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SPACE-TIME VARIATIONAL SADDLE POINT FORMULATIONS OF STOKES AND NAVIER–STOKES EQUATIONS

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Abstract. The instationary Stokes and Navier-Stokes equations are considered in a simultaneously space-time variational saddle point formulation, so involving both velocities and pressure. For the instationary Stokes problem, it is shown that the corresponding operator is a *boundedly invertible* linear mapping between suitable Hilbert spaces, being Cartesian products of (intersections of) Bochner spaces, or duals of those. Based on these results, for the instationary Navier-Stokes equations local uniqueness of solutions is established, as well as existence for sufficiently small data.

1. Introduction

1.1. **Background and motivation.** The classical approach to the existence of weak solutions of the instationary, incompressible Navier-Stokes Equations views these equations as an infinite-dimensional dynamical system (see, e.g., [Tem79] and the references there). From this point of view, most methods for the *numerical solution* of the instationary (Navier–) Stokes equations are *time marching* methods: assuming that some approximate solution on time t is available, for a sufficiently small time increment $\Delta t > 0$, an approximate solution on time $t + \Delta t$ is computed by solving a corresponding stationary problem.

Because of the generally lacking global smoothness of the solution, efficient numerical schemes have to be adaptive. With suitable time-marching schemes, it is possible to adapt both the spatial 'mesh', and the time step Δt depending on t. Combined space-time adaptivity, where Δt is adapted depending on the spatial location, are not easily accomodated by classical time stepping schemes, although some studies on local time stepping have appeared, see e.g. [EL94, FNWW09, Sav08]. In any case, due to the character of time marching, it seems very hard to guarantee a kind of quasi-optimal distribution of the 'grid-points' over space and time, and no results in this direction are presently known to us.

To develop an alternative for time marching schemes, in [SS09, CS11] we studied *simultaneously space-time variational formulations* of linear *parabolic* evolution equations. The operators defined by such variational formulations were shown to be *boundedly invertible* between a Hilbert space H_1 and the dual of another Hilbert space H_2, both H_1 and H_2 being Bochner spaces or intersections of those.

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By equipping H_1 and H_2 with Riesz bases, being *tensor products* of temporal and spatial wavelet collections, the space-time variational problem was written as an *equivalent* well-posed, bi-infinite, symmetric positive definite matrix-vector system by forming normal equations. By running on this system an adaptive wavelet scheme, in its original form being proposed in [CDD01], a sequence of approximations is produced in linear computational complexity that converges with the best possible nonlinear approximation rate from the basis, i.e., the rate of the so-called best N-term approximations.

Because of the application of tensorized wavelet collections in space and time, under mild (Besov) smoothness conditions the latter rate is *equal* (in some situations up to log-factors) to the best possible approximation rate for the solution of a corresponding stationary problem from the spatial wavelet basis, i.e., there is (hardly) no increase in the order of computational complexity as a consequence of the additional time dimension.

Besides the computational realization of the best possible nonlinear approximation rate, the latter property is a major advantage when the approximate solution is needed as function of simultaneously space and time, as it is the case for example in time-dependent optimal control problems, see [GK11]. Indeed, with time marching schemes this would require the availability of the approximate solutions simultaneously at all discrete times, requiring a huge amount of memory.

The results mentioned so far generalize to simultaneously space-time variational formulations of *nonlinear* parabolic evolution equations when they define a mapping from H_1 into H_2', and the Fréchet derivative at the solution is *boundedly invertible* between H_1 and H_2' (see [Ste11a]). The latter condition is satisfied for example for a semi-linear equation with a time-independent spatial operator.

Aiming at the application of space-time variational formulations to the incompressible instationary (*Navier–*) *Stokes* equations, there are two possibilities. The first one is to reduce these equations to problems for the divergence-free velocities only. Then the Stokes equations read as a linear parabolic evolution problem, and the aforementioned results concerning well-posed space-time variational formulations apply. The reduction to divergence-free velocities is also the standard approach followed in the literature for demonstrating existence and uniqueness of solutions (see e.g. [Tem79, DL92]).

In [Ste11b], we investigated the application of the adaptive wavelet scheme to the space-time variational divergence-free velocities formulation of the instationary Stokes problem. With the standard choice of the Hilbert spaces H_1 and H_2, it turns out that it is necessary to construct wavelet collections on the spatial domain $\Omega \subset \mathbb{R}^n$ that are both, properly scaled, a basis for the space of divergence-free $H^1(\Omega)^n$-functions subject to essential boundary conditions, and for its dual space. Since so far we are not able to construct such wavelets, using an H^2-regularity result for the stationary Stokes operator we showed bounded invertibility of the instationary Stokes operator for alternative spaces H_1 and H_2 which are defined by making a shift in the spatial smoothness index of the Bochner spaces. This result is similar in spirit to the regularity result in [Tem79, Ch.III, §1, Prop. 1.2]. Wavelets suitable for this formulation were constructed for *rectangular* domains and free-slip boundary conditions.

