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Abstract. We consider the scattering of acoustic or electromagnetic waves at a penetrable object composed of different homogeneous materials. This problem can be recast as a first-kind boundary integral equation posed on the interface trace spaces through what we call a single trace boundary integral equation formulation (STF). Its Ritz-Galerkin discretization by means of low-order piecewise polynomial boundary elements on fine interface triangulations leads to ill-conditioned linear systems of equations, which defy efficient iterative solution.

Powerful preconditioners for discrete boundary integral equations are provided by the policy of operator preconditioning provided that the underlying trace spaces support a duality pairing with L^2 pivot space. This condition is not met by the STF. As a remedy we have proposed two variants of new multi-trace boundary integral equations (MTF); whereas the STF features unique Cauchy traces on material domain interfaces as unknowns, the multi-trace approach tears apart the traces so that local traces are recovered. Local trace spaces are in duality with respect to the L^2-pairing, and, thus, operator preconditioning becomes available for MTF.

Keywords. Helmholtz equation, Maxwell's equation, transmission problems, boundary integral equations, PMCHWT, operator preconditioning, boundary elements, multi-trace formulations.

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1 Introduction

Boundary integral equations connecting traces of solutions of linear elliptic boundary value problems were a key discovery of classical analysis and potential theory.

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They have also become a pivotal tool for numerical simulation, since they provide the foundation of the boundary element method in all its variants.

No doubt, the study of boundary integral equations is a mature field of applied mathematics. Thus, the reader might be surprised to learn that the field has witnessed recent new developments. At their cradle stood transmission problems arising in the scattering of waves at composite objects. A composite object is finite and composed of a few simple parts with constant material properties inside each of them, see Section 1.1 for details. It goes without saying that this kind of geometry is very common in technical designs.

The use of boundary element methods to tackle such transmission problems numerically is well established and they have become the methods of choice for a range of industrial applications. Yet, for scattering at composite objects scalability of numerical simulations based on traditional boundary element approaches emerged as a challenge; despite the use of sophisticated matrix compression techniques like multipole [15], adaptive cross approximation [2], and panel clustering [47, Ch. 7], computational effort still increases much faster than the number of degrees of freedom. The sole culprit is sluggish convergence of iterative solvers, when applied to high resolution boundary element models. Unfortunately, it seems to be the very structure of the customary first-kind boundary integral equations that denies us the remedy of effective preconditioners. In Section 4 we are going to explain this in detail.

This takes us to the heart of this article: after having understood the basic difficulties inherent in the standard first-kind boundary integral equations, called *single-trace formulation* (STF) hereafter, we will try to convince the reader that those can be overcome by switching to *multi-trace formulations* (MTF). They come in two different varieties, the global MTF to which Section 5 is devoted, and the local MTF treated in Section 6.

Of course, the problem has been attacked from other directions, notably by means of domain decomposition techniques like “Boundary element tearing and interconnecting” (BETI) [36, 51]. Schemes with a multi-trace flavor have also been proposed in computational electromagnetics, see [41, 43, 42] and [49, Sect. 3-4]. They seem to be related to the class of method proposed in this article, but the precise link still awaits disclosure.

In this article we aim to motivate and explain our novel multi-trace boundary integral equations. The emphasis will be rather on their derivation than on comprehensive theoretical analysis. Simplicity will often trump generality and for rigorous proofs and technical details the reader is referred to the original publications

- [19] as concerns the global MTF for acoustic scattering,
- [18], where the global MTF for electromagnetic scattering was introduced,
- [32], which proposed the local MTF.

Discretization will be addressed, because it is a particular difficulty haunting the low order boundary element Ritz-Galerkin discretization of the STF that has triggered the pursuit of new approaches. Yet, we are not going to delve into a full-fledged numerical analysis of a Galerkin discretization of the new MTF. Nor will we report numerical results; these can be found in the seminal publications cited above.

Eventually, we would like to remind the reader that this article reports very recent developments that are unfolding with numerous issues still unresolved. Here we can only provide a snapshot of current research.

List of notations

Ω_i	material sub-domains $\subset \mathbb{R}^d$, Ω_0 unbounded, see Fig. 1
N	number of sub-domains
Ω_*	union of closures of bounded sub-domains
Ω_0	(unbounded) complement of Ω_*
$\boldsymbol{n}, \boldsymbol{n}_i$	exterior unit normal vector fields
Γ_{ij}	interface $\partial\Omega_i \cap \partial\Omega_j$, often assumed to have non-vanishing $d-1$-dimensional measure
Σ	skeleton, union of all interfaces
$\kappa_i, \kappa(\boldsymbol{x})$	(local) wave numbers
$\alpha_i, \alpha(\boldsymbol{x})$	(locally constant) coefficients for PDE operator, see (1.4)
D	first order partial differential operator $\mathbf{grad}$ or $\mathbf{curl}$
D^*	L^2-adjoint of D, either $-\operatorname{div}$ or $\mathbf{curl}$
L	2nd-order partial differential operator $\mathsf{D}^*(\alpha(\boldsymbol{x})\mathsf{D}\cdot) - \kappa^2(\boldsymbol{x})\mathbf{u}$
$\boldsymbol{\mathcal{H}}(\mathsf{D}, \Omega)$	Sobolev spaces, see (1.5)
$\mathbf{u}_{\text{inc}}$	incoming wave, see (1.7)
$\mathcal{T}(\mathsf{D}, \partial\Omega)$	trace space for Sobolev space $\boldsymbol{\mathcal{H}}(\mathsf{D}, \Omega)$
T_D	(Dirichlet) trace operator
$[\cdot,\cdot], [\![\cdot,\cdot]\!]$	bi-linear pairings, usually of L^2-type
$\langle\cdot,\cdot\rangle$	duality pairing between a vector space and its dual
T_N	Neumann trace operator, see (2.3)
$\mathcal{T}(\partial\Omega)$	Cauchy trace space, product of Dirichlet and Neumann trace spaces
$\mathfrak{u}, \mathfrak{v}, \mathfrak{w}, \ldots$	elements of Cauchy trace spaces
$\mathbb{A}, \mathbb{B}, \ldots$	linear operators on Cauchy trace spaces
$\mathbb{T}, \mathbb{T}^+$	Cauchy trace operator, see (2.6)
$\mathbb{X}$	exterior-to-interior trace transfer operator, see (2.8)
SL, DL	single and double layer potentials, see Theorem 2.2
$\mathbb{G}$	total potential, see (2.13)
$\mathbb{P}$	Calderón projector, see Definition 2.4

$\mathcal{CD}(\partial\Omega)$	space of Cauchy data for L, see Definition 2.5
$\mathbb{A}, \mathbb{A}_j[i]$	local compound boundary integral operator, see (2.17)
$\boldsymbol{\mathcal{MT}}(\Sigma)$	multi-trace space, see (3.1)
$\vec{\mathfrak{u}}, \vec{\mathfrak{v}}, \vec{\mathfrak{w}}, \dots$	elements of $\boldsymbol{\mathcal{MT}}(\Sigma)$
$\mathbb{L}_i$	localization operators, $\mathbb{L}_i : \boldsymbol{\mathcal{MT}}(\Sigma) \to \mathcal{T}(\partial\Omega_i)$
$\boldsymbol{\mathcal{ST}}(\Sigma)$	single-trace space, see (3.5)
$\mathbb{S}'_\Sigma$	single-trace boundary integral operator (weak form), see (3.20)
$\Sigma_h, \Gamma_{j,h}$	boundary meshes

1.1 Geometry

The notion of a composite scatterer inspires the geometric setting adopted throughout this article. It is supposed to occupy the bounded domain $\Omega_* \subset \mathbb{R}^d$, $d = 2, 3$, it features a Lipschitz boundary, and it is composed of so-called sub-domains $\Omega_i \subset \mathbb{R}^d$, $i = 1, \dots, N$, that represent open and connected curvilinear Lipschitz polygons ($d = 2$) or polyhedra ($d = 3$). We demand that they do not intersect ($\Omega_i \cap \Omega_j = \emptyset$) for $i \neq j$, and that they form a partition of Ω_* in that

$$\overline{\Omega}_* = \bigcup_{i=1}^{N} \overline{\Omega}_i \ .$$

The unbounded complement of $\overline{\Omega}_*$ should be connected and will provide another sub-domain Ω_0, which means $\Omega_0 := \mathbb{R}^d \setminus \overline{\Omega}_*$.

The generic situation that we have in mind is depicted in Figure 1. The sub-domains will usually be adjacent, which engenders "material junction points", that is, points on which at least three sub-domains abut (marked in Figure 1). In this case, some Ω_i will inevitably have non-smooth boundaries. For each $i = 1, \dots, N$, the boundary $\partial\Omega_i$ is orientable and can be endowed with a unit normal vector field $\boldsymbol{n}_i$ pointing into the exterior of Ω_i.

We write Γ_{ij} for the common interface of Ω_i and Ω_j, $\Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j$, $i \neq j$. Two sub-domains Ω_i and Ω_j are adjacent, if Γ_{ij} is a $d-1$-dimensional Lipschitz manifold (with boundary). In this case we equip Γ_{ij} with an intrinsic orientation that amounts to prescribing a specific transversal direction. Note that, when referring to "all interfaces Γ_{ij}" we only include those that have a positive $d-1$-dimensional measure. We refer to them as "genuine interfaces" in the sequel. The union of all the interfaces Γ_{ij} forms the so-called *skeleton*

$$\Sigma := \bigcup_{ij} \Gamma_{ij} = \bigcup_{i=0}^{N} \partial\Omega_i \ .$$

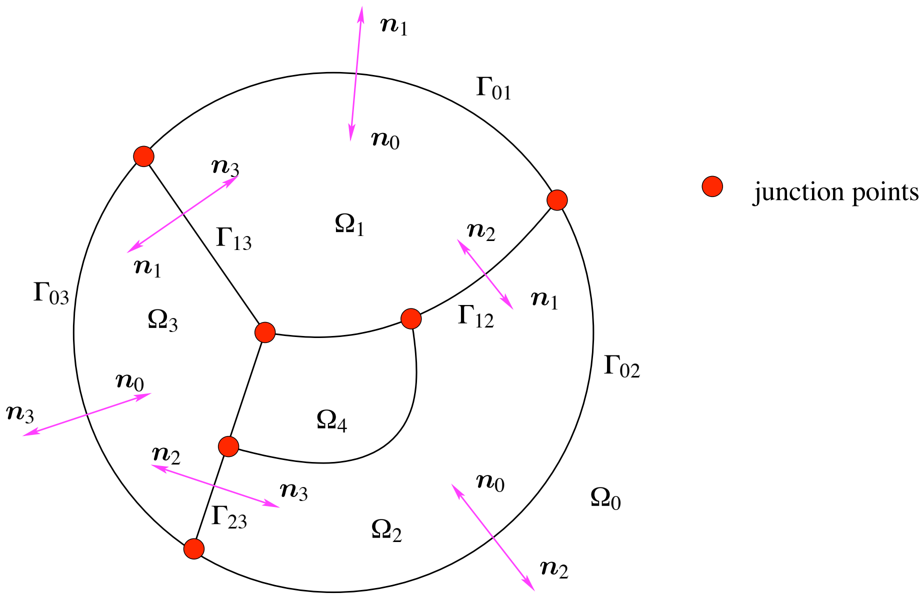

Figure 1. Typical geometry of a 2D "composite object", $N = 4$, induced orientations of some interfaces indicated by normal directions.

For $N > 1$ the skeleton Σ will usually not be orientable, nor be a manifold.

1.2 Transmission Problems

The application that we have in mind for the new multi-trace integral equations are time-harmonic wave scattering problems. The first instance is acoustic scattering with penetrable materials, modelled by a transmission problem for the Helmholtz equation [20, Ch. 2], [40, Ch. 2],

$$\begin{aligned} &-\operatorname{div}(\alpha_i\, \mathbf{grad}\, u) - \kappa_i^2 \beta_i u = 0 \quad \text{in } \Omega_i\ , \quad i = 0, \dots, N\ , \\ &+ \quad \text{transmission conditions across } \Gamma_{ij} \text{ for adjacent sub-domains,} \\ &+ \quad \text{Sommerfeld radiation conditions at } \infty. \end{aligned} \tag{1.1}$$

Here, the $\kappa_i \geq 0$ are the local wave numbers, the coefficients α_i are positive definite $d \times d$ tensors, and $\beta_i > 0$, $i = 0, \dots, N$. The transmission and radiation conditions will be specified in detail in Section 2.1. A related scattering problem can be stated for $d = 3$ and electromagnetic waves based on the time-harmonic Maxwell equations [20, Ch. 6], [40, Ch. 5],

$$\begin{aligned} &\mathbf{curl}(\alpha_i\,\mathbf{curl}\,\mathbf{u}) - \kappa_i^2\beta_i\mathbf{u} = 0 \quad \text{in } \Omega_i \ , \quad i = 0,\dots,N \ , \\ &+ \quad \text{transmission conditions across } \Gamma_{ij} \text{ for adjacent sub-domains,} \\ &+ \quad \text{Silver-Müller radiation conditions at } \infty, \end{aligned} \tag{1.2}$$

where $\mathbf{u}$ stands for the electric or magnetic field. In the former case, α_i is called the magnetic susceptibility and β_i the electric permittivity. Both are positive definite 3×3 tensors.

For the sake of simplicity, in the remainder of this article we will assume $\beta_i \equiv 1$ in (1.1) and β_i to be the identity matrix in (1.2). In both cases α_i is supposed to be a positive number, which describes isotropic materials. Dropping the subscript from α_i and κ_i and writing $\alpha(\boldsymbol{x})$ and $\kappa(\boldsymbol{x})$ indicates that we consider the corresponding piecewise constant functions on $\mathbb{R}^d$.

The transmission problems (1.1) and (1.2) have much in common. Therefore we opt for a unified treatment in an abstract framework. We are aware, that cognitive ease suggests a presentation for a concrete transmission problem. On the other hand, abstraction unveils important general patterns, which, eventually, promotes insight. In our opinion this benefit is worth the extra effort it takes to grasp abstraction. In this spirit, let us introduce the general second-order partial differential operator

$$\mathsf{L}\mathbf{u} := \mathsf{D}^*(\alpha(\boldsymbol{x})\mathsf{D}\mathbf{u}) - \kappa^2(\boldsymbol{x})\mathbf{u} \ , \tag{1.3}$$

where D is a suitable first-order differential operator, for instance, $\mathsf{D} = \mathbf{grad}$ for the Helmholtz equation (1.1), and $\mathsf{D} = \mathbf{curl}$ for Maxwell's equation. The other first-order differential operator occurring in (1.4) is the formal L^2-adjoint D^* of D. Then solutions of both (1.1) and (1.2) satisfy

$$\mathsf{L}\mathbf{u} = 0 \quad \text{locally in } \Omega_i \ , \quad i = 0,\dots,N \ . \tag{1.4}$$

For the variational formulation of the transmission problems associated with (1.4) we have to establish a function space framework. To that end, for the remainder of this section let Ω be a generic domain $\subset \mathbb{R}^d$, $d = 2, 3$. The natural domains of definition of both D and D^* are the Sobolev spaces [1 2]

$$\begin{aligned} \boldsymbol{\mathcal{H}}(\mathsf{D},\Omega) &:= \{\mathbf{u} \in L^2(\Omega) : \ \mathsf{D}\mathbf{u} \in L^2(\Omega)\} \ , \\ \boldsymbol{\mathcal{H}}(\mathsf{D}^*,\Omega) &:= \{\mathbf{u} \in L^2(\Omega) : \ \mathsf{D}^*\mathbf{u} \in L^2(\Omega)\} \ . \end{aligned} \tag{1.5}$$

They become Hilbert spaces when equipped with the natural graph norm. In case Ω is unbounded, a subscript "loc" will tag the Frechet spaces of functions that belong to

[1] Bold roman typeface will mark functions or vector fields defined on a domain $\subset \mathbb{R}^d$.
[2] We use calligraphic font for Hilbert spaces of functions, unless this clashes with firmly established notational conventions for Sobolev spaces.

the corresponding function space on each compact subset. Moreover, in order to deal with the radiation conditions at ∞ for $\Omega = \Omega_0$, the space $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega)$ has to be modified by switching to weaker weighted norms. This yields larger Hilbert spaces $\boldsymbol{\mathcal{H}}_{\mathrm{rad}}(\mathsf{D},\Omega)$ that contain all "physically meaningful outgoing solutions". Details can be found in [47, Sect. 2.9.2.4] and [40, Sects. 2.6.5 & 5.4].

A function $\mathbf{u}$ qualifies as a solution of $\mathsf{L}\mathbf{u} = 0$ on Ω only if it belongs to $\boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{D},\Omega)$. In addition, the differential equation immediately implies $\alpha\mathsf{D}\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{D}^*,\Omega)$ and we conclude that $\mathbf{u}$ is an element of the function space

$$\boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L},\Omega) := \{\mathbf{v} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{D},\Omega) : \ \alpha_i\mathsf{D}\mathbf{v} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{D}^*,\Omega)\} \ . \tag{1.6}$$

Sources will be introduced into our scattering models through a given *incident wave* $\mathbf{u}_{\mathrm{inc}} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L},\mathbb{R}^d)$ that satisfies

$$\mathsf{L}_0\mathbf{u}_{\mathrm{inc}} := \mathsf{D}^*(\alpha_0\mathsf{D}\mathbf{u}) - \kappa_0^2\mathbf{u} = 0 \quad \text{everywhere in } \mathbb{R}^d \ . \tag{1.7}$$

Plane waves provide the most common specimens for $\mathbf{u}_{\mathrm{inc}}$.

Definition 1.1. A function $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L},\Omega) \cap \boldsymbol{\mathcal{H}}_{\mathrm{rad}}(\mathsf{D},\Omega)$ is called a *radiating solution* (on Ω), if it solves $\mathsf{L}\mathbf{u} = 0$ in the sense of distributions.

The function spaces introduced above supply a suitable framework for the variational formulation of the transmission problems that are in the focus of this article: seek $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{D},\mathbb{R}^d)$ such that

$$[\alpha\mathsf{D}\mathbf{u},\mathsf{D}\mathbf{v}]_{\mathbb{R}^d} - \left[\kappa^2\mathbf{u},\mathbf{v}\right]_{\mathbb{R}^d} = 0 \quad \forall \mathbf{v} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{D},\mathbb{R}^d) \ , \tag{1.8}$$

where $[\mathbf{w},\mathbf{q}]_{\mathbb{R}^d} := \int_{\mathbb{R}^d} \mathbf{w}(\boldsymbol{x}) \cdot \mathbf{q}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$, and the *scattered field* $\mathbf{u} - \mathbf{u}_{\mathrm{inc}}$ satisfies

$$(\mathbf{u} - \mathbf{u}_{\mathrm{inc}})_{|\Omega_0} \in \boldsymbol{\mathcal{H}}_{\mathrm{rad}}(\mathsf{D},\Omega_0) \ . \tag{1.9}$$

Obviously, solutions of (1.8) are radiating solutions in each bounded Ω_i. By Rellich's lemma, analytic continuation techniques and Fredholm arguments existence and uniqueness of solutions can be established [1, 28].

Theorem 1.2. *For* $\mathsf{D} = \mathbf{grad}$ *("Helmholtz case") and* $\mathsf{D} = \mathbf{curl}$ *("Maxwell case") the variational problems* (1.8) *possess a unique solution.*

Remark 1.1 (Complex and vanishing wave numbers). Lossy materials, in which propagating waves are damped, can be modelled by wave numbers κ_i with positive imaginary part. All methods discussed in this article can deal with $\mathrm{Im}\,\kappa_i > 0$ without modifications.

In the Helmholtz case $\mathsf{D} = \mathbf{grad}$ all considerations of this article carry over to the case of local pure diffusion $\kappa_i = 0$ for some or all i's. If $\kappa_0 = 0$ the radiation conditions have to be replaced with suitable decay conditions, see [38, Ch. 8].

In the Maxwell case $\mathsf{D} = \mathbf{curl}$ we cannot accommodate $\kappa_i = 0$ for any i due to the notorious low-frequency instability of the representation formulas underlying our methods, see [22, 24] for a discussion. △

2 Boundary Integral Operators

Many boundary value problems for linear second order differential operators with constant coefficients can be recast in the form of boundary integral equations, and this can be achieved through a number of standard steps. This is a venerable insight of applied mathematics dating back to the 19th century and monographs offer comprehensive coverage of the topic, among them [38, 47, 33]. This section is meant to review and summarize relevant formulas and results, introducing key notions and notations along the way.

In this section the exposition will mainly address a generic domain $\Omega \subset \mathbb{R}^d$ with compact Lipschitz boundary $\partial\Omega$ and exterior unit normal vectorfield $\boldsymbol{n} \in L^\infty(\partial\Omega)$. Throughout, α and κ are positive numbers and stand for constant coefficients in the partial differential operator $\mathsf{L} := \mathsf{D}^*\alpha\mathsf{D} - \kappa^2\mathsf{Id}$, *cf.* (1.3).

2.1 Trace spaces and operators

Writing $\boldsymbol{\mathcal{H}}_0(\mathsf{D},\Omega)$ for the closure of smooth compactly supported functions in $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega)$, it turns out that the quotient space $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega)/\boldsymbol{\mathcal{H}}_0(\mathsf{D},\Omega)$ is isomorphic to an intrinsically defined function space $\mathcal{T}(\mathsf{D},\partial\Omega)$ on the boundary $\partial\Omega$, a so-called trace space. "Intrinsic" indicates that we can also establish an isomorphism

$$\mathcal{T}(\mathsf{D},\partial\Omega) \cong \boldsymbol{\mathcal{H}}(\mathsf{D},\mathbb{R}^d \setminus \overline{\Omega})/\boldsymbol{\mathcal{H}}_0(\mathsf{D},\mathbb{R}^d \setminus \overline{\Omega}) \; .$$

The canonical projection $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega) \mapsto \mathcal{T}(\mathsf{D},\partial\Omega)$ gives rise to a surjective and continuous trace operator

$$\mathsf{T}_\mathsf{D} : \boldsymbol{\mathcal{H}}(\mathsf{D},\Omega) \mapsto \mathcal{T}(\mathsf{D},\partial\Omega) \; .$$

It owes its name to the fact that T_D applied to smooth functions in $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega)$ on a domain Ω with piecewise smooth boundary agrees with a particular pointwise restriction almost everywhere on $\partial\Omega$. This relationship is addressed by so-called trace theorems.

