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MULTILEVEL MONTE-CARLO FRONT-TRACKING FOR RANDOM SCALAR CONSERVATION LAWS

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ABSTRACT. We consider random scalar hyperbolic conservation laws (RSCLs) in spatial dimension $d \geq 1$ with bounded random flux functions which are $\mathbb{P}$-a.s. Lipschitz continuous with respect to the state variable, for which there exists a unique random entropy solution (i.e., a strongly measurable mapping from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $C([0,T]; L^1(\mathbb{R}^d))$ with finite second moments). We present a convergence analysis of a Multilevel Monte Carlo Front-Tracking (MLMCFT) algorithm. It is based on "pathwise" application of the Front-Tracking Method from [21] for deterministic SCLs. We compare the MLMCFT algorithms to the Multilevel Monte Carlo Finite-Volume algorithms developed in [25, 26]. Due to the first order convergence of front tracking, we obtain an improved complexity estimate in one space dimension.

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1. Introduction

Many problems in physics and engineering are modeled by hyperbolic systems of conservation or balance laws. The Cauchy problem for such systems takes the form

$$\begin{aligned} \mathbf{U}_t + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(\mathbf{F}_j(\mathbf{U})) = 0 \quad x = (x_1, \dots, x_d) \in \mathbb{R}^d,\ t > 0, \\ \mathbf{U}(x,0) = \mathbf{U}_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \tag{1.1}$$

Here, $\mathbf{U} : \mathbb{R}^d \mapsto \mathbb{R}^m$ is the vector of unknowns and $\mathbf{F}_j : \mathbb{R}^m \mapsto \mathbb{R}^m$ is the flux vector for the j-th direction with m being a positive integer.

This type of partial differential equations are ubiquitous, we mention only the shallow water equations of hydrology, the Euler equations for inviscid, compressible flow and the magnetohydrodynamic (MHD) equations of plasma physics, see, e.g. [5, 12]. In the this paper we focus on the case $m = 1$ in (1.1) which is then called a *scalar conservation law (SCL).*

Solutions of (1.1) develop discontinuities in finite time even when the initial data is smooth. Therefore (1.1) must be interpreted in the weak sense. In order to get uniqueness, (1.1) must be augmented with *entropy conditions*, which at least for scalar conservation laws, makes the initial value problem well posed. The well-posedness the Cauchy problem for scalar conservation laws in several space dimensions ($m = 1$, $d \geq 1$) was first established by Kružkov [22].

For systems ($m > 1$), some well-posedness results for systems in one space dimension exist [2, 3] , but no well-posedness results for systems of conservation laws are available in several space dimensions.

Numerical methods for approximating entropy solutions of systems of conservation laws have undergone extensive development and many efficient methods are available, see [8, 12, 13, 23] and the references there. In particular, finite volume methods are frequently employed for approximating (1.1).

This *classical* paradigm for designing efficient numerical schemes assumes that *data for the SCL* (1.1)*, i.e., initial data* $\mathbf{U}_0$ *and flux are known exactly.*

In many situations of practical interest, however, these data are not known exactly due to inherent uncertainty in modelling and measurements of physical parameters such as, for example, the specific heats in the equation of state for compressible gases, resistivity in MHD etc. Often, the initial data are known only up to certain statistical quantities of interest like the mean, variance, higher moments, and in some cases, the law of the stochastic initial data. In such cases, a mathematical formulation of (1.1) is required which allows for *random data*. The problem of random initial data was considered in [25], and the existence and uniqueness of a random entropy solution was shown, and a convergence analysis for MLMC FV discretizations was given. Efficient MLMC discretization of balance laws with random source terms was investigated in [26].

We mention that the present work as well as [25, 26] consider *correlated random inputs* which typically occur in engineering applications; SCLs with random inputs have been considered before, but generally with *white noise*, that is, spatially and temporally uncorrelated random inputs, see [20, 19, 6, 30, 31].

In [25] a mathematical framework was outlined for deterministic scalar conservation laws with random initial data. This framework was extended to include random flux functions in [24]. Here, we slightly generalize [24] regarding the existence and uniqueness of random entropy solutions for such problems. Furthermore, the efficient numerical approximation of such solutions and, in particular, of their statistics, is the purpose of the present paper.

To this end, we propose and analyze a combination of sampling techniques of the Monte Carlo (MC) type combined with a "pathwise" Front Tracking (FT) solver introduced by Bagrinovskiĭ and Godunov [1] and analyzed, for example, in [21], to approximate random entropy solution of scalar, nonlinear hyperbolic conservation laws.

As the stochastic collocation FVM discretization, and the MLMC FVM algorithms developed in [26] also for the numerical solution of nonlinear, hyperbolic *systems* (1.1), the multilevel version of the Monte-Carlo Front-Tracking method is "non-intrusive" (i.e., it requires only repeated application of existing solvers for input data samples), easy to code and to parallelize, and well-suited for random solutions with low spatial regularity, a situation which is typical in nonlinear hyperbolic conservation laws where discontinuities in realizations of solutions are well known to be generic.

The remainder of this paper is organized as follows: in Section 2, we introduce some preliminary notions from probability theory and functional analysis. The concept of random entropy solutions is introduced and the well-posedness of the scalar hyperbolic conservation law (i.e., (1.1) with $m = 1$) with random initial data is recapitulated in Section 3. The MLMCFT schemes are presented and analyzed in Section 4. Numerical experiments are presented in Section 5.

2. Preliminaries

We recapitulate prerequistes from measure and probability theory which are needed in the subsequent sections. For proofs and further details, we refer for example to [29, Chap. 1].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let E be a Banach space. A mapping $G : \Omega \to E$ is called $\mathbb{P}$-simple function if it is of the form

$$G(\omega) = \sum_{j=1}^{J} g_j \mathbb{1}_{A_j}(\omega), \text{ where } \mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and $g_j \in E$ for $j = 1, \ldots, J$, for some finite J and for $A_j \in \mathcal{F}$. A mapping $f : \Omega \to E$ is strongly $\mathcal{F}$ measurable if there exists a sequence of simple functions f_n converging to f (in the norm of E) $\mathbb{P}$-almost everywhere on Ω.

We call two strongly $\mathbb{P}$-measurable functions $f, g : \Omega \to E$ which agree $\mathbb{P}$-almost everywhere on Ω $\mathbb{P}$-*versions* of each other. We shall need the following lemma.

Lemma 2.1. [29, Corollary 1.13] *Let E_1 and E_2 be Banach spaces, and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. If $f : \Omega \to E_1$ is strongly measurable, and $\phi : E_1 \to E_2$ is continuous, then the composition $\phi \circ f : \Omega \to E_2$ is strongly measurable.*

We define the integral of a simple function $G = \sum g_j \mathbb{1}_{A_j}$ by

$$\int_\Omega G \, d\mathbb{P} = \sum_{j=1}^{N} g_j \mathbb{P}(A_j) \; .$$

If $f : \Omega \to E$ is strongly measurable, we say that f is *Bochner integrable* if there exists a sequence of simple functions $\{f_n\}_{n\geq 0}$ converging to f $\mathbb{P}$-almost everywhere, and

$$\lim_{n\to\infty} \int_\Omega \|f - f_n\|_E \, d\mathbb{P} = 0,$$

([29, Def. 1.15]). We then define the Bochner integral of f by

$$\int_\Omega f \, d\mathbb{P} := \lim_{n\to\infty} \int_\Omega f_n \, d\mathbb{P}. \tag{2.1}$$

A strongly measurable function $f : \Omega \to E$ is Bochner integrable if and only if

$$\int_\Omega \|f\|_E \, d\mathbb{P} < \infty$$

(see for example, [29, Prop. 1.16]) in which case

$$\Big\| \int_\Omega f \, d\mathbb{P} \Big\|_E \le \int_\Omega \|f\|_E \, d\mathbb{P} \, . \tag{2.2}$$

For each $1 \le p < \infty$ we can define the Banach spaces $L^p(\Omega; E)$ to consist of those strongly measurable functions f for which the integrals

$$\int_\Omega \|f\|_E^p \, d\mathbb{P} < \infty \, .$$

These spaces have the natural norm

$$\|f\|_{L^p(\Omega;E)} = \Big(\int_\Omega \|f\|_E^p \, d\mathbb{P} \Big)^{1/p} \, .$$

If $p = \infty$, we define $L^\infty(\Omega; E)$ to be the space of strongly measurable functions $f : \Omega \to E$ for which there exists a number $r \ge 0$ such that $\mathbb{P}(\|f\|_E > r) = 0$. Together with the norm

$$\|f\|_{L^\infty(\Omega;E)} := \inf\{r \ge 0 : \mathbb{P}(\|f\|_E > r) = 0\},$$

this space is a Banach space as well.

If $f : \Omega \to E$ is strongly measurable and $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we call f an *E-valued random variable.*

3. Hyperbolic Conservation Laws with random flux

We review classical results on SCLs with deterministic data, and develop a theory of random entropy solutions for SCLs with a class of random flux flunctions, proving in particular the existence and uniqueness of a random entropy solution with finite second moments.

3.1. **Deterministic scalar hyperbolic conservation laws.** We consider the Cauchy problem for scalar conservation laws (SCL) by setting $m = 1$ in (1.1) and obtaining the SCL in strong form

$$\frac{\partial u}{\partial t} + \sum_{j=1}^d \frac{\partial}{\partial x_j} \left(f_j(u) \right) = 0, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \; t > 0 \, . \tag{3.1}$$

Here the unknown is $u : \mathbb{R}^d \mapsto \mathbb{R}$. Introducing the flux function $f(u)$

$$f(u) = (f_1(u), \ldots, f_d(u)) \in C^1(\mathbb{R}; \mathbb{R}^d) \, , \quad \operatorname{div} f(u) = \sum_{j=1}^d \frac{\partial}{\partial x_j} f_j(u),$$

we may rewrite (3.1) succinctly as

$$\frac{\partial u}{\partial t} + \operatorname{div}(f(u)) = 0 \quad \text{for } (x,t) \in \mathbb{R}^d \times \mathbb{R}_+. \tag{3.2}$$

We supply the SCL (3.2) with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d. \tag{3.3}$$