Bounded invertibility results for *scales* of spaces H_1 and H_2 are demonstrated in [GSSt11], together with mapping properties of the Navier-Stokes operator.

1.2. **This paper.** The approach to tackle the instationary (Navier–) Stokes equations by a reduction to equations for the divergence-free velocities has the obvious disadvantage that no results for the pressure are obtained. Moreover, the numerical solution of these equations by an adaptive wavelet scheme requires a divergence-free wavelet basis, which seems to be realizable in very restricted settings only.

Therefore, in this paper as the second possibility we study *simultaneously space-time variational saddle point formulations* of the (Navier-) Stokes equations for the combined pair of velocities and pressure. For both free-slip and no-slip boundary conditions, we prove that the Stokes operator defined by this variational formulation is *boundedly invertible* between a Hilbert space H_1 and the dual of another Hilbert space H_2. In order to be able to arrive at this result, we have to assume H^2-regularity of the Poisson or of the stationary Stokes operator, which imposes standard smoothness or convexity conditions on the spatial domain.

Both trial- and test-spaces H_1 and H_2 are Cartesian products of spaces for velocities and pressure. Knowing the results for the space-time variational formulations of parabolic evolution equations, the velocity components of the test- and trial-spaces are as expected, and so are the corresponding pressure components of either test- or trial-space. The remaining pressure component, now being fully determined by the instationary Stokes operator, is less standard being the dual of the intersection of two Bochner spaces.

In any case for polytopal spatial domains, suitable wavelet bases can be constructed for all these spaces. Since for free-slip or no-slip boundary conditions, either a pressure or a velocity space requires C^1 wavelets, we expect on non-rectangular spatial domains concrete wavelet constructions to be technically involved.

With the spaces H_1 and H_2 as above, additionally it will be shown that the instationary Navier–Stokes operator maps H_1 into H_2' (for no-slip conditions on two- and three-dimensional domains, and for free-slip conditions on two-dimensional domains). Since our results for the instationary Stokes operator generalize to the linearized instationary Navier–Stokes operator –the difference being a lower order spatial differential operator–, we conclude that any Navier-Stokes solution is locally unique. Moreover, we can use the adaptive wavelet solver to approximate it with the *best possible nonlinear approximation rate in space-time tensorized bases*. Finally, since also Lipschitz continuity of the instationary Navier–Stokes operator will be shown, using a fixed-point argument we conclude existence of a space-time variational Navier–Stokes solution, albeit under a small data hypothesis.

To the best of our knowledge, *well-posedness*, i.e., *bounded invertibility* of the instationary Stokes operator for the combined pair of velocities and pressure has not been established before. Compare the discussion at the end of [Tem79, Ch.III, §1.5] where regularity of the pair of velocities and pressure is established only under *additional* smoothness conditions on the right-hand side. This well-posedness of the instationary Stokes operator, and more generally of the linearized instationary Navier-Stokes operator, is an essential ingredient for any numerical solution method that applies to the simultaneous space-time formulation of the (Navier-) Stokes equations for the combined pair of velocities and pressure (cf. [PR94]).

This paper is organized as follows: In Section 2, necessary and sufficient conditions are recalled for bounded invertibility of generalized linear saddle point problems. In Sections 3 and 4, these conditions are verified for space-time variational formulations of the instationary Stokes problem with free- and no-slip boundary conditions, respectively. In Section 5, the aforementioned mapping properties of the instationary Navier–Stokes operator with homogeneous initial datum are verified.

Throughout, with $C \lesssim D$ we will mean that C can be bounded by a multiple of D, independently of parameters on which C and D may depend. Obviously, $C \gtrsim D$ is defined as $D \lesssim C$, and $C \eqsim D$ as $C \lesssim D$ and $C \gtrsim D$.

2. Generalized saddle point problems

For Banach spaces U, V, P, and Q, and bounded bilinear forms $a : U \times V \to \mathbb{R}$, $b : P \times V \to \mathbb{R}$, and $c : U \times Q \to \mathbb{R}$, we consider the problem of finding $(u, p) \in U \times P$ that, for given $f \in V'$, $g \in Q'$, satisfy

$$a(u, v) + b(p, v) + c(u, q) = f(v) + g(q) \quad (v \in V,\, q \in Q). \tag{2.1}$$

In this section, we collect sufficient and necessary conditions for the corresponding $L : (u, p) \mapsto (f, g) \in \mathcal{L}(U \times P, V' \times Q')$ to be boundedly invertible. These conditions can already be found in [BCM88], and a Hilbert space setting, in [Nic82]. Since some intermediate results will be used in the following sections, we have chosen to include the short arguments.

