(x)\|^2\right] ds
\end{aligned}
\tag{2.158}
$$

for all $t \in [0,\frac{1}{4c}]$, $x \in \mathbb{R}^d$ and all $q \in (n+1,n+2]\cap[0,\frac{p}{2}]$. Next observe that Young's inequality and again assumption (2.147) give

$$
\begin{aligned}
&r\,\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} - \|\mu(x)\|^2 \\
&= \frac{2r}{(p-1)}\left[\langle x,\mu(x)\rangle + \tfrac{(p-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\right] - \frac{2r\,\langle x,\mu(x)\rangle}{(p-1)} - \|\mu(x)\|^2 \\
&\le \frac{2rc\left(1+\|x\|^2\right)}{(p-1)} + \frac{2r\,\|x\|\,\|\mu(x)\|}{(p-1)} - \|\mu(x)\|^2 \le \frac{2rc\left(1+\|x\|^2\right)}{(p-1)} + \frac{r^2\,\|x\|^2}{(p-1)^2} \\
&\le \left(\frac{2rc}{(p-1)} + \frac{r^2}{(p-1)^2}\right)\left(1+\|x\|^2\right) \le \left(2rc+r^2\right)\left(1+\|x\|^2\right)
\end{aligned}
\tag{2.159}
$$

for all $x \in \mathbb{R}^d$ and all $r \in [0,\infty)$. Combining (2.158) and (2.159) implies

$$
\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big] - \Big(1+\|F_t(x)\|^2\Big)^q \\
&\le 2qc\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \left(1+\|x\|^2\right) ds \\
&+ 2q\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \cdot s \cdot \left(4c^2p+4c^2p^2\right)\exp\!\Big(\tfrac{\kappa(q-1)}{2c}\Big)\left(1+\|x\|^2\right) ds \\
&= 2qc\int_0^t \Big(1+\|F_s(x)\|^2\Big)^{(q-1)} \left[1+4cs\left(p+p^2\right)\exp\!\Big(\tfrac{\kappa(q-1)}{2c}\Big)\right]\left(1+\|x\|^2\right) ds
\end{aligned}
\tag{2.160}
$$

and Lemma 2.25 hence gives

$$\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]-\Big(1+\|F_t(x)\|^2\Big)^q\\
&\leq 2qc\left(1+\tfrac{3p^2}{2}\exp\!\Big(\tfrac{\kappa(q-1)}{2c}\Big)\right)\int_0^t\Big(1+\|F_s(x)\|^2\Big)^{(q-1)}\left(1+\|x\|^2\right)ds\\
&\leq pc\left(1+\tfrac{3p^2}{2}\exp\!\Big(\tfrac{\kappa(q-1)}{2c}\Big)\right)\int_0^t e^{(4c(t-s)(q-1)+4ct)}\Big(1+\|F_t(x)\|^2\Big)^q ds \qquad (2.161)\\
&\leq pc\left(1+\tfrac{3p^2}{2}\right)\exp\!\Big(\tfrac{\kappa(q-1)}{2c}\Big)\int_0^t e^{4ctq}\Big(1+\|F_t(x)\|^2\Big)^q ds\\
&\leq 2p^3c\,t\exp\!\Big(\tfrac{\kappa(q-1)}{2c}+\tfrac{p}{2}\Big)\Big(1+\|F_t(x)\|^2\Big)^q
\end{aligned}$$

for all $t \in [0, \frac{1}{4c}]$, $x \in \mathbb{R}^d$ and all $q \in (n+1, n+2] \cap [0, \frac{p}{2}]$. The estimate $1 + r \leq e^r$ for all $r \in \mathbb{R}$ therefore shows

$$\begin{aligned}
&\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]\\
&\leq\Big[1+2p^3c\,t\exp\!\Big(\tfrac{\kappa(q-1)}{2c}+p\Big)\Big]\Big(1+\|F_t(x)\|^2\Big)^q \qquad (2.162)\\
&\leq e^{t\cdot\kappa(q)}\Big(1+\|F_t(x)\|^2\Big)^q
\end{aligned}$$

for all $t \in [0, \frac{1}{4c}]$, $x \in \mathbb{R}^d$ and all $q \in (n+1, n+2] \cap [0, \frac{p}{2}]$. This finishes the induction step and inequality (2.154) thus holds for all $n \in \mathbb{N}_0$. In particular, we get from inequality (2.154) that

$$\mathbb{E}\Big[\Big(1+\|x+\sigma(x)W_t\|^2\Big)^q\Big]\leq\exp\!\Big(t\cdot\sup\nolimits_{r\in[0,\frac{p}{2}]}\kappa(r)\Big)\Big(1+\|x-\mu(x)t\|^2\Big)^q \qquad (2.163)$$

for all $t \in [0, \frac{1}{4c}]$, $x \in \mathbb{R}^d$ and all $q \in [0, \frac{p}{2}]$. This and the estimate $\sup_{r\in[0,\frac{p}{2}]} \kappa(r) < \infty$ then complete the proof of Lemma 2.26. □

Now we apply Lemma 2.26 and Corollary 2.2 to obtain moment bounds for fully drift-implicit Euler approximations.

Corollary 2.27. *Let $d, m \in \mathbb{N}$, $c, T \in (0,\infty)$, $p \in [2,\infty)$ be real numbers, let $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\big)$ be a filtered probability space, let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\xi\colon \Omega \to \mathbb{R}^d$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R}^d)$-measurable function and let $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions with $\mathbb{E}\big[\|\mu(\xi)\|^p\big] < \infty$ and*

$$\langle x-y,\mu(x)-\mu(y)\rangle\leq c\,\|x-y\|^2, \qquad (2.164)$$

$$\langle x,\mu(x)\rangle+\tfrac{(p-1)}{2}\|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}\leq c\left(1+\|x\|^2\right) \qquad (2.165)$$

for all $x, y \in \mathbb{R}^d$. Then there exists a unique family $Y^N\colon \{0,1,\dots,N\}\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}\cap(cT,\infty)$, of stochastic processes satisfying $Y_0^N = \xi$ and

$$Y_{n+1}^N=Y_n^N+\mu(Y_{n+1}^N)\tfrac{T}{N}+\sigma(Y_n^N)\big(W_{(n+1)T/N}-W_{nT/N}\big) \qquad (2.166)$$

for all $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}\cap(cT,\infty)$ and there exists a real number $\rho \in (0,\infty)$ such that

$$\limsup_{N\to\infty}\sup_{n\in\{0,1,\dots,N\}}\mathbb{E}\Big[\big\{1+\|Y_n^N\|^2\big\}^q\Big]\leq e^{\rho T}\cdot\mathbb{E}\Big[\Big\{1+\|\xi\|^2\Big\}^q\Big] \qquad (2.167)$$

for all $q \in [0, \frac{p}{2}]$.

Proof of Corollary 2.27. Throughout this proof, let $F\colon [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ be a function defined through $F_t(x) := x - \mu(x)t$ for all $(t,x) \in [0,\infty)\times\mathbb{R}^d$. Next observe that

$$\langle x-y, (\mu(x)t-x) - (\mu(y)t-y)\rangle \leq (ct-1)\,\|x-y\|^2 \tag{2.168}$$

for all $x, y \in \mathbb{R}^d$ and all $t \in [0,\infty)$. This inequality ensures the unique existence of the stochastic processes $Y^N\colon \{0,1,\dots,N\}\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}\cap(cT,\infty)$. In the next step, note that

$$\big\|F_{T/N}(Y^N_{n+1})\big\|^2 = \big\|Y^N_n + \sigma(Y^N_n)\big(W_{(n+1)T/N} - W_{nT/N}\big)\big\|^2 \tag{2.169}$$

for all $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}\cap(cT,\infty)$. Lemma 2.26 hence implies the existence of a real number $\rho \in \mathbb{R}$ such that

$$\mathbb{E}\Big[\big\{1+\|F_{T/N}(Y^N_{n+1})\|^2\big\}^q \,\big|\, Y^N_n\Big] \leq \exp\!\Big(\tfrac{\rho T}{N}\Big)\cdot\big\{1+\|F_{T/N}(Y^N_n)\|^2\big\}^q \tag{2.170}$$

$\mathbb{P}$-a.s. for all $n \in \{0,1,\dots,N-1\}$, $N \in \mathbb{N}\cap[4cT,\infty)$ and all $q \in [0,\frac{p}{2}]$. Next fix a real number $q \in [0,\frac{p}{2}]$ and we now prove (2.167) for this $q \in [0,\frac{p}{2}]$. If $\mathbb{E}\big[\|\xi\|^{2q}\big] = \infty$, then (2.167) is trivial. We thus assume $\mathbb{E}\big[\|\xi\|^{2q}\big] < \infty$ for the rest of this proof. Hence, we obtain that

$$\mathbb{E}\big[\|\xi\|^{2q} + \|\mu(\xi)\|^{2q}\big] < \infty. \tag{2.171}$$

Now we apply Corollary 2.2 with the Lyapunov-type function $V\colon \mathbb{R}^d \to [0,\infty)$ given by

$$V(x) = \big\{1+\|F_{T/N}(x)\|^2\big\}^q \tag{2.172}$$

for all $x \in \mathbb{R}^d$, with the truncation function $\zeta\colon [0,\infty) \to (0,\infty]$ given by $\zeta(t) = \infty$ for all $t \in [0,\infty)$ and with the sequence $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, given by $t_n = \min(nT/N, T)$ for all $n \in \mathbb{N}_0$ to obtain

$$\sup_{n\in\{0,1,\dots,N\}} \mathbb{E}\Big[\big\{1+\|F_{T/N}(Y^N_n)\|^2\big\}^q\Big] \leq e^{\rho T}\cdot\mathbb{E}\Big[\big\{1+\|F_{T/N}(\xi)\|^2\big\}^q\Big] \tag{2.173}$$

for all $n \in \{0,1,\dots,N\}$ and all $N \in \mathbb{N}\cap[4cT,\infty)$. Lemma 2.25 and the dominated convergence theorem hence give

$$\begin{aligned}
&\limsup_{N\to\infty}\sup_{n\in\{0,1,\dots,N\}} \mathbb{E}\Big[\big\{1+\|Y^N_n\|^2\big\}^q\Big]\\
&\leq \lim_{N\to\infty}\Big(e^{\frac{4cT}{N}}\cdot e^{\rho T}\cdot\mathbb{E}\Big[\big\{1+\|F_{T/N}(\xi)\|^2\big\}^q\Big]\Big)\\
&= e^{\rho T}\cdot\mathbb{E}\Big[\lim_{N\to\infty}\big\{1+\|F_{T/N}(\xi)\|^2\big\}^q\Big] = e^{\rho T}\cdot\mathbb{E}\Big[\big\{1+\|\xi\|^2\big\}^q\Big]
\end{aligned} \tag{2.174}$$

and this completes the proof of Corollary 2.27. □

2.3.2. Partially drift-implicit approximation schemes. In Subsection 2.3.1 above, moment bounds for the fully drift-implicit Euler schemes have been established. This subsection concentrates on partially drift-implicit schemes.

Lemma 2.28 (Partially drift-implicit schemes for SDEs with at most linearly growing diffusion coefficients). *Let $h \in (0,\infty)$, $c \in [1,\infty)$ be real numbers with $hc \leq \frac{1}{4}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let $W\colon [0,\infty)\times\Omega \to \mathbb{R}^m$ be a standard Brownian motion, let $\varphi\colon \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$, $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions with*

$$\langle y, \varphi(x,y)\rangle \leq c\left(2+\|x\|^2+\|y\|^2\right), \qquad \|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1+\|x\|^2\right) \tag{2.175}$$

for all $x, y \in \mathbb{R}^d$ *and let* $Y\colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^d$ *be a stochastic process with*

$$Y_{n+1} = Y_n + \varphi(Y_n, Y_{n+1})\, h + \sigma(Y_n) \left(W_{(n+1)h} - W_{nh}\right) \tag{2.176}$$

for all $n \in \mathbb{N}_0$. *Then*

$$\left\| \sup_{k\in\{0,1,\dots,n\}} \|Y_k\| \right\|_{L^p(\Omega;\mathbb{R})} \leq 2 \left(1 + \|Y_0\|_{L^p(\Omega;\mathbb{R}^d)}\right) \exp\!\left((4 + p^4 \chi_{p/2}) c^2 nh\right) \tag{2.177}$$

for all $p \in [4, \infty)$ *and all* $n \in \mathbb{N}_0$.

Proof of Lemma 2.28. First of all, let $p \in [4, \infty)$ be arbitrary. If $\mathbb{E}\big[\|Y_0\|^p\big] = \infty$, then inequality (2.177) is trivial. Thus we assume $\mathbb{E}\big[\|Y_0\|^p\big] < \infty$ for the rest of this proof. In the sequel, we will show (2.177) by an application of Corollary 2.3. For this we define a stochastic process $Z\colon \mathbb{N} \times \Omega \to \mathbb{R}^d$ through

$$\begin{aligned} Z_n :=& \frac{1}{(1-2hc)} \Big(2 \left\langle Y_{n-1}, \sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\right\rangle \\ &+ \left\|\sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\right\|^2 - h\, \|\sigma(Y_{n-1})\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \Big) \end{aligned} \tag{2.178}$$

for all $n \in \mathbb{N}$ and we then verify inequality (2.22) and inequality (2.23). For inequality (2.22) note that Young's inequality and assumption (2.175) give

$$\begin{aligned} 2\,\|Y_n\|^2 &= 2 \left\langle Y_n, Y_{n-1} + \varphi(Y_{n-1}, Y_n)\, h + \sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\right\rangle \\ &= 2 \left\langle Y_n, Y_{n-1} + \sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\right\rangle + 2h \left\langle Y_n, \varphi(Y_{n-1}, Y_n)\right\rangle \\ &\leq \|Y_n\|^2 + \left\|Y_{n-1} + \sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\right\|^2 \\ &\quad + 2hc \left(2 + \|Y_{n-1}\|^2 + \|Y_n\|^2\right) \end{aligned} \tag{2.179}$$

for all $n \in \mathbb{N}$. Rearranging (2.179) yields

$$\begin{aligned} &1 + \|Y_n\|^2 \\ &\leq \frac{1 + \left\|Y_{n-1} + \sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\right\|^2 + 2hc \left(1 + \|Y_{n-1}\|^2\right)}{(1-2hc)} \\ &= \frac{1 + \|Y_{n-1}\|^2 + h\,\|\sigma(Y_{n-1})\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} + 2hc\left(1 + \|Y_{n-1}\|^2\right)}{(1-2hc)} + Z_n \\ &\leq \frac{\left(1 + \|Y_{n-1}\|^2\right)(1+3hc)}{(1-2hc)} + Z_n \leq e^{7hc} \left(1 + \|Y_{n-1}\|^2\right) + Z_n \end{aligned} \tag{2.180}$$

for all $n \in \mathbb{N}$ where the last inequality follows from the estimate $\frac{1+3x}{1-2x} \leq e^{7x}$ for all $x \in [0, \frac{1}{4}]$. Estimate (2.180) is inequality (2.22) with $\rho = 7c \in [0, \infty)$. Combinig (2.180), (2.178), (2.175) and the assumption that $\mathbb{E}\big[\|Y_0\|^p\big] < \infty$ then shows that $\mathbb{E}\big[\|Y_n\|^p\big] < \infty$ for all $n \in \mathbb{N}_0$ and that $\mathbb{E}\big[|Z_n|\big] < \infty$ for all $n \in \mathbb{N}$. In addition, note that

$$\mathbb{E}\big[Z_n \,|\, (Z_k)_{k\in\{0,1,\dots,n-1\}}\big] = 0 \tag{2.181}$$

$\mathbb{P}$-a.s. for all $n \in \mathbb{N}$. It thus remains to verify inequality (2.23) to complete the proof of Lemma 2.28. For this observe that the estimate

$$\|X - \mathbb{E}[X]\|_{L^q(\Omega;\mathbb{R})} \leq 2\,\|X\|_{L^q(\Omega;\mathbb{R})} \tag{2.182}$$

for all $q \in [1, \infty)$ and all $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings $X \colon \Omega \to \mathbb{R}$ with $\mathbb{E}\big[|X|\big] < \infty$ and Lemma 7.7 in Da Prato & Zabzcyk [**14**] give

$$\begin{aligned}
\|Z_n\|_{L^{p/2}(\Omega;\mathbb{R})} &\leq \frac{\big\|\big\langle Y_{n-1}, \sigma(Y_{n-1})(W_{nh} - W_{(n-1)h})\big\rangle\big\|_{L^{p/2}(\Omega;\mathbb{R})}}{(1/2 - hc)} \\
&\quad + \frac{\big\|\sigma(Y_{n-1})\big(W_{nh} - W_{(n-1)h}\big)\big\|^2_{L^p(\Omega;\mathbb{R}^d)}}{(1/2 - hc)} \\
&\leq \frac{\frac{p\sqrt{h}}{\sqrt{8}} \, \|\langle Y_{n-1}, \sigma(Y_{n-1})(\cdot)\rangle\|_{L^{p/2}(\Omega;HS(\mathbb{R}^m,\mathbb{R}))}}{(1/2 - hc)} \\
&\quad + \frac{\frac{p(p-1)h}{2} \, \|\sigma(Y_{n-1})\|^2_{L^p(\Omega;HS(\mathbb{R}^m,\mathbb{R}^d))}}{(1/2 - hc)}
\end{aligned} \tag{2.183}$$

and hence

$$\begin{aligned}
\|Z_n\|_{L^{p/2}(\Omega;\mathbb{R})} &\leq \frac{\frac{p\sqrt{h}}{\sqrt{2}} \, \big\|\|Y_{n-1}\|\|\sigma(Y_{n-1})\|_{HS(\mathbb{R}^m,\mathbb{R}^d)}\big\|_{L^{p/2}(\Omega;\mathbb{R})}}{(1 - 2hc)} \\
&\quad + \frac{chp(p-1)\,\big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})}}{(1 - 2hc)} \\
&\leq \frac{\frac{p\sqrt{h}}{\sqrt{2}} \, \Big\|\frac{\|Y_{n-1}\|^2}{2} + \frac{c}{2}\left[1 + \|Y_{n-1}\|^2\right]\Big\|_{L^{p/2}(\Omega;\mathbb{R})}}{(1 - 2hc)} \\
&\quad + \frac{chp(p-1)\,\big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})}}{(1 - 2hc)}
\end{aligned} \tag{2.184}$$

and therefore

$$\begin{aligned}
&\|Z_n\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\leq \frac{\frac{p\sqrt{h}c}{\sqrt{2}} \, \big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})} + chp(p-1)\,\big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})}}{(1 - 2hc)} \\
&= pc\sqrt{h}\left[\frac{\frac{1}{\sqrt{2}} + \sqrt{h}\,(p-1)}{(1 - 2hc)}\right]\big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\leq p(p + \sqrt{2} - 1)c\sqrt{h}\,\big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\leq cp^2\sqrt{2h}\,\big\|1 + \|Y_{n-1}\|^2\big\|_{L^{p/2}(\Omega;\mathbb{R})}
\end{aligned} \tag{2.185}$$

for all $n \in \mathbb{N}$. This implies inequality (2.23) with $\nu \colon \mathbb{N} \to [0, \infty)$ given by $\nu_n = cp^2\sqrt{2h}$ for all $n \in \mathbb{N}$. Now we apply Corollary 2.3 with $\rho = 7c$, with the Lyapunov-type function $V \colon \mathbb{R}^d \to [0, \infty)$ given by

$$V(x) = 1 + \|x\|^2 \tag{2.186}$$

for all $x \in \mathbb{R}^d$, with the truncation function $\zeta \colon [0, \infty) \to [0, \infty]$ given by $\zeta(t) = \infty$ for all $t \in [0, \infty)$ and with the sequence $t_n \in \mathbb{R}$, $n \in \mathbb{N}_0$, given by $t_n = nh$ for all

$n \in \mathbb{N}_0$ to obtain

$$\begin{aligned}
&\left\| \sup_{k\in\{0,1,\dots,n\}} \|Y_k\|^2 \right\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\leq e^{7cnh} \left\| \sup_{k\in\{0,1,\dots,n\}} e^{-7ckh} \left(1 + \|Y_k\|^2\right) \right\|_{L^{p/2}(\Omega;\mathbb{R})} \\
&\leq 2e^{7cnh} \left\| 1 + \|Y_0\|^2 \right\|_{L^{p/2}(\Omega;\mathbb{R})} \exp\!\left(\chi_{\frac{p}{2}} \left[\sum_{k=1}^{n} 2p^4 c^2 h \right] \right) \\
&\leq 2\left(1 + \|Y_0\|^2_{L^p(\Omega;\mathbb{R}^d)}\right) \exp\!\left(\left(7 + 2p^4 \chi_{\frac{p}{2}}\right) c^2 nh \right)
\end{aligned} \tag{2.187}$$

for all $n \in \mathbb{N}_0$. This finishes the proof of Lemma 2.28. □

Lemma 2.29 (A class of linear implicit schemes for one-dimensional SDEs). *Let $c \in [0,\infty)$, $T \in (0,\infty)$, let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space, let $W \colon [0,T] \times \Omega \to \mathbb{R}$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion, let $\xi \colon \Omega \to \mathbb{R}$ be an $\mathcal{F}_0/\mathcal{B}(\mathbb{R})$-measurable function and let $a, b, \sigma \colon \mathbb{R} \to \mathbb{R}$ be Borel measurable functions with*

$$x\left(a(x)\,x + b(x)\right) + \tfrac{1}{2}|\sigma(x)|^2 \leq c\left(1 + x^2\right), \quad a(x) \leq c, \quad |b(x)|^2 \leq c\left(1 + x^2\right) \tag{2.188}$$

for all $x \in \mathbb{R}$. Then there exists a unique family $Y^N \colon \{0,1,\dots,N\} \times \Omega \to \mathbb{R}$, $N \in \mathbb{N} \cap (cT,\infty)$, of stochastic processes satisfying $Y_0^N = \xi$ and

$$Y_{n+1}^N = Y_n^N + \left(a(Y_n^N)\,Y_{n+1}^N + b(Y_n^N)\right)\tfrac{T}{N} + \sigma(Y_n^N)\left(W_{(n+1)T/N} - W_{nT/N}\right) \tag{2.189}$$

for all $n \in \{0,1,\dots,N\}$ and all $N \in \mathbb{N} \cap (cT,\infty)$ and it holds

$$\sup_{N\in\mathbb{N}\cap[2cT,\infty)} \sup_{n\in\{0,1,\dots,N\}} \mathbb{E}\Big[1 + \big|Y_n^N\big|^2\Big] \leq e^{(8c+2)T} \cdot \mathbb{E}\big[1 + |\xi|^2\big]\,. \tag{2.190}$$

Proof of Lemma 2.29. First, note that the assumption $a(x) \leq c \in [0,\infty)$ for all $x \in \mathbb{R}$ (see (2.188)) ensures the unique existence of a family $Y^N \colon \{0,1,\dots,N\} \times \Omega \to \mathbb{R}$, $N \in \mathbb{N} \cap (cT,\infty)$, of stochastic processes satisfying $Y_0 = \xi$ and (2.189). The stochastic processes $(Y^N)_{N\in\mathbb{N}}$ thus fulfill

$$Y_{n+1}^N = \frac{Y_n^N + b(Y_n^N)\,\frac{T}{N} + \sigma(Y_n^N)\left(W_{(n+1)T/N} - W_{nT/N}\right)}{\left(1 - a(Y_n^N)\frac{T}{N}\right)} \tag{2.191}$$

for all $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N} \cap (cT,\infty)$. Next observe that (2.188) implies

$$\begin{aligned}
&\mathbb{E}\left[\frac{\left(x + b(x)t + \sigma(x)W_t\right)^2}{\left(1 - a(x)t\right)^2}\right] = \frac{\left(x + b(x)t\right)^2 + |\sigma(x)|^2\, t}{\left(1 - a(x)t\right)^2} \\
&= \frac{x^2\left(1 - 2a(x)t\right)}{\left(1 - a(x)t\right)^2} + \frac{2t\left\{x\left(a(x)\,x + b(x)\right) + \frac{1}{2}|\sigma(x)|^2\right\} + |b(x)|^2\, t^2}{\left(1 - a(x)t\right)^2} \\
&\leq x^2 + \frac{2tc\left(1 + x^2\right) + c\left(1 + x^2\right)t^2}{\left(1 - a(x)t\right)^2} = x^2 + \left(1 + x^2\right)\left(\frac{(2c + ct)\,t}{\left(1 - a(x)t\right)^2}\right)
\end{aligned} \tag{2.192}$$

and therefore

$$\begin{aligned} &\mathbb{E}\left[1 + \frac{\left(x + b(x)t + \sigma(x)W_t\right)^2}{\left(1 - a(x)t\right)^2}\right] \\ &\leq \left(1 + x^2\right)\left(1 + \frac{(2c + ct)\,t}{\left(1 - a(x)t\right)^2}\right) \leq \left(1 + x^2\right)\exp\!\left(\frac{(2c + ct)\,t}{\left(1 - ct\right)^2}\right) \end{aligned} \tag{2.193}$$

for all $t \in [0, \frac{1}{c})$ and all $x \in \mathbb{R}$. Hence, we obtain

$$\mathbb{E}\left[1 + \frac{\left(x + b(x)t + \sigma(x)W_t\right)^2}{\left(1 - a(x)t\right)^2}\right] \leq \left(1 + x^2\right) e^{(8c+2)t} \tag{2.194}$$

for all $t \in [0, \frac{1}{2c}]$ and all $x \in \mathbb{R}$. Note that (2.194) shows that the linear implicit scheme

$$\mathbb{R} \times [0, \tfrac{1}{2c}] \times \mathbb{R} \ni (x, t, y) \mapsto \frac{x + b(x)t + \sigma(x)y}{\left(1 - a(x)t\right)^2} \in \mathbb{R} \tag{2.195}$$

is $(1+x^2)_{x\in\mathbb{R}}$-stable with respect to Brownian motion. Moreover, combining (2.194) and Corollary 2.2 results in (2.190) and this completes the proof of Lemma 2.29. □

CHAPTER 3

Convergence properties of approximation processes for SDEs

With the integrability properties of Chapter 2 at hand, we now prove convergence in probability, strong convergence and weak convergence results for numerical approximation processes for SDEs. For this, our central assumption on the numerical approximation method is a certain consistency property in Definition 3.1 below.

