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Research Report No. 2012-17
July 2012

Seminar für Angewandte Mathematik
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MULTILEVEL MONTE-CARLO FRONT TRACKING FOR RANDOM SCALAR CONSERVATION LAWS

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ABSTRACT. We consider random scalar hyperbolic conservation laws (RSCLs) in spatial dimension $d \geq 1$ with bounded random flux functions which are $\mathbb{P}$-a.s. Lipschitz continuous with respect to the state variable, for which there exists a unique random entropy solution (i.e., a measurable mapping from the probability space into $C(0,T;L^1(\mathbb{R}^d))$ with finite second moments). We present a convergence analysis of a Multi-Level Monte-Carlo Front-Tracking (MLMCFT) algorithm. It is based on "pathwise" application of the Front-Tracking Method from [20] for deterministic SCLs. We compare the MLMCFT algorithms to the Multi-Level Monte-Carlo Finite-Volume methods developed in [25, 26, 27]. Due to the first order convergence of front tracking, we obtain an improved complexity estimate in one space dimension.

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Date: July 2, 2012.

1991 *Mathematics Subject Classification.* 65N30,65M06,35L65.

Acknowledgement. This work is performed as part of ETH interdisciplinary research grant CH1-03 10-1. The work of Schwab was supported in part by ERC FP7 grant no. AdG 247277. The work of Risebro was performed while visiting SAM during a sabbatical in the academic year 2011/12. The authors thank Jonas Šukys for providing a reference solution for a numerical experiment and Siddharth Mishra and Jonas Šukys for valuable discussions.

1. Introduction

Many problems in physics and engineering are modeled by hyperbolic systems of conservation or balance laws. The Cauchy problem for such systems takes the form

$$\mathbf{U}_t + \sum_{j=1}^{d} \frac{\partial}{\partial x_j}(\mathbf{F}_j(\mathbf{U})) = \mathbf{S}, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d,\ t > 0, \tag{1.1}$$
$$\mathbf{U}(x, 0) = \mathbf{U}_0(x), \quad x \in \mathbb{R}^d.$$

Here, $\mathbf{U} : \mathbb{R}^d \mapsto \mathbb{R}^m$ is the vector of unknowns and $\mathbf{F}_j : \mathbb{R}^m \mapsto \mathbb{R}^m$ is the flux vector for the j-th direction with m being a positive integer, and $\mathbf{S} : \mathbb{R}^d \mapsto \mathbb{R}^m$ denotes the so-called *source term.* If $\mathbf{S} = \mathbf{0} \in \mathbb{R}^m$, (1.1) is called *conservation law*, otherwise a *balance law.*

We mention only the shallow water equations of hydrology, the Euler equations for inviscid, compressible flow and the magnetohydrodynamic (MHD) equations of plasma physics, see, e.g. [6, 11]. In the this paper we focus on the case $m = 1$ in (1.1) which is then called a *scalar conservation law (SCL).*

Solutions of (1.1) develop discontinuities in finite time even when the initial data is smooth. Therefore (1.1) must be interpreted in the weak sense. In order to get uniqueness, (1.1) must be augmented with *entropy conditions*, which at least for scalar conservation laws, makes the initial value problem well posed. The well-posedness the Cauchy problem for scalar conservation laws in several space dimensions ($m = 1$, $d \geq 1$) was first established by Kružkov [22].

For systems ($m > 1$), some well-posedness results for systems in one space dimension exist [2, 3] , but no well-posedness results for systems of conservation laws are available in several space dimensions.

Numerical methods for approximating entropy solutions of systems of conservation laws have undergone extensive development and many efficient methods are available, see [8, 11, 12, 23] and the references there. In particular, finite volume methods are frequently employed for approximating (1.1).

This *classical* paradigm for designing efficient numerical schemes assumes that *data for the SCL* (1.1)*, i.e., initial data* $\mathbf{U}_0$ *and flux are known exactly.*

In many situations of practical interest, however, these data are not known exactly due to inherent uncertainty in modelling and measurements of physical parameters such as, for example, the specific heats in the equation of state for compressible gases, resistivity in MHD etc. Often, the initial data are known only up to certain statistical quantities of interest like the mean, variance, higher moments, and in some cases, the law of the stochastic initial data. In such cases, a mathematical formulation of (1.1) is required which allows for *random data*. The problem of random initial data was considered in [25], and the existence and uniqueness of a random entropy solution was shown, and a convergence analysis for MLMC FV discretizations was given. Efficient MLMC discretization of balance laws with random source terms was investigated in [27].

We mention that the present work as well as [25, 27] considered *correlated random inputs* which typically occur in engineering applications; SCLs with random inputs have been considered before, but generally with *white noise*, i.e., spatially and temporally uncorrelated random inputs in [19, 18, 7, 30, 31].

In [27] a mathematical framework was developed for scalar conservation laws with random initial data. This framework was extended to include random flux functions in [24]. Here, we recapitulate result from [24] regarding the existence and uniqueness of random entropy solutions for such problems. The efficient numerical approximation of such solutions and, in particular, of their statistics, is the purpose of the present paper.

To this end , we propose and analyze a combination of sampling techniques of the Monte Carlo (MC) type combined with a "pathwise" Front Tracking (FT) solver introduced by Bagrinovskiĭ and Godunov [1] and analyzed, e.g., in [20], to approximate random entropy solution of scalar, nonlinear hyperbolic conservation laws.

As the stochastic collocation FVM discretization, and the MLMC FVM methods analyzed in [27] also for nonlinear, hyperbolic *systems* (1.1), the multi-level version of the MCThis method is "non-intrusive", easy to code and to parallelize, and well suited for random solutions with low spatial regularity, a situation which is typical in nonlinear hyperbolic conservation laws where discontinuities in realizations of solutions are well known to be generic.

The remainder of this paper is organized as follows: in Section 2, we introduce some preliminary notions from probability theory and functional analysis. The concept of random entropy solutions is introduced and the well-posedness of the scalar hyperbolic conservation law (i.e., (1.1) with $m = 1$) with random initial data is recapitulated in Section 3. The MLMC-FT schemes are presented and analyzed in Section 4. Numerical experiments are presented in Section 5.

2. Preliminaries from Probability

We use the concept of random variables taking values in function spaces. To this end, we recapitulate basic concepts from [4, Chapter 1].

Let $(\Omega, \mathcal{F})$ be a measurable space, with Ω denoting the set of all elementary events, and $\mathcal{F}$ a σ-algebra of all possible events in our probability model. If $(E, \mathcal{G})$ denotes a second measurable space, then an *E-valued random variable* (or random variable taking values in E) is any mapping $X : \Omega \to E$ such that the set $\{\omega \in \Omega$: $X(\omega) \in A\} = \{X \in A\} \in \mathcal{F}$ for any $A \in \mathcal{G}$, i.e., such that X is a $\mathcal{G}$-measurable mapping from Ω into E.

Assume now that E is a metric space; with the Borel σ-field $\mathcal{B}(E)$, $(E, \mathcal{B}(E))$ is a measurable space and we shall always assume that E-valued random variables $X : \Omega \to E$ will be $(\mathcal{F}, \mathcal{B}(E))$ measurable. If E is a separable Banach-space with norm $\| \circ \|_E$ and (topological) dual E^*, then $\mathcal{B}(E)$ is the smallest σ-field of subsets of E containing all sets

$$\{x \in E : \varphi(x) \leq \alpha\},\ \varphi \in E^*,\ \alpha \in \mathbb{R}\,. \tag{2.1}$$

Hence if E is a separable Banach space, $X : \Omega \to E$ is an E-valued random variable iff for every $\varphi \in E^*$, $\omega \longmapsto \varphi(X(\omega)) \in \mathbb{R}^1$ is an $\mathbb{R}^1$-valued random variable. Moreover, we have

Lemma 2.1. *Let E be a separable Banach-space and let $X : \Omega \to E$ be an E-valued random variable on $(\Omega, \mathcal{F})$. Then the mapping $\Omega \ni \omega \longmapsto \|X(\omega)\|_E \in \mathbb{R}^1$ is measurable.*

Proof. Since E is separable, there exists a sequence $\{\varphi_n\} \subset E^*$ such that for all $x \in E$ holds

$$\|x\|_E = \sup_{n\in\mathbb{N}} |\varphi_n(x)| \,. \tag{2.2}$$

Hence we find

$$\forall \omega \in \Omega : \ \|X(\omega)\|_E = \sup_{n\in\mathbb{N}} |\varphi_n(X(\omega))| \tag{2.3}$$

which implies that $\omega \longmapsto \|X(\omega)\|_E$ is an $\mathbb{R}^1$-valued random variable. □

The random variable $X : \Omega \to E$ is called *Bochner integrable* if, for any probability measure $\mathbb{P}$ on the measurable space $(\Omega, \mathcal{F})$,

$$\int_\Omega \|X(\omega)\|_E \, \mathbb{P}(d\omega) < \infty \,. \tag{2.4}$$

A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is any σ-additive set function from Ω into $[0, 1]$ such that $\mathbb{P}(\Omega) = 1$, and the measure space $(\Omega, \mathcal{F}, \mathbb{P})$ is called probability space. We shall assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is complete.

If $X : (\Omega, \mathcal{F}) \to (E, \mathcal{E})$ is a random variable, $\mathcal{L}(X)$ denotes the *law of X under* $\mathbb{P}$, i.e.,

$$\mathcal{L}(X)(A) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \quad \forall A \in \mathcal{E} \,. \tag{2.5}$$

The image measure $\mu_X = \mathcal{L}(X)$ on $(E, \mathcal{E})$ is called law or distribution of X.

A random variable taking values in E is called *simple* if it can take only finitely many values, i.e., if it has the explicit form (with χ_A the indicator function of $A \in \mathcal{F}$)

$$X = \sum_{i=1}^{N} x_i \, \chi_{A_i} , \quad A_i \in \mathcal{F}, \ x_i \in E, \ N < \infty \,. \tag{2.6}$$

We set, for simple random variables X taking values in E and for any $\mathcal{B} \in \mathcal{F}$,

$$\int_B X(\omega) \, \mathbb{P}(d\omega) = \int_B X d\,\mathbb{P} := \sum_{i=1}^{N} x_i \, \mathbb{P}(A_i \cap B) \,. \tag{2.7}$$

By density, for such $X(\cdot)$, and all $B \in \mathcal{F}$,

$$\Big\| \int_B X(\omega) \, \mathbb{P}(d\omega) \Big\|_E \le \int_B \|X(\omega)\|_E \, \mathbb{P}(d\omega) \,. \tag{2.8}$$

For any random variable $X : \Omega \to E$ which is Bochner integrable, there exists a sequence $\{X_m\}_{m\in\mathbb{N}}$ of simple random variables such that, for all $\omega \in \Omega$, $\|X(\omega) - X_m(\omega)\|_E \to 0$ as $m \to \infty$. Therefore, (2.7) and (2.8) extend in the usual fashion by continuity to any E-valued random variable. We denote the integral

$$\int_\Omega X(\omega) \, \mathbb{P}(d\omega) = \lim_{m\to\infty} \int_\Omega X_m(\omega) \, \mathbb{P}(d\omega) \in E \tag{2.9}$$

by $\mathbb{E}[X]$ ("expectation" of X). □

We shall require for $1 \le p \le \infty$ Bochner spaces of p-summable random variables X taking values in the Banach-space E. By $L^1(\Omega, \mathcal{F}, \mathbb{P}; E)$ we denote the set of all (equivalence classes of) integrable, E-valued random variables X. We equip it with the norm

$$\|X\|_{L^1(\Omega;E)} = \int_\Omega \|X(\omega)\|_E \, \mathbb{P}(d\omega) = \mathbb{E}(\|X\|_E) \,. \tag{2.10}$$

