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Fast Convolution Quadrature Based Impedance Boundary Conditions.

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Abstract We consider an eddy current problem in time-domain relying on impedance boundary conditions on the surface of the conductor(s). We pursue its full discretization comprising (i) a finite element Galerkin discretization by means of lowest order edge elements in space, and (ii) temporal discretization based on *Runge-Kutta convolution quadrature* (CQ) for the resulting Volterra integral equation in time. The final algorithm also involves the fast and oblivious approximation of CQ.

For this method we give a comprehensive convergence analysis and establish that the errors of spatial discretization, CQ and of its approximate realization add up to the final error bound.

Keywords eddy current problem, impedance boundary conditions, convolution quadrature, fast and oblivious algorithms

Mathematics Subject Classification (2000) 65M15, 65M60, 65M20, 65R20

1 Transient Eddy Current Model

We consider a linear transient eddy current problem with the conductors occupying the bounded and connected polyhedron $\Omega_C \subset \mathbb{R}^3$. With a finite element discretization in mind we artificially truncate the fields to a simple bounded computational domain $\Omega \subset \mathbb{R}^3$ with $\overline{\Omega}_C \subset \Omega$.

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For rapidly changing fields and high conductivities the skin effect prevents the fields from penetrating deep into the conductors. This permits us to model the effect of the condutor on the fields by means of an *impedance boundary condition* on the surface $\Gamma := \partial\Omega_C$ of the conductor without incurring a severe modeling error. This boundary condition imposes a linear relationship between the tangential components of the electric and magnetic field that is local in space. In *frequency domain* at fixed angular frequency $\omega > 0$ this amounts to the well-known Leontovich boundary condition [6, 21]

$$(\widehat{\mathbf{H}} \times \boldsymbol{n})(\boldsymbol{x}) = \sqrt{\frac{i\omega\sigma(\boldsymbol{x})}{\mu(\boldsymbol{x})}}\widehat{\mathbf{E}}_t(\boldsymbol{x}) \ , \quad \boldsymbol{x} \in \Gamma \ , \tag{1.1}$$

where $\boldsymbol{n} : \Gamma \to \mathbb{R}^3$ is the exterior unit normal vector field on Γ, and $\widehat{\mathbf{H}}$ and $\widehat{\mathbf{E}}$ denote the complex amplitudes of the magnetic and electric field, respectively, and $\widehat{\mathbf{E}}_t := (\boldsymbol{n} \times \widehat{\mathbf{E}}) \times \boldsymbol{n}$ is the tangential component. The material coefficients μ (magnetic permeability) and σ (conductivity) are uniformly positive, but may vary in space. The Leontovic boundary condition (1.1) is the simplest representative of the class of surface impedance boundary conditions (SIBCs) that are hugely popular in computational electromagnetics with entire books devoted to them [10, 22, 23]. More sophisticated "higher order" specimens of SIBCs have been derived, for instance, in [7]. What they all have in common is the structure

$$(\widehat{\mathbf{H}} \times \boldsymbol{n})(\boldsymbol{x}) = Z(i\omega)(\widehat{\mathbf{E}}_t)(\boldsymbol{x}) \ , \boldsymbol{x} \in \Gamma \ , \tag{1.2}$$

where Z may stand for a suitable surface (pseudo-)differential operator. In this article we focus on (1.1), but it should be regarded as a "structural representative" of more general relations of the form (1.2).

Multiplication with an expression in ω as in (1.1) becomes convolution in *time domain.* If we make the assumption that all fields vanish for $t \leq 0$, from (1.1) we arrive at the following *transient impedance boundary condition* for the time-dependent fields

$$\mathbf{H}(\boldsymbol{x}, t) \times \boldsymbol{n}(\boldsymbol{x}) = \int_0^t \eta(\boldsymbol{x}) k(t-\tau) \mathbf{E}_t(\boldsymbol{x}, \tau) \,\mathrm{d}\tau \ , \quad t \geq 0 \ , \quad \boldsymbol{x} \in \Gamma \ , \tag{1.3}$$

with a uniformly positive function $\eta(\boldsymbol{x}) := \sqrt{\sigma(\boldsymbol{x})\mu(\boldsymbol{x})^{-1}}$, $\boldsymbol{x} \in \partial\Omega_c$, and a *convolution kernel* $k : \Gamma \times \mathbb{R}^+ \to \mathbb{R}$, whose temporal Laplace transform is given by

$$K(s) := (\mathcal{L}k(\cdot))(s) = \sqrt{s} \ , \quad s \in \mathbb{C} \setminus (-\infty, 0) \ . \tag{1.4}$$

For the sake of brevity we adopt the "operational calculus notation" for (1.3) [14], expressing it as $\mathbf{H} \times \boldsymbol{n} = \eta K(\partial_t)\mathbf{E}_t$.

Then, the evolution of the (scaled) electromagnetic fields in $D := \Omega \setminus \Omega_C$ is governed by the following initial-boundary value problem that we consider up to a fixed final time $T > 0$:

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{E} = \mathbf{f}(\boldsymbol{x}, t) \quad , \quad \operatorname{div}\mathbf{E} = 0 \quad \text{in } D\times]0,T[\ , \tag{1.5a}$$

$$\mathbf{curl}\,\mathbf{E} \times \boldsymbol{n} = \eta(\boldsymbol{x})K(\partial_t)\mathbf{E}_t \quad \text{on } \Gamma\times]0,T[\ , \tag{1.5b}$$

$$\mathbf{E}_t = 0 \quad \text{on } \partial\Omega\times]0,T[\ , \tag{1.5c}$$

$$\mathbf{E}(\cdot,0) = 0 \quad \text{on } D\ . \tag{1.5d}$$

This is the so-called $\mathbf{E}$-based formulation of an eddy current problem [1, Sect. 2.1]. The zero divergence condition on $\mathbf{E}$ in (1.5a) should be regarded as a *gauging*, which ensures uniqueness of the electric field outside Ω_C. The right hand side $\mathbf{f}$ stands for a source current producing an exciting magnetic field. We assume that it is compatible in the sense that $\mathbf{f}(0) = 0$ and

$$\operatorname{supp}\mathbf{f}(\cdot,t) \subset D \quad \forall 0 \le t \le T \quad , \quad \mathbf{f} \in H^1(]0,T[, \boldsymbol{H}(\operatorname{div}0, D))\ , \tag{1.6}$$

where $\boldsymbol{H}(\operatorname{div}0, D)$ is the space of solenoidal vector fields on D.

Remark 1.1 Symmetries allow the dimensional reduction of (1.5), for instance, in the case of translational invariance, we end up with the so-called TM eddy current model, an initial-boundary value problem for a scalar unknown $u = u(\widetilde{\boldsymbol{x}}, t)$ representing a single component of the electric field

$$-\Delta u = f \quad \text{in } \widetilde{D}\times]0,T[\ , \tag{1.7a}$$

$$\mathbf{grad}\,u \cdot \widetilde{\boldsymbol{n}} = \eta(\widetilde{\boldsymbol{x}})K(\partial_t)u \quad \text{on } \widetilde{\Gamma}\times]0,T[\ , \tag{1.7b}$$

$$u = 0 \quad \text{on } \partial\widetilde{\Omega}\times]0,T[\ , \tag{1.7c}$$

$$u(\cdot,0) = 0 \quad \text{on } \widetilde{D}\ . \tag{1.7d}$$

where the $\tilde{}$ tags two-dimensional cross-sections of the domains/boundaries.

In this article we propose a numerical method for the full discretization of (1.5) in space and time that also allows a highly efficient implementation. Discretization in space will rely on standard finite elements (FE), using edge elements for the approximation of $\mathbf{E}$. A particular challenge for temporal discretization arises from the non-local (in time) character of the convolution in (1.3). The so-called *Convolution Quadrature (CQ)* policy introduced by C. Lubich in [14, 15] addresses this challenge in a uniquely stable fashion. Moreover, it requires only knowledge of the Laplace transform $K(s)$ of the convolution kernel $k(t)$. For the time domain impedance boundary conditions we have this very knowledge, see (1.4). The use of K instead of k is also reflected in the operational calculus notation $K(\partial_t)$. Initially, the CQ methods were based on multistep methods. In [16], the they have been extended to Runge-Kutta methods. This variant will form the foundation of the discretization of (1.5) in time.

Fast algorithms on top of CQ have also been developed in the last decade. A fast "oblivious" algorithm for *approximate* CQ *(FOCQ)* with considerably reduced memory requirements is presented in [19] and we follow these ideas. For *sectorial* decreasing kernels (see (4.2)-(4.3)), N the total number of time steps and ε the target accuracy, the algorithm in [19] reduces the number of multiplications of a naive implementation of CQ from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N \log(\frac{1}{\varepsilon}))$ and the memory requirements from $O(N)$ to $\mathcal{O}(\log N \log(\frac{1}{\varepsilon}))$.

Though we believe that application of CQ to impedance boundary conditions is new, it has become well established for certain kinds of evolution problems, most prominently wave propagation problems in unbounded domains tackled by means of time domain boundary integral equation (TDBIE) methods. We mention in particular [3, 4], where the analysis of CQ based on Runge–Kutta methods is extended to this context and [2] for experimental results and a full list of references. We also mention applications of CQ to boundary element discretizations of visco-elasticity [20].

The focus of this article is on a comprehensive a priori convergence analysis of a fully discrete oblivious finite element Runge–Kutta convolution quadrature algorithm for (1.5). We adopt the "method of lines policy" successively estimating the errors due to spatial and temporal discretization. All the error contributions add up to the total error of the scheme. For spatial and temporal discretization error we find the expected algebraic decay in terms of mesh width and timestep size, respectively. The error due to the oblivious approximation turns out to decay exponentially in a discretization parameter and will usually be negligible compared to the other error contributions.

The paper is organized as follows. In Section 2 we address the spatial variational formulation of (1.5). In Section 3 we examine the spatial error. In Section 4 we review CQ based on Runge–Kutta methods [3, 16], derive error estimates for its application to the spatial semidiscretization of (1.5) (namely (3.3)) and derive an estimate of the full discretization error. In Section 5 we analyze the error introduced by the oblivious approximation of CQ. Since the time integration of (1.5) leads to an intermediate situation where the Laplace transform of the convolution kernel is non decreasing, i.e., $\nu \geq 0$ in (4.3), we briefly show how to extend the theoretical background for the FOCQ to this case, by following closely [12] and [13]. Finally, 2D numerical experiments are provided in Section 6.