Traces occur in the crucial integration by parts formulas ("Green's theorems")

$$[\mathsf{D}\mathbf{u},\mathbf{v}]_\Omega - [\mathbf{u},\mathsf{D}^*\mathbf{v}]_\Omega = [\mathsf{T}_\mathsf{D}\,\mathbf{u},\mathsf{T}_{\mathsf{D}^*}\,\mathbf{v}]_{\partial\Omega} \quad \forall \mathbf{u} \in \boldsymbol{\mathcal{H}}(\mathsf{D},\Omega),\; \mathbf{v} \in \boldsymbol{\mathcal{H}}(\mathsf{D}^*,\Omega) \; , \tag{2.1}$$

with $[\cdot,\cdot]_\Omega$ and $[\cdot,\cdot]_{\partial\Omega}$ denoting (extensions of) the bi-linear L^2-type pairings on Ω and $\partial\Omega$, respectively.

Notations and conventions. Single and double square brackets designate *bi-linear* forms $[\cdot,\cdot]\,,[\![\cdot,\cdot]\!] : V \times W \mapsto \mathbb{C}$ on function spaces, so-called pairings. Usually they will carry an additional subscript for clear identification. Angle brackets $\langle\cdot,\cdot\rangle$ are reserved for the duality pairing $\langle\cdot,\cdot\rangle : V' \times V \to \mathbb{C}$ between a normed space V and its dual V'. Round brackets $(\cdot,\cdot)_V$ are used for (sesqui-linear) inner products in Hilbert spaces. Recall from [47, Thm. 2.1.44] that a pairing $[\cdot,\cdot] : V \times W \to \mathbb{C}$ induces an isomorphism $V \mapsto W'$, if

$$\begin{aligned} \exists c > 0 : \quad \sup_{w \in W} \frac{|\,[v,w]\,|}{\|w\|_W} \geq c\,\|v\|_V \quad \forall v \in V \ , \\ \sup_{v \in V} \frac{|\,[v,w]\,|}{\|v\|_V} > 0 \quad \forall w \in W \ . \end{aligned} \tag{2.2}$$

We phrase this as "W is dual to V with respect to the pairing $[\cdot,\cdot]$", if $[\cdot,\cdot]$ acts on $V \times V$, then V is called "self-dual with respect to the pairing $[\cdot,\cdot]$".

We would like to highlight an important consequence of (2.1).

Theorem 2.1. *The spaces $\mathcal{T}(\mathsf{D},\partial\Omega)$ and $\mathcal{T}(\mathsf{D}^*,\partial\Omega)$ are dual to each other with respect to the pairing $[\cdot,\cdot]_{\partial\Omega}$.*

Two trace operators are induced by the second-order partial differential operator L:

$$\begin{aligned} \mathsf{T}_D : \boldsymbol{\mathcal{H}}(\mathsf{D},\Omega) \to \mathcal{T}(\mathsf{D},\partial\Omega) \quad &, \quad \mathsf{T}_D := \mathsf{T}_{\mathsf{D}} \ , \\ \mathsf{T}_N : \boldsymbol{\mathcal{H}}(\mathsf{L},\Omega) \to \mathcal{T}(\mathsf{D}^*,\partial\Omega) \quad &, \quad \mathsf{T}_N := \mathsf{T}_{\mathsf{D}^*}(\alpha \mathsf{D}\cdot) \ . \end{aligned} \tag{2.3}$$

We are going to refer to the surjective operators T_D and T_N as *Dirichlet trace* and *Neumann trace*, respectively. For the associated trace spaces we adopt the notation $\mathcal{T}_D(\partial\Omega) = \mathcal{T}(\mathsf{D},\partial\Omega)$ and $\mathcal{T}_N(\partial\Omega) := \mathcal{T}(\mathsf{D}^*,\partial\Omega)$ and the parlance "Dirichlet trace space" and "Neumann trace space". These can be merged into the Cauchy trace space

$$\mathcal{T}(\partial\Omega) := \mathcal{T}_D(\partial\Omega) \times \mathcal{T}_N(\partial\Omega) \ , \tag{2.4}$$

which is *self-dual* with respect to the pairing [3]

$$[\![\mathfrak{u},\mathfrak{v}]\!]_{\mathcal{T}(\partial\Omega)} := [u,\varphi]_{\partial\Omega} + [v,\nu]_{\partial\Omega} \ , \quad \mathfrak{u} := \begin{pmatrix} u \\ \nu \end{pmatrix}, \ \mathfrak{v} := \begin{pmatrix} v \\ \varphi \end{pmatrix} \in \mathcal{T}(\partial\Omega) \ . \tag{2.5}$$

A related compact notation is the Cauchy trace operator[4]

$$\mathbb{T} : \boldsymbol{\mathcal{H}}(\mathsf{L},\Omega) \to \mathcal{T}(\partial\Omega) \quad , \quad \mathbb{T}\,\mathbf{u} := \begin{pmatrix} \mathsf{T}_D\,\mathbf{u} \\ \mathsf{T}_N\,\mathbf{u} \end{pmatrix} . \tag{2.6}$$

Of course, traces can also be taken from the exterior of Ω, with the complement $\mathbb{R}^d \setminus \overline{\Omega}$ now playing the role of Ω. The corresponding trace operators are tagged with a superscript "+", for instance,

$$\mathbb{T}^+ : \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L},\mathbb{R}^d \setminus \overline{\Omega}) \to \mathcal{T}(\partial\Omega) \quad , \quad \mathbb{T}^+\,\mathbf{u}^+ := \begin{pmatrix} \mathsf{T}_D^+\,\mathbf{u}^+ \\ \mathsf{T}_N^+\,\mathbf{u}^+ \end{pmatrix} . \tag{2.7}$$

[3] Fraktur font is used to designate functions in the Cauchy trace space, whereas Roman typeface is reserved for Dirichlet traces, and Greek symbols for Neumann traces.

[4] We use blackboard bold typeface in order to distinguish operators acting on or mapping into Cauchy trace spaces.

Note that both Neumann traces T_N and T_N^+ rely on the same coefficient α, but assume opposite orientation of $\partial\Omega$! Thus, even for $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L},\mathbb{R}^d)$ exterior and interior traces agree only up to a change of sign of the Neumann component

$$\mathbb{T}^+\,\mathbf{u} = \mathbb{X}\,\mathbb{T}\,\mathbf{u} \quad \text{where} \quad \mathbb{X}: \begin{cases} \mathcal{T}(\partial\Omega) & \rightarrow & \mathcal{T}(\partial\Omega) \\ \binom{u}{\varphi} & \mapsto & \binom{u}{-\varphi} \end{cases} . \tag{2.8}$$

Concretization 2.1 (Helmholtz case). Our terminology "Dirichlet trace" and "Neumann trace" is borrowed from the standard parlance used for the Helmholtz problem, that is, for $\mathsf{D} = \mathbf{grad}$ and $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega) = H^1(\Omega)$. In this special case the trace operators boil down to

$$\mathsf{T}_D\,u = u_{|\partial\Omega} \quad , \quad \mathsf{T}_N\,u = (\alpha\,\mathbf{grad}\,u)_{|\partial\Omega}\cdot\boldsymbol{n}\ . \tag{2.9}$$

The trace spaces are the well-known Sobolev spaces $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$, see [38, Ch. 3]. △

Concretization 2.2 (Maxwell case). In the Maxwell case $\mathsf{D} = \mathbf{curl}$ and $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega) = \boldsymbol{H}(\mathbf{curl},\Omega)$ we are led to consider the tangential trace operators

$$\mathsf{T}_D\,\mathbf{u} = \boldsymbol{n}\times(\mathbf{u}_{|\partial\Omega}\times\boldsymbol{n}) \quad , \quad \mathsf{T}_N\,\mathbf{u} = (\alpha\,\mathbf{curl}\,\mathbf{u})_{|\partial\Omega}\times\boldsymbol{n}\ .$$

In [10, 7, 11] the related trace space have been characterized and we recall the notations $\mathcal{T}_D(\partial\Omega) = \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma,\Omega)$ and $\mathcal{T}_N(\partial\Omega) = \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Omega)$. △

Remark 2.1. Both, in (2.9) and (2.2), the Neumann traces will change signs, when the orientation of the normal vector $\boldsymbol{n}$ is flipped; the induced orientation of $\partial\Omega$ matters for the Neumann trace! This pattern has a simple explanation, when viewing (1.4) as an equation for differential forms, see [29, 35]. Then it turns out that multiplication with α corresponds to the action of a Hodge-operator, converting straight forms into twisted forms. △

The formula (2.1) can be applied to a solution $\mathbf{u}$ of (1.4) and with $\mathbf{v} := \alpha\mathsf{D}\overline{\mathbf{u}}$, where the overbar effects complex conjugation. Thus, by virtue of the relation (2.1), we obtain

$$[\mathsf{T}_D\,\mathbf{u},\mathsf{T}_N\,\overline{\mathbf{u}}]_{\partial\Omega} = \int_\Omega \alpha\,|\mathsf{D}\mathbf{u}|^2 - \kappa^2\,|\mathbf{u}|^2\,\mathrm{d}\boldsymbol{x}\ . \tag{2.10}$$

Notation. By a subscript index we indicate that a trace operator is applied to a particular sub-domain, e.g., $\mathsf{T}_{D,i}$, $\mathsf{T}_{N,i}$, $\mathbb{T}_i$, etc., $i = 0,\dots,N$. Invariably, the induced orientation of $\partial\Omega_i$ is used.

Remark 2.2 (Radiation conditions). Dirichlet and Neumann traces occur in the formal statement of the radiation conditions mentioned in the statement of the transmission problems. Let us adopt the notation T_D^r and T_N^r for Dirichlet- and Neumann traces onto the boundary of the ball $B_r := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| < r\}$ (from inside). Then $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L}, \Omega)$ will satisfy the radiation condition, if

$$\lim_{r\to\infty} \int\limits_{|\boldsymbol{x}|=r} |(\mathsf{T}_N^r\,\mathbf{u})(\boldsymbol{y}) - \imath\kappa(\mathsf{T}_D^r\,\mathbf{u})(\boldsymbol{y})|^2\,\mathrm{d}S(\boldsymbol{y}) = 0\ . \tag{2.11}$$

△

2.2 Potentials

Potential representations of solutions of (1.4) are the first step towards boundary integral equations. Here, by the term "potential" we refer to a linear operator that maps a function in a trace space on $\partial\Omega$ to a function defined everywhere on $\mathbb{R}^d \setminus \partial\Omega$. Further, potentials are supposed to provide radiating solutions of (1.4) in both Ω and $\mathbb{R}^d \setminus \overline{\Omega}$. Throughout, when acting on sufficiently smooth argument functions, potentials agree with integral operators with singular kernels spawned by fundamental solutions for L. The following key result can be found in [47, Sect. 3.11] and [38, Ch. 6] for Helmholtz' equation (1.1) and in [12] for Maxwell's equation (1.2).

Theorem 2.2 (Single domain representation formula). *There are continuous operators, depending on $\alpha > 0$ and $\kappa > 0$, the*

$$\textit{single layer potential}\quad \mathsf{SL} : \mathcal{T}_N(\partial\Omega) \mapsto \boldsymbol{\mathcal{H}}(\mathsf{L}, \Omega \cup \mathbb{R}^d \setminus \overline{\Omega})\ ,$$
$$\textit{double layer potential}\quad \mathsf{DL} : \mathcal{T}_D(\partial\Omega) \mapsto \boldsymbol{\mathcal{H}}(\mathsf{L}, \Omega \cup \mathbb{R}^d \setminus \overline{\Omega})\ ,$$

such that

(i) $\mathsf{SL}(\varphi)$ and $\mathsf{DL}(u)$ are radiating solutions of (1.4) *in $\Omega \cup \mathbb{R}^d \setminus \overline{\Omega}$ for any $\varphi \in \mathcal{T}_N(\partial\Omega)$, $u \in \mathcal{T}_D(\partial\Omega)$.*
(ii) every solution $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{rad}}(\mathsf{L}, \Omega)$ of $\mathsf{L}\mathbf{u} = 0$ can be written as

$$\mathbf{u} = -\mathsf{DL}(\mathsf{T}_D\,\mathbf{u}) + \mathsf{SL}(\mathsf{T}_N\,\mathbf{u})\ , \tag{2.12}$$

A more compact way to write (2.12) is

$$\mathbf{u} = \mathbb{G}(\mathbb{T}\,\mathbf{u})\ ,\ \text{with}\quad \mathbb{G}(\mathfrak{u}) := -\mathsf{DL}(u) + \mathsf{SL}(\varphi)\ ,\quad \mathfrak{u} := \begin{pmatrix} u \\ \varphi \end{pmatrix} \in \mathcal{T}(\partial\Omega)\ . \tag{2.13}$$

Concretization 2.3 (Helmholtz, $d = 3$). For the Helmholtz problem (1.1) with wave number κ and scalar coefficient $\alpha > 0$ the concrete integral formulas for the potentials are

$$\begin{aligned} \mathsf{SL}(\varphi)(\boldsymbol{x}) &= \tfrac{1}{\alpha}\int_{\partial\Omega} \Phi_{\kappa/\sqrt{\alpha}}(\boldsymbol{x}-\boldsymbol{y})\,\varphi(\boldsymbol{y})\,\mathrm{d}S\ ,\\ \mathsf{DL}(u)(\boldsymbol{x}) &= \int_{\partial\Omega} \mathbf{grad}\,\Phi_{\kappa/\sqrt{\alpha}}(\boldsymbol{x}-\boldsymbol{y})\cdot\boldsymbol{n}(\boldsymbol{y})\,u(\boldsymbol{y})\,\mathrm{d}S\ , \end{aligned} \tag{2.14}$$

for $\boldsymbol{x} \notin \partial\Omega$, u a distribution on $\partial\Omega$, and with the fundamental solution

$$\Phi_\theta(\boldsymbol{z}) = \frac{\exp(\imath\theta\,|\boldsymbol{z}|)}{4\pi\,|\boldsymbol{z}|}\ , \quad \theta \in \mathbb{R}\ . \tag{2.15}$$

△

Concretization 2.4 (Maxwell). In the case of Maxwell's equations (1.2) ($\kappa_i = \kappa$) we find from the fundamental Stratton-Zhu representation formula [12, Sect. 4], [20, Thm. 6.2] for $\boldsymbol{x} \notin \partial\Omega$

$$\begin{aligned}
\mathsf{SL}(\varphi)(\boldsymbol{x}) = {} & \tfrac{1}{\alpha}\int_{\partial\Omega} \Phi_{\kappa/\sqrt{\alpha}}(\boldsymbol{x}-\boldsymbol{y})\varphi(\boldsymbol{y})\,\mathrm{d}S + \\
& \tfrac{1}{\alpha\kappa^2}\,\mathbf{grad}_{\boldsymbol{x}} \int_{\Omega} \Phi_{\kappa/\sqrt{\alpha}}(\boldsymbol{x}-\boldsymbol{y})(\mathrm{div}_\Gamma\varphi)(\boldsymbol{y})\,\mathrm{d}S\ , \\
\mathsf{DL}(u)(\boldsymbol{x}) = {} & -\,\mathbf{curl}_{\boldsymbol{x}} \int_{\partial\Omega} \Phi_{\kappa/\sqrt{\alpha}}(\boldsymbol{x}-\boldsymbol{y})(\boldsymbol{n}\times u)(\boldsymbol{y})\,\mathrm{d}S\ ,
\end{aligned} \tag{2.16}$$

for tangential vector fields $\varphi, u : \partial\Omega \mapsto \mathbb{C}^3$. △

Inherent in the definition of potentials is that they generate a function defined both inside and outside of Ω. This enables us to take traces of potentials also from the exterior of $\partial\Omega$. The famous jump relations tell us that, thus, we will not get any new boundary integral operators.

Theorem 2.3 (Jump relations).

$$(\mathbb{T}^+ - \mathbb{X}\,\mathbb{T}) \circ \mathbb{G} = \begin{pmatrix} \mathsf{T}_D^+ - \mathsf{T}_D \\ \mathsf{T}_N^+ + \mathsf{T}_N \end{pmatrix} \circ \mathbb{G} = -\,\mathbb{X} \quad \text{on} \quad \mathcal{T}(\partial\Omega)\ .$$

We refer to [47, Sect. 3.3.1], [38, Thm. 6.11] (Helmholtz) and [12, Thm. 7] (Maxwell) for proof and more details.

2.3 Calderón projectors

Obviously applying the total trace operator $\mathbb{T}$ to the representation formula (2.12) exactly recovers the traces of a radiating solution of (1.4). The converse is also true. To formalize this, we introduce a pivotal operator, *cf.* [47, Sect. 3.6], [33, Sect. 5.6].

Definition 2.4 (Calderón projector). The Calderón projector $\mathbb{P}$ for L on Ω is defined by

$$\mathbb{P} := \mathbb{T} \circ \mathbb{G} : \mathcal{T}(\partial\Omega) \to \mathcal{T}(\partial\Omega)\ .$$

Thanks to the mapping properties of the trace operators and the potentials, $\mathbb{P}$ is linear and continuous. It has a close link with the subspace of $\mathcal{T}(\partial\Omega)$ that contains all traces of functions in the kernel of L.

Definition 2.5 (Cauchy data). The space of Cauchy data $\mathcal{CD}(\partial\Omega) \subset \mathcal{T}(\partial\Omega)$ for the differential equation (1.4) is

$$\mathcal{CD}(\partial\Omega) := \{\mathfrak{v} \in \mathcal{T}(\partial\Omega) : \exists \mathbf{u} \in \mathcal{H}_{\mathrm{rad}}(\mathsf{L},\Omega),\ \mathsf{L}\mathbf{u} = 0,\ \mathfrak{v} = \mathbb{T}\,\mathbf{u}\} \ .$$

As corollary of Theorem 2.2 we infer the following fact.

Theorem 2.6. *The Calderón projector is a continuous projector* onto $\mathcal{CD}(\partial\Omega)$*, that is,*

$$\mathbb{P}^2 = \mathbb{P} \quad \textit{and} \quad \mathrm{Ker}(\mathsf{Id} - \mathbb{P}) = \mathcal{CD}(\partial\Omega) \ .$$

Boundary integral operators are obtained by letting the trace operators T_D and T_N act on the potentials SL and DL. This is exactly what Definition 2.4 of the Calderón projector boils down to. Hence, we use it to define the continuous compound boundary integral operator

$$\mathbb{A} := \mathbb{P} - \tfrac{1}{2}\mathsf{Id} \overset{\text{(Thm. 2.3)}}{=} \tfrac{1}{2}\begin{pmatrix} \mathsf{T}_D + \mathsf{T}_D^+ \\ \mathsf{T}_N - \mathsf{T}_N^+ \end{pmatrix} \circ \mathbb{G} : \mathcal{T}(\partial\Omega) \to \mathcal{T}(\partial\Omega) \ . \tag{2.17}$$

Recalling the definition of $\mathbb{G}$, (2.17) boils down to applying two traces to two potentials (single and double layer), which yields a total of four boundary integral operators. Yet, we never need to worry about the fine structure of $\mathbb{A}$, which can safely be regarded as a “black box operator” that is at our disposal (in discretized form, of course, see Section 3.3).

Concretization 2.5 (Helmholtz case). In the case of $\mathsf{D} = \mathbf{grad}$, we can disassemble $\mathbb{A}$ into

$$\mathbb{A} = \begin{pmatrix} \mathsf{K} & \mathsf{V} \\ \mathsf{W} & \mathsf{K}' \end{pmatrix} , \tag{2.18}$$

where the individual boundary integral operators are known as single layer boundary integral operator V, double layer boundary integral operator K, adjoint double layer boundary integral operator K', and hypersingular boundary integral operator W. Our notation follows [47, Ch. 3], but the reader should be aware that no universally accepted notational conventions have emerged. △

Concretization 2.6 (Maxwell case). For a splitting of $\mathbb{A}$ according to (2.18) in the case of Maxwell's equations we refer to [30, Sect. 3] for descriptions of the operator building blocks. Beware, that [12] and the monograph [40] adopt conventions for the Neumann trace different from ours, which yields slightly different forms of the boundary integral operators. △

A deep result from the theory of boundary integral operators asserts a generalized Garding inequality for $\mathbb{A}$, see [47, Prop. 3.5.5] for Helmholtz and [12, Lemma 10] for the Maxwell case.

Theorem 2.7 (Coercivity of boundary integral operators). *For any α and κ there exist an isomorphism $\mathbb{F} : \mathcal{T}(\partial\Omega) \to \mathcal{T}(\partial\Omega)$ and a* compact *operator $\mathbb{K} : \mathcal{T}(\partial\Omega) \to \mathcal{T}(\partial\Omega)$ such that for some constant $C > 0$* [5]

$$|[\![\mathbb{A}\,\mathfrak{v}, \mathbb{F}\,\overline{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega)} + [\![\mathbb{K}\,\mathfrak{v}, \overline{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega)}| \geq C\,\|\mathfrak{v}\|^2_{\mathcal{T}(\partial\Omega)} \quad \forall \mathfrak{v} \in \mathcal{T}(\partial\Omega)\ .$$

Remark 2.3 (Weak boundary integral equations). In order to pursue Galerkin discretization of boundary integral equations, these have to be cast in variational form. To this end the self-duality of $\mathcal{T}(\partial\Omega)$ with respect to the pairing $[\![\cdot,\cdot]\!]_{\mathcal{T}(\partial\Omega)}$ comes handy; for instance, to $\mathbb{A}$ we can associate the bi-linear form

$$(\mathfrak{u}, \mathfrak{v}) \mapsto [\![\mathbb{A}\,\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\partial\Omega)}\ , \quad \mathfrak{u}, \mathfrak{v} \in \mathcal{T}(\partial\Omega)\ . \tag{2.19}$$

In turns, the bi-linear form gives rise to an operator $\mathcal{T}(\partial\Omega) \to \mathcal{T}(\partial\Omega)'$ mapping the Cauchy trace space to its dual. Occasionally we will tag such operators with a prime $'$, e.g., $\mathbb{A}'$. △

Notations. Above potentials and boundary integral operators have been introduced for a generic domain Ω and their dependence on the coefficients was suppressed in the notation. In the sequel we will have to consider potentials defined on different boundaries and relying on different coefficients.