More generally, for $1 \le p < \infty$, we define $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$ as the set of p-summable random variables taking values E and equip it with norm

$$\|X\|_{L^p(\Omega;E)} := (\mathbb{E}(\|X\|_E^p))^{1/p}, \ 1 \le p < \infty \,. \tag{2.11}$$

For $p = \infty$, we denote by $L^\infty(\Omega, \mathcal{F}, \mathbb{P}; E)$ the set of all E-valued random variables which are essentially bounded. This set is a Banach space equipped with the norm

$$\|X\|_{L^\infty(\Omega;E)} := \operatorname*{ess\,sup}_{\omega \in \Omega} \|X(\omega)\|_E \,. \tag{2.12}$$

If $T < \infty$ and $\Omega = [0, T]$, $\mathcal{F} = \mathcal{B}([0, T])$, we write $L^p(0, T; E)$. Note that for any separable Banach-space E, and for any $r \ge p \ge 1$,

$$L^r(0, T; E), \ C^0(0, T; E) \in \mathcal{B}(L^p(0, T; E)) \,. \tag{2.13}$$

Here and throughout, $C^0(0, T; E)$ denotes the Bochner space of functions $u : [0, T] \mapsto E$ which are continuous on $[0, T]$ with values in E. More generally, for a time interval $J \subseteq \mathbb{R}$ and for a Banachspace E, we denote by $C^k(J; E)$ the space of continuous functions from J to E which are k-times strongly differentiable with continuous k-th derivative. For unbounded intervals $J \subseteq \mathbb{R}$, we denote by $C_b^k(J; E) \subset C^k(J; E)$ the subset of functions with *bounded* support. For bounded sets $J \subset \mathbb{R}$, $C_b^k(J; E) = C^k(J; E)$. For *any* set $J \subseteq \mathbb{R}$, $C_0^k(J; E) \subset C^k(J; E)$ denotes the set of functions $u : J \mapsto E$ which are, with their derivatives w.r.t. to $t \in J$ up to order k, are *compactly supported* in J. Evidently, for any $k \in \mathbb{N}$ holds $C_0^k(J; E) \subset C_b^k(J; E) \subset C^k(J; E)$. If $k = 0$, we omit the superscript.

3. Hyperbolic Conservation Laws with random flux

We review classical results on SCLs with deterministic data, and develop a theory of random entropy solutions for SCLs with a class of random flux fluctions, proving in particular the existence and uniqueness of a random entropy solution with finite second moments.

We also propose a novel spectral decomposition of the random entropy solutions which is based on a Karhunen–Loève expansion in state space.

3.1. **Deterministic scalar hyperbolic conservation laws.**

We consider the Cauchy problem for scalar conservation laws (SCL) by setting $m = 1$ in (1.1) and obtaining the SCL in strong form

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} \frac{\partial}{\partial x_j} (f_j(u)) = 0, \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \ t > 0 \,. \tag{3.1}$$

Here the unknown is $u : \mathbb{R}^d \mapsto \mathbb{R}$. Introducing the flux function $f(u)$

$$f(u) = (f_1(u), \ldots, f_d(u)) \in C^1(\mathbb{R}; \mathbb{R}^d) \,, \quad \operatorname{div} f(u) = \sum_{j=1}^{d} \frac{\partial}{\partial x_j} f_j(u),$$

we may rewrite (3.1) succinctly as

$$\frac{\partial u}{\partial t} + \operatorname{div}(f(u)) = 0 \quad \text{for } (x,t) \in \mathbb{R}^d \times \mathbb{R}_+. \tag{3.2}$$

We supply the SCL (3.2) with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}^d\,. \tag{3.3}$$

3.2. **Entropy Solutions.** It is well-known that the deterministic Cauchy problem (3.2), (3.3) admits, for each $u_0 \in L^1(\mathbb{R}^d) \cap BV(\mathbb{R})$, a unique entropy solution (see, e.g., [11, 29, 6]). Moreover, for every $t > 0$, $u(\cdot,t) \in L^1(\mathbb{R}^d)$ and several properties of the (nonlinear) data-to-solution operator

$$S : u_0 \longmapsto u(\cdot,t) = S(t)\,u_0, \quad t > 0$$

will be crucial for our subsequent development. To state these properties of $\{S(t)\}_{t\geq 0}$, we introduce some additional notation: for a Banach-space E with norm $\|\circ\|_E$, and for $0 < T \leq +\infty$, denote by $C(0,T;E)$ the space of bounded and continuous functions from $[0,T]$ with values in E, and by $L^p(0,T;E)$, $1 \leq p \leq +\infty$, the space of strongly measurable functions from $(0,T)$ to E such that for $1 \leq p < +\infty$

$$\|v\|_{L^p(0,T;E)} = \Big(\int_0^T \|v(t)\|_E^p \, dt \Big)^{\frac{1}{p}},$$

respectively, if $p = \infty$,

$$\|v\|_{L^\infty(0,T;E)} = \operatorname{ess} \sup_{0\leq t\leq T} \|v(t)\|_E$$

are finite. The following result is classical (we refer to, eg., [11, 12, 21, 8, 23]).

Theorem 3.1.

1) *For every* $u_0 \in L^\infty(\mathbb{R}^d)$, (3.1) - (3.3) *admits a unique entropy solution* $u \in L^\infty(\mathbb{R}^d \times (0,T)) := L^\infty(0,T;L^\infty(\mathbb{R}^d))$.
2) *For every* $t > 0$, *the (nonlinear) data-to-solution map* $S(t)$ *given by*

$$u(\cdot,t) = S(t)\,u_0$$

satisfies

i) $S(t) : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ *is a (contractive) Lipschitz map, i.e.,*

$$\|S(t)u_0 - S(t)v_0\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}\,. \tag{3.4}$$

ii) $S(t)$ *maps* $(L^1 \cap BV)(\mathbb{R}^d)$ *into* $(L^1 \cap BV)(\mathbb{R}^d)$ *and*

$$TV(S(t)u_0) \leq TV(u_0) \quad \forall u_0 \in (L^1 \cap BV)(\mathbb{R}^d)\,. \tag{3.5}$$

iii) *For every* $u_0 \in (L^\infty \cap L^1)(\mathbb{R}^d)$,

$$\|S(t)u_0\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}\,; \tag{3.6}$$

$$\|S(t)u_0\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}\,. \tag{3.7}$$

iv) *The solution operator* $S(t)$ *is a uniformly continuous mapping from* $L^1(\mathbb{R}^d)$ *into* $C_b(0,\infty;L^1(\mathbb{R}^d))$, *and*

$$\|S(\cdot)u_0\|_{C(0,T;L^1(\mathbb{R}^d))} = \max_{0\leq t\leq T} \|S(t)u_0\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)}\,. \tag{3.8}$$

In our analysis of SCLs with random flux, we will require in particular results on the continuous dependence of entropy solutions on the flux function. There holds ([20, Thm. 4.3]).

Theorem 3.2. *Assume* $u_0, v_0 \in BV(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$*, and* $f(\cdot)$*,* $g(\cdot) \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^d)$*.*

Then the unique entropy solutions u *and* v *of the SCL with initial data* u_0*,* v_0 *and with flux functions* f *and* g *satisfy the Kružkov entropy conditions, and the a-priori continuity estimate*

$$\begin{aligned} \|u(\cdot,t) - v(\cdot,t)\|_{L^1(\mathbb{R}^d)} & \\ \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)} &+ t\min\{TV(u_0), TV(v_0)\}\|f-g\|_{\mathrm{Lip}(\mathbb{R};\mathbb{R}^d)} \end{aligned} \tag{3.9}$$

for every $0 \leq t \leq T$*.*

3.3. **Random Flux.** We are in particular interested in the case that the initial data u_0 and the flux functions f_j in (3.1) are uncertain. Existence and uniqueness in the case of random initial data u_0 and random flux f was proved in [25], [24]. To avoid technicalities, we first address *spatially homogeneous random flux functions* whose realizations are elements of the space $E = \mathrm{Lip}(\mathbb{R};\mathbb{R}^d)$. This space being separable, we define random flux functions in the usual fashion.

Definition 3.3. *A* (spatially homogeneous) random flux *for the SCL* (3.1) *is a random field taking values in the separable Banach space* $E = \mathrm{Lip}(\mathbb{R};\mathbb{R}^d)$*, i.e., a measurable mapping from* $(\Omega, \mathcal{F})$ *to* $(\mathrm{Lip}(\mathbb{R};\mathbb{R}^d); \mathcal{B}(\mathrm{Lip}(\mathbb{R};\mathbb{R}^d)))$*. A* bounded random flux *is a random flux whose* $\mathrm{Lip}(\mathbb{R}^1;\mathbb{R}^d)$*-norm is bounded* $\mathbb{P}$*-a.s., i.e.,*

$$\exists 0 < B(f) < \infty : \quad \|f(\omega;\cdot)\|_{\mathrm{Lip}(\mathbb{R}^1;\mathbb{R}^d)} \leq B(f) \quad \mathbb{P}-a.s.\,. \tag{3.10}$$

We observe that a bounded random flux has finite statistical moments of any order. Of particular interest will be the *second moment of a bounded random flux* (i.e., its "two-point correlation in state-space"). We have the following

Lemma 3.4. *Let* f *be a bounded random flux as in Definition* 3.3 *which belongs to* $L^2(\Omega; \mathrm{Lip}(\mathbb{R};\mathbb{R}^d))$*. Then its covariance function, i.e., its centered second moment defined by*

$$\mathrm{Cov}[f](v,v') := \mathbb{E}\left[(f(\cdot;v) - \mathbb{E}[f(\cdot;v)]) \otimes (f(\cdot;v') - \mathbb{E}[f(\cdot;v')])\right]$$

is well-defined for all $v, v' \in \mathbb{R}$ *and there holds*

$$\mathrm{Cov}[f] \in \mathrm{Lip}(\mathbb{R}\times\mathbb{R};\mathbb{R}^{d\times d}_{sym})$$

Consult [24] for a proof of this.

3.4. **Random Entropy Solution.** Based on Theorem 3.1, we will now formulate (3.1) - (3.3) for random initial data $u_0(\omega;\cdot)$ and random flux $f(\omega;\cdot)$. To this end, we denote $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. We assume given a Lipschitz continuous random flux $f(\omega;u)$ as in Definition 3.3 and random initial data u_0, i.e., a $L^1(\mathbb{R}^d)$-valued random variable which is a $L^1(\mathbb{R}^d)$ measurable map

$$u_0 : (\Omega, \mathcal{F}) \longmapsto \left(L^1(\mathbb{R}^d),\ \mathcal{B}(L^1(\mathbb{R}^d))\right). \tag{3.11}$$

We assume further that

$$u_0(\omega;\cdot) \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d) \quad \mathbb{P}\text{-a.s.}, \tag{3.12}$$

which is to say that

$$\mathbb{P}(\{\omega \in \Omega : u_0(\omega;\cdot) \in (L^\infty \cap BV)(\mathbb{R}^d)\}) = 1. \tag{3.13}$$

Since $L^1(\mathbb{R}^d)$ and $\mathrm{Lip}(\mathbb{R};\mathbb{R}^d)$ are separable, (3.11) is well defined and we may impose for $k \in \mathbb{N}$ the *k-th moment condition*

$$\|u_0\|_{L^k(\Omega;L^1(\mathbb{R}^d))} < \infty, \tag{3.14}$$

where the Bochner spaces with respect to the probability measure are defined in Section 2. Then we are interested in random solutions of the *random scalar conservation law* (RSCL)

$$\begin{cases} \partial_t u(\omega;x,t) + \mathrm{div}_x(f(\omega;u(\omega;x,t))) = 0, & t > 0, \\ u(\omega;x,0) = u_0(\omega;x), \end{cases} \qquad x \in \mathbb{R}^d. \tag{3.15}$$

Definition 3.5. *A random field $u : \Omega \ni \omega \to u(\omega;x,t)$, i.e., a measurable mapping from $(\Omega,\mathcal{F})$ to $C(0,T;L^1(\mathbb{R}^d))$, is a* random entropy solution *of the SCL* (3.15) *with random initial data u_0 satisfying* (3.11) - (3.14) *for some $k \geq 2$ and with a* spatially homogeneous *random flux $f(\omega;u)$ as in Definition* 3.3 *that is statistically independent of u_0, if it satisfies the following,*

(i.) Weak solution: For $\mathbb{P}$-a.e $\omega \in \Omega$, $u(\omega;\cdot,\cdot)$ satisfies

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(u(\omega;x,t)\varphi_t(x,t) + \sum_{j=1}^d f_j(\omega;u(\omega;x,t)) \frac{\partial}{\partial x_j}\varphi(x,t) \Big)\, dxdt + \int_{\mathbb{R}^d} u_0(x,\omega)\varphi(x,0)\, dx = 0, \tag{3.16}$$

for all test functions $\varphi \in C_b^1([0,\infty);C_0^0(\mathbb{R}^d)) \cap C_b([0,\infty);C_0^1(\mathbb{R}^d))$.