3.1. Setting and assumptions

Let $T \in (0,\infty)$, $d,m \in \mathbb{N}$, let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion. Moreover, let $D \subset \mathbb{R}^d$ be a non-empty open set, let $\mu\colon D \to \mathbb{R}^d$ and $\sigma\colon D \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions and let $X\colon [0,T]\times\Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths satisfying

$$X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \tag{3.1}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. Note that we assume existence of a solution process staying in the open set D. Next let $\phi\colon \mathbb{R}^d \times [0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ be a Borel measurable function and let $\bar{Y}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes defined through $\bar{Y}_0^N := X_0$ and

$$\bar{Y}_t^N := \bar{Y}^N_{\frac{nT}{N}} + \big(\tfrac{tN}{T} - n\big)\cdot\phi\big(\bar{Y}^N_{\frac{nT}{N}}, \tfrac{T}{N}, W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\big) \tag{3.2}$$

for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\dots,N-1\}$ and all $N \in \mathbb{N}$. A central goal of this chapter is to give sufficient conditions to ensure that the stochastic processes $\bar{Y}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, converge in a suitable sense to the solution process $X\colon [0,T]\times\Omega \to D$ of the SDE (3.1).

3.2. Consistency

This section introduces a consistency property of the increment function $\phi\colon \mathbb{R}^d\times [0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ from Section 3.1 which ensures that the stochastic processes $\bar{Y}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, defined in (3.2) converge in probability to the solution process $X\colon [0,T]\times\Omega \to D$ of the SDE (3.1) (see Theorem 3.3 below).

Definition 3.1 (Consistency of numerical methods for SDEs driven by standard Brownian motions). Let $T \in (0,\infty)$, $d,m \in \mathbb{N}$, let $D \subset \mathbb{R}^d$ be an open set and let $\mu\colon D \to \mathbb{R}^d$ and $\sigma\colon D \to \mathbb{R}^{d\times m}$ be functions. A Borel measurable function $\phi\colon \mathbb{R}^d\times[0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ is then said to be (μ,σ)-consistent with respect to

Brownian motion if

$$\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\sigma(x)W_t-\phi(x,t,W_t)\big\|\Big]\right)=0 \tag{3.3}$$

and

$$\limsup_{t\searrow 0}\left(\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\phi(x,t,W_t)\big]\big\|\right)=0 \tag{3.4}$$

for all non-empty compact sets $K\subset D$ where $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ is an arbitrary standard Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Note that (3.3) in Definition 3.1 assures that the expectation in (3.4) is well-defined. Further consistency notions for numerical approximation schemes for SDEs and results for such schemes in the case of SDEs with globally Lipschitz continuous coefficients can, e.g., be found in Section 9.6 of Kloeden & Platen [**47**], in Chapter 1 of Milstein [**61**], in Beyn & Kruse [**8**], in Kruse [**48**] and in the references therein. The next lemma gives a simple characterization of (μ,σ)-consistency with respect to Brownian motion. Its proof is straightforward and therefore omitted.

LEMMA 3.2. *Let $T\in(0,\infty)$, $d,m\in\mathbb{N}$, let $D\subset\mathbb{R}^d$ be an open set and let $\mu\colon D\to\mathbb{R}^d$ and $\sigma\colon D\to\mathbb{R}^{d\times m}$ be functions. A Borel measurable function $\phi\colon\mathbb{R}^d\times[0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ is then (μ,σ)-consistent with respect to Brownian motion if and only if*

$$\limsup_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup\nolimits_{x\in D_v}\mathbb{E}\big[\|\sigma(x)W_t-\phi(x,t,W_t)\|\big]\right)=0 \tag{3.5}$$

and

$$\limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in D_v}\big\|\mu(x)t-\mathbb{E}\big[\phi(x,t,W_t)\big]\big\|\right)=0 \tag{3.6}$$

for all $v\in\mathbb{N}$ where the sets $D_v\subset D$, $v\in\mathbb{N}$, are given by $D_v:=\{v\in D\colon\|x\|<v$ and $dist(x,D^c)>\frac{1}{v}\}$ for all $v\in\mathbb{N}$ and where $W\colon[0,T]\times\Omega\to\mathbb{R}^m$ is an arbitrary standard Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$.

A list of numerical schemes that are (μ,σ)-consistent with respect to Brownian motion can be found in Section 3.6.

3.3. Convergence in probability

The next theorem shows that the stochastic processes $\bar{Y}^N\colon[0,T]\times\Omega\to\mathbb{R}^d$, $N\in\mathbb{N}$, in (3.2) converge in probability to the solution process $X\colon[0,T]\times\Omega\to D$ of the SDE (3.1) if the increment function $\phi\colon\mathbb{R}^d\times[0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion (see Definition 3.1).

THEOREM 3.3 (Convergence in probability). *Assume that the setting in Section 3.1 is fulfilled and that $\phi\colon\mathbb{R}^d\times[0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion. Then*

$$\lim_{N\to\infty}\mathbb{P}\left[\sup_{t\in[0,T]}\big\|X_t-\bar{Y}^N_t\big\|\geq\varepsilon\right]=0 \tag{3.7}$$

for all $\varepsilon\in(0,\infty)$.

The proof of Theorem 3.3 is presented in the following subsection. More results on convergence in probability and pathwise convergence of temporal numerical approximation processes for SDEs with non-globally Lipschitz continuous coefficients can, e.g., be found in [**49, 24, 22, 23, 64, 65, 43, 41, 12**] and in the references therein.

Theorem 3.3 proves convergence in probability for a class of one-step numerical approximation processes (3.2) in the case of finite dimensional SDEs with locally Lipschitz continuous coefficients μ and σ. The locally Lipschitz assumptions on μ and σ ensure that solutions of the SDE (3.1) are unique up to indistinguishability. We expect that it is possible to generalize Theorem 3.3 to a more general class of possibly infinite dimensional SDEs and also to replace the locally Lipschitz assumptions on μ and σ by weaker conditions such as local monotonicity (see, e.g., Krylov [**49**], Gyöngy & Krylov [**24**] in the finite dimensional case and Liu & Röckner [**52, 53**] in the infinite dimensional case) which ensure that solutions of the considered SDE are unique.

3.3.1. Proof of Theorem 3.3. For the proof of Theorem 3.3, we introduce more notation. First, we define mappings $Y_n^N : \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, through $Y_n^N := \bar{Y}_{nT/N}^N$ for all $n \in \{0, 1, \ldots, N\}$ and all $N \in \mathbb{N}$. In addition, we use the mappings $\bar{\mu} \colon \mathbb{R}^d \to \mathbb{R}^d$, $\bar{\sigma} = (\bar{\sigma}_{i,j}(x))_{i\in\{1,\ldots,d\},j\in\{1,\ldots,m\}} : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and $\bar{\sigma}_i \colon \mathbb{R}^d \to \mathbb{R}^d$, $i \in \{1, 2, \ldots, m\}$, defined by

$$\bar{\mu}(x) := \mu(x) \qquad \text{and} \qquad \bar{\sigma}(x) := \sigma(x) \tag{3.8}$$

for all $x \in D$, by

$$\bar{\mu}(x) := 0 \qquad \text{and} \qquad \bar{\sigma}(x) := 0 \tag{3.9}$$

for all $x \in D^c$ and by $\bar{\sigma}_i(x) := (\bar{\sigma}_{1,i}(x), \ldots, \bar{\sigma}_{d,i}(x))$ for all $x \in \mathbb{R}^d$ and all $i \in \{1, 2, \ldots, m\}$. Next let $D_v \subset D$, $v \in \mathbb{N}$, be a sequence of open sets defined by

$$D_v := \left\{x \in D \colon \|x\| < v \text{ and } \operatorname{dist}(x, D^c) > \tfrac{1}{v}\right\} \tag{3.10}$$

for all $v \in \mathbb{N}$. The assumption that $\phi \colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ is (μ, σ)-consistent with respect to Brownian motion and Lemma 3.2 then ensure that there exists a sequence $t_v \in (0, T]$, $v \in \mathbb{N}$, of real numbers such that

$$\sup_{t\in[0,t_v]} \sup_{x\in D_v} \mathbb{E}\Big[\|\phi(x, t, W_t)\|\Big] < \infty \tag{3.11}$$

for all $v \in \mathbb{N}$ with $D_v \neq \emptyset$. Then let $c_v \in [0, \infty)$, $v \in \mathbb{N}$, be a family of real numbers defined by

(3.12)

$$c_v := \begin{cases} \sup_{\substack{x,y\in\bar{D}_v\\ x\neq y}} \frac{\|\mu(x)-\mu(y)\|}{\|x-y\|} + \sum_{i=1}^m \left(\sup_{\substack{x,y\in\bar{D}_v\\ x\neq y}} \frac{\|\sigma_i(x)-\sigma_i(y)\|}{\|x-y\|} \right) & : D_v \neq \emptyset \\ 0 & : \text{else} \end{cases}$$

for all $v \in \mathbb{N}$. Using Lebesgue's number lemma one can indeed show that $c_v < \infty$ for all $v \in \mathbb{N}$ since $\mu \colon D \to \mathbb{R}^d$ and $\sigma \colon D \to \mathbb{R}^{d\times m}$ are assumed to be locally Lipschitz continuous and since $\bar{D}_v \subset D$, $v \in \mathbb{N}$, is a sequence of compact sets.

Roughly speaking, the consistency condition of Definition 3.1 requires the increment function $\phi \colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ from Section 3.1 to be close to the increment function of the respective Euler-Maruyama approximation method. For this reason, we estimate the distance of the exact solution $X \colon [0, T] \times \Omega \to D$ of the SDE (3.1) and of the Euler-Maruyama approximations (see Lemma 3.4 below) and we estimate the distance of the Euler-Maruyama approximations and of the numerical approximations $Y_n^N \colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, (see Lemma 3.8 below). The triangle inequality will then yield an estimate for $\|X_{\frac{nT}{N}} - Y_n^N\|$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ (see Corollary 3.9 below). For this strategy, we now introduce suitable Euler-Maruyama approximations for the SDE (3.1). More formally, let $Z_n^N \colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, be defined recursively through $Z_0^N := X_0$ and

$$Z_{n+1}^N := Z_n^N + \bar{\mu}(Z_n^N) \cdot \tfrac{T}{N} + \bar{\sigma}(Z_n^N) \left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}} \right) \tag{3.13}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Furthermore, let $\tilde{Z}^N \colon [0, T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be given by

$$\tilde{Z}_t^N = Z_n^N + \bar{\mu}(Z_n^N) \cdot \big(t - \tfrac{nT}{N}\big) + \bar{\sigma}(Z_n^N) \left(W_t - W_{\frac{nT}{N}} \right) \tag{3.14}$$

for all $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Finally, let $\tau_v^N \colon \Omega \to [0, T]$, $v, N \in \mathbb{N}$, and $\delta_v^N \colon \Omega \to \{0, 1, \ldots, N\}$, $v, N \in \mathbb{N}$, be defined by

$$\tau_v^N(\omega) := \inf\Big(\{T\} \cup \big\{ t \in [0, T] \colon X_t(\omega) \notin D_v \big\} \cup \big\{ t \in [0, T] \colon \tilde{Z}_t^N(\omega) \notin D_v \big\} \Big) \tag{3.15}$$

and by

$$\begin{aligned} \delta_v^N(\omega) := \min\Big(\{N\} \cup \big\{ n \in \{0, 1, \ldots, N\} \colon Z_n^N(\omega) \notin D_v \big\} \\ \cup \big\{ n \in \{0, 1, \ldots, N\} \colon Y_n^N(\omega) \notin D_v \big\} \Big) \end{aligned} \tag{3.16}$$

for all $\omega \in \Omega$ and all $v, N \in \mathbb{N}$. Using the notation introduced above, the proof of Theorem 3.3 is divided into the following lemmas.

LEMMA 3.4. *Assume that the setting in Section 3.1 is fulfilled, that the setting in the beginning of Subsection 3.3.1 is fulfilled and that* $\phi \colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ, σ)*-consistent with respect to Brownian motion. Then*

$$\lim_{N \to \infty} \mathbb{P}\left[\sup_{t \in [0,T]} \big\| X_t - \tilde{Z}_t^N \big\| \geq \varepsilon \right] = 0 \tag{3.17}$$

for all $\varepsilon \in (0, \infty)$.

The proof of Lemma 3.4 is literally the same as the proof of Corollary 2.6 of Gyöngy & Krylov [**24**] (replace assumption *(ii)* in [**24**] by the weaker assumption of the existence of an exact solution; see also Section 2 in [**43**]). The proof of Lemma 3.4 is therefore omitted.

LEMMA 3.5. *Assume that the setting in Section 3.1 is fulfilled, that the setting in the beginning of Subsection 3.3.1 is fulfilled and that* $\phi \colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ, σ)*-consistent with respect to Brownian motion. Then*

$$\lim_{v \to \infty} \limsup_{N \to \infty} \mathbb{P}\big[\tau_v^N < T \big] = 0. \tag{3.18}$$

LEMMA 3.6. *Assume that the setting in Section 3.1 is fulfilled, that the setting in the beginning of Subsection 3.3.1 is fulfilled and that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ,σ)*-consistent with respect to Brownian motion. Then*

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{n\in\{0,1,\ldots,\delta_v^N\}} \left\|Z_n^N - Y_n^N\right\|\right] = 0 \tag{3.19}$$

for all $v \in \mathbb{N}$.

LEMMA 3.7. *Assume that the setting in Section 3.1 is fulfilled, that the setting in the beginning of Subsection 3.3.1 is fulfilled and that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ,σ)*-consistent with respect to Brownian motion. Then*

$$\lim_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\big[\delta_v^N < N\big] = 0. \tag{3.20}$$

LEMMA 3.8. *Assume that the setting in Section 3.1 is fulfilled, that the setting in the beginning of Subsection 3.3.1 is fulfilled and that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ,σ)*-consistent with respect to Brownian motion. Then*

$$\lim_{N\to\infty} \mathbb{P}\left[\sup_{n\in\{0,1,\ldots,N\}} \left\|Z_n^N - Y_n^N\right\| \geq \varepsilon\right] = 0 \tag{3.21}$$

for all $\varepsilon \in (0,\infty)$.

The proofs of Lemmas 3.5–3.8 are given below. The next corollary is an immediate consequence of Lemma 3.4 and Lemma 3.8.

COROLLARY 3.9. *Assume that the setting in Section 3.1 is fulfilled, that the setting in the beginning of Subsection 3.3.1 is fulfilled and that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ,σ)*-consistent with respect to Brownian motion. Then*

$$\lim_{N\to\infty} \mathbb{P}\left[\sup_{n\in\{0,1,\ldots,N\}} \left\|X_{\frac{nT}{N}} - Y_n^N\right\| \geq \varepsilon\right] = 0 \tag{3.22}$$

for all $\varepsilon \in (0,\infty)$.

Using Corollary 3.9, the proof of Theorem 3.3 is completed at the end of this subsection. We now present the proofs of Lemmas 3.5–3.8. Let us begin with the proof of Lemma 3.5.

PROOF OF LEMMA 3.5. Observe that subadditivity and monotonicity of the probability measure $\mathbb{P}$ show that

$$\begin{aligned}
&\mathbb{P}\big[\tau_v^N < T\big] \\
&\leq \mathbb{P}\Big[\exists\, t \in [0,T]\colon X_t \notin D_w\Big] + \mathbb{P}\Big[\tau_v^N < T,\, \forall\, t \in [0,T]\colon X_t \in D_w\Big] \\
&\leq \mathbb{P}\Big[\exists\, t \in [0,T]\colon X_t \notin D_w\Big] + \mathbb{P}\Big[\tau_v^N < T,\, X_{\tau_v^N} \in D_w,\, \tilde{Z}_{\tau_v^N}^N \in (D_v)^c\Big] \\
&\leq \mathbb{P}\Big[\exists\, t \in [0,T]\colon X_t \notin D_w\Big] + \mathbb{P}\left[\sup_{t\in[0,T]} \|X_t - \tilde{Z}_t^N\| \geq \operatorname{dist}\big(D_w, (D_v)^c\big)\right]
\end{aligned} \tag{3.23}$$

for all $v, w, N \in \mathbb{N}$ with $w \leq v$. Lemma 3.4 and the estimate $\operatorname{dist}\big(D_w, (D_v)^c\big) > 0$ for all $v, w \in \mathbb{N}$ with $w < v$ hence give

$$\limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\big[\tau_v^N < T\big] \leq \mathbb{P}\Big[\exists\, t \in [0,T]\colon X_t \notin D_w\Big] \tag{3.24}$$

for all $w \in \mathbb{N}$. The continuity of the sample paths of $X \colon [0,T] \times \Omega \to D$ therefore yields

$$\limsup_{v\to\infty} \limsup_{N\to\infty} \mathbb{P}\big[\tau_v^N < T\big] \leq \lim_{w\to\infty} \mathbb{P}\big[\exists\, t \in [0,T] \colon X_t \notin D_w\big] = 0 \tag{3.25}$$

and this completes the proof of Lemma 3.5. □

Proof of Lemma 3.6. Throughout this proof, the mappings $\Delta W_n^N \colon \Omega \to \mathbb{R}^m$, $n \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$, defined by $\Delta W_n^N := W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}$ for all $n \in \{0, 1, \ldots N-1\}$ and all $N \in \mathbb{N}$ are used. This notation, in particular, ensures

$$\begin{aligned} Z^N_{k\wedge\delta_v^N} &= X_0 + \sum_{l=0}^{(k\wedge\delta_v^N)-1} \Big(\bar{\mu}(Z_l^N) \cdot \tfrac{T}{N} + \bar{\sigma}(Z_l^N)\, \Delta W_l^N\Big) \\ &= X_0 + \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta_v^N > l\}} \Big(\bar{\mu}(Z_l^N) \cdot \tfrac{T}{N} + \bar{\sigma}(Z_l^N)\, \Delta W_l^N\Big) \end{aligned} \tag{3.26}$$

and

$$Y^N_{k\wedge\delta_v^N} = X_0 + \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta_v^N > l\}}\, \phi\big(Y_l^N, \tfrac{T}{N}, \Delta W_l^N\big) \tag{3.27}$$

for all $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ and all $v \in \mathbb{N}$. In addition, the mappings $\phi_v \colon D_v \times [0, t_v] \to \mathbb{R}^d$, $v \in \mathbb{N}$, defined through

$$\phi_v(x,t) := \mathbb{E}\big[\phi(x,t,W_t)\big] \tag{3.28}$$

for all $(x,t) \in D_v \times [0, t_v]$ and all $v \in \mathbb{N}$ are used throughout this proof. Observe that the definition of t_v, $v \in \mathbb{N}$, (see (3.11)) ensures that the expectation in (3.28) is well defined and thus that the mappings ϕ_v, $v \in \mathbb{N}$, are well defined. In addition, note that

$$\begin{aligned} &\mathbb{E}\Big[\big\|\mathbb{1}_{\{\delta_v^N > l\}}\, \phi\big(Y_l^N, \tfrac{T}{N}, \Delta W_l^N\big)\big\|\Big] = \mathbb{E}\Big[\mathbb{1}_{\{\delta_v^N > l\}}\, \big\|\phi\big(Y_l^N, \tfrac{T}{N}, \Delta W_l^N\big)\big\|\Big] \\ &\leq \sup_{x\in D_v} \mathbb{E}\Big[\big\|\phi\big(x, \tfrac{T}{N}, \Delta W_l^N\big)\big\|\Big] < \infty \end{aligned} \tag{3.29}$$

for all $l \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \leq t_v$ and all $v \in \mathbb{N}$ and that

$$\mathbb{E}\Big[\mathbb{1}_{\{\delta_v^N > l\}}\, \phi\big(Y_l^N, \tfrac{T}{N}, \Delta W_l^N\big) \,\big|\, \mathcal{F}_{\frac{lT}{N}}\Big] = \mathbb{1}_{\{\delta_v^N > l\}}\, \phi_v\big(Y_l^N, \tfrac{T}{N}\big) \tag{3.30}$$

$\mathbb{P}$-a.s. for all $l \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \leq t_v$ and all $v \in \mathbb{N}$. Combining (3.26) and (3.27) with the triangle inequality then implies

$$\begin{aligned} &\big\|Z^N_{k\wedge\delta_v^N} - Y^N_{k\wedge\delta_v^N}\big\| \\ &\leq \frac{T}{N} \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta_v^N > l\}} \big\|\bar{\mu}(Z_l^N) - \bar{\mu}(Y_l^N)\big\| + \left\| \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta_v^N > l\}} \big(\bar{\sigma}(Z_l^N) - \bar{\sigma}(Y_l^N)\big)\, \Delta W_l^N \right\| \\ &+ \frac{T}{N} \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta_v^N > l\}} \big\|\bar{\mu}(Y_l^N) - \tfrac{N}{T} \cdot \phi_v\big(Y_l^N, \tfrac{T}{N}\big)\big\| \\ &+ \left\| \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta_v^N > l\}} \Big(\bar{\sigma}(Y_l^N)\, \Delta W_l^N + \phi_v\big(Y_l^N, \tfrac{T}{N}\big) - \phi\big(Y_l^N, \tfrac{T}{N}, \Delta W_l^N\big)\Big) \right\| \end{aligned}$$

for all $k \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \leq t_v$ and all $v \in \mathbb{N}$. The definition of $c_v \in [0, \infty)$, $v \in \mathbb{N}$, (see (3.12)) hence yields

$$\begin{aligned}
&\sup_{k \in \{0,1,\ldots,n\}} \left\| Z^N_{k \wedge \delta^N_v} - Y^N_{k \wedge \delta^N_v} \right\| \\
&\leq \frac{Tc_v}{N} \sum_{l=0}^{n-1} \left\| Z^N_{l \wedge \delta^N_v} - Y^N_{l \wedge \delta^N_v} \right\| + \sup_{k \in \{0,1,\ldots,n\}} \left\| \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta^N_v > l\}} \left(\bar{\sigma}(Z^N_l) - \bar{\sigma}(Y^N_l) \right) \Delta W^N_l \right\| \\
&+ T \left(\sup_{x \in D_v} \left\| \mu(x) - \tfrac{N}{T} \cdot \phi_v(x, \tfrac{T}{N}) \right\| \right) \\
&+ \sup_{0 \leq k \leq N} \left\| \sum_{l=0}^{k-1} \mathbb{1}_{\{\delta^N_v > l\}} \Big(\bar{\sigma}(Y^N_l)\, \Delta W^N_l + \phi_v\big(Y^N_l, \tfrac{T}{N}\big) - \phi\big(Y^N_l, \tfrac{T}{N}, \Delta W^N_l\big) \Big) \right\|
\end{aligned}$$

for all $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \leq t_v$ and all $v \in \mathbb{N}$. Combining this and (3.30) with the Burkholder-Davis-Gundy inequality (see, e.g., Theorem 48 in Protter [**68**]) then implies the existence of a real number $\kappa \in [0, \infty)$ such that

$$\begin{aligned}
&\left\| \sup_{k \in \{0,1,\ldots,n\}} \left\| Z^N_{k \wedge \delta^N_v} - Y^N_{k \wedge \delta^N_v} \right\| \right\|^2_{L^1(\Omega;\mathbb{R})} \\
&\leq \frac{4T^2 |c_v|^2}{N} \sum_{l=0}^{n-1} \left\| Z^N_{l \wedge \delta^N_v} - Y^N_{l \wedge \delta^N_v} \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \\
&+ 4\kappa \sum_{l=0}^{n-1} \left\| \mathbb{1}_{\{\delta^N_v > l\}} \left(\bar{\sigma}(Z^N_l) - \bar{\sigma}(Y^N_l) \right) \Delta W^N_l \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \\
&+ 4T^2 \left(\sup_{x \in D_v} \left\| \mu(x) - \tfrac{N}{T} \cdot \phi_v(x, \tfrac{T}{N}) \right\|^2 \right) \\
&+ 4\kappa \sum_{l=0}^{N-1} \left\| \mathbb{1}_{\{\delta^N_v > l\}} \Big(\bar{\sigma}(Y^N_l)\, \Delta W^N_l + \phi_v\big(Y^N_l, \tfrac{T}{N}\big) - \phi\big(Y^N_l, \tfrac{T}{N}, \Delta W^N_l\big) \Big) \right\|^2_{L^1(\Omega;\mathbb{R}^d)}
\end{aligned}$$

for all $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \leq t_v$ and all $v \in \mathbb{N}$. The estimate

$$\begin{aligned}
&\mathbb{E}\bigg[\left\| \mathbb{1}_{\{\delta^N_v > l\}} \Big(\bar{\sigma}(Y^N_l)\, \Delta W^N_l + \phi_v\big(Y^N_l, \tfrac{T}{N}\big) - \phi\big(Y^N_l, \tfrac{T}{N}, \Delta W^N_l\big) \Big) \right\| \bigg] \\
&= \mathbb{E}\bigg[\mathbb{1}_{\{\delta^N_v > l\}} \left\| \bar{\sigma}(Y^N_l)\, \Delta W^N_l + \phi_v\big(Y^N_l, \tfrac{T}{N}\big) - \phi\big(Y^N_l, \tfrac{T}{N}, \Delta W^N_l\big) \right\| \bigg] \\
&\leq \sup_{x \in D_v} \mathbb{E}\bigg[\left\| \sigma(x) W_{T/N} + \phi_v(x, \tfrac{T}{N}) - \phi(x, \tfrac{T}{N}, W_{T/N}) \right\| \bigg]
\end{aligned} \tag{3.31}$$

for all $l \in \{0, 1, \ldots, N-1\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \le t_v$ and all $v \in \mathbb{N}$ hence shows

$$
\begin{aligned}
&\left\| \sup_{k \in \{0,1,\ldots,n\}} \left\| Z^N_{k \wedge \delta^N_v} - Y^N_{k \wedge \delta^N_v} \right\| \right\|^2_{L^1(\Omega;\mathbb{R})} \\
&\le \frac{4T^2 |c_v|^2}{N} \sum_{l=0}^{n-1} \left\| Z^N_{l \wedge \delta^N_v} - Y^N_{l \wedge \delta^N_v} \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \\
&+ \frac{4Tm\kappa}{N} \left(\sum_{l=0}^{n-1} \sum_{i=1}^{m} \left\| \mathbb{1}_{\{\delta^N_v > l\}} \left(\bar{\sigma}_i(Z^N_l) - \bar{\sigma}_i(Y^N_l) \right) \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \right) \\
&+ 4T^2 \left(\sup_{x \in D_v} \left\| \mu(x) - \tfrac{N}{T} \cdot \phi_v(x, \tfrac{T}{N}) \right\|^2 \right) \\
&+ 4\kappa N \left(\sup_{x \in D_v} \left\| \sigma(x) W_{T/N} + \phi_v(x, \tfrac{T}{N}) - \phi(x, \tfrac{T}{N}, W_{T/N}) \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \right)
\end{aligned}
\tag{3.32}
$$

for all $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \le t_v$ and all $v \in \mathbb{N}$. The definition of $c_v \in [0, \infty)$, $v \in \mathbb{N}$, (see (3.12)) hence gives

$$
\begin{aligned}
&\left\| \sup_{k \in \{0,1,\ldots,n\}} \left\| Z^N_{k \wedge \delta^N_v} - Y^N_{k \wedge \delta^N_v} \right\| \right\|^2_{L^1(\Omega;\mathbb{R})} \\
&\le \left(\frac{4Tm^2 |c_v|^2 (T + \kappa)}{N} \right) \left(\sum_{l=0}^{n-1} \left\| Z^N_{l \wedge \delta^N_v} - Y^N_{l \wedge \delta^N_v} \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \right) \\
&+ 4T^2 \left(\sup_{x \in D_v} \left\| \mu(x) - \tfrac{N}{T} \cdot \phi_v(x, \tfrac{T}{N}) \right\|^2 \right) \\
&+ 4T\kappa \left(\frac{N}{T} \cdot \sup_{x \in D_v} \left\| \sigma(x) W_{T/N} + \phi_v(x, \tfrac{T}{N}) - \phi(x, \tfrac{T}{N}, W_{T/N}) \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \right)
\end{aligned}
\tag{3.33}
$$

for all $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$ with $\frac{T}{N} \le t_v$ and all $v \in \mathbb{N}$. Moreover, note that (3.6) in Lemma 3.2 implies

$$
\lim_{t \searrow 0} \left(\frac{1}{\sqrt{t}} \cdot \sup_{x \in D_v} \left\| \phi_v(x, t) \right\| \right) = 0 \tag{3.34}
$$

for all $v \in \mathbb{N}$. This and (3.5) in Lemma 3.2 then yield

$$
\lim_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in D_v} \left\| \sigma(x) W_t + \phi_v(x, t) - \phi(x, t, W_t) \right\|^2_{L^1(\Omega;\mathbb{R}^d)} \right) = 0 \tag{3.35}
$$

for all $v \in \mathbb{N}$. Combining (3.6), (3.33), (3.35) and Gronwall's lemma then shows

$$
\lim_{N \to \infty} \left\| \sup_{n \in \{0,1,\ldots,N\}} \left\| Z^N_{n \wedge \delta^N_v} - Y^N_{n \wedge \delta^N_v} \right\| \right\|^2_{L^1(\Omega;\mathbb{R})} = 0 \tag{3.36}
$$

for all $v \in \mathbb{N}$. This completes the proof of Lemma 3.6. ◻

Proof of Lemma 3.7. Note that subadditivity and monotonicity of the probability measure $\mathbb{P}$ show that

$$\begin{aligned}\mathbb{P}\big[\delta_v^N < N\big] &\leq \mathbb{P}\big[\tau_w^N < T\big] + \mathbb{P}\big[\tau_w^N = T,\, \delta_v^N < N\big] \\ &\leq \mathbb{P}\big[\tau_w^N < T\big] + \mathbb{P}\Big[Z_{\delta_v^N}^N \in D_w,\, Y_{\delta_v^N}^N \notin D_v\Big] \\ &\leq \mathbb{P}\big[\tau_w^N < T\big] + \mathbb{P}\Big[\|Z_{\delta_v^N}^N - Y_{\delta_v^N}^N\| \geq \operatorname{dist}\big(D_w, (D_v)^c\big)\Big]\end{aligned} \tag{3.37}$$

for all $v, w, N \in \mathbb{N}$ with $w \leq v$. Markov's inequality and Lemma 3.6 therefore yield

$$\begin{aligned}&\limsup_{v\to\infty}\limsup_{N\to\infty}\mathbb{P}\big[\delta_v^N < N\big] \\ &\leq \limsup_{N\to\infty}\mathbb{P}\big[\tau_w^N < T\big] + \limsup_{v\to\infty}\limsup_{N\to\infty}\left(\frac{\mathbb{E}\big[\|Z_{\delta_v^N}^N - Y_{\delta_v^N}^N\|\big]}{\operatorname{dist}\big(D_w, (D_v)^c\big)}\right) \\ &= \limsup_{N\to\infty}\mathbb{P}\big[\tau_w^N < T\big]\end{aligned} \tag{3.38}$$

for all $w \in \mathbb{N}$. Combining (3.38) and Lemma 3.5 completes the proof of Lemma 3.7. □

