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Intrinsic Fault Tolerance of Multilevel Monte Carlo Methods

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Abstract

Monte Carlo (MC) and multilevel Monte Carlo (MLMC) methods applied to solvers for Partial Differential Equations with random input data are proved to exhibit intrinsic failure resilience. Sufficient conditions are provided for non-recoverable loss of a random fraction of MC samples not to fatally damage the asymptotic accuracy vs. work of a MC simulation. Specifically, the convergence behavior of MLMC methods on massively parallel hardware with runtime faults is analyzed mathematically and investigated computationally. Our mathematical model assumes node failures which occur uncorrelated of MC sampling and with general sample failure statistics on the different levels and which also assume absence of checkpointing, i.e., we assume irrecoverable sample failures with complete loss of data. Modifications of the MLMC with enhanced resilience are proposed. The theoretical results are obtained under general statistical models of CPU failure at runtime. Particular attention is paid to node failures with so-called Weibull failure models. We discuss the resilience of massively parallel stochastic Finite Volume computational fluid dynamics simulations.

Keywords: Multilevel Monte Carlo, fault tolerance, failure resilience, exascale parallel computing

1. Introduction

Monte Carlo (MC) methods estimate statistical moments of random variables (such as means or so-called "ensemble averages") by sample averages [6]. The goal can, for instance, be to determine the expected solution of a partial differential equation (PDE) with random initial or boundary conditions that follow some statistical law [12–14]. Then, each sample is the solution of the PDE for a random input (such as, in the context of hyperbolic systems of conservation laws, a particular initial/boundary condition). The statistical independence of the input data makes it possible to execute the simulations corresponding to each sample in parallel. The slow convergence of Monte Carlo methods ($M^{-1/2}$ for M draws of input data) entails large numbers of samples. This, in turn, implies good parallel scalability of MC methods to large numbers of processors. Mostly, the simulations take a similar amount of time such that a distribution among large numbers of processors with a balanced load is achieved quite easily. In a parallel setting a serious problem is to guarantee the statistical independence of the random input draws (see, e.g., [22]).

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Multilevel Monte Carlo (MLMC) methods were recently proposed in [7, 12] in order to improve the accuracy versus work. They can be used for efficient numerical simulations of stochastic ordinary or partial differential equations. Unlike MC methods where samples are only computed on one discretization, MLMC methods use a hierarchy of discretization levels hence computations are done on many different discretizations. Based on the expected solution computed on the coarsest discretization level, the expected difference from this level to the next finer one is added, until the finest discretization level is reached. In MLMC methods the expected difference from two consecutive discretization levels is estimated using the MC method. Hence, in this paper the difference of a realization computed on two consecutive discretization levels is referred to as a MLMC sample of a certain "level". Throughout this paper, a "level" does, therefore, not denote a discretization level but a level related to a MLMC sample. Note that the discretization levels need not be related to a grid hierarchy. MLMC can also be applied, e.g., to mesh-free Feynman–Kac problems [19, 21]. It is by now known (see, e.g., [2, 7, 12–14, 19]) that under rather general assumptions, MLMC methods converge faster than MC, in terms of overall computational work, i.e., cumulative execution time. The cumulative execution time and memory consumption of the computation of samples depend on the levels and differ considerably: the computation of a MLMC sample of a 'finer' level may require much more compute resources (execution time, memory space, number of cores) than of a sample of a 'coarser' level. The load of a MLMC simulation is therefore not as easy to balance [28] as in MC, as there are only few samples on the fine levels. Nevertheless, in the context of partial differential equations with random inputs, the MLMC method allows the approximation of ensemble averages of the solution with a complexity analogous to that necessary for one numerical solution of the deterministic problem on the finest mesh [2, 12].

The present study is based on the following assumptions: a) in large scale simulations on emerging, massively parallel computing platforms processor failures at runtime are inevitable [4, 5], and occur, in fact, with increasing frequency as the number of processors increases, respectively the quality of processors decreases; b) processor failures at runtime can, in general, not be predicted, but occur randomly and should therefore be modelled as stochastic processes; c) processor failures at runtime are *not checkpointed* and *not recoverable*; d) the algorithm of interest has *redundancy by design* in order to "survive" a certain number of (non-checkpointed) failure events with random arrivals.

Assumptions c) and d) exclude a large number of currently used standard algorithms, for which any loss of data entails abortion of execution. We mention only standard Gaussian Elimination with loss of one pivot element. Other algorithms may, however, tolerate partial loss of data at runtime. We think of iterative solvers of large, linear systems which may converge even if one or several iterates are "lost" due to hardware failures and so satisfy assumption d); interestingly, assumption b) implies that the *convergence results of deterministic algorithms for deterministic problems on random hardware will necessarily be probabilistic in nature*. Here, we consider the case when the algorithm under consideration is stochastic by design, such as Monte Carlo (MC) methods. As we argue in the present paper, MC methods, being *probabilistic in nature*, are intrinsically fault tolerant: as we prove, the loss of (a "subcritical" fraction of) information by failed samples does not render the whole simulation useless as is the case, e.g., in many matrix computations, such as Gaussian Elimination. MC samples which were lost due to node failures at runtime can be repeated since new, independent samples can be generated to replace the failed ones.

We provide a mathematical argument predicting that the convergence behavior of MC is not affected substantially if the failed samples are simply disregarded, provided there are "not too

many" (made precise in the mathematical analysis, and corroborated in the numerical experiments) of these failures.

Specifically, in the present paper we analyze the performance of both MC and MLMC PDE solvers in the presence of hardware failures at runtime. In particular, we investigate the convergence behavior of these methods if processors fail according to a stochastic failure model; the presently developed mathematical analysis accommodates rather general failure models. Naturally, to arrive at a theory which is amenable to rigorous mathematical treatment, a number of simplifying assumptions had to be made. In particular, in our analysis we do not distinguish among different reasons of processor failure. So, we do not distinguish between node, program, network, or any other type of failure. We assume that the complete MC sample is lost if one of the (maybe multiple) processors fails that are used for its simulation. We disregard all samples affected by a failure and compute the results with the 'surviving' ones.

While in MC all samples are from the (single) finest level (or grid), MLMC gets its statistics also from samples corresponding to coarser grids. (The resolution of the finest level is determined by the required discretization error.) By using information on multiple levels MLMC needs much fewer samples on the finest level than ordinary MC to attain the same quality of answer. MLMC turns out to be much more efficient than MC. To get the optimal MLMC convergence rate (with respect to work), it is crucial to choose properly the numbers M_ℓ of samples on level ℓ.

In the presence of failures without checkpointing MC samples on all levels might be irrevocably lost. The larger (in the sense that they are defined on finer meshes) samples of the finer levels are more vulnerable than the (larger number of) smaller samples on the coarser levels. With very high failure rates it might not be feasible that sufficiently many samples survive on the finer levels. In general, the error components of faulty levels increase and the overall convergence rate is reduced. With a sufficiently high failure rate all samples on a particular level may get lost. In this case, the attainable error is bounded from below by the discretization error of that level.

Our main mathematical results are as follows: we prove convergence of MC and MLMC for the first moment (sample average) provided that sufficiently many samples survive on average. We compute the effect of failures according to existing failure models. Numerical experiments of MLMC for hyperbolic PDE's coupled with the Weibull failure model validate our theory. We further investigate the failure resilience of two and three dimensional time-dependent grid applications, like finite elements, finite differences, or finite volumes. These results are obtained by MLMC simulations treating the sample sizes M_ℓ as random variables.

We also discuss FT issues regarding MPI. In the present standard MPI-3.0 [15], failure of a single MPI process is fatal for the entire MLMC simulation. For our approach to work, MPI would have to be extended by a mechanism to survive losses of MPI processes at runtime, and continue with the remaining ones. Based on this paper the proposed fault tolerant MLMC was implemented [20, 21] using the User Level Failure Mitigation (ULFM) [3], a fault tolerant version of MPI. We compare the theoretical error bound with the measured results from this implementation.

The paper is organized as follows: In Sections 2 and 3 we give error bounds for MC and MLMC, respectively, in the presence of a statistical loss of samples. In Section 4 we discuss the Weibull failure model. In Section 5 we conduct a number of numerical experiments to investigate how convergence is affected by failure. We consider two and three dimensional problems with different convergence rates of the PDE solver. We further discuss the practical applicability of our approach, and outline a possible procedure if the stochastic failure model is unknown.

2. MC with a random number of samples

We first introduce a fault tolerant MC (FT-MC) method. Starting from there the fault tolerant technique used in MLMC is derived.