3.2. **Entropy Solutions.** Solutions to (3.1) are in general not smooth since they can develop discontinuities in finite time. Therefore we look for weak solutions to the equations. In particular, we are interested in distributional solutions in the class of *entropy solutions* which satisfy in addition the entropy condition

$$\eta(u)_t + \operatorname{div} Q(u) \le 0, \quad \text{in } \mathcal{D}(\mathbb{R}^d \times \mathbb{R}^+),$$

for all entropy pairs (η, Q), where η, the *entropy*, is a convex C^2-function and $Q(u) = (Q_1(u), \dots, Q_d(u))$, the *entropy flux*, satisfies $Q_j' = \eta' f_j'$. In this class, uniqueness can be proved, [22]. We will in the following restrict to initial data in $L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$, but results can be proved for more general initial conditions, [28]. By a *function of bounded variation*, or BV-function, we mean a function $f \in L^1(\mathbb{R}^d)$ with

$$TV(f) := \sup\left\{ \int_{\mathbb{R}^d} f \operatorname{div} \varphi \, dx \;\middle|\; \varphi \in C_0^1(\mathbb{R}^d; \mathbb{R}^d),\, |\varphi| \le 1 \right\} < \infty,$$

where $|\varphi|$ denotes absolute value of point-values for φ, see [7, Section 5.1]. We call $TV(f)$ the *total variation* of f. We define the Banach space of functions with bounded variation as the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|f\|_{BV(\mathbb{R}^d)} := \|f\|_{L^1(\mathbb{R}^d)} + TV(f).$$

More details and properties of BV-functions can be found in, for example [7, Chapter 5], [21, Appendix A] or [11, Chapter 1]. Next we introduce the (nonlinear) data-to-solution operator

$$S_t : (u_0, f) \longmapsto u(\cdot, t) =: S_t(u_0, f) \quad t > 0\,.$$

In particular, we shall need the following continuity (with respect to initial data and flux function) result for deterministic scalar conservation laws:

Theorem 3.1. [21, Thm. 2.14, Thm. 4.3] *Assume $u_0, v_0 \in (BV \cap L^\infty)(\mathbb{R}^d)$, and f, $g \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}^d)$. Then there exist unique entropy solutions u and v to (3.1) with initial data u_0 and v_0 respectively and flux functions f and g, which satisfy the a-priori continuity estimates: For all $t \ge 0$ we have*

$$\begin{aligned} &\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} \\ &\qquad \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} + t \min\{\mathrm{TV}(u_0), \mathrm{TV}(v_0)\} \|f - g\|_{W^{1,\infty}(\mathbb{R};\mathbb{R}^d)}, \end{aligned} \tag{3.4}$$

and

$$\|u(\cdot,t) - u(\cdot,s)\|_{L^1(\mathbb{R}^d)} \le (t-s)\, \mathrm{TV}(u_0)\, \|f\|_{W^{1,\infty}(\mathbb{R},\mathbb{R}^d)}\,, \tag{3.5}$$

for all $0 \le s \le t$. In particular, this implies that the soluiton operator S_t is a uniformly continuous mapping from $L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \times W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ into $C([0,T]; L^1(\mathbb{R}^d))$. Moreover, it follows that

$$\|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}\,. \tag{3.6}$$

and

$$\mathrm{TV}(u(\cdot,t)) \le \mathrm{TV}(u_0), \tag{3.7}$$

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \tag{3.8}$$

$$\|u(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)}. \tag{3.9}$$

For a proof, we refer to for example [21], Theorem 2.14 and Theorem 4.3 or other standard references such as [12, 13, 8, 23, 28].

3.3. **Random flux and initial data.** Existence and uniqueness in the case of random initial data u_0 and continuously differentiable random flux f was proved in [25, 24]. Here, we are interested in initial data u_0 and flux functions f_j in (3.1) which are random elements with values in $BV(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $W^{1,\infty}(\mathbb{R};\mathbb{R})$ respectively. To define these, we denote by $(\Omega,\mathcal{F},\mathbb{P})$ a probability space. We consider *spatially homogeneous random flux functions* f, i.e., strongly measurable maps $f:\Omega\to \mathrm{Lip}(\mathbb{R};\mathbb{R}^d)$, and random initial data u_0 being strongly measurable maps from Ω to the intersection of the Banach spaces $BV(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$.

Definition 3.2. *Random data for the SCL* (3.1) *is a random variable taking values in*

$$E_1 = \big(BV(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)\big)\times W^{1,\infty}(\mathbb{R};\mathbb{R}^d).$$

The set E_1 is a Banach space which we equip with the norm

$$\|(u,f)\|_{E_1} = \|u\|_{L^1(\mathbb{R}^d)} + \mathrm{TV}(u) + \|u\|_{L^\infty(\mathbb{R}^d)} + \|f\|_{W^{1,\infty}(\mathbb{R};\mathbb{R}^d)} \ . \tag{3.10}$$

In particular, random data (u_0,f) for the SCL (3.1) - (3.3) *is a strongly measurable map*

$$(u_0,f):(\Omega,\mathcal{F})\longmapsto (E_1,\mathcal{B}(E_1))\ . \tag{3.11}$$

For the ensuing convergence analysis, we shall also require that

$$\|u_0\|_{L^\infty(\Omega;(L^\infty\cap BV)(\mathbb{R}^d))} \le \overline{M} < \infty, \text{ and } \|f\|_{L^\infty(\Omega;W^{1,\infty}([-\overline{M},\overline{M}];\mathbb{R}^d))} \le \overline{M} < \infty\,. \tag{3.12}$$

We shall refer to a random flux f which satisfies (3.12) as *bounded random flux*. By (2.2), for random data with (3.12) the map

$$\Omega\ni\omega\mapsto\Big(\|u_0(\omega;\cdot)\|_{L^1(\mathbb{R}^d)}, \mathrm{TV}(u_0(\omega;\cdot)), \|u_0(\omega;\cdot)\|_{L^\infty(\mathbb{R}^d)}, \|f\|_{W^{1,\infty}(\mathbb{R};\mathbb{R}^d)}\Big) \tag{3.13}$$

is in $L^k(\Omega;\mathbb{R}^4)$ for every $1\le k<\infty$.

3.4. **Random Entropy Solution.** Based on Theorem 3.1, we formulate (3.1) - (3.3) for random data (u_0,f) in the sense of Definition 3.2. We are interested in solutions of the *random scalar conservation law* (RSCL)

$$\begin{cases}\partial_t u(\omega;x,t) + \mathrm{div}_x(f(\omega;u(\omega;x,t))) = 0, & t>0,\\ u(\omega;x,0) = u_0(\omega;x), & \end{cases} \qquad x\in\mathbb{R}^d. \tag{3.14}$$

Definition 3.3. *A random variable $u:\Omega\ni\omega\to u(\omega;x,t)$, i.e., a strongly measurable mapping from $(\Omega,\mathcal{F})$ to $C([0,T];L^1(\mathbb{R}^d))$, is a* random entropy solution *of the SCL* (3.14) *with random data as in* (3.11) - (3.13) *for some $k\ge 2$, if u satisfies the following:*

(i.) Weak solution: For $\mathbb{P}$-a.e $\omega \in \Omega$, $u(\omega;\cdot,\cdot)$ satisfies

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(u(\omega;x,t)\varphi_t(x,t) + \sum_{j=1}^d f_j(\omega;u(\omega;x,t)) \frac{\partial}{\partial x_j}\varphi(x,t) \Big)\, dxdt + \int_{\mathbb{R}^d} u_0(x,\omega)\varphi(x,0)\, dx = 0, \tag{3.15}$$

for all test functions $\varphi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R})$.

(ii.) Entropy condition: For any pair consisting of a (deterministic) entropy η and a (stochastic) entropy flux $Q(\omega;\cdot)$ i.e., η, Q_j with $j = 1,2,\dots,d$ are functions such that η is convex and such that $Q_j'(\omega;\cdot) = \eta' f_j'(\omega;\cdot)$ for all j, and for $\mathbb{P}$-a.e $\omega \in \Omega$, u satisfies the following integral identity,

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(\eta(u(\omega;x,t))\varphi_t(x,t) + \sum_{j=1}^d Q_j(\omega;u(\omega;x,t)) \frac{\partial}{\partial x_j}\varphi(x,t) \Big)\, dxdt + \int_{\mathbb{R}^d} \eta(u_0(\omega;x))\varphi(x,0)\, dx \geq 0, \tag{3.16}$$

for all non-negative test functions $\varphi \in C_0^\infty(\mathbb{R}^d \times \mathbb{R})$.

Theorem 3.4. *Consider the SCL (3.1) - (3.3) with random data (u_0, f) in the sense of Definition 3.2 such that (3.12) holds. Then there exists a random entropy solution u in $C([0,T]; L^1(\mathbb{R}^d))$, which for each $0 \leq t \leq T$ is described by the map*

$$\Omega \ni \omega \mapsto u(\omega;\cdot,t) = S_t(u_0(\omega,\cdot), f(\omega;\cdot)) .$$

For $\mathbb{P}$-almost every $\omega \in \Omega$ we have the bound

$$\|u(\omega;\cdot)\|_{(L^\infty \cap BV)(\mathbb{R}^d)} \leq \|u_0(\omega;\cdot)\|_{(L^\infty \cap BV)(\mathbb{R}^d)} , \tag{3.17}$$

and for all $k \geq 1$, $(u_0, f) \in L^k(\Omega; E_1)$ implies that

$$\|u\|_{L^k(\Omega; C([0,T]; L^1(\mathbb{R}^d)))} \leq \|(u_0, f)\|_{L^k(\Omega;E_1)} . \tag{3.18}$$

Proof. Let $E_2 = C([0,T], L^1(\mathbb{R}^d))$. By (3.12), for almost all ω, the data $u_0(\omega;\cdot)$ and $f(\omega;\cdot)$ are such that there exists a unique entropy solution $u(\omega;\cdot) \in E_2$ to (3.14). Furthermore, from (3.4) it follows that for such ω,

$$\|u(\omega;\cdot,t)\|_{(L^\infty \cap BV)(\mathbb{R}^d)} \leq \|u_0(\omega;\cdot)\|_{(L^\infty \cap BV)(\mathbb{R}^d)} .$$

We have to show that $\omega \mapsto u(\omega;\cdot)$ is a random variable, that is, it is strongly measurable. This will follow from Lemma 2.1 if the mapping $E_1 \ni (u_0, f) \mapsto u \in E_2$ is continuous. This on the other hand, follows from (3.4) and (3.5) in Theorem 3.1.

The inequality (3.17) follows from the corresponding inequality in the deterministic case.