2 Spatial variational formulation

Impedance boundary conditions require the electric fields to belong to the "energy space"

$$U := \{\mathbf{u} \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \ \mathbf{u}_{t\,|\Gamma} \in \boldsymbol{L}^2_t(\Gamma), \ \mathbf{u}_t = 0 \text{ on } \partial\Omega\} \,, \tag{2.1}$$

which is a Hilbert space, when endowed with the usual graph norm $\|\cdot\|_U$. Here, $\boldsymbol{L}^2_t(\Gamma)$ stands for space of square integrable tangential vectorfields on Γ. In order to take into account the gauge condition $\operatorname{div} \mathbf{E} = 0$ the spatial variational formulation of (1.5) is posed on the function space V defined through the U-orthogonal *Helmholtz decomposition*

$$U = V \oplus \mathbf{grad}\, H^1_*(D) \,, \tag{2.2}$$

where $H^1_*(D) := \{\varphi \in H^1(D) : \varphi_{|\Gamma} = \text{const}, \ \varphi_{|\partial\Omega} = 0\}$. Obviously, V is a closed subspace of U and it will be equipped with the same norm. Then the spatial variational formulation of (1.5) reads: seek $\mathbf{E} \in L^2(]0,T[,V)$ such that

$$\underbrace{(\mathbf{curl}\,\mathbf{E}, \mathbf{curl}\,\mathbf{v})_0}_{=: a(\mathbf{E},\mathbf{v})} + K(\partial_t) \underbrace{\int_\Gamma \eta(\boldsymbol{x})\mathbf{E}_t \cdot \mathbf{v}_t \, \mathrm{d}S}_{=: b(\mathbf{E},\mathbf{v})} = (\mathbf{f}, \mathbf{v})_0 \tag{2.3}$$

for all $\mathbf{v} \in V$ and on $[0,T]$. Here $(\cdot,\cdot)_0$ designates the $L^2(D)$ inner product.

Theorem 2.1 *Under the assumptions* (1.6) *on the source term,* (2.9) *has a unique solution* $\mathbf{E} \in L^2(]0,T[,V)$*.*

Proof First, we perform a reduction to the boundary, based on the U-orthogonal decomposition

$$V = V_\partial \oplus (V \cap \boldsymbol{H}_0(\mathbf{curl}, D)) \ .$$

Note that V_∂ can be regarded as a *trace space* of tangential surface vectorfields, because we have, with equivalent norms,

$$V_\partial \cong \boldsymbol{L}^2_t(\Gamma) \cap \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \ ,$$

where $\boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$ is the trace space for $\boldsymbol{H}(\mathbf{curl}, D)$ on Γ, see [5]. In other words, through "**curl curl**-harmonic extension" we may identify functions in V_∂ with their tangential components on Γ.

Now, consider a corresponding splitting of $\mathbf{E}$: $\mathbf{E} = \mathbf{E}_\partial + \mathbf{E}_0$. Applying the splitting to the test function in (2.3), we find that $\mathbf{E}_\partial \in \boldsymbol{L}^2_t(\Gamma) \cap \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma)$ solves

$$s(\mathbf{E}_\partial, \mathbf{v}_\partial) + K(\partial_t) b(\mathbf{E}_\partial, \mathbf{v}_\partial) = (\mathbf{g}, \mathbf{v}_\partial)_{\boldsymbol{L}^2_t(\Gamma)} \quad \forall \mathbf{v}_\partial \in V_\partial \ , \tag{2.4}$$

where $s(\mathbf{E}_\partial, \mathbf{v}_\partial) := a(\mathbf{E}_\partial, \mathbf{v}_\partial)$, using the two different interpretations of functions in V_∂. The right hand side function $\mathbf{g}$ can be obtained as $\mathbf{g}(t) := \mathbf{curl}\,\mathbf{w}(t) \times \boldsymbol{n}$, with

$$\mathbf{w}(t) \in V \cap \boldsymbol{H}_0(\mathbf{curl}, D) \quad , \quad \mathbf{curl}\,\mathbf{curl}\,\mathbf{w}(t) = \mathbf{f}(t) \quad \text{in } D \ .$$

From [9, Lemma 4.2] we conclude $\mathbf{g} \in H^1(]0,T[, \boldsymbol{L}^2_t(\Gamma))$.

Endow $\boldsymbol{L}^2_t(\Gamma)$ with the inner product $b(\cdot,\cdot)$ and write S for the unbounded, self-adjoint, and non-negative operator on $\boldsymbol{L}^2_t(\Gamma)$ induced by the bilinear form $s(\cdot,\cdot)$. Then (2.4) becomes

$$\mathsf{S}\mathbf{E}_\partial + K(\partial_t)\mathbf{E}_\partial = \eta^{-1}\mathbf{g} \quad \text{in } \boldsymbol{L}^2_t(\Gamma) \ , \tag{2.5}$$

$$\Updownarrow$$

$$\mathbf{E}_\partial + K^{-1}(\partial_t)\mathsf{S}\mathbf{E}_\partial = K^{-1}(\partial_t)(\eta^{-1}\mathbf{g}) \ . \tag{2.6}$$

From [5] we know that the first component X of the $\boldsymbol{L}^2_t(\Gamma)$-orthogonal Hodge decomposition

$$\boldsymbol{L}^2_t(\Gamma) \cap \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) = X \oplus \mathbf{curl}_\Gamma\, H^1(\Gamma)$$

is compactly embedded in $\boldsymbol{L}^2_t(\Gamma)$. Taking into account that $\mathsf{S}(\mathbf{curl}_\Gamma\, H^1(\Gamma)) = 0$, we infer that S possesses a compact resolvent.

In addition, note that $K^{-1}(s) = s^{-1/2}$ such that $K^{-1}(\partial_t)$ induces a convolution operator with the 1-regular L^1-kernel $k(t) = \frac{1}{\sqrt{\pi t}}$ of positive type. Thus, we can apply the abstract theory of [18, Ch. 3], which yields the assertion. □

Both bilinear forms a, b defined in (2.3) are clearly symmetric and continuous on U, but we have even stronger properties of $c := a + b : V \times V \to \mathbb{R}$:

Lemma 2.1 *The bilinear form c is positive definite on V.*

Proof The assertion of the lemma is immediate from the definition of the norm

$$\|\mathbf{v}\|_U^2 = \|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(D)} + \|\mathbf{v}\|^2_{L^2(D)} + \|\mathbf{v}_t\|^2_{L^2(\Gamma)}\ , \tag{2.7}$$

and the Poincaré-Friedrichs type inequality

$$\exists C > 0 : \quad \|\mathbf{v}\|_{L^2(D)} \leq C\big(\|\mathbf{curl}\,\mathbf{v}\|^2_{L^2(D)} + \|\mathbf{v}_t\|_{L^2(\Gamma)}\big) \quad \forall \mathbf{v} \in V\ . \tag{2.8}$$

The latter follows from the compact embedding of V in $L^2(D)$, which can be established along the lines of the proof of [9, Thm. 4.1]. □

In order to exploit this useful property of c, we rewrite (2.3) in equivalent form: seek $\mathbf{E} \in L^2(]0, T[, V)$ such that on $[0, T]$

$$c(\mathbf{E}, \mathbf{v}) + \widehat{K}(\partial_t) b(\mathbf{E}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_0 \quad \forall \mathbf{v} \in V\ , \tag{2.9}$$

by defining

$$\widehat{K}(s) := K(s) - 1 = \sqrt{s} - 1\ , \quad s \in \mathbb{C} \setminus (-\infty, 0)\ . \tag{2.10}$$

Remark 2.1 The evolution problem (1.7) can also be cast in the form (2.9) using $V := H^1_{\partial\Omega}(\widetilde{D})$ and

$$a(u, v) := \int_{\widetilde{D}} \mathbf{grad}\, u \cdot \mathbf{grad}\, v\, \mathrm{d}\boldsymbol{x} \quad , \quad b(u, v) := \int_{\widetilde{\Gamma}} \eta(\boldsymbol{x}) uv\, \mathrm{d}S\ . \tag{2.11}$$

3 Spatial Semi-Discretization

3.1 Finite element spaces

We equip D with a tetrahedral mesh $\mathcal{M}$ with mesh-width h and write $\mathcal{E}(\mathcal{M})$ for the finite-dimensional space of lowest order $\boldsymbol{H}(\mathbf{curl}, D)$-conforming edge element functions on $\mathcal{M}$ [9, Sect. 3] and set

$$U_h := U \cap \mathcal{E}(\mathcal{M}) = \{\mathbf{v}_h \in \mathcal{D}(\mathcal{M}) : (\mathbf{v}_h)_t = 0 \text{ on } \partial\Omega\} . \tag{3.1}$$

In analogy to (2.2) the appropriate discrete variational space V_h will be a component of the U-orthogonal *discrete Helmholtz decomposition*

$$U_h = V_h \oplus \mathbf{grad}\, S_h , \tag{3.2}$$

where $S_h \subset H^1_*(D)$ denotes the space of piecewise linear continuous finite element functions on $\mathcal{M}$ that are constant on Γ and vanish on $\partial\Omega$. Then the spatially discrete variational problem reads: seek $\mathbf{E}_h \in L^2([0,T], V_h)$, $\mathbf{E}_h(0) = 0$, such that on $[0,T]$

$$c(\mathbf{E}_h, \mathbf{v}_h) + \widehat{K}(\partial_t) b(\mathbf{E}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in V_h . \tag{3.3}$$

In general, the finite element space V_h is **not** a subspace of V so that (3.3) turns out to be a *non-conforming* Galerkin discretization of (2.9).

Nevertheless, the two spaces are “close” on fine meshes. In order to phrase this in quantitative terms, consider the U-orthogonal projection $\mathsf{Q} : U \to V$ onto $V \subset U$. By definition (2.1), we have $\mathbf{u} - \mathsf{Q}\mathbf{u} \in \mathbf{grad}\, H^1_*(D)$, which implies

$$\mathbf{curl}(\mathbf{u} - \mathsf{Q}\mathbf{u}) = 0 \quad , \quad (\mathbf{u} - \mathsf{Q}\mathbf{u})_t = 0 \quad \text{on } \Gamma . \tag{3.4}$$

The next lemma reveals that Q is a tool for approximating a function $\mathbf{v}_h \in V_h$ in V.

Lemma 3.1 *There is a* $0 < \epsilon \le 1$ *that depends only on* D*, such that*[1]

$$\|\mathbf{v}_h - \mathsf{Q}\mathbf{v}_h\|_U \le C h^\epsilon \|\mathbf{v}_h\|_U \quad \forall \mathbf{v}_h \in V_h . \tag{3.5}$$

Proof Following the ideas in the proof of [9, Lemma 4.5] the proof is reduced to interpolation error estimates.

The “closeness” of V_h and V is also reflected by the fact that the crucial positivity of c is preserved in the discrete setting:

Lemma 3.2 *With a constant* $C > 0$ *depending only on* D *and the shape regularity of* $\mathcal{M}$

$$\|\mathbf{v}_h\|_U^2 \le C |c(\mathbf{v}_h, \mathbf{v}_h)| \quad \forall \mathbf{v}_h \in V_h . \tag{3.6}$$

[1] We write C for generic constants (whose value may differ between different occurrences) that may only depend on D, η, and the shape-regularity of $\mathcal{M}$.