We are interested in the expected value $\mathbb{E}[X]$ of a random variable (RV) X taking values in a Banach space B, on the probability space $(\Omega, \mathscr{A}, \mathbb{P})$, with sample space Ω, σ-algebra $\mathscr{A}$ and probability measure $\mathbb{P}$ [17]. If the 2nd moments of X exist the Monte Carlo method can be used to estimate $\mathbb{E}[X]$. Given a fixed number M of independent, identically distributed samples X^i, $i = 1, 2, \ldots, M$, the MC approximation of $\mathbb{E}[X]$ is defined by

$$E_M[X] := \frac{1}{M} \sum_{i=1}^{M} X^i. \tag{1}$$

The mean square error in the estimator (1) is known to be (see, e.g., [12, 27])

$$\|\mathbb{E}[X] - E_M[X]\|_{L^2(\Omega;B)} \leq M^{-1/2} \|X\|_{L^2(\Omega;B)} := M^{-1/2} \sqrt{\mathbb{E}[\|X\|_B^2]}. \tag{2}$$

The "embarrassingly parallel" evaluation of (1) across a possibly large number of nodes is a common approach to reduce wall clock time in MC simulations [22]. In the present work, we highlight and capitalize on a second feature of the MC estimator (1): software or hardware failures at runtime may cause nodes to fail or even crash, such that some samples get lost. We will give sufficient conditions on the node failure statistics to prove

1. *that it is reasonable to continue the MC simulation without checkpointing,*
2. *that no recovery of samples from failed nodes is required,*
3. *that the statistical quality of the MC simulation is unaffected provided "node failures at runtime do not occur too frequently", and*
4. *that there is a certain critical node failure intensity above which the MC simulation does deteriorate, almost surely.*

Let us now be more specific about the mathematical model from which these assertions are deduced.

In the presence of system failures at runtime, the sample size M in (1) is not a fixed number anymore, but becomes itself a random variable $\hat{M}$. We denote the probability space for the failures by $(\Omega', \mathscr{A}', \mathbb{P}')$ and the respective expectation by $\mathbb{E}'[\cdot]$. We assume throughout the paper that the node failures at runtime occur with statistics that are independent of the realizations of X. This implies that the runtime to compute a solution of a realization of X is independent of, e.g., the realization. In particular, we furthermore assume in the sequel that the *surviving* samples are statistically independent. The N-out-of-M strategy suggested by Li and Mascagni [10] has been designed in such a way that the random numbers generated by *any* N out of M random number streams *are* independent random numbers. The pseudo-random number generators in [11, 24] incorporate this strategy.

For fixed M the approximation $E_M[X]$ is a RV over Ω. For $\hat{M}$ considered as RV over Ω' this approximation becomes

$$E_{\hat{M}}[X] := \frac{1}{\hat{M}} \sum_{i=1}^{\hat{M}} X^i,$$

Therefore, it is a RV over the product probability space $(\Omega \times \Omega', \mathscr{A} \otimes \mathscr{A}', \mathbb{P} \otimes \mathbb{P}')$. Since $(\Omega, \mathscr{A}, \mathbb{P})$ and since $(\Omega', \mathscr{A}', \mathbb{P}')$ are finite measure spaces, there exists (see, e.g., [27] and the references there) an unique product probability measure $\hat{\mathbb{P}} = \mathbb{P} \otimes \mathbb{P}'$ on $\mathscr{A} \otimes \mathscr{A}'$ such that

$$\hat{\mathbb{P}}(A \otimes A') = \mathbb{P}(A)\mathbb{P}'(A'), \qquad \forall A \in \mathscr{A}, A' \in \mathscr{A}' .$$

Theorem 2.1. *The MC error estimate with a random number of samples is given by*

$$\|\mathbb{E}[X] - E_{\hat{M}}[X]\|^2_{L^2(\hat{\Omega};B;\hat{\mathbb{P}})} \leq \mathbb{E}'\left[\hat{M}^{-1/2}\right] \|X\|^2_{L^2(\Omega;B)} . \tag{3}$$

Proof. We substitute the RV $\hat{M}$ for M in equation (2) and integrate over $(\Omega', \mathscr{A}', \mathbb{P}')$ to obtain

$$\begin{aligned}\int_{\Omega'} \|\mathbb{E}[X] - E_{\hat{M}}[X]\|^2_{L^2(\Omega;B)} \mathbb{P}'(d\omega') &\leq \int_{\Omega'} \hat{M}^{-1/2} \|X\|^2_{L^2(\Omega;B)} \mathbb{P}'(d\omega') \\ &= \mathbb{E}'\left[\hat{M}^{-1/2}\right] \|X\|^2_{L^2(\Omega;B)} .\end{aligned}$$

Theorem I.3.17. in [26] then shows that

$$L^2(\Omega'; L^2(\Omega; B)) \cong L^2(\Omega \times \Omega'; B; \mathbb{P} \otimes \mathbb{P}'),$$

which leads to the claimed error estimate (3). □

Once a particular statistical distribution for the node failures has been adopted (and calibrated to the hardware platform of interest), the term $\mathbb{E}'[\hat{M}^{-1/2}]$ can be computed and hence using Theorem 2.1, the a priori error bound for the fault tolerant MC (FT-MC) method as well. The intuition behind Theorem 2.1 is that all failures are accounted for in the error bound of the FT-MC. The number of failures occurring in a FT-MC simulation varies according to a known failure distribution, nevertheless the a priori error bound in Theorem 2.1 remains valid, regardless of the number of failures. Knowing the failure distribution allows to compute an error bound before the number of failures and the number of surviving samples is known.

The condition $\hat{M}(\omega') = 0$ means that no samples remain. If the probability of this event is positive, then obviously $\mathbb{E}'[\hat{M}^{-1/2}]$ tends to infinity and the FT-MC error bound becomes meaningless. *We therefore assume that* $\mathbb{P}'(\{\hat{M} = 0\}) = 0$ *in the ensuing analysis.* In practice, this means that the MC estimation has to be restarted from scratch in the event that all samples are lost at runtime.

3. Abstract multilevel Monte Carlo with sample losses

3.1. Review of the multilevel Monte Carlo method

In this section we consider random variables X that are solutions of problems with random inputs that can be solved only approximately. Prominent examples are PDEs with random initial or boundary conditions as they arise in uncertainty quantification (UQ) in engineering applications. The solutions of these PDEs typically are obtained numerically by a discretization method like the finite element, finite difference, or finite volume method. We assume that we have available a hierarchy of discretizations that is indicated by a measure, e.g., the grid spacing.

In contrast to ordinary Monte Carlo methods, multilevel Monte Carlo (MLMC) methods can exploit this hierarchy of discretizations. If implemented properly, MLMC provides estimates $E(X)$ of higher accuracy than MC for the same amount of work measured in floating point operations.

A hierarchy of discretizations is given, for instance, if the computational domain is rectangular and is discretized by a hierarchy of regular grids with grid spacings (or "mesh widths")

$$h_0 > \cdots > h_L, \qquad h_i = 2h_{i+1}. \tag{4}$$

Such hierarchies are available to most simulations in engineering. Similar to the derivation of Giles [7], the difference between $X(\cdot,\omega)$ of the original problem and its discretization $X_h(\cdot,\omega)$ on the mesh of width h, i.e., the discretization error, is assumed to converge with order α

$$\|\mathbb{E}[X - X_{h_\ell}]\|_B = O(h_\ell^\alpha), \quad \alpha > 0, \tag{5}$$

where $\|\cdot\|_B$ denotes the norm in the Banach space B. We also assume a convergence order β for

$$\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)} = O(h_{\ell-1}^\beta), \quad \beta, \ell > 0, \tag{6}$$

and assume that

$$\|X_{h_0}\|_{L^2(\Omega;B)} = O(1). \tag{7}$$

Equality (6) is a consequence of the bounded variation of X and of the consistency of the discretization scheme, given the almost sure regularity of the sample paths. Furthermore it is assumed that the work needed to compute X_{h_ℓ} satisfies

$$W_\ell = O(h_\ell^{-\gamma}) \,, \quad \gamma > 0 \,. \tag{8}$$

As explained e.g. in [7, 12–14], the MLMC method is based on the following telescopic sum,

$$\mathbb{E}[X_{h_L}] = \mathbb{E}[X_{h_0}] + \sum_{\ell=1}^{L} \mathbb{E}[X_{h_\ell} - X_{h_{\ell-1}}] \,, \tag{9}$$

that allows to estimate $\mathbb{E}[X]$ *levelwise*,

$$E[X_{h_L}] = E_{M_0}[X_{h_0}] + \sum_{\ell=1}^{L} E_{M_\ell}[X_{h_\ell} - X_{h_{\ell-1}}] \,. \tag{10}$$

On level ℓ we use an ordinary MC method with M_ℓ samples $(X^i_{h_\ell} - X^i_{h_{\ell-1}})$, $i = 1, \ldots, M_\ell$, to approximate $\mathbb{E}[X_{h_\ell} - X_{h_{\ell-1}}]$. Note, that in this paper a sample on level ℓ comprises a difference between the same realization X^i computed on two consecutive discretization levels h_ℓ and $h_{\ell-1}$. In order to form this difference the realizations $X^i_{h_\ell}$ and $X^i_{h_{\ell-1}}$ have to be computed with the same ω. On the coarsest level we estimate $\mathbb{E}[X_{h_0}]$ with M_0 samples $X^i_{h_0}$, $i = 1, \ldots, M_0$,. The MLMC error $\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)}$ is the norm of the difference of the true expectation $\mathbb{E}[X]$ in (9) and the MLMC estimate $E[X_{h_L}]$ in (10),

$$\begin{aligned}
\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)} &\le \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \|\mathbb{E}[X_{h_L}] - E[X_{h_L}]\|_{L^2(\Omega;B)} \\
&\le \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \|\mathbb{E}[X_{h_0}] - E_{M_0}[X_{h_0}]\|_{L^2(\Omega;B)} \\
&\qquad + \sum_{\ell=1}^{L} \|\mathbb{E}[X_{h_\ell} - X_{h_{\ell-1}}] - E_{M_\ell}[X_{h_\ell} - X_{h_{\ell-1}}]\|_{L^2(\Omega;B)} \\
&\overset{(2)}{\le} \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + M_0^{-1/2}\|X_{h_0}\|_{L^2(\Omega;B)} \\
&\qquad + \sum_{\ell=1}^{L} M_\ell^{-1/2}\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)}.
\end{aligned} \tag{11}$$