Proof of Lemma 3.8. The identity $\{\delta_v^N < N\} \uplus \{\delta_v^N = N\} = \Omega$ and Markov's inequality imply

$$\begin{aligned}&\mathbb{P}\left[\sup_{n\in\{0,1,\ldots,N\}}\big\|Z_n^N - Y_n^N\big\| \geq \varepsilon\right] \\ &\leq \mathbb{P}\big[\delta_v^N < N\big] + \mathbb{P}\left[\delta_v^N = N,\, \sup_{n\in\{0,1,\ldots,\delta_v^N\}}\big\|Z_n^N - Y_n^N\big\| \geq \varepsilon\right] \\ &\leq \mathbb{P}\big[\delta_v^N < N\big] + \frac{1}{\varepsilon}\cdot\mathbb{E}\left[\sup_{n\in\{0,1,\ldots,\delta_v^N\}}\big\|Z_n^N - Y_n^N\big\|\right]\end{aligned} \tag{3.39}$$

for all $v, N \in \mathbb{N}$ and all $\varepsilon \in (0,\infty)$. Lemma 3.7 and Lemma 3.6 therefore yield

$$\lim_{N\to\infty}\mathbb{P}\left[\sup_{n\in\{0,1,\ldots,N\}}\big\|Z_n^N - Y_n^N\big\| \geq \varepsilon\right] = 0 \tag{3.40}$$

for all $\varepsilon \in (0,\infty)$. This completes the proof of Lemma 3.8. □

Proof of Theorem 3.3. In order to show Theorem 3.3, let $\bar{X}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes defined by

$$\bar{X}_t^N := \big(n + 1 - \tfrac{tN}{T}\big)\, X_{\frac{nT}{N}} + \big(\tfrac{tN}{T} - n\big)\, X_{\frac{(n+1)T}{N}} \tag{3.41}$$

for all $t \in [\frac{nT}{N}, \frac{(n+1)T}{N}]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. Then

$$\sup_{t\in[0,T]}\big\|\bar{X}_t^N - \bar{Y}_t^N\big\| = \sup_{n\in\{0,1,\ldots,N\}}\Big\|X_{\frac{nT}{N}} - Y_n^N\Big\| \tag{3.42}$$

for all $N \in \mathbb{N}$. Moreover, the continuity of the sample paths of $X\colon [0,T]\times\Omega \to D$ yields

$$\lim_{N\to\infty}\left(\sup_{t\in[0,T]}\big\|X_t - \bar{X}_t^N\big\|\right) = 0 \tag{3.43}$$

$\mathbb{P}$-a.s.. This implies $\lim_{N\to\infty} \mathbb{P}\big[\sup_{t\in[0,T]} \|X_t - \bar{X}_t^N\| \geq \varepsilon\big] = 0$ for all $\varepsilon \in (0,\infty)$ and Corollary 3.9 hence shows

$$\begin{aligned}
&\limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \big\|X_t - \bar{Y}_t^N\big\| \geq \varepsilon\right] \\
&\leq \limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \big\|X_t - \bar{X}_t^N\big\| + \sup_{t\in[0,T]} \big\|\bar{X}_t^N - \bar{Y}_t^N\big\| \geq \varepsilon\right] \\
&\leq \limsup_{N\to\infty} \mathbb{P}\left[\sup_{t\in[0,T]} \big\|X_t - \bar{X}_t^N\big\| \geq \tfrac{\varepsilon}{2}\right] \\
&\quad + \limsup_{N\to\infty} \mathbb{P}\left[\sup_{n\in\{0,1,\ldots,N\}} \Big\|X_{\frac{nT}{N}} - Y_n^N\Big\| \geq \tfrac{\varepsilon}{2}\right] = 0
\end{aligned} \tag{3.44}$$

for all $\varepsilon \in (0,\infty)$. This completes the proof of Theorem 3.3. □

3.4. Strong convergence

In this section, we combine the convergence in probability result of Theorem 3.3 with moment bounds for the numerical approximation processes in (3.2) to obtain strong convergence of the numerical approximation processes in (3.2).

3.4.1. Strong convergence based on moment bounds. This subsection presents strong convergence results under the assumption that the numerical approximation processes in (3.2) satisfy suitable moment bounds. Below in Subsections 3.4.3 and 3.4.2, we will give more concrete conditions on the drift coefficient μ, the diffusion coefficient σ and the numerical method which allow us to avoid to impose these moment bound assumptions. The strong convergence results in this subsection use the following well-known modification of Fatou's lemma. For completeness its proof is given below.

Lemma 3.10 (A modified version of Fatou's lemma). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let (E, d_E) be a separable metric space and let $Z_N \colon \Omega \to E$, $N \in \mathbb{N}$, and $Z \colon \Omega \to E$ be $\mathcal{F}/\mathcal{B}(E)$-measurable mappings with $\lim_{N\to\infty} \mathbb{P}\big[d_E(Z_N, Z) \geq \varepsilon\big] = 0$ for all $\varepsilon \in (0,\infty)$. Then*

$$\mathbb{E}\big[\varphi(Z)\big] \leq \liminf_{N\to\infty} \mathbb{E}\big[\varphi(Z_N)\big] \tag{3.45}$$

for all continuous functions $\varphi \colon E \to [0,\infty]$.

Proof of Lemma 3.10. First, let $\varphi \colon E \to [0,\infty]$ be an arbitrary continuous function. Then let $N(k) \in \mathbb{N}$, $k \in \mathbb{N}$, be an increasing sequence of natural numbers such that

$$\liminf_{N\to\infty} \mathbb{E}\big[\varphi(Z_N)\big] = \lim_{k\to\infty} \mathbb{E}\big[\varphi(Z_{N(k)})\big]. \tag{3.46}$$

Next note that $\lim_{k\to\infty} \mathbb{P}\big[d_E(Z_{N(k)}, Z) \geq \varepsilon\big] = 0$ for all $\varepsilon \in (0,\infty)$ by assumption. Consequently, there exists an increasing sequence $k_l \in \mathbb{N}$, $l \in \mathbb{N}$, of natural numbers such that $\lim_{l\to\infty} Z_{N(k_l)} = Z$ $\mathbb{P}$-a.s.. The continuity of φ hence implies $\lim_{l\to\infty} \varphi(Z_{N(k_l)}) = \varphi(Z)$ $\mathbb{P}$-a.s.. Combining this, Fatou's lemma and (3.46) then gives

$$\mathbb{E}\big[\varphi(Z)\big] = \mathbb{E}\bigg[\lim_{l\to\infty} \varphi(Z_{N(k_l)})\bigg] \leq \liminf_{l\to\infty} \mathbb{E}\big[\varphi(Z_{N(k_l)})\big] = \lim_{k\to\infty} \mathbb{E}\big[\varphi(Z_{N(k)})\big] = \liminf_{N\to\infty} \mathbb{E}\big[\varphi(Z_N)\big]. \tag{3.47}$$

The proof of Lemma 3.10 is thus completed. □

Let us now present the promised strong convergence results which use the assumption of moment bounds for the numerical approximation processes in (3.2).

COROLLARY 3.11 (Strong final value convergence based on moment bounds). *Assume that the setting in Section 3.1 is fulfilled, suppose that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ, σ)*-consistent with respect to Brownian motion and let* $p \in (0, \infty)$ *be a real number such that*

$$\limsup_{N\to\infty} \mathbb{E}\Big[\big\|\bar{Y}_T^N\big\|^p\Big] < \infty. \tag{3.48}$$

Then $\mathbb{E}\big[\|X_T\|^p\big] < \infty$ *and*

$$\lim_{N\to\infty} \mathbb{E}\Big[\big\|X_T - \bar{Y}_T^N\big\|^q\Big] = 0 \tag{3.49}$$

for all $q \in (0, p)$.

PROOF OF COROLLARY 3.11. First, let $q \in (0, p)$ be arbitrary. Next observe that inequality (3.48) implies that there exists a natural number $N_0 \in \mathbb{N}$ such that

$$\sup_{N\in\{N_0, N_0+1,\dots\}} \mathbb{E}\Big[\big\|\bar{Y}_T^N\big\|^p\Big] < \infty. \tag{3.50}$$

Theorem 3.3 and Lemma 3.10 hence imply that

$$\mathbb{E}\big[\|X_T\|^p\big] \leq \liminf_{N\to\infty} \mathbb{E}\Big[\big\|\bar{Y}_T^N\big\|^p\Big] \leq \sup_{N\in\{N_0, N_0+1,\dots\}} \mathbb{E}\Big[\big\|\bar{Y}_T^N\big\|^p\Big] < \infty. \tag{3.51}$$

This together with inequality (3.50) shows that the family of random variables

$$\|X_T - \bar{Y}_T^N\|^q, \quad N \in \{N_0, N_0+1, \dots\}, \tag{3.52}$$

is bounded in $L^{p/q}(\Omega; \mathbb{R})$ and, therefore, uniformly integrable (see, e.g., Corollary 6.21 in Klenke [**45**]). Theorem 6.25 in Klenke [**45**] and Theorem 3.3 hence imply (3.49). This completes the proof of Corollary 3.11. □

COROLLARY 3.12 (Strong convergence based on moment bounds). *Assume that the setting in Section 3.1 is fulfilled, suppose that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ, σ)*-consistent with respect to Brownian motion and let* $p \in (0, \infty)$ *be a real number such that*

$$\limsup_{N\to\infty} \sup_{n\in\{0,1,\dots,N\}} \mathbb{E}\Big[\big\|\bar{Y}_{\frac{nT}{N}}^N\big\|^p\Big] < \infty. \tag{3.53}$$

Then $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^p\big] < \infty$ *and*

$$\lim_{N\to\infty} \left(\sup_{t\in[0,T]} \mathbb{E}\Big[\big\|X_t - \bar{Y}_t^N\big\|^q\Big] \right) = 0 \tag{3.54}$$

for all $q \in (0, p)$.

Proof of Corollary 3.12. Inequality (3.53) implies

$$\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\Big[\big\|\bar{Y}_t^N\big\|^p\Big] < \infty \tag{3.55}$$

and this yields that there exists a natural number $N_0 \in \mathbb{N}$ such that

$$\sup_{N\in\{N_0,N_0+1,\dots\}} \sup_{t\in[0,T]} \mathbb{E}\Big[\big\|\bar{Y}_t^N\big\|^p\Big] < \infty. \tag{3.56}$$

Theorem 3.3 and Lemma 3.10 therefore show that

$$\sup_{t\in[0,T]} \mathbb{E}\Big[\big\|X_t\big\|^p\Big] \leq \sup_{t\in[0,T]} \liminf_{N\to\infty} \mathbb{E}\Big[\big\|\bar{Y}_t^N\big\|^p\Big] \leq \limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\Big[\big\|\bar{Y}_t^N\big\|^p\Big] < \infty. \tag{3.57}$$

Moreover, the identity

$$\mathbb{1}_{\left\{\sup_{s\in[0,T]} \|X_s-\bar{Y}_s^N\|<1\right\}} + \mathbb{1}_{\left\{\sup_{s\in[0,T]} \|X_s-\bar{Y}_s^N\|\geq 1\right\}} \equiv 1 \tag{3.58}$$

and Hölder's inequality imply

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[\big\|X_t-\bar{Y}_t^N\big\|^q\Big]\\
&\leq \sup_{t\in[0,T]} \mathbb{E}\Big[\mathbb{1}_{\left\{\sup_{s\in[0,T]} \|X_s-\bar{Y}_s^N\|<1\right\}} \big\|X_t-\bar{Y}_t^N\big\|^q\Big]\\
&+ \sup_{t\in[0,T]} \mathbb{E}\Big[\mathbb{1}_{\left\{\sup_{s\in[0,T]} \|X_s-\bar{Y}_s^N\|\geq 1\right\}} \big\|X_t-\bar{Y}_t^N\big\|^q\Big]\\
&\leq \mathbb{E}\Big[\mathbb{1}_{\left\{\sup_{t\in[0,T]} \|X_t-\bar{Y}_t^N\|<1\right\}} \sup_{t\in[0,T]} \big\|X_t-\bar{Y}_t^N\big\|^q\Big]\\
&+ \left(\mathbb{P}\Big[\sup_{t\in[0,T]} \|X_t-\bar{Y}_t^N\| \geq 1\Big]\right)^{\frac{(p-q)}{p}} \left(\sup_{M\in\{N_0,N_0+1,\dots\}} \sup_{t\in[0,T]} \mathbb{E}\Big[\big\|X_t-\bar{Y}_t^M\big\|^p\Big]\right)^{\frac{q}{p}}
\end{aligned}$$

for all $N \in \{N_0, N_0+1, \dots\}$ and all $q \in (0,p)$. The estimate $|a+b|^p \leq 2^p\,|a|^p + 2^p\,|b|^p$ for all $a, b \in \mathbb{R}$ and (3.56) and (3.57) therefore give

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\Big[\big\|X_t-\bar{Y}_t^N\big\|^q\Big]\\
&\leq \mathbb{E}\Big[\min\Big(1, \sup_{t\in[0,T]} \big\|X_t-\bar{Y}_t^N\big\|^q\Big)\Big]\\
&+ 2^p \left(\mathbb{P}\Big[\sup_{t\in[0,T]} \|X_t-\bar{Y}_t^N\| \geq 1\Big]\right)^{\frac{(p-q)}{p}} \left(\sup_{M\in\{N_0,N_0+1,\dots\}} \sup_{t\in[0,T]} \mathbb{E}\Big[\big\|\bar{Y}_t^M\big\|^p\Big]\right)^{\frac{q}{p}}\\
&+ 2^p \left(\mathbb{P}\Big[\sup_{t\in[0,T]} \|X_t-\bar{Y}_t^N\| \geq 1\Big]\right)^{\frac{(p-q)}{p}} \left(\sup_{t\in[0,T]} \mathbb{E}\Big[\big\|X_t\big\|^p\Big]\right)^{\frac{q}{p}} < \infty
\end{aligned} \tag{3.59}$$

for all $N \in \{N_0, N_0+1, \dots\}$ and all $q \in (0,p)$. Moreover, Theorem 3.3 implies

$$\lim_{N\to\infty} \mathbb{P}\Big[\sup_{t\in[0,T]} \|X_t-\bar{Y}_t^N\| \geq 1\Big] = 0 \tag{3.60}$$

and

$$\lim_{N\to\infty} \mathbb{E}\left[\min\left(1, \sup_{t\in[0,T]} \left\|X_t - \bar{Y}_t^N\right\|^q\right)\right] = 0 \tag{3.61}$$

for all $q \in (0,\infty)$. Combining (3.59)–(3.61) then shows (3.54). This completes the proof of Corollary 3.12. □

Corollary 3.13 (Uniform strong convergence based on a priori moment bounds). *Assume that the setting in Section 3.1 is fulfilled, suppose that $\phi\colon \mathbb{R}^d\times[0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion and let $p \in (0,\infty)$ be a real number such that*

$$\limsup_{N\to\infty} \mathbb{E}\left[\sup_{n\in\{0,1,\ldots,N\}} \left\|\bar{Y}^N_{\frac{nT}{N}}\right\|^p\right] < \infty. \tag{3.62}$$

Then $\mathbb{E}\big[\sup_{t\in[0,T]} \|X_t\|^p\big] < \infty$ and

$$\lim_{N\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left\|X_t - \bar{Y}_t^N\right\|^q\right] = 0 \tag{3.63}$$

for all $q \in (0,p)$.

The proof of Corollary 3.13 is analogous to the proof of Corollary 3.11 and therefore omitted. Corollary 3.13 extends Theorem 2.2 of Higham, Mao & Stuart [**34**] in several ways. First, the moment bound condition on the exact solution in Assumption 2.1 in [**34**] is omitted in Corollary 3.13. This assumption can be omitted in Corollary 3.13 since (3.62) and Lemma 3.10 imply $\mathbb{E}\big[\sup_{t\in[0,T]} \|X_t\|^p\big] < \infty$. Moreover, Theorem 2.2 in [**34**] proves uniform strong mean square convergence for the Euler-Maruyama scheme while Corollary 3.13 proves uniform strong L^q-convergence with $q \in (0,p)$ for (μ,σ)-consistent one-step schemes of the form (3.2).

3.4.2. Strong convergence based on semi stability. Roughly speaking, the next corollary asserts that semi stability with respect to Brownian motion (see Definition 2.8) together with consistency with respect to Brownian motion (see Definition 3.1) and with the a priori growth bound (3.65) implies strong convergence of the numerical approximation processes (3.2) to the solution of the SDE (3.1). Its proof is a direct consequence of Corollary 2.9 and of Corollary 3.12 and is therefore omitted.

Corollary 3.14. *Assume that the setting in Section 3.1 is fulfilled, let $\alpha, r \in (1,\infty]$, $p \in (0,\infty)$, $\theta \in (0,T]$, let $V\colon \mathbb{R}^d \to [0,\infty)$ be a Borel measurable function with $\sup_{x\in\mathbb{R}^d} \frac{\|x\|^p}{1+V(x)} < \infty$ and $\mathbb{E}[V(X_0)] < \infty$, assume that $\phi\colon \mathbb{R}^d\times[0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion, assume that*

$$\mathbb{R}^d \times [0,\theta] \times \mathbb{R}^m \ni (x,t,y) \mapsto x + \phi(t,x,y) \in \mathbb{R}^d \tag{3.64}$$

is α-semi V-stable with respect to Brownian motion and assume that

$$\limsup_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \left(N^{(1-\alpha)(1/p-1/(pr))} \|\bar{Y}^N_{nT/N}\|_{L^{pr}(\Omega;\mathbb{R}^d)}\right) < \infty. \tag{3.65}$$

Then $\limsup_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^p + \|\bar{Y}_t^N\|^p\big] < \infty$ and

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{3.66}$$

for all $q \in (0, p)$.

3.4.3. Strong convergence of an increment-tamed Euler-Maruyama scheme. In Subsection 3.4.1, strong convergence of numerical methods of the form (3.2) has been proved under the assumption of suitable moment bounds for the numerical approximation processes (3.2). The next result proves strong convergence of an increment-tamed Euler method and imposes appropriate assumptions on the coefficients μ and σ of the SDE (3.1).

THEOREM 3.15 (Strong convergence of an increment-tamed Euler-Maruyama scheme). *Assume that the setting in Section 3.1 is fulfilled, assume* $\mathbb{E}\big[\|X_0\|^r\big] < \infty$ *for all* $r \in [0, \infty)$*, let* $p \in [3, \infty)$*,* $c, \gamma_0, \gamma_1 \in [0, \infty)$*, let* $\bar{\mu}\colon \mathbb{R}^d \to \mathbb{R}^d$*,* $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ *be Borel measurable functions with* $\bar{\mu}|_D = \mu$ *and* $\bar{\sigma}|_D = \sigma$*, let* $V \in C_p^3(\mathbb{R}^d, [1, \infty))$ *with*

$$(\mathcal{G}_{\bar{\mu},\bar{\sigma}}V)(x) \leq c \cdot V(x),\ \|\bar{\mu}(x)\| \leq c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]},\ \|\bar{\sigma}(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]} \tag{3.67}$$

for all $x \in \mathbb{R}^d$ *and let* $\bar{Y}^N\colon [0, T] \times \Omega \to \mathbb{R}^d$*,* $N \in \mathbb{N}$*, satisfy*

$$\bar{Y}_t^N = \bar{Y}_{\frac{nT}{N}}^N + \frac{\left(\frac{tN}{T} - n\right)\Big(\bar{\mu}(\bar{Y}_{nT/N}^N)\frac{T}{N} + \bar{\sigma}(\bar{Y}_{nT/N}^N)(W_{(n+1)T/N} - W_{nT/N})\Big)}{\max\big(1, \frac{T}{N}\|\bar{\mu}(\bar{Y}_{nT/N}^N)\frac{T}{N} + \bar{\sigma}(\bar{Y}_{nT/N}^N)(W_{(n+1)T/N} - W_{nT/N})\|\big)} \tag{3.68}$$

for all $t \in (nT/N, (n+1)T/N]$*,* $n \in \{0, 1, \ldots, N-1\}$ *and all* $N \in \mathbb{N}$*. Then* $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^q\big] < \infty$ *and*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{3.69}$$

for all $q \in (0, \infty)$ *with* $q < \frac{p}{2\gamma_1+4\max(\gamma_0,\gamma_1,1/2)} - \frac{1}{2}$ *and* $\limsup_{r\searrow q} \sup_{x\in\mathbb{R}^d} \|x\|^r/V(x) < \infty$*.*

Theorem 3.15 is a direct consequence of Corollary 2.21, Corollary 3.12 and Lemma 3.28 (see (3.141)). The next result is a special case of Theorem 3.15.

COROLLARY 3.16 (Powers of the Lyapunov-type function). *Assume that the setting in Section 3.1 is fulfilled, assume* $\mathbb{E}\big[\|X_0\|^r\big] < \infty$ *for all* $r \in [0, \infty)$*, let* $p \in [3, \infty)$*,* $q \in [1, \infty)$*,* $c, \gamma_0, \gamma_1 \in [0, \infty)$*, let* $\bar{\mu}\colon \mathbb{R}^d \to \mathbb{R}^d$*,* $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ *be Borel measurable functions with* $\bar{\mu}|_D = \mu$ *and* $\bar{\sigma}|_D = \sigma$*, let* $V \in C_p^3(\mathbb{R}^d, [1, \infty))$ *with* $\|\bar{\mu}(x)\| \leq c\,|V(x)|^{\left[\frac{\gamma_0+1}{p}\right]}$*,* $\|\bar{\sigma}(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\,|V(x)|^{\left[\frac{\gamma_1+2}{2p}\right]}$ *and*

$$(\mathcal{G}_{\bar{\mu},\bar{\sigma}}V)(x) + \frac{(q-1)}{2V(x)}\|V'(x)\bar{\sigma}(x)\|_{HS(\mathbb{R}^m,\mathbb{R})}^2 \leq c \cdot V(x) \tag{3.70}$$

for all $x \in \mathbb{R}^d$ *and let* $\bar{Y}^N\colon [0, T] \times \Omega \to \mathbb{R}^d$*,* $N \in \mathbb{N}$*, satisfy* (3.68)*. Then* $\sup_{t\in[0,T]} \mathbb{E}\big[\|X_t\|^r\big] < \infty$ *and*

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^r\big] = 0 \tag{3.71}$$

for all $r \in (0, \infty)$ *with* $r < \frac{pq}{2\gamma_1+4\max(\gamma_0,\gamma_1,1/2)} - \frac{1}{2}$ *and* $\limsup_{v\searrow r} \sup_{x\in\mathbb{R}^d} \frac{\|x\|^v}{V(x)} < \infty$*.*

Corollary 3.16 follows immediately Corollary 2.22, Corollary 3.12 and Lemma 3.28 (see (3.141)). The next corollary gives sufficient conditions for strong L^q-convergence of the increment-tamed Euler-Maruyama method (3.68) for all $q \in (0, \infty)$.