Firstly, to distinguish their incarnations for a sub-domain Ω_j we use a subscript j, e.g., $\mathbb{A}_j$, $\mathbb{G}_j$, etc. Doing so, we tacitly assume that all traces and potentials are based on the coefficients α_j and κ_j associated with Ω_j. Sometimes, though potentials and integral operators are defined on $\partial\Omega_j$ we will nevertheless base them on the coefficients α_i and κ_i from another sub-domain Ω_i. If this is the case, we write $\mathbb{A}_j[i]$, $\mathbb{G}_j[i]$.

3 Classical Single-Trace Integral Equations

This section is dedicated to a review of a well established boundary integral equation formulation for the transmission problems (1.1) and (1.2). For Helmholtz' equation it was first analyzed in [53], but probably used earlier. For Maxwell's equations it agrees with the so-called Poggio-Miller-Chew-Harrington-Wu-Tsai (PMCHWT) integral equations [44, 14, 54, 27]. Its numerical analysis was first accomplished in [12].

3.1 Skeleton trace spaces

The skeleton multi-trace space is simply defined as

$$\boldsymbol{\mathcal{MT}}(\Sigma) := \boldsymbol{\mathcal{MT}}_D(\Sigma) \times \boldsymbol{\mathcal{MT}}_N(\Sigma)\ , \tag{3.1}$$

$$\text{with} \quad \begin{aligned} \boldsymbol{\mathcal{MT}}_D(\Sigma) &:= \mathcal{T}_D(\partial\Omega_0) \times \cdots \times \mathcal{T}_D(\partial\Omega_N)\ , \\ \boldsymbol{\mathcal{MT}}_N(\Sigma) &:= \mathcal{T}_N(\partial\Omega_0) \times \cdots \times \mathcal{T}_N(\partial\Omega_N)\ . \end{aligned} \tag{3.2}$$

[5] We use an overbar to designate complex conjugation.

It owes its name to the fact that on each interface Γ_{ij} a function $\vec{\mathfrak{u}} \in \boldsymbol{\mathcal{MT}}(\Sigma)$ comprises two pairs of Dirichlet and Neumann data, one contributed by the sub-domain on either side. Obviously, $\boldsymbol{\mathcal{MT}}(\Sigma)$ can be identified with the product of local Cauchy trace spaces

$$\boldsymbol{\mathcal{MT}}(\Sigma) \cong \mathcal{T}(\partial\Omega_0) \times \cdots \times \mathcal{T}(\partial\Omega_N) \;, \tag{3.3}$$

which illustrate the localized nature of $\boldsymbol{\mathcal{MT}}(\Sigma)$. We chose the ordering of local trace components given by (3.1) for the sake of simpler notations in Section 3. To isolate the contribution of a single sub-domain we rely on trivial localization operators [6]

$$\mathbb{L}_i : \boldsymbol{\mathcal{MT}}(\Sigma) \to \mathcal{T}(\partial\Omega_i) \quad , \quad \mathbb{L}_i\,\vec{\mathfrak{u}} := \begin{pmatrix} u_i \\ \nu_i \end{pmatrix} , \quad \begin{array}{l} \vec{\mathfrak{u}} = (u_0, \dots, u_N, \nu_0, \dots, \nu_N) \;, \\ u_i \in \mathcal{T}_D(\partial\Omega_i),\ \nu_i \in \mathcal{T}_N(\partial\Omega_i) \;. \end{array}$$

The multi-trace spaces inherit all properties of their local components. For instance, a self-duality of $\boldsymbol{\mathcal{MT}}(\Sigma)$ is obviously induced by the L^2-type bi-linear pairing, *cf.* (2.5),

$$[\![\vec{\mathfrak{u}}, \vec{\mathfrak{v}}]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)} := \sum_{i=0}^{N} [\![\mathbb{L}_i\,\vec{\mathfrak{u}}, \mathbb{L}_i\,\vec{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega_i)} \;, \quad \vec{\mathfrak{u}}, \vec{\mathfrak{v}} \in \boldsymbol{\mathcal{MT}}(\Sigma) \;. \tag{3.4}$$

Now, we introduce an important notion intimately linked to the transmission conditions that connect traces across interfaces. This connection is captured by the so-called *single-trace spaces*

$$\boldsymbol{\mathcal{ST}}(\Sigma) := \boldsymbol{\mathcal{ST}}_D(\Sigma) \times \boldsymbol{\mathcal{ST}}_N(\Sigma) \subset \boldsymbol{\mathcal{MT}}(\Sigma) \;, \tag{3.5}$$

$$\begin{aligned} \boldsymbol{\mathcal{ST}}_D(\Sigma) &:= \{(u_0, \dots, u_N) \in \boldsymbol{\mathcal{MT}}_D(\Sigma) :\ \exists \mathbf{u} \in \boldsymbol{\mathcal{H}}(\mathsf{D}, \mathbb{R}^d),\ u_i = \mathsf{T}_{\mathsf{D},i}\,\mathbf{u}\} \;, \\ \boldsymbol{\mathcal{ST}}_N(\Sigma) &:= \{(\nu_0, \dots, \nu_N) \in \boldsymbol{\mathcal{MT}}_N(\Sigma) :\ \exists \boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}(\mathsf{D}^*, \mathbb{R}^d),\ \nu_i = \mathsf{T}_{\mathsf{D}^*,i}\,\boldsymbol{\phi}\} \;. \end{aligned} \tag{3.6}$$

In words, functions in $\boldsymbol{\mathcal{ST}}_D(\Sigma)$ and $\boldsymbol{\mathcal{ST}}_N(\Sigma)$ are skeleton traces of functions defined on $\mathbb{R}^d$. Further explanations of the term "single trace" are given in Remark 3.2.

Concretization 3.1 (Helmholtz). For the case $\mathsf{D} = \mathbf{grad}$, the space $\boldsymbol{\mathcal{ST}}_D(\Sigma)$ contains restrictions of functions in $\boldsymbol{\mathcal{H}}(\mathsf{D}, \mathbb{R}^d) = H^1(\mathbb{R}^d)$ to the skeleton Σ. We deem the mental image of continuous functions on Σ an appropriate view of $\boldsymbol{\mathcal{ST}}_D(\Sigma)$ in this case.

The space $\boldsymbol{\mathcal{ST}}_N(\Sigma)$ contains the normal component traces of vectorfields in $\boldsymbol{H}(\mathrm{div}, \mathbb{R}^3)$. However, the normal components rely on the local exterior unit vectors $\boldsymbol{n}_i$, which reflect the induced orientation of $\partial\Omega_i$. Note that the two induced orientations of an interface Γ_{ij} are opposite. Thus, an element of $\boldsymbol{\mathcal{ST}}_N(\Sigma)$ will involve two Neumann traces on each interface that have opposite sign. $\triangle$

[6] Functions in a multi-trace space will be distinguished by an overset arrow, e.g., $\vec{\mathfrak{u}}, \vec{\mathfrak{v}}$.

Concretization 3.2 (Maxwell). Consider $d = 3$ and $\mathsf{D} = \mathbf{curl}$. Then $\boldsymbol{\mathcal{ST}}_D(\Sigma)$ will contain tangential components on Σ of vectorfields in $\boldsymbol{H}(\mathbf{curl}, \mathbb{R}^3)$. Imagine these as vectorfields $\Sigma \mapsto \mathbb{R}^3$, tangential to the interfaces in Σ that, in addition, are tangentially continuous across the seams between genuine interfaces.

Again, elements of $\boldsymbol{\mathcal{ST}}_N(\Sigma)$ will no longer support an interpretation as well-defined functions on Σ. They can be obtained as

$$\boldsymbol{\mathcal{ST}}_N(\Sigma) = \{(\nu_1, \dots, \nu_N) \in \boldsymbol{\mathcal{MT}}_N(\Sigma) : \ \exists \vec{\mathfrak{u}} \in \boldsymbol{\mathcal{ST}}_D(\Sigma) : \ \nu_i = u_i \times \boldsymbol{n}_i\} \ , \tag{3.7}$$

that is, by taking a tangential vectorfield on Σ and rotating it counterclockwise (seen from inside Ω_i) on $\partial\Omega_i$. It is clear that the result will be two vectorfields of opposite sign on each interface. △

Remark 3.1. With single trace spaces at our disposal, we can eventually give a rigorous description of the *transmission conditions* already mentioned in (1.1) and (1.2). These are conditions to be satisfied by functions $\mathbf{u}_i$ that provide local solutions of (1.4) on Ω_i so that they qualify as solutions of (1.8). They read

$$(\mathsf{T}_{D,0}\,\mathbf{u}_0, \dots, \mathsf{T}_{D,N}\,\mathbf{u}_N, \mathsf{T}_{N,0}\,\mathbf{u}_0, \dots, \mathsf{T}_{D,N}\,\mathbf{u}_N) \in \boldsymbol{\mathcal{ST}}(\Sigma) \ , \tag{3.8}$$

which means, in the sense of distributions,

$$\mathsf{T}_{D,i}\,\mathbf{u}_i = \mathsf{T}_{D,j}\,\mathbf{u}_j \quad \text{and} \quad \mathsf{T}_{N,i}\,\mathbf{u}_i = -\,\mathsf{T}_{N,j}\,\mathbf{u}_j \quad \text{on } \Gamma_{ij} \ . \tag{3.9}$$

In light of (3.6), the equivalence between (3.8) and (3.9) is known in the form of compatibility conditions inherent in the definition of Sobolev spaces. For instance, in the Helmholtz case, if $\mathbf{u}$ fulfills $\mathbf{u}_{|\Omega_i} \in H^1(\Omega_i)$ it will belong to $H^1(\mathbb{R}^d)$ if and only if it features weak continuity (agreement of Dirichlet traces in the sense of distributions) across all genuine interfaces. Analogous facts are known for $\mathbf{u}_{|\Omega_i} \in \boldsymbol{H}(\mathrm{div}, \Omega_i)$ and $\mathbf{u}_{|\Omega_i} \in \boldsymbol{H}(\mathbf{curl}, \Omega_i)$. △

A fundamental result is the following "polar set" characterization of $\boldsymbol{\mathcal{ST}}(\Sigma)$ as a subspace of $\boldsymbol{\mathcal{MT}}(\Sigma)$ given in [19, Prop. 2.1].

Theorem 3.1.

$$\boldsymbol{\mathcal{ST}}(\Sigma) = \{\mathfrak{u} \in \boldsymbol{\mathcal{MT}}(\Sigma) : \ [\![\mathfrak{u}, \mathfrak{v}]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)} = 0 \ \forall \mathfrak{v} \in \boldsymbol{\mathcal{ST}}(\Sigma)\} \ .$$

Before we give a formal proof, let us explain the intuition behind this result; for sufficiently smooth functions we find

$$[\![\vec{\mathfrak{u}}, \vec{\mathfrak{v}}]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)} = \sum_{i,j} \int_{\Gamma_{ij}} u_i\psi_i + v_i\nu_i + u_j\psi_j + v_j\nu_j \,\mathrm{d}S \ , \tag{3.10}$$

where $\vec{\mathfrak{u}} = (u_0, \dots, u_N, \nu_0, \dots, \nu_N)$, $\vec{\mathfrak{v}} = (v_0, \dots, v_N, \psi_0, \dots, \psi_N)$. This reflects the fact that each interface is visited twice when summing integrals over all sub-domain boundaries. Guided by the insight that functions in $\boldsymbol{\mathcal{ST}}_D(\Sigma)$ are single-valued on each interface, whereas functions in $\boldsymbol{\mathcal{ST}}_N(\Sigma)$ differ in sign on both sides of Γ_{ij}, *cf.* (3.9), we conclude complete cancellation of all terms in (3.10).

Proof of Theorem 3.1. (i) First assume $\vec{\mathfrak{u}} = \binom{\vec{u}}{\vec{\nu}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$, that is, there are associated functions $\mathbf{u} \in \boldsymbol{\mathcal{H}}(\mathsf{D}, \mathbb{R}^d)$ and $\boldsymbol{\phi} \in \boldsymbol{\mathcal{H}}(\mathsf{D}^*, \mathbb{R}^d)$ such that $u_i = \mathsf{T}_{\mathsf{D},i}\,\mathbf{u}$, $\nu_i = \mathsf{T}_{\mathsf{D}^*,i}\,\boldsymbol{\phi}$.

Pick $\vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$ with associated functions $\mathbf{v} \in \boldsymbol{\mathcal{H}}(\mathsf{D}, \mathbb{R}^d)$, $\boldsymbol{\psi} \in \boldsymbol{\mathcal{H}}(\mathsf{D}^*, \mathbb{R}^d)$, so that integration by parts according to (2.1) yields, $\vec{\mathfrak{v}} = \binom{\vec{v}}{\vec{\psi}}$,

$$\begin{aligned}
[\![\vec{\mathfrak{u}}, \vec{\mathfrak{v}}]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)} &= \sum_{i=1}^{N} [u_i, \psi_i]_{\partial\Omega_i} + [v_i, \nu_i]_{\partial\Omega_i} \\
&= \sum_{i=1}^{N} [\mathsf{D}\mathbf{u}, \boldsymbol{\psi}]_{\Omega_i} - [\mathbf{u}, \mathsf{D}^*\boldsymbol{\psi}]_{\Omega_i} + [\mathsf{D}\mathbf{v}, \boldsymbol{\varphi}]_{\Omega_i} - [\mathbf{v}, \mathsf{D}^*\boldsymbol{\varphi}]_{\Omega_i} \\
&= [\mathsf{D}\mathbf{u}, \boldsymbol{\psi}]_{\mathbb{R}^d} - [\mathbf{u}, \mathsf{D}^*\boldsymbol{\psi}]_{\mathbb{R}^d} + [\mathsf{D}\mathbf{v}, \boldsymbol{\varphi}]_{\mathbb{R}^d} - [\mathbf{v}, \mathsf{D}^*\boldsymbol{\varphi}]_{\mathbb{R}^d} = 0\ .
\end{aligned}$$

(ii) Pick $\mathfrak{u} = (u_0, \ldots, u_N, \nu_0, \ldots, \nu_N) \in \boldsymbol{\mathcal{MT}}(\Sigma)$ and local extensions $\mathbf{u}_i \in \boldsymbol{\mathcal{H}}(\mathsf{D}, \Omega_i)$ and $\mathbf{f}_i \in \boldsymbol{\mathcal{H}}(\mathsf{D}^*, \Omega_i)$ such that $u_i = \mathsf{T}_{\mathsf{D},i}\,\mathbf{u}_i$ and $\nu_i = \mathsf{T}_{\mathsf{D}^*,i}\,\mathbf{f}_i$. Then introduce the functions $\mathbf{u} \in L^2(\mathbb{R}^d)$ and $\mathbf{f} \in L^2(\mathbb{R}^d)$ defined piecewise according to

$$\mathbf{u}_{|\Omega_i} := \mathbf{u}_i \quad , \quad \mathbf{f}_{|\Omega_i} = \mathbf{f}_i\ , \quad i = 0, \ldots, N\ . \tag{3.11}$$

We have to show that $\mathbf{u} \in \boldsymbol{\mathcal{H}}(\mathsf{D}, \mathbb{R}^d)$ and $\mathbf{f} \in \boldsymbol{\mathcal{H}}(\mathsf{D}^*, \mathbb{R}^d)$ provided that $[\![\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Sigma)} = 0$ for all $\mathfrak{v} \in \boldsymbol{\mathcal{ST}}(\Sigma)$. This follows, if we can establish

$$\mathsf{D}\mathbf{u}_{|\Omega_i} = \mathsf{D}\mathbf{u}_i \quad , \quad \mathsf{D}^*\mathbf{f}_{|\Omega_i} = \mathsf{D}\mathbf{f}_i\ , \quad i = 0, \ldots, N\ , \tag{3.12}$$

in the sense of distributions, because a function that agrees with $\mathsf{D}\mathbf{u}_i \in L^2(\Omega_i)$ in each sub-domain certainly belongs to $L^2(\mathbb{R}^d)$.

Let $\mathbf{w} \in C_0^\infty(\mathbb{R}^d) \cap \boldsymbol{\mathcal{H}}(\mathsf{D}^*, \mathbb{R}^d)$ and $\mathbf{v} \in C_0^\infty(\mathbb{R}^d) \cap \boldsymbol{\mathcal{H}}(\mathsf{D}, \mathbb{R}^d)$ be smooth compactly supported test functions. Then

$$
\begin{aligned}
&-[\mathbf{u},\mathsf{D}^*\mathbf{w}]_{\mathbb{R}^d}+[\mathbf{f},\mathsf{D}\mathbf{v}]_{\mathbb{R}^d}=\sum_{i=0}^{N}-[\mathbf{u}_i,\mathsf{D}^*\mathbf{w}]_{\Omega_i}+[\mathbf{f}_i,\mathsf{D}\mathbf{v}]_{\Omega_i}\\
&=\sum_{i=0}^{N}-[\mathsf{D}\mathbf{u}_i,\mathbf{w}]_{\Omega_i}+[\mathsf{T}_{\mathsf{D},i}\,\mathbf{u}_i,\mathsf{T}_{\mathsf{D}^*,i}\,\mathbf{w}]_{\partial\Omega_i}+[\mathsf{D}^*\mathbf{f}_i,\mathbf{v}]_{\Omega_i}+[\mathsf{T}_{\mathsf{D},i}\,\mathbf{v},\mathsf{T}_{\mathsf{D}^*,i}\,\mathbf{f}_i]_{\partial\Omega_i}\\
&=\sum_{i=0}^{N}-[\mathsf{D}\mathbf{u}_i,\mathbf{w}]_{\Omega_i}+[\mathsf{D}^*\mathbf{f}_i,\mathbf{v}]_{\Omega_i}+[u_i,\mathsf{T}_{\mathsf{D}^*,i}\,\mathbf{w}]_{\partial\Omega_i}+[\mathsf{T}_{\mathsf{D},i}\,\mathbf{v},\nu_i]_{\partial\Omega_i}\\
&=\ \sum_{i=0}^{N}-[\mathsf{D}\mathbf{u}_i,\mathbf{w}]_{\Omega_i}+[\mathsf{D}^*\mathbf{f}_i,\mathbf{v}]_{\Omega_i}\\
&\qquad+\underbrace{[\![\mathfrak{u},(\mathsf{T}_{\mathsf{D},0}\,\mathbf{v},\dots,\mathsf{T}_{\mathsf{D},N}\,\mathbf{v},\mathsf{T}_{\mathsf{D}^*,0}\,\mathbf{w},\dots,\mathsf{T}_{\mathsf{D}^*,N}\,\mathbf{w})]\!]_{\mathcal{T}(\Sigma)}}_{=0}\,,
\end{aligned}
$$

which amounts to (3.12). □

Let us mention an immediate consequence of Theorem 3.1:

Corollary 3.2. $\mathcal{ST}(\Sigma)$ *is a closed subspace of* $\mathcal{MT}(\Sigma)$.

Remark 3.2. The term "single-trace space" conjures up the image of functions $\Sigma \mapsto \mathbb{C}$. In fact, pairs of such functions can be identified with elements of $\mathcal{ST}(\Sigma)$. First, we are going to elaborate the connection for the Helmholtz problem (1.1), that is $\mathcal{T}_D(\partial\Omega_i) = H^{\frac{1}{2}}(\partial\Omega_i)$ and $\mathcal{T}_N(\partial\Omega_i) = H^{-\frac{1}{2}}(\partial\Omega_i)$. Remember that each interface Γ_{ij} has an intrinsic orientation given by a prescribed transversal direction. A potential mismatch with induced orientation is taken into account by "orientation adjustment functions" $\sigma_j \in L^\infty(\partial\Omega_j)$, $j = 0,\dots,N$, defined as

$$
\sigma_j(\boldsymbol{x}) := \begin{cases} 1 & \text{, if } \boldsymbol{x}\in\Gamma_{ij}\subset\partial\Omega_j \text{ and orientations of } \partial\Omega_j \text{ and } \Gamma_{ij} \text{ match,}\\ -1 & \text{, if } \boldsymbol{x}\in\Gamma_{ij}\subset\partial\Omega_j \text{ and orientations of } \partial\Omega_j \text{ and } \Gamma_{ij} \text{ are opposite.}\end{cases}
$$

Observe that $\sigma_i = -\sigma_j$ on Γ_{ij}. Given a *continuous* function $\hat{u}$ on Σ with $\hat{u}_{|\Gamma_{ij}} \in H^1(\Gamma_{ij})$ and $\hat{\nu} \in L^2(\Sigma)$, we can now define

$$
u_j := \hat{u}_{|\partial\Omega_j} \in H^{\frac{1}{2}}(\partial\Omega_j) \quad , \quad \nu_j := \sigma_j \cdot \hat{\nu}_{|\partial\Omega_j} \in H^{-\frac{1}{2}}(\partial\Omega_j)\ , \tag{3.13}
$$

which gives $\vec{\mathfrak{u}} = (u_0,\dots,u_N,\nu_0,\dots,\nu_N) \in \mathcal{ST}(\Sigma)$.