(ii.) Entropy condition: For any pair consisting of a (deterministic) entropy η and a (stochastic) entropy flux $Q(\omega;\cdot)$ i.e., η, Q_j with $j = 1,2,\dots,d$ are functions such that η is convex and such that $Q_j'(\omega;\cdot) = \eta' f_j'(\omega;\cdot)$ for all j, and for $\mathbb{P}$-a.e $\omega \in \Omega$, u satisfies the following integral identity,

$$\int_0^\infty \int_{\mathbb{R}^d} \Big(\eta(u(\omega;x,t))\varphi_t(x,t) + \sum_{j=1}^d Q_j(\omega;u(\omega;x,t)) \frac{\partial}{\partial x_j}\varphi(x,t) \Big)\, dxdt + \int_{\mathbb{R}^d} \eta(u_0(\omega;x))\varphi(x,0)\, dx \geq 0, \tag{3.17}$$

for all test functions $0 \leq \varphi \in C_b^1([0,\infty);C_0^0(\mathbb{R}^d)) \cap C_b([0,\infty);C_0^1(\mathbb{R}^d))$.

In [24], we proved

Theorem 3.6. *Consider the SCL* (3.1) - (3.3) *with spatially homogeneous, bounded random flux $f : \Omega \to \mathrm{Lip}(\mathbb{R};\mathbb{R}^d)$ as in Definition* 3.3 *and with (independent of f) random initial data $u_0 : \Omega \to L^1(\mathbb{R}^d)$ satisfying* (3.12), (3.13) *and the k-th moment condition* (3.14) *for some integer $k \geq 2$. In particular, then, there exists a constant $\bar{R} < \infty$ such that*

$$\|u_0(\omega;\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \bar{R} \qquad \mathbb{P} - a.e.\ \omega \in \Omega. \tag{3.18}$$

Then there exists a unique random entropy solution $u : \Omega \ni \omega \to C_b(0,T;L^1(\mathbb{R}^d))$ which is "pathwise", i.e., for $\mathbb{P} - a.e.\omega \in \Omega$, described in terms of a nonlinear mapping $S(\omega;t)$ depending only on the random flux, such that

$$u(\omega;\cdot,t) = S(\omega;t)u_0(\omega;\cdot), \quad t > 0,\ \mathbb{P} - a.e.\omega \in \Omega \tag{3.19}$$

such that for every $k \geq m \geq 1$ *and for every* $0 \leq t \leq T < \infty$

$$\|u\|_{L^k(\Omega;C(0,T;L^1(\mathbb{R}^d)))} \leq \|u_0\|_{L^k(\Omega;L^1(\mathbb{R}^d))},$$

$$\|S(\omega;t)u_0(\omega;\cdot)\|_{(L^1\cap L^\infty)(\mathbb{R}^d)} \leq \|u_0(\chi;\cdot)\|_{(L^1\cap L^\infty)(\mathbb{R}^d)}$$

and such that we have $\mathbb{P}$*-a.s.*

$$TV(S(\omega;t)u_0(\omega;\cdot)) \leq TV(u_0(\omega;\cdot)).$$

and, with $\bar{R}$ *as in* (3.18)*,*

$$\sup_{0\leq t\leq T} \|u(\omega;\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leq \bar{R} \quad \mathbb{P}-a.e.\ \omega \in \Omega\ .$$

Theorem 3.6 generalizes the existence of random entropy solutions for random initial data from [25] to the case where the flux function in (3.2) was assumed to be deterministic. It ensures the existence of a unique random entropy solution $u(\omega;x,t)$ with finite k-th moments for bounded random flux and for independent random initial data u_0 provided that $u_0 \in L^k(\Omega,\mathcal{F},\mathbb{P};L^1(\mathbb{R}^d))$ for some $k \geq 2$.

4. Multi Level Monte Carlo Front Tracking

In this section, we present a Multi Level Monte Carlo (MLMC) version of the front tracking approach to the numerical solution of hyperbolic conservation laws with random flux (3.16), (3.17) as developed in [20].

4.1. **The Monte-Carlo Method.** We interpret the Monte-Carlo method as "discretization" of the SCL random data $f(\omega;u)$, $u_0(\omega;x)$ as in (3.11) – (3.13) with respect to ω. We also assume (3.14), i.e., the existence of k-th moments of u_0 for some $k \in \mathbb{N}$, to be specified later. We shall be interested in the statistical estimation of the first and higher moments of u i.e, $\mathcal{M}^k(u) \in (L^1(\mathbb{R}^d))^{(k)}$. For $k = 1$, $\mathcal{M}^1(u) = \mathbb{E}[u]$. The *MC approximation of* $\mathbb{E}[u]$ is defined as follows: given M independent, identically distributed samples $\widehat{u}_0^i$, $i = 1,\ldots,M$, of initial data, the MC estimate of $\mathbb{E}[u(\cdot;\cdot,t)]$ at time t is

$$E_M[u(\cdot,t)] := \frac{1}{M}\sum_{i=1}^{M} \widehat{u}^i(\cdot,t) \tag{4.1}$$

where $\widehat{u}^i(\cdot,t)$ denotes the M unique entropy solutions of the M Cauchy Problems (3.1) – (3.3) with initial data $\widehat{u}_0^i$ and flux samples $\widehat{f}^i(\cdot)$. We observe that by

$$\widehat{u}^i(\cdot,t) = \widehat{S}^i(t)\,\widehat{u}_0^i$$

we have from (3.5) – (3.7) for every M and for every $0 < t < \infty$, by (3.7),

$$\begin{aligned}\|E_M[u(\omega;\cdot,t)]\|_{L^1(\mathbb{R}^d)} &= \Big\|\frac{1}{M}\sum_{i=1}^{M}\widehat{S}^i(t)\widehat{u}_0^i(\cdot;\omega)\Big\|_{L^1(\mathbb{R}^d)}\\ &\leq \frac{1}{M}\sum_{i=1}^{M}\Big\|\widehat{S}^i(t)\,\widehat{u}_0^i(\omega;\cdot)\Big\|_{L^1(\mathbb{R}^d)}\\ &\leq \frac{1}{M}\sum_{i=1}^{M}\big\|\widehat{u}_0^i(\omega;\cdot)\big\|_{L^1(\mathbb{R}^d)}\,.\end{aligned}$$

Using the i.i.d. property of the samples $\{\widehat{u}_0^i\}_{i=1}^M$ of the random initial data u_0, Lemma 2.1 and the linearity of the expectation $\mathbb{E}[\cdot]$, we obtain the bound

$$\mathbb{E}\left[\|E_M[u(\cdot;\cdot,t)]\|_{L^1(\mathbb{R}^d)}\right] \leq \mathbb{E}\left[\|u_0\|_{L^1(\mathbb{R}^d)}\right] = \|u_0\|_{L^1(\Omega;L^1(\mathbb{R}^d))} < \infty.$$

As $M \to \infty$, the MC estimates (4.1) converge and the convergence result from [25] holds as well.

Theorem 4.1. *Assume that in the SCL* (3.1) – (3.3) *the random initial data* u_0 *satisfies*

$$u_0 \in L^2(\Omega; L^1(\mathbb{R}^d))$$

and that the flux $f(\omega; u)$ *is a bounded random flux in the sense of Definition* 3.3. *Assume further that* (3.12) *and* (3.13) *hold.*

Then for every $t > 0$ *the MC estimates* $E_M[u(\cdot,t)]$ *in* (4.1) *converge in* $L^2(\Omega; L^1(\mathbb{R}^d))$ *as* $M \to \infty$, *to* $\mathcal{M}^1(u(\cdot,t)) = \mathbb{E}[u(\cdot,t)]$ *and, for any* $M \in \mathbb{N}$, $0 < t < \infty$, *we have the error bound*

$$\|\mathbb{E}[u(\cdot,t)] - E_M[u(\cdot,t)]\|_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq 2M^{-1/2}\,\|u_0\|_{L^2(\Omega;L^1(\mathbb{R}^d))}\ .$$

4.2. **Front Tracking.** As an exact solution to (3.1) – (3.3) is in general not available, an approximate solution has to be approximated numerically. Here, we investigate using a front tracking method described in [5, 20, 16, 15]. Since the method and the associated convergence analysis differ for the dimensions $d = 1$ and $d > 1$, we treat the two cases separately.

4.2.1. *Front tracking in the one dimensional case.* We start by briefly describing the front tracking algorithm for the deterministic conservation law (3.1) – (3.3) with initial condition u_0 given in $BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Let $\overline{R} := \|u_0\|_{L^\infty(\mathbb{R})}$ and let $\delta > 0$ be a small number. Moreover, set $u_i = \delta i$, for $-\overline{R} \leq i\delta \leq \overline{R}$, and discretize the spatial domain by a grid $\{x_j = j\delta,\, j \in \mathbb{Z}\}$. Then, u_0 is approximated by a piecewise constant function u_0^δ taking in each cell $[j\delta, (j+1)\delta)$ one of the values in $V_\delta := \{u_i \,|\, i \in \mathbb{Z},\, u_i \leq \overline{R}\}$. The flux function f is approximated by the piecewise linear interpolation f^δ,

$$f^\delta(u) = f(u_j) + \frac{f(u_{j+1}) - f(u_j)}{u_{j+1} - u_j}(u - u_j), \qquad u \in [u_j, u_{j+1}), \quad j \in \mathbb{Z}, |j| \leq \overline{R}\delta^{-1}. \tag{4.2}$$

Then we solve the initial value problem

$$u_t^\delta + f^\delta(u^\delta)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T), \tag{4.3a}$$

$$u^\delta(x,0) = u_0^\delta(x), \quad x \in \mathbb{R}, \tag{4.3b}$$

exactly. This means that in each step, we solve the Riemann problems between the states of the piecewise constant function u^δ, then track the discontinuities, called *fronts*, until they interact, solve the emerging Riemann problem and so on. Note that the solution of each Riemann problem is again a piecewise constant function taking values in V_δ because f^δ is piecewise linear with breakpoints $u_i \in V_\delta$. Thus, the (unique) entropy solution $u^\delta(\cdot,t)$ is a piecewise constant function for all $t > 0$. It was shown in [20, Lemma 2.6] that the number of interactions $T(\delta,t)$ between fronts for $t \in (0,\infty)$ is bounded by

$$T(\delta,t) \leq \frac{1}{\delta}(\sharp V_\delta + 1)\ \mathrm{TV}(u^\delta) \leq \frac{1}{\delta}(2\lceil \overline{R}/\delta \rceil + 1)\ \mathrm{TV}(u^\delta) \tag{4.4}$$

so the process terminates. Moreover, the solution u^δ of (4.3) satisfies the Kružkov entropy condition and we have the theorem:

Theorem 4.2 ([20])**.** *For an initial data $u_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and a flux function $f(u) \in \mathrm{Lip}(\mathbb{R})$ which is piecewise C^2, we have*

(i) The solutions u^δ to the differential equation (4.3) are uniformly bounded in δ for all $t \in (0,\infty)$:

$$\|u^\delta(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|u_0\|_{L^\infty(\mathbb{R})}, \quad t \in (0,\infty),$$

(ii) The total variation of u^δ is bounded by the total variation of the initial data for all times $t \in (0,\infty)$,

$$\mathrm{TV}(u^\delta(\cdot,t)) \leq \mathrm{TV}(u_0), \quad t \in (0,\infty),$$

(iii) As the discretization parameter δ goes to zero, the sequence $\{u^\delta\}_{\delta>0}$ converges in $C(0,T;L^1_{\mathrm{loc}}(\mathbb{R}))$ to the unique entropy solution u of (3.1) – (3.3), specifically,

$$\|u(\cdot,t) - u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} \leq \|u_0 - u_0^\delta\|_{L^1(\mathbb{R})} + t\|f - f^\delta\|_{\mathrm{Lip}(\mathbb{R})}\,\mathrm{TV}(u_0) \tag{4.5}$$

Corollary 4.3. *Under the assumptions of Theorem 4.2, we have the following estimate with respect to the discretization parameter δ:*

$$\|u(\cdot,t) - u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} \leq \delta\;\mathrm{TV}(u_0)\left(c + |f|_{W^{2,\infty}(\mathbb{R})}\right). \tag{4.6}$$

Proof. Note first that

$$\|f - f^\delta\|_{\mathrm{Lip}(\mathbb{R})} \leq \delta\|f''\|_{L^\infty(\mathbb{R})} = \delta|f|_{W^{2,\infty}(\mathbb{R})}, \tag{4.7}$$

and

$$\|u_0 - u_0^\delta\|_{L^1(\mathbb{R})} \leq \delta\, c\;\mathrm{TV}(u_0),$$

where $c > 0$ is a constant independent of δ. Then (4.6) follows using (4.7) and (4.5). □

In order to gain optimal convergence rates in the multilevel Monte Carlo front tracking (MCML-FT) algorithm, which we are going to introduce in the next section, it will be useful to have convergence rates of the front tracking algorithm with respect to the amount of computational *work* of the algorithm when the discretization is refined.

Definition 4.4. *By the* (computational) work *or* cost *of an algorithm, we mean the number of floating point operations performed during the execution of the algorithm. We assume that this is proportional to the run time of the algorithm.*

Lemma 4.5 (Work estimate)**.** *Under the assumptions of Theorem 4.2, the front tracking approximation u^δ satisfies the following estimate with respect to the total cost W_δ^{FT} of the front tracking algorithm,*

$$\begin{aligned} \|u(\cdot,t) - u^\delta(\cdot,t)\|_{L^1(\mathbb{R})} &\leq C\;\mathrm{TV}(u_0) \\ &\times \left(1 + \|f\|_{W^{2,\infty}(\mathbb{R})}\right)\left(\left(\|u_0\|_{L^\infty} + 1\right)\left(\mathrm{TV}(u_0) + \|u_0\|_{L^\infty}\right)\right)^{1/2}\left(W_\delta^{FT}\right)^{-1/2}. \end{aligned} \tag{4.8}$$

Proof. Theorem 4.2 implies in particular that we have for the total number of interactions (4.4),

$$T(\delta,t) \leq \frac{1}{\delta}(2\lceil \overline{R}/\delta\rceil + 1)\;\mathrm{TV}(u_0^\delta) \leq \frac{C}{\delta^2}(\|u_0\|_{L^\infty(\mathbb{R})} + 1)\;\mathrm{TV}(u_0), \tag{4.9}$$

and that the number of different Riemann problems that might be solved during the execution of the algorithm is bounded by $4\lceil \overline{R}/\delta \rceil^2$. We use Algorithm 1, which is a modification of Graham's scan [13] used to compute the convex hull of a set of points in the plane, to calculate all the solutions of the Riemann problems with left state $u_i = i\delta$, right state $u_j = j\delta$, $L \le i < j \le R$, where L, R are chosen such that $u_L = \min V_\delta$, $u_R = \max V_\delta$ (a similar algorithm can be used to compute the solutions to the Riemann problems with left state $u_i = i\delta$, right state $u_j = j\delta$, $R \le j < i \le L$). It can easily be verified (see [13]) that the cost of the execution of Algorithm 1 is bounded by $C\,\overline{R}^2\,\delta^{-2}$, where C is a constant independent of $\overline{R}$ and δ, for the input $\delta > 0$, $L = -\lceil \overline{R}/\delta \rceil$, $R = \lceil \overline{R}/\delta \rceil$.

So, if the solutions to all possible Riemann problems are computed and stored in advance, the work W_δ^{FT} to compute the front tracking apprximation $u^\delta(\cdot, t)$ is bounded by $C(\|u_0\|_{L^\infty}+1)(\mathrm{TV}(u_0)+\|u_0\|_{L^\infty})\,\delta^{-2}$, for a constant $C > 0$, uniformly in $t \in (0, \infty)$. We thus obtain (4.8)

□

Algorithm 1 Compute Riemann problems with $u_L \le u_i < u_j \le u_R$

Input: $\delta > 0$, $L < R \in \mathbb{Z}$, (u_L smallest value of u, u_R largest value of u), $\underline{f} = [f_L, \dots, f_R]$, ($f_i = f(u_i)$, $L \le i \le R$)

Output: $U_{i,j} = [u_i, \dots, u_j]$ (states present in solution of RP with left state u_i and right state u_j), $s_{i,j} = [s^1_{i,j}, \dots, s^{k_{ij}}_{i,j}]$ (vector of shock speeds (in increasing order) present in RP with left state u_i and right state u_j, $k_{ij} \in \mathbb{N}$), $L \le i < j \le R$

```
for i = L to R do
    û ← [i, i+1]
    ŝ ← (f_{i+1} − f_i)/δ
    s_{i,i+1} ← ŝ
    U_{i,i+1} ← δ · û
    k ← i + 2
    while k ≤ R do
        sl ← (f_k − f_{û(end)})/(δ(k − û(end)))
        if ŝ = [ ] or sl > ŝ(end) then
            ŝ ← [ŝ, sl]
            û ← [û, k]
            s_{i,k} ← ŝ
            U_{i,k} ← δ · û
            k ← k + 1
        else
            ŝ ← ŝ(1 : end − 1)
            û ← û(1 : end − 1)
        end if
    end while
end for
```

Remark 4.6. *Note that the work W_δ^{FT} to compute the front tracking approximation is of the same order as the work we would need to compute an approximation of the solution by a finite volume scheme on a grid with cells of diameter $\mathcal{O}(\delta)$. But* *due to the better convergence rate with respect to the discretization parameter* δ*, which is of order 1 whereas it is proved to be of order* $1/2$ *for the finite volume approximation, we obtain the improved convergence rate* (4.8) *with respect to work.*

Remark 4.7 (Work estimates for convex flux functions)**.** *If the flux function* f *is convex, the work estimate can be improved. This is because in this case, the number of interactions* $T(\delta, t)$ *can be bounded by the sum of the sizes of the jumps in the initial data. That is, given* u_0 *there holds, for every* $t > 0$ *and* $\delta > 0$*,*

$$T(\delta, t) \leq \frac{1}{\delta} \operatorname{TV}(u_0)$$

(see [20, Lemma 2.6]*), since for a convex flux function, the number of fronts is strictly decreasing at each interaction. Moreover, the solution of each Riemann problem is either a shock wave or a rarefaction wave depending on whether* $u_L > u_R$ *or* $u_L < u_R$*, and we do not need to compute the convex envelope of the flux function.*

So, the solution of one Riemann problem can be computed with a cost proportional to δ*. Thus the total work* W_δ^{FT} *to compute the front tracking approximation reduces to*

$$W_\delta^{FT} \leq C \operatorname{TV}(u_0)\, \delta^{-1}$$

and we obtain the improved convergence rate of the FT method with respect to work,

$$\|u(\cdot, t) - u^\delta(\cdot, t)\|_{L^1(\mathbb{R})} \leq C \operatorname{TV}(u_0)^2 (1 + \|f\|_{W^{2,\infty}(\mathbb{R})})\, (W_\delta^{FT})^{-1}. \tag{4.10}$$

Clearly, the same rate holds also for concave fluxes.

4.2.2. *Front tracking for* $d \geq 2$ *and dimensional splitting.* Front tracking in several space dimensions is based on the method of fractional steps (or dimensional splitting) introduced by Bagrinovskiĭ and Godunov [1] and later on extended by various authors, see e.g. [17] and the references therein. An introduction to splitting methods in general can be found in the book [17]. Here, we will use the dimensional splitting method in combination with the front tracking algorithm for one space dimension as described in the previous subsection 4.2.1. To describe the method, we introduce some notation. We discretize the spatial domain by a Cartesian grid $\{j\Delta x_i, j \in \mathbb{Z}\}$, $i = 1, \dots, d$ in each direction and denote by $I_{j_1,\dots,j_d}$ the grid cell

$$I_{j_1,\dots,j_d} = \left\{(x_1, \dots, x_d) \;\middle|\; j_i \Delta x_i \leq x_1 < (j_i + 1)\Delta x_i \;\text{ for } i = 1, \dots, d\right\}\,.$$

Moreover, we denote the projection operator $\pi_\delta := P_\delta \circ \overline{P}_{\Delta x}$ for a function $u \in L^1(\mathbb{R}^d)$ to be the composition of the projection $\overline{P}_{\Delta x}$ of the function on the cell averages,

$$\overline{P}_{\Delta x} u(x) = \frac{1}{\Delta x_1 \cdots \Delta x_d} \int_{I_{j_1,\dots,j_d}} u \, dx, \quad x = (x_1, \dots, x_d) \in I_{j_1,\dots,j_d}, \tag{4.11}$$

and a projection P_δ of the cell averages onto the values in V_δ. Furthermore, we let f_i^δ, $i = 1, \dots, d$, denote the continuous piecewise linear approximations to f_i, $i = 1, \dots, d$, as in (4.2). We set $\eta = (\delta, \Delta x_1, \dots, \Delta x_d, \Delta t)$ and let u^0 denote the projection of u_0 on the grid, that is $u^0 = \pi_\delta u_0$. Let $S^{f_i^\delta, x_i}(t)$ denote the solution operator of the scalar conservation law in one dimension, viz.,

$$\begin{aligned}
(v_i^\delta)_t + f_i^\delta (v_i^\delta)_{x_i} &= 0, \quad (x_i, t) \in \mathbb{R} \times (0, T), \\
v_i^\delta(x_i, 0) &= v_{i0}^\delta(x_i), \quad x_i \in \mathbb{R},
\end{aligned}$$

that is, we write $v(x_i,t) = S^{f_i^\delta,x_i}(t)v_{i0}^\delta$. Since v_{i0}^δ is piecewise constant, and f_i^δ piecewise linear, the solution can be calculated using front tracking.