COROLLARY 3.17. *Assume that the setting in Section 3.1 is fulfilled, assume* $\mathbb{E}\big[\|X_0\|^r\big] < \infty$ *for all* $r \in [0,\infty)$*, let* $c \in (0,\infty)$*, let* $\bar{\mu}\colon \mathbb{R}^d \to \mathbb{R}^d$*,* $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ *be Borel measurable functions with* $\bar{\mu}|_D = \mu$ *and* $\bar{\sigma}|_D = \sigma$*, let* $V\colon \mathbb{R}^d \to [1,\infty)$ *be a twice differentiable function with a locally Lipschitz continuous second derivative, with* $\limsup_{q\searrow 0}\sup_{x\in\mathbb{R}^d}\frac{\|x\|^q}{V(x)} < \infty$*, with* $\sum_{i=1}^{3}\|V^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \leq c\,|V(x)|^{[1-1/c]}$ *for* $\lambda_{\mathbb{R}^d}$*-almost all* $x \in \mathbb{R}^d$ *and with*

$$\sup_{x\in\mathbb{R}^d}\left[\frac{(\mathcal{G}_{\bar{\mu},\bar{\sigma}}V)(x)}{V(x)} + \frac{r\,\|V'(x)\bar{\sigma}(x)\|^2_{L(\mathbb{R}^m,\mathbb{R})}}{|V(x)|^2}\right] < \infty, \tag{3.72}$$

$$\sup_{x\in\mathbb{R}^d}\left[\frac{\|\bar{\mu}(x)\| + \|\bar{\sigma}(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)}}{(1+\|x\|^c)}\right] < \infty \tag{3.73}$$

for all $r \in [0,\infty)$ *and let* $\bar{Y}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$*,* $N \in \mathbb{N}$*, satisfy* (3.68)*. Then* $\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t\|^q\big] < \infty$ *and* $\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - \bar{Y}^N_t\|^q\big] = 0$ *for all* $q \in (0,\infty)$*.*

Corollary 3.17 follows directly from Corollary 3.16. The next corollary specializes Theorem 3.15 to the function $1 + \|x\|^p$, $x \in \mathbb{R}^d$, with $p \in [3,\infty)$ appropriate as Lyapunov-type function and is the counterpart to Corollary 2.24. It is an immediate consequence of Corollary 2.24, Corollary 3.12 and Lemma 3.28 (see (3.141)).

COROLLARY 3.18 (A special polynomial like Lyapunov-type function). *Assume that the setting in Section 3.1 is fulfilled, assume* $\mathbb{E}\big[\|X_0\|^r\big] < \infty$ *for all* $r \in [0,\infty)$*, let* $c, \gamma_0, \gamma_1 \in [0,\infty)$*,* $p \in [3,\infty)$*, let* $\bar{\mu}\colon \mathbb{R}^d \to \mathbb{R}^d$*,* $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ *be Borel measurable functions with* $\bar{\mu}|_D = \mu$*,* $\bar{\sigma}|_D = \sigma$ *and*

$$\langle x, \bar{\mu}(x)\rangle + \tfrac{(p-1)}{2}\|\bar{\sigma}(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^2\right), \tag{3.74}$$

$$\|\bar{\mu}(x)\| \leq c\left(1 + \|x\|^{[\gamma_0+1]}\right) \quad \text{and} \quad \|\bar{\sigma}(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \leq c\left(1 + \|x\|^{\left[\frac{\gamma_1+2}{2}\right]}\right) \tag{3.75}$$

for all $x \in \mathbb{R}^d$ *and let* $\bar{Y}^N\colon [0,T]\times\Omega \to \mathbb{R}^d$*,* $N \in \mathbb{N}$*, satisfy* (3.68)*. Then* $\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t\|^q\big] < \infty$ *and*

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - \bar{Y}^N_t\|^q\big] = 0 \tag{3.76}$$

for all $q \in (0,\infty)$ *with* $q < \frac{p}{2\gamma_1 + 4\max(\gamma_0,\gamma_1,1/2)} - \frac{1}{2}$*.*

3.5. Weak convergence

Convergence in probability implies stochastic weak convergence. The next corollary is thus an immediate consequence of Theorem 3.3.

COROLLARY 3.19 (Weak convergence with bounded test functions). *Assume that the setting in Section 3.1 is fulfilled, suppose that* $\phi\colon \mathbb{R}^d\times[0,T]\times\mathbb{R}^m \to \mathbb{R}^d$ *is* (μ,σ)*-consistent with respect to Brownian motion and let* $(E, \|\cdot\|_E)$ *be a separable* $\mathbb{R}$*-Banach space. Then*

$$\lim_{N\to\infty}\big\|\mathbb{E}\big[f(X)\big] - \mathbb{E}\big[f(\bar{Y}^N)\big]\big\|_E = 0 \tag{3.77}$$

for all bounded and continuous functions $f\colon C([0,T],\mathbb{R}^d) \to E$*.*

Corollary 3.19 follows directly from Theorem 3.3 since the test functions in Corollary 3.19 are assumed to be bounded. The case of unbounded test functions is more subtle and is analyzed in the sequel. The next lemma proves weak convergence restricted to events whose probabilities converge to one sufficiently fast with possibly unbounded test function under the assumption that the discrete-time stochastic processes $\bar{Y}^N_{nT/N}$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, are 0-semi V-bounded with $V\colon \mathbb{R}^d \to [0,\infty)$ appropriate.

Lemma 3.20 (Semi weak convergence). *Assume that the setting in Section 3.1 is fulfilled, suppose that $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion, let $(E, \|\cdot\|_E)$ be a separable $\mathbb{R}$-Banach space, let $V\colon \mathbb{R}^d \to [0,\infty)$ be a continuous function and assume that the sequence $\bar{Y}^N_{nT/N}\colon \Omega \to \mathbb{R}^d$, $n \in \{0,1,\ldots,N\}$, $N \in \mathbb{N}$, of discrete-time stochastic processes is 0-semi V-bounded, i.e., let $\Omega_N \in \sigma_\Omega(\bar{Y}^N)$, $N \in \mathbb{N}$, be a sequence of events such that*

$$\limsup_{N\to\infty} \sup_{n\in\{0,1,\ldots,N\}} \mathbb{E}\big[\mathbb{1}_{\Omega_N} V\big(\bar{Y}^N_{nT/N}\big)\big] < \infty \quad \text{and} \quad \lim_{N\to\infty} \mathbb{P}\big[\Omega_N\big] = 1. \tag{3.78}$$

Then $\mathbb{E}\big[V(X_T) + \|f(X_T)\|_E\big] < \infty$ and

$$\begin{aligned} &\lim_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{\Omega_N} \|f(X_T) - f(\bar{Y}^N_T)\|_E\big] \\ &= \lim_{N\to\infty} \big\|\mathbb{E}\big[\mathbb{1}_{\Omega_N} f(X_T)\big] - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}^N_T)\big]\big\|_E = 0 \end{aligned} \tag{3.79}$$

for all continuous functions $f\colon \mathbb{R}^d \to E$ with $\limsup_{r\nearrow 1} \sup_{x\in\mathbb{R}^d} \frac{\|f(x)\|_E}{(1+V(x))^r} < \infty$.

Proof of Lemma 3.20. First of all, let $f\colon \mathbb{R}^d \to E$ be a continuous function which satisfies

$$\limsup_{r\nearrow 1} \sup_{x\in\mathbb{R}^d} \frac{\|f(x)\|_E}{(1+V(x))^r} < \infty. \tag{3.80}$$

This ensures that there exists a real number $r \in (0,1)$ such that $\eta := \sup_{x\in\mathbb{R}^d} \frac{\|f(x)\|_E}{(1+V(x))^r} < \infty$. We thus obtain that

$$\|f(x)\|_E \leq \eta\,(1+V(x))^r \tag{3.81}$$

for all $x \in \mathbb{R}^d$. Next observe that Theorem 3.3 and assumption (3.78) show

$$\begin{aligned} &\limsup_{N\to\infty} \mathbb{P}\big[\|X_T - \mathbb{1}_{\Omega_N}\bar{Y}^N_T\| \geq \varepsilon\big] \\ &\leq \limsup_{N\to\infty} \mathbb{P}\big[\|X_T - \mathbb{1}_{\Omega_N} X_T\| \geq \tfrac{\varepsilon}{2}\big] + \limsup_{N\to\infty} \mathbb{P}\big[\mathbb{1}_{\Omega_N}\|X_T - \bar{Y}^N_T\| \geq \tfrac{\varepsilon}{2}\big] \\ &\leq \limsup_{N\to\infty} \mathbb{P}\big[\mathbb{1}_{(\Omega_N)^c}\|X_T\| \geq \tfrac{\varepsilon}{2}\big] + \limsup_{N\to\infty} \mathbb{P}\big[\|X_T - \bar{Y}^N_T\| \geq \tfrac{\varepsilon}{2}\big] \\ &\leq \limsup_{N\to\infty} \mathbb{P}\big[(\Omega_N)^c\big] = 0 \end{aligned} \tag{3.82}$$

for all $\varepsilon \in (0,\infty)$. Lemma 3.10 hence implies that

$$\begin{aligned} \mathbb{E}\big[V(X_T)\big] &\leq \liminf_{N\to\infty} \mathbb{E}\big[V\big(\mathbb{1}_{\Omega_N}\bar{Y}^N_T\big)\big] \\ &\leq V(0) + \limsup_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{\Omega_N} V(\bar{Y}^N_T)\big] < \infty \end{aligned} \tag{3.83}$$

and Jensen's inequality and (3.81) therefore yield that

$$\mathbb{E}\big[\|f(X_T)\|_E\big] \leq \eta \cdot \mathbb{E}\Big[\big(1+V(X_T)\big)^r\Big] \leq \eta \cdot \big(1 + \mathbb{E}[V(X_T)]\big)^r < \infty. \tag{3.84}$$

In addition, observe that Theorem 3.3 implies

$$\lim_{N\to\infty} \mathbb{P}\big[\|f(X_T) - f(\bar{Y}_T^N)\|_E > \varepsilon\big] = 0 \tag{3.85}$$

for all $\varepsilon \in (0,\infty)$. Therefore, we obtain that

$$\lim_{N\to\infty} \mathbb{P}\big[\mathbb{1}_{\Omega_N}\|f(X_T) - f(\bar{Y}_T^N)\|_E > \varepsilon\big] = 0 \tag{3.86}$$

for all $\varepsilon \in (0,\infty)$. Next note that estimate (3.81) ensures that

$$\limsup_{N\to\infty} \big\|\mathbb{1}_{\Omega_N} f(\bar{Y}_T^N)\big\|_{L^{1/r}(\Omega;E)} \leq \eta \cdot \left(1 + \limsup_{N\to\infty} \mathbb{E}\Big[\mathbb{1}_{\Omega_N} V\big(\bar{Y}_T^N\big)\Big]\right)^r < \infty. \tag{3.87}$$

This implies that there exists a natural number $N_0 \in \mathbb{N}$ such that

$$\sup_{N\in\{N_0,N_0+1,\dots\}} \big\|\mathbb{1}_{\Omega_N} f(\bar{Y}_T^N)\big\|_{L^{1/r}(\Omega;E)} < \infty. \tag{3.88}$$

Inequality (3.88), the fact that $\frac{1}{r} > 1$ and Corollary 6.21 of Klenke [**45**] show that the family $\mathbb{1}_{\Omega_N}\|f(\bar{Y}_T^N)\|_E$, $N \in \{N_0, N_0+1,\dots\}$, of $\mathcal{F}/\mathcal{B}(\mathbb{R})$-measurable mappings is uniformly integrable. This together with (3.84) yields that the sequence $\mathbb{1}_{\Omega_N}\|f(X_T) - f(\bar{Y}_T^N)\|_E$, $N \in \{N_0, N_0+1,\dots\}$, is uniformly integrable. Equation (3.86) hence shows (3.79). The proof of Lemma 3.20 is thus completed. □

3.5.1. Convergence of Monte Carlo methods. In (3.79) in Lemma 3.20, the convergence holds restricted to a sequence of events whose probabilities converge to one. In Proposition 3.22 below, we will get rid of the restriction to these events and prove convergence of the corresponding Monte Carlo method on an event of probability one. For proving this result, we first need a minor generalization of Lemma 2.1 in Kloeden & Neuenkirch [**46**]. More precisely, Lemma 2.1 in Kloeden & Neuenkirch [**46**] proves that convergence in the p-th mean with order $\beta \in (0,\infty)$ for all $p \in (0,\infty)$ implies almost sure convergence with order $\beta - \varepsilon$ where $\varepsilon \in (0,\infty)$ is arbitrarily small. The next result is a minor generalization of this result and, in particular, proves for every arbitrary fixed $p \in (0,\infty)$ that convergence in the p-th mean with order $\beta \in (\frac{1}{p},\infty)$ implies almost sure convergence with order $\beta - 1/p - \varepsilon$ for every arbitrarily small $\varepsilon \in (0,\infty)$.

Lemma 3.21 (L^p-convergence with order $\beta \in (1/p,\infty)$ implies almost sure convergence). *Let $M \in \mathbb{N}$, let $(E,\mathcal{E},\mu)$ be a measurable space and let $Y_N\colon E \to \mathbb{R}$, $N \in \{M, M+1,\dots\}$, be a family of $\mathcal{E}/\mathcal{B}(\mathbb{R})$-measurable mappings. Then*

$$\begin{aligned} &\Bigg\|\sup_{N\in\{M,M+1,\dots\}} N^\alpha \cdot |Y_N|\Bigg\|_{L^p(E;\mathbb{R})} \\ &\leq \left[\sum_{N=M}^{\infty} N^{(\alpha-\beta)p}\right]^{1/p} \left[\sup_{N\in\{M,M+1,\dots\}} N^\beta \cdot \|Y_N\|_{L^p(E;\mathbb{R})}\right] \end{aligned} \tag{3.89}$$

for all $\alpha,\beta \in \mathbb{R}$, $p \in (0,\infty)$. In particular, if $\sup_{N\in\{M,M+1,\dots\}}(N^\beta\,\|Y_N\|_{L^p(E;\mathbb{R})}) < \infty$ *for one $p \in (0,\infty)$ and one $\beta \in \mathbb{R}$, then*

$$\int_E \left\{\sup_{N\in\{M,M+1,\dots\}} (N^\alpha \cdot |Y_N|)\right\}^p d\mu < \infty, \quad \sup_{N\in\{M,M+1,\dots\}} \big(N^\alpha \cdot |Y_N|\big) < \infty \ \ \mu\text{-a.s.} \tag{3.90}$$

for all $\alpha \in (-\infty, \beta - \frac{1}{p})$. *Moreover, if* $\sup_{N\in\{M,M+1,\dots\}}(N^{\beta}\,\|Y_N\|_{L^p(E;\mathbb{R})}) < \infty$ *for all* $p \in (0,\infty)$ *and one* $\beta \in \mathbb{R}$, *then*

$$\int_E \left\{ \sup_{N\in\{M,M+1,\dots\}} (N^{\alpha}\cdot|Y_N|) \right\}^p d\mu < \infty, \qquad \sup_{N\in\{M,M+1,\dots\}} \big(N^{\alpha}\cdot|Y_N|\big) < \infty \;\; \mu\text{-a.s.} \tag{3.91}$$

for all $\alpha \in (-\infty, \beta)$, $p \in (0,\infty)$.

Proof of Lemma 3.21. Note that

$$\begin{aligned} &\int_E \left\{ \sup_{N\in\{M,M+1,\dots\}} (N^{\alpha}\cdot|Y_N|) \right\}^p d\mu \\ &= \int_E \left\{ \sup_{N\in\{M,M+1,\dots\}} (N^{\alpha p}\cdot|Y_N|^p) \right\} d\mu \\ &\leq \int_E \left\{ \sum_{N=M}^{\infty} (N^{\alpha p}\cdot|Y_N|^p) \right\} d\mu = \sum_{N=M}^{\infty} \left(N^{\alpha p}\cdot \int_E |Y_N|^p \, d\mu \right) \\ &\leq \left(\sum_{N=M}^{\infty} N^{(\alpha-\beta)p} \right) \left(\sup_{N\in\{M,M+1,\dots\}} N^{\beta p}\cdot \int_E |Y_N|^p \, d\mu \right) \end{aligned} \tag{3.92}$$

for all $\alpha, \beta \in \mathbb{R}$ and all $p \in (0,\infty)$. This proves inequality (3.89). The assertions in (3.90) and (3.91) then follow immediately from inequality (3.89). The proof of Lemma 3.21 is thus completed. □

Note also that the last assertion in Lemma 3.21, i.e., the assertion in (3.91), is essentially Lemma 2.1 in [**46**] generalized to arbitrary measure spaces.

Proposition 3.22 (Convergence of Monte Carlo methods). *Assume that the setting in Section 3.1 is fulfilled, suppose that* $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ,σ)*-consistent with respect to Brownian motion, let* $n \in \mathbb{N}$, $\alpha \in (n+1,\infty)$, *let* $V\colon \mathbb{R}^d \to [0,\infty)$ *be a continuous function, assume that the sequence* $(\bar{Y}^N_{kT/N})_{k\in\{0,1,\dots,N\}}$, $N \in \mathbb{N}$, *of discrete-time stochastic processes is* α*-semi* V*-bounded and for every* $N \in \mathbb{N}$ *let* $\bar{Y}^{N,m}\colon [0,T]\times\Omega \to \mathbb{R}^d$, $m \in \mathbb{N}$, *be independent stochastic processes with* $\mathbb{P}_{\bar{Y}^N} = \mathbb{P}_{\bar{Y}^{N,m}}$ *for all* $m \in \mathbb{N}$. *Then*

$$\lim_{N\to\infty} \left| \mathbb{E}\big[f(X_T)\big] - \frac{\sum_{m=1}^{N^n} f(\bar{Y}^{N,m}_T)}{N^n} \right| = 0 \tag{3.93}$$

$\mathbb{P}$*-a.s. for all continuous functions* $f\colon \mathbb{R}^d \to \mathbb{R}$ *with* $\limsup_{r\nearrow\frac{n\wedge 2}{2}} \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1+V(x))^r} < \infty$.

Proof of Proposition 3.22. Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function which satisfies

$$\limsup_{r\nearrow\frac{n\wedge 2}{2}} \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1+V(x))^r} < \infty. \tag{3.94}$$

Next let $M \in \mathbb{N}$ be a natural number and let $A_N \in \mathcal{B}(\mathbb{R}^d)^{\otimes\{0,1,\dots,N\}}$, $N \in \mathbb{N}$, be a sequence of sets such that

$$c := \sup_{N\in\{M,M+1,\dots\}} \bigg(\mathbb{E}\Big[\mathbb{1}_{\big\{(\bar{Y}^N_{kT/N})_{k\in\{0,1,\dots,N\}}\in A_N\big\}} V(\bar{Y}^N_T)\Big] + N^\alpha \cdot \mathbb{P}\Big[(\bar{Y}^N_{kT/N})_{k\in\{0,1,\dots,N\}} \in (A_N)^c \Big] \bigg) < \infty. \tag{3.95}$$

Such a sequence of sets indeed exists since the sequence $(\bar{Y}^N_{kT/N})_{k\in\{0,1,\dots,N\}}$, $N \in \mathbb{N}$, of discrete-time stochastic processes is assumed to be α-semi V-bounded. We define now two families $\Omega_N \in \mathcal{F}$, $N \in \mathbb{N}$, and $\Omega_{N,m} \in \mathcal{F}$, $N, m \in \mathbb{N}$, of events by

$$\Omega_N := \Big\{ (\bar{Y}^N_{kT/N})_{k\in\{0,1,\dots,N\}} \in A_N \Big\} \quad \text{and} \quad \Omega_{N,m} := \Big\{ (\bar{Y}^{N,m}_{kT/N})_{k\in\{0,1,\dots,N\}} \in A_N \Big\} \tag{3.96}$$

for all $N, m \in \mathbb{N}$. Note that $\mathbb{P}[\Omega_N] = \mathbb{P}[\Omega_{N,m}]$ for all $N, m \in \mathbb{N}$. Moreover, observe that Lemma 3.20 implies

$$\lim_{N\to\infty} \mathbb{E}\big[\mathbb{1}_{(\Omega_N)^c} |f(X_T)| \big] = 0 \quad \text{and} \quad \lim_{N\to\infty} \big| \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(X_T)\big] - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}^N_T)\big] \big| = 0. \tag{3.97}$$

In the next step note that condition (3.94) shows the existence of a real number $r \in (0, \frac{n\wedge 2}{2})$ such that

$$\eta := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1+V(x))^r} < \infty. \tag{3.98}$$

The Burkholder-Davis-Gundy inequality (see, e.g., Theorem 48 in Protter [**68**]) then shows the existence of a real number $\kappa \in [0,\infty)$ such that

$$\begin{aligned}
&\left\| \frac{\sum_{m=1}^{N^n} \big\{ \mathbb{1}_{\Omega_{N,m}} f(\bar{Y}^{N,m}_T) - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}^N_T)\big]\big\}}{N^n} \right\|_{L^{1/r}(\Omega;\mathbb{R})} \\
&\leq \frac{\kappa}{N^n} \left(\sum_{m=1}^{N^n} \big\| \mathbb{1}_{\Omega_{N,m}} f(\bar{Y}^{N,m}_T) - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}^N_T)\big] \big\|^2_{L^{1/r}(\Omega;\mathbb{R})} \right)^{1/2} \\
&\leq \frac{2\kappa \big\| \mathbb{1}_{\Omega_N} f(\bar{Y}^N_T) \big\|_{L^{1/r}(\Omega;\mathbb{R})}}{\sqrt{N^n}} \leq \frac{2\kappa\eta \,\big\| \mathbb{1}_{\Omega_N} \big|1+V(\bar{Y}^N_T)\big|^r \big\|_{L^{1/r}(\Omega;\mathbb{R})}}{\sqrt{N^n}} \\
&= \frac{2\kappa\eta \big\{ \mathbb{E}\big[\mathbb{1}_{\Omega_N} \big(1+V(\bar{Y}^N_T)\big)\big]\big\}^r}{N^{n/2}} \leq \frac{2\kappa\eta \big(1+\mathbb{E}\big[\mathbb{1}_{\Omega_N} V(\bar{Y}^N_T)\big]\big)^r}{N^{n/2}} \\
&\leq \frac{2\kappa\eta\,(1+c)^r}{N^{n/2}}
\end{aligned} \tag{3.99}$$

for all $N \in \{M, M+1, \dots\}$. Therefore, we obtain

$$\sup_{N\in\{M,M+1,\dots\}} \left(N^{\frac{n}{2}} \left\| \frac{\sum_{m=1}^{N^n} \big\{ \mathbb{1}_{\Omega_{N,m}} f(\bar{Y}^{N,m}_T) - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}^N_T)\big]\big\}}{N^n} \right\|_{L^{1/r}(\Omega;\mathbb{R})} \right) < \infty. \tag{3.100}$$

The condition $\frac{n}{2} > r$ and the second inequality in (3.90) in Lemma 3.21 hence yield

$$\lim_{N\to\infty} \left(\frac{\sum_{m=1}^{N^n} \big\{ \mathbb{1}_{\Omega_{N,m}} f(\bar{Y}^{N,m}_T) - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}^N_T)\big]\big\}}{N^n} \right) = 0 \tag{3.101}$$

$\mathbb{P}$-a.s.. Furthermore, observe that

$$\begin{aligned}&\lim_{N\to\infty} \mathbb{1}_{\left[\cup_{K=N}^{\infty}\cup_{m=1}^{K^n}(\Omega_{K,m})^c\right]}(\omega)\\ &= \mathbb{1}_{\left[\cap_{N=1}^{\infty}\cup_{K=N}^{\infty}\cup_{m=1}^{K^n}(\Omega_{K,m})^c\right]}(\omega) = \mathbb{1}_{\left[\limsup_{N\to\infty}\left(\cup_{m=1}^{N^n}(\Omega_{N,m})^c\right)\right]}(\omega)\end{aligned} \tag{3.102}$$

for all $\omega \in \Omega$. In addition, the condition $\alpha > n+1$ gives

$$\sum_{N=M}^{\infty} \mathbb{P}\Big[\cup_{m=1}^{N^n}(\Omega_{N,m})^c\Big] \leq \sum_{N=M}^{\infty} N^n \cdot \mathbb{P}[(\Omega_N)^c] \leq c \sum_{N=M}^{\infty} N^{(n-\alpha)} < \infty \tag{3.103}$$

and the Borel-Cantelli Lemma thus shows

$$\mathbb{P}\left[\limsup_{N\to\infty}\left(\cup_{m=1}^{N^n}(\Omega_{N,m})^c\right)\right] = 0. \tag{3.104}$$

Combining (3.102) with the estimate

$$\begin{aligned}&\frac{\sum_{m=1}^{N^n} \mathbb{1}_{(\Omega_{N,m})^c}|f(\bar{Y}_T^{N,m})|}{N^n}\\ &\leq \frac{\left(\sum_{m=1}^{N^n} |f(\bar{Y}_T^{N,m})|\right)\left(\max_{m\in\{1,2,\dots,N^n\}} \mathbb{1}_{(\Omega_{N,m})^c}\right)}{N^n}\\ &\leq \frac{\left[\sum_{m=1}^{N^n} |f(\bar{Y}_T^{N,m})|\right] \mathbb{1}_{\cup_{K=N}^{\infty}\cup_{m=1}^{K^n}(\Omega_{K,m})^c}}{N^n}\end{aligned} \tag{3.105}$$

for all $N \in \mathbb{N}$ results in

$$\lim_{N\to\infty}\left(\frac{\sum_{m=1}^{N^n} \mathbb{1}_{(\Omega_{N,m})^c}(\omega)\,|f(\bar{Y}_T^{N,m}(\omega))|}{N^n}\right) = 0 \tag{3.106}$$

for all $\omega \in \left[\cap_{N=1}^{\infty} \cup_{K=N}^{\infty} \cup_{m=1}^{K^n}(\Omega_{K,m})^c\right]^c = \left[\limsup_{N\to\infty}\left(\cup_{m=1}^{N^n}(\Omega_{N,m})^c\right)\right]^c$. This and (3.104) then show

$$\lim_{N\to\infty}\left(\frac{\sum_{m=1}^{N^n} \mathbb{1}_{(\Omega_{N,m})^c}|f(\bar{Y}_T^{N,m})|}{N^n}\right) = 0 \tag{3.107}$$

$\mathbb{P}$-a.s.. In the next step observe that the triangle inequality implies

(3.108)

$$\begin{aligned}&\left|\mathbb{E}\big[f(X_T)\big] - \frac{\sum_{m=1}^{N^n} f(\bar{Y}_T^{N,m})}{N^n}\right|\\ &\leq \left|\mathbb{E}\big[f(X_T)\big] - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(X_T)\big]\right| + \left|\mathbb{E}\big[\mathbb{1}_{\Omega_N} f(X_T)\big] - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}_T^N)\big]\right|\\ &\quad + \left|\mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}_T^N)\big] - \frac{\sum_{m=1}^{N^n} \mathbb{1}_{\Omega_{N,m}} f(\bar{Y}_T^{N,m})}{N^n}\right| + \left|\frac{\sum_{m=1}^{N^n}\left(\mathbb{1}_{\Omega_{N,m}} - 1\right) f(\bar{Y}_T^{N,m})}{N^n}\right|\\ &\leq \mathbb{E}\big[\mathbb{1}_{(\Omega_N)^c}|f(X_T)|\big] + \left|\mathbb{E}\big[\mathbb{1}_{\Omega_N} f(X_T)\big] - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}_T^N)\big]\right|\\ &\quad + \frac{\left|\sum_{m=1}^{N^n}\left\{\mathbb{1}_{\Omega_{N,m}} f(\bar{Y}_T^{N,m}) - \mathbb{E}\big[\mathbb{1}_{\Omega_N} f(\bar{Y}_T^N)\big]\right\}\right|}{N^n} + \frac{\sum_{m=1}^{N^n} \mathbb{1}_{(\Omega_{N,m})^c}|f(\bar{Y}_T^{N,m})|}{N^n}\end{aligned}$$

for all $N \in \mathbb{N}$. Combining (3.97), (3.101), (3.107) and (3.108) finally implies (3.93). The proof of Proposition 3.22 is thus completed. $\square$

3.5.2. Convergence of the Monte Carlo Euler method. Next we prove almost sure convergence of the Monte Carlo Euler method. This result generalizes Theorem 2.1 in [**37**] which assumes μ to be globally one-sided Lipschitz continuous, σ to be globally Lipschitz continuous and μ and σ to grow at most polynomially fast. Corollary 3.23 is a direct consequence of Theorem 2.13, of Corollary 2.6 and of Proposition 3.22 and its proof is therefore omitted.