Proof The proof is straightforward from the following variant of the discrete Poincaré-Friedrichs inequality [9, Cor. 4.4]

$$\|\mathbf{v}_h\|_{L^2(D)} \leq C\big(\|\mathbf{curl}\,\mathbf{v}_h\|_{L^2(D)} + \|(\mathbf{v}_h)_t\|_{L^2(\Gamma)}\big) \quad \forall \mathbf{v}_h \in V_h \ . \tag{3.7}$$

Its proof may invoke Lemma 3.1 and Lemma 2.1. □

As a consequence, c induces a norm $\|\cdot\|_c$ on V_h, which is equivalent to the U-norm uniformly in h.

3.2 Estimation of spatial error

We retain the notations $\mathbf{E} \in H^1(]0,T[,V)$ and $\mathbf{E}_h \in H^1(]0,T[,V_h)$ for the solutions of (2.9) and (3.3), respectively, and aim to bound $t \to \|\mathbf{E}(t) - \mathbf{E}_h(t)\|_U$. As usual, such estimates rely on a Galerkin projection $\mathsf{P}_h : V \to V_h$, here defined according to

$$c(\mathsf{P}_h\mathbf{v}, \mathbf{w}_h) = c(\mathbf{v}, \mathbf{w}_h) \quad \forall \mathbf{w}_h \in V_h \ , \tag{3.8}$$

which, thanks to Lemma 3.2, is a valid definition. Standard finite element error estimates from [9, Sect. 6.1] give the approximation property

$$\|\mathbf{u} - \mathsf{P}_h\mathbf{u}\|_U \leq Ch\big(\|\mathbf{u}\|_{H^1(D)} + \|\mathbf{curl}\,\mathbf{u}\|_{H^1(D)} + \|\mathbf{u}_t\|_{H^1(\Gamma)}\big) \tag{3.9}$$

for all $\mathbf{u} \in (H^1(D))^3$ with $\mathbf{curl}\,\mathbf{u} \in (H^1(D))^3$. This allows to control $\|\mathbf{E} - \mathsf{P}_h\mathbf{E}\|_U$ so that it remains to estimate $\|\mathbf{E}_h - \mathsf{P}_h\mathbf{E}\|_U$, which is achieved through a stability argument for an evolution problem with a "residual type" right hand side. Denote $\mathbf{e}_h(t) := \mathbf{E}_h(t) - \mathsf{P}_h\mathbf{E}(t)$ and compute

$$\begin{aligned}
&c(\mathbf{e}_h, \mathbf{v}_h) + \widehat{K}(\partial_t)b(\mathbf{e}_h, \mathbf{v}_h) \\
&\overset{(3.8)}{=} (\mathbf{f}, \mathbf{v}_h)_0 - c(\mathbf{E}, \mathbf{v}_h) - \widehat{K}(\partial_t)b(\mathsf{P}_h\mathbf{E}, \mathbf{v}_h) \\
&= (\mathbf{f}, \mathbf{v}_h)_0 - c(\mathbf{E}, \mathsf{Q}\mathbf{v}_h) + c(\mathbf{E}, \mathsf{Q}\mathbf{v}_h - \mathbf{v}_h) - \widehat{K}(\partial_t)b(\mathsf{P}_h\mathbf{E}, \mathbf{v}_h) \\
&= \underbrace{(\mathbf{f}, \mathbf{v}_h - \mathsf{Q}\mathbf{v}_h)_0}_{=0 \text{ by } (3.4)} + \widehat{K}(\partial_t)b(\mathbf{E}, \mathsf{Q}\mathbf{v}_h) + \underbrace{c(\mathbf{E}, \mathsf{Q}\mathbf{v}_h - \mathbf{v}_h)}_{=0 \text{ by } (3.4)} - \widehat{K}(\partial_t)b(\mathsf{P}_h\mathbf{E}, \mathbf{v}_h) \\
&= \widehat{K}(\partial_t)b(\mathbf{E} - \mathsf{P}_h\mathbf{E}, \mathbf{v}_h) \ .
\end{aligned}$$

Here, the two marked terms vanish due to (3.4). This yields the discrete evolution equation for the error

$$c(\mathbf{e}_h, \mathbf{v}_h) + \widehat{K}(\partial_t)b(\mathbf{e}_h, \mathbf{v}_h) = \widehat{K}(\partial_t)b(\mathbf{E} - \mathsf{P}_h\mathbf{E}, \mathbf{v}_h) \ . \tag{3.10}$$

The stability analysis of (3.10) is achieved by means of simultaneous "diagonalization"; since both c and b are symmetric and semi-definite, and c is even positive definite on V_h, see Lemma 3.2, there exists a sequence of non-negative eigenvalues $\{\lambda_{h,\ell}\}_{\ell=1}^{M}$, $M := \dim V_h$, and an *c-orthonormal* basis $\{\mathbf{u}_{h,\ell}\}_{\ell=1}^{M}$ of V_h so that

$$b(\mathbf{u}_{h,\ell}, \mathbf{v}_h) = \lambda_{h,\ell} c(\mathbf{u}_{h,\ell}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h \ . \tag{3.11}$$

Besides, it is immediate from the the definition of c on Page 6 that $0 \leq \lambda_{h,\ell} \leq 1$ for all h and ℓ.

By expanding $\mathbf{e}_h(t) = \sum_{\ell=1}^{M} \alpha_\ell(t)\mathbf{u}_{h,\ell}$ we obtain the following system of Volterra integral equations for the expansion coefficients

$$\alpha_\ell(t) + \lambda_{h,\ell}\widehat{K}(\partial_t)\alpha_\ell(t) = \widehat{K}(\partial_t)p_\ell(t) \quad \ell = 1, \ldots, M, \tag{3.12}$$

where $p_\ell(t) := b(\mathbf{E}(t) - \mathsf{P}_h\mathbf{E}(t), \mathbf{u}_{h,\ell})$. Note that by orthonormality $\|\mathbf{e}_h(t)\|_c^2 = \sum_{\ell=1}^{M} \alpha_\ell^2(t)$ so that we may target the $\alpha_\ell(t)$'s in order to gauge $\|\mathbf{e}_h(t)\|_U$. To do so we need the following identity:

Lemma 3.3 (Parseval's formula) *Let $f : [0, \infty[\to \mathbb{C}$ be a function whose Laplace transform $F : \mathbb{C} \to \mathbb{C}$ is analytic in the half plane $\mathrm{Re}\,(z) > \sigma_0$ for some $\sigma_0 \geq 0$. Then for every $\sigma > \sigma_0$ there holds true*

$$\|e^{-\sigma t} f(t)\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{2\pi i} \int_{\sigma + i\mathbb{R}} |F(s)|^2 ds \ .$$

Proof First we note that for $\sigma > \sigma_0$ the complex conjugate $\overline{f(t)}$ satisfies

$$\begin{aligned}
\overline{f(t)} &= \overline{\frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{st}F(s)\,\mathrm{d}s} = \overline{\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{(\sigma+i\omega)t}F(\sigma+i\omega)\,i\mathrm{d}\omega} \\
&= \frac{e^{\sigma t}}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega t}\overline{F(\sigma+i\omega)}\,\mathrm{d}\omega = \frac{e^{\sigma t}}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{-(s-\sigma)t}\overline{F(s)}\,\mathrm{d}s \\
&= \frac{e^{2\sigma t}}{2\pi i}\int_{\sigma+i\mathbb{R}} e^{-st}\overline{F(s)}\,\mathrm{d}s \ ,
\end{aligned}$$

where in the second and in the fourth step we use the substitution $s = \sigma + i\omega$. Thus we have

$$\begin{aligned}
\|e^{-\sigma t} f(t)\|^2_{L^2(\mathbb{R}_+)} &= \int_0^{\infty} e^{-2\sigma t} f(t)\overline{f(t)}\,\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_0^{\infty} e^{-2\sigma t} f(t) e^{2\sigma t}\int_{\sigma+i\mathbb{R}} e^{-st}\overline{F(s)}\,\mathrm{d}s\,\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \overline{F(s)}\int_0^{\infty} f(t)e^{-st}\,\mathrm{d}t\,\mathrm{d}s = \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} \overline{F(s)}F(s)\,\mathrm{d}s \ .
\end{aligned}$$

□

Lemma 3.4 *If $p_\ell(t) \in H^1(]0,T[)$, then for every $\sigma > 1$ there is a constant $C > 0$ depending only on D, σ, and η such that*

$$\|\mathbf{e}_h\|^2_{L^2([0,T];V)} \leq Ce^{2\sigma T}\left\| e^{-\sigma t}\frac{\partial}{\partial t} b(\mathbf{E} - \mathsf{P}_h\mathbf{E}, \cdot)\right\|^2_{L^2([0,T];V_h')} \ . \tag{3.13}$$

Proof By the associativity of the convolution the solution of the integral equation (3.12) can be rewritten as

$$\alpha_\ell(t) = \tilde{K}_\ell(\partial_t)\hat{K}(\partial_t)p_\ell(t) ,$$

where the Laplace transform $\tilde{K}_\ell$ of the convolution kernel $\tilde{k}_\ell$ satisfies

$$\tilde{K}_\ell(s) = \frac{1}{1+\lambda_{h,\ell}(\sqrt{s}-1)} .$$

Since the real part of $\sqrt{s}$ is positive and $\lambda_{h,\ell}$ is non-negative we have that

$$\tilde{K}_\ell(s)\hat{K}(s) = \frac{\sqrt{s}-1}{1+\lambda_{h,\ell}(\sqrt{s}-1)} \tag{3.14}$$

is analytic in $\mathbb{C}\setminus\{(-\infty,1]\}$. As $\Re\sqrt{s}>1$, if $\Re s>1$, we find the elementary bound

$$|\tilde{K}_\ell(s)\hat{K}(s)| \leq |s|^{1/2} , \quad \forall \Re s \geq 1 , \tag{3.15}$$

uniformly in h and ℓ.

