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CONVERGENCE OF FINITE DIFFERENCE SCHEMES FOR SYMMETRIC KEYFITZ-KRANZER SYSTEM

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Abstract. We are concerned with the convergence of numerical schemes for the initial value problem associated to the Keyfitz-Kranzer system of equations. This system is a toy model for several important models such as in elasticity theory, magnetohydrodynamics, and enhanced oil recovery. In this paper we prove the convergence of two difference schemes. One of these schemes is shown to converge to the unique entropy solution. Finally, the convergence is illustratred by several examples.

1. Introduction

In this paper, we consider the Cauchy problem for the $n \times n$ symmetric system of Keyfitz-Kranzer type

$$\begin{cases} u_t + (u\phi(|u|))_x = 0, & x \in \Omega = \mathbb{R} \times (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, $u = \left(u^{(1)}, \dots, u^{(n)}\right) : \mathbb{R} \times [0,T) \to \mathbb{R}^n$ is the unknown vector map with $|u| = \sqrt{{u^{(1)}}^2 + \cdots + {u^{(n)}}^2}$, $u_0 = \left(u_0^{(1)}, \dots, u_0^{(n)}\right)$ the initial data, and $\phi : \mathbb{R} \to \mathbb{R}$ is given (sufficiently smooth) scalar function (see Section 2 for the complete list of assumptions). Systems of this type was first considered in [10, 12] and later on by several other authors [4], as a prototypical example of a non-strictly hyperbolic system. This type of system is a model system for some phenomena in magnetohydrodynamics, elasticity theory and enhanced oil-recovery. This system also has similarities to a model of chromatography [1] and to a model describing polymer flooding in porous media [16]. For the flux function $F(u) = u\phi(|u|)$, a straightforward calculation shows $B(u) = dF(u)$ is the matrix with entries

$$B_{i,j}(u) = \phi(|u|)\delta_{i,j} + \phi'(|u|)\frac{u_i u_j}{|u|}, \quad i,j = 1,2,\cdots,n,$$

where $\delta_{i,j}$ is the Kronecker delta, given by

$$\delta_{i,j} = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

The matrix $B(u)$ is symmetric, therefore the system (1.1) is hyperbolic, that is, all the eigenvalues of $B(u)$ are real and the corresponding collection of eigenvectors is complete. It is easy to see that the first eigenvalue of $B(u)$ is $\lambda_1 = \phi(|u|) + \phi'(|u|)\,|u|$ and other $n-1$ eigenvalues are $\lambda_i = \phi(|u|)$, $i = 1,2,\cdots,n-1$. The presence of multiple eigenvalues shows that the system (1.1) is not strictly hyperbolic.

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Due to the nonlinearity, discontinuities in the solution may appear independently of the smoothness of the initial data and weak solution must be sought. A weak solution is defined as follows:

Definition 1.1. *We say $u(x,t)$ a weak solution to (1.1) if*

***D**.1 $u(x,t) \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$.*

***D**.2 For all test functions $\psi \in C_0^\infty(\mathbb{R}\times[0,\infty))$*

$$\iint_{\mathbb{R}\times\mathbb{R}^+} u\psi_t + u\phi(|u|)\psi_x\,dxdt + \int_{\mathbb{R}} u_0\psi(x,0)\,dx = 0, \tag{1.2}$$

It is well known that weak solutions may be discontinuous and they are not uniquely determind by their initial data. Consequently, an entropy condition must be imposed to single out the physically correct solution. Therefore the Cauchy problem is viewed in the framework of entropy solutions. For (1.1), an entropy formulation was first introduced by Freistühler [5, 6], and seemingly independently, by Panov [14]. An entropy solution to (1.1) is defined as follows:

Definition 1.2. *A bounded measurable function $u(x,t)$ is called an entropy solution to (1.1) if*

***D**.1 For all test functions $\psi \in C_0^\infty(\mathbb{R}\times[0,\infty))$*

$$\iint_{\mathbb{R}\times\mathbb{R}^+} u\psi_t + u\phi(|u|)\psi_x\,dxdt + \int_{\mathbb{R}} u_0\psi(x,0)\,dx = 0, \tag{1.3}$$

***D**.2 $r = |u|$ is an entropy solution (in the sense of Kružkov [11]) of the scalar conservation law*

$$\begin{cases} r_t + (r\phi(r))_x = 0, & t>0, \\ r(x,0) = |u_0(x)|. \end{cases} \tag{1.4}$$

Regarding the existence, uniqueness of solutions and continuous dependence of solutions on the initial data we have the following result

Theorem 1.1. *The system (1.1) has the following properties:*

- *(E) The system has a solution for $u_0 \in L^\infty(\mathbb{R})$.*
- *(U) For any such u_0, there is precisely one solution u with the property that $r =: |u|$ satisfies the scalar conservation laws (1.4) and Kružkov's entropy criterion.*
- *(S) This solution u depends $L^1_{\mathrm{loc}}(\mathbb{R})$ continuously on the initial data u_0.*

This theorem was first proved in [6] by using the famous equivalence result of Wagner [17]. The key idea behind this proof is to view the system (1.1) as an extended system, consisting of (1.1) and an additional conservation law by r (1.4), with Wagner's transformation theory. On the other hand, in [14], Panov gave a "direct" proof of both existence and uniqueness. The existence was proved by showing the convergence of the singularly perturbed problems

$$u^\varepsilon_t + (u^\varepsilon\phi(|u^\varepsilon|))_x = \varepsilon u^\varepsilon_{xx},$$

to an entropy solution as $\varepsilon \to 0$. The idea behind the existence proof was first to show the existence of a *measure-valued* solution $\nu_{(t,x)}$ of the Cauchy problem (1.1). Then he showed that indeed $\nu_{(t,x)}$ is regular: $\nu_{(t,x)}(u) = \delta(u - u(t,x)), u(t,x) \in L^\infty(\mathbb{R}\times\mathbb{R}^+,\mathbb{R}^n)$ and consequently this gives existence of a solution to (1.1).

In view of the analytic properties of the solutions of (1.1), several different methods for computing the solution suggest themselves. Foremost among these methods is Glimm's scheme [8]. Regarding other numerical methods, it is tempting to use the equation satisfied by r, and view r as an independent variable. Defining $v \in S^{n-1}$ by $vr = u$, we formally have that

$$r_t + (r\phi(r))_x = 0 \tag{1.5}$$

$$(rv)_t + (r\phi(r)v)_x = 0 \tag{1.6}$$

or

$$v_t + \phi(r)v_x = 0. \tag{1.7}$$

As a strategy, one can then solve (1.5) first, and then either (1.6) or (1.7). These should then hold subject to the constraint $|v| = 1$. Without this constraint, (1.5)–(1.6) is a "triangular" system of conservation laws, see [3]. Using any monotone scheme for (1.5) and (1.6) will ensure the strong convergence of the approximate solutions to (1.5) and the weak-star convergence of the approximate solutions to (1.6). This approach was used in [7]. To show that $u = rv$ is an entropy solution to (1.1), one must show (for the approximations) that the limit of $|v| = 1$ if $|v_0| = 1$. In this paper, for the semi discrete scheme, we discretize (1.1) in space and show the convergence of approximate solution to a weak solution of (1.1). But we are unable to extend our analysis to the fully discrete scheme based on discretizing (1.1). To overcome this difficulty, we propose another scheme based on discretizing (1.5)–(1.7) and prove the convergence of approximate solution to unique entropy solution of (1.1).

The present paper can be divided into four parts:

1. In Section 2, we present the mathematical framework used in this paper. In particular, we used a compensated compactness result in the spirit of Tartar [15] but the proof is based on div-curl lemma and does not rely on the Young measure.

2. In section 3, we propose an upwind semi discrete finite difference scheme and prove the convergence of the approximate solution to the weak solution of (1.1). Main idea behind this proof is to prove first the strong convergence of approximate solution $r_{\Delta x} = |u_{\Delta x}|$ using compensated compactness technique [15, 2]. Next step is to prove a BV estimate of $\tau^i_{\Delta x}$ for $i = 1, \cdots, n$, where $\tau^i_{\Delta x} = \tan(\phi^i_{\Delta x})$ and $\phi^i_{\Delta x}$ denotes the angle between $u^i_{\Delta x}$ and $r_{\Delta x}$. Then Helly's theorem combined with the strong convergence of $r_{\Delta x}$ gives the strong convergence of approximate solution $u_{\Delta x}$.

3. In section 4, for a fully discrete scheme, we are only able to conclude that u is only a distributional solution of

$$u_t + (u\,\phi(r))_x = 0,$$

with $|u| \le r$. To overcome this difficulty, i.e., to prove that u is an unique entropy solution, we propose another fully discrete scheme relying on explicit decoupling of the variables r and v expressed by the "nonconservative" formulation (1.5)–(1.7)

$$\begin{cases} r_t + (r\phi(r))_x &= 0, \\ v_t + \phi(r)\,v_x &= 0, \end{cases}$$

with $r(0) = |u(0)|$. It is not difficult to show the convergence of $r_{\Delta x}$ to r, r being the unique entropy solution, and the strong convergence of $v_{\Delta x}$. In order to conclude that $u = r\,v$ is the unique entropy solution of (1.1), one has to show $|w(x,t)| = 1$ and this has been achieved in this paper using Wagner transformation [17].

4. Finally, in Section 5, we test our numerical schemes and provide some numerical results.

2. Mathematical Framework

In this section we present some mathematical tools that we shall use in the analysis. To start with the basic assumptions on the initial data and the funtion $\phi(r)$, we assume that ϕ is a twice differentiable function $\phi : [0,\infty) \to [0,\infty)$ so that

A.1 $\phi(r) > 0$ and $\phi'(r) \geq 0$ for all relevant r;
A.2 $\phi(r), \phi'(r)$ and $\phi''(r)$ are bounded for all relevant r;
A.3 meas $\Big\{ r \;\Big|\; 2\phi'(r) + r\phi''(r) = 0 \Big\} = 0$;
A.4 $|u_0| \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap B.V.(\mathbb{R})$.