COROLLARY 3.23 (Convergence of the Monte Carlo Euler method). *Assume that the setting in Section 3.1 is fulfilled, let $p \in [3, \infty)$, $c, \gamma_0, \gamma_1 \in [0, \infty)$ be real numbers with $\gamma_1 + 2(\gamma_0 \vee \gamma_1) < p/4$, let $\bar{\mu} \colon \mathbb{R}^d \to \mathbb{R}^d$, $\bar{\sigma} \colon \mathbb{R}^d \to \mathbb{R}^{d \times m}$ be Borel measurable functions with $\bar{\mu}|_D = \mu$, $\bar{\sigma}|_D = \sigma$ and $\phi(x, t, y) = \bar{\mu}(x)t + \bar{\sigma}(x)y$ for all $x \in \mathbb{R}^d$, $t \in [0, T]$, $y \in \mathbb{R}^m$, let $V \in C^3_p(\mathbb{R}^d, [1, \infty))$ with $\mathbb{E}[V(X_0)] < \infty$ and with*

$$(\mathcal{G}_{\bar{\mu},\bar{\sigma}} V)(x) \le c \cdot V(x), \; \|\bar{\mu}(x)\| \le c\,|V(x)|^{\left[\frac{\gamma_0 + 1}{p}\right]}, \; \|\bar{\sigma}(x)\|_{L(\mathbb{R}^m, \mathbb{R}^d)} \le c\,|V(x)|^{\left[\frac{\gamma_1 + 2}{2p}\right]}$$

for all $x \in \mathbb{R}^d$ and for every $N \in \mathbb{N}$ let $\bar{Y}^{N,m} \colon [0, T] \times \Omega \to \mathbb{R}$, $m \in \mathbb{N}$, be independent stochastic processes with $\mathbb{P}_{\bar{Y}^N} = \mathbb{P}_{\bar{Y}^{N,m}}$ for all $m \in \mathbb{N}$. Then

$$\lim_{N \to \infty} \left| \mathbb{E}\big[f(X_T)\big] - \frac{\sum_{m=1}^{N^2} f(\bar{Y}_T^{N,m})}{N^2} \right| = 0 \tag{3.109}$$

$\mathbb{P}$-a.s. for all continuous functions $f \colon \mathbb{R}^d \to \mathbb{R}$ with $\limsup_{r \nearrow 1} \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{(1 + V(x))^r} < \infty$.

More results for approximating statistical quantities of solutions of SDEs with non-globally Lipschitz continuous coefficients can, e.g., be found in Yan [**83**], Milstein & Tretyakov [**63**], Dörsek [**16**] and in the references therein.

3.6. Numerical schemes for SDEs

The purpose of this section is to present a few examples of numerical schemes which are (μ, σ)-consistent with respect to Brownian motion. Theorem 3.3 then shows that the approximation processes (3.2) corresponding to such schemes converge in probability to the exact solution of the SDE (3.1). We now go into detail and describe the setting that we use in this section. Throughout this section, assume that the setting in Section 3.1 is fulfilled and let $\bar{\mu} \colon \mathbb{R}^d \to \mathbb{R}^d$ and $\bar{\sigma} \colon \mathbb{R}^d \to \mathbb{R}^{m \times d}$ be two arbitrary Borel measurable functions satisfying

$$\bar{\mu}(x) = \mu(x) \qquad \text{and} \qquad \bar{\sigma}(x) = \sigma(x) \tag{3.110}$$

for all $x \in D$. Moreover, define mappings $Y_n^N \colon \Omega \to \mathbb{R}^d$, $n \in \{0, 1, \ldots, N\}$, $N \in \mathbb{N}$, by $Y_n^N := \bar{Y}_{nT/N}^N$ for all $n \in \{0, 1, \ldots, N\}$ and all $N \in \mathbb{N}$ and define mappings $\Delta W_n^N \colon \Omega \to \mathbb{R}^m$, $n \in \{0, 1, \ldots, N - 1\}$, $N \in \mathbb{N}$, by $\Delta W_n^N := W_{(n+1)T/N} - W_{nT/N}$ for all $n \in \{0, 1, \ldots N - 1\}$ and all $N \in \mathbb{N}$. Using this notation, we get from (3.2) that

$$Y_{n+1}^N = Y_n^N + \phi\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big) \tag{3.111}$$

for all $n \in \{0, 1, \ldots, N - 1\}$ and all $N \in \mathbb{N}$. The following subsections provide examples of Borel measurable functions $\phi \colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ and numerical approximation schemes of the form (3.111), respectively, which are (μ, σ)-consistent with respect to Brownian motion.

3.6.1. A few Euler-type schemes for SDEs. Let $\eta_0\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$, $\eta_1\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d\times d}$ and $\eta_2\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d\times d}$ be Borel measurable functions. The next lemma then gives sufficient conditions to ensure that schemes of the form

$$\begin{aligned} Y_{n+1}^N = Y_n^N &+ \eta_0\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big) + \eta_1\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big)\, \bar{\mu}(Y_n^N)\, \tfrac{T}{N} \\ &+ \eta_2\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big)\, \bar{\sigma}(Y_n^N)\, \Delta W_n^N \end{aligned} \tag{3.112}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$ are (μ, σ)-consistent with respect to Brownian motion.

Lemma 3.24. *Assume that the setting in Section 3.1 is fulfilled, let $\bar{\mu}\colon \mathbb{R}^d \to \mathbb{R}^d$, $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}^{m\times d}$, $\eta_1, \eta_2\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d\times d}$ and $\eta_0\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be Borel measurable functions with $\bar{\mu}|_D = \mu$ and $\bar{\sigma}|_D = \sigma$ and assume that*

$$\limsup_{t \searrow 0} \left(\sup_{x \in K} \mathbb{E}\big[\, \|\eta_2(x,t,W_t)\sigma(x)W_t\|\,\big] \right) < \infty, \tag{3.113}$$

$$\limsup_{t \searrow 0} \left(\sup_{x \in K} \mathbb{E}\Big[\, \|\eta_1(x,t,W_t) - I\|_{L(\mathbb{R}^d)} + \|\eta_2(x,t,W_t) - I\|^2_{L(\mathbb{R}^d)}\,\Big] \right) = 0, \tag{3.114}$$

$$\limsup_{t \searrow 0} \left(\frac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\, \|\eta_0(x,t,W_t)\|\,\big] \right) = \limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \big\|\mathbb{E}\big[\eta_0(x,t,W_t)\big]\big\| \right) = 0, \tag{3.115}$$

$$\limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \big\|\mathbb{E}\big[\eta_2(x,t,W_t)\sigma(x)W_t\big]\big\| \right) = 0 \tag{3.116}$$

for all non-empty compact sets $K \subset D$. Then $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ given by

$$\phi(x,t,y) = \eta_0(x,t,y) + \eta_1(x,t,y)\bar{\mu}(x)t + \eta_2(x,t,y)\bar{\sigma}(x)y \tag{3.117}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $y \in \mathbb{R}^m$ is (μ, σ)-consistent with respect to Brownian motion.

Proof of Lemma 3.24. The triangle inequality and the Hölder inequality give

$$\begin{aligned} &\frac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\|\sigma(x)W_t - \phi(x,t,W_t)\|\big] \\ &\leq \frac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\|\eta_0(x,t,W_t)\|\big] + \frac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\|\eta_1(x,t,W_t)\mu(x)t\|\big] \\ &\quad + \frac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\|(I - \eta_2(x,t,W_t))\sigma(x)W_t\|\big] \\ &\leq \frac{1}{\sqrt{t}} \cdot \sup_{x \in K} \mathbb{E}\big[\|\eta_0(x,t,W_t)\|\big] + \sqrt{t} \left(\sup_{x \in K} \mathbb{E}\big[\|\eta_1(x,t,W_t)\|_{L(\mathbb{R}^d)}\big] \right) \left(\sup_{x \in K} \|\mu(x)\| \right) \\ &\quad + \frac{\|W_t\|_{L^2(\Omega;\mathbb{R}^m)}}{\sqrt{t}} \left(\sup_{x \in K} \|\eta_2(x,t,W_t) - I\|_{L^2(\Omega;L(\mathbb{R}^d))} \right) \left(\sup_{x \in K} \|\sigma(x)\|_{L(\mathbb{R}^m,\mathbb{R}^d)} \right) \end{aligned} \tag{3.118}$$

for all $t \in (0,T]$ and all non-empty compact sets $K \subset D$. Combining (3.118), (3.114) and (3.115) then shows (3.3). It thus remains to establish (3.4) in order to complete the proof of Lemma 3.24. To this end note that

$$
\begin{aligned}
&\limsup_{t\searrow 0}\left(\sup_{x\in K}\left\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\phi(x,t,W_t)\big]\right\|\right)\\
&\leq \limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in K}\left\|\mathbb{E}\big[\eta_0(x,t,W_t)\big]\right\|\right)\\
&\quad+\left(\limsup_{t\searrow 0}\sup_{x\in K}\mathbb{E}\big[\|\eta_1(x,t,W_t)-I\|_{L(\mathbb{R}^d)}\big]\right)\left(\sup_{x\in K}\|\mu(x)\|\right)\\
&\quad+\limsup_{t\searrow 0}\left(\tfrac{1}{t}\cdot\sup_{x\in K}\left\|\mathbb{E}\big[\eta_2(x,t,W_t)\sigma(x)W_t\big]\right\|\right)
\end{aligned}
\tag{3.119}
$$

for all non-empty compact sets $K\subset D$. Inequality (3.119) and equations (3.114), (3.115) and (3.116) then show (3.4). This completes the proof of Lemma 3.24. □

In the remainder of this subsection, a few numerical schemes of the form (3.112) are presented.

The Euler-Maruyama scheme. In the case $\eta_0(x,t,y)=0$, $\eta_1(x,t,y)=I$ and $\eta_2(x,t,y)=I$ for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and all $y\in\mathbb{R}^m$, the numerical scheme (3.112) is the well-known Euler-Maruyama scheme

$$Y_{n+1}^N = Y_n^N + \bar{\mu}(Y_n^N)\tfrac{T}{N} + \bar{\sigma}(Y_n^N)\Delta W_n^N \tag{3.120}$$

for all $n\in\{0,1,\ldots,N-1\}$ and all $N\in\mathbb{N}$ (see Maruyama [**58**]). Of course, this choice satisfies the assumptions of Lemma 3.24. Combining Lemma 3.24 and Theorem 3.3 thus shows that the Euler-Maruyama scheme (3.120) converges in probability to the exact solution of the SDE (3.1). In the literature convergence in probability and also pathwise convergence of the Euler-Maruyama scheme has already been proved even in a more general setting than the setting considered here; see, e.g., Krylov [**49**] and Gyöngy [**22**]. Strong convergence of the Euler-Maruyama scheme, however, often fails to hold if the coefficients μ and σ of the SDE (3.1) grow more than linearly (see [**40**]) and therefore, we are interested in appropriately modified Euler-Maruyama schemes which are truncated or tamed in a suitable way and which therefore do converge strongly even for SDEs with superlinearly growing coefficients. Although strong convergence fails to hold, the corresponding Monte Carlo Euler method does converge with probability one for a large class of SDEs with possibly superlinearly growing coefficients; see Corollary 3.23 above.

A drift-truncated Euler scheme. In the case $\eta_0(x,t,y)=0$, $\eta_1(x,t,y)=\frac{1}{\max(1,t\|\bar{\mu}(x)\|)}I$ and $\eta_2(x,t,y)=I$ for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and all $y\in\mathbb{R}^m$, the numerical scheme (3.112) reads as

$$Y_{n+1}^N = Y_n^N + \frac{\bar{\mu}(Y_n^N)\frac{T}{N}}{\max\big(1,\frac{T}{N}\|\bar{\mu}(Y_n^N)\|\big)} + \bar{\sigma}(Y_n^N)\Delta W_n^N \tag{3.121}$$

for all $n\in\{0,1,\ldots,N-1\}$ and all $N\in\mathbb{N}$. In the case where the drift $\bar{\mu}$ in (3.121) is of gradient type and the noise is additive, i.e., $\bar{\sigma}(x)=\bar{\sigma}(0)$ and $\bar{\mu}(x)=-(\nabla U)(x)$ for all $x\in\mathbb{R}^d$ and some appropriately smooth function $U\colon\mathbb{R}^d\to\mathbb{R}$, the scheme (3.121) has been used as proposal for the Metropolis-Adjusted-Truncated-Langevin-Algorithm (MATLA; see Roberts & Tweedie [**69**]). The choice in (3.121) also satisfies the assumptions of Lemma 3.24. Combining Lemma 3.24 and Theorem 3.3 hence proves that the drift-truncated Euler scheme (3.121) converges in probability to the exact solution of the SDE (3.1). If the diffusion coefficient $\bar{\sigma}$ in (3.121) grows at most linearly, then moment bounds and strong convergence of the drift-truncated Euler scheme (3.121) can be studied by combining Theorem 2.13, Lemma 2.18, Corollary 2.9, Lemma 3.24 and Corollary 3.14. More precisely, Theorem 2.13 and Lemma 2.18 can be used to prove, under suitable assumptions on $\bar{\mu}$ and $\bar{\sigma}$ (see Theorem 2.13 for more details), that the drift-truncated Euler scheme (3.121) is α-semi V-stable with respect to Brownian motion with $\alpha \in (0, \infty)$ and $V\colon \mathbb{R}^d \to [0, \infty)$ appropriate. Combining this and Corollary 3.14 with the fact that (3.121) is (μ, σ)-consistent with respect to Brownian motion according to Lemma 3.24 finally proves, under the additional assumption that $\bar{\sigma}$ grows at most linearly, strong convergence of the drift-truncated Euler scheme (3.121).

A drift-tamed Euler scheme. A slightly different variant of the drift-truncated Euler scheme (3.121) is the drift-tamed Euler-type method considered in [**38**]. More precisely, in the case $\eta_0(x, t, y) = 0$, $\eta_1(x, t, y) = \frac{1}{1+t\|\bar{\mu}(x)\|} I$ and $\eta_2(x, t, y) = I$ for all $x \in \mathbb{R}^d$, $t \in [0, T]$ and all $y \in \mathbb{R}^m$, the numerical scheme (3.112) reads as

$$Y_{n+1}^N = Y_n^N + \frac{\bar{\mu}(Y_n^N)\frac{T}{N}}{1 + \frac{T}{N}\left\|\bar{\mu}(Y_n^N)\right\|} + \bar{\sigma}(Y_n^N)\Delta W_n^N \tag{3.122}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. If $D = \mathbb{R}^d$, if σ is globally Lipschitz continuous and if μ is continuously differentiable and globally one-sided Lipschitz continuous with an at most polynomially growing derivative, then strong convergence of the drift-tamed Euler scheme (3.122) with the standard rate $\frac{1}{2}$ has been proved in [**38**]. Moment bounds and strong convergence of the drift-tamed Euler scheme (3.122) in a more general setting can be obtained in precisely the same way as illustated for the drift-truncated Euler scheme (3.121).

The Milstein scheme. In addition to the setting described in the beginning of Section 3.6, assume in this paragraph that $\bar{\sigma} = (\bar{\sigma}_{i,j})_{i\in\{1,\ldots,d\}, j\in\{1,\ldots,m\}} = (\bar{\sigma}_j)_{j\in\{1,\ldots,m\}}\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is continuously differentiable. In the case $\eta_1(x, t, y) = \eta_2(x, t, y) = I$ and

$$\begin{aligned}&\eta_0(x, t, y)\\ &= \frac{1}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{m}\left(\tfrac{\partial}{\partial x_k}\bar{\sigma}_i\right)(x)\cdot\bar{\sigma}_{k,j}(x)\cdot y_i\cdot y_j - \frac{t}{2}\sum_{k=1}^{d}\sum_{i=1}^{m}\left(\tfrac{\partial}{\partial x_k}\bar{\sigma}_i\right)(x)\cdot\bar{\sigma}_{k,i}(x)\end{aligned} \tag{3.123}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $t \in [0, T]$ and all $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, the numerical scheme (3.112) reads as

$$\begin{aligned}Y_{n+1}^N &= Y_n^N + \bar{\mu}(Y_n^N)\tfrac{T}{N} + \bar{\sigma}(Y_n^N)\Delta W_n^N\\ &\quad + \frac{1}{2}\sum_{k=1}^{d}\sum_{i,j=1}^{m}\left(\tfrac{\partial}{\partial x_k}\bar{\sigma}_i\right)(Y_n^N)\cdot\bar{\sigma}_{k,j}(Y_n^N)\cdot\Delta W_n^{N,i}\cdot\Delta W_n^{N,j}\\ &\quad - \frac{T}{2N}\sum_{k=1}^{d}\sum_{i=1}^{m}\left(\tfrac{\partial}{\partial x_k}\bar{\sigma}_i\right)(Y_n^N)\cdot\sigma_{k,i}(Y_n^N)\end{aligned} \tag{3.124}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$ where $(\Delta W_n^{N,1}, \ldots, \Delta W_n^{N,m}) = \Delta W_n^N$ for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. This choice satisfies the assumptions of Lemma 3.24. Moreover, if the commutativity condition (see, e.g., (3.13) in Section 10.3 in Kloeden & Platen [**47**])

$$\sum_{k=1}^{d} \big(\tfrac{\partial}{\partial x_k}\bar{\sigma}_i\big)(x)\cdot\bar{\sigma}_{k,j}(x)=\sum_{k=1}^{d}\big(\tfrac{\partial}{\partial x_k}\bar{\sigma}_j\big)(x)\cdot\bar{\sigma}_{k,i}(x) \tag{3.125}$$

for all $x\in\mathbb{R}^d$ and all $i,j\in\{1,2,\dots,m\}$ is fulfilled, then the scheme (3.124) is nothing else but the well-known Milstein scheme (see Milstein [**60**] or, e.g., (3.16) in Section 10.3 in Kloeden & Platen [**47**]). Combining Lemma 3.24 and Theorem 3.3 thus shows that the scheme (3.124) converges in probability to the exact solution of the SDE (3.1). In the literature, almost sure convergence with rate $1-\varepsilon$ for $\varepsilon\in(0,1)$ arbitrarily small and thus also convergence in probability of the Milstein scheme has already been proved in [**43**]. However, as in the case of the Euler-Maruyama scheme (see (3.120)), strong convergence of the Milstein scheme often fails to hold if at least one of the coefficients μ and σ of the SDE (3.1) grows more than linearly (see [**40**]). Nonetheless, if the drift term is tamed appropriately as in (3.122), then strong convergence of the corresponding drift-tamed Milstein scheme has been established in Gang & Wang [**19**] for a class of SDEs with possibly superlinearly growing drift coefficients.

Balanced implicit methods. Milstein, Platen and Schurz [**62**] introduced the following class of balanced implicit methods. Let $c_0,c_1,\dots,c_m\colon\mathbb{R}^d\to\mathbb{R}^{d\times d}$ be Borel measurable functions such that the matrix

$$I+c_0(x)t+\sum_{j=1}^{m}c_j(x)|y_j|\in\mathbb{R}^{d\times d} \tag{3.126}$$

is invertible for all $x\in\mathbb{R}^d$, $t\in[0,T]$ and all $y=(y_1,\dots,y_m)\in\mathbb{R}^m$. This condition is, e.g., satisfied if the matrices $c_0(x,t,y),\dots,c_m(c,t,y)$, $(x,t,y)\in\mathbb{R}^d\times[0,T]\times\mathbb{R}^m$, are positive semi-definite. The associated balanced implicit method is then given through

$$\begin{aligned} &Y_{n+1}^N\\ &=Y_n^N+\bar{\mu}(Y_n^N)\tfrac{T}{N}+\bar{\sigma}(Y_n^N)\Delta W_n^N\\ &\quad+\Big(c_0(Y_n^N)\tfrac{T}{N}+\textstyle\sum_{j=1}^{m}c_j(Y_n^N)\left|\Delta W_n^{N,j}\right|\Big)\left(Y_n^N-Y_{n+1}^N\right)\\ &=Y_n^N\\ &\quad+\Big(I+c_0(Y_n^N)\tfrac{T}{N}+\textstyle\sum_{j=1}^{m}c_j(Y_n^N)\left|\Delta W_n^{N,j}\right|\Big)^{-1}\Big(\bar{\mu}(Y_n^N)\tfrac{T}{N}+\bar{\sigma}(Y_n^N)\Delta W_n^N\Big)\end{aligned} \tag{3.127}$$

for all $n\in\{0,1,\dots,N-1\}$ and all $N\in\mathbb{N}$ where $(\Delta W_n^{N,1},\dots,\Delta W_n^{N,m})=\Delta W_n^N$ for all $n\in\{0,1,\dots,N-1\}$ and all $N\in\mathbb{N}$. For the case of at most linearly growing coefficients $\bar{\mu}$ and $\bar{\sigma}$ and uniformly bounded matrices $c_0,c_1,\dots,c_m$, Theorem 4.1 of Schurz [**75**] implies uniformly bounded moments of the balanced implicit method (3.127). Moreover, under additional assumptions such as global Lipschitz continuity of $\bar{\mu}$ and $\bar{\sigma}$, Theorem 5.1 of Schurz [**75**] implies strong mean square convergence of the balanced implicit method with convergence order $\frac{1}{2}$. In order to apply the above theory, write (3.127) in the form (3.112) with $\eta_0(x,t,y)=0$ and

$$\eta_1(x,t,y) = \eta_2(x,t,y) = \left(I + c_0(x)t + \sum_{j=1}^{m} c_j(x)|y_j| \right)^{-1} \tag{3.128}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $y = (y_1,\ldots,y_m) \in \mathbb{R}^m$. Lemma 3.24 can be applied to derive conditions on the functions $c_0, c_1, \ldots, c_m$ which imply (μ,σ)-consistency with respect to Brownian motion. Theorem 3.3 then shows convergence in probability of the balanced implicit method (3.127). Moreover, similar as described above for the drift-truncated Euler scheme, moment bounds and strong convergence of the balanced implicit method (3.127) can be studied by combining Theorem 2.13, Lemma 2.18, Corollary 2.9, Lemma 3.24 and Corollary 3.14.

3.6.2. Comparison results for numerical schemes for SDEs. The next lemma considers two numerical approximation schemes of the form (3.2) and shows that if one of the two schemes is (μ,σ)-consistent with respect to Brownian motion and if the two schemes are close to each other in an appropriate sense (see (3.129)), then the other scheme is also (μ,σ)-consistent with respect to Brownian motion. Its proof is straightforward and hence omitted.

Lemma 3.25 (A comparison result for consistency). *Assume that the setting in Section 3.1 is fulfilled and let $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be a function which is (μ,σ)-consistent with respect to Brownian motion and which satisfies*

$$\limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \mathbb{E}\left[\left\| \phi(x,t,W_t) - \hat{\phi}(x,t,W_t) \right\| \right] \right) = 0 \tag{3.129}$$

for all non-empty compact sets $K \subset D$. Then $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ is (μ,σ)-consistent with respect to Brownian motion too.

The next lemma, in particular, illustrates that convex combinations of schemes that are (μ,σ)-consistent with respect to Brownian motion are (μ,σ)-consistent with respect to Brownian motion too. More precisely, the next lemma shows that schemes of the form

$$Y_{n+1}^N = Y_n^N + \eta_1 \cdot \phi_1\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big) + \eta_2 \cdot \phi_2\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big) \tag{3.130}$$

for all $n \in \{0,1,\ldots,N\}$ and all $N \in \mathbb{N}$ are (μ,σ)-consistent with respect to Brownian motion provided that $\eta_1, \eta_2 \in \mathbb{R}$ are real numbers with $\eta_1 + \eta_2 = 1$ and provided that $\phi_1, \phi_2\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m$ are (μ,σ)-consistent with respect to Brownian motion. Its proof is clear and therefore omitted.

Lemma 3.26 (Generalized convex combinations of numerical schemes for SDEs). *Assume that the setting in Section 3.1 is fulfilled, let $\phi_1, \phi_2\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be (μ,σ)-consistent functions with respect to Brownian motion and let $\eta_1, \eta_2 \in \mathbb{R}$ be two real numbers with $\eta_1 + \eta_2 = 1$. Then $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ given by*

$$\phi(x,t,y) = \eta_1 \cdot \phi_1(x,t,y) + \eta_2 \cdot \phi_2(x,t,y) \tag{3.131}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $y \in \mathbb{R}^m$ is (μ,σ)-consistent with respect to Brownian motion.

Finally, the next lemma gives a simple characterization of (μ,σ)-consistency with respect to Brownian motion. Its proof is straightforward and hence omitted.