Next, by the Sobolev extension theorem we can find an extended function p_ℓ^{ext} of p_ℓ so that $p_\ell^{\mathrm{ext}}(t) = p_\ell(t)$ for every $t\in[0,T]$ and

$$\|e^{-\sigma t}p_\ell^{\mathrm{ext}}(t)\|_{H^1(\mathbb{R}_+)} \leq C\|e^{-\sigma t}p_\ell(t)\|_{H^1([0,T])} .$$

for some $C>0$ depending on T and σ. We also define $\alpha_\ell^{\mathrm{ext}}(t) := \tilde{K}_\ell(\partial_t)\hat{K}(\partial_t)p_\ell^{\mathrm{ext}}(t)$, which is a true extension of α_ℓ. Then Lemma 3.3 with $\sigma_0=1$, the bound (3.15) on $|\tilde{K}_\ell\hat{K}(s)|$, $|s|\geq 1$, and the assumptions on p_ℓ imply

$$\begin{aligned}
&\|e^{-\sigma t}\alpha_\ell\|^2_{L^2([0,T]))} \leq \|e^{-\sigma t}\alpha_\ell^{\mathrm{ext}}\|^2_{L^2([0,+\infty))} \\
&\quad = \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} |\tilde{K}_\ell(s)\hat{K}(s)\mathcal{L}(p_\ell^{\mathrm{ext}})(s)|^2 \,\mathrm{d}s \leq \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} |s||\mathcal{L}(p_\ell^{\mathrm{ext}})(s)|^2 \,\mathrm{d}s \\
&\quad \leq \frac{1}{2\pi i}\int_{\sigma+i\mathbb{R}} |s\mathcal{L}(p_\ell^{\mathrm{ext}})(s)|^2 \,\mathrm{d}s \overset{\text{Lemma (3.3)}}{=} \left\| e^{-\sigma t}\frac{\partial}{\partial t}p_\ell^{\mathrm{ext}}\right\|^2_{L^2([0,+\infty))} \\
&\quad \leq C^2 \left\| e^{-\sigma t}\frac{\partial}{\partial t}p_\ell\right\|^2_{L^2([0,T])} ,
\end{aligned}$$

and thus

$$\|\alpha_\ell\|^2_{L^2([0,T])} \leq e^{2\sigma T}C^2\left\| e^{-\sigma t}\frac{\partial}{\partial t}p_\ell\right\|^2_{L^2([0,T])} . \tag{3.16}$$

As remarked above $\|\mathbf{e}_h(t)\|_c^2 = \sum_{\ell=1}^M \alpha_\ell(t)^2$ and, for every $t\in[0,T]$,

$$\begin{aligned}
&\left\| b\left(\frac{\partial}{\partial t}\mathbf{E}(t) - \mathsf{P}_h\frac{\partial}{\partial t}\mathbf{E}(t),\cdot\right)\right\|^2_{V_h'} \\
&= \sup_{\mathbf{v}_h\in V_h} \frac{|b(\frac{\partial}{\partial t}\mathbf{E}(t)-\mathsf{P}_h\frac{\partial}{\partial t}\mathbf{E}(t),\mathbf{v}_h)|^2}{\|\mathbf{v}_h\|_V^2} = \sup_{\mathbf{v}_h\in V_h}\frac{|\frac{\partial}{\partial t}b(\mathbf{E}(t)-\mathsf{P}_h\mathbf{E}(t),\mathbf{v}_h)|^2}{\|\mathbf{v}_h\|_V^2} \\
&= \sup_{(\gamma_k)\in\mathbb{R}^M}\frac{|\frac{\partial}{\partial t}\sum_{\ell=1}^N p_\ell(t)\gamma_k|^2}{\|(\gamma_k)\|_{\ell^2(\mathbb{R}^M)}} = \sup_{(\gamma_k)\in\mathbb{R}^M}\frac{|\sum_{\ell=1}^N \frac{\partial}{\partial t}p_\ell(t)\gamma_k|^2}{\|(\gamma_k)\|_{\ell^2(\mathbb{R}^M)}} = \sum_{\ell=1}^M\left(\frac{\partial}{\partial t}p_\ell(t)\right)^2 .
\end{aligned}$$

Thus summing inequality (3.16) over ℓ gives (3.13). □

Taking for granted sufficient spatial and temporal regularity of the field solution $\mathbf{E}(t)$, we can combine the estimate of Lemma 3.4 with the projection error bound (3.9) and end up with first order convergence: for fixed $\sigma > 1$,

$$\|\mathbf{E} - \mathbf{E}_h\|_{L^2(]0,T[,U)} \leq C e^{\sigma T} h \, , \tag{3.17}$$

where C may depend on $\mathbf{E}$, σ, and the shape regularity of the finite element mesh.

4 Temporal Discretization and Error Estimate of Full discretization

4.1 Runge-Kutta Convolution Quadrature

For $g \in H^1(]0,T[,U)$, $g(0) = 0$, the Convolution Quadrature method approximates the continuous convolution

$$K(\partial_t)g = \int_0^t k(t-\tau)g(\tau)\, d\tau \tag{4.1}$$

by using only the Laplace transform K of the convolution kernel k [14–17]. We zero in on Convolution Quadrature based on Runge-Kutta methods. This method was developed in [16] for sectorial K, that is, for K being analytic in a sector

$$\Sigma(\varphi) := \{s \in \mathbb{C} : \ |\arg(s-\sigma)| < \pi - \varphi, \quad \text{with } \varphi < \frac{1}{2}\pi\} \, , \tag{4.2}$$

and satisfying in this sector,

$$|K(s)| \leq C|s|^{\nu}, \tag{4.3}$$

for some real C and $\nu < 0$. Later, in [4], the CQ has been extended to more general kernels, namely to the case when K is analytic only on a half plane $\Re z > \sigma_0$, for some $\sigma_0 > 0$, and the growth condition (4.3) is satisfied for some $\nu \in \mathbb{R}$, allowing for $\nu \geq 0$.

In what follows we will assume that the underlying m-stage Runge–Kutta method is A-stable [8, Chapter IV.3], with order p, stage order q, and is described by the Butcher Tableau

$$\begin{array}{c|c} \mathbf{c} & \mathcal{A} \\ \hline & \mathbf{b}^T \end{array},$$

where $\mathcal{A} \in \mathbb{R}^{m,m}$ and both $\mathbf{c}, \mathbf{b} \in \mathbb{R}^m$. We will also assume, cf. [19], that the row of weights $\mathbf{b}^T$ equals the last row of the coefficient matrix $\mathcal{A}$, that is,

$$b_j = a_{m,j}, \qquad j = 1, \ldots, m.$$

Relevant examples of such Runge-Kutta methods are the m-stage RadauIIA methods, of order $p = 2m$ and of stage order $q = m$.

Under these assumptions, for a fixed step-size $\Delta t > 0$, the continuous convolution (4.1) at time $t = (n+1)\Delta t$ is approximated by the last component of the sum

$$\big(K(\underline{\partial_{\Delta t}})\mathbf{g}\big)_n := \sum_{j=0}^{n} \mathbf{W}_{n-j}\mathbf{g}_j \ , \tag{4.4}$$

where

$$\mathbf{g}_j := (g(t_j + c_1\Delta t), \ldots, g(t_j + c_m\Delta t))^T \in U^m$$

and the convolution weights $\mathbf{W}_n \in \mathbb{R}^{m,m}$ are defined by the power series expansion [16, Section 2]

$$\sum_{n=0}^{\infty} \mathbf{W}_n\zeta^n := K\left(\frac{\boldsymbol{\Delta}(\zeta)}{\Delta t}\right) , \quad \boldsymbol{\Delta}(\zeta) := \left(\mathcal{O} + \frac{\zeta}{1-\zeta}\mathbb{1}\boldsymbol{b}^T\right)^{-1} , \tag{4.5}$$

with $\mathbb{1} = (1, \ldots, 1)^T$. In this way,

$$\left(K(\partial_t)g\right)((n+1)\Delta t\,) \approx \left(K(\partial_{\Delta t})g\right)_{n+1} := \sum_{j=0}^{n} \boldsymbol{\omega}_{n-j}\mathbf{g}_j \ , \tag{4.6}$$

with $\boldsymbol{\omega}_n = (\omega_n^1, \ldots, \omega_n^m)$ the last row of $\mathbf{W}_n$.

The matrix function $\boldsymbol{\Delta}(\zeta)$ plays a key role in the derivation of the method. Its properties are gathered in [4, Lemma 3], which basically ensures that for $\zeta > 0$ small enough (4.5) is well defined.

The approximation in (4.6) can be extended to all $0 \le t \le T$ by using the zero extension of g to negative times and defining ($t_j := j\Delta t$)

$$\left(K(\partial_{\Delta t})g\right)(t) := \sum_{j=0}^{\infty} \boldsymbol{\omega}_j (g(t - t_j + c_l\Delta t\,))_{l=1}^m \ . \tag{4.7}$$

The properties of (4.7) have been analyzed in [4, Theorem 3]. More precisely, for K satisfying (4.3), the following error estimate holds

$$\begin{aligned} &\left\| \left(K(\partial_t)g\right)(t) - \left(K(\partial_{\Delta t})g\right)(t) \right\|_U \\ &\qquad \le C(\Delta t\,)^{\min(p,q+1-\nu)} \left(\left\| g^{(r)}(0) \right\|_U + \int_0^t \left\| g^{(r+1)}(\tau) \right\|_U \mathrm{d}\,\tau \right) , \end{aligned} \tag{4.8}$$

for $0 \le t \le T$, $r > \max(p+\nu, p, q+1)$, $g \in C^r([0,T], U)$ with $g(0) = g'(0) = \cdots = g^{(r-1)}(0) = 0$ and Δt small enough. The constant $C > 0$ depends on the Runge-Kutta method, on the final time T and on the constants in (4.3).

4.2 Application Convolution Quadrature to Fredholm convolution equations

As a direct consequence of the Cauchy product of series, the Convolution Quadrature method inherits the associativity of the continuous convolution at stage level [3], that is,

$$K_1(\underline{\partial_{\Delta t}})K_2(\underline{\partial_{\Delta t}}) = (K_1K_2)(\underline{\partial_{\Delta t}})\ .$$

This property is particularly useful when applying the CQ to solve integral equations. In particular, as described in Section 4.3, we will be concerned with scalar *Fredholm convolution equations* of the form

$$\mu(t) + \lambda\widehat{K}(\partial_t)\mu(t) = f(t), \quad t \geq 0, \tag{4.9}$$

for a parameter $\lambda \geq 0$. Provided that $1 + \lambda\widehat{K}(s) \neq 0$ for every s in the analyticity domain of $\widehat{K}$, the associativity of the (continuous) convolution implies

$$\mu = \tilde{K}(\partial_t)f, \tag{4.10}$$

with

$$\tilde{K}(s) := \frac{1}{1+\lambda\widehat{K}(s)}. \tag{4.11}$$

Thus, the application of the Convolution Quadrature to solve (4.9), this is

$$\mu_{\Delta t}(t) + \lambda\widehat{K}(\partial_{\Delta t})\mu_{\Delta t}(t) = f(t), \quad t \geq 0, \tag{4.12}$$

is equivalent to the evaluation of

$$\mu_{\Delta t} = \tilde{K}(\partial_{\Delta t})f. \tag{4.13}$$

The following Lemma provides a convergence estimate which is *uniform in* λ for the approximation of y by $y_{\Delta t}$, when $\widehat{K}(s) = \sqrt{s} - 1$. This result will be used in the proof of convergence for the fully discrete method in Section 4.3.

Lemma 4.1 *Let $\lambda \geq 0$, $\widehat{K}(s) = \sqrt{s} - 1$, and $f \in C^{r+1}([0,T])$ satisfy $f(0) = f'(0) = \cdots = f^{(r)}(0) = 0$ for $r > \max(p, q+1)$, where p and q are respectively the order and the stage order of an A-stable Runge-Kutta method. Let $\mu(t)$ be the solution of (4.9) at time $t \in [0,T]$ and $y_{\Delta t}$ the solution of (4.12).*

Then there exists $\overline{\Delta t} > 0$ and $C = C(\overline{\Delta t}, T, f)$ such that for $0 < \Delta t \leq \overline{\Delta t}$ and any $t \in [0,T]$ it holds

$$|\mu(t) - \mu_{\Delta t}(t)| \leq C\Delta t^{\min(p,q+1)} \int_0^t |f^{(r+1)}(\tau)|\, d\tau, \tag{4.14}$$

uniformly in $\lambda > 0$.