With (6) and (7) this bound becomes

$$\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)} = O(h_L^\alpha) + M_0^{-1/2} O(1) + \sum_{\ell=1}^{L} M_\ell^{-1/2} O(h_{\ell-1}^\beta). \tag{12}$$

This result is valid for any numbers L and M_ℓ. However, to benefit from MLMC, a clever choice is crucial. One may balance the expected error terms on the right side of (12) [12] or minimize the computational work with respect to the expected error, see [7]. In either case, when expressed in terms of work, the convergence rates of MLMC are always as good as the ones from standard MC methods. Giles [7] shows that for $\alpha \geq 1/2$ and any $\epsilon < e^{-1}$ there is an L and values M_ℓ, $0 \leq \ell \leq L$, such that the MLMC (discretization and sampling) error is bounded by

$$\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)} < \epsilon.$$

with the total work for computing $E[X_{h_L}]$ being given by

$$\text{work} = \begin{cases} O(\epsilon^{-2}), & \beta > 1, \\ O(\epsilon^{-2}(\log \epsilon)^2), & \beta = 1, \\ O(\epsilon^{-2-(1-\beta)/\alpha}), & 0 < \beta < 1. \end{cases}$$

3.2. Sample losses in MLMC

In the previous section 2 we considered the MC method with a random number of samples, given the realistic assumption that at least one MC sample survives. Within each level of the MLMC method a standard MC simulation is executed. This allows to reuse most parts of the FT-MC approach in the FT-MLMC method. In the FT-MLMC method, in contrast to the FT-MC method, it is feasible to compute an MLMC estimate even in the case that all samples of one level $\ell \in [0, L]$ are lost. A level on which all MC samples are lost will be referred to as a *lost level*. In MLMC $\hat{M}_\ell = 0$ on some level ℓ should not lead to an infinite error bound. Therefore, the MLMC error bound has to be modified such that it can handle lost levels.

3.3. MLMC error bound with lost levels

We derive a MLMC error bound assuming that $M_\ell \geq 0$, and hence $M_\ell = 0$ might appear. We shall refer to this case as an "entirely lost level". In *the discussion of this section, the number of samples M_ℓ are not random.*

The MLMC estimate (10) is modified such that lost levels ($M_\ell = 0$) are not taken into account,

$$E[X_{h_L}] = E_{M_0}[X_{h_0}] + \sum_{\ell=1}^{L} E_{M_\ell}[X_{h_\ell} - X_{h_{\ell-1}}], \quad E_{M_\ell}[\cdot] = 0 \text{ if } \hat{M}_\ell = 0. \tag{13}$$

The error bound

$$\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)}$$

is analyzed.

In the following derivation we use a bound for a MC simulation with $M = 0$ when nothing is computed ($E_M[X] = 0$, $M = 0$):

$$\|\mathbb{E}[X] - E_0[X]\|_{L^2(\Omega;B)} = \|\mathbb{E}[X]\|_{L^2(\Omega;B)} = \sqrt{\|\mathbb{E}[X]\|_B^2} \leq \sqrt{\mathbb{E}[\|X\|_B^2]} = \|X\|_{L^2(\Omega;B)}. \tag{14}$$

Following the derivation (11) provides an error bound for the MLMC method where levels might be lost ($M_\ell = 0$)

$$\begin{aligned}
\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)} &\leq \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \|\mathbb{E}[X_{h_L}] - E[X_{h_L}]\|_{L^2(\Omega;B)} \\
&\leq \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \|\mathbb{E}[X_{h_0}] - E_{M_0}[X_{h_0}]\|_{L^2(\Omega;B)} \\
&\quad + \sum_{\ell=1}^{L} \|\mathbb{E}[X_{h_\ell} - X_{h_{\ell-1}}] - E_{M_\ell}[X_{h_\ell} - X_{h_{\ell-1}}]\|_{L^2(\Omega;B)} \\
&\overset{(2),(14)}{\leq} \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \min(1, M_0^{-1/2})\|X_{h_0}\|_{L^2(\Omega;B)} \\
&\quad + \sum_{\ell=1}^{L} \min(1, M_\ell^{-1/2})\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)} \,.
\end{aligned} \tag{15}$$

This inequality allows to bound the MLMC error even if all samples are lost on some levels.

Theorem 3.1. *The multilevel Monte Carlo error with possible loss of all samples on certain levels (i.e. when $M_\ell \geq 0$) is bounded by*

$$\begin{aligned}
\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)} &\leq \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \min(1, M_0^{-1/2})\|X_{h_0}\|_{L^2(\Omega;B)} \\
&\quad + \sum_{\ell=1}^{L} \min(1, M_\ell^{-1/2})\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)},
\end{aligned}$$

or, with assumptions (5), (6) and (7), by

$$\|\mathbb{E}[X] - E[X_{h_L}]\|_{L^2(\Omega;B)} \leq O(h_L^\alpha) + \min(1, M_0^{-1/2})O(1) + \sum_{\ell=1}^{L} \min(1, M_\ell^{-1/2})O(h_{\ell-1}^\beta) \,.$$

The loss of all samples of level ℓ ($M_\ell = 0$) increases the MLMC error bound in Theorem 3.1 substantially, leading to an error $O(h_{\ell-1}^\beta)$ whatever happens on higher levels. The loss of all samples of a low discretization level is particularly severe. However, in MLMC methods, the M_ℓ are very large on the coarsest levels, implying a very small probability of losing them all.

3.4. Fault Tolerant MLMC error bound with random failures

As in the fault tolerant MC method we work under the "all or nothing paradigm", i.e., a sample is either correctly computed or irrecoverably lost when nodes fail at runtime. We do not distinguish between node, program, network, or any other cause of failure at runtime. We discard all samples affected by a failure, and compute the result with the remaining ones. Then the number M_ℓ of samples per level is not a fixed number anymore, but rather a random number denoted by $\hat{M}_\ell$.

In accordance with Section 2 the probability space for the random solution X is denoted as $(\Omega, \mathscr{A}, \mathbb{P})$. The probability space to model runtime failures is denoted as $(\Omega', \mathscr{A}', \mathbb{P}')$. Evidently, the number of MC samples per level $\hat{M}_\ell$ depends on the runtime failures. We model $\hat{M}_\ell$, $\hat{M}_\ell \leq M_\ell$, as a measurable mapping

$$\hat{M}_\ell : (\Omega', \mathscr{A}', \mathbb{P}') \to (\mathbb{N}_0, 2^{\mathbb{N}_0}).$$

We derive an error bound for an MLMC sample average estimate which is computed based on a *random number of samples* $\hat{M}_\ell$ which models the MLMC method in the presence of failures at runtime. Specifically, the following fault tolerant MLMC (FT-MLMC) estimator is used:

$$\hat{E}[X_{h_L}] = E_{\hat{M}_0}[X_{h_0}] + \sum_{\ell=1}^{L} E_{\hat{M}_\ell}[X_{h_\ell} - X_{h_{\ell-1}}] \,, \quad E_{\hat{M}_\ell}[\cdot] = 0 \text{ if } \hat{M}_\ell = 0.$$

Theorem 3.2. *The FT-MLMC error under influence of failure is bounded by*

$$\begin{aligned}\|\mathbb{E}[X] - \hat{E}[X_{h_L}]\|_{L^1(\Omega';L^2(\Omega;B))} \leq \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \mathbb{E}'[\min(1, \hat{M}_0^{-1/2})]\|X_{h_0}\|_{L^2(\Omega;B)} \\ + \sum_{\ell=1}^{L} \mathbb{E}'[\min(1, \hat{M}_\ell^{-1/2})]\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)},\end{aligned}$$

or given the assumptions (5), (6) *and* (7)

$$\|\mathbb{E}[X] - \hat{E}[X_{h_L}]\|_{L^1(\Omega';L^2(\Omega;B))} \leq O(h_L^\alpha) + \mathbb{E}'[\min(1, \hat{M}_0^{-1/2})]O(1) + \sum_{\ell=1}^{L} \mathbb{E}'[\min(1, \hat{M}_\ell^{-1/2})]O(h_{\ell-1}^\beta).$$

Here, we assume that the $\hat{M}_\ell$ *are statistically independent of* $(\Omega, \mathscr{A}, \mathbb{P})$.