Then we obtain an approximation of the solution to (3.1) – (3.3) by successively applying the front tracking solution operator $S^{f_i^\delta,x_i}(t)$ followed by the projection operator π_δ (in order to prevent the number of discontinuities from growing excessively). We denote the approximate solutions at the timesteps $t_r = r\Delta t$, $t \in \mathbb{Q}$ by

$$u^{n+i/d} = \pi_\delta \circ S^{f_i^\delta,x_i}(\Delta t)\, u^{n+(i-1)/d}, \quad i = 1,\dots,d,\; n \in \mathbb{N},$$

and

(4.12)
$$u^\eta(x,t) = \begin{cases} S^{f_i^\delta,x_i}(d(t - t_{n+(i-1)/d}))\, u^{n+(i-1)/d}, & t \in [t_{n+(i-1)/d}, t_{n+i/d}), \\ u^{n+i/d}, & t = t_{n+i/d}, \end{cases}$$

$i = 1,\dots,d$ and $n \in \mathbb{N}$. The approximation u^η satisfies (see [20][Chapter 4]):

Theorem 4.8. *Let $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $f_i(u) \in \mathrm{Lip}(\mathbb{R})$ and piecewise C^2. Then the function u^η defined in* (4.12) *satisfies*

(i) Uniform bound in $\eta = (\delta, \Delta x_1, \dots, \Delta x_d, \Delta t)$ for all $t \in (0,\infty)$:
$$\|u^\eta(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad t \in (0,\infty),$$

(ii) The total variation of u^η is bounded by the total variation of the initial data for all times $t \in (0,\infty)$,
$$\mathrm{TV}(u^\eta(\cdot,t)) \le \mathrm{TV}(u_0), \quad t \in (0,\infty),$$

(iii) For any sequence $\{\eta_j\}_{j\in\mathbb{N}}$, where $\eta_j \to 0$ when $j \to \infty$, satisfying
$$\max_{i=1,\dots,d} \Delta x_i/\Delta t \le K < \infty,$$
the corresponding sequence $\{u^{\eta_j}\}_{j\in\mathbb{N}}$ converges in $C(0,T; L^1_{\mathrm{loc}}(\mathbb{R}^d))$ to the unique entropy solution u of (3.1)–(3.3)*. Specifically, we have, denoting $\|f\|_{\mathrm{Lip}} = \max_{i=1,\dots,d} \|f_i\|_{\mathrm{Lip}(\mathbb{R})}$ and $\Delta x = \max_{i=1,\dots,d} \Delta x_i$,*

(4.13)
$$\begin{aligned} &\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \\ &\quad\le \|u_0 - u^0\|_{L^1(\mathbb{R}^d)} + t\|f - f^\delta\|_{\mathrm{Lip}(\mathbb{R})}\,\mathrm{TV}(u_0) \\ &\quad + 2\,\mathrm{TV}(u_0)\,\sqrt{2t}\,(\sqrt{d}+1)\sqrt{d\Delta x^2/\Delta t + \Delta x\|f\|_{\mathrm{Lip}} + \Delta t\|f\|_{Lip}^2}\,. \end{aligned}$$

Corollary 4.9. *Under the assumptions of Theorem* 4.8 *and choosing the parameters Δx, Δt and δ as*

$$\Delta x = k_1 \Delta t = k_2 \delta^2, \tag{4.14}$$

where $k_1, k_2 > 0$ are constants, the dimensional splitting front tracking algorithm converges at rate 1 *in the parameter δ, specifically,*

$$\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\,\delta\,(1+t)\Big(1 + \|f\|_{W^{2,\infty}(\mathbb{R};\mathbb{R}^d)}\Big)\,\mathrm{TV}(u_0), \tag{4.15}$$

where $C > 0$ is a constant depending at most linearly on d.

Proof. Using similarly as in Corollary 4.3 that the approximation u^0 of the initial data u_0 satisfies

$$\|u_0 - u^0\|_{L^1(\mathbb{R}^d)} \le c\,d\,\delta\,\mathrm{TV}(u_0),$$

and (4.7), (4.13) yields a convergence rate with respect to the parameters Δx, Δt and δ,

$$\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq \Big(c\, d\, \delta + t\, \delta\, \|f\|_{W^{2,\infty}(\mathbb{R};\mathbb{R}^d)} + 2\sqrt{2t}\,(\sqrt{d}+1)\sqrt{d\Delta x^2/\Delta t + \Delta x\|f\|_{\mathrm{Lip}} + \Delta t\|f\|_{\mathrm{Lip}}^2}\,\Big)\,\mathrm{TV}(u_0).$$

We see that this yields 4.15 if we choose Δx, Δt and δ as in (4.14). □

We next estimate the convergence rate of the dimensional splitting front tracking algorithm with respect to the work needed to compute one approximation of the solution.

Lemma 4.10. *(Work estimate for $d \geq 2$) Under the assumptions of Theorem* 4.8 *and* (4.14)*, the front tracking approximation satisfies,*

$$\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq C\,(1 + t^{(2d+3)/(2d+2)})\Big(1 + \|f\|_{W^{2,\infty}}\Big) \times \mathrm{TV}(u_0)\,\Big((\|u_0\|_{L^\infty}+1)\,(\|u_0\|_{L^\infty} + \mathrm{TV}(u_0))\Big)^{1/(2(d+1))} (W_{\delta,d}^{FT})^{-1/(2(d+1))}\,, \tag{4.16}$$

where $C > 0$ is a constant depending only on d.

Proof. The work done in one time interval $(t_{n+(i-1)/d}, t_{n+i/d}]$ consists of two components, the front tracking approximation in $(t_{n+(i-1)/d}, t_{n+i/d})$ and the projections at time $t = t_{n+i/d}$. As in the one-dimensional case, we can solve all possible Riemann problems beforehand and store the solutions, the work to do this is of order $C\,\overline{R}^2\, d\,\delta^{-2}$, where $\overline{R} = \|u_0\|_{L^\infty}$, since the flux f has d components f_i (see Remark 4.5). Then the work for the front tracking approximation in $(t_{n+(i-1)/d}, t_{n+i/d})$ is of the order of the number of interactions of fronts $T(\eta,t)$ in that time interval. This number is bounded by

$$T(\eta,t) \leq C\,(\|u_0\|_{L^\infty}+1)\,(\mathrm{TV}(u_0) + \|u_0\|_{L^\infty})\,\delta^{-2}(\Delta x)^{-(d-1)},$$

which is (4.9) multiplied by $(\Delta x)^{-(d-1)}$, because we do the front tracking in each segment $I^i_{j_1,\dots,j_d} := [j_1\Delta x, (j_1+1)\Delta x) \times \cdots \times [j_{i-1}\Delta x, (j_{i-1}+1)\Delta x) \times \mathbb{R} \times \cdots \times [j_d\Delta x, (j_d+1)\Delta x)$. The work $W^{\pi_\delta}_{t_{n+i/d}}$ needed to do the projections at time $t_{n+i/d}$ is of the same order,

$$W^{\pi_\delta}_{t_{n+i/d}} = C(\|u_0\|_{L^\infty(\mathbb{R})}+1)\,(\mathrm{TV}(u_0) + \|u_0\|_{L^\infty})\,\delta^{-2}(\Delta x)^{-(d-1)},$$

as it is proportional to the number of fronts in the x_i-direction and the number of segments $I^i_{j_1,\dots,j_d}$. Hence the total work $W^{FT}_{\delta,d}$ needed to compute the front tracking approximation $u^\eta(\cdot,t)$ is of order

$$W^{FT}_{\delta,d} = C\,t\,d\,(\|u_0\|_{L^\infty(\mathbb{R})}+1)\,(\mathrm{TV}(u_0) + \|u_0\|_{L^\infty})\,\delta^{-2}(\Delta x)^{-(d-1)}(\Delta t)^{-1}\,.$$

Now using (4.14), we obtain the convergence estimate with respect to work, (4.16). □

Remark 4.11. *Observe that the convergence rate* (4.16) *is of the same order with respect to the work $W^{FT}_{\delta,d}$ as the one for the approximation by a finite volume scheme (see e.g.* [25]*). So in contrast to the one-dimensional case we do not get an improvement of the rate by using the front tracking method.*

Remark 4.12 (Work estimate for convex flux functions)**.** *As in the case $d = 1$, the estimate on the total work $W^{FT}_{\delta,d}$ can be improved if the components f_i, $i = 1, \dots, d$ of the flux function are convex. Again, solving a Riemann problem with left state u_L and right state u_R reduces to checking whether $u_L > u_R$. Moreover, the total number of interactions in each time interval $t \in (t_{n+(i-1)/d}, t_{n+i/d})$ is bounded by $T(\eta, t) \le \mathrm{TV}(u_0)\delta^{-1}$ and therefore,*

$$\|u(\cdot,t) - u^\eta(\cdot,t)\|_{L^1(\mathbb{R}^d)} \le C\left(1 + t^{(2d+2)/(2d+1)}\right) \times \left(1 + \|f\|_{W^{2,\infty}}\right) \mathrm{TV}(u_0)^{(2d+2)/(2d+1)} \left(W^{FT}_{\delta,d}\right)^{-1/(2d+1)},$$

for convex or concave flux functions.

4.2.3. *Front tracking for RSCLs.* Having described the convergence properties of the front tracking algorithm for deterministic scalar conservation laws, we are ready to state the convergence result for the approximation of the random scalar conservation law (3.15):

Theorem 4.13. *Assume that $f(\omega; u) \in \mathrm{Lip}(\mathbb{R}; \mathbb{R}^d)$ in the sense of Definition 3.3, so that $f(\omega; \cdot) \in C^2([-\overline{R}, \overline{R}])$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, and the initial data $u_0 : (\Omega, \mathcal{F}) \to (L^1(\mathbb{R}^d, \mathcal{B}(L^1(\mathbb{R}^d))$ satisfying (3.12) and (3.18). Let f_i^δ denote the piecewise linear interpolations to f_i as in (4.2).*

For $\delta > 0$, f_i^δ denotes the piecewise linear interpolations of the flux function component f_i, as defined in (4.2). Let the discretization parameter vector $\eta = \delta$ if $d = 1$, and $\eta = (\delta, \Delta x_1, \dots, \Delta x_d, \Delta t)$ if $d > 1$, and let $u^\eta(\omega; \cdot, \cdot)$ denote the corresponding approximate solution defined by (4.3a) if $d = 1$ and (4.12) if $d > 1$, with initial data $u_0(\omega; \cdot)$ and flux functions $f_1(\omega; \cdot), \dots, f_d(\omega; \cdot)$. Then the approximations u^η satisfy

$$\|u^\eta(\cdot; \cdot, t)\|_{L^\infty(\Omega; L^\infty(\mathbb{R}^d))} \le \overline{R}, \quad t \in (0, \infty),$$

the total variation is bounded $\mathbb{P}$-almost surely,

$$TV(u^\eta(\omega; \cdot, t)) \le \mathrm{TV}(u_0(\omega; \cdot)), \quad t \in (0, \infty),\ \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

And, as $\eta \to 0$, the sequence $(u^\eta)_{\eta > 0}$ converges $\mathbb{P}$-almost surely and in $C(0, T; L^1(\mathbb{R}^d))$, to the unique random entropy solution of the RSCL (3.15). Moreover, if $d = 1$, we have the error bound

$$\|u(\omega; \cdot, t) - u^\eta(\omega; \cdot, t)\|_{L^1(\mathbb{R})} \le \|u_0(\omega; \cdot) - u^0(\omega; \cdot)\|_{L^1(\mathbb{R})} + t\|f(\omega; \cdot) - f^\delta(\omega; \cdot)\|_{\mathrm{Lip}(\mathbb{R})} \mathrm{TV}(u_0(\omega; \cdot)),$$

and if $d > 1$, we have

$$\begin{aligned} &\|u(\omega; \cdot, t) - u^\eta(\omega; \cdot, t)\|_{L^1(\mathbb{R}^d)} \\ &\le \|u_0(\omega; \cdot) - u^0(\omega; \cdot)\|_{L^1(\mathbb{R}^d)} + t \max_{i=1,\dots,d} \|f_i(\omega; \cdot) - f_i^\delta(\omega; \cdot)\|_{\mathrm{Lip}(\mathbb{R})} \mathrm{TV}(u_0(\omega; \cdot)) \\ &+ 2\ \mathrm{TV}(u_0(\omega; \cdot)) \sqrt{2t}\,(\sqrt{d} + 1)\sqrt{d\Delta x^2/\Delta t + \Delta x\|f(\omega; \cdot)\|_{\mathrm{Lip}} + \Delta t\|f(\omega; \cdot)\|^2_{Lip}}. \end{aligned} \tag{4.17}$$