Proof The proof follows straightforwardly from (4.10) and [4, Theorem 3] by noticing that for every s with $\Re s > 1$

$$|\tilde{K}(s)| = \left|\frac{1}{1+\lambda(\sqrt{s}-1)}\right| \leq 1\ ,$$

uniformly in $\lambda > 0$. □

4.3 Time-Stepping Error Estimates

In this Section we analyze the time discretization error of the Convolution Quadrature applied to the semidiscrete problem (3.3),

$$c(\mathbf{E}_h, \mathbf{v}_h) + \widehat{K}(\partial_t) b(\mathbf{E}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)_0 \quad \forall \mathbf{v}_h \in V_h \ , \tag{4.15}$$

with $\widehat{K}(s) = \sqrt{s} - 1$.

It goes without saying that convolution quadrature commutes with diagonalization of the spatial operators as performed in Section 3. Thus, let $\{\mathbf{u}_{h,\ell}\}_{\ell=1}^M$ be the c-orthonormal basis of V_h defined in (3.11). By expanding $\mathbf{E}_h(t) = \sum_{\ell=1}^M \mu_\ell(t)\mathbf{u}_{h,\ell}$ and $f_\ell(t) := (\mathbf{f}(t), ,)_0 \, \mathbf{u}_{h,\ell}$ we can reduce (3.3) to the following system of integral equations

$$\mu_\ell(t) + \lambda_{h,\ell} \hat{K}(\partial_t) \mu_\ell(t) = f_\ell(t) \ , \quad \ell = 1, \ldots, M. \tag{4.16}$$

The time stepping error is derived from the above decomposition.

Theorem 4.1 *Let $\mathbf{E}_{h,\Delta t} := \sum_{\ell=1}^M \mu_{\ell,\Delta t} \, \mathbf{u}_{h,\ell}$ be the convolution quadrature approximation of the solution to* (3.3) *with $\mu_{\ell,\Delta t}$ according to* (4.13) *applied to* (4.16)*. Then*

$$\|\mathbf{E}_h - \mathbf{E}_{h,\Delta t}\|_{L^2(]0,T[,U)} = C\Delta t^{\,\min(p,q+1)} \|\mathbf{f}\|_{H^{r+1}(]0,T[,U')} \ , \tag{4.17}$$

with $C > 0$ independent of Δt, the discretization in space, and $\mathbf{f}$.

Proof By Lemma 4.1 the Convolution Quadrature approximation $\mu_{\ell,\Delta t}$ of μ_ℓ in (4.16) satisfies

$$|\mu_\ell(t) - \mu_{\ell,\Delta t}(t)| \le C\Delta t^{\min(p,q+1)} \int_0^t |f_\ell^{(r+1)}(\tau)| \, d\tau \quad \text{for } t \in [0,T] \ . \tag{4.18}$$

Note that, in particular, $C > 0$ does not depend on ℓ. Then we can estimate

$$\begin{aligned}
\|\mathbf{E}_h - \mathbf{E}_{h,\Delta t}\|^2_{L^2(]0,T[,U)} &= \int_0^T \|\mathbf{E}_h(s) - \mathbf{E}_{h,\Delta t}(s)\|_U^2 \, ds \\
&= \int_0^T \sum_{\ell=1}^M |\mu_\ell(s) - \mu_{\ell,\Delta t}(s)|^2 \, ds \\
&\overset{(4.18)}{\le} C^2 \Delta t^{2\min(p,q+1)} \int_0^T \sum_{\ell=1}^M \left(\int_0^s |f_l^{(r+1)}(\tau)| \, d\tau \right)^2 ds \\
&\le C^2 \Delta t^{2\min(p,q+1)} \int_0^T s \sum_{\ell=1}^M \int_0^s |f_\ell^{(r+1)}(\tau)|^2 \, d\tau \, ds \\
&\le C^2 \Delta t^{2\min(p,q+1)} \frac{T^2}{2} \|\mathbf{f}\|^2_{H^{r+1}(]0,T[,U')},
\end{aligned}$$

where the Cauchy–Schwarz inequality has been used in the fourth step. □

The convergence of the fully-discrete approximation of (2.9) is immediate from (3.17) and Theorem 4.1.

Theorem 4.2 *Let* $\mathbf{E}_{h,\Delta t}$ *be the fully-discrete approximation of* (2.9) *introduced in Theorem 4.1. Then*

$$\|\mathbf{E} - \mathbf{E}_{h,\Delta t}\|_{L^2(]0,T[,U)} \leq C_1 e^{\sigma T} h + C_2 \Delta t^{\min(p,q+1)} \|\mathbf{f}\|_{H^{r+1}(]0,T[,U)} .$$

5 Fast and Oblivious Convolution Quadrature

The effect of the fast implementation of the CQ amounts to an special quadrature approximation of the CQ weights, see [12]. The analysis of this error for the general case $\nu > 0$ in (4.3) is the subject of the next subsection.

5.1 Quadrature estimates for approximate CQ weights

Given K analytic in a sector (4.2) and satisfying (4.3) for some $\nu \in \mathbb{R}$, the convolution weights in (4.4) can be expressed as contour integrals in the complex plane as follows

$$\mathbf{W}_n = \frac{\Delta t}{2\pi i} \int_\gamma K(s) R(\Delta t\, s)^{n-1} (\mathbf{I} - \Delta t\, s\mathcal{Q})^{-1} \mathbb{1} \mathbf{b}^T (\mathbf{I} - \Delta t\, s\mathcal{Q})^{-1}\, ds,$$

for γ a contour beginning and ending in the left half of the complex plane [19], R the stability function of the Runge-Kutta method, $\mathbf{I}$ the identity matrix and $\mathbb{1} = (1, \dots, 1)^T$. If $\nu \geq 0$ in (4.3), this representation is valid for $n \geq n_0$, with n_0 big enough.

The choice and the parametrization of γ plays an important role in the numerical approximation of the $\mathbf{W}_n$. Following [12,19], we choose γ as the left branch of a hyperbola and parameterize by

$$\mathbb{R} \to \gamma : x \mapsto \gamma(x) := \mu\, (1 - \sin(\alpha + ix)) + \sigma\ , \tag{5.1}$$

for a certain parameter $\mu > 0$, which will depend on n, $0 < \alpha < \frac{\pi}{2} - \varphi$ and φ and σ from (4.2).

After parametrization, the convolution weights read

$$\mathbf{W}_n = \Delta t \int_{\mathbb{R}} \mathbf{G}_{\Delta t\,, n-1}(x)\; dx, \tag{5.2}$$

with

$$\begin{aligned} \mathbf{G}_{\Delta t\,,n}(x) = \frac{1}{2\pi i} & K(\gamma(x)) R(\Delta t\, \gamma(x))^n\, (\mathbf{I} - \Delta t\, \gamma(x)\mathcal{Q})^{-1} \mathbb{1} \\ & \cdot \mathbf{b}^T (\mathbf{I} - \Delta t\, \gamma(x)\mathcal{Q})^{-1}\, \gamma'(x)\ . \end{aligned} \tag{5.3}$$

The approximation of $\mathbf{W}_n$ is then carried out by means of the composite trapezoidal rule on $2N_Q - 1$ intervals of size $\tau > 0$ applied to (5.2). An essential feature of the quadrature approximation in order to achieve an oblivious algorithm is the possibility of using the same contour γ for different values of n, varying along geometrically growing intervals of the form $[B^{\ell-1}, B^{\ell}]$, for some prescribed ratio $B > 1$.

In order to analyze the error, we consider the class $S(D_d, \mathbb{R})$ of analytic functions $G: D_d \to \mathbb{R}$ defined on the horizontal strip

$$D_d := \{s \in \mathbb{C} |\ |\mathrm{Im}(s)| \le d\} ,$$

which satisfy the following two conditions

$$\int_{-d}^{d} |G(x+iy)| dy \to 0, \qquad \text{as } |x| \to \infty , \tag{5.4}$$

$$N(G, D_d) := \int_{\mathbb{R}} \{|G(x+id)| + |G(x-id)|\}\ dx < \infty . \tag{5.5}$$

For $G \in S(D_d, \mathbb{R})$ we denote the quadrature error due to the composite trapezoidal rule

$$E_{\tau,N_Q}(G) := \int_{\mathbb{R}} G(x)\ dx - \tau \sum_{k=-N_Q}^{N_Q} G(k\tau) . \tag{5.6}$$

Theorem 5.1 *Assume that $G \in S(D_d, \mathbb{R})$ for some $d > 0$, and that there exist C, $a > 0$, $\theta \in (0, 1)$ and $n \ge 1$ such that*

$$|G(x)| \le C \left(1 + \frac{a}{n} \cosh x\right)^{-\theta n}, \quad x \in \mathbb{R}. \tag{5.7}$$

Then, for $\tau > 0$, $N_Q \ge 1$, there holds

$$\begin{aligned} |E_{\tau,N_Q}(G)| \le{}& \frac{N(G, D_d)}{e^{2\pi d/\tau} - 1} \\ &+ C \left(\phi(a\theta)\, e^{-a\theta \cosh(N_Q \tau)/2} + \left(1 + \frac{a}{n} \cosh(N_Q \tau)\right)^{-(\theta n - 1)} \right), \end{aligned}$$

with $\phi(a) = 2 + |\log(1 - e^{-a/2})|$.

Proof The proof follows the one of [12, Theorem 2] and is based on Lemma A.1, which is a modified version of [12, Lemma 2]. □

For the estimate of $E_{\tau,N_Q}(G_{\Delta t,n})$ we set

$$t = n\Delta t\, ,\ a_0 = 2 + \frac{4-\theta}{2} b,\ a_1 = 2 + 2b,\ a_2 = \frac{\theta}{2} b. \tag{5.8}$$

Theorem 5.2 *For $\frac{1}{\theta} \le b\mu t \le \frac{n}{2}$ the quadrature error (5.6) for $G_{\Delta t,n}$ of (5.3) satisfies*

$$\begin{aligned} |E_{\tau,N_Q}(G_{\Delta t,n})| \le C\mu^{1+\nu} \Big(& \frac{e^{a_0 \mu t}}{e^{2\pi d/\tau} - 1} + e^{(a_1 - a_2 \cosh(N_Q \tau))\mu t} \\ & + e^{a_1 \mu t} \left(1 + \frac{b\mu t}{n} \cosh(N_Q \tau)\right)^{-(\theta n - 1))} \Big), \end{aligned} \tag{5.9}$$

where ν is the exponent of (4.3).