Proof. By Theorem 3.1 we have

$$\begin{aligned}\|\mathbb{E}[X] - \hat{E}[X_{h_L}]\|_{L^2(\Omega;B)} \leq \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \min(1, \hat{M}_0^{-1/2})\|X_{h_0}\|_{L^2(\Omega;B)} \\ + \sum_{\ell=1}^{L} \min(1, \hat{M}_\ell^{-1/2})\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)},\end{aligned}$$

Both sides are integrated over the probability space $(\Omega', \mathscr{A}', \mathbb{P}')$ leading to

$$\begin{aligned}&\int_{\Omega'} \|\mathbb{E}[X] - \hat{E}[X_{h_L}]\|_{L^2(\Omega;B)}\mathbb{P}'(d\omega') \\ &\qquad \leq \int_{\Omega'} \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B + \min(1, \hat{M}_0^{-1/2})\|X_{h_0}\|_{L^2(\Omega;B)} \\ &\qquad\qquad + \sum_{\ell=1}^{L} \min(1, \hat{M}_\ell^{-1/2})\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)}\mathbb{P}'(d\omega') \,.\end{aligned}$$

With the linearity of the mathematical expectation the error estimate becomes

$$\begin{aligned}&\int_{\Omega'} \|\mathbb{E}[X] - \hat{E}[X_{h_L}]\|_{L^2(\Omega;B)}\mathbb{P}'(d\omega') \\ &\qquad \leq \int_{\Omega'} \|\mathbb{E}[X] - \mathbb{E}[X_{h_L}]\|_B\mathbb{P}'(d\omega') + \int_{\Omega'} \min(1, \hat{M}_0^{-1/2})\mathbb{P}'(d\omega')\|X_{h_0}\|_{L^2(\Omega;B)} \\ &\qquad\qquad + \sum_{\ell=1}^{L} \int_{\Omega'} \min(1, \hat{M}_\ell^{-1/2})\mathbb{P}'(d\omega')\|X_{h_\ell} - X_{h_{\ell-1}}\|_{L^2(\Omega;B)} \,.\end{aligned}$$

□

We derived a general error bound for a MLMC method in the presence of faults. Regarding the implementation some aspects should be considered. A (single level) MC method is used to estimate $\mathbb{E}[X_{h_\ell} - X_{h_{\ell-1}}]$ by $E_{\hat{M}_\ell}[X_{h_\ell} - X_{h_{\ell-1}}] = \sum_{i=1}^{\hat{M}_\ell}(X^i_{h_\ell} - X^i_{h_{\ell-1}})$, where $X^i_{h_\ell}$ and $X^i_{h_{\ell-1}}$ approximate the same realization X^i with two different discretization parameters h_ℓ and $h_{\ell-1}$. This implies a strong statistical correlation between $X^i_{h_\ell}$ and $X^i_{h_{\ell-1}}$. Failures influence the random number of samples per level $\hat{M}_\ell$. It is emphasized that a sample on a level always consists of the difference between the two computed solutions $X^i_{h_\ell} - X^i_{h_{\ell-1}}$. In other words, whenever a sample $X^i_{h_{\ell-1}}$ is lost due to a failure, it is required that the corresponding $X^i_{h_\ell}$ is disregarded as well, and vice versa. We propose therefore to do all computations for a sample $X^i_{h_\ell} - X^i_{h_{\ell-1}}$ on a single unit. A failure in this unit leads automatically to a loss of both $X^i_{h_\ell}$ and $X^i_{h_{\ell-1}}$. Only entirely computed samples $X^i_{h_\ell} - X^i_{h_{\ell-1}}$ are accumulated or communicated to other units. This procedure has the slight disadvantage that two different discretizations have to be computed on the same unit.

4. Statistical model of node failure at runtime

System failures can interfere with the computation of the MLMC estimate. They can be due to errors in software, hardware or network but also due to environmental effects. These failures are manifestations of various types of errors which can be roughly classified as permanent, transient, or silent [5]. Permanent errors do not disappear, whereas transient errors are short term errors. Silent errors are undetected (permanent or transient) errors, and hence they may lead to undetected erroneous results. Other classifications into hard and soft errors are conceivable and for instance described in [8].

In the previous section we have extended the MLMC theory to cover random numbers of samples. This allows to disregard all samples affected by detected errors (permanent or transient, soft or hard). Our fault tolerant MLMC method simply ignores these lost samples and tries to make the best out of the surviving ones. Silent (undetected) errors however are not covered by this theory.

To specify the error bound of the FT-MLMC method the statistics of failures leading to sample losses has to be known. At present only rather rough failure models are available and, in our opinion, the development of more detailed, and realistic models is needed. For new HPC systems first empirical statistical failure studies [16, 25] are available. Schroeder and Gibson [25] derived a somewhat realistic failure model that has been adopted in the present paper. The authors studied the failures which occurred over 9 years in more than 20 different systems at Los Alamos National Laboratory. Their raw data [30] contains an entry for any failure that required the attention of a system administrator. The authors conclude that the time interval between two failures of an individual node, as well as for the entire system is well fitted by the *Weibull distribution* with the Weibull shape parameter k between 0.7 and 0.8. The probability density function of the Weibull random variable x is given by (see e.g. [17])

$$f(x;\lambda,k) = \begin{cases} \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}, & \text{if } x \geq 0, \\ 0, & \text{if } x < 0, \end{cases}$$

where $\lambda > 0$ is the scale parameter and $k > 0$ is the shape parameter. Schroeder and Gibson also found the failure rates to be roughly proportional to the number of processors in the system. Therefore, we use the Weibull distribution to model the time between two consecutive failures on

one node. This model assumes that node failures are statistically independent, which is one of the rather rough approximations made. The data of Schroeder and Gibson [25] also gives evidence of a correlation between the failure rate and the type and intensity of the workload running on a machine, as the failure rate is considerably lower during the night and the weekend. However, in the present model we had to idealize, since more accurate node failure data and models are hard to come by, at least at present. Nevertheless we point out that our theory accommodates also other, more sophisticated parametric failure models, as long as they satisfy the general assumptions in the present paper, and we hope that failure models with better quantitative accuracy will become available in the near future.

4.1. Time to next failure as a renewal process

When a new simulation on a computer is started, generally the last time of failure of this computer is not known. Thus, when applying the above failure model, we are not only interested in the time between two failures but also in the distribution of the time from the start of the computation to the first failure on a node. This time is modeled with an ordinary renewal process (RP). An ordinary RP is a sequence of i.i.d. nonnegative RVs $Y_1, Y_2, \ldots$ In the applied model these RVs are the time intervals between two failures, hence they are Weibull distributed. After a failure the failed component is assumed to be replaced within a negligible time. Therefore, the time to the next failure is again Weibull distributed. The RP is started at $t = 0$. We are interested in the so called forward recurrence time until the next failure. F_t is the time from t up to and including the moment of the next failure. The MLMC computation is started at time t. Rinne [23] states that, as long as t is finite, F_t can only be evaluated numerically. In the case $t \to \infty$ however [23, eq. (4.41a)] provides the analytic distribution function of F_∞.

A procedure to draw realizations of the forward recurrence time F_∞ is given in [29]:

1. Draw $S \sim \Gamma(1+\frac{1}{k}, \lambda^k)$, where λ and k are the Weibull scale and shape parameter, respectively.
2. Compute $V = S^{1/k}$.
3. Draw $F_\infty \sim \text{Uniform}(0, V)$.

4.2. Calibration of the Weibull parameters

The two Weibull scale and shape parameters λ and k are used in the above drawing of the forward recurrence times F_∞. They depend on the machinery the FT-MLMC method runs on. In [23] several methods are presented to estimate both Weibull parameters.

In our analysis we use the parameters given in [25], $k \approx 0.7$ and $\lambda \approx 7.6 \cdot 10^5$ for a single node. Note that $k \approx 0.7$ is explicitly given in [25] for an individual node. λ is estimated from [25, Fig. 6(b)]. The time between failures, such that the cumulative distribution function is equal to 0.5, is given approximately by $x \approx 4.5 \cdot 10^5$s. Therefore, λ can be approximated by $0.5 \approx 1 - e^{-(4.5 \cdot 10^5/\lambda)^k}$, yielding $\lambda \approx 7.6 \cdot 10^5$.

These parameters lead to a mean time between failure (MTBF) of approximately 11 days for a typical node. We use nodes with 128 cores, as in [25] a node has 80–128 cores. Whenever a node is hit by a failure, results of all involved 128 cores are assumed to be irrevocably lost.

4.3. Avoiding a parametric failure model

We briefly discuss an approach to avoid the parametric statistical failure model and instead directly work with the failure log of the computer at hand. To this end, we choose time instants t_i, $i = 1, \ldots, I$, in the past at which we start synthetic MLMC runs, where for each sample the

participating cores are specified, cf. Fig. 1. Based on the failure log we determine for each MLMC level the number of samples, $\hat{M}_\ell(t_i)$, $\ell = 1, \dots, L$, that would have survived. The (in general non-integer) expected values $\mathbb{E}'[\min(1, \hat{M}_\ell^{-1/2})] \approx I^{-1} \sum_{i=1}^{I} \min(1, \hat{M}_\ell(t_i)^{-1/2})$ are estimated by the Monte Carlo method. These expected values are then used in the FT-MLMC error bound of Theorem 3.1.

The time instants t_i can be chosen according to the desired start of the FT-MLMC run (e.g. Monday 5 pm). Then the time of day and the day of the week correlation is taken into account. Such correlations are prominently present in the failure statistics of Schroeder and Gibson [25].

The downside of this approach is, that the failure log of the computer at hand will at most span the time since the start of its operation. Hence the number of time instants t_i leading to nonoverlapping synthetic MLMC runs is typically limited, in particular, if additional requests, as the time of the day, are imposed. Furthermore, the presented procedure may lead to correlated "failure samples", which would reduce the Monte Carlo convergence rate, and thus increase the demand for time instants. Advantages of this approach are its simplicity, its practical applicability, the fact that many failure correlations are taken into account, and that no failure model has to be constructed.