For electromagnetic scattering, see (1.2), a similar approach works. We start from two *tangential* vector fields $\Sigma \mapsto \mathbb{C}^3$, again denoted by $\hat{u}$, and $\hat{\nu}$, that are piecewise smooth on the interfaces and *tangentially continuous* across the seams of the skeleton, that is, they have a unique tangential component on the boundaries of the interfaces, recall Concretization 3.2. Then define

$$\begin{aligned} u_j &:= \hat{u}_{|\partial\Omega_j} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \partial\Omega_j) , \\ \nu_j &:= \hat{\nu}_{|\partial\Omega_j} \times \boldsymbol{n}_j \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \partial\Omega_j) . \end{aligned} \tag{3.14}$$

Here, a sign flip as in (3.13) is induced by the opposite directions of $\boldsymbol{n}_i$ and $\boldsymbol{n}_j$ on Γ_{ij}. One easily verifies that $(u_0, \dots, u_N, \nu_0, \dots, \nu_N) \in \boldsymbol{\mathcal{ST}}(\Sigma)$. △

3.2 A first-kind boundary integral equation

Assume that $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\mathrm{rad}}(\mathsf{L}, \mathbb{R}^d)$ solves the transmission problem (1.8). Obviously, $\mathbf{u}_{|\Omega_j}$ provides local solutions of the homogeneous PDE, see Definition 1.1, on the bounded sub-domains $\Omega_1, \dots, \Omega_N$, and Theorem 2.6 permits us to conclude

$$(\mathsf{Id} - \mathbb{P}_j)\, \mathbb{T}_j\, \mathbf{u} = 0 \quad \text{in } \mathcal{T}(\partial\Omega_j) , \quad j = 1, \dots, N . \tag{3.15}$$

As $\mathbf{u} - \mathbf{u}_{\mathrm{inc}}$ is a radiating solution on Ω_0, owing to (1.9) the same argument gives

$$(\mathsf{Id} - \mathbb{P}_0)\, \mathbb{T}_0(\mathbf{u} - \mathbf{u}_{\mathrm{inc}}) = 0 \tag{3.16}$$

which implies that for all $\vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$

$$\sum_{j=0}^{N} \left[\!\left[(\mathsf{Id} - \mathbb{P}_j)\, \mathbb{T}_j\, \mathbf{u}, \mathbb{L}_j\, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_j)} = \left[\!\left[(\mathsf{Id} - \mathbb{P}_0)\, \mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}}, \mathbb{L}_0\, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_0)} . \tag{3.17}$$

A key observation is that by definition $\mathbb{T}_\Sigma\, \mathbf{u} \in \boldsymbol{\mathcal{ST}}(\Sigma)$ so that, thanks to Theorem 3.1, the variational formulation (3.17) is *equivalent* to

$$\sum_{j=0}^{N} \left[\!\left[\mathbb{A}_j\, \mathbb{T}_j\, \mathbf{u}, \mathbb{L}_j\, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_j)} = -\left[\!\left[(\mathsf{Id} - \mathbb{P}_0)\, \mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}}, \mathbb{L}_0\, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_0)} , \tag{3.18}$$

for all $\vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$. Another simplification results from the property (1.7), which implies $\mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}} \in \mathcal{CD}(\partial\Omega_*)$, and by Theorem 2.6 we can conclude that $(\mathsf{Id} - \mathbb{P}_0)\, \mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}} = \mathbb{T}_0\, \mathbf{u}_0$. Thus, the integral operators on the right hand side of (3.18) can be dropped.

We have derived a variational equation satisfied by $\mathbb{T}_\Sigma\, \mathbf{u} \in \boldsymbol{\mathcal{ST}}(\Sigma)$. Regarding the skeleton Cauchy traces as unknowns, this yields the classical single-trace boundary integral equation (STF): seek $\vec{\mathfrak{u}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$ such that

$$\boxed{\sum_{j=0}^{N} \left[\!\left[\mathbb{A}_j\, \mathbb{L}_j\, \vec{\mathfrak{u}}, \mathbb{L}_j\, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_j)} = -\left[\!\left[\mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}}, \mathbb{L}_0\, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_0)} \quad \forall \vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma) .} \tag{3.19}$$

This STF variational problem can equivalently be expressed as the operator equation[7]

$$\mathbb{S}'_\Sigma \vec{\mathfrak{u}} := \sum_{j=0}^{N} \mathbb{L}_j^* \, \mathbb{A}_j \, \mathbb{L}_j \, \vec{\mathfrak{u}} = -\, \mathbb{L}_0^* \, \mathbb{T}_0 \, \mathbf{u}_{\mathrm{inc}} \;, \tag{3.20}$$

with $\mathbb{S}'_\Sigma : \boldsymbol{\mathcal{ST}}(\Sigma) \to \boldsymbol{\mathcal{ST}}(\Sigma)'$. Here, $\mathbb{L}_j^* : \mathcal{T}(\partial\Omega_j) \to \boldsymbol{\mathcal{ST}}(\Sigma)'$ stands for an adjoint of the localization operator $\mathbb{L}_j : \boldsymbol{\mathcal{ST}}(\Sigma) \to \mathcal{T}(\partial\Omega_j)$ with respect to the respective L^2-type duality pairings

$$\left\langle \mathbb{L}_j^* \, \mathfrak{u}_j, \vec{\mathfrak{v}} \right\rangle_{\boldsymbol{\mathcal{ST}}(\Sigma)} := \left[\!\left[\mathfrak{u}_j, \mathbb{L}_j \, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_j)} \quad \forall \mathfrak{u}_j \in \mathcal{T}(\partial\Omega_j),\ \vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma) \;. \tag{3.21}$$

A consequence of Theorem 2.7 is that the bi-linear form on the left hand side of (3.19) fulfills a generalized Gårding inequality.

Theorem 3.3. *There is an isomorphism* $\mathbb{F} : \boldsymbol{\mathcal{ST}}(\Sigma) \mapsto \boldsymbol{\mathcal{ST}}(\Sigma)$ *and a compact operator* $\mathbb{K} : \boldsymbol{\mathcal{ST}}(\Sigma) \mapsto \boldsymbol{\mathcal{MT}}(\Sigma)$ *such that for some* $C > 0$

$$\left| \left\langle \mathbb{S}'_\Sigma \vec{\mathfrak{v}}, \mathbb{F}\,\overline{\vec{\mathfrak{v}}} \right\rangle + \left[\!\left[\mathbb{K}\,\vec{\mathfrak{v}}, \overline{\vec{\mathfrak{v}}} \right]\!\right]_{\mathcal{T}(\Sigma)} \right| \geq C \left\| \vec{\mathfrak{v}} \right\|^2_{\boldsymbol{\mathcal{MT}}(\Sigma)} \quad \forall \vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma) \;. \tag{3.22}$$

Concretization 3.3. For the case of the Helmholtz equation we find $\mathbb{F} = \mathsf{Id}$ and the assertion of Theorem 3.3 is a corollary of Theorem 2.7. Note that in this case (3.22) is a standard Garding inequality.

For electromagnetic wave propagation (1.2) a sophisticated $\mathbb{F}$ is needed based on a direct splitting of $\boldsymbol{\mathcal{ST}}(\Sigma)$, see [8]. △

The variational problem (3.19) will always possess unique solutions, *cf.* [53] and [19, Prop. A.1].

Lemma 3.4 (Uniqueness of solutions of STF).
$\mathbb{S}'_\Sigma$ *from* (3.20) *is injective.*

Proof. Let $\vec{\mathfrak{w}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$ satisfy

$$\left\langle \mathbb{S}'_\Sigma \vec{\mathfrak{w}}, \vec{\mathfrak{v}} \right\rangle = \sum_{j=0}^{N} \left[\!\left[\mathbb{A}_j \, \mathbb{L}_j \, \vec{\mathfrak{w}}, \mathbb{L}_j \, \vec{\mathfrak{v}} \right]\!\right]_{\mathcal{T}(\partial\Omega_j)} = 0 \quad \forall \vec{\mathfrak{v}} \in \boldsymbol{\mathcal{ST}}(\Sigma) \;. \tag{3.23}$$

Define (radiating) local solutions

$$\mathbf{u}_j := \mathbb{G}_j(\mathbb{L}_j \, \vec{\mathfrak{w}})_{|\Omega_j} \in \boldsymbol{\mathcal{H}}_{\mathrm{rad}}(\mathsf{L}, \Omega_j) \;, \tag{3.24}$$

and recall that Definition 2.4 means

$$\mathbb{T}_j \, \mathbf{u}_j = \mathbb{P}_j(\mathbb{L}_j \, \vec{\mathfrak{w}}) = (\mathbb{A}_j + \tfrac{1}{2}\mathsf{Id}) \, \mathbb{L}_j \, \vec{\mathfrak{w}} \;. \tag{3.25}$$

[7] The prime in the notation $\mathbb{S}'_\Sigma$ is intended to highlight that this operator is defined via a bi-linear form and, thus, must be regarded as a mapping of a space into a dual space.

Hence, (3.23) implies

$$\sum_{j=0}^{N} [\![\mathbb{T}_j\,\mathbf{u}_j, \mathbb{L}_j\,\vec{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega_j)} = \sum_{j=0}^{N} [\![(\mathbb{A}_j + \tfrac{1}{2}\mathsf{Id})\,\mathbb{L}_j\,\vec{\mathfrak{w}}, \mathbb{L}_j\,\vec{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega_j)} = 0 \tag{3.26}$$

for all $\vec{\mathfrak{v}} \in \mathcal{ST}(\Sigma)$ by virtue of (3.23) and Theorem 3.1. This latter theorem, using its other inclusion, permits us to conclude

$$(\mathbb{T}_j\,\mathbf{u}_j)_{j=0}^{N} \in \mathcal{ST}(\Sigma)\ . \tag{3.27}$$

In other words, the compound function

$$\mathbf{u} \in L^2_{\mathrm{loc}}(\mathbb{R}^d) \quad , \quad \mathbf{u}_{|\Omega_j} := \mathbf{u}_j\ ,$$

satisfies the transmission conditions, which renders $\mathbf{u} \in \mathcal{H}(\mathsf{L}, \mathbb{R}^d)$ an entire radiating solution. Thus, we conclude $\mathbf{u} = 0$ from the uniqueness result of Theorem 1.2.

Next, we introduce "exterior" radiating solutions

$$\mathbf{u}_j^+ := \mathbb{G}_j(\mathbb{L}_j\,\vec{\mathfrak{w}})_{|\mathbb{R}^d\setminus\overline{\Omega}_j} \in \mathcal{H}_{\mathrm{rad}}(\mathsf{L}, \mathbb{R}^d\setminus\overline{\Omega}_j)\ . \tag{3.28}$$

and examine their exterior traces

$$\mathbb{T}_j^+\,\mathbf{u}_j^+ = (\mathsf{Id} - \mathbb{P}_j)(\mathbb{L}_j\,\vec{\mathfrak{w}}) = (\tfrac{1}{2}\mathsf{Id} - \mathbb{A}_j)\,\mathbb{L}_j\,\vec{\mathfrak{w}}\ . \tag{3.29}$$

Thus, the same arguments as in (3.26) imply

$$\left(\mathbb{T}_j^+\,\mathbf{u}_j^+\right)_{j=0}^{N} \in \mathcal{ST}(\Sigma)\ . \tag{3.30}$$

Relying on "Rellich's lemma" [20, Lemma .2.11] we are going to show $\mathbf{u}_j^+ = 0$ for all $j = 0, \ldots, N$. This is a standard technique, but important enough to justify a detailed presentation. Rellich's lemma asserts that

$$\mathbf{u} \in \mathcal{H}_{\mathrm{rad}}(\mathsf{L}, \Omega_0)\ , \quad \mathsf{L}\mathbf{u} = 0\ , \quad \lim_{r\to\infty} \int\limits_{|\boldsymbol{x}|=r} |\,\mathsf{T}_D^r\,\mathbf{u}|^2\,\mathrm{d}S = 0 \tag{3.31}$$

implies $\mathbf{u} = 0$ in Ω_0. Recall the notation T_D^r and T_N^r for Dirichlet- and Neumann traces onto the boundary of the ball $B_r := \{\boldsymbol{x} \in \mathbb{R}^d : |\boldsymbol{x}| < r\}$. Applying integration by parts (2.10) we get for r large enough and $j = 1, \ldots, N$

$$\int\limits_{|\boldsymbol{x}|=r} \mathsf{T}_D^r\,\mathbf{u}_j^+ \cdot \mathsf{T}_N^r\,\mathbf{u}_j^+\,\mathrm{d}S =$$

$$\int\limits_{B_r\setminus\overline{\Omega}_j} \alpha\left|\mathsf{D}\mathbf{u}_j^+\right|^2 - \kappa^2\left|\mathbf{u}_j^+\right|^2\,\mathrm{d}\boldsymbol{x} - \int\limits_{\partial\Omega_j} \mathsf{T}_{D,j}^+\,\mathbf{u}_j^+ \cdot \mathsf{T}_{N,j}^+\,\mathbf{u}_j^+\,\mathrm{d}S\ , \tag{3.32}$$

and for $j = 0$

$$0 = \int\limits_{\mathbb{R}^d\setminus\overline{\Omega}_0} \alpha \left|\mathsf{D}\mathbf{u}_0^+\right|^2 - \kappa^2 \left|\mathbf{u}_0^+\right|^2 \,\mathrm{d}\boldsymbol{x} - \int\limits_{\partial\Omega_0} \mathsf{T}_{D,0}^+\,\mathbf{u}_0^+ \cdot \mathsf{T}_{N,0}^+\,\mathbf{u}_0^+\,\mathrm{d}S \ . \tag{3.33}$$

Summing these equations for $j = 0, \ldots, N$ and taking the imaginary part yields

$$\sum_{j=1}^{N} - \operatorname{Im} \int\limits_{|\boldsymbol{x}|=r} \mathsf{T}_D^r\,\mathbf{u}_j^+ \cdot \mathsf{T}_N^r\,\mathbf{u}_j^+\,\mathrm{d}S = \sum_{j=0}^{N} \operatorname{Im} \int\limits_{\partial\Omega_j} \mathsf{T}_{D,j}^+\,\mathbf{u}_j^+ \cdot \mathsf{T}_{N,j}^+\,\mathbf{u}_j^+\,\mathrm{d}S =$$

$$- \tfrac{1}{2} \operatorname{Im} \left[\!\left[\left(\mathbb{T}_j^+\,\mathbf{u}_j^+\right)_{j=0}^{N}, \left(\mathbb{T}_j^+\,\mathbf{u}_j^+\right)_{j=0}^{N} \right]\!\right]_{\mathcal{T}(\Sigma)} = 0 \ , \tag{3.34}$$

because of (3.30), which allows to apply Theorem 3.1.

Simple algebra shows that the radiation conditions (2.11) for $\mathbf{u}_j^+$ combined with (3.34) gives $\lim_{r\to\infty} \sum_{i=1}^{N} \int_{|\boldsymbol{x}|=r} |\,\mathsf{T}_D^r\,\mathbf{u}_j^+|^2\,\mathrm{d}S = 0$ as required by Rellich's lemma, and, consequently, $\mathbf{u}_j^+ = 0$, $j = 1, \ldots, N$. Next, we appeal to the jump relations of Theorem 2.3 to see

$$\mathbb{X}_j\,\mathbb{L}_j\,\vec{\mathfrak{w}} = \mathbb{T}_j\,\mathbf{u}_j - \mathbb{T}_j^+\,\mathbf{u}_j^+ = 0 \ . \tag{3.35}$$

As $\vec{\mathfrak{w}} \in \boldsymbol{\mathcal{ST}}(\Sigma)$ this immediately implies $\mathbb{L}_0\,\vec{\mathfrak{w}} = 0$, that is, $\vec{\mathfrak{w}} = 0$. □

Combining Theorem 3.3 and Lemma 3.4 provides the essential prerequisite for applying the Fredholm alternative to (3.19), which immediately confirms existence and uniqueness of solutions of (3.19): using the STF operator $\mathbb{S}'_\Sigma$, we can give a concise statement of this main result.

Corollary 3.5. $\mathbb{S}'_\Sigma : \boldsymbol{\mathcal{ST}}(\Sigma) \mapsto \boldsymbol{\mathcal{ST}}(\Sigma)'$ *from* (3.20) *is an* isomorphism.

3.3 Boundary element Galerkin discretization

Abstract Galerkin discretization of (3.19) restricts the variational equation to a finite dimensional subspace $\boldsymbol{\mathcal{ST}}_h(\Sigma) \subset \boldsymbol{\mathcal{ST}}(\Sigma)$ [47, Sect. 4.2.2]. Here we take a closer look only at piecewise polynomial constructions with respect to a triangulation Σ_h of the skeleton Σ. For $d = 2$ this amounts to partitioning all interfaces Γ_{ij} into (curved) segments, separated by so-called nodes. We demand that multiple junctions of interfaces will always coincide with nodes. In three dimensions, each interface is equipped with a (curved) triangular mesh. These meshes are supposed to be compatible across interface boundaries ("no hanging nodes"). The requirements imply that the restriction of Σ_h to $\partial\Omega_j$ is a valid triangulation $\Gamma_{j,h} := \Sigma_{h\,|\partial\Omega_j}$ of this boundary.

Suitable piecewise polynomial (w.r.t. $\Gamma_{j,h}$) subspaces $\mathcal{T}_{D,h}(\partial\Omega_j)$ and $\mathcal{T}_{N,h}(\partial\Omega_j)$ of $\mathcal{T}_D(\partial\Omega_j)$ and $\mathcal{T}_N(\partial\Omega_j)$, respectively, can be obtained by applying the trace operators

$\mathsf{T}_{\mathsf{D},j}$ and $\mathsf{T}_{\mathsf{D}^*,j}$ to piecewise polynomial finite element subspaces of $\boldsymbol{\mathcal{H}}(\mathsf{D},\Omega_j)$ and $\boldsymbol{\mathcal{H}}(\mathsf{D}^*,\Omega_j)$, respectively. The finite element spaces are built upon some triangular ($d=2$) or tetrahedral ($d=3$) mesh of Ω_j, whose restriction to $\partial\Omega_j$ coincides with $\Gamma_{j,h}$.

Concretization 3.4 (Helmholtz). For the Helmholtz transmission problem (1.1) we have $\mathcal{T}_D(\partial\Omega_j) = H^{\frac{1}{2}}(\partial\Omega_j)$ and the simplest choice is continuous piecewise linear functions on $\Gamma_{j,h}$. Obviously, this space can be obtained by taking the point trace of continuous piecewise linear functions on a suitable finite element mesh. The natural tent function basis of this $\mathcal{T}_{D,h}(\partial\Omega_j)$ is dual to the point evaluations in the nodes of $\Gamma_{j,h}$.

The Neumann trace space for the Helmholtz transmission problem (1.1) is $\mathcal{T}_N(\partial\Omega_j) = H^{-\frac{1}{2}}(\partial\Omega_j)$, which can be approximated by means of piecewise constant discontinuous functions on $\Gamma_{j,h}$. The characteristics functions of mesh cells provide a natural basis.

This and more general constructions of piecewise polynomial boundary element subspaces of $H^{\frac{1}{2}}(\partial\Omega_j)$ and $H^{-\frac{1}{2}}(\partial\Omega_j)$ are discussed in [47, Sect. 4.1]. $\triangle$

Concretization 3.5 (Maxwell). In the Maxwell case, that is, $d=3$ and $\mathcal{T}_D(\partial\Omega_j) = \boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma,\partial\Omega_j)$, surface edge elements supply the simplest space $\mathcal{T}_{D,h}(\partial\Omega_j)$. They are piecewise linear tangentially continuous tangential vector fields. Locally supported basis functions are associated with edges of the surface mesh $\Gamma_{j,h}$. In computational electromagnetics these are known as Rao-Wilton-Glisson (RWG) basis functions [45].

As for the approximation of Neumann traces, we have the simple formula

$$\mathcal{T}_{N,h}(\partial\Omega_j) = \mathcal{T}_{D,h}(\partial\Omega_j) \times \boldsymbol{n}_j \ ,$$

that is, Neumann traces can be approximated by the same finite dimensional spaces of tangential vector fields apart from a rotation by $\frac{\pi}{2}$. This matches the identity $\boldsymbol{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma,\partial\Omega_j) = \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\partial\Omega_j) \times \boldsymbol{n}_j$. The space thus constructed is closely related to the so-called lowest order Raviart-Thomas finite element space [5, 6, 46]. $\triangle$

With $\mathcal{T}_{D,h}(\partial\Omega_j)$ and $\mathcal{T}_{N,h}(\partial\Omega_j)$, $j=0,\dots,N$, at our disposal, finite dimensional subspaces of $\boldsymbol{\mathcal{MT}}(\Sigma)$ can be obtained by taking products. The construction of $\boldsymbol{\mathcal{ST}}_h(\Sigma)$ is not so straightforward and is guided by Remark 3.2. We describe it separately for the Helmholtz and Maxwell case.

Concretization 3.6 (Helmholtz). In the case of the Helmholtz transmission problem (1.1) $\boldsymbol{\mathcal{ST}}_h(\Sigma)$ can be obtained by combining two spaces,

(i) a space $\mathcal{F}_{D,h}$ of *continuous* Σ_h-piecewise polynomial functions,

(ii) and a space $\mathcal{F}_{N,h}$ of *possibly discontinuous* Σ_h-piecewise polynomial functions,

from which we get

$$\boldsymbol{\mathcal{ST}}_h(\Sigma) = \bigotimes_{j=0}^{N} \mathcal{F}_{D,h}{}_{|\partial\Omega_j} \times \bigotimes_{j=0}^{N} \sigma_j\,\mathcal{F}_{N,h}{}_{|\partial\Omega_j} \ ,$$

where the "orientation adjustment functions" σ_j are defined in Remark 3.2. We point out that even if $\mathcal{F}_{N,h}$ contained only continuous functions, we will usually wind up with a discontinuous approximation space for Neumann data in $\boldsymbol{\mathcal{ST}}_h(\Sigma)$, because the orientation adjustment function σ_j may jump.

Clearly, for this simplest choice of piecewise linear and piecewise constant boundary element spaces as explained in Concretization 3.4, the dimension of $\boldsymbol{\mathcal{ST}}_h(\Sigma)$ will agree with the sum of the number of vertices and cells of Σ_h. △

Concretization 3.7 (Maxwell). The construction of $\boldsymbol{\mathcal{ST}}_h(\Sigma)$ for the Maxwell case $\mathsf{D} = \mathbf{curl}$ and $d = 3$ starts from a space $\boldsymbol{\mathcal{F}}_h$ of Σ_h-piecewise polynomial tangential and tangentially continuous vector fields, *cf.* Concretization 3.2. This means that for an $\vec{u}_h \in \boldsymbol{\mathcal{F}}_h$ its orthogonal projection onto any edge of Σ_h must be unique. The simplest specimen is the edge element skeleton space, whose functions are piecewise linear and uniquely characterized by their path integrals along edges, *cf.* Concretization 3.5.