The proof of this theorem follows the one of [12, Theorem 3] and can be found in the Appendix.

Now, given a target accuracy ε and assuming that $t = n\Delta t \in [B^{\ell-1}\Delta t\,, B^{\ell}\Delta t\,]$, for some $B > 1$, we select $\tilde{\varepsilon}$ small enough so that

$$\left(\frac{1}{B^{\ell-1}\Delta t}\log\left(\frac{1}{\tilde{\varepsilon}}\right)\right)^{1+\nu}\tilde{\varepsilon} < \varepsilon.$$

The same arguments as in [12] show that the bracket in (5.9) is smaller than $\tilde{\varepsilon}$ if the following asymptotic proportionalities hold

$$\begin{aligned} &\frac{1}{\tau} \sim \mu t + \log\left(\frac{1}{\tilde{\varepsilon}}\right), \qquad && \frac{c_1}{B}\log\left(\frac{1}{\tilde{\varepsilon}}\right) \le \mu t \le c_1 \log\left(\frac{1}{\tilde{\varepsilon}}\right), \\ &N_Q \sim \log\left(\frac{1}{\tilde{\varepsilon}}\right), && n \ge c\log\left(\frac{1}{\tilde{\varepsilon}}\right), \end{aligned}$$

for an arbitrary constant c_1 and c big enough. These relations imply

$$\mu \sim t^{-1}\log\left(\frac{1}{\tilde{\varepsilon}}\right),$$

which justifies the choice of $\tilde{\varepsilon}$.

Unfortunately, the constants involved in the asymptotic proportionalities above are difficult to quantify explicitly. For practical computations, the same choice of parameters proposed in [13] for the inversion of the Laplace transform $K(s)$ at time $t \in [B^{\ell-1}\Delta t\,, B^{\ell}\Delta t\,]$ can be used. The exponential rate of convergence with respect to N_Q and the effect of increasing the ratio B can be observed in Figure 5.1, where we show the error in the approximation of the convolution weights associated to $\hat{K}(s) = \sqrt{s} - 1$. The error is measured with respect to a reference solution computed by using the same method with $B = 10$ and $N_Q = 150$.

5.2 Analysis of the error introduced by the FOCQ

The application of the Fast and Oblivious Algorithm for the Convolution Quadrature (FOCQ) introduces another source of error in the approximation of (3.3). In order to analyze this additional perturbation, again we use the spectral decomposition (4.16). Following the notation in (4.7) and Theorem 4.1, we recall that the $\mu_{\ell,\Delta t}(t)$ are the solutions of

$$\mu_{\ell,\Delta t}(t) + \lambda_{h,\ell}\hat{K}(\partial_{\Delta t})\mu_{\ell,\Delta t}(t) = f_\ell(t), \quad \ell = 1, \ldots, M\ .$$

The FOCQ essentially boils down to approximating the convolution weight matrices $\mathbf{W}_j$ of the CQ. Thus, temporarily, we adopt a matrix perspective and write

$$\underline{\boldsymbol{\mu}_\ell} := (\boldsymbol{\mu}_{\ell,0}, \ldots, \boldsymbol{\mu}_{\ell,N})^T \in \mathbb{R}^{mN}\ , \quad \boldsymbol{\mu}_{\ell,n} := (\mu_{\ell,\Delta t}(t_n + c_l\Delta t\,))_{l=1}^{m}\ .$$

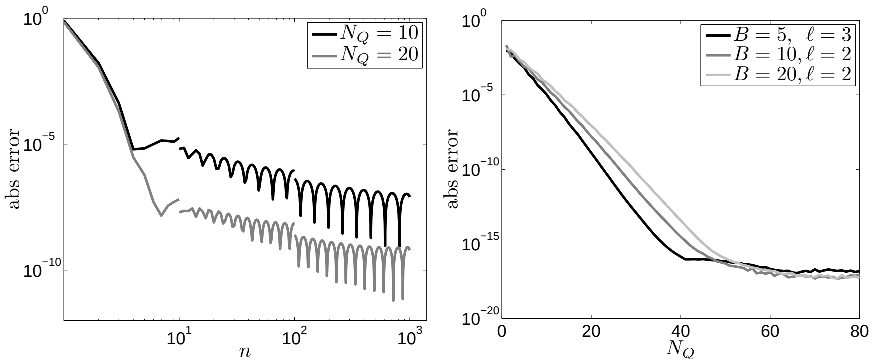

Fig. 5.1 *Left:* Approximation error in the Euclidean norm with respect to the index n of the convolution weight $\mathbf{W}_n$, for $B = 10$ and $\Delta t = 0.2$. *Right:* Maximum error in the Euclidean norm with respect to N_Q in the window of indices $[25, 125]$, for $B = 5$, $[25, 100]$, for $B = 10$, and $[25, 400]$, for $B = 20$.

Using (4.4), we find

$$\boldsymbol{\mu}_{\ell,n} + \lambda_{h,\ell} \left(\widehat{K}(\underline{\partial_{\Delta t}}) \underline{\boldsymbol{\mu}_\ell} \right)_n = \mathbf{f}_{\ell,n}, \quad \ell = 1, \ldots, M \ , \quad n = 0, \ldots, N \ , \tag{5.10}$$

for

$$\mathbf{f}_{\ell,n} := (f_\ell(t_n + c_1 \Delta t\,), \ldots, f_\ell(t_n + c_m \Delta t\,))^T \in \mathbb{R}^m \ .$$

Tagging the approximate convolution weight matrices with $\widetilde{\ }$, the FOCQ can be taken into account by replacing the convolution in (5.10) with

$$\left(\widehat{K}(\widetilde{\underline{\partial_{\Delta t}}}) \mathbf{g} \right)_n := \sum_{j=0}^{n} \widetilde{\mathbf{W}}_{n-j} \mathbf{g}_j \ , \tag{5.11}$$

for $\widetilde{\mathbf{W}}_j$ the perturbed convolution weights, which are supposed to satisfy

$$\|\mathbf{W}_j - \widetilde{\mathbf{W}}_j\| \leq \varepsilon, \qquad \forall j = 0, \ldots, N \ , \tag{5.12}$$

for some target accuracy ε, which is essentially given by the estimate of Theorem 5.2. Consequently, we denote by $\widetilde{\boldsymbol{\mu}_{\boldsymbol{\ell},\boldsymbol{n}}}$ the solution of

$$\widetilde{\boldsymbol{\mu}_{\boldsymbol{\ell},\boldsymbol{n}}} + \lambda_{h,\ell} \left(\widehat{K}(\widetilde{\underline{\partial_{\Delta t}}}) \widetilde{\underline{\boldsymbol{\mu}_\ell}} \right)_n = \mathbf{f}_{\ell,n}, \quad \ell = 1, \ldots, M, \tag{5.13}$$

Next, we reformulate (5.10) and (5.13) as linear systems of size $m \times (N + 1)$ as follows. As above, we define the 'super' (column) vectors

$$\underline{\boldsymbol{\mu}_\ell} = (\boldsymbol{\mu}_{\boldsymbol{\ell},\boldsymbol{n}})_{n=0}^N, \qquad \widetilde{\underline{\boldsymbol{\mu}_\ell}} = (\widetilde{\boldsymbol{\mu}_{\boldsymbol{\ell},\boldsymbol{n}}})_{n=0}^N, \qquad \underline{\mathbf{f}_\ell} = (\mathbf{f}_{\ell,n})_{n=0}^N,$$

and the block matrices

$$\boldsymbol{\mathcal{M}} := \begin{pmatrix} \mathbf{W}_0 & 0 & \cdots & 0 \\ \mathbf{W}_1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ \mathbf{W}_N & \cdots & \mathbf{W}_1 & \mathbf{W}_0 \end{pmatrix}, \qquad \widetilde{\boldsymbol{\mathcal{M}}} := \begin{pmatrix} \widetilde{\mathbf{W}}_0 & 0 & \cdots & 0 \\ \widetilde{\mathbf{W}}_1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ \widetilde{\mathbf{W}}_N & \cdots & \widetilde{\mathbf{W}}_1 & \widetilde{\mathbf{W}}_0 \end{pmatrix}.$$

With this notation, (5.10) and (5.13) become

$$\boldsymbol{\mathcal{A}}_{h,\ell}\underline{\boldsymbol{\mu}_\ell} = \underline{\mathbf{f}_\ell} \quad \text{and} \quad \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}\underline{\widetilde{\boldsymbol{\mu}_\ell}} = \underline{\mathbf{f}_\ell}$$

with

$$\boldsymbol{\mathcal{A}}_{h,\ell} = (\boldsymbol{\mathcal{I}} + \lambda_{h,\ell}\boldsymbol{\mathcal{M}}), \qquad \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell} = (\boldsymbol{\mathcal{I}} + \lambda_{h,\ell}\widetilde{\boldsymbol{\mathcal{M}}}),$$

for $\boldsymbol{\mathcal{I}}$ the identity matrix of size $m(N+1)$. By elementary linear algebra we obtain

$$\boldsymbol{\mathcal{A}}_{h,\ell}^{-1} = \begin{pmatrix} \mathbf{W}_0^* & 0 & \cdots & 0 \\ \mathbf{W}_1^* & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ \mathbf{W}_N^* & \cdots & \mathbf{W}_1^* & \mathbf{W}_0^* \end{pmatrix}, \quad \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}^{-1} = \begin{pmatrix} \widetilde{\mathbf{W}}_0^* & 0 & \cdots & 0 \\ \widetilde{\mathbf{W}}_1^* & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ \widetilde{\mathbf{W}}_N^* & \cdots & \widetilde{\mathbf{W}}_1^* & \widetilde{\mathbf{W}}_0^* \end{pmatrix}, \qquad (5.14)$$

for certain blocks $\mathbf{W}_j^*$ and $\widetilde{\mathbf{W}}_j^*$, $j = 0, \ldots, N$. Then the expansion coefficients of the final FOCQ solution are defined according to

$$\widetilde{\mu}_{\ell,\Delta t}(t) := \sum_{j=0}^{\infty} \widetilde{\boldsymbol{\omega}}_j^* (f(t - t_j + c_l \Delta t))_{l=1}^m, \quad 0 \le t \le T, \qquad (5.15)$$

for $\widetilde{\boldsymbol{\omega}}_j^*$ the last row of $\widetilde{\mathbf{W}}_j^*$, cf. (4.7). These coefficients are to be compared with

$$\mu_{\ell,\Delta t}(t) := \sum_{j=0}^{\infty} \boldsymbol{\omega}_j^* (f(t - t_j + c_l \Delta t))_{l=1}^m, \quad 0 \le t \le T, \qquad (5.16)$$

for $\boldsymbol{\omega}_j^*$ the last row of $\mathbf{W}_j^*$.