Next, we recapitulate the results we shall use from the compensated compactness method due to Murat and Tartar [13, 15]. For a nice overview of applications of the compensated compactness method to hyperbolic conservation laws, we refer to Chen [2]. Let $\mathcal{M}(\mathbb{R})$ denote the space of bounded Radon measures on $\mathbb{R}$ and

$$C_0(\mathbb{R}) = \{\psi \in C(\mathbb{R}) : \lim_{|x|\to\infty} \psi(x) = 0\}.$$

If $\mu \in \mathcal{M}(\mathbb{R})$, then

$$\langle \mu, \psi \rangle = \int_{\mathbb{R}} \psi\, d\mu, \quad \text{for all} \quad \psi \in C_0(\mathbb{R}).$$

Recall that $\mu \in \mathcal{M}(\mathbb{R})$ if and only if $|\langle \mu, \psi\rangle| \leq C\,\|\psi\|_{L^\infty(\mathbb{R})}$ for all $\psi \in C_0(\mathbb{R})$. We define

$$\|\mu\|_{\mathcal{M}(\mathbb{R})} = \sup\{|\langle \mu, \psi\rangle| : \psi \in C_0(\mathbb{R}), \|\psi\|_{L^\infty(\mathbb{R})} \leq 1\}.$$

The space $\Big(\mathcal{M}(\mathbb{R}), \|\cdot\|_{\mathcal{M}(\mathbb{R})}\Big)$ is a Banach space and it is isometrically isomorphic to the dual space of $\Big(C_0(\mathbb{R}), \|\cdot\|_{L^\infty(\mathbb{R})}\Big)$, while we define the space of probablity measures

$$\mathrm{Prob}(\mathbb{R}) = \{\mu \in \mathcal{M}(\mathbb{R}) : \mu \,\text{is nonnegative and}\; \|\mu\|_{\mathcal{M}(\mathbb{R})} = 1\}.$$

Then we can state the fundamental theorem in the theory of compensated compactness.

Theorem 2.1. *Let $K \subset \mathbb{R}$ be a bounded open set and $u^\varepsilon : \mathbb{R}\times[0,T] \to K$. Then there exists a family of probablity measures $\{\nu_{(x,t)}(\lambda) \in Prob(\mathbb{R})\}_{(x,t)\in\mathbb{R}\times[0,T]}$ (depending weak-star measurably on (x,t)) such that*

$$\operatorname{supp} \nu_{(x,t)} \subset \overline{K} \text{ for a.e. } (x,t) \in \mathbb{R}\times[0,T].$$

Furthermore, for any continuous function $\Phi : K \to \mathbb{R}$, we have along a subsequence

$$\Phi(u^\varepsilon) \overset{\star}{\rightharpoonup} \overline{\Phi} \text{ in } L^\infty(\mathbb{R}\times[0,T]) \text{ as } \varepsilon \downarrow 0,$$

where (the exceptional set depends possibly on Φ)

$$\overline{\Phi}(x,t) := \langle \nu_{(x,t)}, \Phi\rangle = \int_{\mathbb{R}} \Phi(\lambda) d\nu_{(x,t)}(\lambda) \text{ for a.e. } (x,t) \in \mathbb{R}\times[0,T].$$

In the literature, $\nu_{(x,t)}$ is often referred to as a Young measure. Theorem 2.1 provides us with a representation formula for weak limits of nonlinear functions and Young measures. A uniformly bounded sequence $\{u^\varepsilon\}_{\varepsilon>0}$ converges to u a.e. on $\mathbb{R}\times[0,T]$ if and only if the corresponding Young measure $\nu_{(x,t)}$ reduces to a Dirac measure located at $u(x,t)$, i.e., $\nu_{(x,t)} = \delta_{u(x,t)}$.

Before we state the compensated compactness theorem, we recall the celebrated div-curl lemma.

Lemma 2.1 (div-curl lemma)**.** *Let Ω be a bounded open subset of $\mathbb{R}^2$. With $\varepsilon > 0$ denoting a parameter taking its value in a sequence which tends to zero, suppose*

$$D^\varepsilon \rightharpoonup D \text{ in } (L^2(\Omega))^2, \qquad E^\varepsilon \rightharpoonup E \text{ in } (L^2(\Omega))^2,$$
$$\{\operatorname{div} D^\varepsilon\}_{\varepsilon>0} \text{ lies in a compact subset of } W^{-1,2}_{\mathrm{loc}}(\Omega),$$
$$\{\operatorname{curl} E^\varepsilon\}_{\varepsilon>0} \text{ lies in a compact subset of } W^{-1,2}_{\mathrm{loc}}(\Omega).$$

Then along a subsequence

$$D^\varepsilon \cdot E^\varepsilon \to D\cdot E \text{ in } \mathcal{D}'(\Omega).$$

We shall use the following compensated compactness result.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}\times\mathbb{R}^+$ be a bounded open set, and assume that $\{u^\varepsilon\}$ is a sequence of uniformly bounded functions such that $|u^\varepsilon| \le M$ for all ε. Also assume that $f : [-M, M] \to \mathbb{R}$ is a twice differentiable function. Let $u^\varepsilon \overset{\star}{\rightharpoonup} u$ and $f(u^\varepsilon) \overset{\star}{\rightharpoonup} v$, and set*

$$(2.1)\qquad \begin{aligned}(\eta_1(s), q_1(s)) &= (s-k, f(s)-f(k)),\\ (\eta_2(s), q_2(s)) &= \left(f(s)-f(k), \int_k^s (f'(\theta))^2\, d\theta\right),\end{aligned}$$

where k is an arbitrary constant. If

$$\eta_i(u^\varepsilon)_t + q_i(u^\varepsilon)_x \text{ is in a compact set of } H^{-1}_{\mathrm{loc}}(\Omega) \text{ for } i = 1,\ 2,$$

then

(1) $v = f(u)$, a.e. (x,t),
(2) $u^\varepsilon \to u$, a.e. (x,t) if $\operatorname{meas}\{u \mid f''(u) = 0\} = 0$.

For a proof of this theorem, see the monograph of Lu [18]. A feature of the compensated compactness result above is that it avoids the use of the Young measure by following an approach developed by Chen and Lu [18, 2] for the standard scalar conservation law. This is preferable as the fundamenmtal theorem of Young measures applies most easily to functions that are continuous in all variables.

The following compactness interpolation result (known as Murat's lemma [13]) is useful in obtaining the H^{-1}_{loc} compactness needed in Theorem 2.2.

Lemma 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^2$. Suppose that the sequence $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ of distributions is bounded in $W^{-1,\infty}(\Omega)$. Suppose also that*

$$\mathcal{L}_\varepsilon = \mathcal{L}_{1,\varepsilon} + \mathcal{L}_{2,\varepsilon},$$

where $\{\mathcal{L}_{1,\varepsilon}\}_{\varepsilon>0}$ is in a compact subset of $H^{-1}(\Omega)$ and $\{\mathcal{L}_{2,\varepsilon}\}_{\varepsilon>0}$ is in a bounded subset of $\mathcal{M}_{\mathrm{loc}}(\Omega)$. Then $\{\mathcal{L}_\varepsilon\}_{\varepsilon>0}$ is in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$.

Finally, we shall need the following Kolmogorov's compactness lemma.

Lemma 2.3 (L^1_{loc} compactness, see [9]). *Let $u^\varepsilon : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ be a family of functions such that for each positive T,*

$$|u^\varepsilon(x,t)| \le C_T,\ (x,t) \in \mathbb{R} \times [0,T]$$

for a constant C_T independent of ε. Assume in addition that for all compact $B \subset \mathbb{R}$ and for $t \in [0,T]$

$$\sup_{|\xi|\le|\rho|} \int_B |u^\varepsilon(x+\xi,t) - u^\varepsilon(x,t)|\, dx \le \nu_{B,T}(|\rho|),$$

for a modulus of continuity ν. Furthermore, assume for s and t in $[0,T]$ that

$$\int_B |u^\varepsilon(x,t) - u^\varepsilon(x,s)|\, dx \le \omega_{B,T}(|t-s|) \text{ as } \varepsilon \downarrow 0,$$

for some modulus of continuity ω_T. Then there exists a sequence $\varepsilon_j \to 0$ such that for each $t \in [0,T]$ the function $\{u^{\varepsilon_j}(t)\}$ converges to a function $u(t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. The convergence is in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$.

3. A Semi discrete Finite difference scheme

We start by introducing some notation needed to define the semi discrete finite difference schemes. Throughout this paper we reserve Δx to denote a small positive numbers that represent the spatial discretization parameter of the numerical schemes. Given $\Delta x > 0$, we set $x_j = j\Delta x$ for $j \in \mathbb{Z}$ and for any function $u = u(x)$ admitting pointvalues we write $u_j = u(x_j)$. Furthermore, let us introduce the spatial grid cells

$$I_j = [x_{j-1/2}, x_{j+1/2}),$$

where $x_{j\pm1/2} = x_j \pm \Delta x/2$. Let $D_\pm$ denote the discrete forward and backward differences, i.e.,

$$D_\pm u_j = \mp \frac{u_j - u_{j\pm1}}{\Delta x}.$$

Also, a discrete Leibnitz rule holds:

$$D_\pm(u_j v_j) = u_j D_\pm v_j + v_{j\pm1} D_\pm u_j$$

Furthermore, for any C^2 function f, using the Taylor expansion on the sequence $f(u_j)$ we obtain

$$D_\pm f(u_j) = f'(u_j) D_\pm u_j \pm \frac{\Delta x}{2} f''(\xi_j^\pm)(D_\pm u_j)^2,$$

for some $\xi_j^\pm$ between $u_{j\pm1}$ and u_j. We will make frequent use of this, which states that a discrete chain rule holds up to an error term of order $\Delta x (D_\pm u_j)^2$. To a sequence $\{u_j\}_{j\in\mathbb{Z}}$ we associate the function $u_{\Delta x}$ defined by

$$u_{\Delta x}(x) = \sum_{j\in\mathbb{Z}} u_j \mathbb{1}_{I_j}(x),$$

similarly, we also define $r_{\Delta x}$ as

$$r_{\Delta x}(x) = \sum_{j\in\mathbb{Z}} r_j \mathbb{1}_{I_j}(x),$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A, viz.