To prove (3.18), we compute

$$\begin{aligned} \|u\|^k_{L^k(\Omega; C([0,T]; L^1(\mathbb{R}^d)))} &= \int_\Omega \sup_{t \leq T} \|u(\omega;\cdot,t)\|^k_{L^1(\mathbb{R}^d)} \, d\mathbb{P} \\ &\leq \int_\Omega \|u_0(\omega;\cdot)\|^k_{L^1(\mathbb{R}^d)} \, d\mathbb{P} \\ &\leq \|(u_0, f)\|^k_{L^k(\Omega;E_1)} . \end{aligned}$$

□

Remark 3.5. *The random entropy solution $u : \Omega \to C([0,T]; L^1(\mathbb{R}^d))$ is unique in the sense that if a random variable $(\widetilde{u}_0, \widetilde{f})$ is a $\mathbb{P}$-version of (u_0, f), then the solution $\widetilde{u}(\cdot;\cdot,t) := S_t(\widetilde{u}_0, \widetilde{f})$ corresponding to it is a $\mathbb{P}$-version of $u(\cdot;\cdot,t) := S_t(u_0, f)$, that is, they agree everywhere on Ω except on a set with $\mathbb{P}$-measure zero. To see this, we note that by the continuity of the operator S_t, (3.4), we have for any $t \in (0,T]$,*

$$\begin{aligned}
&\|u(\cdot;\cdot,t) - \widetilde{u}(\cdot;\cdot,t)\|_{L^\infty(\Omega;L^1(\mathbb{R}^d))} \\
&\quad \le \|\widetilde{u}_0 - u_0\|_{L^\infty(\Omega;L^1(\mathbb{R}^d))} \\
&\qquad + t\min\{\|TV(u_0)\|_{L^\infty(\Omega)}, \|TV(\widetilde{u}_0)\|_{L^\infty(\Omega)}\}\|f - \widetilde{f}\|_{L^\infty(\Omega;W^{1,\infty}([-\overline{M},\overline{M}];\mathbb{R}^d))} \\
&\quad = 0,
\end{aligned}$$

and therefore it follows also that u is unique in $L^\infty(\Omega; L^1(\mathbb{R}^d); d\mathbb{P})$.

4. Multi Level Monte Carlo Front Tracking

In this section, we present a Multi Level Monte Carlo (MLMC) version of the front tracking approach to the numerical solution of hyperbolic conservation laws with random flux (3.15), (3.16) as developed in [21].

4.1. The Monte-Carlo Method.

We interpret the Monte-Carlo method as "discretization" of the SCL random data $f(\omega; u)$, $u_0(\omega; x)$ as in (3.11) – (3.13) with respect to ω. We assume in particular the existence of k-th moments of u_0 for some $k \in \mathbb{N}$. We shall be interested in the statistical estimation of the first and higher moments of u, i.e, $\mathcal{M}^k(u) \in (L^1(\mathbb{R}^d))^{(k)}$. For $k = 1$, $\mathcal{M}^1(u) = \mathbb{E}[u]$. The *MC approximation of* $\mathbb{E}[u]$ is defined as follows: given M independent, identically distributed samples $(\widehat{u}_0^i, \widehat{f}^i)$, $i = 1, \dots, M$, of random data, the MC estimate of $\mathbb{E}[u(\cdot;\cdot,t)]$ at time t is

$$E_M[u(\cdot,t)] := \frac{1}{M}\sum_{i=1}^{M} \widehat{u}^i(\cdot,t) \tag{4.1}$$

where $\widehat{u}^i(\cdot,t)$ denotes the M unique entropy solutions of the M Cauchy Problems (3.1) – (3.3) with initial data $\widehat{u}_0^i$ and flux samples $\widehat{f}^i(\cdot)$. We observe that by

$$\widehat{u}^i(\cdot,t) = S_t(\widehat{u}_0^i, \widehat{f}^i)$$

we have for every M and for every $0 < t < \infty$, by (3.9),

$$\begin{aligned}
\|E_M[u(\omega;\cdot,t)]\|_{L^1(\mathbb{R}^d)} &= \Big\|\frac{1}{M}\sum_{i=1}^{M} S_t((\widehat{u}_0^i, \widehat{f}^i)(\omega))\Big\|_{L^1(\mathbb{R}^d)} \\
&\le \frac{1}{M}\sum_{i=1}^{M} \Big\|S_t((\widehat{u}_0^i, \widehat{f}^i)(\omega))\Big\|_{L^1(\mathbb{R}^d)} \\
&\le \frac{1}{M}\sum_{i=1}^{M} \big\|\widehat{u}_0^i(\omega;\cdot)\big\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Using the i.i.d. property of the samples $\{\widehat{u}_0^i, \widehat{f}^i\}_{i=1}^M$, Theorem 3.4 and and the linearity of the expectation $\mathbb{E}[\cdot]$, we obtain the bound

$$\mathbb{E}\left[\|E_M[u(\cdot;\cdot,t)]\|_{L^1(\mathbb{R}^d)}\right] \leq \mathbb{E}\left[\|u_0\|_{L^1(\mathbb{R}^d)}\right] = \|u_0\|_{L^1(\Omega;L^1(\mathbb{R}^d))} < \infty.$$

As $M \to \infty$, the MC estimates (4.1) converge and the convergence result from [25] holds as well.

Theorem 4.1. *Assume that in the SCL* (3.1) – (3.3) *the random data* (u_0, f) *satisfies* (3.12).

Then for every $t > 0$ *the MC estimates* $E_M[u(\cdot,t)]$ *in* (4.1) *converge in* $L^2(\Omega; L^1(\mathbb{R}^d))$ *as* $M \to \infty$, *to* $\mathcal{M}^1(u(\cdot,t)) = \mathbb{E}[u(\cdot,t)]$ *and, for any* $M \in \mathbb{N}$, $0 < t < \infty$, *we have the error bound*

$$\|\mathbb{E}[u(\cdot,t)] - E_M[u(\cdot,t)]\|_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq 2M^{-1/2}\|u_0\|_{L^2(\Omega;L^1(\mathbb{R}^d))}\ .$$

4.2. **Front Tracking.** As an exact solution to (3.1) – (3.3) is in general not available, an approximate solution has to be computed numerically. Here, we investigate using a front tracking method described in [4, 21, 17, 16]. Since the method and the associated convergence analysis differ for the dimensions $d = 1$ and $d > 1$, we treat the two cases separately.

4.2.1. *Front tracking in the one dimensional case.* We start by briefly describing the front tracking algorithm for the deterministic conservation law (3.1) – (3.3) with initial condition u_0 given in $BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $\overline{M} := \|u_0\|_{L^\infty(\mathbb{R})}$ and let $\delta > 0$ be a small number. Moreover, set $u_i = \delta i$, for $-\overline{M} \leq i\delta \leq \overline{M}$, and discretize the spatial domain by a grid $\{x_j = j\delta,\ j \in \mathbb{Z}\}$. Then, u_0 is approximated by a piecewise constant function u_0^δ taking in each cell $[j\delta, (j+1)\delta)$ one of the values in $V_\delta := \{u_i \,|\, i \in \mathbb{Z},\ |u_i| \leq \overline{M}\}$. The flux function f is approximated by the piecewise linear interpolation f^δ,

$$f^\delta(u) = f(u_j) + \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j}(u - u_j), \qquad u \in [u_j, u_{j+1}), \quad j \in \mathbb{Z}, |j| \leq \overline{M}\delta^{-1}\ . \tag{4.2}$$

Then we solve the initial value problem

$$u_t^\delta + f^\delta(u^\delta)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{4.3a}$$

$$u^\delta(x,0) = u_0^\delta(x), \quad x \in \mathbb{R}, \tag{4.3b}$$

exactly. This means that in each step, we solve the Riemann problems between the states of the piecewise constant function u^δ, then track the discontinuities, called *fronts*, until they interact, solve the emerging Riemann problem and so on. Note that the solution of each Riemann problem is again a piecewise constant function taking values in V_δ because f^δ is piecewise linear with breakpoints $u_i \in V_\delta$. Thus, the (unique) entropy solution $u^\delta(\cdot,t)$ is a piecewise constant function for all $t > 0$. It was shown in [21, Lemma 2.6] that the number of interactions $T(\delta,t)$ between fronts for $t \in (0,\infty)$ is bounded by

$$T(\delta,t) \leq \frac{1}{\delta}(|V_\delta| + 1)\ \mathrm{TV}(u^\delta) \leq \frac{1}{\delta}(2\lceil\overline{M}/\delta\rceil + 1)\ \mathrm{TV}(u^\delta) \tag{4.4}$$

where we denoted $|V_\delta|$ the cardinality of the set V_δ which is bounded for all $t > 0$ by $2\lceil\overline{M}/\delta\rceil$ due to (4.5). Hence the process terminates. Moreover, the solution u^δ of (4.3) satisfies the Kružkov entropy condition and we have the theorem:

Theorem 4.2 ([21]). *For initial data $u_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and flux function $f(u) \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ we have*

(i) The solutions u^δ to the differential equation (4.3) are uniformly bounded in δ for all $t \in (0,\infty)$:

$$\|u^\delta(\cdot,t)\|_{L^\infty(\mathbb{R})} \le \|u_0\|_{L^\infty(\mathbb{R})}, \quad t \in (0,\infty), \tag{4.5}$$

(ii) The total variation of u^δ is bounded by the total variation of the initial data for all times $t \in (0,\infty)$,

$$\mathrm{TV}(u^\delta(\cdot,t)) \le \mathrm{TV}(u_0), \quad t \in (0,\infty),$$

(iii) As the discretization parameter δ goes to zero, the sequence $\{u^\delta\}_{\delta>0}$ converges in $C([0,T];L^1(\mathbb{R}))$ to the unique entropy solution u of (3.1) – (3.3). Specifically,

$$\|u(\cdot,t) - u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} \le \|u_0 - u_0^\delta\|_{L^1(\mathbb{R})} + t\|f - f^\delta\|_{\mathrm{Lip}(\mathbb{R})}\,\mathrm{TV}(u_0) \tag{4.6}$$

Corollary 4.3. *Under the assumptions of Theorem 4.2, we have the following estimate with respect to the discretization parameter δ:*

$$\|u(\cdot,t) - u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} \le \delta\ \mathrm{TV}(u_0)\left(c + |f|_{W^{2,\infty}(\mathbb{R})}\right). \tag{4.7}$$

Proof. Note first that

$$\|f - f^\delta\|_{\mathrm{Lip}(\mathbb{R})} \le \delta\|f''\|_{L^\infty(\mathbb{R})} = \delta|f|_{W^{2,\infty}(\mathbb{R})}, \tag{4.8}$$
$$\|u_0 - u_0^\delta\|_{L^1(\mathbb{R})} \le \delta\, c\ \mathrm{TV}(u_0),$$

where $c > 0$ is a constant independent of δ. Then (4.7) follows using (4.8) and (4.6). □

In order to obtain convergence rate bounds in the Multilevel Monte Carlo front tracking (MCMLFT) algorithm, which we are going to introduce in the next section, it will be useful to have convergence rates of the front tracking algorithm with respect to the amount of computational *work* of the algorithm when the discretization is refined.