LEMMA 3.27 (A characterization of consistency). *Let $T \in (0, \infty)$, $d, m \in \mathbb{N}$, let $D \subset \mathbb{R}^d$ be an open set and let $\mu\colon D \to \mathbb{R}^d$ and $\sigma\colon D \to \mathbb{R}^{d\times m}$ be locally Lipschitz continuous functions. A Borel measurable function $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ is then (μ,σ)-consistent with respect to Brownian motion if and only if there exists a Borel measurable function $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ which is (μ,σ)-consistent with respect to Brownian motion and which satisfies*

$$\lim_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}} \cdot \sup_{x\in K} \mathbb{E}\Big[\big\|\hat{\phi}(x,t,W_t) - \phi(x,t,W_t)\big\|\Big]\right) = 0 \tag{3.132}$$

and

$$\lim_{t\searrow 0}\left(\tfrac{1}{t} \cdot \sup_{x\in K} \big\|\mathbb{E}\big[\hat{\phi}(x,t,W_t)\big] - \mathbb{E}\big[\phi(x,t,W_t)\big]\big\|\right) = 0 \tag{3.133}$$

for all non-empty compact sets $K \subset D$ where $W\colon [0,T]\times\Omega \to \mathbb{R}^m$ is an arbitrary standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

3.6.3. Taming principles for numerical schemes for SDEs. Let $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be a function which is (μ,σ)-consistent with respect to Brownian motion and consider a sequence $Z^N\colon \{0,1,\ldots N\}\times\Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, of stochastic processes given by $Z_0^N = X_0$ and

$$Z_{n+1}^N = Z_n^N + \hat{\phi}\big(Z_n^N, \tfrac{T}{N}, \Delta W_n^N\big) \tag{3.134}$$

for all $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. The next lemma then gives sufficient conditions to ensure that numerical approximation schemes of the form

$$Y_{n+1}^N = Y_n^N + \eta\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big)\,\hat{\phi}\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big) \tag{3.135}$$

for all $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$ are (μ,σ)-consistent with respect to Brownian motion where $\eta\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d\times d}$ is a suitable Borel measurable function.

LEMMA 3.28 (Increment taming principle). *Assume that the setting in Section 3.1 is fulfilled, let $\eta\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d\times d}$ be a Borel measurable function, let $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be a function which is (μ,σ)-consistent with respect to Brownian motion and assume that*

$$\limsup_{t\searrow 0}\left(\frac{1}{t} \cdot \sup_{x\in K} \mathbb{E}\Big[\big\|\hat{\phi}(x,t,W_t)\big\|^2\Big]\right) < \infty \tag{3.136}$$

and

$$\limsup_{t\searrow 0}\left(\frac{1}{t} \cdot \sup_{x\in K} \mathbb{E}\Big[\big\|\eta(x,t,W_t) - I\big\|^2_{L(\mathbb{R}^d)}\Big]\right) = 0 \tag{3.137}$$

for all non-empty compact sets $K \subset D$. Then $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ given by

$$\phi(x,t,y) = \eta(x,t,y)\,\hat{\phi}(x,t,y) \tag{3.138}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$, $y \in \mathbb{R}^m$ is (μ,σ)-consistent with respect to Brownian motion.

PROOF OF LEMMA 3.28. Combining Hölder's inequality, equation (3.136) and equation (3.137) implies

$$
\begin{aligned}
&\limsup_{t\searrow 0}\left(\frac{1}{t}\cdot\sup_{x\in K}\mathbb{E}\Big[\big\|\hat{\phi}(x,t,W_t)-\phi(x,t,W_t)\big\|\Big]\right)\\
&\leq \limsup_{t\searrow 0}\left(\frac{1}{t}\cdot\sup_{x\in K}\mathbb{E}\Big[\|\eta(x,t,W_t)-I\|_{L(\mathbb{R}^d)}\cdot\|\hat{\phi}(x,t,W_t)\|\Big]\right)\\
&\leq \left(\limsup_{t\searrow 0}\frac{\sup_{x\in K}\mathbb{E}\big[\|\eta(x,t,W_t)-I\|^2_{L(\mathbb{R}^d)}\big]}{t}\right)^{1/2}\\
&\quad\cdot\left(\limsup_{t\searrow 0}\frac{1}{t}\cdot\sup_{x\in K}\mathbb{E}\left[\|\hat{\phi}(x,t,W_t)\|^2\right]\right)^{\frac{1}{2}}=0
\end{aligned}
\tag{3.139}
$$

for all non-empty compact sets $K\subset D$. Inequality (3.139) and Lemma 3.25 then complete the proof of Lemma 3.28. ☐

Let us illustrate Lemma 3.28 by an example. More precisely, in the special case $\eta(x,t,y)=\frac{1}{\max(1,t\|\hat{\phi}(x,t,y)\|)}I$ for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $y\in\mathbb{R}^m$, the scheme (3.135) reads as

$$
Y^N_{n+1}=Y^N_n+\frac{\hat{\phi}\big(Y^N_n,\frac{T}{N},\Delta W^N_n\big)}{\max\big(1,\frac{T}{N}\big\|\hat{\phi}\big(Y^N_n,\frac{T}{N},\Delta W^N_n\big)\big\|\big)}
\tag{3.140}
$$

for all $n\in\{0,1,\ldots,N-1\}$ and all $N\in\mathbb{N}$.

If we now additionally assume that the Z^N, $N\in\mathbb{N}$, in (3.134) are Euler-Maruyama approximations, i.e., that $\hat{\phi}(x,t,y)=\bar{\mu}(x)t+\bar{\sigma}(x)y$ for all $x\in\mathbb{R}^d$, $t\in[0,T]$, $y\in\mathbb{R}^m$, then (3.140) reads as

$$
Y^N_{n+1}=Y^N_n+\frac{\bar{\mu}(Y^N_n)\frac{T}{N}+\bar{\sigma}(Y^N_n)\Delta W^N_n}{\max\big(1,\frac{T}{N}\big\|\bar{\mu}(Y^N_n)\frac{T}{N}+\bar{\sigma}(Y^N_n)\Delta W^N_n\big\|\big)}
\tag{3.141}
$$

for all $n\in\{0,1,\ldots,N-1\}$ and all $N\in\mathbb{N}$. This increment-tamed Euler-Maruyama scheme clearly satisfies the assumptions of Lemma 3.28. Theorem 3.3 hence shows that the scheme (3.141) convergences in probability to the exact solution of the SDE (3.1). Note that this scheme is frequently studied in this article. Strong convergence of the scheme (3.141) is studied in Subsection 3.4.3 above (see also Chapter 4 for a list examples of SDEs in case of which the scheme (3.141) has been shown to converge strongly).

In the case of spatially discretized semilinear stochastic partial differential equations, another choice for the increment function $\hat{\phi}\colon\mathbb{R}^d\times[0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ in (3.134) naturally arises. More precisely, suppose that

$$
\bar{\mu}(x)=Ax+F(x)
\tag{3.142}
$$

for all $x\in\mathbb{R}^d$ where $A\in\mathbb{R}^{d\times d}$ is a $d\times d$-matrix with $\det(I-tA)\neq 0$ for all $t\in[0,\infty)$ and where $F\colon\mathbb{R}^d\to\mathbb{R}^d$ is a Borel measurable function and suppose that

$$
\hat{\phi}(x,t,y)=(I-tA)^{-1}\,(x+F(x)t+\sigma(x)y)-x
\tag{3.143}
$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$ and all $y \in \mathbb{R}^m$. The approximations processes Z^N, $N \in \mathbb{N}$, in equation (3.134) thus reduce to the linear implicit Euler approximations

$$\begin{aligned} Z_{n+1}^N &= \left(I - \tfrac{T}{N}A\right)^{-1} \left(Z_n^N + F(Z_n^N)\tfrac{T}{N} + \sigma(Z_n^N)\Delta W_n^N\right) \\ &= Z_n^N + \left[\left(I - \tfrac{T}{N}A\right)^{-1} \left(Z_n^N + F(Z_n^N)\tfrac{T}{N} + \sigma(Z_n^N)\Delta W_n^N\right) - Z_n^N\right] \end{aligned} \tag{3.144}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$ and the scheme in (3.140) then reads as

(3.145)
$$Y_{n+1}^N = Y_n^N + \frac{\left(I - \frac{T}{N}A\right)^{-1}\left(Y_n^N + F(Y_n^N)\frac{T}{N} + B(Y_n^N)\Delta W_n^N\right) - Y_n^N}{\max\!\left(1, \frac{T}{N}\left\|\left(I - \frac{T}{N}A\right)^{-1}\left(Y_n^N + F(Y_n^N)\frac{T}{N} + B(Y_n^N)\Delta W_n^N\right) - Y_n^N\right\|\right)}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$.

In (3.135), (3.140), (3.141) and (3.145), respectively, the increment function $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ is tamed in a suitable way so that the scheme does not diverge strongly (see [**40**]) and moment bounds can be obtained (see Subsections 2.1.3, 2.1.4 and 2.2.3). Instead of the increment function, also the whole scheme can be tamed in an appropriate way. This is the subject of the next lemma. More precisely, the next lemma gives sufficient conditions to ensure that schemes of the form

$$Y_{n+1}^N = \eta\left(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\right)\left(Y_n^N + \hat{\phi}\left(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\right)\right) \tag{3.146}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$ are (μ, σ)-consistent with respect to Brownian motion where $\eta\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d \times d}$ is a Borel measurable function and where $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ is a function which is (μ, σ)-consistent with respect to Brownian motion.

LEMMA 3.29 (Full taming principle). *Assume that the setting in Section 3.1 is fulfilled, let $\eta\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d \times d}$ be a Borel measurable function, let $\hat{\phi}\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be a function which is (μ, σ)-consistent with respect to Brownian motion and assume that*

$$\limsup_{t \searrow 0} \left(\sup_{x \in K} \mathbb{E}\Big[\|\hat{\phi}(x, t, W_t)\|^2\Big] \right) < \infty \tag{3.147}$$

and

$$\limsup_{t \searrow 0} \left(\frac{1}{t^2} \cdot \sup_{x \in K} \mathbb{E}\Big[\|\eta(x, t, W_t) - I\|_{L(\mathbb{R}^d)}^2\Big] \right) = 0 \tag{3.148}$$

for all non-empty compact sets $K \subset D$. Then $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ given by

$$\phi(x, t, y) = \eta(x, t, y)(x + \hat{\phi}(x, t, y)) - x \tag{3.149}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$ and all $y \in \mathbb{R}^m$ is (μ, σ)-consistent with respect to Brownian motion.

Proof of Lemma 3.29. Hölder's inequality, the estimate $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \in \mathbb{R}$ and equations (3.147) and (3.148) imply

$$\begin{aligned}
&\limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \mathbb{E}\Big[\big\| \hat{\phi}(x,t,W_t) - \phi(x,t,W_t) \big\| \Big] \right) \\
&\leq \limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \mathbb{E}\Big[\| \eta(x,t,W_t) - I \|_{L(\mathbb{R}^d)} \cdot \big\| x + \hat{\phi}(x,t,W_t) \big\| \Big] \right) \\
&\leq 2 \left(\limsup_{t \searrow 0} \frac{1}{t^2} \cdot \sup_{x \in K} \mathbb{E}\Big[\big\| \eta(x,t,W_t) - I \big\|^2_{L(\mathbb{R}^d)} \Big] \right)^{1/2} \\
&\quad \cdot \left(\sup_{x \in K} \|x\|^2 + \limsup_{t \searrow 0} \sup_{x \in K} \mathbb{E}\Big[\big\| \hat{\phi}(x,t,W_t) \big\|^2 \Big] \right)^{1/2} = 0
\end{aligned} \tag{3.150}$$

for all non-empty compact sets $K \subset D$. Inequality (3.150) and Lemma 3.25 then complete the proof of Lemma 3.29. □

As an example of Lemma 3.29, let $r \in (1, \infty)$ be a real number and consider the choice $\eta \colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d \times d}$ given by

$$\eta(x,t,y) = \mathbb{1}_{[0,1]}\big(t^r \| x + \hat{\phi}(x,t,y) \| \big) \cdot I \tag{3.151}$$

for all $x \in \mathbb{R}^d$, $t \in [0,T]$ and all $y \in \mathbb{R}^m$. Note that the choice (3.151) satisfies (3.148) in Lemma 3.29 provided that (3.147) is fulfilled. Indeed, observe that Markov's inequality and (3.147) imply that

$$\begin{aligned}
&\limsup_{t \searrow 0} \left(\frac{1}{t^2} \cdot \sup_{x \in K} \mathbb{E}\Big[\| \eta(x,t,W_t) - I \|^2_{L(\mathbb{R}^d)} \Big] \right) \\
&= \limsup_{t \searrow 0} \left(\frac{1}{t^2} \cdot \sup_{x \in K} \mathbb{E}\Big[1 - \mathbb{1}_{[0,1]}\big(t^r \| x + \hat{\phi}(x,t,W_t) \| \big) \Big] \right) \\
&= \limsup_{t \searrow 0} \left(\frac{1}{t^2} \cdot \sup_{x \in K} \mathbb{P}\Big[t^r \| x + \hat{\phi}(x,t,W_t) \| > 1 \Big] \right) \\
&= \limsup_{t \searrow 0} \left(t^{-2} \cdot \sup_{x \in K} \mathbb{P}\Big[\| x + \hat{\phi}(x,t,W_t) \|^2 > t^{-2r} \Big] \right) \\
&\leq \limsup_{t \searrow 0} \left(t^{(2r-2)} \cdot \sup_{x \in K} \mathbb{E}\Big[\| x + \hat{\phi}(x,t,W_t) \|^2 \Big] \right) = 0
\end{aligned} \tag{3.152}$$

for all non-empty compact sets $K \subset D$. Next note that in the case (3.151) the scheme (3.146) reads as

$$Y^N_{n+1} = \mathbb{1}_{\left\{ \| Y^N_n + \hat{\phi}(Y^N_n, \frac{T}{N}, \Delta W^N_n) \| \leq \frac{N^r}{T^r} \right\}} \Big(Y^N_n + \hat{\phi}\big(Y^N_n, \tfrac{T}{N}, \Delta W^N_n \big) \Big) \tag{3.153}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. A similiar class of approximations has been proposed in Milstein & Tretyakov [**63**] in which the truncation barrier $\frac{N^r}{T^r}$ in the indicator set in (3.153) is replaced by a possibly large real number $R \in (0, \infty)$ which does not depend on $N \in \mathbb{N}$.

3.6.4. Linear implicit numerical schemes for SDEs. The next lemma gives sufficient conditions that ensure that linear implicit numerical schemes of the form

$$
\begin{aligned}
&Y_{n+1}^N \\
&= Y_n^N + \mathbb{1}_{\{\det(I - A(Y_n^N, \frac{T}{N}, \Delta W_n^N)) \neq 0\}} A\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big)\, Y_{n+1}^N + b\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big) \\
&= \Big(I - \mathbb{1}_{\{\det(I - A(Y_n^N, \frac{T}{N}, \Delta W_n^N)) \neq 0\}} A\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big)\Big)^{-1} \Big(Y_n^N + b\big(Y_n^N, \tfrac{T}{N}, \Delta W_n^N\big)\Big)
\end{aligned} \tag{3.154}
$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$ are (μ, σ)-consistent with respect to Brownian motion where $A\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d \times d}$ and $b\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ are suitable Borel measurable functions (see Lemma 3.30 for the detailed assumptions).

Lemma 3.30 (Linear implicit numerical schemes for SDEs driven by Brownian motions). *Assume that the setting in Section 3.1 is fulfilled, let $A\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^{d \times d}$ and $b\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ be Borel measurable functions and let $t_K \in (0,T]$, $K \subset D$ non-empty compact set, be a family of real numbers such that $I - A(x,t,y) \in \mathbb{R}^{d \times d}$ is invertible for all $(x,t,y) \in K \times [0, t_K] \times \mathbb{R}^m$ and all non-empty compact sets $K \subset D$. Moreover, assume that*

$$
\mathbb{R}^d \times [0,T] \times \mathbb{R}^m \ni (x,t,y) \mapsto A(x,t,y)\, x + b(x,t,y) \in \mathbb{R}^d \tag{3.155}
$$

is (μ, σ)-consistent with respect to Brownian motion, that

$$
\phi(x,t,y) = \big(I - A(x,t,y)\big)^{-1} \big(x + b(x,t,y)\big) - x \tag{3.156}
$$

for all $(x,t,y) \in K \times [0, t_K] \times \mathbb{R}^m$ and all non-empty compact sets $K \subset D$ and that

$$
\limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \mathbb{E}\Big[\big\| (I - A(x,t,W_t))^{-1} A(x,t,W_t) \big\|^2 \Big] \right) = 0, \tag{3.157}
$$

$$
\limsup_{t \searrow 0} \left(\frac{1}{t} \cdot \sup_{x \in K} \mathbb{E}\Big[\big\| A(x,t,W_t)\, x + b(x,t,W_t) \big\|^2 \Big] \right) < \infty \tag{3.158}
$$

for all non-empty compact sets $K \subset D$. Then $\phi\colon \mathbb{R}^d \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^d$ is (μ, σ)-consistent with respect to Brownian motion.

Proof of Lemma 3.30. Note that

$$
\begin{aligned}
&\phi(x,t,y) - (A(x,t,y)\, x + b(x,t,y)) \\
&= \Big((I - A(x,t,y))^{-1} - I \Big) (x + b(x,t,y)) - A(x,t,y)\, x \\
&= (I - A(x,t,y))^{-1} A(x,t,y)\, (x + b(x,t,y)) - A(x,t,y)\, x \\
&= \Big((I - A(x,t,y))^{-1} - I \Big) A(x,t,y)\, x + (I - A(x,t,y))^{-1} A(x,t,y)\, b(x,t,y) \\
&= (I - A(x,t,y))^{-1} A(x,t,y)\, (A(x,t,y)\, x + b(x,t,y))
\end{aligned} \tag{3.159}
$$

for all $(x,t,y) \in K \times [0, t_K] \times \mathbb{R}^m$ and all non-empty compact sets $K \subset D$. Combining equation (3.159), Hölder's inequality, equation (3.157), inequality (3.158) and Lemma 3.25 then completes the proof of Lemma 3.30. □

3.6.5. Fully drift-implicit numerical schemes for SDEs. The next lemma gives sufficient conditions to ensure that the fully drift-implicit Euler scheme described by

$$Y_{n+1}^N = Y_n^N + \mu\big(Y_{n+1}^N\big)\,\tfrac{T}{N} + \sigma\big(Y_n^N\big)\,\Delta W_n^N \tag{3.160}$$

for all $n \in \{0, 1, \ldots, N-1\}$ and all sufficiently large $N \in \mathbb{N}$ is (μ, σ)-consistent with respect to Brownian motion where $D = \mathbb{R}^d$ and where $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfies a one-sided linear growth bound (see Lemma 3.31 for the detailed assumptions).

LEMMA 3.31 (Fully drift-implicit Euler-Maruyama scheme for SDEs driven by Brownian motions). *Assume that the setting in Section 3.1 is fulfilled, assume that* $D = \mathbb{R}^d$*, let* $c, \kappa \in [0, \infty)$*,* $\theta \in (0, T]$ *be real numbers with* $\langle x, \mu(x)\rangle \leq c\big(1 + \|x\|^2\big)$ *and* $\|\mu(x) - \mu(y)\| \leq c\,(1 + \|x\|^\kappa + \|y\|^\kappa)\,\|x - y\|$ *for all* $x, y \in \mathbb{R}^d$ *and assume that*

$$\phi(x, t, y) = \mu\big(x + \phi(x, t, y)\big)\,t + \sigma(x)\,y \tag{3.161}$$

for all $(x, t, y) \in \mathbb{R}^d \times [0, \theta] \times \mathbb{R}^m$*. Then* $\phi\colon \mathbb{R}^d \times [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ *is* (μ, σ)*-consistent with respect to Brownian motion.*

PROOF OF LEMMA 3.31. First of all, observe that the Cauchy-Schwarz inequality and the assumption that $\langle v, \mu(v)\rangle \leq c\big(1 + \|v\|^2\big)$ for all $v \in \mathbb{R}^d$ imply that

$$\begin{aligned} &\|x + \phi(x, t, y)\|^2 = \langle x + \phi(x, t, y), x + \phi(x, t, y)\rangle \\ &= \langle x + \phi(x, t, y), x + \sigma(x)y\rangle + \langle x + \phi(x, t, y), \mu(x + \phi(x, t, y))\rangle\, t \\ &\leq \|x + \phi(x, t, y)\|\,\|x + \sigma(x)y\| + ct\,\big(1 + \|x + \phi(x, t, y)\|^2\big) \end{aligned} \tag{3.162}$$

and Young's inequality hence gives that

$$\begin{aligned} &\|x + \phi(x, t, y)\|^2 \\ &\leq \|x + \phi(x, t, y)\|\,\|x + \sigma(x)y\| + ct + ct\,\|x + \phi(x, t, y)\|^2 \\ &\leq \tfrac{1}{2}\,\|x + \phi(x, t, y)\|^2 + \tfrac{1}{2}\,\|x + \sigma(x)y\|^2 + ct + ct\,\|x + \phi(x, t, y)\|^2 \end{aligned} \tag{3.163}$$

for all $(x, t, y) \in \mathbb{R}^d \times [0, \theta] \times \mathbb{R}^m$. Rearranging therefore yields

$$\big(1 - ct - \tfrac{1}{2}\big)\,\|x + \phi(x, t, y)\|^2 \leq \tfrac{1}{2}\,\|x + \sigma(x)y\|^2 + ct \tag{3.164}$$

for all $(x, t, y) \in \mathbb{R}^d \times [0, \theta] \times \mathbb{R}^m$ and this shows that

$$\begin{aligned} \|x + \phi(x, t, y)\|^2 &\leq \frac{\frac{1}{2}\,\|x + \sigma(x)y\|^2 + ct}{\big(1 - ct - \frac{1}{2}\big)} \leq \frac{\|x\|^2 + \|\sigma(x)y\|^2 + cT}{\big(1 - ct - \frac{1}{2}\big)} \\ &\leq \frac{\|x\|^2 + \|\sigma(x)\|^2_{L(\mathbb{R}^m, \mathbb{R}^d)}\,\|y\|^2 + cT}{\big(1 - ct - \frac{1}{2}\big)} \end{aligned} \tag{3.165}$$

for all $(x, t, y) \in \mathbb{R}^d \times [0, \theta] \times \mathbb{R}^m$ with $ct < \frac{1}{2}$ and hence

$$\limsup_{t \searrow 0}\left(\mathbb{E}\left[\sup_{x \in K}\|x + \phi(x, t, W_t)\|^r\right]\right) < \infty \tag{3.166}$$

for all $r \in [0,\infty)$ and all non-empty compact sets $K \subset \mathbb{R}^d$. Combining this with the estimate

$$\begin{aligned} &\|\mu(x)\| \leq \|\mu(x)-\mu(0)\| + \|\mu(0)\| \leq c\,(2+\|x\|^\kappa)\,\|x\| + \|\mu(0)\| \\ &\leq c\left(2\,\|x\| + \|x\|^{(\kappa+1)}\right) + \|\mu(0)\| \leq 3c\left(1+\|x\|^{(\kappa+1)}\right) + \|\mu(0)\| \\ &\leq (3c+\|\mu(0)\|)\left(1+\|x\|^{(\kappa+1)}\right) \end{aligned} \tag{3.167}$$

for all $x \in \mathbb{R}^d$ results in

$$\begin{aligned} &\limsup_{t\searrow 0}\left(\mathbb{E}\Big[\sup_{x\in K}\big\|\mu\big(x+\phi(x,t,W_t)\big)\big\|^r\Big]\right) \\ &\leq (6c+2\,\|\mu(0)\|)^r\left(1+\limsup_{t\searrow 0}\mathbb{E}\Big[\sup_{x\in K}\|x+\phi(x,t,W_t)\|^{r(\kappa+1)}\Big]\right) < \infty \end{aligned} \tag{3.168}$$

for all $r \in [0,\infty)$ and all non-empty compact sets $K \subset \mathbb{R}^d$. This implies that

$$\begin{aligned} &\limsup_{t\searrow 0}\left(\mathbb{E}\Big[\sup_{x\in K}\|\phi(x,t,W_t)\|^r\Big]\right) \\ &= \limsup_{t\searrow 0}\left(\mathbb{E}\Big[\sup_{x\in K}\big\|\mu\big(x+\phi(x,t,W_t)\big)\,t+\sigma(x)W_t\big\|^r\Big]\right) \\ &\leq 2^r\limsup_{t\searrow 0}\left(t^r\cdot\mathbb{E}\Big[\sup_{x\in K}\big\|\mu\big(x+\phi(x,t,W_t)\big)\big\|^r\Big]\right) \\ &\quad + 2^r\limsup_{t\searrow 0}\left(\sup_{x\in K}\|\sigma(x)\|^r_{L(\mathbb{R}^m,\mathbb{R}^d)}\,\mathbb{E}\big[\,\|W_t\|^r\,\big]\right) = 0 \end{aligned} \tag{3.169}$$

for all $r \in (0,\infty)$ and all non-empty compact sets $K \subset \mathbb{R}^d$. In addition, (3.168) shows that

$$\begin{aligned} &\lim_{t\searrow 0}\left(\tfrac{1}{\sqrt{t}}\cdot\sup_{x\in K}\mathbb{E}\big[\,\|\phi(x,t,W_t)-\sigma(x)W_t\|\,\big]\right) \\ &= \lim_{t\searrow 0}\left(\sqrt{t}\cdot\sup_{x\in K}\mathbb{E}\big[\,\|\mu(x+\phi(x,t,W_t))\|\,\big]\right) \\ &\leq \left(\lim_{t\searrow 0}\sqrt{t}\right)\left(\limsup_{t\searrow 0}\sup_{x\in K}\mathbb{E}\big[\,\|\mu(x+\phi(x,t,W_t))\|\,\big]\right) = 0 \end{aligned} \tag{3.170}$$

for all non-empty compact sets $K \subset \mathbb{R}^d$. In the next step we note that

$$\begin{aligned} &\limsup_{t\searrow 0}\sup_{x\in K}\big\|\mu(x)-\tfrac{1}{t}\cdot\mathbb{E}\big[\phi(x,t,W_t)\big]\big\| \\ &= \limsup_{t\searrow 0}\sup_{x\in K}\big\|\mathbb{E}\big[\mu(x)-\mu\big(x+\phi(x,t,W_t)\big)\big]\big\| \\ &\leq \limsup_{t\searrow 0}\mathbb{E}\Big[\sup_{x\in K}\big\|\mu(x)-\mu\big(x+\phi(x,t,W_t)\big)\big\|\Big] \\ &\leq \limsup_{t\searrow 0}\mathbb{E}\Big[c\left(1+\sup_{x\in K}\|x\|^\kappa+\sup_{x\in K}\|x+\phi(x,t,W_t)\|^\kappa\right)\sup_{x\in K}\|\phi(x,t,W_t)\|\Big] \end{aligned} \tag{3.171}$$

and Hölder's inequality together with (3.166) and (3.169) hence implies that

$$\begin{aligned}
&\limsup_{t\searrow 0} \sup_{x\in K} \left\|\mu(x) - \tfrac{1}{t}\cdot \mathbb{E}\big[\phi(x,t,W_t)\big]\right\| \\
&\leq 3c \left(\limsup_{t\searrow 0} \mathbb{E}\left[\left(1 + \sup_{x\in K}\|x\|^{2\kappa} + \sup_{x\in K}\|x+\phi(x,t,W_t)\|^{2\kappa}\right)\right]\right)^{\frac{1}{2}} \\
&\quad \cdot \left(\limsup_{t\searrow 0} \mathbb{E}\left[\sup_{x\in K}\|\phi(x,t,W_t)\|^2\right]\right)^{\frac{1}{2}} = 0
\end{aligned} \tag{3.172}$$

for all non-empty compact sets $K \subset \mathbb{R}^d$. Combining (3.170) and (3.172) completes the proof of Lemma 3.31. □

CHAPTER 4

Examples of SDEs

In this chapter, we apply the strong convergence results of Chapter 3 to a selection of examples of SDEs with non-globally Lipschitz continuous coefficients. In Section 4.1, we first describe the general setting in which these examples appear and then treat each example separately in the subsequent sections.