$$\mathbb{1}_A(x) = \begin{cases} 1, & \text{for } x \in A \\ 0, & \text{for } x \notin A. \end{cases}$$

Throughout this paper we use the notations $u_{\Delta x}, r_{\Delta x}$ to denote the functions associated with the sequence $\{u_j\}_{j\in\mathbb{Z}}$ and $\{r_j\}_{j\in\mathbb{Z}}$ respectively. For later use, recall that the $L^\infty(\mathbb{R})$ norm, the $L^1(\mathbb{R})$ norm, the $L^2(\mathbb{R})$ norm, and the $BV(\mathbb{R})$ semi-norm of a lattice function $u_{\Delta x}$ is defined respectively as

$$\begin{aligned} \|u_{\Delta x}\|_{L^\infty(\mathbb{R})} &= \sup_{j\in\mathbb{Z}} |u_j|\,, \\ \|u_{\Delta x}\|_{L^1\mathbb{R})} &= \Delta x \sum_{j\in\mathbb{Z}} |u_j|\,, \\ \|u_{\Delta x}\|_{L^2\mathbb{R})} &= \Delta x \sum_{j\in\mathbb{Z}} |u_j|^2\,, \\ |u_{\Delta x}|_{BV(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} |u_j - u_{j-1}|\,. \end{aligned}$$

Observe that all the eigenvalues of the system (1.1) are positive by our assumptions. We consider the following semi discrete upwind finite difference scheme of the form

$$u_j'(t) + D_-\left(\phi(r_j(t))u_j(t)\right) = 0, \text{ for } j \in \mathbb{Z},\ t > 0, \tag{3.1}$$

with initial values

$$u_j(0) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx. \tag{3.2}$$

Here $r_j(t) = |u_j(t)|$. Now it is easy to see that $\{u_j(t)\}_{j\in\mathbb{Z}}$ satisfy the (infinite) system of ordinary differential equations and it is natural to view (3.1) as an ordinary differential equation in $L^2(\mathbb{R})^n$. To show the local (in time) existence and uniqueness of differentiable solutions we must show that the right hand side of (3.1) is Lipschitz continuous in $L^2(\mathbb{R})^n$. Set

$$F(u_{\Delta x})_j = D_-\left(\phi(r_j)u_j\right).$$

The infinite system of differential equations (3.1) can then be written

$$\frac{d}{dt}\left(u_{\Delta x}(t)\right) = F(u_{\Delta x})_{\Delta x}.$$

We view $F(u_{\Delta x})_{\Delta x}$ as an element in $L^2(\mathbb{R})^n$. We must show that

$$\|F(u_{\Delta x})_{\Delta x} - F(v_{\Delta x})_{\Delta x}\|_{L^2(\mathbb{R})^n} \le \gamma \|u_{\Delta x} - v_{\Delta x}\|_{L^2(\mathbb{R})^n} \tag{3.3}$$

for some locally bounded $\gamma = \gamma(u_{\Delta x}, v_{\Delta x})$ and for a fixed $\Delta x > 0$. Set $\tilde r_j = |v_j|$, note that

$$|r_j - \tilde r_j| \le \frac{|u_j + v_j|}{r_j + \tilde r_j}\,|u_j - v_j|\,.$$

Then

$$\begin{aligned} \|F(u_{\Delta x}) - F(v_{\Delta x})\|_{L^2(\mathbb{R})^n} &\le \frac{2}{\Delta x}\Big(\sup_j |u_j|\,\|\phi'\|_{L^\infty}\,\|r_{\Delta x} - \tilde r_{\Delta x}\|_{L^2(\mathbb{R})} \\ &\qquad + \|\phi\|_{L^\infty}\,\|u_{\Delta x} - v_{\Delta x}\|_{L^2(R)^n}\Big) \\ &\le \gamma\,\|u_{\Delta x} - v_{\Delta x}\|_{L^2(R)^n} \end{aligned}$$

Therefore F is locally Lipschitz continuous, and there is a $\tau > 0$ so that the initial value problem (3.1) has a unique differentiable solution for $t \in [0, \tau)$, if $\tau < \infty$, then

$$\lim_{t \uparrow \tau} \|u_{\Delta x}(t)\|_{L^2(\mathbb{R})^n} = \infty.$$

We shall proceed to show that the L^2 norm remains bounded if it is bounded initially, so the solution can be defined up to any time.

Lemma 3.1. *Let $\{u_j(t)\}$ be defined by (3.1), and let $r_j(t) = |u_j(t)|$. Then*

$$\|r_{\Delta x}(t)\|_{L^1(\mathbb{R})} \le \|r_{\Delta x}(0)\|_{L^1(\mathbb{R})},$$
$$\|r_{\Delta x}(t)\|_{L^2(\mathbb{R})} \le \|r_{\Delta x}(0)\|_{L^2(\mathbb{R})},$$
$$\|r_{\Delta x}(t)\|_{L^\infty(\mathbb{R})} \le \|r_{\Delta x}(0)\|_{L^\infty(\mathbb{R})}.$$

Furthermore, there is a constant C, independent of Δx and T, such that

$$\int_0^T \sum_j \int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2\right) \phi'(s)\, ds + \Delta x \sum_j \phi_{j-1} \Delta x \left|D_- u_j\right|^2 \, dt \le C. \tag{3.4}$$

Proof. Let η be a differentiable function $\eta : \mathbb{R}^n \to \mathbb{R}$, take the inner product of (3.1) with $\nabla_u \eta(u_j)$ to get

$$\begin{aligned} &\frac{d}{dt}\eta(u_j) + D_-\left(\phi_j \eta(u_j)\right) \\ &\qquad + \left[(\nabla_u \eta(u_j), u_j) - \eta(u_j)\right] D_- \phi_j + \phi_{j-1} \frac{\Delta x}{2} d_u^2 \eta_{j-1/2}\left(D_- u_j, D_- u_j\right) = 0. \end{aligned} \tag{3.5}$$

Here $\phi_j = \phi(r_j)$, and $d^2\eta$ denotes the Hessian matrix of η, so that

$$d_u^2 \eta_{j-1/2} = d_u^2 \eta(u_{j-1/2})$$

for some $u_{j-1/2}$ between u_j and u_{j-1}. By a limiting argument, the function $\eta(u) = |u|$ can be used. This function is convex, i.e., $d_u^2 |u| \ge 0$. This means that

$$\frac{d}{dt} r_j + D_-(r_j \phi(r_j) \le 0. \tag{3.6}$$

Multiplying by Δx and summing over j we get

$$\|r_{\Delta x}(t)\|_{L^1(\mathbb{R})} \le \||u_0|\|_{L^1(\mathbb{R})}. \tag{3.7}$$

Furthermore, if $r_j(t) \ge r_{j-1}(t)$, then $dr_j(t)/dt \le 0$. This shows that $0 \le r_j(t) \le \sup_j |u_j(0)|$. Hence, if $\||u_0|\|_{L^\infty(\mathbb{R})} < \infty$, then $r_{\Delta x}$ is bounded independently of t and Δx.

Choosing $\eta(u) = |u|^2$ in (3.5) we get

$$\frac{d}{dt} r_j^2(t) + D_-\left(r_j^2 \phi_j\right) + r_j^2 D_- \phi_j + \phi_{j-1} \Delta x \left|D_- u_j\right|^2 = 0.$$

We have that

$$\begin{aligned} D_-\left(r_j^2 \phi_j\right) + r_j^2 D_- \phi_j &= \frac{2}{\Delta x} \int_{r_{j-1}}^{r_j} s\phi(s) + s^2 \phi'(s)\, ds + \frac{1}{\Delta x} \int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2\right) \phi'(s)\, ds \\ &= D_- g(r_j) + \frac{1}{\Delta x} \int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2\right) \phi'(s)\, ds, \end{aligned}$$

where

$$g(r) = 2\int_0^r s\phi(s) + s^2\phi'(s)\,ds. \tag{3.8}$$

Using this we find that

$$\|r_{\Delta x}(t)\|_{L^2(\mathbb{R})} \le \| |u_0| \|_{L^2(\mathbb{R})}, \tag{3.9}$$

since, by the assumption that $\phi' \ge 0$,

$$\int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2\right)\phi'(s)\,ds \ge 0.$$

Hence $\|u_{\Delta x}(t)\|_{L^2(\mathbb{R})^n}$ is bounded independently of Δx and t. Therefore, the exists a differentiable solution $u_{\Delta x}(t)$ to (3.1) for all $t > 0$. Furthermore, we have the bound

$$\int_0^T \sum_j \int_{r_{j-1}}^{r_j} \left(r_j^2 - s^2\right)\phi'(s)\,ds + \Delta x \sum_j \phi_{j-1}\Delta x \left|D_- u_j\right|^2\,dt \le C,$$

for some constant C which is independent of t and Δx.

□

Now let δ be a positive constant, and let e be some unit vector in $\mathbb{R}^n$. Choose

$$\eta(u) = \max\left\{\delta\,|u| - (e,u), 0\right\}.$$

and observe that $(\nabla\eta(u), u) - \eta(u) = 0$. Furthermore η is convex, so that

$$\frac{d}{dt}\eta(u_j) + D_-(\eta(u_j)\phi_j) \le 0,$$

which implies that

$$\sum_j \eta(u_j(t)) \le \sum_j \eta(u_j(0)).$$

We have that $\eta(u) = 0$ if u is in the cone $\Gamma_\delta = \{u \,|\, \delta\,|u| \le (e,u)\}$. Hence this cone is positively invariant for (3.1). Observe that there is no loss of generality in choosing the coordinates such that $e = (1,\dots,1)/\sqrt{n}$. If $\delta < 1$, then the invariant cone is in the first n-tant in $\mathbb{R}^n$, so that $u_j^{(i)}(t) \ge 0$ for all $t > 0$ if $u_0 \in \Gamma_\delta$. In particular $u_j^{(i)}(t) = 0$ if and only if $r_j(t) = 0$.