Definition 4.4. *By the* (computational) work *or* cost *of an algorithm, we mean the number of floating point operations performed during the execution of the algorithm. We assume that this is proportional to the run time of the algorithm.*

Lemma 4.5 (Work estimate)**.** *Under the assumptions of Theorem 4.2, the front tracking approximation u^δ satisfies the following estimate with respect to the total cost W_δ^{FT} of the front tracking algorithm,*

$$\begin{aligned}\|u(\cdot,t) - u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} &\le C\ \mathrm{TV}(u_0) \\ &\times \left(1 + \|f\|_{W^{2,\infty}(\mathbb{R})}\right)\left((\|u_0\|_{L^\infty} + 1)\left(\mathrm{TV}(u_0) + \|u_0\|_{L^\infty}\right)\right)^{1/2}\left(W_\delta^{FT}\right)^{-1/2}.\end{aligned} \tag{4.9}$$

Proof. Theorem 4.2 implies in particular that we have for the total number of interactions (4.4), (due to (3.12), in the case of random initial data holds $\mathrm{TV}(u_0) \le \overline{M}$ $\mathbb{P}$-as.)

$$T(\delta,t) \le \frac{1}{\delta}(2\lceil \overline{M}/\delta\rceil + 1)\ \mathrm{TV}(u_0^\delta) \le \frac{C}{\delta^2}(\|u_0\|_{L^\infty(\mathbb{R})} + 1)\ \mathrm{TV}(u_0), \tag{4.10}$$

and that the number of different Riemann problems that might be solved during the execution of the algorithm is bounded by $4\lceil \overline{M}/\delta\rceil^2$. We use Algorithm 1, which is a modification of Graham's scan [14] used to compute the convex hull of a set of points in the plane, to calculate all the solutions of the Riemann problems with left state $u_i = i\delta$, right state $u_j = j\delta$, $L \leq i < j \leq R$, where L, R are chosen such that $u_L = \min V_\delta$, $u_R = \max V_\delta$ (a similar algorithm can be used to compute the solutions to the Riemann problems with left state $u_i = i\delta$, right state $u_j = j\delta$, $R \leq j < i \leq L$). It can easily be verified (see [14]) that the cost of the execution of Algorithm 1 is bounded by $C\,\overline{M}^2\,\delta^{-2}$, where C is a constant independent of $\overline{M}$ and δ, for the input $\delta > 0$, $L = -\lceil \overline{M}/\delta \rceil$, $R = \lceil \overline{M}/\delta \rceil$.

So, if the solutions to all possible Riemann problems are computed and stored in advance, the work W_δ^{FT} to compute the front tracking apprximation $u^\delta(\cdot,t)$ is bounded by $C(\|u_0\|_{L^\infty}+1)(\mathrm{TV}(u_0)+\|u_0\|_{L^\infty})\,\delta^{-2}$, for a constant $C > 0$, uniformly in $t \in (0,\infty)$. We thus obtain (4.9)

□

Algorithm 1 Compute Riemann problems with $u_L \leq u_i < u_j \leq u_R$

Input: $\delta > 0$, $L < R \in \mathbb{Z}$, (u_L smallest value of u, u_R largest value of u), $\underline{f} = [f_L, \ldots, f_R]$, ($f_i = f(u_i)$, $L \leq i \leq R$)

Output: $U_{i,j} = [u_i, \ldots, u_j]$ (states present in solution of RP with left state u_i and right state u_j), $s_{i,j} = [s^1_{i,j}, \ldots, s^{k_{ij}}_{i,j}]$ (vector of shock speeds (in increasing order) present in RP with left state u_i and right state u_j, $k_{ij} \in \mathbb{N}$), $L \leq i < j \leq R$

```
for i = L to R do
    û ← [i, i+1]
    ŝ ← (f_{i+1} − f_i)/δ
    s_{i,i+1} ← ŝ
    U_{i,i+1} ← δ · û
    k ← i + 2
    while k ≤ R do
        sl ← (f_k − f_{û(end)})/(δ(k − û(end)))
        if ŝ = [ ] or sl > ŝ(end) then
            ŝ ← [ŝ, sl]
            û ← [û, k]
            s_{i,k} ← ŝ
            U_{i,k} ← δ · û
            k ← k + 1
        else
            ŝ ← ŝ(1 : end − 1)
            û ← û(1 : end − 1)
        end if
    end while
end for
```

Remark 4.6. *Note that the work W_δ^{FT} to compute the front tracking approximation is of the same order as the work we would need to compute an approximation of the solution by a finite volume scheme on a grid with cells of diameter $\mathcal{O}(\delta)$. But due to the better convergence rate with respect to the discretization parameter δ,* *which is of order 1 whereas it is proved to be of order 1/2 for the finite volume approximation, we obtain the improved convergence rate* (4.9) *with respect to work.*

Remark 4.7 (Work estimates for convex flux functions)**.** *If the flux function f is convex, the work estimate can be improved. This is because in this case, the number of interactions $T(\delta,t)$ can be bounded by the sum of the sizes of the jumps in the initial data. That is, given u_0 there holds, for every $t>0$ and $\delta>0$,*

$$T(\delta,t) \le \frac{1}{\delta}\,\mathrm{TV}(u_0)$$

(see [21, Lemma 2.6]*), since for a convex flux function, the number of fronts is strictly decreasing at each interaction. Moreover, the solution of each Riemann problem is either a shock wave or a rarefaction wave depending on whether $u_L > u_R$ or $u_L < u_R$, and we do not need to compute the convex envelope of the flux function.*

So, the solution of one Riemann problem can be computed with a cost proportional to δ. Thus the total work W_δ^{FT} to compute the front tracking approximation reduces to

$$W_\delta^{FT} \le C\ \mathrm{TV}(u_0)\,\delta^{-1}$$

and we obtain the improved convergence rate of the FT method with respect to work,

$$\|u(\cdot,t)-u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} \le C\,\mathrm{TV}(u_0)^2(1+\|f\|_{W^{2,\infty}(\mathbb{R})})\,(W_\delta^{FT})^{-1}. \tag{4.11}$$

Clearly, the same rate holds also for concave fluxes.

4.2.2. *Front tracking for $d\ge 2$ and dimensional splitting.* Front tracking in several space dimensions is based on the method of fractional steps (or dimensional splitting) introduced by Bagrinovskiĭ and Godunov [1] and later on extended by various authors, see e.g. [18] and the references therein. Here, we will use the dimensional splitting method in combination with the front tracking algorithm for one space dimension as described in the previous subsection 4.2.1. To describe the method, we introduce some notation. We discretize the spatial domain by a Cartesian grid $\{j\Delta x_i, j\in\mathbb{Z}\}$, $i=1,\dots,d$ in each direction and denote by $I_{j_1,\dots,j_d}$ the grid cell

$$I_{j_1,\dots,j_d} = \left\{(x_1,\dots,x_d)\ \middle|\ j_i\Delta x_i \le x_1 < (j_i+1)\Delta x_i \ \text{ for } i=1,\dots,d\right\}\ .$$

Moreover, we denote the projection operator $\pi_\delta := P_\delta\circ \overline{P}_{\Delta x}$ for a function $u\in L^1(\mathbb{R}^d)$ to be the composition of the projection $\overline{P}_{\Delta x}$ of the function on the cell averages,

$$\overline{P}_{\Delta x}u(x) = \frac{1}{\Delta x_1\cdots\Delta x_d}\int_{I_{j_1,\dots,j_d}} u\,dx, \quad x=(x_1,\dots,x_d)\in I_{j_1,\dots,j_d}, \tag{4.12}$$

and a projection P_δ of the cell averages onto the values in V_δ. Furthermore, we let f_i^δ, $i=1,\dots,d$, denote the continuous piecewise linear approximations to f_i, $i=1,\dots,d$, as in (4.2). We set $\eta=(\delta,\Delta x_1,\dots,\Delta x_d,\Delta t)$ and let u^0 denote the projection of u_0 on the grid, that is $u^0=\pi_\delta u_0$. Let $S^{f_i^\delta,x_i}(t)$ denote the solution operator of the scalar conservation law in one dimension, viz.,

$$\begin{aligned}(v_i^\delta)_t + f_i^\delta(v_i^\delta)_{x_i} &= 0, \quad (x_i,t)\in\mathbb{R}\times(0,T),\\ v_i^\delta(x_i,0) &= v_{i0}^\delta(x_i), \quad x_i\in\mathbb{R},\end{aligned}$$

that is, we write $v(x_i,t) = S^{f_i^\delta,x_i}(t)v_{i0}^\delta$. Since v_{i0}^δ is piecewise constant, and f_i^δ piecewise linear, the solution can be calculated using front tracking.

Then we obtain an approximation of the solution to (3.1) – (3.3) by successively applying the front tracking solution operator $S^{f_i^\delta,x_i}(t)$ followed by the projection operator π_δ (in order to prevent the number of discontinuities from growing excessively). We denote the approximate solutions at the timesteps $t_r = r\Delta t$, $t \in \mathbb{Q}$ by

$$u^{n+i/d} = \pi_\delta \circ S^{f_i^\delta,x_i}(\Delta t)\, u^{n+(i-1)/d}, \quad i = 1,\dots,d,\ n \in \mathbb{N},$$

and

(4.13)
$$u^\eta(x,t) = \begin{cases} S^{f_i^\delta,x_i}(d(t - t_{n+(i-1)/d}))\, u^{n+(i-1)/d}, & t \in [t_{n+(i-1)/d}, t_{n+i/d}), \\ u^{n+i/d}, & t = t_{n+i/d}, \end{cases}$$

$i = 1,\dots,d$ and $n \in \mathbb{N}$. The approximation u^η satisfies (see [21][Chapter 4]):

Theorem 4.8. *Let $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $f_i(u) \in \mathrm{Lip}(\mathbb{R})$ and piecewise C^2. Then the function u^η defined in (4.13) satisfies*

(i) Uniform bound in $\eta = (\delta, \Delta x_1, \dots, \Delta x_d, \Delta t)$ for all $t \in (0,\infty)$:

$$\|u^\eta(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad t \in (0,\infty),$$

(ii) The total variation of u^η is bounded by the total variation of the initial data for all times $t \in (0,\infty)$,

$$\mathrm{TV}(u^\eta(\cdot,t)) \le \mathrm{TV}(u_0), \quad t \in (0,\infty),$$

(iii) For any sequence $\{\eta_j\}_{j\in\mathbb{N}}$, where $\eta_j \to 0$ when $j \to \infty$, satisfying

$$\max_{i=1,\dots,d} \Delta x_i/\Delta t \le K < \infty,$$

the corresponding sequence $\{u^{\eta_j}\}_{j\in\mathbb{N}}$ converges in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to the unique entropy solution u of (3.1)–(3.3). Specifically, we have, denoting $\|f\|_{\mathrm{Lip}} = \max_{i=1,\dots,d} \|f_i\|_{\mathrm{Lip}(\mathbb{R})}$ and $\Delta x = \max_{i=1,\dots,d} \Delta x_i$,

$$\begin{aligned} &\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \\ &\quad\le \|u_0 - u^0\|_{L^1(\mathbb{R}^d)} + t\|f - f^\delta\|_{\mathrm{Lip}(\mathbb{R})}\,\mathrm{TV}(u_0) \\ &\quad + 2\ \mathrm{TV}(u_0)\sqrt{2t}\,(\sqrt{d}+1)\sqrt{d\Delta x^2/\Delta t + \Delta x\|f\|_{\mathrm{Lip}} + \Delta t\|f\|^2_{Lip}}\,. \end{aligned} \tag{4.14}$$

Corollary 4.9. *Under the assumptions of Theorem 4.8 and choosing the parameters Δx, Δt and δ as*

$$\Delta x = k_1 \Delta t = k_2 \delta^2, \tag{4.15}$$

where k_1 and k_2 are positive constants, the dimensional splitting front tracking algorithm converges at rate 1 in the parameter δ, specifically,

$$\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\,\delta\,(1+t)\Big(1 + \|f\|_{W^{2,\infty}(\mathbb{R};\mathbb{R}^d)}\Big)\,\mathrm{TV}(u_0), \tag{4.16}$$

where $C > 0$ is a constant depending at most linearly on d.