Proof. The assertion follows from Theorems 4.2 and 4.8 upon noting that the assumptions given there are satisfied pathwise, i.e., for $\mathbb{P}$-a.e. $\omega \in \Omega$. □

Corollary 4.14. *Choosing $\Delta x = k_1\delta$ for $d = 1$ and $\Delta x = k_1\Delta t = k_2\delta^2$ for $d \geq 2$,* (4.17) *implies the $\mathbb{P}$-a.s. convergence estimate with respect to δ (c.f. Corollaries* 4.3 *and* 4.9*)*

$$\|u(\omega;\cdot,t) - u^\eta(\omega;\cdot,t)\|_{L^1(\mathbb{R}^d)} \leq C\,\delta\,(1+t)\Big(1 + \|f(\omega;\cdot)\|_{W^{2,\infty}(\mathbb{R};\mathbb{R}^d)}\Big)\,\mathrm{TV}(u_0(\omega;\cdot))\;. \tag{4.18}$$

If in addition $u_0 \in L^p(\Omega; BV(\mathbb{R}^d))$ and $f \in L^q(\Omega; W^{2,\infty}(\mathbb{R};\mathbb{R}^d))$ for some $1 \leq p, q \leq \infty$ with $1/p + 1/q = 1$, we have

$$\begin{aligned}\|\mathbb{E}[u(t)] - \mathbb{E}[u^\eta(t)]\|_{L^1(\mathbb{R}^d)} &\leq \|u(t) - u^\eta(t)\|_{L^1(\Omega;L^1(\mathbb{R}^d))}\\ &\leq C\,\delta\,(1+t)\Big(1 + \|f\|_{L^q(\Omega;W^{2,\infty})}\Big)\|\,\mathrm{TV}(u_0)\|_{L^p(\Omega)},\end{aligned} \tag{4.19}$$

for all δ and $t > 0$.

Proof. The inequality (4.19) is proved by an application of Hölder's inequality to (4.18). □

4.3. **Multilevel Flux Decomposition.** The approximate, continuous, piecewise linear flux functions f_i^δ defined by (4.2) are particular useful in connection with empirical flux data (such as typically arise in Buckley-Leverett models where flux functions are built from empirical data) and with MLMC, as will be seen in the next subsection.

To this end, we from now on assume that

$$f(\omega;\cdot) \in W^{2,\infty}([-\overline{R},\overline{R}]) \quad \mathbb{P} - a.e.\ \omega \in \Omega, \tag{4.20}$$

where $\overline{R}$ is a bound on $\|u_0\|_{L^\infty(\mathbb{R}^d)}$. Next we choose $\delta_0 > 0$ and let $\delta_\ell = 2^{-\ell}\delta_0$. Let also $f_i^\ell(\omega;\cdot) := f_i^{\delta_\ell}(\omega;\cdot)$ denote the continuous piecewise linear interpolant of $f_i(\omega;\cdot)$, for $i = 1,\dots,d$, as defined by (4.2), and similarly set $f^\ell := (f_1^\ell,\dots,f_d^\ell)$.

Lemma 4.15. *Under assumption* (4.20)*, for $\ell = 0,1,2,\dots$, the continuous, piecewise linear flux interpolants $f_i^\ell(\omega;\cdot) = f_i^{\delta_\ell}(\omega;\cdot)$ are bounded random flux functions in the sense of Defition* 3.3 *which, moreover, satisfy the bound* (3.10) *with some finite constant $B(f)$ which is bounded independently of ℓ, and which satisfy for $\mathbb{P}$-a.e. $\omega \in \Omega$ the error bound*

$$\|f_i(\omega;\cdot) - f_i^\ell(\omega;\cdot)\|_{\mathrm{Lip}([-\overline{R},\overline{R}];\mathbb{R}^d)} \leq C2^{-\ell}\|\partial_u^2 f_i(\omega;\cdot)\|_{L^\infty([-\overline{R},\overline{R}])} \tag{4.21}$$

Proof. The proof of (4.21) follows from standard approximation estimates for the nodal interpolation. □

The following corollary is a direct consequence of (4.21).

Corollary 4.16. *Under the assumptions of Lemma* 4.15*, we have*

$$\|(f_i^\ell - f_i^{\ell-1})(\omega;\cdot)\|_{\mathrm{Lip}([-\overline{R},\overline{R}];\mathbb{R}^d)} \leq 2C2^{-\ell}\|\partial_u^2 f_i(\omega;\cdot)\|_{L^\infty([-\overline{R},\overline{R}];\mathbb{R})}.$$

Here, the constant $C > 0$ is independent of ℓ and of the flux f.

4.4. **MLMC Front Tracking.** The MLMC discretization of differential equations with random inputs was proposed by M. Giles in [9, 10], upon earlier work by Heinrich on numerical integration in [14]. For random scalar conservation laws (RSCLs), the MLMC Finite Volume discretizations were proposed and analyzed, in the case of deterministic flux and random initial conditions, in [25], and for RSCLs with random flux, in [24].

Here, we analyze the convergence of MLMC in conjunction with Front Tracking (FT) discretizations. Although the analysis proceeds, broadly speaking, along the lines of what was done in [25, 24], there are notable differences: First, unlike [24], there is no need for a principal component analysis of the random flux, e.g. via a Karhunen–Loève expansion. Secondly, we propose the use of a *multiresolution decomposition of the random flux* on the phase space of the solution. Finally, the error bounds which we shall obtain relate, in a rather explicit fashion, the number M_ℓ of MC samples on different discretization levels to the flux variance at resolution ℓ, i.e., to $\|f^\ell - f^{\ell-1}\|^2_{L^2(\Omega;\mathrm{Lip}(\mathbb{R},\mathbb{R}^d))}$. Since f^ℓ is piecewise linear, this quantity can easily be computed for empirically calibrated random flux functions and, thereby, the number M_ℓ of "samples" (which are approximate solutions of the RSCL with flux functions f^ℓ and $f^{\ell-1}$, obtained by front tracking), can be scaled accordingly.

We start the analysis by introducing some notation. For $d = 1$, we let $\Delta x_\ell = \delta_l = 2^{-\ell}\delta_0$ for some $\delta_0 > 0$. For $d \geq 2$, $\ell = 0, 1, 2, \ldots$, we set

$$\eta_\ell = (\delta_\ell, \Delta x_\ell, \Delta t_\ell) = \left(2^{-\ell}\delta_0, 2^{-2\ell}\Delta x_0, 2^{-2\ell}\Delta t_0\right).$$

Moreover, we let $u_0^\ell(\omega;\cdot) := \pi_\ell u_0(\omega;\cdot)$ where $\pi_\ell = P_{\Delta x_\ell} \circ \overline{P}_{\Delta x_\ell}$, cf. (4.11). Note that we set $\Delta x_1 = \cdots = \Delta x_d = \Delta x_\ell$.

Then we denote for $\ell = 0, 1, 2, ...,$ by $u^\ell(\omega; x, t)$ the approximations of $u(\omega; x, t)$ obtained by the front tracking method with initial data u_0^ℓ and f^ℓ.

As in [25], $E_M[\cdot]$ denotes the sample average of M i.i.d. samples of a random quantity. We are interested in the computation of the statistical mean

$$\mathbb{E}[u(t)] \in C(0, T; L^1(\mathbb{R}^d))$$

of the random entropy solution of the RSCL (3.1) - (3.3). To this end, the MLMC-FT approximation is defined as follows: for a given level $L \in \mathbb{N}$ of refinement, we use the linearity of the mathematical expectation $\mathbb{E}[\cdot]$ to write

$$\mathbb{E}[u(t)] \simeq \mathbb{E}[u^L(t)] = \sum_{\ell=0}^{L} \mathbb{E}\left[u^\ell - u^{\ell-1}\right].$$

Here, and in the following, we adopt the convention that $u^{-1} \equiv 0$.

We next *estimate the expectations of increments for each level of refinement by a level-dependent number M_ℓ of samples*, which results in the MLMC estimate

$$E_L^{MLMC}[u^L(t)] := \sum_{\ell=0}^{L} E_{M_\ell}\left[u^\ell - u^{\ell-1}\right]. \tag{4.22}$$

Here, u^ℓ are the approximations obtained by front tracking for the initial data u_0^ℓ and the flux functions f^ℓ.

4.5. **Convergence Analysis.** We are now interested in estimating

$$\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)].$$

To this end, we write

$$\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)] = \underbrace{\mathbb{E}[u(t)] - \mathbb{E}[u^L(t)]}_{A} + \underbrace{\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]}_{B}.$$

We have already estimated the $L^1(\mathbb{R}^d)$-norm of term A in equation (4.19). In this setting, it is of order $\mathcal{O}(2^{-L})$ under the additional assumption that $u_0 \in L^p(\Omega; BV(\mathbb{R}^d))$ and $f \in L^q(\Omega; W^{2,\infty}(\mathbb{R}; \mathbb{R}^d))$, where $1/p + 1/q = 1$. Consider now the term B. To estimate it, we write, with $\Delta u^\ell := u^\ell - u^{\ell-1}$ for $\ell = 0, 1, 2, ..., L$ and with the convention that $u^{-1} \equiv 0$,

$$\begin{aligned}
&\|\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\
&\qquad = \Big\|\mathbb{E}\Big[\sum_{\ell=0}^{L}(u^\ell - u^{\ell-1})\Big] - E_L^{MLMC}[u^L(t)]\Big\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\
&\qquad = \Big\|\sum_{\ell=0}^{L}\left\{\mathbb{E}[\Delta u^\ell] - E_{M_\ell}[\Delta u^\ell]\right\}\Big\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}.
\end{aligned}$$

Expanding the square, and interpreting the M_ℓ samples as i.i.d. copies of the random variable $u^\ell(\omega; x, t)$, we obtain

$$\|\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} = \sum_{\ell=0}^{L}\big\|\mathbb{E}[\Delta u^\ell] - E_{M_\ell}[\Delta u^\ell]\big\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}.$$

Next we estimate each term in the sum as follows:

$$\begin{aligned}
B_\ell &:= \big\|\mathbb{E}[\Delta u^\ell] - E_{M_\ell}[\Delta u^\ell]\big\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\
&= \frac{1}{M_\ell}\mathbb{E}\left[\|\mathbb{E}[\Delta u^\ell(t)] - \Delta u^\ell(t)\|^2_{L^1(\mathbb{R}^d)}\right] \\
&\leq \frac{1}{M_\ell}\|\Delta u^\ell(t)\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}.
\end{aligned}$$

We use the elementary estimate

$$\|\Delta u^\ell(\omega;\cdot,t)\|^2_{L^1(\mathbb{R}^d)} \leq 2\|u(\omega;\cdot,t) - u^\ell(\omega;\cdot,t)\|^2_{L^1(\mathbb{R}^d)} + 2\|u(\omega;\cdot,t) - u^{\ell-1}(\omega;\cdot,t)\|^2_{L^1(\mathbb{R}^d)}$$

and the convergence rate (4.18), to obtain

$$\|u(t) - u^\ell(t)\|_{L^2(\Omega;L^2(\mathbb{R}^d))} \leq C\,\delta_\ell\,(1+t)\Big(1 + \|f\|_{L^2(\Omega;W^{2,\infty})}\Big)\|\operatorname{TV}(u_0)\|_{L^\infty(\Omega)}.$$

under the assumption that $u_0 \in L^\infty(\Omega; BV(\mathbb{R}^d))$ and $f \in L^2(\Omega; W^{2,\infty}(\mathbb{R}; \mathbb{R}^d))$. Thus,

$$B_\ell \leq \frac{1}{M_\ell}C\,\delta_\ell^2\,(1+t^2)\Big(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\Big)\|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)},$$

where $C > 0$ is a constant which depends on d but which is independent of t. Summing over $\ell = 0, \ldots, L$, we arrive at

$$\begin{aligned}
&\|\mathbb{E}[u^L(t)] - E_L^{MLMC}[u^L(t)]\|^2_{L^2(\Omega;L^1(\mathbb{R}))} \\
&\qquad \leq C\,(1+t^2)\sum_{\ell=0}^{L}\frac{1}{M_\ell}\,\delta_\ell^2\,\Big(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\Big)\|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)}.
\end{aligned}$$

We can now state our basic MLMC-FT error bound.