Therefore, if $u_0 \in \{u \,|\; |u| \le R\} \cap \Gamma_\delta$, then $u_{\Delta x}(x,t)$ is also in this set. This enables us to deduce the weak-$*$ convergence of a subsequence (which we do not relabel) of $\{u_{\Delta x}\}_{\Delta x > 0}$.

Let now $\eta_i(r)$ and $q_i(r)$ be given by (2.1) for $i = 1$, 2. We then have that

$$\frac{d}{dt}\eta_1(u_j) + D_-(q_1(r_j)) + e_{1,j} = 0, \tag{3.10}$$

where

$$f(r) = r\phi(r), \quad q_1(r) = f(r) - f(k) \quad \text{and}$$
$$e_{1,j} = \phi_{j-1}\Delta x\,(D_- u_j)^T \frac{1}{r_{j-1/2}}\left(I - \frac{u_{j-1/2}\otimes u_{j-1/2}}{r_{j-1/2}^2}\right)(D_- u_j).$$

For any vector u, the matrix $u \otimes u$ is defined as $u \otimes u = u_i u_j$. We shall now find an equation satisfied by η_2. First observe that

$$\frac{d}{dt} r_j + f'(r_j) D_- r_j - \frac{\Delta x}{2} f''\left(r_{j-1/2}\right) \left(D_- r_j\right)^2 + e_{1,j} = 0.$$

Multiplying this with $f'(u_j)$ we get

$$\frac{d}{dt} f(r_j) + q_2'(r_j) D_- r_j - f'(r_j) \frac{\Delta x}{2} f''\left(r_{j-1/2}\right) \left(D_- r_j\right)^2 + f'(r_j) e_{1,j} = 0.$$

Set

$$e_{2,j} = \frac{\Delta x}{2} f''\left(r_{j-1/2}\right) \left(D_- r_j\right)^2.$$

This can be rewritten as

$$\frac{d}{dt} \eta_2(r_j) + D_- q_2(r_j) + \frac{\Delta x}{2} q_2''(r_{j-1/2}) \left(D_- r_j\right)^2 - f'(r_j) \left(e_{2,j} - e_{1,j}\right) = 0. \tag{3.11}$$

Finally set

$$e_{3,j} = \frac{\Delta x}{2} q_2''(r_{j-1/2}) \left(D_- r_j\right)^2,$$

and

$$e_i(x,t) = e_{i,j}(t) \quad \text{for } x \in (x_{j-1/2}, x_{j+1/2}] \text{ and } i = 1, 2, 3.$$

Lemma 3.2. *We have that $e_i \in \mathcal{M}_{\mathrm{loc}}(\mathbb{R} \times [0,T))$ for $i = 1,\ 2,\ 3$.*

Proof. Set $\Omega = \mathbb{R} \times [0,T)$, and let ψ be a test function in $L^\infty(\Omega)$. Note that from (3.7) and (3.10) it follows that

$$\iint_\Omega e_1(x,t)\, dxdt \le C,$$

where the constant C does not depend on Δx or T. Since $e_1 \ge 0$, this means that

$$\left|\langle e_1, \psi \rangle\right| \le \iint_\Omega |\psi|\, e_1\, dxdt \le C \left\|\psi\right\|_{L^\infty(\Omega)},$$

and thus $e_1 \in \mathcal{M}_{\mathrm{loc}}(\Omega)$. To show the same for e_2 and e_3 observe that

$$\left|D_- r_j\right| \le \left|D_- u_j\right|.$$

Since $\phi(r) > 0$, (3.4) implies that

$$\int_0^T \Delta x \sum_j \Delta x \left|D_- u_j\right|^2 \le C,$$

for some constant C which is independent of Δx and T. We also have that f' and f'' are locally bounded, and $r_{\Delta x}$ is bounded, this means that, for $i = 2, 3$,

$$\iint_\Omega e_i(x,t)\, dxdt \le C \int_0^T \Delta x \sum_j \Delta x \left(D_- r_j\right)^2\, dt \le \int_0^T \Delta x \sum_j \Delta x \left|D_- u_j\right|^2 \le C.$$

Thus also e_2 and e_3 are in $\mathcal{M}_{\mathrm{loc}}(\Omega)$. □

Observe that, Lemma 3.2 implies that also $f'(r_j)(e_{1,j} - e_{2,j})$ is in $\mathcal{M}_{\mathrm{loc}}(\Omega)$.

Lemma 3.3. *Let $u_{\Delta x}$ be generated by the scheme (3.1) and $r_{\Delta x} = |u_{\Delta x}|$. Then*

$$\left\{\eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})\right\}_{\Delta x > 0} \text{ is compact in } H^{-1}_{\mathrm{loc}},$$

where η_i and q_i are given by (2.1) for $i = 1, 2$.

Proof. Let $i = 1$ or $i = 2$, and ψ is a test function in $H^1_{\mathrm{loc}}(\Omega)$. we define

$$\begin{aligned}
\langle \mathcal{L}_i, \psi\rangle &= \langle \eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})_x, \psi\rangle \\
&= \int_0^T \int_{\mathbb{R}} \left(\eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})_x\right) \psi(x,t)\, dxdt \\
&= \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{d}{dt}\eta_i(r_j)\psi(x,t) - q_i(r_j)\psi_x(x,t)\, dx\, dt \\
&= \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{d}{dt}\eta_i(r_j)\psi(x,t)\, dx - q_i(r_j)\, \Delta x D_-\psi(x_{j+1/2},t)\, dt \\
&= \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{d}{dt}\eta_i(r_j)\psi(x,t) + D_-q_i(r_j)\psi(x_{j-1/2},t)\, dx\, dt \\
&= \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\frac{d}{dt}\eta_i(r_j) + D_-q_i(r_j)\right)\psi(x,t)\, dx\, dt \\
&\qquad + \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\psi(x_{j-1/2},t) - \psi(x,t)\right) D_-q_i(r_j)\, dx\, dt \\
&=: \langle \mathcal{L}_{i,1}, \psi\rangle + \langle \mathcal{L}_{2,i}, \psi\rangle.
\end{aligned}$$

By (3.10), (3.11) and Lemma 3.2 we know that $\mathcal{L}_{i,1} \in \mathcal{M}_{\mathrm{loc}}(\Omega)$. Regarding $\mathcal{L}_{i,2}$ we have

$$\begin{aligned}
|\langle \mathcal{L}_{2,i}, \psi\rangle| &= \Big|\int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \int_{x_{j-1/2}}^{x} \psi_x(y,t)\, dy\, D_-q_i(r_j)\, dx\, dt\Big| \\
&\le \int_0^T \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} \sqrt{x - x_{j-1/2}}\Big(\int_{x_{j-1/2}}^{x} \left(\psi_x(y,t)\right)^2\, dy\Big)^{1/2} |D_-q_i(r_j)|\, dx\, dt \\
&\le \int_0^T \sum_j \Delta x^{3/2}\Big(\int_{x_{j-1/2}}^{x_{j+1/2}} \left(\psi_x(x,t)\right)^2\, dx\Big)^{1/2} \|q_i'\|_{L^\infty} |D_-r_j|\, dt \\
&\le \|q_i'\|_{L^\infty} \int_0^T \Big(\sum_j \Delta x \int_{x_{j-1/2}}^{x_{j+1/2}} \left(\psi_x(x,t)\right)^2\, dx\Big)^{1/2} \Big(\Delta x^2 \sum_j \left(D_-r_j\right)^2\Big)^{1/2} dt \\
&\le \|q_i'\|_{L^\infty} \sqrt{\Delta x}\Big(\iint_{\Omega} \left(\psi_x(x,t)\right)^2\, dxdt\Big)^{1/2} \Big(\int_0^T \Delta x \sum_j \Delta x \left(D_-r_j\right)^2\, dt\Big)^{1/2} \\
&\le C\sqrt{\Delta x}\, \|\psi\|_{H^1(\Omega)}\, .
\end{aligned}$$

Therefore the above estimate shows that $\mathcal{L}_{2,i}$ is compact in $H^{-1}(\Omega)$. By Lemma 2.2, we conclude the sequence $\{\eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})\}_{\Delta x>0}$ is compact in $H^{-1}_{\mathrm{loc}}(\Omega)$.

□

Lemma 3.4. *If*

$$\operatorname{meas}\left\{r \;\middle|\; 2\phi'(r) + r\phi''(r) = 0\right\} = 0,$$

then there is a subsequence of $\{\Delta x\}$ *(not relabeled) and a function* r *such that* $r_{\Delta x} \to r$ *a.e.* $(x,t) \in \Omega$. *We have that* $r \in C([0,T]; L^1(\mathbb{R}))$. *Furthermore,* r

satisfies

$$\begin{cases} r_t + f(r)_x \le 0, & x \in \mathbb{R},\ t > 0, \\ r = |u_0|, & x \in \mathbb{R},\ t = 0, \end{cases}$$

in the distributional sense.

Proof. The strong convergence of $r_{\Delta x}$ follows from the compensated compactness theorem, Theorem 2.2 and the compactness of $\{\eta_i(r_{\Delta x})_t + q_i(r_{\Delta x})_x\}_{\Delta x>0}$ for $i = 1, 2$ i.e., Lemma 3.3.

To show the L^1 continuity, first note that $t \mapsto \|r_{\Delta x}(\cdot,t)\|_{L^1(\mathbb{R})}$ is a non-increasing function. Thus for $0 \le s \le t \le T$

$$\begin{aligned} \|r_{\Delta x}(\cdot,t) - r_{\Delta x}(\cdot,s)\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} r_{\Delta x}(x,s) - r_{\Delta x}(x,t)\,dx \\ &= -\int_{\mathbb{R}}\int_s^t \partial_t r_{\Delta x}(x,\tau)\,d\tau\,dx \\ &\le \int_s^t \int_{\mathbb{R}} e_1(x,\tau)\,dx\,d\tau \end{aligned}$$

Now, e_1 is in $L^1(\Omega)$, with norm bounded independently of Δx. Thus $t \mapsto r_{\Delta x}(\cdot,t)$ is in $C([0,T]; L^1(\mathbb{R}))$ with a modulus of continuity bounded independently of Δx. Then

$$\begin{aligned} \|r(\cdot,t) - r(\cdot,s)\|_{L^1(\mathbb{R})} \le \|r(\cdot,t) - r_{\Delta x}(\cdot,t)\|_{L^1(\mathbb{R})} &+ \|r_{\Delta x}(\cdot,t) - r_{\Delta x}(\cdot,s)\|_{L^1(\mathbb{R})} \\ &+ \|r_{\Delta x}(\cdot,s) - r(\cdot,s)\|_{L^1(\mathbb{R})}. \end{aligned}$$

The first and the last term above can be made arbitrarily small by choosing Δx small, and the middle term is small if s is close to t. Hence $t \mapsto \|r(\cdot,t)\|_{L^1(\mathbb{R})}$ is continuous.