Proof. Using similarly as in Corollary 4.3 that the approximation u^0 of the initial data u_0 satisfies

$$\|u_0 - u^0\|_{L^1(\mathbb{R}^d)} \le c\,d\,\delta\,\mathrm{TV}(u_0),$$

and (4.8), (4.14) yields a convergence rate with respect to the parameters Δx, Δt and δ,

$$\|u(\cdot,t)-u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le \Big(c\,d\,\delta + t\,\delta\,\|f\|_{W^{2,\infty}(\mathbb{R};\mathbb{R}^d)} + 2\sqrt{2t}\,(\sqrt{d}+1)\sqrt{d\Delta x^2/\Delta t + \Delta x\|f\|_{\mathrm{Lip}} + \Delta t\|f\|_{\mathrm{Lip}}^2}\,\Big)\,\mathrm{TV}(u_0).$$

We see that this yields 4.16 if we choose Δx, Δt and δ as in (4.15). □

We next estimate the convergence rate of the dimensional splitting front tracking algorithm with respect to the work needed to compute one approximation of the solution.

Lemma 4.10. *(Work estimate for $d \ge 2$) Under the assumptions of Theorem* 4.8 *and* (4.15)*, the front tracking approximation satisfies,*

$$\|u(\cdot,t)-u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\,(1+t^{(2d+3)/(2d+2)})\Big(1+\|f\|_{W^{2,\infty}}\Big) \times \mathrm{TV}(u_0)\,\Big((\|u_0\|_{L^\infty}+1)\,(\|u_0\|_{L^\infty}+\mathrm{TV}(u_0))\Big)^{1/(2(d+1))}\,(W^{FT}_{\delta,d})^{-1/(2(d+1))}\,, \tag{4.17}$$

where $C>0$ is a constant depending only on d.

Proof. The work done in one time interval $(t_{n+(i-1)/d}, t_{n+i/d}]$ consists of two components, the front tracking approximation in $(t_{n+(i-1)/d}, t_{n+i/d})$ and the projections at time $t = t_{n+i/d}$. As in the one-dimensional case, we can solve all possible Riemann problems beforehand and store the solutions, the work to do this is of order $C\,\overline{R}^2\,d\,\delta^{-2}$, where $\overline{R} = \|u_0\|_{L^\infty}$, since the flux f has d components f_i (see Remark 4.5). Then the work for the front tracking approximation in $(t_{n+(i-1)/d}, t_{n+i/d})$ is of the order of the number of interactions of fronts $T(\eta,t)$ in that time interval. This number is bounded by

$$T(\eta,t) \le C\,(\|u_0\|_{L^\infty}+1)\,(\mathrm{TV}(u_0)+\|u_0\|_{L^\infty})\,\delta^{-2}(\Delta x)^{-(d-1)},$$

which is (4.10) multiplied by $(\Delta x)^{-(d-1)}$, because we do the front tracking in each segment $I^i_{j_1,\dots,j_d} := [j_1\Delta x,(j_1+1)\Delta x)\times\cdots\times[j_{i-1}\Delta x,(j_{i-1}+1)\Delta x)\times\mathbb{R}\times\cdots\times[j_d\Delta x,(j_d+1)\Delta x)$. The work $W^{\pi_\delta}_{t_{n+i/d}}$ needed to do the projections at time $t_{n+i/d}$ is of the same order,

$$W^{\pi_\delta}_{t_{n+i/d}} = C(\|u_0\|_{L^\infty(\mathbb{R})}+1)\,(\mathrm{TV}(u_0)+\|u_0\|_{L^\infty})\,\delta^{-2}(\Delta x)^{-(d-1)},$$

as it is proportional to the number of fronts in the x_i-direction and the number of segments $I^i_{j_1,\dots,j_d}$. Hence the total work $W^{FT}_{\delta,d}$ needed to compute the front tracking approximation $u^\eta(\cdot,t)$ is of order

$$W^{FT}_{\delta,d} = C\,t\,d\,(\|u_0\|_{L^\infty(\mathbb{R})}+1)\,(\mathrm{TV}(u_0)+\|u_0\|_{L^\infty})\,\delta^{-2}(\Delta x)^{-(d-1)}(\Delta t)^{-1}\,.$$

Now using (4.15), we obtain the convergence estimate with respect to work, (4.17). □

Remark 4.11. *Observe that the convergence rate* (4.17) *is of the same order with respect to the work $W^{FT}_{\delta,d}$ as the one for the approximation by a finite volume scheme (see e.g.* [25]*). So in contrast to the one-dimensional case we do not get an improvement of the rate by using the front tracking method.*

Remark 4.12 (Work estimate for convex flux functions)**.** *As in the case $d=1$, the estimate on the total work $W^{FT}_{\delta,d}$ can be improved if the components f_i, $i=1,\dots,d$ of the flux function are convex. Again, solving a Riemann problem with left state u_L and right state u_R reduces to checking whether $u_L > u_R$. Moreover, the total* *number of interactions in each time interval* $t \in (t_{n+(i-1)/d}, t_{n+i/d})$ *is bounded by* $T(\eta, t) \leq \mathrm{TV}(u_0)\delta^{-1}$ *and therefore,*

$$\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq C\left(1 + t^{(2d+2)/(2d+1)}\right) \times \left(1 + \|f\|_{W^{2,\infty}}\right)\mathrm{TV}(u_0)^{(2d+2)/(2d+1)} \left(W^{FT}_{\delta,d}\right)^{-1/(2d+1)},$$

for convex or concave flux functions.

4.2.3. *Front tracking for RSCLs.* Having described the convergence properties of the front tracking algorithm for deterministic scalar conservation laws, we are ready to state the convergence result for the approximation of the random scalar conservation law (3.14):

Theorem 4.13. *Assume that the random (as in Definition* 3.2*) initial data* u_0 *and flux function* f *satisfy* (3.12)*.*

For $\delta > 0$*, let* $f_i^\delta(\omega,\cdot)$ *denote the piecewise linear interpolations to the random flux component functions* $f_i(\omega,\cdot)$ *as defined in* (4.2)*.*

Let the discretization parameter vector $\eta = \delta$ *if* $d = 1$*, and* $\eta = (\delta, \Delta x_1, \ldots, \Delta x_d, \Delta t)$ *if* $d > 1$*, and let* $u^\eta(\omega;\cdot,\cdot)$ *denote the corresponding approximate solution defined by* (4.3a) *if* $d = 1$ *and* (4.13) *if* $d > 1$*, with initial data* $u_0(\omega;\cdot)$ *and flux functions* $f_1(\omega;\cdot), \ldots, f_d(\omega;\cdot)$*. Then the approximations* u^η *satisfy*

$$\|u^\eta(\cdot;\cdot,t)\|_{L^\infty(\Omega;L^\infty(\mathbb{R}^d))} \leq \overline{M}, \quad t \in (0,\infty),$$

the total variation is bounded $\mathbb{P}$*-almost surely,*

$$\mathrm{TV}(u^\eta(\omega;\cdot,t)) \leq \mathrm{TV}(u_0(\omega;\cdot)), \quad t \in (0,\infty),\ \mathbb{P}\text{-a.e. } \omega \in \Omega\,.$$

As $\eta \to 0$*, the sequence* $(u^\eta)_{\eta>0}$ *converges* $\mathbb{P}$*-almost surely and in* $C([0,T];L^1(\mathbb{R}^d))$*, to the unique random entropy solution of the RSCL* (3.14)*. Moreover, if* $d = 1$*, we have* $\mathbb{P}$*-a.s. the error bound*

$$\|u(\omega;\cdot,t) - u^\eta(\omega;\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0(\omega;\cdot) - u^0(\omega;\cdot)\|_{L^1(\mathbb{R})} + t\|f(\omega;\cdot) - f^\delta(\omega;\cdot)\|_{\mathrm{Lip}(\mathbb{R})}\,\mathrm{TV}(u_0(\omega;\cdot)),$$

and if $d > 1$*, we have* $\mathbb{P}$*-a.s.*

$$\begin{aligned}\|u(\omega;\cdot,t) - u^\eta(\omega;\cdot,t)\|_{L^1(\mathbb{R}^d)} &\leq \|u_0(\omega;\cdot) - u^0(\omega;\cdot)\|_{L^1(\mathbb{R}^d)} + t\max_{i=1,\ldots,d}\|f_i(\omega;\cdot) - f_i^\delta(\omega;\cdot)\|_{\mathrm{Lip}(\mathbb{R})}\,\mathrm{TV}(u_0(\omega;\cdot)) \\ &\quad + 2\,\mathrm{TV}(u_0(\omega;\cdot))\sqrt{2t}\,(\sqrt{d}+1)\sqrt{d\Delta x^2/\Delta t + \Delta x\|f(\omega;\cdot)\|_{\mathrm{Lip}} + \Delta t\|f(\omega;\cdot)\|^2_{Lip}}.\end{aligned} \tag{4.18}$$

Proof. The assertion follows from Theorems 4.2 and 4.8 upon noting that the assumptions given there are satisfied pathwise, i.e., for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

From now on we assume that

$$f(\omega;\cdot) \in L^\infty(\Omega; W^{2,\infty}([-\overline{M},\overline{M}];\mathbb{R}^d)) \tag{4.19}$$

where $\overline{M}$ is as in (3.12).