5. Numerical experiments

All MLMC methods satisfying assumptions (5)–(8) are suited for the FT-MLMC method. We assess the quality of our FT-MLMC error bounds by means of the grid-based finite volume code ALSVID-UQ [1] for solving hyperbolic systems of conservation laws. We set $\alpha = \beta = s$ in eqs. (5)–(6). With this choice, the work to compute a sample $X^i_{h_\ell} - X^i_{h_{\ell-1}}$ on level ℓ is

$$W_\ell = 2^{d+1} W_{\ell-1} = 2^{(d+1)\ell} W_0, \qquad \ell \geq 0, \tag{16}$$

where the exponent originates from d space and 1 time dimensions. Note that samples on low levels require only a small fraction of the execution time of samples on high levels. The same holds for memory space. Memory usage of a grid-based PDE solver is given by

$$\mathrm{mem}_\ell = 2^d \mathrm{mem}_{\ell-1} = 2^{d\ell} \mathrm{mem}_0, \qquad \ell \geq 0. \tag{17}$$

As suggested in [12] for $s < (d+1)/2$, the number of samples M_ℓ on level ℓ, is set to be

$$M_\ell = 2^{-2s} M_{\ell-1} = 2^{2(L-\ell)s} M_L, \qquad \ell \geq 0. \tag{18}$$

On the finest level, we choose a moderate number of samples, e.g., $M_L = 8$.

We parallelize our simulation level-wise and across levels. A core deals with samples of only one level, i.e, it computes solutions on two resolutions used for a sample $X^i_{h_\ell} - X^i_{h_{\ell-1}}$ on level ℓ.

In our implementation we choose an intermediate level b on which individual samples are executed on a single core. This level is somewhat arbitrary. It is determined typically based on memory requirements or execution times of a sample. The subdomains are chosen large enough such that the communication over subdomain-interfaces plays a minor role. On levels $\ell > b$, a single sample is executed in parallel on multiple cores. The number of cores is determined by the memory requirement of the sample, as given in equation (17). On levels $\ell \leq b$ samples are executed sequentially. So, the number N_ℓ of cores assigned to a sample on level ℓ is

$$N_\ell = \lceil 2^{d(\ell-b)} \rceil, \qquad \ell \geq 0. \tag{19}$$

We collect samples in tasks of equal execution times. Tasks are the units of work that are submitted to the compute nodes. A sample of the finest level forms a task. Tasks of coarser levels consist of multiple samples. On very coarse levels, all respective samples are included in a single task that may have a shorter execution time than (most) other tasks. Tasks of the same level are always computed on independent nodes. Any single node may compute tasks from different levels.

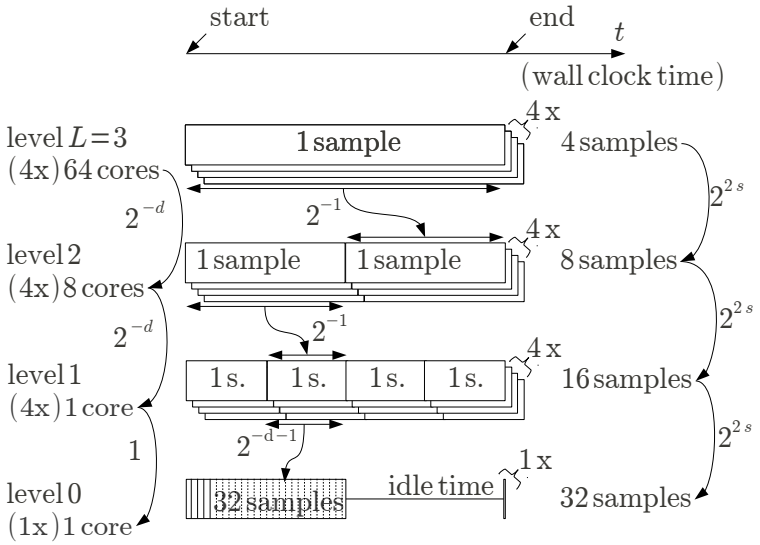


Figure 1: Computing an MLMC estimate with parameters $L = 3$, $M_L = 4$, $d = 3$, $s = 1/2$ and only one core per task on levels 0 and 1.

In Fig. 1 a 3D example is given with 4 levels, $b = 1$, $M_L = 4$, and $s = 1/2$. The MLMC estimate is computed in parallel on 293 cores. The 4 samples of level L are each computed in a task of 64 cores. Hence, 256 cores are involved in the computation of this level. On level $b = 1$ the number of cores in a task is one. This number increases by the factor 8 from one level to the next finer, according to (19). On level 0, one core deals with a single task that comprises all 32 samples of this level. The execution time of this task is approximately half of that of the others, causing a small load imbalance.

As mentioned in Section 3.4, in our analysis it is crucial that samples $X^i_{h_\ell} - X^i_{h_{\ell-1}}$ be lost as a whole, i.e., it must never happen that one part (either $X^i_{h_\ell}$ or $X^i_{h_{\ell-1}}$) survives and is used in the end result while the other part is lost. This is achieved by computing an entire sample $X^i_{h_\ell} - X^i_{h_{\ell-1}}$ on cores belonging to the same node. In case of node failure the entire sample is lost. Large samples however use many cores such that multiple nodes have to be used to compute them. In case of node failure, the processes running on unaffected cores have to realize that they should stop computing their part of the sample. How this is solved is implementation dependent. In our FT-MLMC implementation [20] we used User Level Failure Mitigation (ULFM) [3], a fault tolerant extension of MPI. With ULFM a MPI process trying to communicate with a failed one is notified about the failure. This process can then prohibit any further communication on the communicator used for the computation of the affected sample. This ensures that the information of failures is received by all involved MPI processes. Further details are available in [20].

5.1. Lowering the risk of losing all samples of a level

A high probability of losing all samples on one level increases the FT-MLMC error bound in Theorem 3.2 substantially. This applies particularly to the coarse levels where the impact of a loss is highest. We investigate three ways of storing the results of the samples to reduce the risk of losing information:

- "late save/n tasks": The samples of a level are split into at least n tasks each of which is computed on independent nodes. The samples are stored only after the completion of the entire task. So, according to our assumptions, a failure at runtime of a task leads to the total loss of all its samples. With n=4 in the example shown in Fig. 1 only level 0 is affected as all other levels already run on at least four tasks, see Fig. 2. This strategy reduces the risk of losing all samples on level 0 by reducing the execution time of a task as well as by using multiple tasks. The execution time of the now 4 tasks is reduced by a factor of 4 compared to the original one which increases load imbalance.

Figure 2: The "late save/4 tasks" strategy requests at least 4 tasks per level. In the example of Fig. 1 this leads to a large number of very small tasks on level 0.

- "immediate save": Immediately after the completion of a sample, its result is safely stored, and hence will not get lost.

- "intermediate save": The samples of a level are split into at least two tasks each of which is computed on independent nodes. The results are exchanged among the partner tasks up to 4 times in a FT-MLMC run.

The three strategies have a different behavior in case of failure. In the "immediate save" strategy, a larger part of the data on a node survives the failure of the node, leading to higher failure resilience. This advantage is offset by a higher communication overhead. In the "late save" strategy the statistical quality of the remaining samples is equivalent to the N-out-of-M strategy suggested by Li and Mascagni [10]. How to achieve statistical independence of the remaining samples in the "immediate save" strategy is to our knowledge presently an open question. With this strategy only the first entries (up to the failure) of a random number stream are used. Also the resilience of massive parallel RNG (such as WELL [9]) in the presence of partial loss of streams remains to be addressed. The "intermediate save" strategy is a compromise between the "late save" and the "immediate save" strategies. It reduced the communication overhead, compared to the "immediate save" strategy, as the results are stored less frequently and only locally, and fewer tasks are used, compared to the "late save" strategy.

5.2. Euler equation of gas dynamics

We show results in two and three spacial dimensions for the finite volume method (FT-MLMC-FVM) that solves the Euler equations of gas dynamics with stochastic initial data. Following [13,

28], the d-dimensional Euler equations of gas dynamics are given by

$$\begin{cases} \rho_t + \operatorname{div}(\rho \boldsymbol{u}) = 0, \\ (\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u} + p\boldsymbol{I}) = 0, \\ E_t + \operatorname{div}((E+p)\boldsymbol{u}) = 0, \end{cases}$$

in the domain $D = (0,1)^d$, with outflow boundary conditions, where ρ is the density, $\boldsymbol{u}$ the velocity vector and $\boldsymbol{I}$ the identity matrix of order d. The pressure p and the total energy E are related by the equation of state of an ideal gas.