To see that r is a distributional subsolution of the conservation law, multiply (3.6) with a non-negative test function ψ and integrate over x and t to obtain

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} r_{\Delta x}\psi_t + f(r_{\Delta x})\psi_x\,dxdt + \int_{\mathbb{R}} r_{\Delta x}(0,x)\psi(0,x)\,dx \\ &\qquad\ge \int_0^T \sum_j f(r_j)\frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}}\int_x^{x+\Delta x} \left(\psi_x(x,t) - \psi_x(z,t)\right)\,dz\,dx\,dt. \end{aligned}$$

The term on the right tends to 0 as $\Delta x \to 0$, which shows that r is a subsolution. $\square$

Let now

$$\tau_j^{(i)} = \frac{u_j^{(i)} - (u_j, e)}{(u_j, e)},$$

if $u_j \ne 0$. If $u_j = 0$ set $\tau_j^{(i)} = \tau_{j+1}^{(i)}$. Observe that this makes sense since $u_j^{(i)} = 0$ only if $r_j = 0$. We have

$$\frac{d}{dt}u_j^{(i)}(t) + u_j^{(i)}(t)D_-\phi_j + \phi_{j-1}(t)D_-u_j^{(i)} = 0.$$

If $u_j^{(i)}(t) > 0$ for $t < t_0$ and $u_j^{(i)}(t_0) = 0$ then $du_j^{(i)}/dt(t_0) \le 0$. If $u_{j-1}^{(i)}(t_0) > 0$ then $du_j^{(i)}/dt(t_0) > 0$, which is a contradiction. Thus if $r_{j_0}(t_0) = 0$, then $r_j(t) = 0$ for all $j < j_0$ and $t > t_0$. Thus the definition of $\tau_j^{(i)}$ makes sense.

First note that

$$D_-\tau_j^{(i)} = \frac{\left(D_-u_j^{(i)}\right)(u_j,e) - u_j^{(i)}\left(D_-u_j,e\right)}{(u_j,e)\,(u_{j-1},e)}.$$

Using this, we find

$$\begin{aligned}
\frac{d}{dt}\tau_j^{(i)} &= \frac{\left(du_j^{(i)}/dt - (du_j/dt,e)\right)(u_j,e) - \left(u_j^{(i)} - (u_j,e)\right)(du_j/dt,e)}{(u_j,e)^2} \\
&= -\frac{D_-\left(u_j^{(i)}\phi_j\right)(u_j,e) - u_j^{(i)}\left(D_-\left(u_j\phi\right),e\right)}{(u_j,e)^2} \\
&= -\phi_{j-1}\frac{\left(D_-u_j^{(i)}\right)(u_j,e) - u_j^{(i)}\left(D_-u_j,e\right)}{(u_j,e)^2} \\
&= -\phi_{j-1}\frac{(u_{j-1},e)}{(u_j,e)}D_-\tau_j^{(i)}.
\end{aligned}$$

Set

$$\lambda_j = \begin{cases} \phi_{j-1}\frac{(u_{j-1},e)}{(u_j,e)} & \text{if } r_j > 0, \\ \lambda_{j+1} & \text{if } r_j = 0. \end{cases}$$

Now $\tau_j^{(i)}$ satisfies

$$\frac{d}{dt}\tau_j^{(i)} + \lambda_j D_-\tau_j^{(i)} = 0. \tag{3.12}$$

Lemma 3.5. *If*

$$\left|\tau_{\Delta x}^{(i)}(\cdot,0)\right|_{B.V.(\mathbb{R})} \le C$$

for some constant C which is independent of Δx, then there is a subsequence of $\{\Delta x\}$ (not relabeled) and functions $\tau^{(i)}$ in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$ such that $\tau_{\Delta x}^{(i)}(\cdot,t) \to \tau^{(i)}(\cdot,t)$ in $L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof. Note that $\lambda_j \ge 0$, and that λ_j is bounded. Set $\theta_j = D_-\tau_j^{(i)}$. Then θ_j satisfies

$$\frac{d}{dt}\theta_j + \lambda_{j-1}D_-\theta_j + \theta_j D_-\lambda_j = 0. \tag{3.13}$$

Let $\eta_\alpha(\theta)$ be a smooth approximation to $|\theta|$ such that

$$\eta_\alpha''(\theta) \ge 0 \quad \text{and} \quad \lim_{\alpha\to 0}\eta_\alpha(\theta) = \lim_{\alpha\to 0}\left(\theta\eta_\alpha'(\theta)\right) = |\theta|\,.$$

We multiply (3.13) by $\eta_\alpha'(\theta_j)$ to get an equation satisfied by $\eta_\alpha(\theta_j)$. Observe that

$$\begin{aligned}
\lambda_{j-1}\eta_\alpha'(\theta_j)D_-\theta_j + \theta_j\eta_\alpha'(\theta_j)D_-\lambda_j &= \lambda_{j-1}D_-\eta_\alpha(\theta_j) + \theta_j\eta_\alpha'(\theta_j)D_-\lambda_j \\
&\qquad + \frac{\Delta x}{2}\lambda_{j-1}\eta_\alpha''(\theta_{j-1/2})\left(D_-\theta_j\right)^2 \\
&\ge D_-\left(\lambda_j\eta_\alpha(\theta_j)\right) \\
&\qquad + \left(\theta_j\eta_\alpha'(\theta_j) - \eta_\alpha(\theta_j)\right)D_-\lambda_j.
\end{aligned}$$

Hence

$$\frac{d}{dt}\eta_\alpha(\theta_j) + D_-\left(\lambda_j\eta_\alpha(\theta_j)\right) \le \left(\eta_\alpha(\theta_j) - \theta_j\eta_\alpha'(\theta_j)\right)D_-\lambda_j.$$

Now let $\alpha \to 0$ to obtain

$$\frac{d}{dt}\left|\theta_j\right| + D_-\left(\lambda_j\left|\theta_j\right|\right) \le 0. \tag{3.14}$$

If we multiply this with Δx, sum over j and integrate in t, we find that

$$\left|\tau_{\Delta x}^{(i)}(\cdot,t)\right|_{B.V} \le \left|\tau_{\Delta x}^{(i)}(\cdot,0)\right|_{B.V.} \le C \tag{3.15}$$

Note that, since $\tau_{\Delta x}^{(i)}(\cdot,t)$ has bounded variation and satisfies (3.12), it is L^1_{loc} Lipschitz continuous in t, that is

$$\left\|\tau_{\Delta x}^{(i)}(\cdot,t) - \tau_{\Delta x}^{(i)}(\cdot,s)\right\|_{L^1_{\mathrm{loc}}(\mathbb{R})} \le \sup_j \lambda_j \left|\tau^{(i)}(\cdot,0)\right|_{B.V.} |t-s|\,. \tag{3.16}$$

Hence, the above estimates (3.15), (3.16) and an application of Kolmogorov's compactness criterion (Lemma 2.3) shows that $\tau^{(i)} = \lim_{\Delta x\to 0}\tau_{\Delta x}^{(i)}$ is continuous in t, with values in $L^1_{\mathrm{loc}}(\mathbb{R})$. In other words, the convergence is in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$. $\square$

Now we have the strong convergence of $r_{\Delta x}$ and of $\tau_{\Delta x}^{(i)}$. This means that also $u_{\Delta x}$ converges strongly to some function u in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$ since we have

$$u_{\Delta x}^{(i)} = r_{\Delta x}\sin\left(\varphi_{\Delta x}^{(i)}\right), \tag{3.17}$$

where

$$\varphi_{\Delta x}^{(i)} = \tan^{-1}\left(\tau_{\Delta x}^{(i)}\right)$$

and $\tau \mapsto \tan^{-1}(\tau)$ is a continuous function.

Theorem 3.1. *Let ϕ be a twice differentiable function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(r) > 0$ and $\phi'(r) \ge 0$, and*

$$\mathrm{meas}\left\{r \;\middle|\; 2\phi'(r) + r\phi''(r) = 0\right\} = 0.$$

Let $u_{\Delta x}$ be defined by (3.1)–(3.2). If $u_0 \in \Gamma_\delta$ for some $\delta < 1$, $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and

$$\left|\tau_{\Delta x}^{(i)}\right|_{B.V.(\mathbb{R})} \le C \quad \text{for } i = 1,\dots,n$$

and C is independent of Δx, then there exists a function u in $C([0,T];L^1_{\mathrm{loc}}(\mathbb{R}))$ such that $u_{\Delta x} \to u$ as $\Delta x \to 0$. The function u is a weak solution to (1.1).

Proof. We have already established convergence. Regarding the continuity, since $r_{\Delta x}$ and $\tau_{\Delta x}^{(i)}$ are $L^1_{\mathrm{loc}}(\mathbb{R})$ continuous in time, (3.17) implies that also u has this continuity.

It remains to show that u is a weak solution. To this end, observe that[1]

$$\int_0^T\int_{\mathbb{R}} D_-\left(u_{\Delta x}\phi(r_{\Delta x})\right)\psi(x,t)\,dxdt = -\int_0^T\int_{\mathbb{R}} u_{\Delta x}\phi(r_{\Delta x})D_+\psi(x,t)\,dxdt.$$

As $\Delta x \to 0$, $D_+\psi \to \psi_x$ for any $\psi \in C_0^1(\Omega)$. This means that u is a weak solution. $\square$

[1]Here we "extend" the definition of D_- and D_+ to arbitrary functions in the obvious manner.