Corollary 4.14. *Under the assumption* (4.19)*, choose* $\Delta x = k_1\delta$ *for* $d = 1$ *and* $\Delta x = k_1\Delta t = k_2\delta^2$ *for* $d \geq 2$*. Then*

(4.20) $\|u(\omega;\cdot,t) - u^\eta(\omega;\cdot,t)\|_{L^1(\mathbb{R}^d)}$

$$\leq C\,\delta\,(1+t)\left(1+\|f(\omega;\cdot)\|_{W^{2,\infty}([-\overline{M},\overline{M}];\mathbb{R}^d)}\right)\mathrm{TV}(u_0(\omega;\cdot))\,.$$

If in addition $u_0 \in L^p(\Omega; BV(\mathbb{R}^d))$ *and* $f \in L^q(\Omega; W^{2,\infty}(\mathbb{R};\mathbb{R}^d))$ *for some* $1 \leq p, q \leq \infty$ *with* $1/p + 1/q = 1$*, we have*

$$\begin{aligned}\|\mathbb{E}[u(t)] - \mathbb{E}[u^\eta(t)]\|_{L^1(\mathbb{R}^d)} &\leq \|u(t) - u^\eta(t)\|_{L^1(\Omega;L^1(\mathbb{R}^d))}\\ &\leq C\,\delta\,(1+t)\Big(1+\|f\|_{L^q(\Omega;W^{2,\infty})}\Big)\|\,\mathrm{TV}(u_0)\|_{L^p(\Omega)},\end{aligned} \tag{4.21}$$

for all δ *and* $t > 0$.

Proof. The bound (4.20) follows from the regularity assumption on $f(\omega,\cdot)$, and the inequality (4.21) is proved by an application of Hölder's inequality to (4.20), and by using (2.2). □

4.2.4. *Multilevel Flux Decomposition.* The approximate, continuous, piecewise linear flux functions f_i^δ defined by (4.2) are particular useful in connection with empirical flux data (such as typically arise in Buckley-Leverett models where flux functions are built from empirical data) and with MLMC, as will be seen in the next subsection.

We choose $\delta_0 > 0$ and let $\delta_\ell = 2^{-\ell}\delta_0$. Let also $f_i^\ell(\omega;\cdot) := f_i^{\delta_\ell}(\omega;\cdot)$ denote the continuous piecewise linear interpolant of $f_i(\omega;\cdot)$, for $i = 1,\ldots,d$, as defined by (4.2), and similarly set $f^\ell := (f_1^\ell,\ldots,f_d^\ell)$.

Lemma 4.15. *Under assumption* (4.19)*, for* $\ell = 0,1,2,...$*, the continuous, piecewise linear flux interpolants* $f_i^\ell(\omega;\cdot) = f_i^{\delta_\ell}(\omega;\cdot)$ *are bounded random flux functions in the sense of Definition* 3.2 *which satisfy the bound* (3.12) *with constant* $\overline{M}$ *which is independent of* ℓ*, and which satisfy for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *the error bound*

$$\|f_i(\omega;\cdot) - f_i^\ell(\omega;\cdot)\|_{W^{1,\infty}([-\overline{M},\overline{M}];\mathbb{R}^d)} \leq C2^{-\ell}\|\partial_u^2 f_i(\omega;\cdot)\|_{L^\infty([-\overline{M},\overline{M}])} \tag{4.22}$$

Proof. The proof of (4.22) follows from standard approximation estimates for the nodal interpolation. □

The following corollary is a direct consequence of (4.22).

Corollary 4.16. *Under the assumptions of Lemma* 4.15*, we have*

$$\|(f_i^\ell - f_i^{\ell-1})(\omega;\cdot)\|_{\mathrm{Lip}([-\overline{M},\overline{M}];\mathbb{R}^d)} \leq 2C2^{-\ell}\|\partial_u^2 f_i(\omega;\cdot)\|_{L^\infty([-\overline{M},\overline{M}];\mathbb{R})}\,.$$

Here, the constant $C > 0$ *is independent of* ℓ *and of the flux* f.

4.3. **MLMC Front Tracking.** The MLMC discretization of differential equations with random inputs was proposed by M. Giles in [9, 10], upon earlier work by Heinrich on numerical integration in [15]. For random scalar conservation laws (RSCLs), the MLMC Finite Volume discretizations were proposed and analyzed, in the case of deterministic flux and random initial conditions, in [25], and for RSCLs with random flux, in [24].

Here, we analyze the convergence of MLMC in conjunction with Front Tracking (FT) discretizations. Although the analysis proceeds, broadly speaking, along the lines of what was done in [25, 24], there are notable differences: First, unlike [24], there is no need for a principal component analysis of the random flux, e.g. via a Karhunen–Loève expansion. Secondly, we propose the use of a *multiresolution decomposition of the random flux* on the phase space of the solution. Finally, the error bounds which we shall obtain relate, in a rather explicit fashion, the number

M_ℓ of MC samples on different discretization levels to the flux variance at resolution ℓ, i.e., to $\|f^\ell - f^{\ell-1}\|^2_{L^2(\Omega;\mathrm{Lip}(\mathbb{R},\mathbb{R}^d))}$. Since f^ℓ is piecewise linear, this quantity can easily be computed for empirically calibrated random flux functions and, thereby, the number M_ℓ of "samples" (which are approximate solutions of the RSCL with flux functions f^ℓ and $f^{\ell-1}$, obtained by front tracking), can be scaled accordingly.

We start the analysis by introducing some notation. For $d = 1$, we let $\Delta x_\ell = \delta_l = 2^{-\ell}\delta_0$ for some $\delta_0 > 0$. For $d \geq 2$, $\ell = 0, 1, 2, \ldots$, we set

$$\eta_\ell = (\delta_\ell, \Delta x_\ell, \Delta t_\ell) = \left(2^{-\ell}\delta_0, 2^{-2\ell}\Delta x_0, 2^{-2\ell}\Delta t_0\right).$$

Moreover, we let $u_0^\ell(\omega;\cdot) := \pi_\ell u_0(\omega;\cdot)$ where $\pi_\ell = P_{\Delta x_\ell} \circ \overline{P}_{\Delta x_\ell}$, cf. (4.12). Note that we set $\Delta x_1 = \cdots = \Delta x_d = \Delta x_\ell$.

Then we denote for $\ell = 0, 1, 2, ...$, by $u^\ell(\omega; x, t)$ the approximations of $u(\omega; x, t)$ obtained by the front tracking method with initial data u_0^ℓ and f^ℓ.

As in [25], $E_M[\cdot]$ denotes the sample average of M i.i.d. samples of a random quantity. We are interested in the computation of the statistical mean

$$\mathbb{E}[u(t)] \in C(0, T; L^1(\mathbb{R}^d))$$

of the random entropy solution of the RSCL (3.1) - (3.3). To this end, the MLMC-FT approximation is defined as follows: for a given level $L \in \mathbb{N}$ of refinement, we use the linearity of the mathematical expectation $\mathbb{E}[\cdot]$ to write

$$\mathbb{E}[u(t)] \simeq \mathbb{E}[u^L(t)] = \sum_{\ell=0}^{L} \mathbb{E}\left[u^\ell - u^{\ell-1}\right].$$

Here, and in the following, we adopt the convention that $u^{-1} \equiv 0$.

We next *estimate the expectations of increments for each level of refinement by a level-dependent number M_ℓ of samples*, which results in the MLMC estimate

$$E_L^{MLMC}[u^L(t)] := \sum_{\ell=0}^{L} E_{M_\ell}\left[u^\ell - u^{\ell-1}\right]. \tag{4.23}$$

Here, u^ℓ are the approximations obtained by front tracking for the initial data u_0^ℓ and the flux functions f^ℓ.

4.4. **Convergence Analysis.** We are now interested in estimating

$$\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)].$$

To this end, we write

$$\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)] = \underbrace{\mathbb{E}[u(t)] - \mathbb{E}[u^L(t)]}_{A} + \underbrace{\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]}_{B}.$$

We have already estimated the $L^1(\mathbb{R}^d)$-norm of term A in equation (4.21). In this setting, it is of order $\mathcal{O}(2^{-L})$ under the additional assumption that $u_0 \in L^p(\Omega; BV(\mathbb{R}^d))$ and $f \in L^q(\Omega; W^{2,\infty}(\mathbb{R};\mathbb{R}^d))$, where $1/p + 1/q = 1$. Consider now the term B. To estimate it, we write, with $\Delta u^\ell := u^\ell - u^{\ell-1}$ for $\ell = 0, 1, 2, ..., L$ and with the convention that $u^{-1} \equiv 0$,

$$\begin{aligned}&\|\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}\\ &\qquad = \left\|\mathbb{E}\Big[\sum_{\ell=0}^{L}(u^\ell - u^{\ell-1})\Big] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}\end{aligned}$$

$$= \Big\|\sum_{\ell=0}^{L} \left\{\mathbb{E}[\Delta u^\ell] - E_{M_\ell}[\Delta u^\ell]\right\} \Big\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}.$$

Expanding the square, and interpreting the M_ℓ samples as i.i.d. copies of the random variable $u^\ell(\omega; x, t)$, we obtain

$$\|\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} = \sum_{\ell=0}^{L} \left\|\mathbb{E}[\Delta u^\ell] - E_{M_\ell}[\Delta u^\ell]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}.$$

Next we estimate each term in the sum as follows:

$$\begin{aligned} B_\ell &:= \left\|\mathbb{E}[\Delta u^\ell] - E_{M_\ell}[\Delta u^\ell]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\ &= \frac{1}{M_\ell}\mathbb{E}\left[\|\mathbb{E}[\Delta u^\ell(t)] - \Delta u^\ell(t)\|^2_{L^1(\mathbb{R}^d)}\right] \\ &\le \frac{1}{M_\ell}\|\Delta u^\ell(t)\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}. \end{aligned}$$

We use the elementary estimate

$$\|\Delta u^\ell(\omega;\cdot,t)\|^2_{L^1(\mathbb{R}^d)} \le 2\|u(\omega;\cdot,t) - u^\ell(\omega;\cdot,t)\|^2_{L^1(\mathbb{R}^d)} + 2\|u(\omega;\cdot,t) - u^{\ell-1}(\omega;\cdot,t)\|^2_{L^1(\mathbb{R}^d)}$$

and the convergence rate (4.20), to obtain

$$\|u(t) - u^\ell(t)\|_{L^2(\Omega;L^2(\mathbb{R}^d))} \le C\,\delta_\ell\,(1+t)\Big(1 + \|f\|_{L^2(\Omega;W^{2,\infty})}\Big)\|\operatorname{TV}(u_0)\|_{L^\infty(\Omega)}.$$

under the assumption that $u_0 \in L^\infty(\Omega; BV(\mathbb{R}^d))$ and $f \in L^2(\Omega; W^{2,\infty}(\mathbb{R};\mathbb{R}^d))$. Thus,

$$B_\ell \le \frac{1}{M_\ell} C\,\delta_\ell^2\,(1+t^2)\Big(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\Big)\|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)},$$

where $C > 0$ is a constant which depends on d but which is independent of t. Summing over $\ell = 0, \ldots, L$, we arrive at

$$\begin{aligned} &\|\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]\|^2_{L^2(\Omega;L^1(\mathbb{R}))} \\ &\qquad \le C\,(1+t^2)\sum_{\ell=0}^{L}\frac{1}{M_\ell}\,\delta_\ell^2\,\Big(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\Big)\|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)}. \end{aligned}$$

We can now state our basic MLMC-FT error bound.