$$E := \frac{p}{\gamma - 1} + \frac{1}{2}\rho|\boldsymbol{u}|^2,$$

with γ the ratio of specific heats. We apply random initial conditions which render the state vector $\boldsymbol{X}(x,t,\omega) = \{\rho(x,t,\omega), \boldsymbol{u}(x,t,\omega), p(x,t,\omega)\}$ a random variable. The variable of interest is the state vector computed at $t = .6\,\mathrm{s}$. In [13] a MLMC Finite Volume Method (MLMC-FVM) approach is proposed to estimate $\mathbb{E}[\boldsymbol{X}(\cdot,t)]$. We choose the Banach space B in Sections 2 and 3 as $L^1(D)$. A MLMC-FVM error bound is shown for scalar solutions in [12] and assumed to hold for nonlinear hyperbolic systems of conservation laws in [13, 14] by

$$\|\mathbb{E}[\boldsymbol{X}(\cdot,t)] - E^{\hat{\mathcal{L}}}[\boldsymbol{X}(\cdot,t)]\|_{L^2(\Omega;L^1(D))} \leq C_1 h_L^s + C_2 \hat{M}_0^{*-\frac{1}{2}} + C_3 \left\{ \sum_{\ell=1}^{L} \hat{M}_\ell^{*-\frac{1}{2}} h_{\ell-1}^s \right\}, \tag{20}$$

which corresponds to the error bound (12). We set the constants as $10 \cdot C_1 = C_2 = 10 \cdot C_3 = .5$. The convergence rate

$$\|\mathbb{E}[\boldsymbol{X}(\cdot,t)] - E^{\mathcal{L}}[\boldsymbol{X}(\cdot,t)]\|_{L^2(\Omega;L^1(D))} \lesssim W^{-s/(d+1)} \cdot \log(W), \qquad s < (d+1)/2,$$

is empirically shown in [13]. Here, $W = \sum W_\ell M_\ell$ is the total work. Sample errors on level ℓ are bounded by [12–14]

$$\|\boldsymbol{X}_\ell(\cdot,t) - \boldsymbol{X}_{\ell-1}(\cdot,t)\|_{L^2(\Omega;L^1(D))} \leq C_4 h_{\ell-1}^s. \tag{21}$$

This bound is related to assumption (6). The bounds (20) and (21) imply the validity of Theorems 3.1 and 3.2 for the FT-MLMC-FVM method.

5.3. Assessment of the FT-MLMC error bound

The error bound is compared with the measured error of a FT-MLMC implementation. In [20] we report on the implementation of a MPI-parallelized FT-MLMC method in ALSVID-UQ [1]. In this implementation we used the User Level Failure Mitigation (ULFM) [3], a fault tolerant extension of MPI. A synthetic failure generator is used to simulate MPI process failures. For this purpose a timed asynchronous interrupt is set, which kills the MPI process using the exit system call. This assures that failures can happen at any time, during the computation, the communication, or while ULFM is recovering from previous failures. Our implementation in [20] demonstrates that the FT-MLMC algorithm proposed in this paper can be successfully implemented.

In [20] we computed the FT-MLMC results on Brutus, a large compute cluster at ETH Zurich, where we used one node with four 12-core AMD Opteron 6174 CPUs and 64 GB of RAM. Due to incompatibility of the ULFM developer implementation with the batch system of Brutus it is

currently not possible to run the code on multiple nodes. Therefore, in this Section 5.3, we use a modified version of the failure model in Section 4 where failures do not affect an entire node but only a process.

The exact parameters used for the simulation are described in [20] and are summarized in Table 1. All measurements are averages over multiple FT-MLMC runs. We executed 100 runs if $6\,\mathrm{s} < \mathrm{MTBF} < 100\,\mathrm{s}$, otherwise just 30. The error is computed using a MLMC reference solution $E[X_{\mathrm{ref}}]$ with $L = 8$, $M_L = 8$ and $h_L = 2^{-11}$, the absolute FT-MLMC error is measured as $\sqrt{E_M[\|E[X_{h_L}] - E[X_{\mathrm{ref}}]\|_{L^1}^2]}$, with $M = 100$ or $M = 30$ respectively.

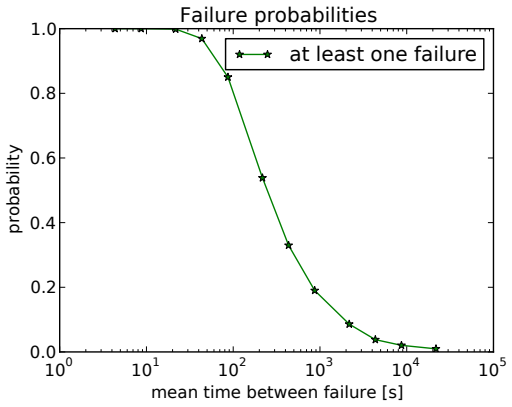
M_L	2
d; s	2; 1/2
# cells on level 0 ($\propto 1/h_0$)	2^4
intermediate save	4 times, using 2 tasks
late save	using 2 tasks
W_5 (time for one sample)	$16 \times 94\,\mathrm{s} = 1504\,\mathrm{s}$
simulations with levels	$L = 5$
one core per task	$b = 3$
Weibull parameters λ; k	variable; 0.5

Table 1: The parameters used for the 2D FT-MLMC runs, cf. Fig. 3.

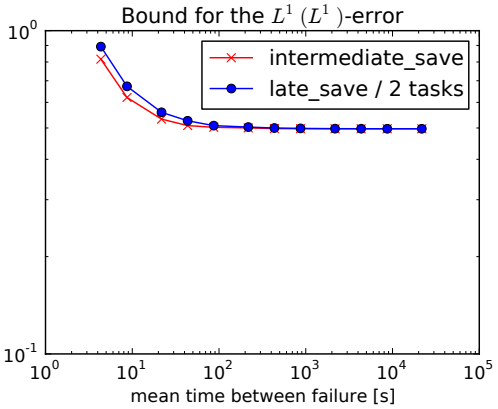
In Fig. 3 we compare the measurements from a FT-MLMC implementation with the derived FT-MLMC error bound. The same FT-MLMC problem was simulated with different mean time between failures (MTBF), by varying the Weibull scale parameter λ of the failure model. In order to save computing time rather large failure rates (small MTBFs) where used. We determine the *entry failure rate*, i.e., the failure rate starting at which fault tolerance is useful. This point is reached when at least 10% of all FT-MLMC runs encounter a failure, and hence 10% of all fault intolerant implementations (using standard MPI-3.0) would terminate without a result. This is predicted by the theory derived in this paper and shown in Fig. 3(a) by the "at least one failure" probability. In Fig. 3(c) the equivalent measurements for the ULFM implementation [20] are shown. In both subfigures the entry failure rate is around $\mathrm{MTBF} = 10^3\,\mathrm{s}$ for the given problem. Fig. 3(c) additionally presents the measured "process failure" probability, which shows that in average 20% of all started processes fail at $\mathrm{MTBF} \approx 6\,\mathrm{s}$. However, at the entry failure rate ($\mathrm{MTBF} = 10^3\,\mathrm{s}$) the process failure probability is still small.

In Figs. 3(b) and 3(d) the FT-MLMC error bound and the measured relative FT-MLMC error, respectively, are shown for the "intermediate save" and the "late save" strategies. Additionally we present the measured reference error from the fault free ALSVID-UQ [1] in Fig. 3(d). We measure the *critical failure rate*, the failure rate beyond which the FT-MLMC method does not perform well anymore. In the FT-MLMC implementation, shown in Fig. 3(d), the critical failure rate is at $\mathrm{MTBF} \approx 40\,\mathrm{s}$. In the FT-MLMC error bound, shown in Fig. 3(b), the critical failure rate is similar. There the "intermediate save" strategy provides a small benefit over the "late save" strategy, whereas in the measurement this benefit is negligible. At $\mathrm{MTBF} \approx 40\,\mathrm{s}$ the "intermediate save" strategy performs even slightly better compared to the "late save" strategy. This statistical artifact appears since we are only averaging over 100 FT-MLMC runs.

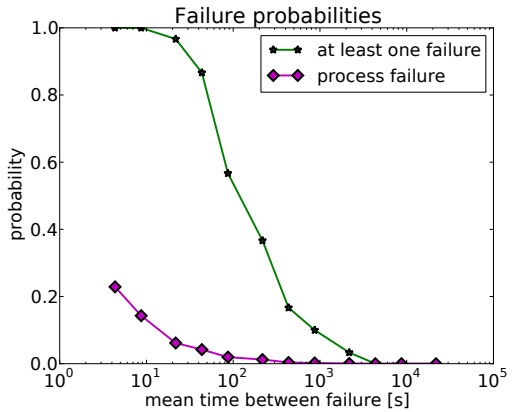
Both the FT-MLMC implementation as well as the FT-MLMC error bound indicate that the error only increases slightly between the entry failure rate and the critical failure rate. In our



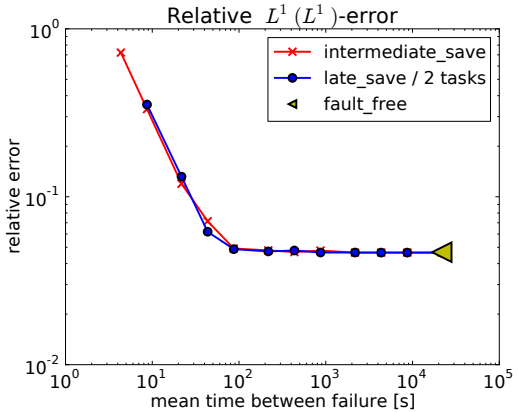
(a) Failure probabilities of the 2D FT-MLMC bound.



(b) 2D FT-MLMC error bound.



(c) Measured failure probabilities of the 2D FT-MLMC implementation [20].



(d) Measured relative errors of the 2D FT-MLMC implementation [20].

Figure 3: Behavior of the 2D FT-MLMC-FVM runs with Weibull distributed failures.

opinion, this is a fundamental insight, as one could also expect a "gradual" loss of performance, rather than a large range of failure intensities where only a minor degradation is to be observed. Only after the critical failure rate is reached, the error strongly increases and the numerical quality of the simulation results is lost.