Theorem 4.17. *Consider the RSCL* (3.11) - (3.19) *with random initial data* u_0 *satisfying* (3.11) -(3.13)*, and with a bounded random flux in the sense of Definition* 3.3 *such that* $f \in L^2(\Omega; W^{2,\infty}([-\overline{R}, \overline{R}]))$ *where* $\overline{R}$ *is defined in* (3.18)*, such that* f *is statistically independent of* u_0*.*

Then, for any $L \in \mathbb{N}$ *and for any choice of samples sizes* $\{M_\ell\}_{\ell=0}^L$ *in the MLMC-FT estimator* $E_L^{MLMC}[u^L(t)]$ *in* (4.22) *we have the error bound*

$$
\begin{aligned}
\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))}
&\leq 2\,C(1+t^2)\delta_L^2 \left(1 + \|f\|^2_{L^1(\Omega;W^{2,\infty})}\right) \|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)} \\
&\quad + C\,(1+t^2) \sum_{\ell=0}^L \frac{1}{M_\ell}\,\delta_\ell^2\, \left(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\right) \|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)} \\
&\leq C\Big[2^{-2L} + \sum_{\ell=0}^L M_\ell^{-1} 2^{-2\ell}\Big](1+t^2) \\
&\qquad \times \left(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\right) \|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)}\,.
\end{aligned}
$$

With the particular choice

$$M_\ell = 2^{2(L-\ell)}\,, \quad \ell = 0, \dots, L,$$

we find for any $0 \leq t \leq T < \infty$ *the bound*

$$
\begin{aligned}
\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \\
\leq C\,L 2^{-2L}\,(1+t^2)\Big(1 + \|f\|^2_{L^2(\Omega;W^{2,\infty})}\Big) \|\operatorname{TV}(u_0)\|^2_{L^\infty(\Omega)}.
\end{aligned}
\tag{4.23}
$$

Proof. The proof follows from the foregoing analysis. □

If we denote the work for one FT solution at mesh level ℓ by W_ℓ^{FT}, and use the front tracking work estimates in Lemmas 4.5 and 4.10, we obtain the work estimate $W_{L,MLMC}^{FT}$ for the MLMC front tracking method,

$$
W_{L,MLMC}^{FT} = C \sum_{\ell=0}^L M_\ell W_\ell^{FT} = \begin{cases} \mathcal{O}(W_L^{FT}\,\log W_L^{FT}) = \mathcal{O}(L\,\delta_L^{-2}) & \text{if } d = 1, \\ \mathcal{O}(W_L^{FT}) = \mathcal{O}(\delta_L^{-2(d+1)}) & \text{if } d \geq 2. \end{cases}
\tag{4.24}
$$

This gives us the convergence rates for the MLMC-FT estimator $E_L^{MLMC}[u^L(t))]$ with respect to work:

Corollary 4.18. *Under the assumptions of Theorem* 4.17*, the MLMC-FT estimator* $E_L^{MLMC}[u^L(t))]$ *converges with the following rates to the ensemble average* $\mathbb{E}[u(t)]$ *of the random entropy solution*

$$
\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}))} \leq C\,(\log W_{L,MLMC}^{FT})^2\,(W_{L,MLMC}^{FT})^{-1},
\tag{4.25}
$$

for $d = 1$*, and*

$$
\left\|\mathbb{E}[u(t)] - E_L^{MLMC}[u^L(t)]\right\|^2_{L^2(\Omega;L^1(\mathbb{R}^d))} \leq C\,(\log W_{L,MLMC}^{FT})\,(W_{L,MLMC}^{FT})^{-1/(d+1)}
$$

for $d \geq 2$*, where* $C > 0$ *is a constant depending on* d *and* t*, and on* $\|u_0\|_{L^\infty(\Omega;BV(\mathbb{R}^d))}$ *and* $\|f\|_{L^2(\Omega;W^{2,\infty}(\mathbb{R};\mathbb{R}^d))}$*.*

Remark 4.19. *We have seen in Lemma* 4.7 *that the convergence rate of the deterministic front tracking algorithm for $d = 1$ is one with respect to work, if the flux function f is convex. However, this does not show up as an improvement of the convergence rate of the MLMC-FT method, since in this case the work of the Monte Carlo method dominates. Specifically, in the case of a convex flux and $d = 1$, we have*

$$\begin{aligned} W_{L,MLMC}^{FT} &= C\sum_{\ell=0}^{L} M_\ell\, W_\ell^{FT} \leq C\sum_{\ell=0}^{L} M_\ell\, \delta_\ell^{-1} \\ &\leq C\, 2^{2L}\sum_{\ell=0}^{L} 2^{-2\ell}\, 2^{\ell} \leq C\, 2^{2L} = \mathcal{O}(\delta_L^{-2}), \end{aligned} \tag{4.26}$$

which is the same effort as in the general case (4.24) *apart from the missing factor L.*

This is to be contrasted to several space dimensions, where we have a small gain in convergence rate if all the flux components f_j, $j = 1, \ldots, d$ are convex, since the convergence rate of the deterministic dimensional splitting front tracking method is worse than that of the Monte Carlo method:

$$\begin{aligned} W_{L,MLMC}^{FT} &= C\sum_{\ell=0}^{L} M_\ell\, W_\ell^{FT} \leq C\sum_{\ell=0}^{L} M_\ell\, \delta_\ell^{-(2d+1)} \\ &\leq C\, 2^{2L}\sum_{\ell=0}^{L} 2^{(-1+2d)\ell} \leq C\, 2^{(2d+1)L} = \mathcal{O}(\delta_L^{-(2d+1)}). \end{aligned}$$

5. Numerical Experiments

In this section, we test the performance of the MLMC-FT method on several examples with random fluxes in one and two space dimensions.

5.1. Convex random flux in one space dimension.

We consider the random scalar conservation law,

$$u_t + f(\omega; u)_x = 0, \quad x \in [-1, 1],\, t \in (0, \infty), \tag{5.1a}$$

$$u(\omega; x, 0) = -\sin(\pi x), \quad x \in [-1, 1],\, t = 0, \tag{5.1b}$$

with periodic boundary conditions and the random flux $f(\omega; u)$ given by

$$f(\omega; u) = \frac{1}{p(\omega)}|u|^{p(\omega)}, \quad p(\omega) \sim \mathcal{U}(1.5, 2.5). \tag{5.2}$$

This flux function is a bounded random flux and for $\mathbb{P}$-a.e., $f(\omega; \cdot) \in \mathrm{Lip}([-\overline{R}, \overline{R}])$, where $\overline{R} := \|u_0\|_{L^\infty(\mathbb{R})}$. An approximation of the mean of the random entropy solution at time $t = 1$, computed by the MLMC-FT method for $L = 9$, with $\delta_0 = 2^{-4}$ at the coarsest level, and $M_L = 8$ samples at the level with the finest resolution, is shown in Figure 1. In order to compute an estimate on the error of the approximation of the mean by the MLMC estimator $E_L^{MLMC}[u^L(t)]$ in the $L^2(\Omega; L^1(\mathbb{R}))$-norm, we use the relative error estimator introduced in [25] based on a Monte Carlo quadrature in the stochastic domain: We denote U_{ref} a reference solution and $\{U_k\}_{k=1,\ldots,K}$ a sequence of independent approximate solutions

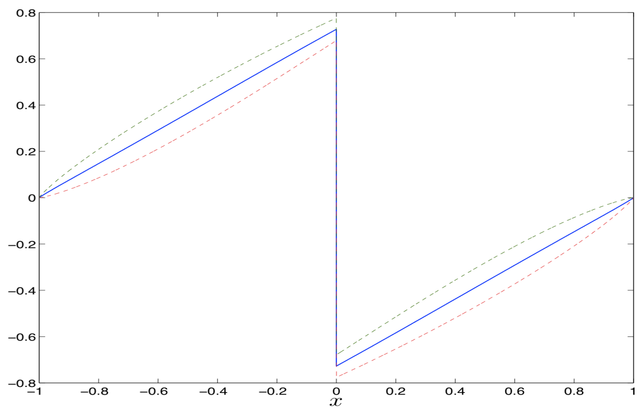

FIGURE 1. The estimator $E_L^{MLMC}[u^L(t)]$ computed by the MLMC-FT method at time $t = 1$ with $L = 9$ for problem (5.1), (5.2). The dashed lines denote the mean with ± standard deviation.

$E_L^{MLMC}[u^L(t)]$ obtained by running the MLMC-FT solver K times and corresponding to K realizations in the stochastic domain. Then we estimate the relative error by

$$\mathcal{R}E = \sqrt{\sum_{k=1}^{K} (\mathcal{R}E_k)^2 / K}, \tag{5.3}$$

where

$$\mathcal{R}E_k = 100 \times \frac{\|U_{\text{ref}} - U_k\|_{l^1}}{\|U_{\text{ref}}\|_{l^1}}. \tag{5.4}$$

In [25] the sensitivity of the error with respect to the parameter K is investigated. For this example, we will use $K = 30$ which was shown to be sufficient for most problems [25, 28]. To compute a reference solution U_{ref}, we have made use of the symmetry properties of the each realization (a shock at $x = 0$, smoothness away from the shock) and used the characteristics of the differential equation to compute an accurate approximation of $\mathbb{E}[u(t)]$. In Figure 2 the errors (5.3) versus the resolution δ_L at the finest level L of the MLMC estimator and versus the run time (in seconds) are shown ($L = 0, \ldots, 6$). We observe that the convergence rates are ≈ 0.9 with respect to the resolution and ≈ 0.4 with respect to work, which is approximately what we would expect from the theoretical results: Equation (4.23) implies that the error estimator (5.3) is asymptotically of order $\mathcal{O}(\sqrt{L}\, 2^{-L}) = \mathcal{O}(2^{-\alpha(L)\, L}) = \mathcal{O}(\delta_L^{-\alpha(L)})$ with respect to the resolution at the finest level, where

$$\alpha(L) = 1 - \frac{\log L}{2\, L \log 2} \xrightarrow{L\to\infty} 1. \tag{5.5}$$

For $L = 6$, we have $\alpha(L + 1) \approx 0.8$. Due to (4.26), the estimator (5.3) is of order $\mathcal{O}((W_{L,MLMC}^{FT})^{-\alpha(L)/2})$ with respect to work, hence for $L = 6$, $\alpha(L + 1)/2 \approx 0.4$.