4. FULLY DISCRETE SCHEMES

In this section, we propose two different fully discrete schemes and shown one of them converges to the unique entropy solution of (1.1). We start by introducing some notations needed to define the fully discrete finite difference schemes. We reserve Δt to denote a small positive number that represent the temporal discretization parameter of the numerical schemes. For $n = 0, 1, \cdots, N$, where $N\Delta t = T$, for some fixed time horizon $T > 0$, we set $t^n = n\Delta t$. For any function $v(t)$, admitting pointvalues, we let D_+^t denote the discrete forward difference operator in the time direction, i.e.,

$$D_+^t v(t) = \frac{v(t+\Delta t) - v(t)}{\Delta t}.$$

Furthermore, we introduce the spatial-temporal grid cells

$$I_j^n = [x_{j-1/2}, x_{j+1/2}) \times [t^n, t^{n+1}).$$

As before, to a sequence $\left\{u_j^n\right\}_{j\in\mathbb{Z}, n\geq 0}$ we associate the function $u_{\Delta x}$ defined by

$$u_{\Delta x}(x,t) = \sum_{j\in\mathbb{Z}, n\geq 0} u_j^n \mathbb{1}_{I_j^n}(x,t),$$

similarly, we also define $r_{\Delta x}$ as

$$r_{\Delta x}(x,t) = \sum_{j\in\mathbb{Z}, n\geq 0} r_j^n \mathbb{1}_{I_j^n}(x,t),$$

where $\mathbb{1}_A$ denotes the characteristic function of the set A.

First, we consider the following fully discrete finite difference scheme

$$D_+^t u_j^n + D_-\left(u_j^n \phi\left(\left|u_j^n\right|\right)\right) = 0, \tag{4.1}$$

with initial values

$$u_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u_0(x)\,dx.$$

As before, it is not difficult to prove the following lemma:

Lemma 4.1. *If* $u_0 \in L^2(\mathbb{R})^n \cap L^\infty(\mathbb{R})^n$,

$$\begin{aligned} \left\|u_{\Delta x}(\cdot, t_n)\right\|_{L^2(\mathbb{R})^n}^2 &\leq \left\|u_0\right\|_{L^2(\mathbb{R})^n}^2, \\ \left\|u_{\Delta x}(\cdot, t_n)\right\|_{L^\infty(\mathbb{R})^n}^2 &\leq \left\|u_0\right\|_{L^\infty(\mathbb{R})^n}^2, \end{aligned} \tag{4.2}$$

for all $n > 0$, *furthermore*

$$\Delta t \Delta x \sum_{n=0, j\in\mathbb{Z}}^{N-1} \Delta x \left|D_- u_j^n\right|^2 \leq 2\left\|u_0\right\|_{L^2(\mathbb{R})^n}^2, \tag{4.3}$$

where $N = \operatorname{ceil}(T/\Delta t)$.

Next, regarding the strong convergence of $R := r^2$, we have the following result

Lemma 4.2. *Suppose*

$$\begin{aligned} &\lim_{r\downarrow 0} \frac{\phi'(r)}{r} \leq \infty, \\ &\operatorname{meas}\left\{r \;\middle|\; 3\phi'(r) + 2r\phi''(r) = 0\right\} = 0, \end{aligned} \tag{4.4}$$

then there is a subsequence of $\{R_{\Delta x}\}$ *(not relabeled) and a function* R *such that* $R_{\Delta x} \to R$ *a.e.* $(x,t) \in \Omega$. *We have that* $R \in L^\infty([0,T]; L^1(\mathbb{R}))$. *Furthermore,* R *satisfies*

$$\begin{cases} R_t + G(R)_x \leq 0, & x \in \mathbb{R},\ t > 0, \\ R = |u_0|^2, & x \in \mathbb{R},\ t = 0, \end{cases}$$

in the distributional sense.

Since $R_{\Delta x}$ converges strongly to R, also $r_{\Delta x}$ will converge strongly to $r := \sqrt{R}$. The sequence $\{u_{\Delta x}\}_{\Delta x > 0}$ is uniformly bounded, so a subsequence will converge weak $*$ to some function $u \in L^\infty(\Omega)$. Hence the limit u is a distributional solution of

$$u_t + \left(u\phi(r)\right)_x = 0.$$

In order to conclude that u is a weak solution to (1.1), we would have to show that $|u| = r$. We have not been able to prove this, and merely conclude that $|u| \leq r$. The reason for this is that $v \mapsto |v|$ is convex, and that weak limits of a convex function are not less than the convex function of the weak limit.

To overcome this difficulty, we propose another fully discrete scheme based on explicit decoupling of the variables r and v.

4.1. **A scheme which enforces the entropy condition.** Let us define $w_{\Delta x} = \frac{u_{\Delta x}}{r}$ and let $r_{\Delta x}$ and $w_{\Delta x}$ satisfies

$$\begin{cases} r_j^{n+1} = r_j^n - \Delta t D_- f_j^n, & n \geq 0, \\ r_j^0 = \left|u_j^0\right|, \end{cases} \tag{4.5}$$

and

$$\begin{cases} w_j^{n+1} = w_j^n - \Delta t \phi_j^n D_- w_j^n, & n \geq 0, \\ r_j^0 w_j^0 = u_j^0. \end{cases} \tag{4.6}$$

Regarding the convergence of the approximations $\{r_{\Delta x}\}$ we choose

$$\Delta t \left\| f' \right\|_{L^\infty(\mathbb{R})} \leq \Delta x. \tag{4.7}$$

We list some useful properties of $r_{\Delta x}$ in the next lemma [9].

Lemma 4.3. *Assume that the CFL condition (4.7) holds and* $r_0 \in BV(\mathbb{R}) \cap L^\infty(\mathbb{R})$. *Then for each* $\Delta x > 0$ *we have that*

(a) $-M \leq r_{\Delta x}(x,t) \leq M$, *for all* x *and* $t > 0$.

(b) *For* $n \geq 0$ *the functions*

$$n \mapsto \Delta x \sum_{j\in\mathbb{Z}} \left|r_j^n\right|, \quad n \mapsto \sum_{j\in\mathbb{Z}} \left|r_j^n - r_{j-1}^n\right|, \quad n \mapsto \sum_{j\in\mathbb{Z}} \left|r_j^{n+1} - r_j^n\right|$$

are non-increasing. In particular this means that the family $\{r^{\Delta x}\}_{\Delta x > 0}$ *is (uniformly in* Δx*) bounded in* $L^\infty(\mathbb{R}^+; L^1(\mathbb{R})) \cap \mathrm{BV}(\mathbb{R} \times \mathbb{R}^+)$.

(c) *Moreover* $r_{\Delta x}(\cdot,t) \to r(\cdot,t)$ *strongly in* $L^1(\mathbb{R})$ *for all* $t \geq 0$, *where* $r \in \mathrm{Lip}([0,T]; L^1(\mathbb{R}))$ *and is the unique entropy (in the sense of Kružkov) solution of the conservation law*

$$\begin{cases} r_t + f(r)_x = 0, \\ r(x,0) = |u(x,0)|. \end{cases} \tag{4.8}$$

Observe that, we can write the scheme (4.6) as

$$w_j^{n+1} = \left(1 - \lambda\phi_j^n\right) w_j^n + \lambda\phi_j^n w_{j-1}^n.$$

If $\lambda\phi_j^n < 1$ for all j, then w_j^{n+1} is a convex combination of w_j^n and w_{j-1}^n. Thus

$$\inf_j w_j^0 \le w_j^n \le \sup_j w_j^0, \quad n > 0. \tag{4.9}$$

and

$$\left|w_{\Delta x}(\cdot,t)\right|_{B.V.(\mathbb{R})^n} \le \left|w_{\Delta x}(\cdot,0)\right|_{B.V.(\mathbb{R})^n}. \tag{4.10}$$

Furthermore

$$\Delta x \sum_j \left|w_j^{n+1} - w_j^n\right| \le \Delta t \left\|\phi\right\|_{L^\infty} \sum_j \left|w_j^n - w_{j-1}^n\right| \le C\Delta t \left|w_{\Delta x}(\cdot,t_n)\right|_{B.V.(\mathbb{R})^n}. \tag{4.11}$$

Hence the map $t \mapsto w_{\Delta x}(\cdot,t)$ is L^1- Lipschitz continuous. Finally, the above estimates (4.10), (4.11) and an application of Kolmogorov's compactness criterion (Lemma 2.3) shows that $w = \lim_{\Delta x\to 0} w_{\Delta x}$ is continuous in t, with values in $(L^1_{\mathrm{loc}}(\mathbb{R}))^n$.

Multiply the equation (4.5) for r_j^{n+1} with that (4.6) for w_j^{n+1} to get

$$\begin{aligned}
r_j^{n+1}w_j^{n+1} &= \left(r_j^n - \Delta t D_- f_j^n\right)\left(w_j^n - \Delta t\phi_j^n D_- w_j^n\right)\\
&= r_j^n w_j^n - \Delta t\left(w_j^n D_- f_j^n + f_j^n D_- w_j^n\right) + \Delta t^2 \phi_j^n D_- f_j^n D_- w_j^n\\
&= r_j^n w_j^n - \Delta t\left(w_j^n D_- f_j^n + f_{j-1}^n D_- w_j^n\right) - \Delta t\left(f_j^n - f_{j-1}^n\right) D_- w_j^n\\
&\qquad + \Delta t^2 \phi_j^n D_- f_j^n D_- w_j^n\\
&= r_j^n w_j^n - \Delta t D_-\left(f_j^n w_j^n\right) + \Delta t\left(f_j^n - f_{j-1}^n\right) D_- w_j^n\left(\lambda\phi_j^n - 1\right).
\end{aligned}$$

Then we have

$$D_+^t\left(r_j^n w_j^n\right) + D_-\left(f_j^n w_j^n\right) = \Delta t\left(\lambda\phi_j^n - 1\right)\left(f_j^n - f_{j-1}^n\right) D_- w_j^n =: e_j^n. \tag{4.12}$$