4.1. Setting and assumptions

The following setting is used throughout Chapter 4. Let $T \in (0,\infty)$, $d, m \in \mathbb{N}$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathcal{F}_t)_{t\in[0,T]}$ and let $W = (W^{(1)}, \ldots, W^{(m)}) \colon [0,T] \times \Omega \to \mathbb{R}^m$ be a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Brownian motion. Moreover, let $D \subset \mathbb{R}^d$ be an open set, let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma \colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ be Borel measurable functions such that $\mu|_D \colon D \to \mathbb{R}^d$ and $\sigma|_D \colon D \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous and let $X = (X^{(1)}, \ldots, X^{(d)}) \colon [0,T] \times \Omega \to D$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths satisfying $\mathbb{E}\big[\|X_0\|^p\big] < \infty$ for all $p \in [1,\infty)$ and

$$X_t = X_0 + \int_0^t \mu(X_s)\,ds + \int_0^t \sigma(X_s)\,dW_s \tag{4.1}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$. Thus we assume the existence of a solution process of the SDE (4.1). Our goal is then to approximate the solution process $X \colon [0,T]\times\Omega \to D$ of the SDE (4.1) in the strong sense. For this we concentrate for simplicity on the increment-tamed Euler-Maruyama approximations in Subsection 3.4.3. More precisely, let $\bar{Y}^N \colon [0,T] \times \Omega \to \mathbb{R}^d$, $N \in \mathbb{N}$, be a sequence of stochastic processes defined through $\bar{Y}_0^N := X_0$ and

$$\bar{Y}_t^N = \bar{Y}^N_{\frac{nT}{N}} + \frac{\big(\frac{tN}{T} - n\big)\,\big(\mu(\bar{Y}^N_{nT/N})\frac{T}{N} + \sigma(\bar{Y}^N_{nT/N})\,(W_{(n+1)T/N} - W_{nT/N})\big)}{\max\big(1, \frac{T}{N}\|\mu(\bar{Y}^N_{nT/N})\frac{T}{N} + \sigma(\bar{Y}^N_{nT/N})\,(W_{(n+1)T/N} - W_{nT/N})\|\big)} \tag{4.2}$$

for all $t \in \big(\frac{nT}{N}, \frac{(n+1)T}{N}\big]$, $n \in \{0,1,\ldots,N-1\}$ and all $N \in \mathbb{N}$. In the next sections, we present several examples from the literature for the SDE (4.1) and we show under suitable assumptions that the numerical approximation processes (4.2) converge strongly to the solution process of each of the following examples. To the best of our knowledge, this is the first result in the literature that proves strong convergence for the stochastic van der Pol oscillator (4.4), for the stochastic Duffing-van der Pol oscillator (4.13), for the stochastic Lorenz equation (4.20), for the stochastic Brusselator (4.23), for the stochastic SIR model (4.31), for the SDE (4.40) from experimental psychology and for the Lotka-Volterra predator-prey model (4.48).

4.2. Stochastic van der Pol oscillator

The van der Pol oscillator was proposed to describe stable oscillation; see van der Pol [**81**] and the references therein. As, for instance, in Timmer et. al [**78**], we consider a stochastic version with additive noise (see also Leung [**50**] for more general stochastic versions of the van der Pol oscillator and also equation (4.1) in Schurz [**74**] for a generalized stochastic van der Pol oscillator with multiplicative noise). More formally, assume that the setting in Section 4.1 is fulfilled, let $\alpha, \beta, \gamma, \delta \in (0, \infty)$ be real numbers and suppose that $d = 2$, $m = 1$, $D = \mathbb{R}^2$ and

$$\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \alpha\left(\gamma - (x_1)^2\right) x_2 - \delta x_1 \end{pmatrix}, \qquad \sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix} \tag{4.3}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then the SDE (4.1) is the stochastic van der Pol equation

$$\begin{aligned} dX_t^{(1)} &= X_t^{(2)}\, dt, \\ dX_t^{(2)} &= \left[\alpha\left(\gamma - (X_t^{(1)})^2\right) X_t^{(2)} - \delta X_t^{(1)}\right] dt + \beta\, dW_t \end{aligned} \tag{4.4}$$

for $t \in [0, \infty)$. In an abbreviated form, the SDE (4.4) can also be written as

$$\ddot{X}_t - \alpha\left(\gamma - (X_t)^2\right)\dot{X}_t + \delta X_t = \beta \dot{W}_t \tag{4.5}$$

for $t \in [0, \infty)$. Clearly, the diffusion coefficient $\sigma\colon \mathbb{R}^2 \to \mathbb{R}^2$ in (4.4) is globally Lipschitz continuous and the drift coefficient $\mu\colon \mathbb{R}^2 \to \mathbb{R}^2$ in (4.4) is not globally Lipschitz continuous. It is also well known that the drift coefficient $\mu\colon \mathbb{R}^2 \to \mathbb{R}^2$ in (4.4) is not globally one-sided Lipschitz continuous. Indeed, note that

$$\begin{aligned} &\left\langle \begin{pmatrix} u \\ u \end{pmatrix} - \begin{pmatrix} 0 \\ 2u \end{pmatrix}, \mu\begin{pmatrix} u \\ u \end{pmatrix} - \mu\begin{pmatrix} 0 \\ 2u \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} u \\ -u \end{pmatrix}, \begin{pmatrix} u \\ \alpha\left(\gamma - u^2\right) u - \delta u \end{pmatrix} - \begin{pmatrix} 2u \\ 2\alpha\gamma u \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} u \\ -u \end{pmatrix}, \begin{pmatrix} -u \\ -\alpha\gamma u - \alpha u^3 - \delta u \end{pmatrix} \right\rangle = (\alpha\gamma + \delta - 1)\, u^2 + \alpha u^4 \end{aligned} \tag{4.6}$$

for all $u \in \mathbb{R}$ and hence there exists no real number $c \in \mathbb{R}$ such that it holds for every $x, y \in \mathbb{R}^2$ that

$$\langle x - y, \mu(x) - \mu(y) \rangle \leq c \, \|x - y\|^2 . \tag{4.7}$$

However, the drift coefficient in (4.4) satisfies the global one-sided linear growth condition

$$\langle x, \mu(x) \rangle \leq c\left(1 + \|x\|^2\right) \tag{4.8}$$

for all $x \in \mathbb{R}^d$ and some $c \in [0, \infty)$ (see, e.g., Example 16 in Subsection 3.2.2 in Boccara [**6**]). Therefore, Corollary 3.17 applies here with the Lyapunov-type function $V\colon \mathbb{R}^2 \to [1, \infty)$ given by

$$V(x) = 1 + \|x\|^2 \tag{4.9}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ (cf., e.g., page 75 in Boccara [**6**]) and we obtain

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.10}$$

for all $q \in (0, \infty)$.

4.3. Stochastic Duffing-van der Pol oscillator

The Duffing equation is a further model for an oscillator. The Duffing-van der Pol equation unifies both the Duffing equation and the van der Pol equation and has been used, e.g., in certain aeroelasticity problems; see Holmes & Rand [**35**]. As, for instance, in Schenk-Hoppé [**70**], we consider a stochastic version with an affine-linear noise term (see, e.g., also Arnold, Namachchivaya & Schenk-Hoppé [**4**], Section 9.4 in Arnold [**3**] and Section 13.1 in Kloeden & Platen [**47**]; we also refer to [**4**] for details and references on the physical background of the stochastic Duffing-van der Pol oscillator). More precisely, assume that the setting in Section 4.1 is fulfilled, let $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ be two triples of real numbers with $\alpha_3 \geq 0$ and suppose that $d = 2$, $m = 3$, $D = \mathbb{R}^2$ and

$$\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ \alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_2 \, (x_1)^2 - (x_1)^3 \end{pmatrix} \tag{4.11}$$

and

$$\sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \beta_1 x_1 & \beta_2 x_2 & \beta_3 \end{pmatrix} \tag{4.12}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then the SDE (4.1) is the stochastic Duffing-van der Pol equation

$$\begin{aligned} dX_t^{(1)} &= X_t^{(2)} \, dt, \\ dX_t^{(2)} &= \Big[\alpha_1 X_t^{(1)} - \alpha_2 X_t^{(2)} - \alpha_3 X_t^{(2)} \big(X_t^{(1)}\big)^2 - \big(X_t^{(1)}\big)^3\Big] \, dt \\ &\quad + \beta_1 X_t^{(1)} dW_t^{(1)} + \beta_2 X_t^{(2)} dW_t^{(2)} + \beta_3 \, dW_t^{(3)} \end{aligned} \tag{4.13}$$

for $t \in [0, \infty)$. In an abbreviated form, the SDE (4.13) can also be written as

$$\ddot{X}_t - \alpha_1 X_t + \alpha_2 \dot{X}_t + \alpha_3 \dot{X}_t (X_t)^2 + (X_t)^3 = \beta_1 X_t \dot{W}_t^{(1)} + \beta_2 \dot{X}_t \dot{W}_t^{(2)} + \beta_3 \dot{W}_t^{(3)} \tag{4.14}$$

for $t \in [0, \infty)$. Clearly, the diffusion coefficient σ in (4.13) is globally Lipschitz continuous. In addition, it is well know that the drift coefficient μ in (4.13) is not globally one-sided Lipschitz continuous and also fails to satisfy the global one-sided linear growth condition (4.8). Indeed, note that

$$\begin{aligned} \left\langle \begin{pmatrix} u \\ -1 \end{pmatrix}, \mu\begin{pmatrix} u \\ -1 \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} u \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ \alpha_1 u + \alpha_2 + \alpha_3 u^2 - u^3 \end{pmatrix} \right\rangle \\ &= -\,(1 + \alpha_1)\, u - \alpha_2 - \alpha_3 u^2 + u^3 \end{aligned} \tag{4.15}$$

for all $u \in \mathbb{R}$ and hence that there exists no real number $c \in \mathbb{R}$ such that it holds for every $x \in \mathbb{R}^2$ that $\langle x, \mu(x) \rangle \leq c\,\big(1 + \|x\|^2\big)$. The function

$$\mathbb{R}^2 \ni (x_1, x_2) \mapsto 1 + (x_1)^2 + (x_2)^2 \in [1, \infty) \tag{4.16}$$

is thus no Lyapunov-type function for the SDE (4.13). However, Corollary 3.17 applies here with the Lyapunov-type function $V \colon \mathbb{R}^2 \to [1, \infty)$ given by

$$V(x_1, x_2) = 1 + (x_1)^4 + 2\,(x_2)^2 \tag{4.17}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ (cf., e.g., (8) in Holmes & Rand [**35**]) and we hence obtain $\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0$ for all $q \in (0, \infty)$.

4.4. Stochastic Lorenz equation

Lorenz [**54**] derived a three-dimensional system as a simplified model of convection rolls in the atmosphere and this equation became famous for its chaotic behaviour. As, for instance, in Schmalfuß [**72**], we consider a stochastic version hereof with multiplicative noise. Assume that the setting in Section 4.1 is fulfilled, let $(\alpha_1, \alpha_2, \alpha_3), (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$ be two triples of real numbers and suppose that $d = m = 3$, $D = \mathbb{R}^3$ and

$$\mu\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 (x_2 - x_1) \\ \alpha_2 x_1 - x_2 - x_1 x_3 \\ x_1 x_2 - \alpha_3 x_3 \end{pmatrix} \tag{4.18}$$

and

$$\sigma\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \beta_1 x_1 & 0 & 0 \\ 0 & \beta_2 x_2 & 0 \\ 0 & 0 & \beta_3 x_3 \end{pmatrix} \tag{4.19}$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Under these assumptions, the SDE (4.1) is thus the stochastic Lorenz equation

$$\begin{aligned} dX_t^{(1)} &= \left[\alpha_1 X_t^{(2)} - \alpha_1 X_t^{(1)}\right] dt + \beta_1 X_t^{(1)}\, dW_t^{(1)}, \\ dX_t^{(2)} &= \left[\alpha_2 X_t^{(1)} - X_t^{(2)} - X_t^{(1)} X_t^{(3)}\right] dt + \beta_2 X_t^{(2)}\, dW_t^{(2)}, \\ dX_t^{(3)} &= \left[X_t^{(1)} X_t^{(2)} - \alpha_3 X_t^{(3)}\right] dt + \beta_3 X_t^{(3)}\, dW_t^{(3)} \end{aligned} \tag{4.20}$$

for $t \in [0, \infty)$. Clearly, the diffusion coefficient in (4.20) is globally Lipschitz continuous. In addition, it is well know that the drift coefficient in (4.20) is not globally one-sided Lipschitz continuous but fulfills the global one-sided linear growth condition (4.8) (see, e.g., (29) in Schmalfuß [**72**]). Therefore, Corollary 3.17 applies here with $V\colon \mathbb{R}^3 \to [1, \infty)$ given by

$$V(x) = 1 + \|x\|^2 \tag{4.21}$$

for all $x \in \mathbb{R}^3$ to obtain $\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0$ for all $q \in (0, \infty)$.

4.5. Stochastic Brusselator in the well-stirred case

The Brusselator is a model for a trimolecular chemical reaction and has been studied in Prigogine & Lefever [**67**] and by other scientists from Brussels (Tyson [**80**] proposed the name "Brusselator" to appreciate the innovative contribution of these scientists from Brussel on this model). The following stochastic version hereof in the "well-stirred case" (see, e.g., Section 1 in Scheutzow [**71**] for further details) has been proposed by Dawson [**15**] (see also Scheutzow [**71**]). Assume that the setting in Section 4.1 is fulfilled, let $\alpha, \delta \in (0, \infty)$ be two real numbers, let $g_1, g_2\colon [0, \infty) \to \mathbb{R}$ be two globally Lipschitz continuous functions with $g_1(0) = g_2(0) = 0$ and $\sup_{x\in[0,\infty)} |g_2(x)| < \infty$ and suppose that $d = m = 2$, $D = (0, \infty)^2$, $\mu(x) = 0$ and $\sigma(x) = 0$ for all $x \in D^c$ and

$$\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \delta - (\alpha + 1)\, x_1 + x_2 \cdot (x_1)^2 \\ \alpha x_1 - x_2 \cdot (x_1)^2 \end{pmatrix}, \quad \sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} g_1(x_1) & 0 \\ 0 & g_2(x_2) \end{pmatrix} \tag{4.22}$$

for all $x = (x_1, x_2) \in D$. Then the SDE (4.1) is the stochastic Brusselator equation

$$(4.23)\qquad \begin{aligned} dX_t^{(1)} &= \Big[\delta - (\alpha+1)X_t^{(1)} + X_t^{(2)}(X_t^{(1)})^2\Big]\,dt + g_1(X_t^{(1)})\,dW_t^{(1)} \\ dX_t^{(2)} &= \Big[\alpha X_t^{(1)} - X_t^{(2)}(X_t^{(1)})^2\Big]\,dt + g_2(X_t^{(2)})\,dW_t^{(2)} \end{aligned}$$

for $t \in [0,\infty)$. By assumption the diffusion coefficient in (4.23) is globally Lipschitz continuous. Clearly, the drift coefficient in (4.23) is not globally one-sided Lipschitz continuous and also fails to satisfy the one-sided linear growth condition (4.8). Instead here the squared sum of the coordinates is a Lyapunov-type function on the state space $(0,\infty)^2$; see, e.g., the proof of Theorem 2.1 a) in Scheutzow [**71**]. However, in order to apply Corollary 3.17 in this example, we need to construct a Lyapunov-type function $V\colon \mathbb{R}^2 \to [1,\infty)$ on the space $\mathbb{R}^2$. For this let $\phi\colon \mathbb{R} \to [0,1]$ and $\psi\colon \mathbb{R}^2 \to [0,1]$ be two infinitely often differentiable functions with $\phi(x) = 0$ for all $x \in (-\infty, 0]$, with $\phi(x) = 1$ for all $x \in [1,\infty)$ and with

$$(4.24)\qquad \psi(x_1,x_2) = \phi(x_1)\cdot\phi(-x_2) + \phi(-x_1)\cdot\phi(x_2)$$

for all $x = (x_1,x_2) \in \mathbb{R}^2$. Then consider $V\colon \mathbb{R}^3 \to [1,\infty)$ given by

$$(4.25)\qquad \begin{aligned} &V(x_1,x_2) \\ &= \frac{5}{2} + (x_1+x_2)^2 - 2\cdot x_1\cdot x_2\cdot\psi(x_1,x_2) \\ &= \frac{5}{2} + \|x\|^2 + 2x_1x_2\,(1-\psi(x_1,x_2)) \\ &= \frac{5}{2} + \begin{cases} (x_1+x_2)^2 & : x_1x_2 \geq 0 \\ \|x\|^2 & : (x_1x_2 < 0) \wedge (\min(|x_1|,|x_2|) \geq 1) \\ \|x\|^2 + 2x_1x_2\,(1-\psi(x_1,x_2)) & : \text{else} \end{cases} \end{aligned}$$

for all $x = (x_1,x_2) \in \mathbb{R}^2$ and observe that $V\colon \mathbb{R}^2 \to [1,\infty)$ is infinitely often differentiable and satisfies

$$(4.26)\qquad \sup_{x\in\mathbb{R}^3}\left(\frac{\sum_{i=1}^3 \left\|V^{(i)}(x)\right\|_{L^{(i)}(\mathbb{R}^2,\mathbb{R})}}{(1+\|x\|)}\right) < \infty.$$

Next note that the fact $2 + a^2 - 2a \geq \frac{a^2}{2}$ for all $a \in [0,\infty)$ implies

$$(4.27)\qquad \begin{aligned} V(x_1,x_2) &= \tfrac{5}{2} + \|x\|^2 + 2x_1x_2\,(1-\psi(x_1,x_2)) \\ &\geq \begin{cases} \tfrac{5}{2} + \|x\|^2 & : (x_1x_2 \geq 0) \vee (\min(|x_1|,|x_2|) \geq 1) \\ \tfrac{5}{2} + \|x\|^2 - 2\,|x_1|\,|x_2| & : (x_1x_2 < 0) \wedge (\min(|x_1|,|x_2|) < 1) \end{cases} \\ &\geq \tfrac{5}{2} + \|x\|^2 - 2\max(|x_1|,|x_2|) \\ &= \Big[\tfrac{1}{2} + |\min(|x_1|,|x_2|)|^2\Big] + \Big[2 + |\max(|x_1|,|x_2|)|^2 - 2\max(|x_1|,|x_2|)\Big] \\ &\geq \frac{1 + |\min(|x_1|,|x_2|)|^2 + |\max(|x_1|,|x_2|)|^2}{2} = \frac{1+\|x\|^2}{2} \end{aligned}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. In addition, observe that $\|V'(x)\sigma(x)\|_{L(\mathbb{R}^2,\mathbb{R})} = 0$ for all $x \in D^c$ and hence

$$\sup_{x=(x_1,x_2)\in\mathbb{R}^2} \left(\frac{\|V'(x)\sigma(x)\|_{L(\mathbb{R}^2,\mathbb{R})}}{V(x)} \right) \leq 2 \left[\sup_{x=(x_1,x_2)\in D} \left(\frac{(x_1 + x_2)\,(|g_1(x_1)| + |g_2(x_1)|)}{(1 + \|x\|^2)} \right) \right] < \infty. \tag{4.28}$$

Combining (4.26)–(4.28) shows that Corollary 3.17 applies here with the function $V \colon \mathbb{R}^2 \to [1, \infty)$ given in (4.25) and we therefore get

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.29}$$

for all $q \in (0, \infty)$. Please also note that other choices than (4.25) are possible for the Lyapunov-type function $V \colon \mathbb{R}^2 \to [1, \infty)$ and that the choice (4.25) simply ensures that the smoothness and growth assumptions of Corollary 3.17 are met.

4.6. Stochastic SIR model

The SIR model from epidemiology for the total number of susceptibles, infected and recovered individuals has been introduced by Anderson & May [**2**]. The following stochastic version has been studied first by Tornatore, Buccellato & Vetro [**79**]. Assume that the setting in Section 4.1 is fulfilled, let $\alpha, \beta, \gamma, \delta \in (0, \infty)$ be real numbers and suppose that $d = 3$, $m = 1$, $D = (0, \infty)^3$, $\mu(x) = \sigma(x) = 0$ for all $x \in D^c$ and

$$\mu\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\alpha x_1 x_2 - \delta x_1 + \delta \\ \alpha x_1 x_2 - (\gamma + \delta) x_2 \\ \gamma x_2 - \delta x_3 \end{pmatrix}, \qquad \sigma\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\beta x_1 x_2 \\ \beta x_1 x_2 \\ 0 \end{pmatrix} \tag{4.30}$$

for all $x = (x_1, x_2, x_3) \in D$. Under these assumptions, the solution process $(S_t, I_t, R_t) := (X_t^{(1)}, X_t^{(2)}, X_t^{(3)})$, $t \in [0, \infty)$, of the SDE (4.1) fulfills

$$\begin{aligned} dS_t &= \big[-\alpha S_t I_t - \delta S_t + \delta \big] dt - \beta S_t I_t \, dW_t \\ dI_t &= \big[\alpha S_t I_t - (\gamma + \delta) I_t \big] dt + \beta S_t I_t \, dW_t \\ dR_t &= \big[\gamma I_t - \delta R_t \big] dt \end{aligned} \tag{4.31}$$

for $t \in [0, \infty)$. Clearly, both the drift coefficient and the diffusion coefficient grow superlinearly in this example. In addition, it is also obvious that the drift coefficient fails to satisfy the one-sided linear growth condition (4.8). We now construct an appropriate Lyapunov-type function $V \colon \mathbb{R}^3 \to [1, \infty)$ for this example so that Corollary 3.17 can be applied. The Lyapunov-type function here is constructed similarly as in in the case of the stochastic Brusselator in Section 4.5 above. More formally, let $\phi \colon \mathbb{R} \to [0, 1]$ and $\psi \colon \mathbb{R}^2 \to [0, 1]$ be two infinitely often differentiable functions with $\phi(x) = 0$ for all $x \in (-\infty, 0]$, with $\phi(x) = 1$ for all $x \in [1, \infty)$ and with

$$\psi(x_1, x_2) = \phi(x_1) \cdot \phi(-x_2) + \phi(-x_1) \cdot \phi(x_2) \tag{4.32}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then consider $V \colon \mathbb{R}^3 \to [1, \infty)$ given by

$$V(x_1, x_2, x_3) = \frac{5}{2} + (x_1 + x_2)^2 + (x_3)^2 - 2 \cdot x_1 \cdot x_2 \cdot \psi(x_1, x_2) \tag{4.33}$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ (cf., e.g., Tornatore, Buccellato & Vetro [**79**]) and observe that $V\colon \mathbb{R}^3 \to [1,\infty)$ is infinitely often differentiable and satisfies

$$\sup_{x\in\mathbb{R}^3} \left(\frac{\sum_{i=1}^{3} \|V^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^3,\mathbb{R})}}{(1+\|x\|)} \right) < \infty. \tag{4.34}$$

As in (4.27) it follows that

$$V(x) \geq \tfrac{1}{2}\left(1 + \|x\|^2\right) \tag{4.35}$$

for all $x \in \mathbb{R}^3$. In addition, observe that $|V'(x)\sigma(x)| = 0$ for all $x \in D^c$ and

$$\begin{aligned} &|V'(x)\sigma(x)| \\ &= \Big| -\big(\tfrac{\partial}{\partial x_1} V\big)(x_1,x_2,x_3) \cdot \beta \cdot x_1 \cdot x_2 + \big(\tfrac{\partial}{\partial x_2} V\big)(x_1,x_2,x_3) \cdot \beta \cdot x_1 \cdot x_2 \Big| \\ &= \beta\,|x_1|\,|x_2| \,\Big| \big(\tfrac{\partial}{\partial x_1} V\big)(x_1,x_2,x_3) - \big(\tfrac{\partial}{\partial x_2} V\big)(x_1,x_2,x_3) \Big| \\ &= \beta\,|x_1|\,|x_2|\,|2(x_1+x_2) - 2(x_1+x_2)| = 0 \end{aligned} \tag{4.36}$$

for all $x = (x_1, x_2, x_3) \in D$ and we thus get

$$|V'(x)\sigma(x)| = 0 \tag{4.37}$$

for all $x \in \mathbb{R}^3$. Combining (4.34)–(4.37) shows that Corollary 3.17 applies here with the Lyapunov-type function $V\colon \mathbb{R}^3 \to [1,\infty)$ given in (4.33) and we therefore get that

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.38}$$

for all $q \in (0,\infty)$. Note that other choices than (4.33) are possible for the Lyapunov-type function $V\colon \mathbb{R}^3 \to [1,\infty)$ and that the choice (4.33) simply ensures that the smoothness and growth assumptions of Corollary 3.17 are met.