Then, taking the cue from (3.14), we find

$$\boldsymbol{\mathcal{ST}}_h(\Sigma) = \bigotimes_{j=0}^{N} \boldsymbol{\mathcal{F}}_{h\,|\partial\Omega_j} \times \bigotimes_{j=0}^{N} \big(\boldsymbol{\mathcal{F}}_{h\,|\partial\Omega_j} \times \boldsymbol{n}_j\big) .$$

Using surface edge elements to define $\boldsymbol{\mathcal{F}}_h$, we thus end up with a space $\boldsymbol{\mathcal{ST}}_h(\Sigma)$, whose dimension is twice the number of edges in Σ_h. △

Remark 3.3. Galerkin discretization of (3.19) using the standard choices for $\boldsymbol{\mathcal{ST}}_h(\Sigma)$ explained above will lead to a linear system of equations, whose matrix can be obtained by adding the matrices arising from a boundary element Galerkin discretization of the boundary integral operators $\mathbb{A}_j$, $j = 0, \dots, N$, after suitably padding them with zero rows and columns, in order to take into account the localization to $\partial\Omega_j$. △

It turns out that all piecewise polynomial boundary element spaces give rise to asymptotically quasi-optimal Galerkin approximations. This result is stated as a discrete inf-sup condition in the next theorem. In the case of Helmholtz' equation (1.1) its proof boils down to a classical duality argument, see [48] and [47, Sect. 4.2.3]. For Maxwell's equation (1.2) it requires more subtle considerations centering around a uniform gap property of a discrete Hodge-type decomposition. We are not going to give details on this technique, but refer to [12, Sect. 9.1] for principal ideas and [8] for the application to STF.

Theorem 3.6. *Let $\boldsymbol{\mathcal{ST}}_h(\Sigma) \subset \boldsymbol{\mathcal{ST}}(\Sigma)$ be a space of piecewise polynomial boundary element functions of fixed polynomial degree based on a skeleton mesh Σ_h with mesh-width $h > 0$.*

There is $h_0 > 0$ and a constant $c > 0$ depending only on Σ, the type and polynomial degree of the boundary element space, and the shape-regularity of Σ_h, such that the following discrete inf-sup condition *holds*

$$\sup_{\vec{\mathfrak{v}}_h \in \boldsymbol{\mathcal{ST}}_h(\Sigma)} \frac{|\langle \mathbb{S}'_\Sigma \vec{\mathfrak{u}}_h, \vec{\mathfrak{v}}_h \rangle|}{\|\vec{\mathfrak{v}}_h\|_{\boldsymbol{\mathcal{MT}}(\Sigma)}} \geq c\, \|\vec{\mathfrak{u}}_h\|_{\boldsymbol{\mathcal{MT}}(\Sigma)} \quad \forall \vec{\mathfrak{u}}_h \in \boldsymbol{\mathcal{ST}}_h(\Sigma) , \quad \forall h < h_0 . \tag{3.36}$$

This implies that the trace norm of the Galerkin discretization error is proportional to the best approximation error of $\boldsymbol{\mathcal{ST}}_h(\Sigma)$ in $\boldsymbol{\mathcal{ST}}(\Sigma)$ provided that Σ_h is *sufficiently fine*, hence the attribute "asymptotic" attached to this estimate. More details can be found in [47, Sect. 4.2.2].

4 Preconditioning

According to the results reported in the previous section, the STF operator $\mathbb{S}'_\Sigma$ defined in (3.20) induces an isomorphism between the trace space $\boldsymbol{\mathcal{ST}}(\Sigma)$ and its dual. This is typical of *first kind* boundary integral equations and the STF belongs to this class. Unfortunately, the standard local bases of the piecewise polynomial boundary element spaces discussed in Section 3.3 are not stable with respect to the norm of $\boldsymbol{\mathcal{ST}}(\Sigma)$. Indeed, writing $\left\{\vec{\mathfrak{b}}_i\right\}_{i=1}^M$, $M := \dim \boldsymbol{\mathcal{ST}}_h(\Sigma)$, for this basis of the boundary element space $\boldsymbol{\mathcal{ST}}_h(\Sigma) \subset \boldsymbol{\mathcal{ST}}(\Sigma)$, we find $r, s \in \mathbb{Z}$ satisfying $s - r = 2$, such that

$$\underline{C}h^s \left\| \sum_{i=1}^M \gamma_i \vec{\mathfrak{b}}_i \right\|^2_{\boldsymbol{\mathcal{MT}}(\Sigma)} \leq \sum_{i=1}^M \gamma_i^2 \left\| \vec{\mathfrak{b}}_i \right\|^2_{\boldsymbol{\mathcal{MT}}(\Sigma)} \leq \overline{C}h^r \left\| \sum_{i=1}^M \gamma_i \vec{\mathfrak{b}}_i \right\|^2_{\boldsymbol{\mathcal{MT}}(\Sigma)} , \tag{4.1}$$

for all $\gamma_i \in \mathbb{C}$, where $h > 0$ stands for the meshwidth of the skeleton mesh Σ_h and the constants $\underline{C}, \overline{C} > 0$ depend only on the geometry of Σ, the ratio of the sizes of the largest and smallest panel, and, for $d = 3$, on the shape regularity measure of the triangles of Σ_h. For both Helmholtz and Maxwell the estimate (4.1) can be proved by means of local inverse inequalities and interpolation in Sobolev scales [47, Sect. 6.4.1].

As a consequence, on a sequence of meshes created by uniform regular refinement the Galerkin matrix $\mathbf{A} \in \mathbb{C}^{M,M}$ arising from a standard boundary element Galerkin discretization of (3.19) will display a growth of its condition number (defined through the Euclidean norm on $\mathbb{C}^M$) like

$$\boxed{\mathrm{cond}_2(\mathbf{A}) = O(h^{-2})} , \tag{4.2}$$

cf. [47, Cor. 6.4.14 & Lemma 4.51]. This can be blamed for the commonly observed severely degraded performance of iterative Krylov-subspace solvers like GMRES or BiCGStab on fine meshes, though a rigorous theoretical link between (4.2) and speed of convergence has not yet been established, *cf.* [13, 4], unless more precise information about the spectrum of $\mathbf{A}$ is available.

Slow convergence of iterative solvers has become a major issue, since local low rank matrix compression algorithms have paved the way for using boundary element discretizations with many degrees of freedoms (10^4–10^6). These algorithms are known as fast multipole methods [25, 21], $\mathcal{H}$-matrix compression [26], or adaptive cross approximation [3, Ch. 3].

Compressed matrices merely allow fast matrix$\times$vector multiplication, which is sufficient for applying iterative solvers, but precludes direct elimination techniques. The bottom line is that it becomes essential to accelerate the convergence of the Krylov-subspace iterative solvers, which is usually achieved by preconditioning. The next two sections will outline a powerful heuristics that, nowadays, plays an important role in boundary element implementations.

4.1 Operator Products

Temporarily we focus on the single-trace formulation (3.19) for the case $N = 1$, that is, the case of a single homogeneous scatterer. We elaborate how in this very special situation the discrete variational STF can be preconditioned efficiently. A first particular feature of the $N = 1$ case is that $\Sigma = \partial\Omega_0 = \partial\Omega_1$; from now we are going to write Γ for this single interface. We endow it with a positive orientation with respect to Ω_1. Then the single-trace variational formulation (3.19) becomes

$$\left[\!\left[\mathbb{A}_0\,\mathbb{L}_0\,\vec{\mathfrak{u}},\mathbb{L}_0\,\vec{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\Gamma)}+\left[\!\left[\mathbb{A}_1\,\mathbb{L}_1\,\vec{\mathfrak{u}},\mathbb{L}_1\,\vec{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\Gamma)}=-\left[\!\left[\mathbb{T}_\Gamma\,\mathbf{u}_{\mathrm{inc}},\mathbb{L}_0\,\vec{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\Gamma)}\quad\forall\vec{\mathfrak{v}}\in\boldsymbol{\mathcal{ST}}(\Sigma)\,, \tag{4.3}$$

with equivalent operator form, *cf.* (3.20),

$$\mathbb{S}'_\Gamma\vec{\mathfrak{u}}:=(\mathbb{L}_0^*\,\mathbb{A}_0\,\mathbb{L}_0+\mathbb{L}_1^*\,\mathbb{A}_1\,\mathbb{L}_1)\,\vec{\mathfrak{u}}=-\,\mathbb{L}_0^*\,\mathbb{T}_\Gamma\,\mathbf{u}_{\mathrm{inc}}\,. \tag{4.4}$$

The particular situation makes possible the straightforward identification of $\boldsymbol{\mathcal{ST}}(\Sigma)$ with the Cauchy trace space $\mathcal{T}(\Gamma)$ via

$$\boldsymbol{\mathcal{ST}}(\Sigma)=\{(u,u,-\varphi,\varphi):\ u\in\mathcal{T}_D(\Gamma),\ \varphi\in\mathcal{T}_N(\Gamma)\}\,, \tag{4.5}$$

which introduces the isometric isomorphism

$$\mathbb{C}:\mathcal{T}(\Gamma)\mapsto\boldsymbol{\mathcal{ST}}(\Sigma)\quad,\quad\mathfrak{u}=\begin{pmatrix}u\\\varphi\end{pmatrix}\mapsto\tfrac{1}{2}(u,u,-\varphi,\varphi)\,. \tag{4.6}$$

Thus, the L^2-type pairing providing self-duality of $\mathcal{T}(\Gamma)$ indirectly induces a self-duality of $\boldsymbol{\mathcal{ST}}(\Sigma)$. The related isometric isomorphism is denoted by $\mathbb{D}':\boldsymbol{\mathcal{ST}}(\Sigma)\to\boldsymbol{\mathcal{ST}}(\Sigma)'$ and defined through

$$\left\langle\mathbb{D}'\vec{\mathfrak{u}},\vec{\mathfrak{v}}\right\rangle_{\boldsymbol{\mathcal{MT}}(\Sigma)}:=\left[\!\left[\mathbb{C}^{-1}\,\vec{\mathfrak{u}},\mathbb{C}^{-1}\,\vec{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\Gamma)}\,,\quad\forall\vec{\mathfrak{u}},\vec{\mathfrak{v}}\in\boldsymbol{\mathcal{ST}}(\Sigma)\,. \tag{4.7}$$

As is evident from (4.6) and (2.5), $\mathbb{D}'$ is spawned by an L^2-type local pairing on Γ. Since $\mathbb{S}'_\Sigma:\boldsymbol{\mathcal{ST}}(\Sigma)\mapsto\boldsymbol{\mathcal{ST}}(\Sigma)'$ is continuous, we find that the operator product

$$(\mathbb{D}'^{-1})^*\mathbb{S}'_\Gamma\mathbb{D}'^{-1}\mathbb{S}'_\Gamma:\boldsymbol{\mathcal{ST}}(\Sigma)\mapsto\boldsymbol{\mathcal{ST}}(\Sigma) \tag{4.8}$$

gives rise to a *continuous* mapping within the *same* space. The relevance of this observation for accelerating iterative solvers will be elaborated in Section 4.3.

4.2 Calderón identities

Now we explore an interesting property of the operator product from (4.8) in the special case $\alpha_1 = \alpha_0$; equal coefficients $\alpha_0 = \alpha_1$ will be taken for granted throughout this subsection. The starting point is the projector property of the Calderón projectors stated in Theorem 2.6. It can be used to extract from (2.17) that

$$\mathbb{A}^2 = \tfrac{1}{4}\mathsf{Id} \, . \tag{4.9}$$

This equation comprises several so-called *Calderón identities* for boundary integral operators, see [47, Prop. 3.6.4]. Similar relationships hold for the operator product from (4.8) and we will also call them "Calderón identities". To derive these we rely on the identification of $\boldsymbol{\mathcal{ST}}(\Sigma)$ and $\mathcal{T}(\Gamma)$ according to (4.6). It enables us to convert (4.3) into: seek $\mathfrak{u} \in \mathcal{T}(\Gamma)$ such that

$$\left\langle \mathbb{S}'_\Gamma \, \mathbb{C}\, \mathfrak{u}, \mathbb{C}\, \mathfrak{v} \right\rangle_{\boldsymbol{\mathcal{ST}}(\Sigma)} = [\![\mathbb{A}_0 \, \mathbb{X}\, \mathfrak{u}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} + [\![\mathbb{A}_1 \, \mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} = -[\![\mathbb{T}_\Gamma \, \mathbf{u}_{\mathrm{inc}}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} \, , \tag{4.10}$$

for all $\mathfrak{v} \in \mathcal{T}(\Gamma)$.

Now, we aim to relate $\mathbb{A}_0$ and $\mathbb{A}_1$. To do so, it is important to note that both potentials $\mathbb{G}_0$ and $\mathbb{G}_1$ map between the same spaces and satisfy

$$\mathbb{G}_0 = -\mathbb{G}_1[0] \circ \mathbb{X} \quad \text{on } \mathcal{T}(\Gamma) \, , \tag{4.11}$$

due to the fact that the orientation of the boundary matters for the kernel of the double layer potential, *cf.* the formulas (2.14) and (2.16). In addition, we resort to the *jump relations* of Theorem 2.3 that tell us

$$\mathbb{T}_0 \, \mathbb{G}_1[0] = \begin{pmatrix} \mathsf{T}^+_{D,1} \\ \mathsf{T}^+_{N,1} \end{pmatrix} \mathbb{G}_1[0] = \begin{pmatrix} \mathsf{T}_{D,1} \\ -\mathsf{T}_{N,1} \end{pmatrix} \mathbb{G}_1[0] - \mathbb{X} = \mathbb{X}\, \mathbb{T}_1 \, \mathbb{G}_1[0] - \mathbb{X} \, . \tag{4.12}$$

Since $\mathbb{X}$ is an involution and $[\![\mathbb{X}\, \mathfrak{u}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\partial\Omega)} = -[\![\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\partial\Omega)}$ (from (2.8) and (2.5)) we have the equalities

$$\begin{aligned}
[\![\mathbb{A}_0 \, \mathbb{X}\, \mathfrak{u}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} &\overset{(2.17)}{=} \left[\!\left[(\mathbb{T}_0 \, \mathbb{G}_0 - \tfrac{1}{2}\mathsf{Id})\, \mathbb{X}\, \mathfrak{u}, \mathbb{X}\, \mathfrak{v} \right]\!\right]_{\mathcal{T}(\Gamma)} \\
&\overset{(4.11)}{=} [\![-\mathbb{T}_0 \, \mathbb{G}_1[0]\mathfrak{u}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} - \tfrac{1}{2} [\![\mathbb{X}\, \mathfrak{u}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} \\
&\overset{(4.12)}{=} [\![-(\mathbb{X}\, \mathbb{T}_1 \, \mathbb{G}_1[0] - \mathbb{X})\mathfrak{u}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} + \tfrac{1}{2} [\![\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} \\
&= [\![(\mathbb{T}_1 \, \mathbb{G}_1[0] - \mathsf{Id})\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} + \tfrac{1}{2} [\![\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} \\
&\overset{(2.17)}{=} [\![\mathbb{A}_1[0]\mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} \, .
\end{aligned} \tag{4.13}$$

This means that the single trace boundary integral equations for a homogeneous scatterer can be rephrased as: seek $\mathfrak{u} \in \mathcal{T}(\Gamma)$ such that

$$\left[\!\left[\mathbb{S}'_\Sigma \mathbb{C}\, \mathfrak{u}, \mathbb{C}\, \mathfrak{v} \right]\!\right]_{\mathcal{T}(\Gamma)} = [\![(\mathbb{A}_1 + \mathbb{A}_1[0])\, \mathfrak{u}, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} = -[\![\mathbb{T}_\Gamma \, \mathbf{u}_{\mathrm{inc}}, \mathbb{X}\, \mathfrak{v}]\!]_{\mathcal{T}(\Gamma)} \, , \tag{4.14}$$

for all $\mathfrak{v} \in \mathcal{T}(\Gamma)$. Here we relied on the identification (4.6) through the isomorphism $\mathbb{C}$ again.

A closer inspection of $\mathbb{A}_1$ and $\mathbb{A}_1[0]$ for $\boxed{\alpha_0 = \alpha_1}$ reveals that these boundary integral operators feature kernels with the same singular behavior for $\boldsymbol{x} = \boldsymbol{y}$, *cf.* (2.14). Forming their difference, these singularities will cancel, resulting in an operator that is smoothing compared to $\mathbb{A}_1$ and $\mathbb{A}_1[0]$. Hence, the Rellich embedding theorem [38, Thm. 3.27] allows the following conclusion, see [47, Lemma 3.9.8].

Lemma 4.1 (Compact modification). *Provided that $\alpha_j = \alpha_k$, the operator $\mathbb{A}_i[j] - \mathbb{A}_i[k] : \mathcal{T}(\partial\Omega_i) \mapsto \mathcal{T}(\partial\Omega_i)$ is compact.*

As a consequence of this result and (4.9) applied to $\mathbb{A}_1$ we find the desired Calderón identity for the operator product from (4.8):

$$\mathbb{A}_1 \circ (\mathbb{A}_1 + \mathbb{A}_1[0]) = \tfrac{1}{2}\mathsf{Id} + \{\text{compact}\} \quad \text{in} \quad \mathcal{T}(\Gamma) \tag{4.15}$$

$$\Downarrow \quad (4.14)$$

$$(\mathbb{D}'^{-1})^* \mathbb{S}'_\Gamma \mathbb{D}'^{-1} \mathbb{S}'_\Gamma = \mathsf{Id} + \{\text{compact}\} \quad \text{in} \quad \boldsymbol{\mathcal{ST}}(\Sigma) \; . \tag{4.16}$$

We learn that $\mathbb{B} := (\mathbb{D}'^{-1})^* \mathbb{S}'_\Gamma \mathbb{D}'^{-1}$ is an inverse of $\mathbb{S}'_\Gamma$ up to a compact perturbation, which will make the eigenvalues of the operator product in (4.16) cluster around 1. In a sense, $\mathbb{B}$ qualifies as a *preconditioning operator* for $\mathbb{S}'_\Gamma$! Why this parlance is justified and how to transfer this insight to Ritz-Galerkin discretization will be explained in the next section.

4.3 Operator preconditioning

The policy of operator preconditioning is best understood in an abstract framework that we present following [31, Sect. 2]. We consider a Hilbert space V with dual V' and two bounded operators $\mathsf{A} : V \mapsto V'$ and $\mathsf{B} : V \mapsto V'$ with associated sesqui-linear forms $\mathsf{a} : V \times V \mapsto \mathbb{C}$ and $\mathsf{b} : V \times V \mapsto \mathbb{C}$. The spaces V and V' are also dual with respect to the pairing $[\cdot,\cdot] : V \times V \mapsto \mathbb{C}$, which is supposed to give rise to an *isometric* isomorphism $V \mapsto V'$. In other words, the norm on V' is defined via $[\cdot,\cdot]$.

Galerkin discretization of A and B is based on finite-dimensional subspaces $V_h \subset V$ and $\widetilde{V}_h \subset V$. Stability of the discretization is assumed, which amounts to discrete inf-sup-conditions of the form

$$\sup_{v_h \in V_h} \frac{|\,\mathsf{a}(u_h, v_h)|}{\|v_h\|_V} \geq c_A \, \|u_h\|_V \quad \forall u_h \in V_h \; . \tag{4.17}$$

$$\sup_{\widetilde{v}_h \in \widetilde{V}_h} \frac{|\,\mathsf{b}(\widetilde{u}_h, \widetilde{v}_h)|}{\|\widetilde{v}_h\|_V} \geq c_B \, \|\widetilde{u}_h\|_V \quad \forall \widetilde{u}_h \in \widetilde{V}_h \; . \tag{4.18}$$

The crucial assumption is the stability of the duality pairing for the Galerkin trial spaces

$$\sup_{\widetilde{v}_h \in \widetilde{V}_h} \frac{[\widetilde{v}_h, u_h]}{\|\widetilde{v}_h\|_V} \geq c_D \, \|u_h\|_V \quad \forall u_h \in V_h \; . \tag{4.19}$$

A necessary condition for (4.19) is $M := \dim V_h = \dim \widetilde{V}_h$. Choosing *arbitrary* bases of V_h and $\widetilde{V}_h$ we can represent the sesqui-linear forms a, b, and $[\cdot,\cdot]$ by means of their $M \times M$ Galerkin matrices $\mathbf{A}, \mathbf{B}, \mathbf{D} \in \mathbb{C}^{M,M}$. Then [31, Theorem 2.1] asserts that

$$\kappa(\mathbf{D}^{-H}\mathbf{B}\mathbf{D}^{-1}\mathbf{A}) \leq \delta := \frac{\|\mathsf{A}\| \, \|\mathsf{B}\|}{c_A c_B c_D^2} \; , \tag{4.20}$$

where κ stands for the spectral condition number of a matrix, that is, the ratio of the moduli of the largest (in modulus) and smallest eigenvalue.

Estimate (4.20) suggests that $\mathbf{D}^{-H}\mathbf{B}\mathbf{D}^{-1}$ can serve as a preconditioner of $\mathbf{A}$. Indeed, for symmetric sesqui-linear forms a and b the estimate (4.20) immediately implies that the rate of linear convergence of a preconditioned iterative Krylov-subspace solver (CG or MINRES) can be bounded in terms of δ alone. Provided that the stability constants c_A, c_B, and c_D can be uniformly bounded away from zero for a family of trial spaces, "uniformly fast"/"h-independent" convergence of the iterative solver is ensured.

Without symmetry of a and b, (4.20) does not permit us to predict the speed of convergence of an iterative solver like GMRES. In this case using $\mathbf{D}^{-H}\mathbf{B}\mathbf{D}^{-1}$ as a preconditioner is heuristics, which has proved to be highly successful in practice, though.

4.4 Stable duality pairing for boundary elements

The single trace formulation for $N = 1$, discretized by means of lowest order piecewise polynomial boundary elements as discussed in Section 4.1, can be fit into the abstract framework of operator preconditioning as introduced in the previous section in a straightforward way; the incarnations of the operators and pairings are immediate from (4.8) and (4.16). Based on the Hilbert space $V := \boldsymbol{\mathcal{ST}}(\Sigma)$, the role of A and B is played by the continuous STF operator $\mathbb{S}'_\Gamma : \boldsymbol{\mathcal{ST}}(\Sigma) \to \boldsymbol{\mathcal{ST}}(\Sigma)'$. The isometric duality pairing is provided by the operator $\mathbb{D}'$ according to (4.7).