The sharp rate of failure increase at the critical failure rate does not match well when comparing the error bound and the measured relative error. One reason is that the constants C_1, C_2, and C_3 in the error bound are unknown or can only be estimated very conservatively. In reality many additional constants are subsumed in these three constants. Also keep in mind, that we are comparing two different things, a possibly pessimistic asymptotic error bound with an actual discretization, or sampling error, respectively, which may not yet be in the asymptotic range.

5.4. Analysis of FT-MLMC error bounds

Four error bounds with different parameters are analyzed, see Table 2. The error bounds for the 2-dimensional problem are shown in Fig. 4 and for the three dimensional problems in Fig. 5. Note that the evaluation of the FT-MLMC error bound does not involve the computation of actual

samples (in contrast to the FT-MLMC error). It only invloves the error bound (20) based on the parameters of Table 2, the failure distribution and the work assigned to the cores. Two plots are shown for each case. On the left the problem size varies and the failure rate remains fixed, while on the right the failure rate varies while the problem size remains fixed. For easier comparison of the two plots on the left and on the right, we indicate a particular run by a yellow diamond.

	Fig. 4(a,b)	Fig. 4(c,d)	Fig. 5(a,b)	Fig. 5(c,d)
M_L	8	8	8	8
d; s	2; 1/2	2; 1	3; 1/2	3; 1
# cells on level 0 ($\propto 1/h_0$)	32×32	16×16	$8 \times 8 \times 8$	$4 \times 4 \times 4$
W_0 (time for one sample)	0.05 s	0.04 s	0.01 s	0.007 s
simulations with levels	$L = 5, \ldots, 13$	$L = 5, \ldots, 13$	$L = 4, \ldots, 11$	$L = 4, \ldots, 11$
one core per task	$b = 4$	$b = 3$	$b = 4$	$b = 3$
Weibull parameters λ; k	$7.6 \cdot 10^5$; 0.7	$7.6 \cdot 10^5$; 0.7	$7.6 \cdot 10^5$; 0.7	$7.6 \cdot 10^5$; 0.7

Table 2: The parameters used for the 2D and 3D FT-MLMC runs, of Fig. 4 and 5 (a)–(d).

We determine the entry failure rate and the critical failure rate in all the measurements of Figs. 4(b,d) and 5(b,d). In all these cases the entry failure rate is at around MTBF $= 5 \cdot 10^7$ s and the critical failure rate at around MTBF $= 10^5$ s. Again we observe that between the entry failure rate and the critical failure rate the error bound increases only slightly. Hence, this desirable property is found also in large runs, and in multiple dimensional runs. Only after the critical failure rate the error explodes.

Similar to the entry failure rate, the failure rate from which on fault tolerance is useful, there is an *entry problem size*, the problem size from which on fault tolerance is useful, when operating on a given computer. Again we define this point as reached when 10% of all FT-MLMC runs encounter a failure. The entry problem size corresponds to a total execution time of around $5 \cdot 10^6$ s for Figs. 4(a) and 5(a,c). In Fig. 4(c) the entry problem size is around $5 \cdot 10^5$ s. Similar to the critical failure rate, we use the term of *critical problem size* in Figs. 4(a,c) and 5(a,c). This problem size corresponds in our measurements to an execution time of around 10^9 s. Also in these figures we observe that the convergence behavior of FT-MLMC is only negligibly affected by failures between the *entry problem size* and the critical problem size. After the critical problem size however, the FT-MLMC method experiences many sample losses, such that the performance drops significantly. This is apparent for the increasing disparity between the two FT-MLMC graphs "late save/4 tasks" and "immediate save" with the fault-free MLMC "reference" graph. Hence, when using the failure distribution of Section 4 the FT-MLMC method allows to solve problems that take more than 100 times longer compared to the largest problem solved with a standard MLMC implementation.

The "late save/4 tasks" error does not only stop decreasing but grows again after the critical problem size, Fig. 4(a,c) and 5(a). At first this may surprise. However, as the runtime of the tasks is determined by the runtime of the samples of the finest level, the vulnerability of all tasks increases as the number of levels increases.

In Fig. 4(a,c) and 5(a,c) it is further observed that the performance of the FT-MLMC method drops significantly as soon as the probability of losing all samples on the finest level approaches values close to one. This can be expected, since then much of the computational work is used attempting to compute samples of which only very few to none survive. This leads to a high error to work ratio, or, in other words, to a poor FT-MLMC performance.

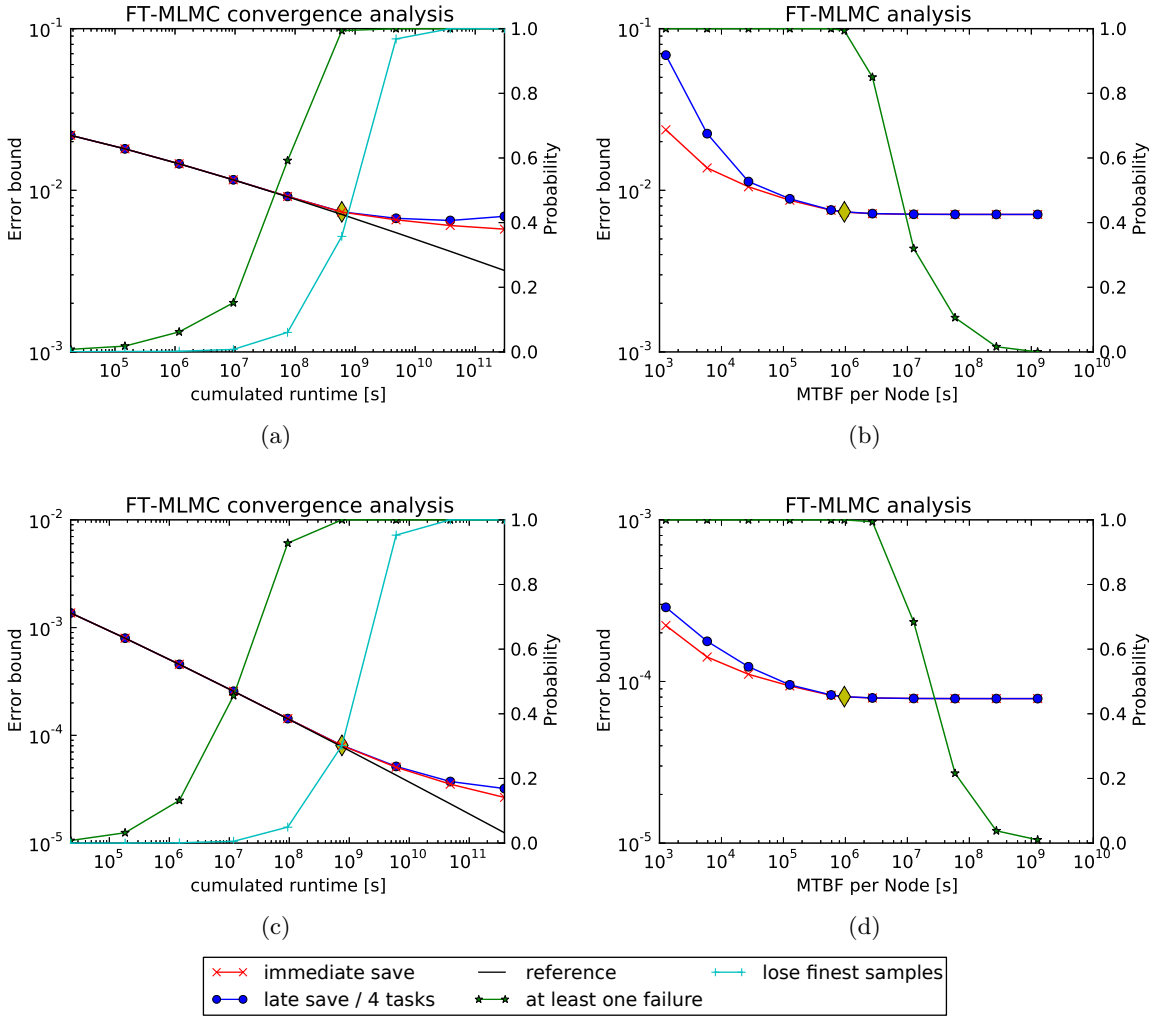


Figure 4: Behavior of the 2D FT-MLMC method with parameters from Table 2.

5.5. Practical applicability

To directly employ the theoretical error bounds derived in this paper, it is essential to have an accurate, parametric statistical node failure model of the compute platform used. Once such a model has been verified and calibrated to the particular hardware of interest, the computational work can be redistributed among nodes and cores in order to minimize the overall impact of node failures on the accuracy of the computation. Based on a statistical node failure model of the type considered here, a FT-MLMC error bound can be computed, which allows to infer precise statements on the contributions to the overall error stemming from failures, and on whether the problem of interest can be solved using FT-MLMC or plain MLMC on the given system.