We set $B \in \mathcal{L}(V, P')$, $C \in \mathcal{L}(U, Q')$ by

$$(Bv)(p) = b(p, v) = (B'p)(v) \quad \text{and} \quad (Cu)(q) = c(u, q) = (C'q)(u).$$

For a closed subspace Z of some Banach space X, the polar set $Z^0 \subset X'$ is defined by $\{f \in X' : f(Z) = 0\}$.

Theorem 2.1. *Recalling a, b and c being bounded, the variational problem* (2.1) *defines boundedly invertible linear mapping $U \times P \to V' \times Q'$ if and only if the following three conditions are satisfied:*

(i) *for all $f \in (\ker B)'$, there exists a unique $u \in \ker C$ such that $a(u, v) = f(v)$ $(v \in \ker B)$, and $\|u\|_U \lesssim \|f\|_{(\ker B)'}$,*
(ii) *for all $g \in Q'$, there exists a $u \in U$ such that $c(u, q) = g(q)$ $(q \in Q)$, and $\|u\|_U \lesssim \|g\|_{Q'}$,*
(iii) *for all $f \in (\ker B)^0$, there exists a unique $p \in P$ such that $b(p, v) = f(v)$ $(v \in V)$, and $\|p\|_P \lesssim \|f\|_{V'}$.*

Proof. For completeness we give the proof.

Suppose (i)–(iii) are satisfied, and let $f \in V'$, $g \in Q'$. Condition (ii) shows that there exists a $\bar{u} \in U$ with

$$c(\bar{u}, q) = g(q) \quad (q \in Q), \tag{2.2}$$

with $\|\bar{u}\|_U \lesssim \|g\|_{Q'}$. Condition (i) shows that there exist a unique $u_0 \in \ker C$ that solves

$$a(u_0, v) = f(v) - a(\bar{u}, v) \quad (v \in \ker B),$$

with $\|u_0\|_U \lesssim \|f\|_{V'} + \|\bar{u}\|_U$. With $u := \bar{u} + u_0$, Condition (iii) now shows that there exists a unique $p \in P$ that solves

$$b(p, v) = f(v) - a(u, v) \quad (v \in V,\, q \in Q), \tag{2.3}$$

with $\|p\|_P \lesssim \|f\|_{V'} + \|u\|_U$. From (2.2), (2.3), and $u_0 \in \ker C$, we conclude that there exists a unique solution $(u, p) \in V \times Q$ of (2.1) with $\|u\|_U + \|p\|_P \lesssim \|f\|_{V'} + \|g\|_{Q'}$.

Conversely, let (2.1) define a boundedly invertible linear mapping. Choose $g = 0$ and $f \in (\ker B)'$ and extend f to an element of V' with equal norm. Then necessarily $u \in \ker C$ and it solves $a(u,v) = f(v)$ $(v \in \ker B)$, which shows (i).

By taking $f = 0$, one infers (ii).

By taking $f \in (\ker B)^0$ and $g = 0$, (i) shows that $u = 0$, and (iii) follows. □

Proposition 2.2. *Recalling b and c being bounded, conditions equivalent to* (ii) *and* (iii) *are*

(ii)′ $\inf_{0\neq q\in Q} \sup_{0\neq u\in U} \frac{c(u,q)}{\|q\|_Q \|u\|_U} > 0$,
(iii)′ $\inf_{0\neq p\in P} \sup_{0\neq v\in V} \frac{b(p,v)}{\|p\|_P \|v\|_V} > 0$,

respectively.

The equivalence of (ii) and (ii)′ follows from the equivalence of (a) and (e) in Lemma 2.3 stated below. Another application of Lemma 2.3 shows that (iii)′ implies that $B' \in \mathcal{L}(P, V')$ is a homeomorphism onto $(\ker B)^0$, which means (iii). Conversely, since $\operatorname{ran} B' \subset (\ker B)^0$ by definition, (iii) implies that $(\ker B)^0 = \operatorname{ran} B'$ and that B' is injective, and so, by Lemma 2.3, that (iii)′ is valid.

Lemma 2.3. *For two separable Banach spaces X and Y, let $T \in \mathcal{L}(X, Y')$. Then the following are equivalent:*

(a) $\inf_{0\neq y\in Y} \sup_{0\neq x\in X} \frac{(Tx)(y)}{\|x\|_X \|y\|_Y} > 0$,
(b) $T' \in \mathcal{L}(Y, X')$ *is a homeomorphism onto its range,*
(c) T' *injective and* $\operatorname{ran} T'$ *is closed,*
(d) T' *injective and* $\operatorname{ran} T' = (\ker T)^0$,
(e) T *has a bounded right-inverse,*
(f) T *is surjective.*

Proof. (a)⇔(b) and (b)⇒(c) follow easily.