4.7. Experimental psychology model

The motivation for the following example from experimental psychology is explained in Section 7.2 in Kloeden & Platen [**47**]. Assume that the setting in Section 4.1 is fulfilled, let $\alpha, \delta \in (0,\infty)$, $\beta \in \mathbb{R}$ be real numbers and suppose that $d = 2$, $m = 1$, $D = \mathbb{R}^2$ and

$$\mu\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (x_2)^2\,(\delta + 4\alpha x_1) - \frac{\beta^2 x_1}{2} \\ -x_1 x_2\,(\delta + 4\alpha x_1) + \frac{\beta^2 x_2}{2} \end{pmatrix}, \qquad \sigma\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\beta x_2 \\ \beta x_1 \end{pmatrix} \tag{4.39}$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. The SDE (4.1) is then the Stratonovich stochastic differential equation

$$\begin{aligned} dX_t^{(1)} &= \Big[\big(X_t^{(2)}\big)^2\big(\delta + 4\alpha X_t^{(1)}\big)\Big]\,dt - \beta X_t^{(2)} \circ dW_t \\ dX_t^{(2)} &= \Big[-X_t^{(1)} X_t^{(2)}\big(\delta + 4\alpha X_t^{(1)}\big)\Big]\,dt + \beta X_t^{(1)} \circ dW_t \end{aligned} \tag{4.40}$$

for $t \in [0,\infty)$. The SDE (4.40) is a transformed version of a model proposed in Haken, Kelso & Bunz [**27**] in the deterministic case and in Schöner, Haken & Kelso [**73**] in the stochastic case (see Section 7.2 in Kloeden & Platen [**47**] for details). The diffusion coefficient in (4.40) is clearly globally Lipschitz continuous. The drift coefficient in (4.40) is not globally one-sided Lipschitz continuous but fulfills the global one-sided linear growth bound (4.8). The drift coefficient in (4.40) even fulfills $\langle x, \mu(x)\rangle = 0$ for all $x \in \mathbb{R}^2$ and therefore the function $V\colon \mathbb{R}^2 \to [1,\infty)$ given by

$$V(x) = 1 + \|x\|^2 \tag{4.41}$$

is a Lyapunov-type function for the SDE (4.40) (see, e.g., Section 7.2 in Kloeden & Platen [**47**]). Hence, Corollary 3.17 applies here with $V\colon \mathbb{R}^2 \to [1,\infty)$ given by (4.41) and we therefore get $\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0$ for all $q \in (0,\infty)$.

4.8. Scalar stochastic Ginzburg-Landau equation

The Ginzburg-Landau equation is from the theory of superconductivity and has been introduced by Ginzburg & Landau [**21**] to describe a phase transition. As, for instance, in (4.52) in Section 4.4 in Kloeden & Platen [**47**], we consider in this section a simplified scalar version of the Ginzburg-Landau equation pertubed by a multiplicative noise term. More precisely, assume that the setting in Section 4.1 is fulfilled, let $\alpha, \beta, \delta \in (0,\infty)$ be real numbers and suppose that $d = m = 1$, $D = \mathbb{R}$, $\mu(x) = \alpha x - \delta x^3$ and $\sigma(x) = \beta x$ for all $x \in \mathbb{R}$. Under these assumptions, the SDE (4.1) reads as

$$dX_t = \big[\alpha X_t - \delta X_t^3\big]\,dt + \beta X_t\,dW_t \tag{4.42}$$

for $t \in [0,\infty)$ (see, e.g., (4.52) in Section 4.4 in Kloeden & Platen [**47**]). Clearly, Corollary 3.17 applies here with $V\colon \mathbb{R} \to [1,\infty)$ given by

$$V(x) = 1 + x^2 \tag{4.43}$$

for all $x \in \mathbb{R}$ to obtain

$$\lim_{N\to\infty}\sup_{t\in[0,T]}\mathbb{E}\big[|X_t - \bar{Y}_t^N|^q\big] = 0 \tag{4.44}$$

for all $q \in (0,\infty)$. Obviously, this example has a globally one-sided Lipschitz continuous drift coefficient and a globally Lipschitz continuous diffusion coefficient. So, the convergence results, e.g., in [**36, 34, 74, 38, 19**] apply here (see Chapter 1 for more details).

4.9. Stochastic Lotka-Volterra equations

The Lotka-Volterra predator-prey model (see Lotka [**55**] and Volterra [**82**]) and the Lotka-Volterra model for competing species have a quadratic drift term. Here we study the following more general Lotka-Volterra model as considered, for instance, in Section 7.1 in Kloeden & Platen [**47**] (see also Dobrinevski & Frey [**18**]). Assume that the setting in Section 4.1 is fulfilled, let $A, c = (c_1,\dots,c_d) \in \mathbb{R}^d$, $v = (v_1,\dots,v_d) \in (0,\infty)^d$, $B \in \mathbb{R}^{d\times d}$ and suppose that $d = m$, $D = (0,\infty)^d$, $\mu(x) = \sigma(x) = 0$ for all $x \in D^c$, that

$$\left\langle x, \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_d \end{pmatrix} Bx \right\rangle \leq 0 \tag{4.45}$$

for all $x \in \mathbb{R}^d$ and that

$$\mu\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_d \end{pmatrix}(A + Bx) \tag{4.46}$$

and

$$\sigma\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} c_1 x_1 & & 0 \\ & \ddots & \\ 0 & & c_d x_d \end{pmatrix} \tag{4.47}$$

for all $x = (x_1, \ldots, x_d) \in D$. The SDE (4.1) is then the d-dimensional stochastic Lotka-Volterra system

$$dX_t = \begin{pmatrix} X_t^{(1)} & & 0 \\ & \ddots & \\ 0 & & X_t^{(d)} \end{pmatrix} (A + BX_t)\, dt + \begin{pmatrix} c_1 X_t^{(1)} & & 0 \\ & \ddots & \\ 0 & & c_d X_t^{(d)} \end{pmatrix} dW_t \tag{4.48}$$

for $t \in [0, \infty)$ (see, e.g., (1.6) in Section 7.1 in Kloeden & Platen [**47**]). The drift coefficient of the SDE (4.48) contains a quadratic term and is therefore clearly not globally Lipschitz continuous. In order to apply Corollary 3.17, a Lyapunov-type function $V : \mathbb{R}^d \to [1, \infty)$ needs to be constructed. For this let $\phi : \mathbb{R} \to [0, 1]$ and $\psi : \mathbb{R}^2 \to [0, 1]$ be two infinitely often differentiable functions with $\phi(x) = 0$ for all $x \in (-\infty, 0]$, with $\phi(x) = 1$ for all $x \in [1, \infty)$ and with

$$\psi(x, y) = \phi(x) \cdot \phi(-y) + \phi(-x) \cdot \phi(y) \tag{4.49}$$

for all $x, y \in \mathbb{R}$. Note that $\psi(x, y) = \psi(y, x)$ for all $x, y \in \mathbb{R}$ and $\psi(x, y) = 0$ for all $x, y \in \mathbb{R}$ with $x \cdot y \geq 0$. Then consider $V : \mathbb{R}^d \to [1, \infty)$ given by

(4.50)
$$\begin{aligned} V(x) &= 1 + \tfrac{d^4 \|v\|^4}{|\min(v_1,\ldots,v_d)|^2} + |\langle v, x\rangle|^2 - \sum_{\substack{i,j \in \{1,2,\ldots,d\} \\ i \neq j}} v_i \cdot v_j \cdot x_i \cdot x_j \cdot \psi(x_i, x_j) \\ &= 1 + \tfrac{d^4 \|v\|^4}{|\min(v_1,\ldots,v_d)|^2} + \sum_{i=1}^{d} |v_i x_i|^2 + \sum_{\substack{i,j \in \{1,2,\ldots,d\} \\ i \neq j}} v_i \cdot v_j \cdot x_i \cdot x_j \cdot (1 - \psi(x_i, x_j)) \end{aligned}$$

for all $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and observe that $V : \mathbb{R}^d \to [1, \infty)$ is infinitely often differentiable and satisfies

$$\sup_{x \in \mathbb{R}^d} \left(\tfrac{\sum_{i=1}^{3} \|V^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d, \mathbb{R})}}{(1 + \|x\|)} \right) < \infty. \tag{4.51}$$

In the next step note that

$$\begin{aligned} &V(x) \\ &\geq 1 + \tfrac{d^4 \|v\|^4}{|\min(v_1,\ldots,v_d)|^2} + |\min(v_1, \ldots, v_d)|^2 \|x\|^2 \\ &\quad - |\max(v_1, \ldots, v_d)|^2 \left(\sum_{i,j \in \{1,2,\ldots,d\}} \max(|x_i|, |x_j|) \right) \\ &\geq 1 + \tfrac{d^4 \|v\|^4}{|\min(v_1,\ldots,v_d)|^2} + |\min(v_1, \ldots, v_d)|^2 \|x\|^2 - |\max(v_1, \ldots, v_d)|^2 d^2 \|x\| \end{aligned} \tag{4.52}$$

for all $x \in \mathbb{R}^d$ and the estimate $a^2 + b^2 - ab \geq \frac{a^2+b^2}{2}$ for all $a, b \in \mathbb{R}$ therefore proves that

$$\begin{aligned} V(x) &\geq 1 + \tfrac{d^4\|v\|^4}{|\min(v_1,\ldots,v_d)|^2} + |\min(v_1, \ldots, v_d)|^2 \|x\|^2 - d^2 \|v\|^2 \|x\| \\ &= 1 + |\min(v_1, \ldots, v_d)|^2 \left[\tfrac{d^4\|v\|^4}{|\min(v_1,\ldots,v_d)|^4} + \|x\|^2 - \tfrac{d^2\|v\|^2}{|\min(v_1,\ldots,v_d)|^2} \|x\| \right] \\ &\geq 1 + \tfrac{|\min(v_1,\ldots,v_d)|^2}{2} \left[\tfrac{d^4\|v\|^4}{|\min(v_1,\ldots,v_d)|^4} + \|x\|^2 \right] \geq 1 + \tfrac{|\min(v_1,\ldots,v_d)|^2}{2} \|x\|^2 \end{aligned} \tag{4.53}$$

for all $x \in \mathbb{R}^d$. This shows

$$\sup_{x \in \mathbb{R}^d} \left(\frac{\|x\|^2}{V(x)} \right) < \infty. \tag{4.54}$$

In addition, observe that

$$\sup_{x \in \mathbb{R}^d} \left(\frac{\|V'(x)\sigma(x)\|_{L(\mathbb{R}^d,\mathbb{R})}}{\left(1 + \|x\|^2\right)} \right) < \infty. \tag{4.55}$$

Next note that assumption (4.45) gives that

$$\sup_{x \in \mathbb{R}^d} \left(\frac{(\mathcal{G}_{\mu,\sigma} V)(x)}{V(x)} \right) < \infty \tag{4.56}$$

Combining (4.51), (4.54), (4.55) and (4.56) shows that Corollary 3.17 applies here with the function $V \colon \mathbb{R}^d \to [1, \infty)$ given in (4.50) and we therefore get

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.57}$$

for all $q \in (0, \infty)$.

Finally, let us describe two more specific examples of the stochastic Lotka-Volterra system (4.48).

4.9.1. Stochastic Verhulst equation. In addition to the assumptions above, let in this subsection $\eta, \lambda \in (0, \infty)$ be real numbers and suppose that $d = m = 1$, $A = \eta + \frac{c^2}{2}$ and $B = -\lambda$. The stochastic Lotka-Volterra system (4.48) thus simplifies to the one-dimensional Stratonovich SDE

$$dX_t = \left[\eta \cdot X_t - \lambda \cdot (X_t)^2 \right] dt + c \cdot X_t \circ dW_t \tag{4.58}$$

for $t \in [0, \infty)$. Equation (4.58) is referred to as stochastic Verhulst equation in the literature (see, e.g., Section 4.4 in Kloeden & Platen [**47**]). Note also that (4.45) is fulfilled here with $v = 1$, for instance. Clearly, this example has a globally one-sided Lipschitz continuous drift coefficient and a globally Lipschitz continuous diffusion coefficient. Hence, the convergence results, e.g., in [**36, 34, 74**] can be applied here (see Chapter 1 for more details).

4.9.2. Predator-prey model. In addition to the assumptions above, let in this subsection $\alpha, \beta, \gamma, \delta \in (0, \infty)$ be real numbers and suppose that $d = m = 2$, $A = (\alpha, -\delta)$ and

$$B = \begin{pmatrix} 0 & -\beta \\ \gamma & 0 \end{pmatrix}. \tag{4.59}$$

The stochastic Lotka-Volterra system (4.48) is then the two-dimensional SDE

$$\begin{aligned} dX_t^{(1)} &= X_t^{(1)} \left(\alpha - \beta \cdot X_t^{(2)}\right) dt + c_1 \cdot X_t^{(1)} \, dW_t^{(1)} \\ dX_t^{(2)} &= X_t^{(2)} \left(\gamma \cdot X_t^{(1)} - \delta\right) dt + c_2 \cdot X_t^{(2)} \, dW_t^{(2)} \end{aligned} \tag{4.60}$$

for $t \in [0, \infty)$. Note that (4.45) is fulfilled here with $v = (\gamma, \beta)$, for instance. The deterministic case ($c_1 = c_2 = 0$) of this model has been introduced by Lotka [**55**] and Volterra [**82**].

4.10. Volatility processes

There are a number of models in the literature on computational finance which generalize the Black-Scholes model with a stochastic volatility process. To unify some squared volatility processes of these models, we consider the following SDE. Assume that the setting in Section 4.1 is fulfilled, let $a \in [1, \infty)$, $b \in [\frac{1}{2}, \infty)$, $\alpha, \beta \in (0, \infty)$, $\gamma \in \mathbb{R}$, $\delta \in [0, \infty)$ be real numbers with

$$a + 1 \geq 2b \qquad \text{and} \qquad \delta \geq \mathbb{1}_{\{\frac{1}{2}\}}(b) \cdot \frac{\beta^2}{2} \tag{4.61}$$

and suppose that $d = m = 1$, $D = (0, \infty)$, $\mu(x) = \delta$ and $\sigma(x) = 0$ for all $x \in (-\infty, 0]$ and

$$\mu(x) = \delta + \gamma \cdot x - \alpha \cdot x^a \qquad \text{and} \qquad \sigma(x) = \beta \cdot x^b \tag{4.62}$$

for all $x \in (0, \infty)$. The SDE (4.1) then reads as

$$dX_t = \left[\delta + \gamma X_t - \alpha \, (X_t)^a\right] dt + \beta \, (X_t)^b \, dW_t \tag{4.63}$$

for $t \in [0, \infty)$. The assumption $\delta \geq \mathbb{1}_{\{1/2\}}(b) \cdot \frac{\beta^2}{2}$ in (4.61) ensures the existence of an up to indistinguishability unique strictly positive solution of (4.63). The drift coefficient $\mu \colon \mathbb{R} \to \mathbb{R}$ in this example is globally one-sided Lipschitz continuous. Indeed, note that

$$\begin{aligned} &\langle x - y, \mu(x) - \mu(y) \rangle \\ &= \begin{cases} \gamma \, (x-y)^2 - \alpha \, (x^a - y^a) \, (x - y) \leq \gamma \, (x-y)^2 & : x, y > 0 \\ \langle x - y, \mu(x) - \delta \rangle \leq (x - y) \, \gamma x \leq \max(\gamma, 0) \, (x-y)^2 & : x > 0, y \leq 0 \\ \langle y - x, \mu(y) - \delta \rangle \leq (x - y) \, \gamma x \leq \max(\gamma, 0) \, (x-y)^2 & : x \leq 0, y > 0 \\ \langle x - y, \delta - \delta \rangle = 0 \leq \max(\gamma, 0) \, (x-y)^2 & : x, y \leq 0 \end{cases} \end{aligned} \tag{4.64}$$

for all $x, y \in \mathbb{R}$ and therefore

$$\langle x - y, \mu(x) - \mu(y) \rangle \leq \max(\gamma, 0) \, (x-y)^2 \tag{4.65}$$

for all $x, y \in \mathbb{R}$. In the sequel, the application of Corollary 3.18 for the SDE (4.63) is illustrated. For this define $p_0 \in (1, \infty]$ and $q_0 \in (-\frac{1}{2}, \infty]$ by

$$p_0 := \begin{cases} \infty & : b \in [\frac{1}{2}, 1] \cup [\frac{1}{2}, \frac{a+1}{2}) \\ \frac{2\alpha + \beta^2}{\beta^2} & : \text{otherwise} \end{cases} \tag{4.66}$$

and

$$q_0 := \begin{cases} 0 & : p_0 \in (1, 3) \\ \frac{p_0}{4\{b - 2 + \max(a, 3/2)\}} - \frac{1}{2} & : p_0 \in [3, \infty) \\ \infty & : p_0 = \infty \end{cases} \tag{4.67}$$

and note that this definition ensures $p_0 \geq q_0$. In addition, observe that

$$
\begin{aligned}
&\langle x, \mu(x)\rangle + \tfrac{(p-1)}{2} \|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} = x \cdot \mu(x) + \tfrac{(p-1)}{2} |\sigma(x)|^2 \\
&= x\left(\delta + \gamma x - \alpha x^a\right) + \tfrac{(p-1)\beta^2 x^{2b}}{2} \\
&\leq (\delta + \max(\gamma, 0))\left(1 + x^2\right) - \alpha x^{(a+1)} + \left(\tfrac{(p-1)\beta^2}{2}\right) x^{2b} \\
&= (\delta + \max(\gamma, 0))\left(1 + x^2\right) + x^{2b}\left(\tfrac{(p-1)\beta^2}{2} - \alpha \cdot x^{(a+1-2b)}\right)
\end{aligned} \tag{4.68}
$$

for all $x \in (0, \infty)$ and all $p \in [0, \infty)$ and

$$
\langle x, \mu(x)\rangle + \tfrac{(p-1)}{2} \|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)} = x \cdot \delta \leq \delta\left(1 + x^2\right) \tag{4.69}
$$

for all $x \in (-\infty, 0]$ and all $p \in [0, \infty)$. Combining (4.68) and (4.69) then results in

$$
\sup_{x \in \mathbb{R}} \left(\frac{\langle x, \mu(x)\rangle + \frac{(p-1)}{2} \|\sigma(x)\|^2_{HS(\mathbb{R}^m,\mathbb{R}^d)}}{\left(1 + \|x\|^2\right)} \right) < \infty \tag{4.70}
$$

for all $p \in [0, p_0] \cap [0, \infty)$. Next note for every $p \in [0, \infty)$ that

$$
\mathbb{E}\big[|X_t|^p\big] < \infty \tag{4.71}
$$

for all $t \in [0, \infty)$ if and only if $p \leq p_0$. Furthermore, observe that if $p_0 \geq 3$, then

$$
\begin{aligned}
&\sup_{p \in [3,\infty) \cap [0,p_0]} \left(\frac{p}{2\max(0, 2(b-1)) + 4\max\big(a-1, \max(0, 2(b-1)), \frac{1}{2}\big)} - \frac{1}{2} \right) \\
&= \sup_{p \in [3,\infty) \cap [0,p_0]} \left(\frac{p}{4\max(0, b-1) + 4\max\big(a-1, 2b-2, \frac{1}{2}\big)} - \frac{1}{2} \right) \\
&= \sup_{p \in [3,\infty) \cap [0,p_0]} \left(\frac{p}{4\left\{\max(0, b-1) + \max\big(a-1, \frac{1}{2}\big)\right\}} - \frac{1}{2} \right) \\
&= \sup_{p \in [3,\infty) \cap [0,p_0]} \left(\frac{p}{4\left\{\max(-1, b-2) + \max(a, 3/2)\right\}} - \frac{1}{2} \right) = q_0.
\end{aligned} \tag{4.72}
$$

Combining (4.70) and (4.72) with Corollary 3.18 then implies

$$
\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.73}
$$

for all $q \in (0, \infty)$ with $q < q_0$. Let us illustrate this by three more specific examples.

4.10.1. Cox-Ingersoll-Ross process.

In addition to the assumptions above, suppose that $a = 1$, $b = \frac{1}{2}$ and $\gamma = 0$. The SDE (4.63) is then the Cox-Ingersoll-Ross process

$$
dX_t = \big[\delta - \alpha X_t\big]\, dt + \beta\sqrt{X_t}\, dW_t \tag{4.74}
$$

for $t \in [0, \infty)$ which has been introduced in Cox, Ingersoll & Ross [**11**] as model for instantaneous interest rates. Later Heston [**30**] proposed this process as a model for the squared volatility in a Black-Scholes type market model. Here we have $p_0 = q_0 = \infty$ and hence get $\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0$ for all $q \in (0, \infty)$. Both the drift and the diffusion coefficient clearly grow at most linearly. Therefore, strong convergence of the Euler-Maruyama approximations is well-known (see, e.g., Krylov [**49**] and Gyöngy [**22**] for convergence in probability and pathwise convergence respectively). Strong convergence rates of a drift-implicit Euler method for the SDE (4.74) are established in Theorem 1.1 in Dereich, Neuenkirch & Szpruch [**17**].

4.10.2. Simplified Ait-Sahalia interest rate model. In addition to the assumptions above, suppose that $a = 2$ and $b < \frac{3}{2}$. Under these additional assumptions, the SDE (4.63) reads as

$$dX_t = \left[\delta + \gamma X_t - \alpha (X_t)^2\right] dt + \beta (X_t)^b \, dW_t \tag{4.75}$$

for $t \in [0, \infty)$. A more general version hereof has been used in Ait-Sahalia [**1**] for testing continuous-time models of the spot interest rate. Here we also have $p_0 = q_0 = \infty$ and therefore

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.76}$$

for all $q \in (0, \infty)$. A strong convergence result for this SDE is Theorem 6.2 in Szpruch et al. [**77**].

4.10.3. Volatility process in the Lewis stochastic volatility model. In addition to the assumptions above, suppose that $a = 2$, $b = \frac{3}{2}$, $\gamma \in [0, \infty)$ and $\delta = 0$. The SDE (4.63) is then the instantaneous variance process in the Lewis stochastic volatility model (see Lewis [**51**])

$$dX_t = \alpha X_t \left(\tfrac{\gamma}{\alpha} - X_t\right) dt + \beta (X_t)^{\frac{3}{2}} \, dW_t \tag{4.77}$$

for $t \in [0, \infty)$. Here we get

$$p_0 = \tfrac{2\alpha+\beta^2}{\beta^2} \in (1, \infty) \tag{4.78}$$

and

$$q_0 = \max\!\left(\tfrac{p_0}{6} - \tfrac{1}{2}, 0\right) = \max\!\left(\tfrac{\alpha-\beta^2}{3\beta^2}, 0\right) \in [0, \infty). \tag{4.79}$$

In the case $\alpha > \beta^2$ we thus get

$$\lim_{N\to\infty} \sup_{t\in[0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.80}$$

for all $q \in (0, q_0)$. The stochastic volatility model associated to (4.77) is also known as 3/2-stochastic volatility model (see also [**29, 32**]). Furthermore, we note that Theorem 4.4 in Mao & Szpruch [**56**] proves strong L^2-convergence of drift-implicit Euler methods for the SDE (4.77) in the case $2\alpha \geq \beta^2$. More formally, (4.65) ensures that there exist unique stochastic processes $Z^N \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}$, $N \in \mathbb{N} \cap (\max(0, \gamma)\, T, \infty)$, satisfying $Z_0^N = X_0$ and

$$\begin{aligned} Z_{n+1}^N &= Z_n^N + \mu(Z_{n+1}^N)\frac{T}{N} + \sigma(Z_n^N)\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right) \\ &= Z_n^N + \mathbb{1}_{\{Z_{n+1}^N > 0\}}\left(\gamma Z_{n+1}^N - \alpha\left(Z_{n+1}^N\right)^2\right)\frac{T}{N} + \sigma(Z_n^N)\left(W_{\frac{(n+1)T}{N}} - W_{\frac{nT}{N}}\right) \end{aligned} \tag{4.81}$$

for all $n \in \mathbb{N}_0$ and all $N \in \mathbb{N} \cap (\max(0, \gamma)\, T, \infty)$ and Theorem 4.4 in Mao & Szpruch [**56**] then, in particular, proves that in the case $2\alpha \geq \beta^2$ it holds that

$$\lim_{N\to\infty} \mathbb{E}\big[\|X_T - \bar{Z}_T^N\|^p\big] = 0 \tag{4.82}$$

for all $p \in (0,2)$ where $\bar{Z}^N \colon [0,T] \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, are defined through

$$\bar{Z}_t^N := \left(n + 1 - \tfrac{tN}{T}\right) Z_n^N + \left(\tfrac{tN}{T} - n\right) Z_{n+1}^N \tag{4.83}$$

for all $t \in [nT/N, (n+1)T/N]$, $n \in \{0, 1, \ldots, N-1\}$ and all $N \in \mathbb{N}$. Next we observe that Corollary 2.27 and Lemma 2.29 can be applied here to prove moment bounds and strong convergence of implicit numerical approximations methods for the SDE (4.77). In particular, Corollary 2.27 implies that

$$\sup_{N \in \mathbb{N}} \sup_{n \in \{0,1,\ldots,N\}} \mathbb{E}\big[\|Z_n^N\|^p\big] < \infty \tag{4.84}$$

for all $p \in [0, p_0]$ in the case $2\alpha \geq \beta^2$. This, Corollary 3.12 and Lemma 3.31 then show in the case $2\alpha \geq \beta^2$ that

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\big[\|X_t - \bar{Z}_t^N\|^p\big] = 0 \tag{4.85}$$

for all $p \in (0, p_0)$. Equation (4.85) improves Theorem 4.4 in Mao & Szpruch [**56**] in the case $2\alpha > \beta^2$.

4.11. Langevin equation

A commonly used model for the motion of molecules in a potential is the Langevin equation (see, e.g., Subsection 2.1 in Beskos & Stuart [**7**]). Corollary 3.17 does not apply to completely arbitrary Langevin equations but requires the following assumptions on the potential. Assume that the setting in Section 4.1 is fulfilled, let $\varepsilon, c \in (0,\infty)$ be real numbers, let $U \colon \mathbb{R}^d \to \mathbb{R}$ be a continuously differentiable function with a locally Lipschitz continuous derivative and suppose that $D = \mathbb{R}^d$, $d = m$ and

$$\mu(x) = -(\nabla U)(x) \qquad \text{and} \qquad \sigma(x) = \sqrt{2\varepsilon} I \tag{4.86}$$

for all $x \in \mathbb{R}^d$. Moreover, let $V \colon \mathbb{R}^d \to [1,\infty)$ be a twice differentiable function with a locally Lipschitz continuous second derivative, with

$$\limsup_{q \searrow 0} \sup_{x \in \mathbb{R}^d} \left(\frac{\|x\|^q}{V(x)} \right) < \infty \tag{4.87}$$

and with

$$\langle \nabla U(x), \nabla V(x) \rangle \geq -c \cdot V(x) \quad \text{and} \quad \sum_{i=1}^{3} \|V^{(i)}(x)\|_{L^{(i)}(\mathbb{R}^d,\mathbb{R})} \leq c\,|V(x)|^{[1-1/c]} \tag{4.88}$$

for $\lambda_{\mathbb{R}^d}$-almost all $x \in \mathbb{R}^d$. Under these assumptions, the SDE (4.1) reads as

$$dX_t = -(\nabla U)(X_t)\,dt + \sqrt{2\varepsilon}\,dW_t \tag{4.89}$$

for $t \in [0,\infty)$. Corollary 3.17 then implies

$$\lim_{N \to \infty} \sup_{t \in [0,T]} \mathbb{E}\big[\|X_t - \bar{Y}_t^N\|^q\big] = 0 \tag{4.90}$$

for all $q \in (0,\infty)$. If the force $(-\nabla U)$ in the Langevin equation is globally one-sided Lipschitz continuous and satisfies suitable growth and regularity conditions, then the strong convergence results, e.g., in [**36, 34, 74, 38, 19**] apply here (see Chapter 1 for more details).

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