Theorem 4.17. *Consider the RSCL with random data (u_0, f) (3.11) in the sense of Definition 3.2 and satisfying (3.12). Assume for $\overline{M}$ as in (3.12) that (4.19) holds.*

Then, for any $L \in \mathbb{N}$ and for any choice of samples sizes $\{M_\ell\}_{\ell=0}^{L}$ in the MLMC-FT estimator $E_L^{MLMC}[u^L(t)]$ in (4.23) we have the error bound

$$\begin{aligned} &\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\ &\quad \le 2\,C(1+t^2)\delta_L^2\left(1 + \|f\|^2_{L^1(\Omega;W^{2,\infty})}\right)\|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)} \\ &\qquad + C\,(1+t^2)\sum_{\ell=0}^{L}\frac{1}{M_\ell}\,\delta_\ell^2\,\left(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\right)\|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)} \\ &\quad \le C\Big[2^{-2L} + \sum_{\ell=0}^{L} M_\ell^{-1}2^{-2\ell}\Big](1+t^2) \end{aligned}$$

$$\times \left(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\right) \|\mathrm{TV}(u_0)\|^2_{L^\infty(\Omega)} .$$

With the particular choice

$$M_\ell = 2^{2(L-\ell)} , \quad \ell = 0, \dots, L,$$

we find for any $0 \le t \le T < \infty$ *the bound*

$$\begin{aligned} &\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\ &\qquad \le C\, L 2^{-2L}\,(1+t^2)\left(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\right)\|\,\mathrm{TV}(u_0)\|^2_{L^\infty(\Omega)}. \end{aligned} \tag{4.24}$$

Proof. The proof follows from the foregoing analysis. □

If we denote the work for one FT solution at mesh level ℓ by W_ℓ^{FT}, and use the front tracking work estimates in Lemmas 4.5 and 4.10, we obtain the work estimate $W^{FT}_{L,MLMC}$ for the MLMC front tracking method,

$$W^{FT}_{L,MLMC} = C\sum_{\ell=0}^{L} M_\ell W_\ell^{FT} = \begin{cases} \mathcal{O}(W_L^{FT}\,\log W_L^{FT}) = \mathcal{O}(L\,\delta_L^{-2}) & \text{if } d = 1, \\ \mathcal{O}(W_L^{FT}) = \mathcal{O}(\delta_L^{-2(d+1)}) & \text{if } d \ge 2. \end{cases} \tag{4.25}$$

This gives us the convergence rates for the MLMC-FT estimator $E_L^{MLMC}[u^L(t))]$ with respect to work:

Corollary 4.18. *Under the assumptions of Theorem* 4.17*, the MLMC-FT estimator* $E_L^{MLMC}[u^L(t))]$ *converges with the following rates to the ensemble average* $\mathbb{E}[u(t)]$ *of the random entropy solution*

$$\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}))} \le C\,(\log W^{FT}_{L,MLMC})^2\,(W^{FT}_{L,MLMC})^{-1}, \tag{4.26}$$

for $d = 1$*, and*

$$\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \le C\,(\log W^{FT}_{L,MLMC})\,(W^{FT}_{L,MLMC})^{-1/(d+1)}$$

for $d \ge 2$*, where* $C > 0$ *is a constant depending on* d *and* t*, and on* $\|u_0\|_{L^\infty(\Omega;BV(\mathbb{R}^d))}$ *and* $\|f\|_{L^2(\Omega;W^{2,\infty}(-\overline{M},\overline{M};\mathbb{R}^d))}$.

Remark 4.19. *We have seen in Lemma* 4.7 *that the convergence rate of the deterministic front tracking algorithm for* $d = 1$ *is one with respect to work, if the flux function* f *is convex. However, this does not show up as an improvement of the convergence rate of the MLMC-FT method, since in this case the work of the Monte Carlo method dominates. Specifically, in the case of a convex flux and* $d = 1$*, we have*

$$\begin{aligned} W^{FT}_{L,MLMC} &= C\sum_{\ell=0}^{L} M_\ell\, W_\ell^{FT} \le C\sum_{\ell=0}^{L} M_\ell\, \delta_\ell^{-1} \\ &\le C\, 2^{2L}\sum_{\ell=0}^{L} 2^{-2\ell}\, 2^{\ell} \le C\, 2^{2L} = \mathcal{O}(\delta_L^{-2}), \end{aligned} \tag{4.27}$$

which is the same effort as in the general case (4.25) *apart from the missing factor* L*.*

This is to be contrasted to several space dimensions, where we have a small gain in convergence rate if all the flux components f_j*,* $j = 1, \dots, d$ *are convex, since the* *convergence rate of the deterministic dimensional splitting front tracking method is worse than that of the Monte Carlo method:*

$$W^{FT}_{L,MLMC} = C \sum_{\ell=0}^{L} M_\ell \, W^{FT}_\ell \le C \sum_{\ell=0}^{L} M_\ell \, \delta_\ell^{-(2d+1)}$$
$$\le C\, 2^{2L} \sum_{\ell=0}^{L} 2^{(-1+2d)\ell} \le C\, 2^{(2d+1)L} = \mathcal{O}(\delta_L^{-(2d+1)}).$$

5. Numerical Experiments

In this section, we test the performance of the MLMC-FT method on several examples with random fluxes in one and two space dimensions.

5.1. **Convex random flux in one space dimension.**

We consider the random scalar conservation law,

$$u_t + f(\omega; u)_x = 0, \quad x \in [-1,1],\, t \in (0,\infty), \tag{5.1a}$$

$$u(\omega; x, 0) = -\sin(\pi x), \quad x \in [-1,1],\, t = 0, \tag{5.1b}$$

with periodic boundary conditions and the random flux $f(\omega; u)$ given by

$$f(\omega; u) = \frac{1}{p(\omega)} |u|^{p(\omega)}, \quad p(\omega) \sim \mathcal{U}(1.5, 2.5). \tag{5.2}$$

This flux function is a bounded random flux and for $\mathbb{P}$-a.e., $f(\omega;\cdot) \in \mathrm{Lip}([-\overline{M}, \overline{M}]; \mathbb{R})$, where $\overline{M} \ge \|u_0\|_{L^\infty(\mathbb{R})}$ is as in (3.12). An approximation of the mean of the random entropy solution at time $t = 1$, computed by the MLMC-FT method for $L = 9$, with $\delta_0 = 2^{-4}$ at the coarsest level, and $M_L = 8$ samples at the level with the finest resolution, is shown in Figure 1. In order to compute an estimate on the error

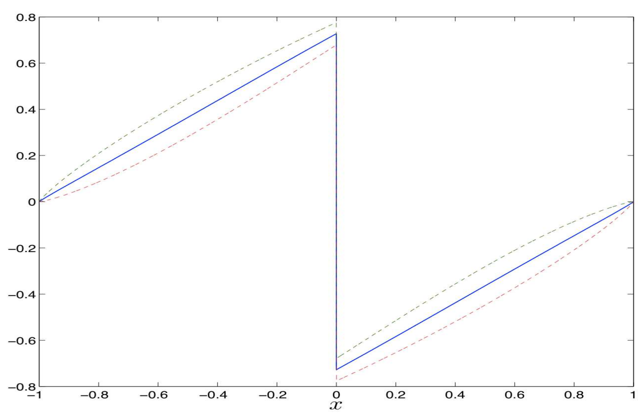


Figure 1. The estimator $E_L^{MLMC}[u^L(t)]$ computed by the MLMCFT method at time $t = 1$ with $L = 9$ for problem (5.1), (5.2). The dashed lines denote the mean with ± standard deviation.

of the approximation of the mean by the MLMC estimator $E_L^{MLMC}[u^L(t)]$ in the $L^2(\Omega; L^1(\mathbb{R}))$-norm, we use the relative error estimator introduced in [25] based on a Monte Carlo quadrature in the stochastic domain: We denote by U_{ref} a reference solution and by $\{U_k\}_{k=1,\dots,K}$ a sequence of statistically independent approximate solutions $E_L^{MLMC}[u^L(t)]$ obtained by running the MLMC-FT solver K times and corresponding to K realizations in the stochastic domain. Then we estimate the relative error by

$$\mathcal{R}E = \sqrt{\sum_{k=1}^{K} (\mathcal{R}E_k)^2/K}, \tag{5.3}$$

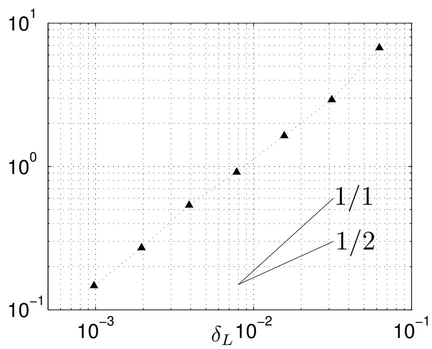
where

$$\mathcal{R}E_k = 100 \times \frac{\|U_{\text{ref}} - U_k\|_{l^1}}{\|U_{\text{ref}}\|_{l^1}}. \tag{5.4}$$

In [25] the sensitivity of the error with respect to the parameter K is investigated. For this example, we will use $K = 30$ which was shown to be sufficient for most problems [25, 27]. To compute a reference solution U_{ref}, we have made use of the symmetry properties of the each realization (a shock at $x = 0$, smoothness away from the shock) and used the characteristics of the differential equation to compute an accurate approximation of $\mathbb{E}[u(t)]$. In Figure 2 the errors (5.3) versus the resolution δ_L at the finest level L of the MLMC estimator and versus the run time (in seconds) are shown ($L = 0, \dots, 6$). We observe that the convergence rates are ≈ 0.9 with respect to the resolution and ≈ 0.4 with respect to work, which is approximately what we would expect from the theoretical results: Equation (4.24) implies that the error estimator (5.3) is asymptotically of order $\mathcal{O}(\sqrt{L}\, 2^{-L}) = \mathcal{O}(2^{-\alpha(L)\, L}) = \mathcal{O}(\delta_L^{-\alpha(L)})$ with respect to the resolution at the finest level, where

$$\alpha(L) = 1 - \frac{\log L}{2\, L \log 2} \xrightarrow{L\to\infty} 1. \tag{5.5}$$

For $L = 6$, we have $\alpha(L+1) \approx 0.8$. Due to (4.27), the estimator (5.3) is of order $\mathcal{O}((W_{L,MLMC}^{FT})^{-\alpha(L)/2})$ with respect to work, hence for $L = 6$, $\alpha(L+1)/2 \approx 0.4$.