The weak point of the presently proposed approach is that *accurate and detailed, validated parametric statistical failure models and the data to calibrate them* are currently not available. Hopefully, the situation will improve over time, as the awareness of faults grows. For the time being we opt to use failure models that allow to determine rough estimates of the entry failure

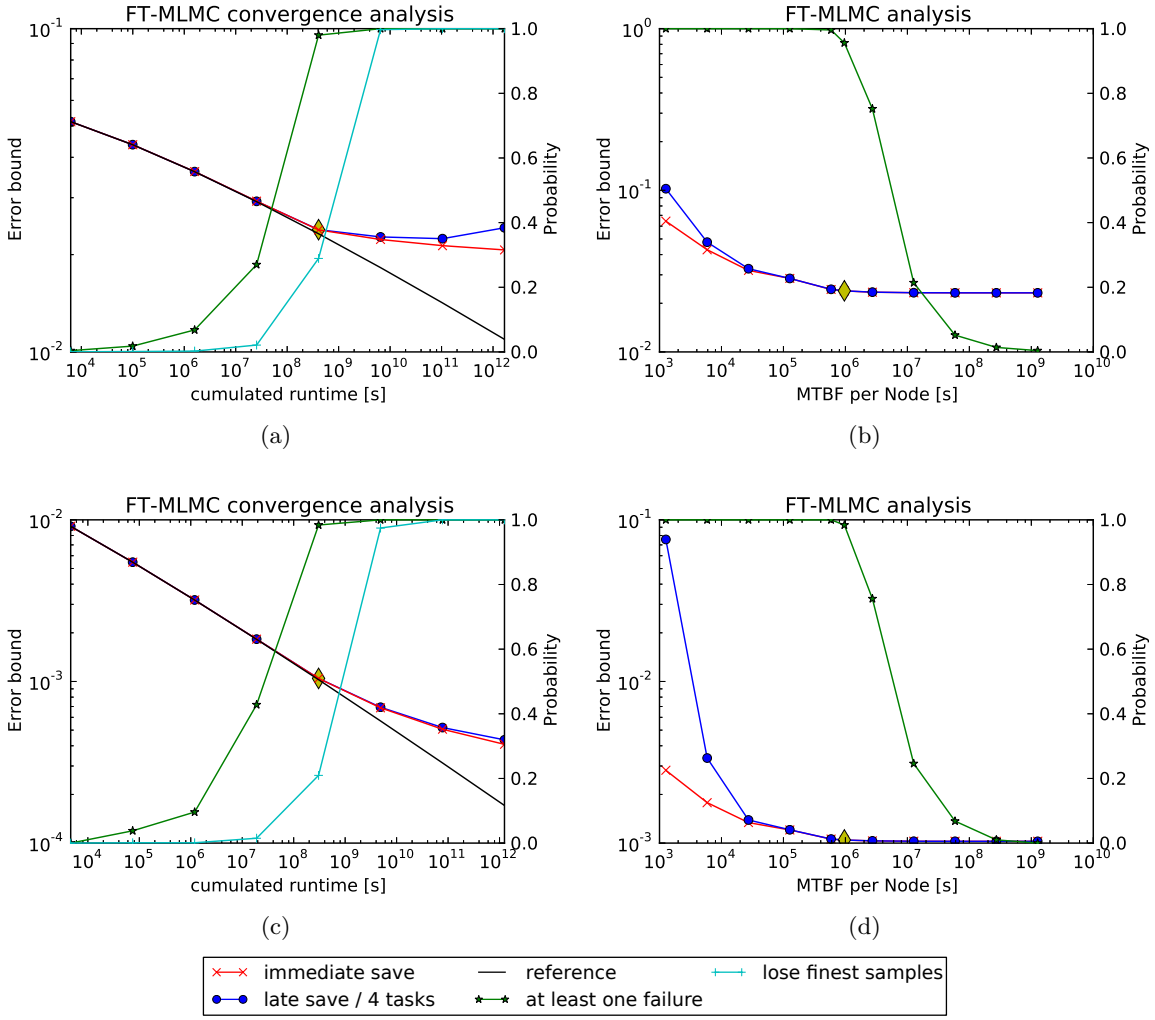


Figure 5: Behavior of the 3D FT-MLMC method with parameters from Table 2.

rate, critical failure rate, entry problem size, and critical problem size. Alternatively, in Section 4.3, we outline how to bypass the failure models and directly work on failure logs.

We expect that in practice the entry problem size and the entry failure rate are not computed with a known failure distribution, but rather that users will be bothered when failures terminate the application prematurely in at least 10% of the runs, which coincides with the mentioned quantities. Only then fault tolerance appears to become an issue for the user (This figure might, however, be problem and application dependent.) It is not practical for the user to decide when the performance of FT-MLMC deteriorates, i.e., the critical problem size and the critical failure rate cannot be determined *without some knowledge* of the failure distribution. In our measurements we observed that there is a range between the entry failure rate and the critical failure rate, and a range between the entry problem size and the critical problem size where the performance of FT-MLMC is hardly unaffected by faults. In our problems, obviating checkpointing in this observed fault tolerance extends the range of applicability by two orders of magnitude: on a given computer the

FT-MLMC method is applicable to problems that take around 100 times longer compared to the "plain" fault-free MLMC method, or, equivalently, a given problem can be solved on a computer with around 100 times smaller MTBF thanks to the inherent fault tolerance of MLMC.

The performance of our algorithms depends directly on the parameters d, s, W_0, and M_L of Tables 1 and 2. They characterize the problem and its numerical approximation in terms of the resolution of the discretization. Methods different from the finite volume method considered here lead to similar values of these parameters. Therefore, we are confident that the qualitative behavior of a FT-MLMC method, in particular, the considerable gap between entry failure rate and the critical failure rate, remains similar for other applications.

Our failure model does not take into account that random node failures may in practice be correlated among different nodes. This has adverse effects on the error bound. For large samples, running on multiple nodes, correlated node failures could be beneficial in the following sense: as soon as one node of a sample failed, the computation of this sample is stopped anyway, hence it does not matter if additional nodes fail simultaneously. For most problems which occur in engineering practice, and in particular for the Finite Volume simulations considered in the present paper, MC samples on low levels easily fit on a single node. In the "late save/4 tasks" strategy at least 4 cores on different nodes are used to compute samples of these levels. Node failure correlation might increase the risk of losing all 4 involved nodes simultaneously, which has a clearly negative effect on the error bound. But we do not see any parameter which would ruin the nice property that the error only increases slightly between the entry failure rate and the critical failure rate, and between the entry problem size and the critical problem size, and that there is a considerably large range where FT-MLMC is applicable.

The measurement of the number of failures in one single FT-MLMC run on a computer with unknown failure distribution does not provide any statistical information: *it is not possible to estimate the range of applicability of the FT-MLMC method with a single run.* The same holds for attempts to estimate the constants in the a priori error estimate from data obtained with one single run.

6. Conclusions

We introduced and analyzed a checkpoint-free and fault-tolerant multilevel Monte Carlo strategy, termed FT-MLMC. The approach is based on disregarding all samples affected by a node failure at run-time. It is assumed that MPI is extended such that processes unaffected by a failure can continue working and communicating. The MLMC estimate is computed with the remaining samples. By incorporating general statistical models of sample losses into the mathematical failure model and into the error bound of the FT-MLMC method a new MLMC error bound conditional on prescribed node-failure statistics is derived.

The principal conclusion of the present analysis is that *up to a certain rate of node failures the FT-MLMC error bound and, hence, the performance of the FT-MLMC, is provably of the same type as that of the standard, fault-free MLMC.* We therefore conclude that *in this range of failure rates*, in MC and MLMC PDE simulations there is no need for additional fault tolerance strategies such as, e.g., checkpoint/restart or re-computation of lost samples.

Due to the increasing likelihood of node-failures at run-time in emerging massively parallel hardware, obviation of checkpointing may afford substantial increases in parallel efficiency and scalability. In our analysis, we paid particular attention to the case when failures are Weibull distributed. We considered, exemplarily, a hyperbolic system of conservation laws with random input

data where samples were computed with the Finite Volume Method (FVM). We emphasize, however, that neither the Weibull distribution nor the FVM are essential for the principal conclusions about the performance of the FT-MLMC method.

We compared the derived FT-MLMC error bound with measured errors of a fault tolerant implementation [20, 21]. We showed by a number of examples that the FT-MLMC method compared to the standard MLMC method can solve, on a given hardware platform, considerably larger and more time-consuming problems as compared to the standard fault-free MLMC method.

The proposed approach of simply discarding samples affected by a node failure at run-time has the additional benefit that failures do not extend the run-time as does for instance checkpoint/restart, or the re-computation of lost samples. This leads to a bounded run-time, even if failures occur.

We conclude with the need for further research to develop better and more realistic stochastic and/or deterministic failure models for massively parallel (in particular, exascale) computing platforms. In the best case a failure model would be provided by hardware vendors, ideally with failure intensity parameters. Another line of research is the development of effective self-calibrating statistical failure models which, when run on a given computing platform, would "probe" a given hardware for failures and accumulate estimates of failure parameters at runtime.

The presently proposed models and their analysis completely disregard silent-errors. In a further work we may cover some silent-errors by detecting them by statistical means [10].

We further emphasize that the presently proposed failure models are uncorrelated across the hardware platform: in the presently proposed models, failures occur independently of relative position of the node within the compute platform. There is, however, evidence that the 'aging' of hardware as well as the time of day and the day of the week can affect fault arrival intensity parameters [25], and it is plausible that spatial correlation (e.g. due to local overheating) can play a role as well.

We extended the FT-MLMC algorithm towards a dynamic setting, where each FT-MLMC simulation generates its individual failure statistics and reacts accordingly [18]. In this approach an a priori parametric failure model is no longer required. However, a fault tolerant implementation becomes even more involved.

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