Lemma 5.1 *Let $\mu_{\ell,\Delta t}$, $\ell = 1, \ldots, M$ be the approximation of the coefficient μ_ℓ in* (4.16) *by the CQ. Let $\widetilde{\mu}_{\ell,\Delta t}$ be the result of computing the $\mu_{\ell,\Delta t}$ by means of the perturbed CQ satisfying* (5.12)*. Then there exists $C > 0$, depending on the Runge–Kutta method but not on ℓ and any discretization parameter, such that*

$$\|\mu_{\ell,\Delta t} - \widetilde{\mu}_{\ell,\Delta t}\|_{L^2(]0,T[)} \le \frac{mC^2N^3e^{2\sigma T}\varepsilon}{1 - CN^2e^{\sigma T}\varepsilon}\|f_\ell\|_{L^2(]0,T[)}. \qquad (5.17)$$

Proof From (5.15) and (5.16) it follows, for $0 \le t \le T$,

$$
\begin{aligned}
\|\mu_{\ell,\Delta t} - \widetilde{\mu}_{\ell,\Delta t}\|_{L^2(]0,T[)} &\le \sum_{j=0}^{\infty} \left\| \left(\boldsymbol{\omega}_j^* - \widetilde{\boldsymbol{\omega}}_j^*\right) \left(f_\ell(\cdot - t_j + c_l \Delta t)\right)_{l=1}^m \right\|_{L^2(]0,T[)} \\
&\le \max_{0\le j\le N} \|(f_\ell(\cdot - t_j + c_l\Delta t))_{l=1}^m\|_{L^2(]0,T[)} \sum_{j=0}^{N} \left\|\boldsymbol{\omega}_j^* - \widetilde{\boldsymbol{\omega}}_j^*\right\|_\infty \\
&\le m\|f_\ell\|_{L^2(]0,T[)} \sum_{j=0}^{N} \|\mathbf{W}_j^* - \widetilde{\mathbf{W}}_j^*\|_\infty \\
&= m\|f_\ell\|_{L^2(]0,T[)} \|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}^{-1}\|_{\infty\to\infty}.
\end{aligned}
$$

We write

$$
\begin{aligned}
\|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}^{-1}\|_{\infty\to\infty} &= \|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{A}}_{h,\ell}\widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}^{-1})\|_{\infty\to\infty} \\
&\le \|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}\|_{\infty\to\infty} \, \|\boldsymbol{\mathcal{I}} - (\boldsymbol{\mathcal{I}} - (\boldsymbol{\mathcal{A}}_{h,\ell} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell})\boldsymbol{\mathcal{A}}_{h,\ell}^{-1})^{-1}\|_{\infty\to\infty} \\
&= \|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}\|_{\infty\to\infty} \sum_{j=1}^{\infty} \left\|(\boldsymbol{\mathcal{A}}_{h,\ell} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell})\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}\right\|_{\infty\to\infty}^{j} \\
&\le \frac{\|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}\|_{\infty\to\infty}^2 \, \|\boldsymbol{\mathcal{A}}_{h,\ell} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}\|_{\infty\to\infty}}{1 - \|\boldsymbol{\mathcal{A}}_{h,\ell} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}\|_{\infty\to\infty} \, \|\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}\|_{\infty\to\infty}}.
\end{aligned}
$$

From (5.12) it follows

$$
\|\boldsymbol{\mathcal{A}}_{h,\ell} - \widetilde{\boldsymbol{\mathcal{A}}}_{h,\ell}\|_{\infty\to\infty} \le \lambda_{\ell,h} N\varepsilon \le N\varepsilon,
$$

since the eigenvalues $\lambda_{\ell,h}$ are all in $[0,1]$, see the remark after (3.11).

In order to estimate the norm of $\boldsymbol{\mathcal{A}}_{h,\ell}^{-1}$, we use the associativity of the CQ at the stage level. This property implies that the blocks $\mathbf{W}_j^*$ in (5.14) are the convolution weights associated to kernel with Laplace transform

$$
K^*(s) = \frac{1}{1 + \lambda_{h,\ell}\widehat{K}(s)} \ ,
$$

which satisfies

$$
|K^*(s)| \le 1, \quad \forall \lambda_{h,\ell} > 0, \ \operatorname{Re} s > 1 \ .
$$

On the one hand, by [3, Lemma 5.2] we have that for every $\sigma > 1$ there exists $\Delta t_0 > 0$ and a constant C, depending on the Runge–Kuta method, such that

$$
\sup_{|\zeta| \le e^{-\Delta t\, \sigma}} \left\| K^*\left(\frac{\boldsymbol{\Delta}(\zeta)}{\Delta t}\right)\right\| \le C \quad \text{for} \quad \Delta t < \Delta t_0 \ .
$$

On the other hand, for $0 < \rho < 1$, the convolution weights can be written as the Cauchy integrals

$$
\mathbf{W}_j^* = \frac{1}{2\pi i} \int_{|\zeta|=\rho} \zeta^{-1-j} K^*\left(\frac{\boldsymbol{\Delta}(\zeta)}{\Delta t}\right) d\zeta. \tag{5.18}
$$

Then, by taking $\rho = e^{-\Delta t\,\sigma}$, we deduce the bound

$$\|\mathbf{W}_j^*\| \leq Ce^{j\Delta t\,\sigma} \leq Ce^{\sigma T}.$$

Now we can estimate, for every $n = 0, \dots, N$,

$$\|\boldsymbol{\mu}_{\boldsymbol{\ell},\boldsymbol{n}}\|_\infty = \left\| \sum_{j=0}^{n} \mathbf{W}_{n-j}^* \mathbf{f}_{\ell,j} \right\|_\infty \leq C \sum_{j=0}^{n} e^{(n-j)\Delta t\,\sigma} \|\mathbf{f}_{\ell,j}\|_\infty \leq C(N+1)e^{\sigma T}\|\underline{\mathbf{f}_\ell}\|_\infty .$$

and thus

$$\|\underline{\boldsymbol{\mu}_\ell}\|_\infty \leq C(N+1)e^{\sigma T}\|\underline{\mathbf{f}_\ell}\|_\infty, \tag{5.19}$$

so that

$$\|\boldsymbol{\mathcal{A}}_{\Delta t\,,\ell}^{-1}\|_{\infty\to\infty} \leq C(N+1)e^{\sigma T}$$

and (5.17) follows. □

Theorem 5.3 *Adopting the notations from Theorem 4.1, we can bound the impact of the fast and oblivious implementation of our CQ methods by*

$$\|\mathbf{E}_{h,\Delta t} - \widetilde{\mathbf{E}}_{h,\Delta t}\|_{L^2(]0,T[,U)} \leq M \frac{C^2 N^3 e^{2\sigma T}\varepsilon}{1 - CN^2 e^{\sigma T}\varepsilon} \|\mathbf{f}\|_{L^2(]0,T[,U')} \,, \tag{5.20}$$

for $\widetilde{\mathbf{E}}_{h,\Delta t} = \sum_{\ell=1}^{M} \widetilde{\mu}_{\ell,\Delta t}\,\mathbf{u}_{h,\ell}$. *The constant* $C > 0$ *depends only on the underlying Runge-Kutta method.*

6 Numerical Experiments

For numerical tests we restrict ourselves to the scalar TM eddy current model (1.7) in two dimensions. As trial spaces U_h we use spaces of piecewise linear continuous functions on triangular meshes on $\widetilde{D}$. Writing $M := \dim U_h$ we end up with the Fredholm integral equation

$$\mathbf{A}\cdot\boldsymbol{\mu}(t) + \mathbf{B}\cdot K(\partial_t)\boldsymbol{\mu}(t) = \mathbf{f}(t) \qquad \text{for } t \in]0,T] \,, \tag{6.1}$$

where the vector $\boldsymbol{\mu}(t) \in \mathbb{R}^M$ contains the time-dependent coefficients of an approximation in space of u with respect to the standard nodal basis, $\mathbf{A}$ and $\mathbf{B}$ are the Galerkin matrices associated with the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ from (2.11) and $\mathbf{f}(t)$ is the load vector associated with the source function f. Note that in the 2-dimensional case the matrix $\mathbf{A}$ is symmetric positive definite. Thus, we need not introduce the modified convolution kernel $\hat{K}$, cf (2.10).

We discretize the convolution in (6.1) by means of Runge-Kutta CQ as explained in Section 4.1. As we have seen there, the convolution quadrature algorithm provides an approximation of the convolution simultaneously at Runge-Kutta internal times. To write the time discretization of equation of (6.1) we thus rely on vectors

$$\tilde{\boldsymbol{\mu}}_i \approx (\boldsymbol{\mu}(i\Delta t + c_1\Delta t), \cdots, \boldsymbol{\mu}(i\Delta t + c_m\Delta t))^T \in \mathbb{R}^{mM} \,,$$

which contain approximations of the spatially semi-discrete solution at Runge-Kutta internal times. The fully discrete approximation of (1.7) can be then computed by successively solving

$$(\mathbf{I}_m \otimes \mathbf{A}) \cdot \tilde{\boldsymbol{\mu}}_i + (\mathbf{I}_m \otimes \mathbf{B}) \sum_{j=0}^{i} (\mathbf{W}_{i-j} \otimes \mathbf{I}_M))\, \tilde{\boldsymbol{\mu}}_j = \tilde{\mathbf{f}}_i \tag{6.2}$$

for $i = 0, .., (T - \Delta t)/\Delta t$, where $\otimes$ is the Kronecker product, $\mathbf{I}_n$ is an $n \times n$ identity matrix, and $\tilde{\mathbf{f}}_i = (\mathbf{f}(i\Delta t + c_1 \Delta t), \dots, \mathbf{f}(i\Delta t + c_m \Delta t))^T$. Regarding the first 20 terms of the sum in (6.2), we approximate the convolution weights $\mathbf{W}_i$ as in [16, Section 2] and compute the sum classically. For the rest we exploit the FOCQ approximation along suitable hyperbolae with the range parameter $B = 10$ and compute the sum efficiently with the FOCQ, see Section 5. The contour parameter are chosen accordingly to [13, Section 4]. Already for moderate numbers of quadrature points the FOCQ introduces a negligible error in the approximation, as confirmed by the experiments (see Figure 6.2).

In our numerical tests[2] we choose $\widetilde{D}$ to be an annulus around the origin with radii 0.5 and 2 and we include the source function by imposing the Dirichlet boundary condition $g(x, y, t) := \frac{32}{105\sqrt{\pi}} t^{7/2} + \frac{t^3}{6} \log(4)$ on $\partial\widetilde{\Omega}$. The analytical solution is then

$$u(x, y, t) := \frac{32}{105\sqrt{\pi}} t^{7/2} + \frac{t^3}{6} \left(\frac{1}{2} \log(x^2 + y^2) + \log(2) \right).$$

A first numerical test is performed by choosing the fast convolution quadrature based on the implicit Euler method, which is the 1-step RadauIIA method (FOCQ of order 1). For 6 different spatial grids and 12 different time steps we measured the time-discrete $\ell^2_{\Delta t}([0, 4], H^1(\widetilde{D}))$-error as well as the $L^2(\widetilde{D})$-error in space[3] at a fixed time $t = 4$. The spatial triangular meshes have been created through uniform refinement while the timesteps by repetitively halving an initial timestep.