The discrete inf-sup conditions (4.17) and (4.18) are guaranteed by Theorem 3.6 on sufficiently fine interface meshes with constants independent of the meshwidth h. However, no h-uniform discrete inf-sup condition (4.19) is available for the pairing on $\boldsymbol{\mathcal{ST}}_h(\Sigma) \times \boldsymbol{\mathcal{ST}}_h(\Sigma)$ effected by the operator $\mathbb{D}'$!

In the case of $\mathsf{D} = \mathbf{grad}$ (Helmholtz) and lowest degree boundary elements, see Concretization 3.4, this is obvious, because in general $\dim \mathcal{T}_{N,h}(\Gamma) \neq \dim \mathcal{T}_{D,h}(\Gamma)$, whereas the duality pairing relies on products of functions of these two spaces. Even in the case of $\mathsf{D} = \mathbf{curl}$ (Maxwell) and surface edge elements (4.19) fails to hold uniformly in h, though in this case $\dim \mathcal{T}_{N,h}(\Gamma)$ and $\dim \mathcal{T}_{D,h}(\Gamma)$ both agree with the number of edges of Γ_h. This was demonstrated in [17, Prop. 3.1].

A remedy for this conundrum is to use *different meshes* for the construction of $\mathcal{T}_{N,h}(\Gamma)$ and $\mathcal{T}_{D,h}(\Gamma)$, thereby ensuring stability and, of course, matching dimensions $\dim \mathcal{T}_{N,h}(\Gamma) = \dim \mathcal{T}_{D,h}(\Gamma)$. For 2nd-order diffusion problems the pioneering work [50] investigated a suitable construction for lowest order polynomial BEM. It relies on *dual meshes*. An application to acoustic scattering (Helmholtz case) is covered in [16]. The idea was adapted to surface edge elements in [9] and we refer to this article for a comprehensive analysis and the proof of uniform stability of standard duality pairings on the primal-dual boundary element spaces. Thus, all pieces are in place for operator preconditioning of the discretized STF for $N = 1$!

Remark 4.1 (Riesz pairing). The alert reader will have realized that on the Hilbert space $\mathcal{ST}(\Sigma)$ a suitable alternative candidate for $\mathbb{D}'$ in (4.7) would be the Riesz isomorphism $\mathcal{ST}(\Sigma) \to \mathcal{ST}(\Sigma)'$ induced by the inner product of $\mathcal{ST}(\Sigma)$. This choice would preserve the continuity properties of $(\mathbb{D}'^{-1})^* \mathbb{S}'_\Gamma \mathbb{D}'^{-1} \mathbb{S}'_\Gamma$.

At second glance, the Riesz isomorphism turns out to be utterly useless for the purpose of operator preconditioning; the matrix $\mathbf{D}$ would be the Galerkin matrix of another boundary integral operator of the first kind, structurally similar to $\mathbb{S}'_\Sigma$. Therefore, applying the preconditioner would entail solving two ill-conditioned linear systems of equations as hard as the one tackled by the preconditioner. △

Remark 4.2 (Sparse pairing matrix). Duality inducing pairings based on the pivot space L^2 are inherently local; inserting two functions with non-overlapping supports will yield zero. Thus, using boundary element spaces on dual meshes for the stable discretization of the pairing along with standard local basis functions will lead to *sparse* Galerkin matrices $\mathbf{D}$, *cf.* (4.20). Thus, direct Gaussian elimination becomes a competitive option for computing the action of $\mathbf{D}^{-1}$ and $\mathbf{D}^{-H}$ in the preconditioner. The efficiency of iterative solvers also hinges on sparsity. Thus, strengthening Remark 4.1 we can make the statement that

> operator preconditioning of Galerkin matrices arising from boundary integral equations will be feasible, only if the underlying pairing is of L^2-type, hence local.

△

Remark 4.3 (Varying coefficients α_i). Provided that the uniform stability assumptions are met, operator preconditioning for the discretized STF will be robust with respect to the choice of skeleton meshes. No assumptions on α_i other than $\alpha_i > 0$ are necessary. Of course, the heights of the jumps of $\alpha(\boldsymbol{x})$ will affect the stability constants.

In the case $\alpha_i = \alpha_0$, $i = 1, \ldots, N$, we conclude from the results of Section 4.2 the eigenvalues of the operator product from (4.16) have $1 \in \mathbb{C}$ as sole accumulation point. Therefore, one may hope that a similar property holds for the spectrum of the matrix product from (4.20). Yet, so far there is no rigorous proof that this is the case.

△

4.5 The challenge

Unfortunately, the case of a homogeneous scatterer $N = 1$ is very special. In this case $\mathcal{T}(\Gamma)$ offers an isomorphic model of $\boldsymbol{\mathcal{ST}}(\Sigma)$, see (4.6), and, thus, a local L^2-type duality pairing has become available. Whenever $N > 1$ and there are material junction points, it seems impossible to find a way to express the self-duality of $\boldsymbol{\mathcal{ST}}(\Sigma)$ through an L^2-type pairing. Also note that Theorem 3.1 rules out that $[\![\cdot,\cdot]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)}$ can serve as a duality pairing for $\boldsymbol{\mathcal{ST}}(\Sigma)$.

Hence, efficient operator preconditioning of the single-trace boundary integral equations becomes elusive, remember Remark 4.2. Hitherto, this problem has not been overcome. This prompted us to

> seek alternatives to the single trace formulation that are amenable to operator preconditioning.

Our search has been trained on formulations with straightforward L^2-type duality pairings. It led to the development of new boundary integral equations belonging to the class of multi-trace formulations (MTF). They will be presented in he remainder of the paper.

5 Global Multi-Trace Formulation

5.1 Separated sub-domains

In Section 4.1 we have seen that the classical single trace formulation allows operator preconditioning for $N = 1$. Asking the question, what was exceptional about this setting, we arrived at the conclusion that it is the availability of a simple L^2-pairing bringing about the duality of $\boldsymbol{\mathcal{ST}}(\Sigma)$ and $\boldsymbol{\mathcal{ST}}(\Sigma)'$. This was the gift of the isomorphism $\mathbb{C}$ from (4.6).

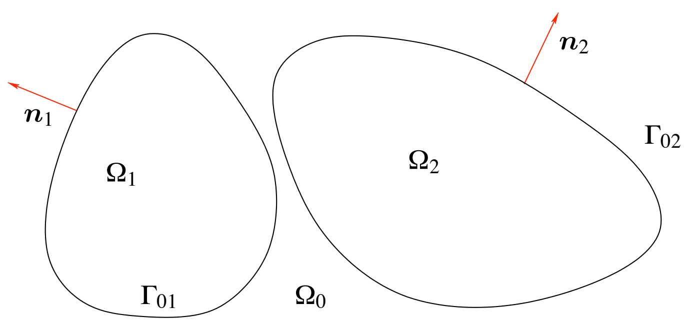

Figure 2. $N = 2$, isolated sub-domains

Obviously, an analogous isometric isomorphism

$$\boldsymbol{\mathcal{ST}}(\Sigma) \cong \mathcal{T}(\partial\Omega_1) \times \cdots \times \mathcal{T}(\partial\Omega_N)\ , \tag{5.1}$$

still exists for $N > 1$, if all sub-domains Ω_j are *separated* as illustrated in Figure 2 for $N = 2$. In this setting the isomorphism $\mathbb{C} : \mathcal{T}(\partial\Omega_1) \times \cdots \times \mathcal{T}(\partial\Omega_N) \to \boldsymbol{\mathcal{ST}}(\Sigma)$ underlying (5.1) reads

$$\mathbb{C} : \begin{cases} \mathcal{T}(\partial\Omega_1) \times \mathcal{T}(\partial\Omega_2) & \to \quad \boldsymbol{\mathcal{ST}}(\Sigma) \\ \begin{pmatrix} \binom{u_1}{\varphi_1} \\ \binom{u_2}{\varphi_2} \end{pmatrix} & \mapsto \quad (u_1 \vee u_2, u_1, u_2, -(\varphi_1 \vee \varphi_2), \varphi_1, \varphi_2) \end{cases}, \tag{5.2}$$

where $\vee$ indicates that two functions on $\partial\Omega_1$ and $\partial\Omega_2$ are combined into a function on $\partial\Omega_0 = \partial\Omega_1 \cup \partial\Omega_2$:

$$(\mathbf{u}_1 \vee \mathbf{u}_2)(\boldsymbol{x}) := \mathbf{u}_i(\boldsymbol{x})\ ,\ \text{if} \quad \boldsymbol{x} \in \Omega_i\ , \quad i = 1, 2\ .$$

We remark that this isomorphism relies on the induced orientation of $\partial\Omega_j$ from Ω_j, see Figure 2.

A duality pairing for the right hand side of (5.1) is furnished by the local L^2-type pairings on the sub-domain boundaries $\partial\Omega_j$. As pointed out in Section 4.4, related h-uniformly stable discrete duality pairings for standard BEM can be achieved by employing dual meshes on each $\partial\Omega_j$. Summing up, standard operator preconditioning remains feasible for separated sub-domains.

Instead of using $\mathbb{C}$ as a tool to graft a duality pairing onto $\boldsymbol{\mathcal{ST}}(\Sigma)$ we may as well use it to define a "new" operator

$$\mathbb{M}_G' := \mathbb{C}^*\, \mathbb{S}_\Sigma'\, \mathbb{C} : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)'\ , \tag{5.3}$$

where we wrote

$$\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) := \mathcal{T}(\partial\Omega_1) \times \cdots \times \mathcal{T}(\partial\Omega_N)\ , \tag{5.4}$$

for a "clipped multi-trace space", that differs from $\boldsymbol{\mathcal{MT}}(\Sigma)$ by a missing contribution from $\mathbb{T}_{\partial\Omega_0}$. The operator $\mathbb{C}^*$ is the formal adjoint of $\mathbb{C}$. By means of $\mathbb{M}_G'$ we can state a variational problem completely equivalent to (3.19): seek $\widehat{\mathfrak{u}} \in \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$ such that

$$\begin{aligned} \left\langle \mathbb{M}_G'\, \widehat{\mathfrak{u}}, \widehat{\mathfrak{v}} \right\rangle_{\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)} &= -[\![\mathbb{T}_0\, \mathbf{u}_{\text{inc}}, \mathbb{X}_1\, \mathfrak{v}_1 \vee \mathbb{X}_2\, \mathfrak{v}_2 \vee \cdots \vee \mathbb{X}_N\, \mathfrak{v}_N]\!]_{\mathcal{T}(\partial\Omega_0)} \\ &\overset{(2.8)}{=} \sum_{i=1}^{N} [\![\mathbb{T}_i[0]\mathbf{u}_{\text{inc}}, \mathfrak{v}_i]\!]_{\mathcal{T}(\partial\Omega_i)}\ . \end{aligned} \tag{5.5}$$

for all $\widehat{\mathfrak{v}} = (\mathfrak{v}_1, \ldots, \mathfrak{v}_N) \in \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$.

Restricting the pairing $[\![\cdot,\cdot]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)}$ to $\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$ gives another L^2*-type pairing*

$$[\![\vec{\mathfrak{u}},\vec{\mathfrak{v}}]\!]_{\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)} := \sum_{i=1}^{N} [\![\mathbb{L}_i\,\vec{\mathfrak{u}},\mathbb{L}_i\,\vec{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega_i)}\ , \quad \mathfrak{u},\mathfrak{v}\in\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)\ , \tag{5.6}$$

that will induce a self-duality of $\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$. Harking back to (4.7), let us designate the isomorphism induced by the pairing $[\![\cdot,\cdot]\!]_{\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)}$ by $\widehat{\mathbb{D}}' : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)'$. It can be used to convert $\mathbb{M}_G'$ into a genuine boundary integral operator $\mathbb{M}_G : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$ according to $\mathbb{M}_G := (\widehat{\mathbb{D}}')^{-1}\circ\mathbb{M}_G'$.

Working with $\mathbb{M}_G$ has the benefit that its structure can be made explicit and we do so for the case $N=2$. To begin with we recast $\mathbb{A}_0$, which acts on functions in $\mathcal{T}(\partial\Omega_0)=\mathcal{T}(\partial\Omega_1)\times\mathcal{T}(\partial\Omega_2)=\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$. For $\widehat{\mathfrak{u}}=(\mathfrak{u}_1,\mathfrak{u}_2)$, $\widehat{\mathfrak{v}}=(\mathfrak{v}_1,\mathfrak{v}_2)$ we find

$$\begin{aligned}\left[\!\!\left[\mathbb{A}_0\,\mathbb{L}_0\,\mathbb{C}\begin{pmatrix}\mathfrak{u}_1\\ \mathfrak{u}_2\end{pmatrix},\mathbb{L}_0\,\mathbb{C}\begin{pmatrix}\mathfrak{v}_1\\ \mathfrak{v}_2\end{pmatrix}\right]\!\!\right]_{\mathcal{T}(\partial\Omega_0)} &= \\ &[\![\mathbb{A}_1[0]\mathfrak{u}_1,\mathfrak{v}_1]\!]_{\mathcal{T}(\partial\Omega_1)} + [\![\mathbb{T}_1[0]\mathbb{G}_2[0]\mathfrak{u}_2,\mathfrak{v}_1]\!]_{\mathcal{T}(\partial\Omega_1)} + \\ &[\![\mathbb{T}_2[0]\mathbb{G}_1[0]\mathfrak{u}_1,\mathfrak{v}_2]\!]_{\mathcal{T}(\partial\Omega_2)} + [\![\mathbb{A}_2[0]\mathfrak{u}_2,\mathfrak{v}_2]\!]_{\mathcal{T}(\partial\Omega_2)}\ .\end{aligned} \tag{5.7}$$

To see this, we advise the reader to study the derivation of (4.13) by means of the jump relations again, because similar manipulations yield the identity

$$\begin{aligned}\mathbb{A}_0\,\mathbb{L}_0\,\mathbb{C}\begin{pmatrix}\mathfrak{u}_1\\ \mathfrak{u}_2\end{pmatrix} &= (\mathbb{T}_0\,\mathbb{G}_0-\tfrac{1}{2}\mathsf{Id})(\mathbb{X}_1\,\mathfrak{u}_1\vee\mathbb{X}_2\,\mathfrak{u}_2)\\ &\overset{(4.11)}{=} \mathbb{T}_1^+(-\mathbb{G}_1[0]\mathfrak{u}_1-\mathbb{G}_2[0]\mathfrak{u}_2)\ \vee\ \mathbb{T}_2^+(-\mathbb{G}_1[0]\mathfrak{u}_1-\mathbb{G}_2[0]\mathfrak{u}_2)-\\ &\quad\tfrac{1}{2}(\mathbb{X}_1\,\mathfrak{u}_1\vee\mathbb{X}_2\,\mathfrak{u}_2)\\ &\overset{(4.12)}{=} (-\,\mathbb{X}_1\,\mathbb{T}_1[0]\mathbb{G}_1[0]+\tfrac{1}{2}\,\mathbb{X}_1)\mathfrak{u}_1\ \vee\ (-\,\mathbb{X}_2\,\mathbb{T}_2[0]\mathbb{G}_2[0]+\tfrac{1}{2}\,\mathbb{X}_2)\mathfrak{u}_2-\\ &\quad\mathbb{X}_1\,\mathbb{T}_1[0]\mathbb{G}_2[0]\mathfrak{u}_2\ \vee\ \mathbb{X}_2\,\mathbb{T}_2[0]\mathbb{G}_1[0]\mathfrak{u}_1\\ &= -\,\mathbb{X}_1(\mathbb{T}_1[0]\mathbb{G}_1[0]-\tfrac{1}{2}\mathsf{Id})\mathfrak{u}_1\ \vee\ \mathbb{X}_2(\mathbb{T}_2[0]\mathbb{G}_2[0]-\tfrac{1}{2}\mathsf{Id})\mathfrak{u}_2\\ &\quad\mathbb{X}_1\,\mathbb{T}_1[0]\mathbb{G}_2[0]\mathfrak{u}_2\ \vee\ \mathbb{X}_2\,\mathbb{T}_2[0]\mathbb{G}_1[0]\mathfrak{u}_1\ .\\ &\overset{(2.17)}{=} -\,\mathbb{X}_1(\mathbb{A}_1[0]\mathfrak{u}_1+\mathbb{T}_1[0]\mathbb{G}_2[0]\mathfrak{u}_2)-\mathbb{X}_2(\mathbb{T}_2[0]\mathbb{G}_1[0]\mathfrak{u}_1+\mathbb{A}_2[0]\mathfrak{u}_2)\ .\end{aligned} \tag{5.8}$$

We conclude (5.7) because of $[\![\mathbb{X}_i\,\mathfrak{u},\mathbb{X}_i\,\mathfrak{v}]\!]_{\mathcal{T}(\partial\Omega_i)} = -[\![\mathfrak{u},\mathfrak{v}]\!]_{\mathcal{T}(\partial\Omega_i)}$. For the sake of a compact operator notation we may abbreviate the "remote coupling integral operators"

$$\mathbb{R}_i^j := \mathbb{T}_j[0]\mathbb{G}_i[0]\ ,\quad i\neq j\ , \tag{5.9}$$

which evaluate a potential defined on one sub-domain boundary on the other. Then, for $N = 2$, $\mathbb{M}_G$ can be rephrased as

$$\mathbb{M}_G \begin{pmatrix} \mathfrak{u}_1 \\ \mathfrak{u}_2 \end{pmatrix} = \begin{pmatrix} \mathbb{A}_1[0] + \mathbb{A}_1 & \mathbb{R}_2^1 \\ \mathbb{R}_1^2 & \mathbb{A}_2[0] + \mathbb{A}_2 \end{pmatrix} \begin{pmatrix} \mathfrak{u}_1 \\ \mathfrak{u}_2 \end{pmatrix} , \tag{5.10}$$

for $\mathfrak{u}_1 \in \mathcal{T}(\partial\Omega_1), \mathfrak{u}_2 \in \mathcal{T}(\partial\Omega_2)$.

In Section 4.2 we established the Calderón identity (4.16) for the STF operator in the case $N = 1$ and $\alpha_0 = \alpha_1$. Hardly surprising, a similar result remains valid for $N > 1$ and separated sub-domains. For the sake of simplicity we largely restrict the discussion to $N = 2$, see Figure 2.

First, we recall the null field property of the total potential $\mathbb{G}$ belonging to a generic domain Ω with Lipschitz boundary $\partial\Omega$.

Lemma 5.1 (Null field property). *If* $\mathfrak{u} \in \mathcal{T}(\partial\Omega)$ *belongs to the space of Cauchy data* $\mathcal{CD}(\partial\Omega)$ *for the differential equation* (1.4)*, see Definition 2.5, then* $\mathbb{G}\mathfrak{u} = 0$ *in* $\mathbb{R}^d \setminus \overline{\Omega}$.

An immediate consequence are important identities for the remote coupling operators.

Lemma 5.2 (Annihilation of remote coupling operators). *The remote coupling operators* $\mathbb{R}_i^j$ *defined in* (5.9) *satisfy for* $i \neq j$

$$\begin{aligned} \mathbb{R}_i^j\, \mathbb{R}_j^i &= 0 , \\ \mathbb{R}_i^j\, \mathbb{A}_i[0] &= -\tfrac{1}{2}\, \mathbb{R}_i^j , \\ \mathbb{A}_j[0]\, \mathbb{R}_i^j &= \tfrac{1}{2}\, \mathbb{R}_i^j . \end{aligned} \tag{5.11}$$

Proof. The product of two different remote coupling operators vanishes:

$$\mathbb{R}_i^j\, \mathbb{R}_j^i = \mathbb{T}_j[0]\mathbb{G}_i[0]\, \mathbb{T}_i[0]\mathbb{G}_j[0] = \mathbb{T}_j[0](0) = 0 ,$$

because Theorem 2.2 asserts $\mathbb{T}_i[0]\mathbb{G}_j[0](\mathcal{T}(\partial\Omega_j)) \subset \mathcal{CD}(\partial\Omega_i)$ and, thus, by Lemma 5.1

$$\mathbb{G}_i[0]\, \mathbb{T}_i[0]\mathbb{G}_j[0]_{|\overline{\Omega}_j} = 0 .$$

The same arguments can be applied to the other operator products, for instance,

$$\mathbb{R}_i^j\, \mathbb{A}_i[0] = \mathbb{T}_j[0]\mathbb{G}_i[0] \left(\mathbb{T}_i[0]\mathbb{G}_i[0] - \tfrac{1}{2}\mathsf{Id}\right) = -\tfrac{1}{2}\, \mathbb{R}_i^j .$$

□

Returning to $N = 2$, owing to the assumption $\alpha_0 = \alpha_1 = \alpha_2$, we can appeal to Lemma 4.1 and obtain from (5.10)

$$\mathbb{M}_G = \mathbb{M}_G[0] + \{\text{compact}\} , \quad \mathbb{M}_G[0] := \begin{pmatrix} 2\,\mathbb{A}_1[0] & \mathbb{R}_2^1 \\ \mathbb{R}_1^2 & 2\,\mathbb{A}_2[0] \end{pmatrix} . \tag{5.12}$$

An immediate consequence of Lemma 5.2 and (4.9) is that $(\mathbb{M}_G[0])^2 = \mathsf{Id}$, which implies the Calderón identity

$$\mathbb{M}_G \circ \mathbb{M}_G = \mathsf{Id} + \{\text{compact}\}\ , \tag{5.13}$$

provided that $\boxed{\alpha_i = \alpha_0}$ for all $i = 1, \dots, N$.

5.2 The gap idea

The situation of separated sub-domains is of little interest in itself, but it will serve as a crucial stepping stone for the generalization of the STF to a first type of multi-trace boundary integral formulation.