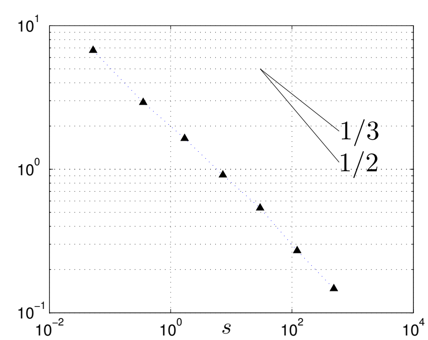


FIGURE 2. Left: Error (5.3) versus the resolution. Right: Error versus the run time of the MLMC-FT solver in seconds for the problem (5.1), (5.2). At the coarsest level, we have used $\delta_0 = 2^{-4}$ and at the finest level, we have used $M_L = 8$ samples.

Remark 5.1. *For exponents $p \in [1.5, 2)$, the second derivative of the flux function $f(u,p)$ in* (5.2) *is not uniformly bounded. Therefore the bound* (4.8) *does not apply. However, by a careful refinement of the estimates in* [21, Chapter 2], *it is possible to show that the (deterministic) front tracking method converges at rate one with respect to the discretization parameter δ if the flux function f is in $W^{2,1}([-\overline{M}, \overline{M}]; \mathbb{R})$ and the initial data $u_0 \in BV(\mathbb{R})$ has a bounded number of local maxima and minima.*

5.2. **Nonconvex random flux in one space dimension.** In a second experiment, we test the performance of the MLMC-FT method on the initial value problem (5.1) with periodic boundary conditions and the nonconvex random flux function

$$f(\omega; u) = \operatorname{sgn}(u) \frac{|u|^{p(\omega)}}{p(\omega)}, \quad p(\omega) \sim \mathcal{U}(2.5, 3.5). \tag{5.6}$$

For $\overline{M} > 0$ as in (3.12), we have $f \in L^2(\Omega; W^{2,\infty}([-\overline{M}, \overline{M}]; \mathbb{R}))$, hence the assumptions in Theorem 4.17 are satisfied for this problem. In Figure 3, we show an approximation of the mean of the solution computed by the MLMC-FT-solver at time $t = 1$ with $L = 9$, $\delta_0 = 2^{-5}$ at the coarsest level and $M_L = 4$ samples at the finest level. We see that the mean of the solution is continuous, whereas all

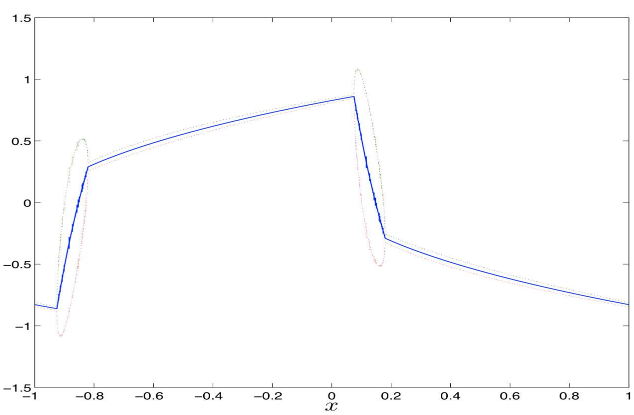


FIGURE 3. The estimator $E_L^{MLMC}[u^L(t)]$ for problem (5.1), (5.6) computed by the MLMC-FT method at time $t = 1$ with $L = 9$. The dashed lines denote the mean with $\pm$ standard deviation.

computed pathwise, approximate realizations $u(\omega; \cdot)$ of random entropy solutions of (5.1), (5.6) develop shocks.

This is not unexpected, because while each realization has discontinuities, the location of these discontinuities is random, and disappear upon taking the expectation. However, for each realization, the solution varies (very) rapidly at the shock location, hence the variance will be larger around in the regions where shocks are typically located, than in regions where each realization is continuous. For our example, each realization has two shocks, one around $x = 0.1$ and one around $x = -0.9$. We see that the variance is indeed much larger in around $x = 0.1$ and $x = -0.9$.

We use this approximation as a reference solution and compute the error estimators (5.3), (5.4) for $L = 0, \ldots, 5$, $\delta_0 = 2^{-5}$, $M_L = 4$ and $K = 30$. The results are shown in Figure 4. Similarly as for the first example in Section 5.1, the experimentally observed convergence rates validate the a priori estimates (4.24) and (4.26) as we are not yet in the asymptotic regime and for $L = 5$, $\alpha(L+1) \approx 0.78$, c.f. (5.5) (we observe ≈ 0.85 versus resolution and ≈ 0.35 versus run time).

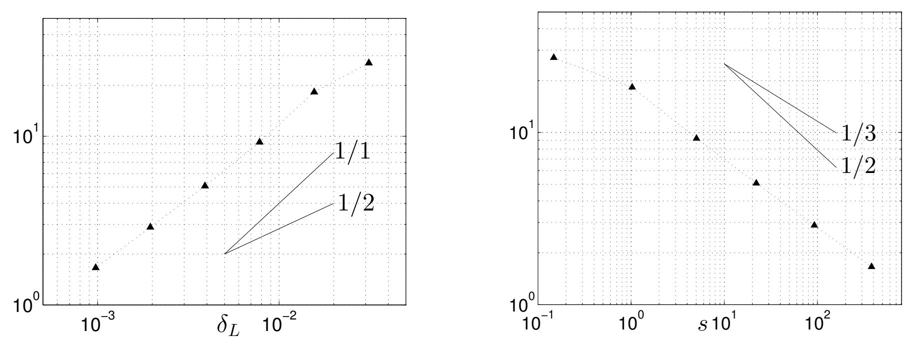


FIGURE 4. Left: Error (5.3) versus the resolution. Right: Error versus the run time of the MLMC-FT solver in seconds for the problem (5.1), (5.6). At the coarsest level, we have used $\delta_0 = 2^{-5}$ and at the finest level, we have used $M_L = 4$ samples.

5.3. **Random fluxes in two space dimensions.** We test the performance of the MLMC-FT algorithm in several space dimensions on the following test problem,

$$u_t + f(\omega; u)_x + g(\omega; u)_y = 0, \quad (x, y) \in [0, 2]^2,\ t \in (0, \infty), \tag{5.7a}$$

$$u(\omega; x, y, 0) = \begin{cases} 1, & 0.1 < x, y < 0.9, \\ -1, & (x - 1.5)^2 + (y - 1.5)^2 < 0.16, \\ 0, & \text{otherwise}, \end{cases} \tag{5.7b}$$

with periodic boundary conditions and random fluxes f and g given by

$$f(\omega; u) = g(\omega; u) = \frac{|u|^{p(\omega)}}{p(\omega)}, \quad p(\omega) \sim \mathcal{U}(1, 3). \tag{5.8}$$

In Section 4.2.2 we have seen that in order to have the optimal convergence rate of the front tracking/dimensional splitting method, we have to choose the grid size Δx, the time step Δt and the refinement parameter δ of the flux function interpolations as

$$\Delta x = k_1 \Delta t = k_2 \delta^2.$$

We call k_1 a *CFL-number* in analogy to finite volume methods, although no restriction needs to be imposed on k_1 since dimensional splitting combined with front tracking method has been shown to converge for any choice of constants $k_1 > 0$.

Due to the increased computational effort of the multidimensional problem compared with the one dimensional problems, we have chosen to refine with respect to the grid size Δx. Therefore we set $\Delta x_\ell = 2^{-\ell} \Delta x_0$ and $\delta_\ell = 2^{-\ell/2} \delta_0$ and use at level $\ell = 0, \ldots, L$, $M_\ell = 2^{L-\ell} M_L$ samples. In Figure 5 we show an approximation of the mean of (5.7), (5.8) by the MLMC-FT method computed at time $t = 1$ for $L = 8$ with $M_L = 4$, $\Delta x_0 = 2^{-3}$ and CFL-number $k_1 = 20$. As a reference solution,

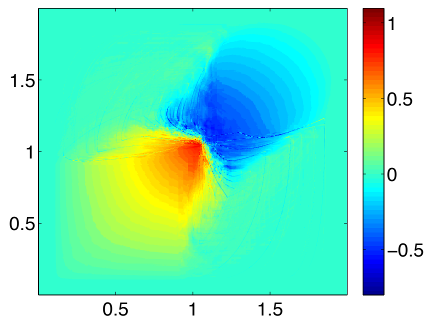


FIGURE 5. Mean and variance of (5.7), (5.8) computed by the MLMC-FT method for $L = 8$, $t = 1$, $M_L = 4$, $\Delta x_0 = 2^{-3}$, CFL-condition $k_1 = 20$ (number of grid cells: $2^{12} \times 2^{12}$). Left: Estimated mean of the solution. Right: Estimated variance of the solution.

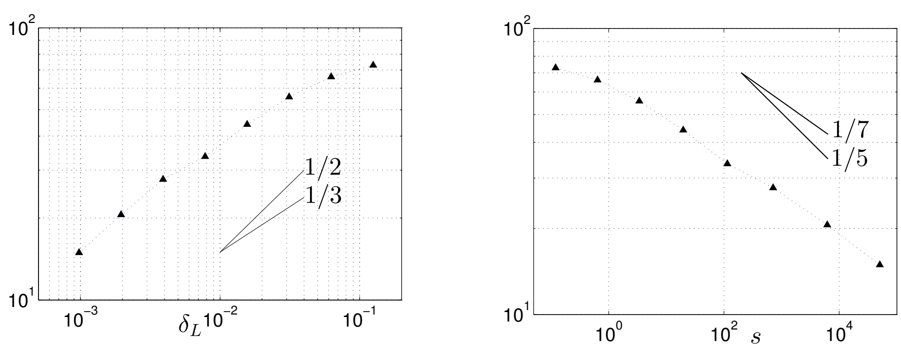


FIGURE 6. Left: Error (5.3) versus the resolution. Right: Error versus the run time of the MLMC-FT solver in seconds (x-axis figure right hand side) for the problem (5.1), (5.2). At the coarsest level, we have used $\Delta x_0 = 2^{-3}$ and at the finest level, we have used $M_L = 4$ samples, $K = 5$.

we use an approximation of the mean of the solution computed by a MLMC-FVM scheme as in [26], with an HLL-solver and second order WENO reconstruction, $L = 8$, $M_L = 4$, $\Delta x_0 = 2^{-2}$, on a mesh with $2^{11} \times 2^{11}$ grid cells. We compute the error estimators (5.3), (5.4) for $K = 5$, $L = 0, \ldots, 7$, $M_L = 4$, $M_\ell = 2^{L-\ell} M_L$, $\Delta x_0 = 0.125$, $\Delta x_\ell = 2^{-\ell} \Delta x_0$. The errors are shown in Figure 6. We measure convergence rates of ≈ 0.45 with respect to the grid size Δx and ≈ 0.15 with respect to the run time of the MLMC-FT solver. From the a priori estimates we would expect rates of 1/2 versus the grid size and 1/5 versus work asymptotically, so our rates are slightly below that. This could indicate that we are not yet in the asymptotic regime for our values of L.

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