The expected linear convergence both in time and space in the $\ell^2_{\Delta t}([0, 4], H^1(\widetilde{D}))$-norm is observed in Figure 6.1. The rates of algebraic convergence become more conspicuous when we examine the $L^2(\widetilde{D})$-norm in space at a fixed time, where we have quadratic convergence in space; see Figure 6.1.

The impact of the FOCQ on the algorithm is investigated in Figure 6.2. We compute the error in the $\ell^2_{\Delta t}([0, 4], H^1(\widetilde{D}))$-norm and in the $L^2(\widetilde{D})$-norm in space at a fixed time for the fourth finest spatial grid and the timestep $\Delta t = 2^{-8}$ (see the dots in Figure 6.1). In both cases few quadrature nodes on the contours are enough to render the perturbation due to the FOCQ approximation of the convolution weights $\mathbf{W}_i$ negligible.

We perform a second numerical test and this time the convolution is approximated by using the FOCQ based on the 2-stage RadauIIA method

[2] The experiments are perfomed in MATLAB and are based on the library LehrFEM developed at the ETHZ.

[3] Both the $H^1(\widetilde{D})$- and the $L^2(\widetilde{D})$-norm are computed approximately with 7 point quadrature rules on triangles.

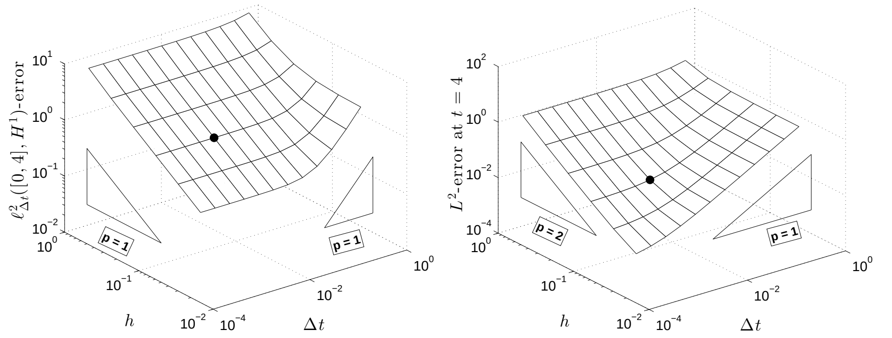

Fig. 6.1 Error in the $\ell^2_{\Delta t}([0,4], H^1(\widetilde{D}))$-norm (*left*) and in the $L^2(\widetilde{D})$ at a fixed time $t = 4$ (*right*) for the coupling of FEM and FOCQ base on the implicit Euler method. The two dots denote the spatial mesh and timestep used in Figure 6.2.

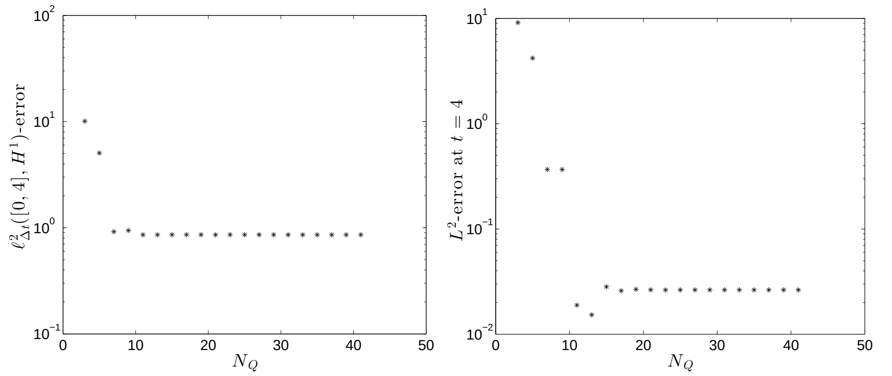

Fig. 6.2 Impact of FOCQ on the total error in the $\ell^2_{\Delta t}([0,4], H^1(\widetilde{D}))$-norm (*left*) and in the $L^2(\widetilde{D})$ at a fixed time $t = 4$ (*right*) for the coupling of FEM and FCQ base on the implicit Euler method.

(FOCQ of order 3). Again we measure both the $\ell^2_{\Delta t}([0,4], H^1(\widetilde{D}))$-error and the $L^2(\widetilde{D})$-error in space at a fixed time $t = 4$ for several meshes and timesteps. We expect that the convolution quadrature error contributes to the total error with a cubic algebraic rate in Δt. This is only partially confirmed by the experiment because the total error is almost always dominated by the discretization error in space, as we can see in Figure 6.3.

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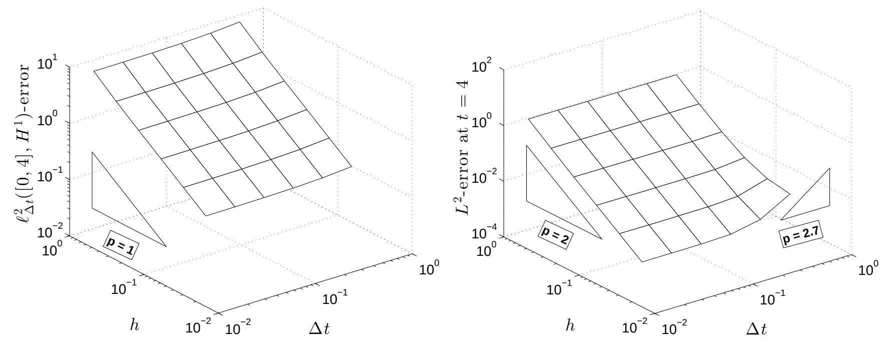

Fig. 6.3 Error in the $\ell^2_{\Delta t}\,([0,4], H^1(\widetilde{D}))$-norm (*left*) and in the $L^2(\widetilde{D})$ at a fixed time $t = 4$ (*right*) for the coupling of FEM and FOCQ based on the 2-stage RadauIIA method.

A Proof of Theorem 5.2

Proof The integrand $G_{\Delta t\,,n}$ is analytic in the strip D_d. Moreover, by [12, Lemma 1] it holds

$$|G_{\Delta t\,,n}(x+iy)| \le \frac{C\mu^{1+\nu}e^{2\mu\Delta t\, n}}{(1-b\Delta t\,\mu)^n}\frac{\cosh(x)^{1-\nu}}{(1+b\Delta t\mu\cosh(x))^n},$$

for some $b,\mu > 0$. It is easy to show that for any $\nu \in \mathbb{R}$, $b > 0$ and $\theta \in (0,1)$ there exist $x_0 \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that

$$x^{1-\nu}\left(1+\frac{bx}{n}\right)^{-(1-\theta)n} < C(n_0,x_0), \qquad \text{for} \quad x > x_0 \quad \text{and} \quad n > n_0.$$

Then we can estimate

$$|G_{\Delta t\,,n}(x+iy)| \le \frac{C\mu^{1+\nu}e^{2\mu\Delta t\, n}}{(1-b\Delta t\,\mu)^n}\frac{1}{(1+b\Delta t\mu\cosh(x))^{\theta n}}, \tag{A.1}$$

for $\theta \in (0,1)$, x and n big enough. The bound (A.1) implies that $G_{\Delta t\,,n}$ satisfies (5.4) and (5.7).

We now estimate $N(G_{\Delta t\,,n}, D_d)$. By Lemma A.1 below, it is

$$N(G_{\Delta t\,,n}, D_d) \le \frac{C\mu^{1+\nu}e^{2\mu t}}{(1-b\mu t/n)^n}\left(\phi\left(b\mu t\theta\right)e^{-b\mu t\theta/2} + \left(1+\frac{b\mu t}{n}\right)^{-(\theta n-1)}\right).$$

Since for $0 \le bt\mu \le n/2$ it holds

$$(1-b\mu t/n)^{-n} \le e^{2bt\mu}, \tag{A.2}$$

$$(1+b\mu t/n)^{-(\theta n-1)} \le \frac{3}{2}e^{-b\mu t\theta/2},$$

and $\phi(x) \le 3$ for $x \ge 1$, it follows

$$N(G_{\Delta t\,,n}, D_d) \le C\mu^{1+\nu}e^{\mu t(2+(4-\theta)b/2)}.$$

Then Theorem 5.1 and inequality (A.2) give the result, with the notation (5.8). $\square$

The technical lemma below is a modified version of [12, Lemma 2].

Lemma A.1 *For $R \geq 0$, $a > 0$, $\theta \in (0,1)$ and $n \geq 1$ there holds*

$$\int_R^{+\infty} \Big(1 + \frac{a}{n}\cosh(x)\Big)^{-\theta n} dx \leq \phi\,(a\theta)\, e^{-a\theta\cosh(R)/2} + \Big(1 + \frac{a}{n}\cosh(R)\Big)^{-(\theta n - 1)}.$$

Proof With the change of variables $u = \cosh(x)$ we have

$$\int_R^{+\infty} \Big(1 + \frac{a}{n}\cosh(x)\Big)^{-\theta n} dx = \int_{\cosh(R)}^{+\infty} \Big(1 + \frac{a}{n}u\Big)^{-\theta n} \frac{du}{\sqrt{u^2-1}}$$

With the following inequality, which holds for $0 \leq y \leq n$,

$$\Big(1 + \frac{y}{n}\Big)^{-n} \leq e^{-y/2},$$

we can choose $\beta = \max\{\cosh(R), \frac{n}{a}\}$ and use [11, Lemma 1] to bound

$$\begin{aligned}\int_{\cosh(R)}^{\beta} \Big(1 + \frac{a}{n}u\Big)^{-\theta n} \frac{du}{\sqrt{u^2-1}} &\leq \int_{\cosh(R)}^{\beta} e^{-au\theta/2} \frac{du}{\sqrt{u^2-1}} \\ &\leq \int_R^{+\infty} e^{-a\theta\cosh(x)/2} dx \leq \phi\,(a\theta)\, e^{-a\theta\cosh(R)/2}.\end{aligned}$$

Since

$$\Big(1 + \frac{a}{n}u\Big)^{-1} \leq \Big(1 + \frac{a}{n}\cosh(R)\Big)^{-1} \qquad \text{for } u \geq \cosh(R),$$

we have

$$\begin{aligned}\int_\beta^{+\infty} \Big(1 + \frac{a}{n}u\Big)^{-\theta n} \frac{du}{\sqrt{u^2-1}} &\leq \Big(1 + \frac{a}{n}\cosh(R)\Big)^{-(\theta n - 1)} \int_\beta^{+\infty} \Big(1 + \frac{a}{n}u\Big)^{-1} \frac{du}{\sqrt{u^2-1}} \\ &\leq \Big(1 + \frac{a}{n}\cosh(R)\Big)^{-(\theta n - 1)}\end{aligned}$$

because the integral on the right hand side is bounded by 1, see [12, Lemma 2]. □

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