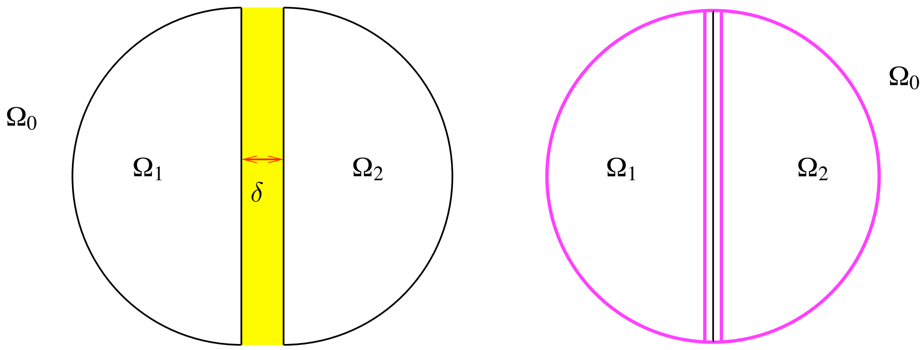

Figure 3. Illustration of the "gap idea" for $N = 2$. Left: sub-domains separated by gap of width $\delta > 0$. Right: closed gap

Consider the situation sketched in Figure 3 (left) with the sub-domains separated by a narrow gap of width $\delta > 0$. We investigate what will happen to the operator $\mathbb{R}_2^1[0] := \mathbb{T}_2[0]\mathbb{G}_1[0]$ as $\delta \to 0$. Keep in mind that

$$(\mathbb{G}_1[0]\mathfrak{u}_1)_{|\mathbb{R}^d \setminus \Omega_1} \in \boldsymbol{\mathcal{H}}_{\mathrm{loc}}(\mathsf{L}, \mathbb{R}^d \setminus \Omega_1)\ , \quad \mathfrak{u}_1 \in \mathcal{T}(\partial\Omega_1)\ . \tag{5.14}$$

Hence, $\mathbb{T}_2[0]\mathbb{G}_1[0]\mathfrak{u}_1$ is well defined as a function in $\mathcal{T}(\partial\Omega_2)$ whenever $\partial\Omega_2 \subset \mathbb{R}^d \setminus \Omega_1$. These conditions are still met even when $\delta = 0$. Thus the operator $\mathbb{M}_G' : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)'$ as defined in (5.10) for $N = 2$ and in (5.3) for general separated sub-domains remains meaningful and will give rise to the *global multi-trace integral equation formulation* (global MTF) for the transmission problem (1.8). Manipulations completely parallel to the derivation of (5.8) give the final variational form of the global MTF [19, Sect. 8.2]: seek $\widehat{\mathfrak{u}} = (\mathfrak{u}_1, \dots, \mathfrak{u}_N) \in \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$

$$\boxed{\left[\!\left[\mathbb{M}_G\,\widehat{\mathfrak{u}}, \widehat{\mathfrak{v}}\right]\!\right]_{\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)} = \sum_{i=1}^{N} \left[\!\left[\mathbb{T}_i[0]\mathbf{u}_{\mathrm{inc}}, \mathbb{L}_i\,\widehat{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\partial\Omega_i)}}\ , \tag{5.15}$$

for all $\widehat{\mathfrak{v}} = (\mathfrak{v}_1, \dots, \mathfrak{v}_N) \in \widehat{\mathcal{MT}}(\Sigma)$ with $\mathbb{M}_G : \widehat{\mathcal{MT}}(\Sigma) \to \widehat{\mathcal{MT}}(\Sigma)$ given by

$$\mathbb{M}_G := \begin{pmatrix} \mathbb{A}_1[0] + \mathbb{A}_1 & \mathbb{R}_2^1 & \dots & \dots & \mathbb{R}_N^1 \\ \mathbb{R}_1^2 & \mathbb{A}_2 + \mathbb{A}_2 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \mathbb{A}_{N-1}[0] + \mathbb{A}_{N-1} & \mathbb{R}_N^{N-1} \\ \mathbb{R}_1^N & \dots & \dots & \mathbb{R}_{N-1}^N & \mathbb{A}_N[0] + \mathbb{A}_N \end{pmatrix} . \tag{5.16}$$

5.3 Properties of global MTF

Owing to the derivation of the global MTF from the STF via a "zero gap reasoning", as explained in Section 5.2, we expect that

> all properties of the classical STF for separated sub-domains carry over to the global MTF.

The global MTF lives up to this expectation. Concrete results are given in the following theorems. Firstly, we adapt Theorem 3.3 to the global MTF:

Theorem 5.3 (Coercivity of global MTF, [19, Sect. 10], [18, Sect. 9.5]). *The bi-linear form of the global MTF from* (5.15) *satisfies a (generalized) Gårding inequality on* $\widehat{\mathcal{MT}}(\Sigma)$.

The assertion made in Lemma 3.4 for the STF remains valid for the global MTF.

Theorem 5.4 (Uniqueness for global MTF, [19, Sect. 9], [18, Thm. 8.1]). *The global MTF operator from* (5.16) *is injective.*

By virtue of a Fredholm alternative argument, the last two theorems imply existence of solutions of (5.15) and the existence of a bounded inverse of $\mathbb{M}_G$. The special Calderón identity (5.13) holds for the global MTF without restrictions on the geometry.

Theorem 5.5 (Calderón identity, [19, Sect. 11], [18, Sect. 11]). *Provided that* $\alpha_i = \alpha_0$, $i = 1, \dots, N$, *the global MTF operator from* (5.16) *satisfies*

$$(\mathbb{M}_G)^2 = \mathsf{Id} + \{compact\} .$$

5.4 Galerkin discretization

In Section 3.3 we introduced finite dimensional subspaces $\mathcal{T}_{D,h}(\partial\Omega_i) \subset \mathcal{T}_D(\partial\Omega_i)$ and $\mathcal{T}_{N,h}(\partial\Omega_i) \subset \mathcal{T}_N(\partial\Omega_i)$ generated by piecewise polynomial functions on meshes of $\partial\Omega_i$. Writing $\mathcal{T}_h(\partial\Omega_i) := \mathcal{T}_{D,h}(\partial\Omega_i) \times \mathcal{T}_{N,h}(\partial\Omega_i)$ the definition (5.4) immediately suggests that we use

$$\widehat{\mathcal{MT}}_h(\Sigma) := \mathcal{T}_h(\partial\Omega_1) \times \dots \times \mathcal{T}_h(\partial\Omega_N) \tag{5.17}$$

as trial and test space for a boundary element Galerkin discretization of (5.15). Again, the assertion of Theorem 3.6 for the STF has a counterpart for the global MTF.

Theorem 5.6 (Stability of Galerkin discretization, [19, Prop. 10.4], [18, Prop. 10.1]). *The Galerkin discretization of the global MTF* (5.15) *based on standard piecewise polynomial boundary element spaces satisfies an h-uniform asymptotic discrete inf-sup condition.*

Remark 5.1. We emphasize that, in principle, the spaces $\mathcal{T}_h(\partial\Omega_i)$ need not match at all, which allows the combinations of different types of boundary elements on different sub-domains. △

Remark 5.2. It goes without saying that the system matrix $\mathbf{M}$ of the resulting linear systems of equations inherits the structure of (5.16); its i-th diagonal block will agree with the usual Galerkin matrix corresponding to $\mathbb{A}_i[0] + \mathbb{A}_i$ on $\boldsymbol{\mathcal{ST}}_h(\partial\Omega_i)$. For the off-diagonal block related to the operators $\mathbb{R}_i^j[0]$ recall that, in case Ω_i and Ω_j have a common interface, according to (5.9) the traces to be used on Γ_{ij} are exterior traces with respect to Ω_i. Hence, the jump relations of Theorem 2.3 have to be taken into account and the integral representation formulas have to be supplemented by $\frac{1}{2}$-weighted L^2-pairings. Details can be found in [34, Sect. 5]. △

The global MTF (5.15) readily lends itself to efficient operator preconditioning as discussed in Section 4.3:

(i) Obviously, $\mathbb{M}_G : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$ is continuous, and so will be $(\mathbb{M}_G)^2 : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$. In terms of $\mathbb{M}_G^{'} := \widehat{\mathbb{D}'} \circ \mathbb{M}_G$ this states the continuity of $(\widehat{\mathbb{D}'}^{-1})^* \mathbb{M}_G^{'} \widehat{\mathbb{D}'}^{-1} \mathbb{M}_G^{'} : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)$. Further, by Theorem 5.5 the latter operator will be a compact perturbation of the identity in the case of constant α, compare (5.13).

(ii) The duality pairing $[\![\cdot,\cdot]\!]_{\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)}$ from (5.6) is a local L^2-type pairing, which will yield a sparse Galerkin matrix $\mathbf{D}$, *cf.* Remark 4.2 for an explanation why this is essential.

(iii) Uniformly stable discretizations of the pairing on $\widehat{\boldsymbol{\mathcal{MT}}}_h(\partial\Omega_i)$ can be constructed based on dual meshes, see Section 4.4.

Remark 5.3. In the context of operator preconditioning we may replace $\mathbb{M}_G^{'}$ with the fully decoupled local integral operators, that is, we may use the "diagonal preconditioner"

$$\widehat{\mathbb{D}'}^{-1} \mathbb{A}_G' \widehat{\mathbb{D}'}^{-1} : \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)' \to \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) ,$$

with

$$\left\langle \mathbb{A}_G' \widehat{\mathfrak{u}}, \widehat{\mathfrak{v}} \right\rangle_{\widehat{\boldsymbol{\mathcal{MT}}}(\Sigma)} := \sum_{i=1}^{N} [\![\mathbb{A}_i\, \mathfrak{u}_i, \mathfrak{v}_i]\!]_{\mathcal{T}(\partial\Omega_i)} , \quad \widehat{\mathfrak{u}}, \widehat{\mathfrak{v}} \in \widehat{\boldsymbol{\mathcal{MT}}}(\Sigma) . \tag{5.18}$$

Given an uniformly stable discrete duality pairing this will also provide a preconditioner that is robust with respect to the resolution of the surface meshes. △

6 Local Multi-Trace Formulation

The idea to decouple the skeleton Cauchy traces into local contributions that guided the development of the global multi-trace formulation of Section 5 can also be pursued in a fairly different spirit. This led to the so-called local multi-trace formulation, which will be derived and explored in this section.

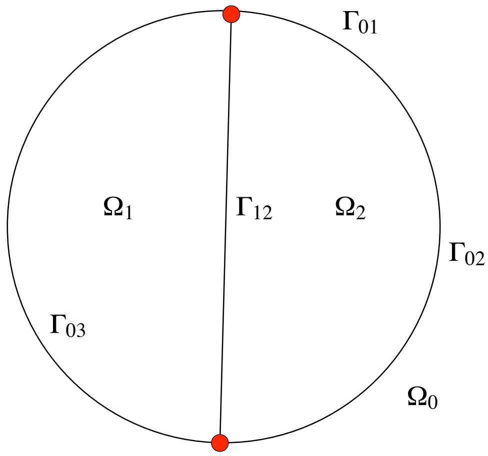

Figure 4. Generic geometric setting for $N = 2$

The local MTF was first presented in [32] for the Helmholtz transmission problems (1.1). Its extension to electromagnetic transmission problems (1.2) is work in progress and has not been published yet (as of summer 2012). Formally, the adaptation of the method to Maxwell's transmission problems is straightforward and in the next section we will also include this case.

6.1 Partial transmission conditions

First, let us examine the situation $N = 2$, that is a scatterer composed of only two sub-domains, as sketched in Figure 4 for $d = 2$. For the purpose of presenting the local multi-trace formulation this case is generic, completely captures the essence of the methods, but still helps to avoid excessive notational complexity. With $\mathbf{u} \in \boldsymbol{\mathcal{H}}_{\text{rad}}(\mathsf{L}, \mathbb{R}^d)$ denoting the weak solution of the transmission problem (1.8) we write $\vec{\mathfrak{u}} := \mathbb{T}_\Sigma \mathbf{u} \in \boldsymbol{\mathcal{ST}}(\Sigma) \subset \boldsymbol{\mathcal{MT}}(\Sigma)$ for its skeleton Cauchy trace as defined in Section 3.1. As in (3.15), from Theorem 2.6 we immediately conclude

$$(\tfrac{1}{2}\mathsf{Id} - \mathbb{A}_i)\,\mathbb{L}_i\,\vec{\mathfrak{u}} = 0 \qquad \text{in } \mathcal{T}(\partial\Omega_i)\ , \quad i = 1,2\ , \tag{6.1}$$

$$(\tfrac{1}{2}\mathsf{Id} - \mathbb{A}_0)(\mathbb{L}_0\,\vec{\mathfrak{u}} - \mathbb{T}_0\,\mathbf{u}_{\text{inc}}) = 0 \qquad \text{in } \mathcal{T}(\partial\Omega_0)\ . \tag{6.2}$$

The same simplification that permitted us to switch from (3.18) to (3.19) can be applied to (6.2) and yields

$$(\tfrac{1}{2}\mathsf{Id} - \mathbb{A}_0)\,\mathbb{L}_0\,\vec{\mathfrak{u}} = -\,\mathbb{T}_0\,\mathbf{u}_{\text{inc}}\ . \tag{6.3}$$

The next key manipulation of (6.1)–(6.2) starts from the local transmission conditions

$$\mathsf{T}_{D,i}\,\mathbf{u} = \mathsf{T}_{D,j}\,\mathbf{u} \quad \text{and} \quad \mathsf{T}_{N,i}\,\mathbf{u} = -\,\mathsf{T}_{N,j}\,\mathbf{u} \quad \text{on } \Gamma_{ij} \quad \text{in } \mathcal{T}(\partial\Omega_0)\ . \tag{3.9}$$

They are used to substitute the terms $\frac{1}{2}\,\mathbb{L}_i\,\vec{\mathfrak{u}}$ in (6.1) and (6.3) with Cauchy traces from adjacent sub-domains. This is done on each interface Γ_{ij} and yields

$$\begin{array}{rcccccl}
\mathbb{A}_0\,\mathbb{L}_0\,\vec{\mathfrak{u}} & - & \frac{1}{2}\,\mathbb{X}_{1\to 0}\,\mathbb{L}_1\,\vec{\mathfrak{u}} & - & \frac{1}{2}\,\mathbb{X}_{2\to 0}\,\mathbb{L}_2\,\vec{\mathfrak{u}} & = & -\,\mathbb{T}_0\,\mathbf{u}_{\text{inc}}\ , \\
-\frac{1}{2}\,\mathbb{X}_{0\to 1}\,\mathbb{L}_0\,\vec{\mathfrak{u}} & + & \mathbb{A}_1\,\mathbb{L}_1\,\vec{\mathfrak{u}} & - & \frac{1}{2}\,\mathbb{X}_{2\to 1}\,\mathbb{L}_2\,\vec{\mathfrak{u}} & = & 0\ , \\
-\frac{1}{2}\,\mathbb{X}_{0\to 2}\,\mathbb{L}_0\,\vec{\mathfrak{u}} & - & \frac{1}{2}\,\mathbb{X}_{1\to 2}\,\mathbb{L}_1\,\vec{\mathfrak{u}} & + & \mathbb{A}_2\,\mathbb{L}_2\,\vec{\mathfrak{u}} & = & 0\ .
\end{array} \tag{6.4}$$

Here, we have introduced the *local transmission operators* $\mathbb{X}_{i\to j}$, which, if Γ_{ij} is a genuine interface,

(i) take a pair of functions on $\partial\Omega_i$ as argument, corresponding to Dirichlet and Neumann traces,

(ii) restrict both functions to Γ_{ij},

(iii) flip the sign of the second function in order to take into account the transmission conditions (3.9),

(iv) and, finally, extend both functions by zero to functions on $\partial\Omega_j$.

Formally, we can define

$$\mathbb{X}_{i\to j} : \left\{ \begin{array}{ccc}
L^2(\partial\Omega_i) \times L^2(\partial\Omega_i) & \to & L^2(\partial\Omega_j) \times L^2(\partial\Omega_j) \\
\mathfrak{u} := \binom{u}{\varphi} & \mapsto & (\mathbb{X}_{i\to j}\,\mathfrak{u})(\boldsymbol{x}) := \begin{cases} \binom{u(\boldsymbol{x})}{-\varphi(\boldsymbol{x})} & \text{for } \boldsymbol{x} \in \Gamma_{ij}\ , \\ 0 & \text{elsewhere on } \partial\Omega_j\ . \end{cases}
\end{array} \right. \tag{6.5}$$

The reader might be wondering, what made us rely on L^2-spaces here; we are going to answer this question in the next section.

Temporarily glossing over issues of proper function spaces, we have found that Cauchy traces of solutions of the transmission problems (1.8) satisfy the boundary integral equation (6.4). Turning the tables, now we boldly convert (6.4) into novel boundary integral equations for unknown *local* Cauchy traces: find $\mathfrak{u}_i \in \mathcal{T}(\partial\Omega_i)$ satisfying

$$\begin{array}{ccccccc} \mathbb{A}_0\,\mathfrak{u}_0 & - & \frac{1}{2}\,\mathbb{X}_{1\to 0}\,\mathfrak{u}_1 & - & \frac{1}{2}\,\mathbb{X}_{2\to 0}\,\mathfrak{u}_2 & = & -\,\mathbb{T}_0\,\mathbf{u}_{\text{inc}}\ , \\ -\frac{1}{2}\,\mathbb{X}_{0\to 1}\,\mathfrak{u}_0 & + & \mathbb{A}_1\,\mathfrak{u}_1 & - & \frac{1}{2}\,\mathbb{X}_{2\to 1}\,\mathfrak{u}_2 & = & 0\ , \\ -\frac{1}{2}\,\mathbb{X}_{0\to 2}\,\mathfrak{u}_0 & - & \frac{1}{2}\,\mathbb{X}_{1\to 2}\,\mathfrak{u}_1 & + & \mathbb{A}_2\,\mathfrak{u}_2 & = & 0\ . \end{array} \tag{6.6}$$

We have arrived at a boundary integral equation, which is set on the skeleton multi-trace space $\boldsymbol{\mathcal{MT}}(\Sigma)$! We call (6.6) the *local multi-trace formulation*, due to the fact, that, in contrast to the global multi-trace formulation (5.15) introduced in Section 5, coupling of sub-domains relies on local transmission operators alone.

It is not difficult to state the local MTF for an arbitrary number of sub-domains: find $\vec{\mathfrak{u}} \in \boldsymbol{\mathcal{MT}}(\Sigma)$ that solves

$$\mathbb{A}_i\,\mathbb{L}_i\,\vec{\mathfrak{u}} - \frac{1}{2}\sum_{\substack{j=0\\ j\neq i}}^{N} \mathbb{X}_{j\to i}\,\mathbb{L}_j\,\vec{\mathfrak{u}} = \begin{cases} -\,\mathbb{T}_0\,\mathbf{u}_{\text{inc}} & \text{for } i=0\ , \\ 0 & \text{else}\ , \end{cases} \qquad i=0,\dots,N\ . \tag{6.7}$$

Of course, we have set $\mathbb{X}_{i\to j} := 0$ in case there is no genuine interface Γ_{ij}.

Having departed from a solution of the transmission problem tells us that its Cauchy traces will satisfy (6.7). There is no guarantee yet that a solution of (6.7) has anything to do with a solution of the transmission problem. This reverse conclusion holds thanks to uniqueness of solutions of (6.7).

Theorem 6.1 (Uniqueness of solutions of local MTF boundary integral equations, [32, Sect. 3.2.6]).

For $\mathbf{u}_{\text{inc}} = 0$ *the local MTF integral equations* (6.7) *have the only solution* $\vec{\mathfrak{u}} = 0$.

6.2 Local MTF: variational formulation

A rigorous statement of the local MTF (6.6) in trace spaces encounters massive obstacles, because the local transmission operators $\mathbb{X}_{i\to j}$ fail to be continuous mappings $\mathcal{T}(\partial\Omega_i) \to \mathcal{T}(\partial\Omega_j)$. We elaborate this for the case of the Helmholtz transmission problem and the corresponding trace spaces given in Concretization 2.1. First, we remind that the mapping

$$L^2(\partial\Omega_i) \to L^2(\partial\Omega_i)\ , \quad w \mapsto \chi_{ij} w\ , \quad \chi_{ij}(\mathbf{x}) = \begin{cases} 1 & \text{, if } \boldsymbol{x} \in \Gamma_{ij}\ , \\ 0 & \text{, elsewhere,} \end{cases} \tag{6.8}$$

does not have a continuous restriction or extension as a mapping $H^{\frac{1}{2}}(\partial\Omega_i) \to H^{\frac{1}{2}}(\partial\Omega_i)$ or $H^{-\frac{1}{2}}(\partial\Omega_i) \to H^{-\frac{1}{2}}(\partial\Omega_i)$, respectively, if $\Gamma_{ij} \neq \partial\Omega_i$. This peculiar property of the trace spaces forces us to resort to a slightly modified function space framework in order to obtain a meaningful variational formulation of (6.7).