Let now $\psi \in C_0^\infty(\Omega)$ be a test function, multiply the above equation by ψ and integrate over Ω to get

$$\begin{aligned}
&\sum_{n=1,j}^{\infty} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} r_j^n w_j^n D_-^t\psi + f_j^n w_j^n D_+\psi \,dxdt\\
&\qquad + \frac{1}{\Delta t}\int_0^{\Delta t} \sum_j \int_{x_{j-1/2}}^{x_{j+1/2}} r_j^0 w_j^0 \psi \,dxdt = \sum_{n,j} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} e_j^n \psi \,dxdt.
\end{aligned}$$

Since we have the convergence of $r_{\Delta x}$ and $w_{\Delta x}$, the left hand side of this converges to

$$\iint_\Omega rw\psi_t + f(r)w\psi_x \,dxdt + \int_{\mathbb{R}} r(x,0)w(x,0)\psi(x,0)\,dx.$$

Regarding the right hand side we have

$$\begin{aligned}
\Big|\sum_{n,j} \int_{t_n}^{t_{n+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} e_j^n \psi \,dxdt\Big| &\le \Delta t \left\|\psi\right\|_{L^\infty(\Omega)} \left(\lambda\left\|\phi\right\|_{L^\infty} + 1\right) \Delta t \sum_{n,j} \left|f_j^n - f_{j-1}^n\right| \left|w_j^n - w_{j-1}^n\right|\\
&\le \Delta t C \left\|\psi\right\|_{L^\infty(\mathbb{R})} \left\|w_{\Delta x}(x,0)\right\|_{L^\infty(\mathbb{R})^n} T \left\|r_{\Delta x}(x,0)\right\|_{B.V.(\mathbb{R})},
\end{aligned}$$

where T is such that $\mathrm{supp}\psi \subset [0,T]$. Hence

$$\iint_\Omega rw\psi_t + f(r)w\psi_x \, dxdt + \int_{\mathbb{R}} r(x,0)w(x,0)\psi(x,0)\, dx = 0.$$

Hence, we see that rw is a weak solution to the Cauchy problem

$$(rw)_t + (f(r)w)_x = 0.$$

In other words, (r, rw) is a weak solution to

$$\begin{aligned} r_t + f(r)_x &= 0, \\ (r\,w)_t + (\phi(r) r\, w)_x &= 0. \end{aligned} \tag{4.13}$$

Next, we follow the argument given in [6] and conclude that $|w| = 1$ using Wagner transformation.

Finally, collecting all the results above, we have proved the following theorem.

Theorem 4.1. *Assume that $u_0 \in B.V.(\mathbb{R})$. If $\lambda = \Delta t/\Delta x$ satisfies the CFL-condition $\lambda < \sup_x f'(|u_0(x)|)$, and $u_{\Delta x}$ is defined by (4.5), (4.6), then $u = \lim_{\Delta x \to 0} u_{\Delta x}$ is the unique entropy (in the sense of Definition 1.2) solution to (1.1).*

Remark 4.1. *We propose another scheme based on discretizing the "conservative" form (1.5)–(1.6). Let $r_{\Delta x}$ and $u_{\Delta x}$ satisfy*

$$\begin{cases} r_j^{n+1} = r_j^n - \Delta t D_- f(r_j^n), & n \geq 0, \\ r_j^0 = \left|u_j^0\right|, \end{cases} \tag{4.14}$$

and

$$u_j^{n+1} = u_j^n - \Delta t D_- \left(u_j^n \phi(r_j^n)\right), \tag{4.15}$$

for $n \geq 0$ and $f(r) = r\phi(r)$, with u_j^0 given by (3.2). In this case, using the strong convergence of $r_{\Delta x}$, it is easy to show that $u = \lim_{\Delta x \to 0} u_{\Delta x}$ is a distributional solution of

$$u_t + (u\phi(r))_x = 0.$$

In this case, again using Wagner transformation, it is easy to prove $|u| = r$. In other words, $u = \lim_{\Delta x \to 0} u_{\Delta x}$ is the unique entropy solution of (1.1).

5. Numerical experiments

We close this paper by demonstrating how these schemes work in practice. We perform all the computations for 2×2 system with $\phi(r) = r^2$.

5.1. **Numerical Experiment-1.**

In this case we approximate the system (1.1) with initial data

$$U_0(x) = \begin{cases} U_l, & x < 0, \\ U_r, & x > 0. \end{cases}$$

It is not difficult to find the exact solution of (1.1) in this case. For the sake of completeness we write the explicit form of the exact solutions $U(x,t) = \bar{U}(x/t)$.

If $|U_l| < |U_r|$, then

$$\bar{U}(\xi) = \begin{cases} U_l, & \xi \leq |U_l|^2, \\ U_m, & |U_l|^2 \leq \xi \leq 3\,|U_l|^2, \\ (\frac{\xi}{3})^{1/2} \frac{U_r}{|U_r|}, & 3\,|U_l|^2 \leq \xi \leq 3\,|U_r|^2 \\ U_r, & \xi \geq 3\,|U_r|^2, \end{cases}$$

If $|U_l| > |U_r|$, then

$$\bar{U}(\xi) = \begin{cases} U_l, & \xi \leq |U_l|^2, \\ U_m, & |U_l|^2 \leq \xi \leq |U_l|^2 + |U_l|\,|U_r| + |U_r|^2, \\ U_r, & \xi \geq |U_l|^2 + |U_l|\,|U_r| + |U_r|^2, \end{cases}$$

with $U_m = \frac{|U_l|}{|U_r|} U_r$ in both cases.

In what follows, we test the fully discrete explicit numerical scheme (4.1) with initial data

$$U_0(x) = \begin{cases} U_-, & x < 0, \\ U_+, & x > 0, \end{cases}$$

where

$$U_- = (0.5, 1.5), \qquad U_+ = (1.5, 2.0),$$

for the first experiment and

$$U_- = (1.5, 2.0), \qquad U_+ = (0.5, 1.5),$$

for the second experiment. The computations are performed on a computational domain $[-5, 20]$ with 4000 mesh points. To enforce the CFL condition we set the time step $\Delta t = (CFL)\Delta x / 3 \sup |U_0|^2$, where we use a CFL number 0.75. Although we do not plot the exact solutions, a comparison of the computational results displayed in Figs 5.1 with the exact solution shows good agreement.

5.2. **Numerical experiment- 2.** In this case, we test our fully discrete explicit numerical scheme (4.5)–(4.15) with initial data $U_0 = r_0 w_0$, where

$$r_0(x) = \begin{cases} r_-, & x < 0, \\ r_+, & x > 0, \end{cases}$$

with

$$r_- = 1.0, \qquad r_+ = 0.75,$$

for the first and third numerical experiments and

$$r_- = 0.75, \qquad r_+ = 1.0,$$

for the second and fourth numerical experiments. Similarly, for w_0 we take

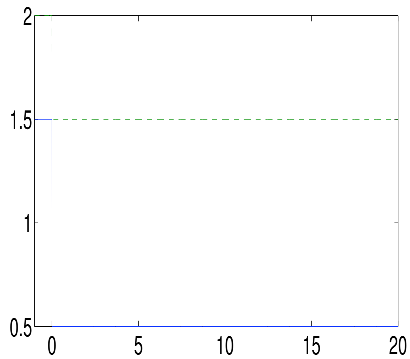
$$w_0(x) = \begin{cases} (1.0, 0.0), & x < 0.2 \\ (\cos(8\pi(x - 0.2)), \sin(8\pi(x - 0.2))), & 0.2 \leq x \leq 0.7, \\ (1.0, 0.0), & x \geq 0.7, \end{cases}$$

for the first and second numerical experiments and

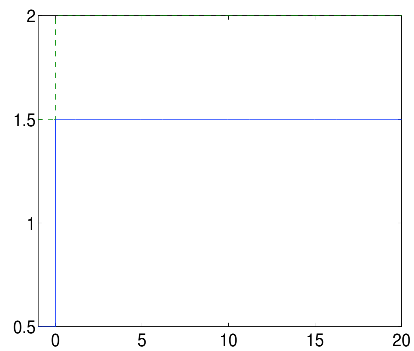
$$w_0(x) = \begin{cases} (1.0, 0.0), & x \leq 0.2, \\ (-1.0, 0.0), & x \geq 0.2, \end{cases}$$

In this case also, it is easy to find the exact solution. Although we do not plot the exact solutions, we give the explicit form of the exact solution. The exact solution is given by $U = rw$ with

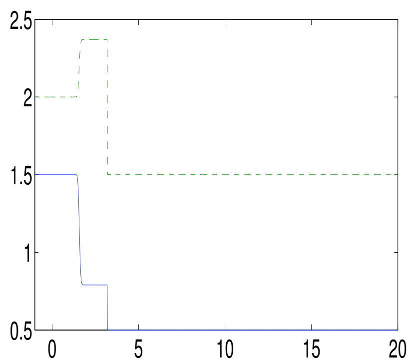
$$r(x,t) = \begin{cases} r_-, & x \leq s\,t, \\ r_+, & x \geq s\,t, \end{cases} \quad \text{with } s = r_-^2 + r_- r_+ + r_+^2,$$



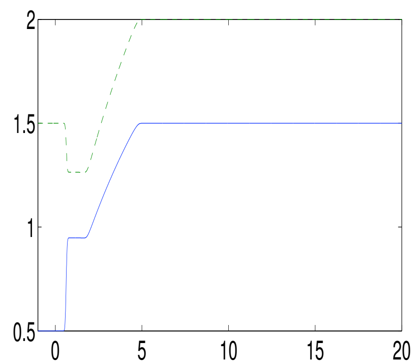
(a) $T = 0$



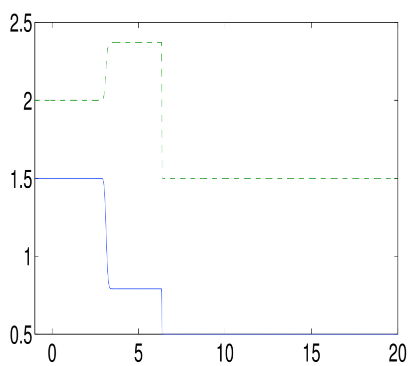
(b) $T = 0$



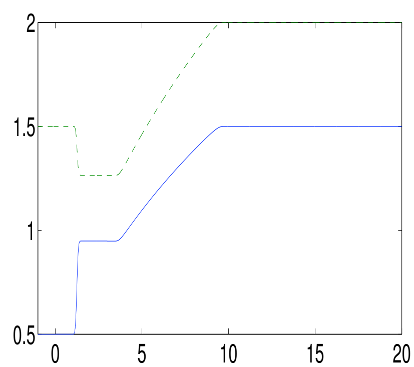
(c) $T = 0.25$



(d) $T = 0.25$



(e) $T = 0.50$



(f) $T = 0.50$

FIGURE 5.1. Left column: Experiment-1: The dotted-dashed curve represents the first component of U, the solid curve represents the second component. Right column: Experiment-2: The dotted-dashed curve represents the first component of U, the solid curve represents the second component.