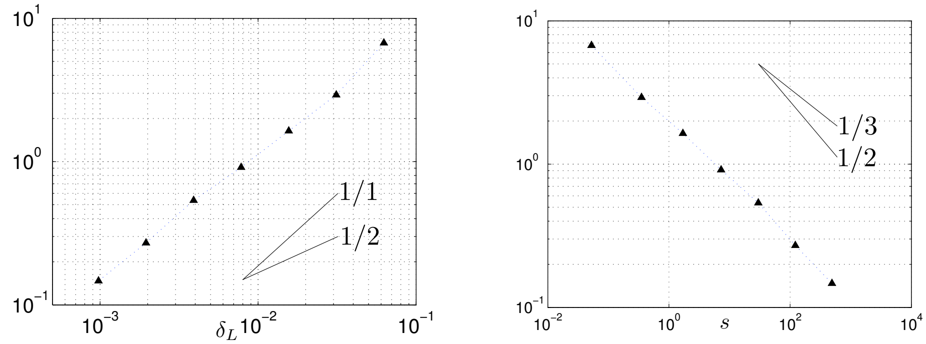

FIGURE 2. Left: Error (5.3) versus the resolution. Right: Error versus the run time of the MLMC-FT solver in seconds for the problem (5.1), (5.2). At the coarsest level, we have used $\delta_0 = 2^{-4}$ and at the finest level, we have used $M_L = 8$ samples.

Remark 5.1. *For exponents* $p \in [1.5, 2)$*, the second derivative of the flux function* $f(u,p)$ *in* (5.2) *is not uniformly bounded. Therefore the bound* (4.7) *does not apply. However, by a careful refinement of the estimates in* [20, Chapter 2]*, it is possible to show that the (deterministic) front tracking method converges at rate one with respect to the discretization parameter* δ *if the flux function* f *is in* $W^{2,1}([-\overline{R}, \overline{R}])$ *and the initial data* $u_0 \in BV(\mathbb{R})$ *has a bounded number of local maxima and minima.*

5.2. **Nonconvex random flux in one space dimension.** In a second experiment, we test the performance of the MLMC-FT method on the initial value problem (5.1) with periodic boundary conditions and the nonconvex random flux function

$$f(\omega; u) = \operatorname{sgn}(u)\frac{|u|^{p(\omega)}}{p(\omega)}, \quad p(\omega) \sim \mathcal{U}(2.5, 3.5). \tag{5.6}$$

For any $\overline{R} > 0$, we have $f \in L^2(\Omega; W^{2,\infty}([-\overline{R}, \overline{R}]))$, hence the assumptions in Theorem 4.17 are satisfied for this problem. In Figure 3, we show an approximation of the mean of the solution computed by the MLMC-FT-solver at time $t = 1$ with $L = 9$, $\delta_0 = 2^{-5}$ at the coarsest level and $M_L = 4$ samples at the finest level. We see that the mean of the solution is continuous, whereas all computed pathwise, approximate realizations $u(\omega; \cdot)$ of random entropy solutions of (5.1), (5.6) develop shocks.

This is not unexpected, because while each realization has discontinuities, the location of these discontinuities is random, and disappear upon taking the expectation. However, for each realization, the solution varies (very) rapidly at the shock location, hence the variance will be larger around in the regions where shocks are typically located, than in regions where each realization is continuous. For our example, each realization has two shocks, one around $x = 0.1$ and one around $x = -0.9$. We see that the variance is indeed much larger in around $x = 0.1$ and $x = -0.9$.

We use this approximation as a reference solution and compute the error estimators (5.3), (5.4) for $L = 0, \ldots, 5$, $\delta_0 = 2^{-5}$, $M_L = 4$ and $K = 30$. The results are

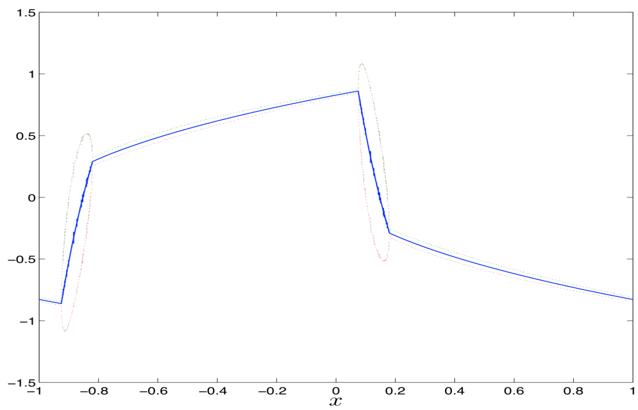

Figure 3. The estimator $E_L^{MLMC}[u^L(t)]$ for problem (5.1), (5.6) computed by the MLMC-FT method at time $t = 1$ with $L = 9$. The dashed lines denote the mean with ± standard deviation.

shown in Figure 4. Similarly as for the first example in Section 5.1, the experimentally observed convergence rates validate the a priori estimates (4.23) and (4.25) as we are not yet in the asymptotic regime and for $L = 5$, $\alpha(L+1) \approx 0.78$, c.f. (5.5) (we observe ≈ 0.85 versus resolution and ≈ 0.35 versus run time).

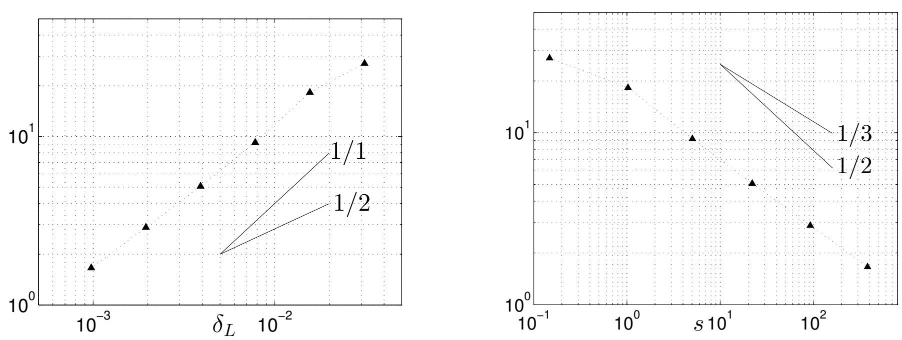

Figure 4. Left: Error (5.3) versus the resolution. Right: Error versus the run time of the MLMC-FT solver in seconds for the problem (5.1), (5.6). At the coarsest level, we have used $\delta_0 = 2^{-5}$ and at the finest level, we have used $M_L = 4$ samples.

5.3. **Random fluxes in two space dimensions.** We test the performance of the MLMC-FT algorithm in several space dimensions on the following test problem,

$$u_t + f(\omega; u)_x + g(\omega; u)_y = 0, \quad (x, y) \in [0,2]^2,\ t \in (0, \infty), \tag{5.7a}$$

$$u(\omega; x, y, 0) = \begin{cases} 1, & 0.1 < x, y < 0.9, \\ -1, & (x-1.5)^2 + (y-1.5)^2 < 0.16, \\ 0, & \text{otherwise}, \end{cases} \tag{5.7b}$$

with periodic boundary conditions and random fluxes f and g given by

$$f(\omega; u) = g(\omega; u) = \frac{|u|^{p(\omega)}}{p(\omega)}, \quad p(\omega) \sim \mathcal{U}(1, 3). \tag{5.8}$$

In Section 4.2.2 we have seen that in order to have the optimal convergence rate of the front tracking/dimensional splitting method, we have to choose the grid size Δx, the time step Δt and the refinement parameter δ of the flux function interpolations as

$$\Delta x = k_1 \Delta t = k_2 \delta^2.$$

We call k_1 a *CFL-number* in analogy to finite volume methods, although no restriction needs to be imposed on k_1 since dimensional splitting combined with front tracking method has been shown to converge for any choice of constants $k_1 > 0$.

Due to the increased computational effort of the multidimensional problem compared to the one dimensional problems, we have chosen to refine with respect to the grid size Δx. Therefore we set $\Delta x_\ell = 2^{-\ell} \Delta x_0$ and $\delta_\ell = 2^{-\ell/2} \delta_0$ and use at level $\ell = 0, \dots, L$, $M_\ell = 2^{L-\ell} M_L$ samples. In Figure 5 we show an approximation of the mean of (5.7), (5.8) by the MLMC-FT method computed at time $t = 1$ for $L = 8$ with $M_L = 4$, $\Delta x_0 = 2^{-3}$ and CFL-number $k_1 = 20$. As a reference solution,

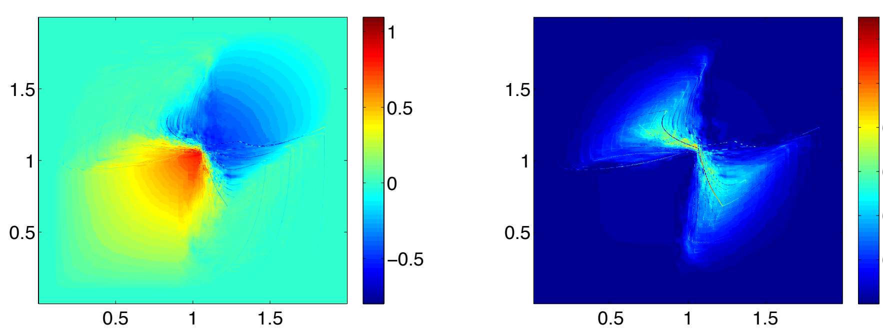

FIGURE 5. Mean and variance of (5.7), (5.8) computed by the MLMC-FT method for $L = 8$, $t = 1$, $M_L = 4$, $\Delta x_0 = 2^{-3}$, CFL-condition $k_1 = 20$ (number of grid cells: $2^{12} \times 2^{12}$). Left: Estimated mean of the solution. Right: Estimated variance of the solution.

we use an approximation of the mean of the solution computed by a MLMC-FVM scheme as in [26], with an HLL-solver and second order WENO reconstruction, $L = 8$, $M_L = 4$, $\Delta x_0 = 2^{-2}$, on a mesh with $2^{11} \times 2^{11}$ grid cells. We compute the error estimators (5.3), (5.4) for $K = 5$, $L = 0, \dots, 7$, $M_L = 4$, $M_\ell = 2^{L-\ell} M_L$, $\Delta x_0 = 0.125$, $\Delta x_\ell = 2^{-\ell} \Delta x_0$. The errors are shown in Figure 6. We measure convergence rates of ≈ 0.45 with respect to the grid size Δx and ≈ 0.15 with respect to the run time of the MLMC-FT solver. From the a priori estimates we would expect rates of $1/2$ versus the grid size and $1/5$ versus work asymptotically, so our rates are slightly below that. This could indicate that we are not yet in the asymptotic regime for our values of L.

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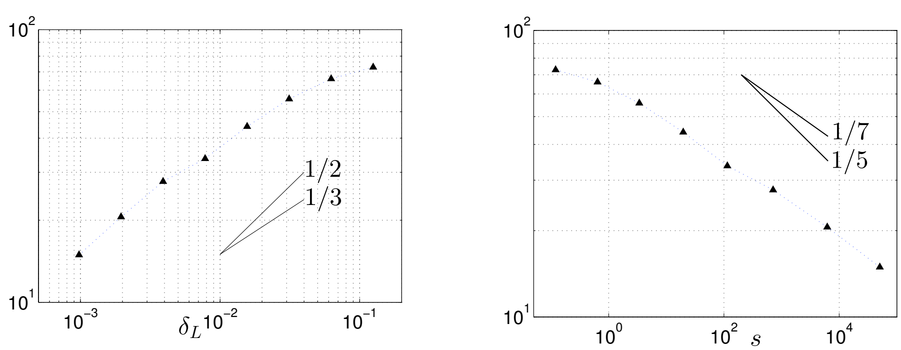

FIGURE 6. Left: Error (5.3) versus the resolution. Right: Error versus the run time of the MLMC-FT solver in seconds (x-axis figure right hand side) for the problem (5.1), (5.2). At the coarsest level, we have used $\Delta x_0 = 2^{-3}$ and at the finest level, we have used $M_L = 4$ samples, $K = 5$.

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