To begin with, let the space $\mathcal{T}_*(\Gamma_{ij})$, $* = D, N$, contain the restrictions of Dirichlet/Neumann traces in $\mathcal{T}_*(\partial\Omega_i)$ to Γ_{ij}. It is endowed with the natural norm

$$\|u\|_{\mathcal{T}_*(\Gamma_{ij})} := \inf\{\|v\|_{\mathcal{T}_*(\partial\Omega_i)} : v \in \mathcal{T}_*(\partial\Omega_i),\ v_{|\Gamma_{ij}} = u\}\ , \quad * = D, N\ .$$

These spaces can be assembled into "broken trace spaces" on $\partial\Omega_i$

$$\mathcal{T}_{*,\mathrm{pw}}(\partial\Omega_i) := \bigotimes\nolimits_j \mathcal{T}_*(\Gamma_{ij})\ , \quad * = D, N\ ,$$

where, of course, the product of spaces covers only genuine interfaces. We also write $\mathcal{T}_{\mathrm{pw}}(\partial\Omega_i) := \mathcal{T}_{D,\mathrm{pw}}(\partial\Omega_i) \times \mathcal{T}_{N,\mathrm{pw}}(\partial\Omega_i)$ for a space of "broken Cauchy traces" on $\partial\Omega_i$ and it is convenient to introduce the "broken multi-trace space"

$$\boldsymbol{\mathcal{MT}}_{\mathrm{pw}}(\Sigma) := \mathcal{T}_{\mathrm{pw}}(\partial\Omega_i) \times \cdots \times \mathcal{T}_{\mathrm{pw}}(\partial\Omega_N)\ . \tag{6.9}$$

By restriction to interfaces, $\mathcal{T}_*(\partial\Omega_i)$ can be embedded into $\mathcal{T}_{*,\mathrm{pw}}(\partial\Omega_i)$. The latter may be a strictly larger space, and the L^2 duality pairing between $\mathcal{T}_D(\partial\Omega_i)$ and $\mathcal{T}_N(\partial\Omega_i)$ usually cannot be extended to a continuous bi-linear form on $\mathcal{T}_{D,\mathrm{pw}}(\partial\Omega_i) \times \mathcal{T}_N(\partial\Omega_i)$ or $\mathcal{T}_D(\partial\Omega_i) \times \mathcal{T}_{N,\mathrm{pw}}(\partial\Omega_i)$, respectively. When feeding broken traces into one slot of the L^2 duality pairing, the other slot has to be supplied with traces satisfying certain constraints. These constraints are met by localized Dirichlet and Neumann traces belonging to the spaces

$$\widetilde{\mathcal{T}}_*(\Gamma_{ij}) = \{\phi \in \mathcal{T}_*(\Gamma_{ij}) :\ \widetilde{\phi} \in \mathcal{T}_*(\partial\Omega_j)\}\ , \tag{6.10}$$

with norm

$$\|\phi\|_{\widetilde{\mathcal{T}}_*(\Gamma_{ij})} = \left\|\widetilde{\phi}\right\|_{\mathcal{T}_*(\partial\Omega_i)}\ , \quad * = D, N\ , \quad * = D, N\ , \tag{6.11}$$

where $\widetilde{\phi}$ is the extension of ϕ to $\partial\Omega_j$ by zero. In general, the norm of $\widetilde{\mathcal{T}}_*(\Gamma_{ij})$ is strictly stronger than that of $\mathcal{T}_*(\Gamma_{ij})$. Most importantly, the pairs of spaces $\mathcal{T}_D(\Gamma_{ij})$–$\widetilde{\mathcal{T}}_N(\Gamma_{ij})$ and $\widetilde{\mathcal{T}}_D(\Gamma_{ij})$–$\mathcal{T}_N(\Gamma_{ij})$ are in duality with respect to the L^2-pairing on Γ_{ij}, *cf.* (2.2). Gluing the spaces of localized Dirichlet and Neumann traces yields spaces with "zero boundary conditions at the boundaries of the interfaces"

$$\widetilde{\mathcal{T}}_*(\partial\Omega_i) = \bigotimes_j \widetilde{\mathcal{T}}_*(\Gamma_{ij}) \subset \mathcal{T}_*(\partial\Omega_i)\ , \quad * = D, N\ , \tag{6.12}$$

which can be assembled into the constrained Cauchy trace space $\widetilde{\mathcal{T}}(\partial\Omega_i) := \widetilde{\mathcal{T}}_D(\partial\Omega_i) \times \widetilde{\mathcal{T}}_N(\partial\Omega_i)$. In analogy to (3.1) we set

$$\widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma) := \widetilde{\mathcal{T}}(\partial\Omega_0) \times \cdots \times \widetilde{\mathcal{T}}(\partial\Omega_N)\ . \tag{6.13}$$

The relationships of the above spaces can be summarized as

$$\widetilde{\mathcal{T}}_D(\partial\Omega_i) \subset \mathcal{T}_D(\partial\Omega_i) \subset \mathcal{T}_{D,\mathrm{pw}}(\partial\Omega_i) \qquad \widetilde{\mathcal{T}}_N(\partial\Omega_i) \subset \mathcal{T}_N(\partial\Omega_i) \subset \mathcal{T}_{N,\mathrm{pw}}(\partial\Omega_i)\ , \tag{6.14}$$

where the arrows connect spaces that are in duality with respect to the L^2-pairing on $\partial\Omega_i$. Combine this with the continuity of the transmission operators

$$\mathbb{X}_{i\to j} : \mathcal{T}(\partial\Omega_i) \to \mathcal{T}_{\mathrm{pw}}(\partial\Omega_j)\ , \tag{6.15}$$

to see that

$$(\vec{\mathfrak{u}}, \vec{\mathfrak{v}}) \to \left[\!\left[\mathbb{X}_{i\to j}\, \mathbb{L}_i\, \vec{\mathfrak{u}}, \mathbb{L}_j\, \vec{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\partial\Omega_j)}$$

becomes a continuous bi-linear form on $\boldsymbol{\mathcal{MT}}(\Sigma) \times \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)$.

Thus, $\widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)$ is a suitable *test space* for the weak variational formulation of (6.6), because the resulting bi-linear form will be continuous. For the special situation $N = 2$ we arrive at the variational equation: seek $(\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3) \in \boldsymbol{\mathcal{MT}}(\Sigma)$ such that for all $(\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3) \in \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)$

$$\begin{array}{ccccc}
[\![\mathbb{A}_0\, \mathfrak{u}_0, \mathfrak{v}_0]\!]_{\mathcal{T}(\partial\Omega_0)} & - & \left[\!\left[\frac{1}{2}\, \mathbb{X}_{1\to 0}\, \mathfrak{u}_1, \mathfrak{v}_0\right]\!\right]_{\mathcal{T}(\partial\Omega_0)} & - & \left[\!\left[\frac{1}{2}\, \mathbb{X}_{2\to 0}\, \mathfrak{u}_2, \mathfrak{v}_0\right]\!\right]_{\mathcal{T}(\partial\Omega_0)} \\
 & & & = & -[\![\mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}}, \mathfrak{v}_0]\!]_{\mathcal{T}(\partial\Omega_0)}\ , \\
-\left[\!\left[\frac{1}{2}\, \mathbb{X}_{0\to 1}\, \mathfrak{u}_0, \mathfrak{v}_1\right]\!\right]_{\mathcal{T}(\partial\Omega_1)} & + & [\![\mathbb{A}_1\, \mathfrak{u}_1, \mathfrak{v}_1]\!]_{\mathcal{T}(\partial\Omega_1)} & - & \left[\!\left[\frac{1}{2}\, \mathbb{X}_{2\to 1}\, \mathfrak{u}_2, \mathfrak{v}_1\right]\!\right]_{\mathcal{T}(\partial\Omega_1)} \\
 & & & = & 0\ , \\
-\left[\!\left[\frac{1}{2}\, \mathbb{X}_{0\to 2}\, \mathfrak{u}_0, \mathfrak{v}_2\right]\!\right]_{\mathcal{T}(\partial\Omega_2)} & - & \left[\!\left[\frac{1}{2}\, \mathbb{X}_{1\to 2}\, \mathfrak{u}_1, \mathfrak{v}_2\right]\!\right]_{\mathcal{T}(\partial\Omega_2)} & + & [\![\mathbb{A}_2\, \mathfrak{u}_2, \mathfrak{v}_2]\!]_{\mathcal{T}(\partial\Omega_2)} \\
 & & & = & 0\ .
\end{array} \tag{6.16}$$

In a general situation the variational problem for the local MTF reads [32, Problem 6] as follows.

Seek $\vec{\mathfrak{u}} \in \boldsymbol{\mathcal{MT}}(\Sigma)$ that satisfies

$$\left\langle \mathbb{M}_L'\, \vec{\mathfrak{u}}, \vec{\mathfrak{v}} \right\rangle = -\left[\!\left[\mathbb{T}_0\, \mathbf{u}_{\mathrm{inc}}, \mathbb{L}_0\, \vec{\mathfrak{v}}\right]\!\right]_{\mathcal{T}(\partial\Omega_0)} \quad \forall \vec{\mathfrak{v}} \in \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)\ , \tag{6.17}$$

with a *continuous* operator

$$\mathbb{M}_L' := \mathbb{L}_i^*\, \mathbb{A}_i\, \mathbb{L}_i - \tfrac{1}{2} \sum_{\substack{j=0 \\ j\neq i}}^{N} \mathbb{L}_i^*\, \mathbb{X}_{j\to i}\, \mathbb{L}_j : \boldsymbol{\mathcal{MT}}(\Sigma) \to \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)'\ .$$

Continuity of $\mathbb{M}_L'$ is an immediate consequence of the L^2-type dualities expressed in (6.14).

Corollary 6.2 (Continuity of local MTF bi-linear form, [32, Sect 3.2.4]). *There is a constant $C > 0$ such that*

$$\left|\left\langle \mathbb{M}_L' \, \vec{\mathfrak{u}}, \vec{\mathfrak{v}} \right\rangle\right| \leq C \, \|\vec{\mathfrak{u}}\|_{\mathcal{MT}(\Sigma)} \left\|\vec{\mathfrak{v}}\right\|_{\widetilde{\mathcal{MT}}(\Sigma)} \quad \forall \vec{\mathfrak{u}} \in \mathcal{MT}(\Sigma), \; \vec{\mathfrak{v}} \in \widetilde{\mathcal{MT}}(\Sigma) \; .$$

6.3 Local MTF: Stability

A result in the theory of Sobolev spaces states that the function space $\mathcal{H}(\mathsf{D}, \Omega)$ as introduced in (1.5) does not support a continuous trace onto the "wire basket", that is, onto the union of all boundaries of the interfaces. In other words, smooth functions whose supports do not intersect $\bigcup_{i,j} \partial\Gamma_{ij}$ are dense in $\mathcal{H}(\mathsf{D}, \Omega)$. This translates into following density result for trace spaces.

Lemma 6.3. *The constrained trace space $\widetilde{\mathcal{MT}}(\Sigma)$ is a dense subspace of $\mathcal{MT}(\Sigma)$.*

Thus, sloppily speaking, the space $\widetilde{\mathcal{MT}}(\Sigma)$ provides enough test functions to let the variational equation (6.17) imply the integral equations (6.7). Solutions of the latter are unique and so this remains true for the variational equation.

Theorem 6.4 (Uniqueness of solutions of variational local MTF). *Solutions of the variational problem* (6.17) *are unique*.

The local MTF bi-linear form also enjoys surprising coercivity properties.

Theorem 6.5 (Coercivity of local MTF bi-linear form, [32, Sect 3.2.7]). *There exist an isomorphism $\mathbb{F} : \mathcal{MT}(\Sigma) \to \mathcal{MT}(\Sigma)$ and a* compact *operator $\mathbb{K} : \mathcal{MT}(\Sigma) \to \mathcal{MT}(\Sigma)$ such that for some constant $C > 0$ the following (generalized) Gårding inequality holds*

$$\mathrm{Re}\left\{\left\langle \mathbb{M}_L' \, \vec{\mathfrak{u}}, \mathbb{F} \, \overline{\vec{\mathfrak{u}}} \right\rangle + + [\![\mathbb{K} \, \mathfrak{v}, \overline{\mathfrak{v}}]\!]_{\mathcal{T}(\partial\Omega)}\right\} \geq C \left\|\vec{\mathfrak{v}}\right\|^2_{\mathcal{MT}(\Sigma)} \quad \forall \vec{\mathfrak{u}} \in \widetilde{\mathcal{MT}}(\Sigma) \; . \tag{6.18}$$

Proof. We restrict ourselves to the Helmholtz case, for which $\mathbb{F} = \mathsf{Id}$. Then note that for $\mathfrak{u}_i = \binom{u_i}{\varphi_i} \in \widetilde{\mathcal{T}}(\partial\Omega_i)$ and $\mathfrak{u}_j = \binom{u_j}{\varphi_j} \in \widetilde{\mathcal{T}}(\partial\Omega_j)$

$$[\![\mathbb{X}_{i\to j} \, \mathfrak{u}_i, \overline{\mathfrak{u}}_j]\!]_{\mathcal{T}(\partial\Omega_j)} = \left[u_i, \overline{\varphi}_j\right]_{\Gamma_{ij}} - \left[u_j, \overline{\varphi}_i\right]_{\Gamma_{ij}} = -\overline{[\![\mathbb{X}_{j\to i} \, \mathfrak{u}_j, \overline{\mathfrak{u}}_i]\!]_{\mathcal{T}(\partial\Omega_i)}} \; .$$

We observe a cancellation of the real parts of "off-diagonal terms" in the local MTF bi-linear form when written as in (6.16), which permits us to conclude

$$\mathrm{Re}\left\langle \mathbb{M}_L' \, \vec{\mathfrak{u}}, \overline{\vec{\mathfrak{u}}} \right\rangle = \sum_{j=1}^{N} \mathrm{Re} \, [\![\mathbb{A}_i \, \mathfrak{u}_i, \overline{\mathfrak{u}}_i]\!]_{\mathcal{T}(\partial\Omega_i)} \; ,$$

for $\vec{\mathfrak{u}} = (\mathfrak{u}_1, \ldots, \mathfrak{u}_N) \in \widetilde{\mathcal{MT}}(\Sigma)$, and the assertion of the theorem is implied by Theorem 2.7. □

At first glance, Theorems 6.4 and 6.5 seem to pave the way for an application of a Fredholm alternative argument as in Section 3.2. A closer scrutiny, however, reveals a fundamental *mismatch* between the norms that support continuity of $\mathbb{M}'_L$, see Corollary 6.2, and the $\boldsymbol{\mathcal{MT}}(\Sigma)$-norm, for which we have coercivity. More sophisticated tools are needed to cope with this mismatch and they are summarized in the following lemma, which is an extension of a results first obtained by J.L. Lions [37, Ch. III, Thm. 1.1], see also [23].

Lemma 6.6 (Lemma 10 in [32]). *Let H be a Hilbert space and Φ a subspace of H (not necessarily closed). Moreover, let $\mathsf{m} : H \times \Phi \to \mathbb{C}$ and $\mathsf{k} : H \times H \to \mathbb{C}$ be bi-linear forms satisfying the following properties:*

(i) *For every $\varphi \in \Phi$, the linear form $u \mapsto \mathsf{m}(u, \varphi)$ is continuous in H.*

(ii) *The linear operator $\mathsf{K} : H \to H'$ associated to the bi-linear form $\mathsf{k}(\cdot, \cdot)$ is compact and continuous.*

(iii) *There exists $\alpha > 0$ such that*

$$\mathrm{Re}\{\mathsf{m}(\varphi, \overline{\varphi}) + \mathsf{k}(\varphi, \overline{\varphi})\} \geq \alpha \|\varphi\|_H^2 , \quad \forall\, \varphi \in \Phi. \tag{6.19}$$

(iv) *The bi-linear form $u \mapsto \mathsf{m}(u, \varphi)$ is injective, i.e. $\mathsf{m}(u, \varphi) = 0$ for all $\varphi \in \Phi$, implies $u = 0$.*

Then, for $l \in H'$ there exists $u_0 \in H$ solution of

$$\mathsf{m}(u, \varphi) \;=\; \langle l, \varphi \rangle , \quad \forall\, \varphi \in \Phi, \tag{6.20}$$

satisfying the stability estimate

$$\|u_0\|_H \leq \frac{C_{\mathsf{m}}}{\alpha} \|l\|_{H'} . \tag{6.21}$$

where $C_{\mathsf{m}} > 0$ is independent of l.

We apply this abstract lemma with

- the spaces $H := \boldsymbol{\mathcal{MT}}(\Sigma), \Phi := \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma) \subset \boldsymbol{\mathcal{MT}}(\Sigma)$,
- the bi-linear form m given by $(\vec{\mathfrak{u}}, \vec{\mathfrak{v}}) \to \left\langle \mathbb{M}'_L \,\vec{\mathfrak{u}}, \vec{\mathfrak{v}} \right\rangle$,
- the operator $\mathbb{K}$ from Theorem 6.5 providing K,
- (6.18) in place of the estimate (6.19),
- Theorem 6.4 guaranteeing injectivity of m.

This gives existence of solutions of the local MTF variational problem (6.17). In addition, Theorem 6.4 ensures uniqueness, which permits us to conclude the following main result, which amounts to the well-posedness of the local MTF variational problem.

Theorem 6.7. *The local MTF operator $\mathbb{M}'_L : \boldsymbol{\mathcal{MT}}(\Sigma) \to \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)'$ possesses a continuous inverse $(\mathbb{M}'_L)^{-1} : \boldsymbol{\mathcal{MT}}(\Sigma)' \to \boldsymbol{\mathcal{MT}}(\Sigma)$.*

6.4 Boundary element Galerkin discretization

On an algorithmic level the boundary element Galerkin discretization of the local MTF variational formulation (6.17) largely runs parallel to that of the global MTF explained in Section 3.3. Both trial and test space in (6.17) are replaced with the piecewise polynomial boundary element spaces introduced in Section 3.3, that is, we rely on

$$\boldsymbol{\mathcal{MT}}_h(\Sigma) := \mathcal{T}_h(\partial\Omega_0) \times \cdots \times \mathcal{T}_h(\partial\Omega_N)\ . \tag{6.22}$$

Note the difference with the multi-trace boundary element space $\widehat{\boldsymbol{\mathcal{MT}}}_h(\Sigma)$ from (5.17) used for the global MTF: now $\partial\Omega_0$ also bears two discrete Cauchy traces.

Also be aware that $\boldsymbol{\mathcal{MT}}_h(\Sigma) \not\subset \widetilde{\boldsymbol{\mathcal{MT}}}(\Sigma)$, because $\mathcal{T}_{D,h}(\partial\Omega_i) \subset \widetilde{\mathcal{T}}_D(\partial\Omega_i)$ would entail setting all basis functions associated with vertices or edges on boundaries of interfaces Γ_{ij} to zero. Nevertheless, we can just go ahead and plug functions in $\boldsymbol{\mathcal{MT}}_h(\Sigma)$ into the variational equations (6.17), since they are all square integrable and the terms involving the transmission operators remain well defined. In fact, all we have to integrate are products of piecewise polynomials.

Eventually, we arrive at a linear system of equations, whose system matrix is of size $\sum_{i=0}^{N} m_i$, $m_i := \dim \mathcal{T}_{D,h}(\partial\Omega_i) + \dim \mathcal{T}_{N,h}(\partial\Omega_i)$, and has the form

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{X}_{21} & \cdots & & \cdots & \mathbf{X}_{N1} \\ \mathbf{X}_{12} & \mathbf{A}_2 & \mathbf{X}_{32} & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \mathbf{X}_{N-2,N-1} & \mathbf{A}_{N-1} & \mathbf{X}_{N,N-1} \\ \mathbf{X}_{1N} & \cdots & & \cdots & \mathbf{X}_{N-1,1} & \mathbf{A}_N \end{pmatrix}, \tag{6.23}$$

where the matrices $\mathbf{A}_i \in \mathbb{C}^{m_i,m_i}$, $i = 0, \ldots, N$, arise from standard Galerkin BEM discretization of the boundary integral operators $\mathbb{A}_i$ as defined in (2.17). The matrices $\mathbf{X}_{ij} \in \mathbb{R}^{m_j,m_i}$ are *sparse* and $\mathbf{X}_{ij} = 0$, if Ω_i and Ω_j have no common genuine interface.

Remark 6.1. Contrasting (6.23) with (5.16) reveals a substantial advantage of the local MTF compared with the global version: since the global MTF relies on non-local boundary integral operators to take into account the coupling between sub-domains, its final system matrix will feature fully populated off-diagonal blocks throughout. Even though they will allow efficient compression, handling them requires much more effort than dealing with the sparse $\mathbf{X}_{ij}$ blocks in (6.23). △

The downside of the local MTF is its failure to fit natural trace space as discussed at length in Section 6.2. This also compounds difficulties for the a priori convergence analysis, which is still incomplete. A preliminary result is given in [32, Sect. 4] for the case of the Helmholtz transmission problem in 2D only.

Theorem 6.8 (Convergence of Galerkin discretization of local MTF, [32, Thm. 13]). *Consider the Galerkin discretization of* (6.17) *in the Helmholtz case with lowest order piecewise polynomial (piecewise linear continuous/piecewise constant,* cf. *Concretization 3.4) boundary elements.*

If $\mathbb{M}_L'$ preserves the regularity of Cauchy traces (see [32, Conjecture 1] for a precise statement), then the bi-linear form of the local MTF variational formulation satisfies a discrete inf-sup condition uniform in the meshwidth, provided that the resolution of the boundary element spaces is large enough.

Operator preconditioning of the discretized local MTF is a straightforward application of the recipes outlined in Section 4 and, in particular, Section 4.4. It naturally relies on the self-duality of $\boldsymbol{\mathcal{MT}}(\Sigma)$ with respect to the L^2-type pairing $[\![\cdot,\cdot]\!]_{\boldsymbol{\mathcal{MT}}(\Sigma)}$ from (3.4). As explained in Section 4.4, uniformly stable (with respect to the standard trace norm of $\boldsymbol{\mathcal{MT}}(\Sigma)$) discrete duality pairings are available for $\boldsymbol{\mathcal{MT}}_h(\Sigma)$ according to (6.22). The role of the operator B can be played by

$$\mathsf{B} \sim \sum_{i=0}^{N} \mathbb{L}_i^* \, \mathbb{A}_i \, \mathbb{L}_i : \boldsymbol{\mathcal{MT}}(\Sigma) \to \boldsymbol{\mathcal{MT}}(\Sigma)' \,, \tag{6.24}$$

which amounts to the operator of the STF, see (3.20), but now considered on the fully decoupled multi-trace space $\boldsymbol{\mathcal{MT}}(\Sigma)$, recall Remark 5.3.

Yet, operator preconditioning is also haunted by the non-standard function space framework needed for local MTF; since A in (4.20) is the local MTF operator $\mathbb{M}_L'$ it is actually unbounded on $\boldsymbol{\mathcal{MT}}(\Sigma)$. Thus A has to be replaced with $\mathsf{A}_h : V_h \to V_h'$ in (4.20) and we will inevitably incur a blow-up of $\|\mathsf{A}_h\|$ as the meshwidth $h \to 0$. Inverse inequalities linking the $\mathcal{T}(\partial\Omega_i)$-norm and $\widetilde{\mathcal{T}}(\partial\Omega_i)$-norm of boundary element functions allow to quantify this blow-up. In the case of $H^{\frac{1}{2}}$-conforming boundary elements such estimates are known, see [39] and the so-called edge and face lemmas in the theory of domain decomposition methods, see [55, Sect. 4.2.4] and [52, Sect. 4.6]. They suggest a behavior like

$$\|\mathsf{A}_h\| \approx O(|\log h|^{\alpha}) \,, \quad \alpha \text{ a small integer.} \tag{6.25}$$

Hence, a slight dependence of the performance of the preconditioner on the meshwidth can be expected, but will be very moderate and hardly noticeable in numerical computations. This expectation is bolstered by 2D numerical results reported in [32, Sect. 5].

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