and

$$w(x,t) = \begin{cases} w_0(x - r_-^2 t), & x \le r_-^2 t, \\ w_0(\frac{r_-}{r_+}(x - r_-^2 t)), & r_-^2 t \le x \le s\,t, \\ w_0(x - r_+^2 t), & x \ge s\,t, \end{cases}$$

for the first and third numerical experiments. Similarly,

$$r(x,t) = \begin{cases} r_-, & x \le 3r_-^2 t, \\ (x/3t)^{1/2}, & 3r_-^2 t \le x \le 3r_+^2 t \\ r_+, & x \ge 3r_+^2 t, \end{cases}$$

and

$$w(x,t) = \begin{cases} w_0(x - r_-^2 t), & x \le r_-^2 t, \\ w_0(\frac{r_-}{r_+}(x - r_-^2 t)), & r_-^2 t \le x \le 3r_-^2 t, \\ w_0(\frac{2}{3\sqrt{3}r_+} x^{3/2} t^{-1/2}), & 3r_-^2 t \le x \le 3r_+^2 t \\ w_0(x - r_+^2 t), & x \ge 3r_+^2 t, \end{cases}$$

for the second and fourth numerical experiments.

In all the experiments computational domain is $[-1, 4]$ and we use Neumann boundary conditions at the left boundary. We also use a CFL number 0.75 and 4000 mesh points for all the experiments. A comparison of the computational results displayed in Figs 5.2–5.3 with the exact solution shows good agreement.

Below we show the computational results for four different qualitatively significant sets of data: a compression or an expansion wave in r inisiated slightly behind a continuous pulse or a discontinuous contact wave in w. Fig 5.2–5.3 display the computed solution at three different times. In the plots, the dot-dash curve represents the first component of U and the dotted curve represents the second component, while the solid curve represents the r-component of (r, U).

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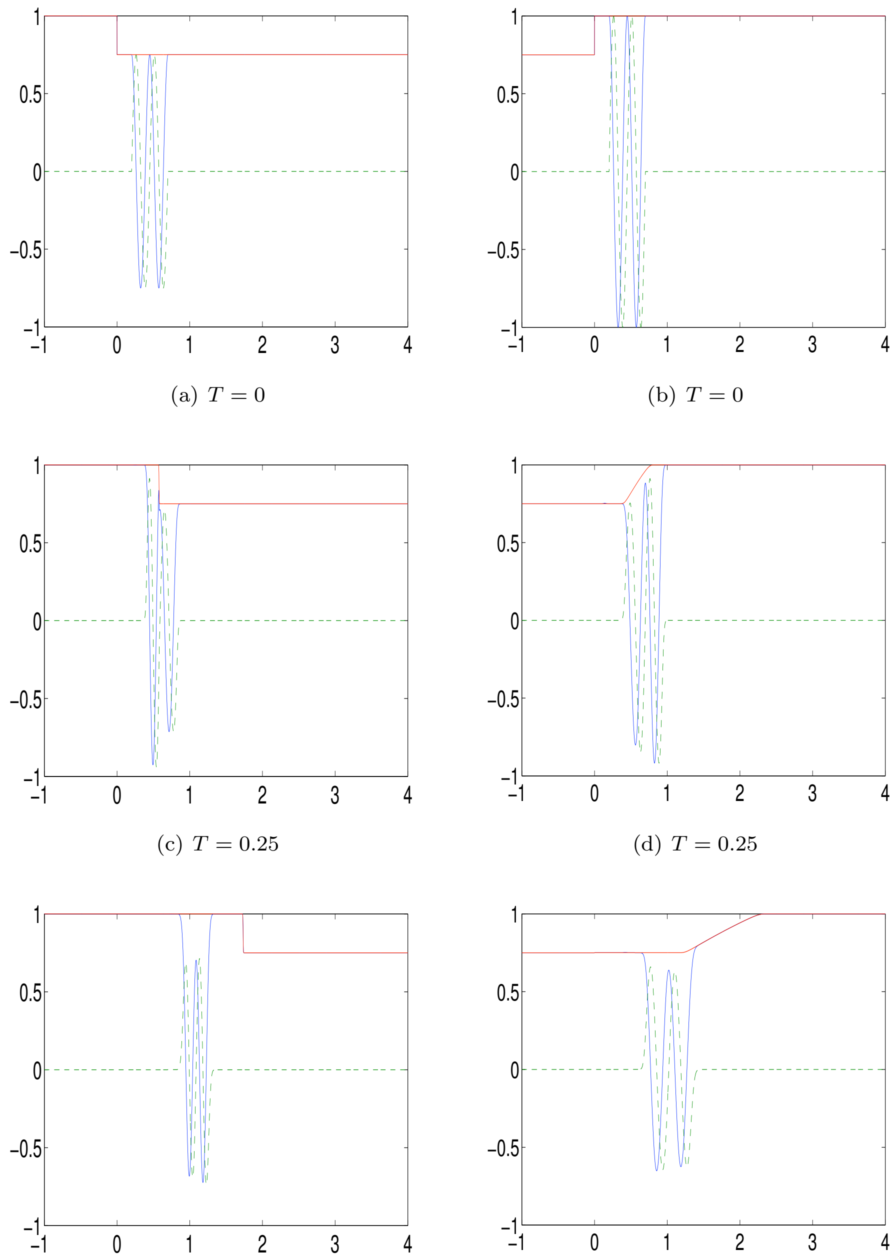
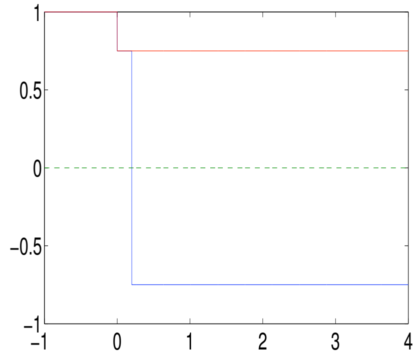


(a) $T = 0$ (b) $T = 0$

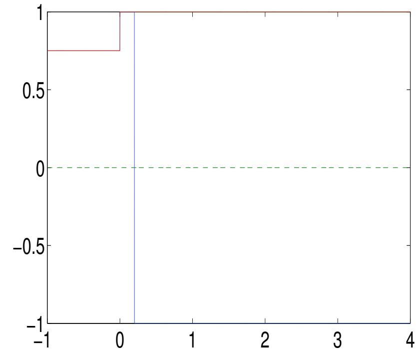
(c) $T = 0.25$ (d) $T = 0.25$

(e) $T = 0.75$ (f) $T = 0.75$

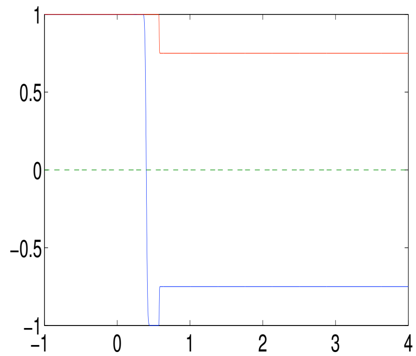
FIGURE 5.2. Left column: Experiment-1: A shock wave initiated behind a continuous rotational wave. The dotted-dashed curve represents the first component of U, the dotted curve represents the second component and the solid curve represents r. Right column: Experiment-2: An expansion wave initiated behind a continuous rotational wave. The dotted-dashed curve represents the first component of U, the dotted curve represents the second component and the solid curve represents r.



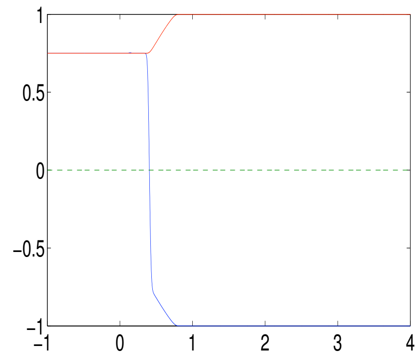
(a) $T = 0$



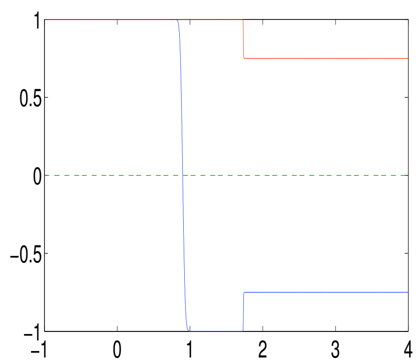
(b) $T = 0$



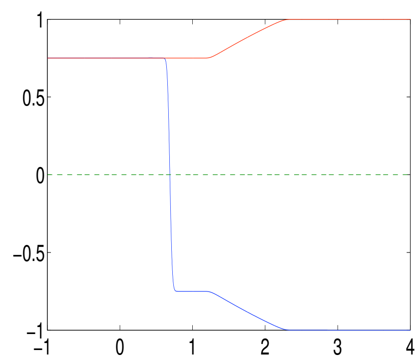
(c) $T = 0.25$



(d) $T = 0.25$



(e) $T = 0.75$



(f) $T = 0.75$

FIGURE 5.3. Left column: Experiment-3: A shock wave initiated behind a discontinuous rotational wave. The dotted-dashed curve represents the first component of U, the dotted curve represents the second component and the solid curve represents r. Right column: Experiment-4: An expansion wave initiated behind a discontinuous rotational wave. The dotted-dashed curve represents the first component of U, the dotted curve represents the second component and the solid curve represents r.

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