(c)⇒(b) is a consequence of the *open mapping theorem.*

(c)⇔(d) follows from the *closed range theorem.*

(f)⇒(e) follows from $T \in \mathcal{L}(X \setminus \ker T, Y')$ being bijective, and so by the open mapping theorem, being boundedly invertible.

(e)⇒(f) is obvious.

(f)⇒(c): Since $\operatorname{ran} T$ is closed, the closed range theorem shows that $\operatorname{ran} T'$ is closed, and that $(\ker T')^0 = \operatorname{ran} T = Y'$, so that, by an application of the Hahn-Banach theorem, $\ker T' = \emptyset$.

(c)⇒(f): Since $\operatorname{ran} T'$ is closed, the closed range theorem shows that $\operatorname{ran} T = (\ker T')^0 = Y'$ because T' is injective. □

3. The instationary Stokes problem with free-slip boundary conditions, as a well-posed operator equation

Let $\Omega \subset \mathbb{R}^n$ be a *bounded Lipschitz* domain. Later in this section, we will have to add the condition that Ω is such that the Poisson problem is $H^2(\Omega)$-regular.

Given vector fields $\tilde{\mathbf{f}}$ on $[0,T] \times \Omega$ and $\mathbf{u}_0$ on Ω, and functions g on $[0,T] \times \Omega$, and g_i $(1 \leq i \leq n-1)$ on $[0,T] \times \partial\Omega$, we consider the instationary inhomogeneous Stokes problem with *free-slip* boundary conditions of finding for some $\nu > 0$ a velocity field

$\mathbf{u}$ and corresponding pressure p that satisfy

$$\left\{\begin{aligned} \tfrac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p &= \tilde{\mathbf{f}} && \text{on } [0,T]\times\Omega,\\ \operatorname{div}_{\mathbf{x}} \mathbf{u} &= g && \text{on } [0,T]\times\Omega,\\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } [0,T]\times\partial\Omega,\\ \tfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\cdot\boldsymbol{\tau}_i &= g_i && \text{on } [0,T]\times\partial\Omega,\ 1\le i\le n-1,\\ \mathbf{u}(0,\cdot) &= \mathbf{u}_0 && \text{on } \Omega, \end{aligned}\right. \tag{3.1}$$

where $\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_{n-1}$ is an orthonormal set of tangent vectors.

By taking the canonical scalar product of the first equation with smooth test functions $\mathbf{v}$, that as function of $\mathbf{x}$ have vanishing normals at $\partial\Omega$, and that as function of t vanish at $t=T$, and by applying integration by parts in space and time, and by multiplying the second equation by smooth functions q, we arrive at a variational problem of the form (2.1), where

$$\left\{\begin{aligned} a(\mathbf{u},\mathbf{v}) &= -\int_0^T\!\!\int_\Omega \mathbf{u}\cdot\tfrac{\partial \mathbf{v}}{\partial t}\,d\mathbf{x}dt + \int_0^T\!\!\int_\Omega \nu\nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\mathbf{v}\,d\mathbf{x}dt,\\ b(p,\mathbf{v}) &= -\int_0^T\!\!\int_\Omega p\operatorname{div}\mathbf{v}\,d\mathbf{x}dt,\\ c(\mathbf{u},q) &= \int_0^T\!\!\int_\Omega q\operatorname{div}\mathbf{u}\,d\mathbf{x}dt,\\ \mathbf{f}(\mathbf{v}) &= \int_0^T\!\!\int_\Omega \tilde{\mathbf{f}}\cdot\mathbf{v}\,d\mathbf{x}dt + \int_0^T\!\!\int_{\partial\Omega}\sum_{i=1}^{n-1}(\mathbf{v}\cdot\boldsymbol{\tau}_i)g_i d\mathbf{x}dt + \int_\Omega \mathbf{u}_0\cdot\mathbf{v}(0,\cdot)\,d\mathbf{x},\\ g(q) &= \int_0^T\!\!\int_\Omega g\,q\,d\mathbf{x}dt. \end{aligned}\right. \tag{3.2}$$

Theorem 3.1. *With* $L_{2,0}(\Omega) := L_2(\Omega)/\mathbb{R}$, $\tilde{H}^2(\Omega) := \{p\in H^2(\Omega) : \frac{\partial p}{\partial n} = 0$ *on* $\partial\Omega\}/\mathbb{R}$, $\mathbf{H}^1(\Omega) := \{\mathbf{w}\in H^1(\Omega)^n : \mathbf{w}\cdot\mathbf{n} = 0$ *on* $\partial\Omega\}$ *and*

$$\begin{aligned} \mathbf{U} &:= L_2((0,T);\mathbf{H}^1(\Omega)),\\ P &:= \Big(L_2((0,T);L_{2,0}(\Omega)) \cap H^1_{0,\{T\}}\big((0,T);\tilde{H}^2(\Omega)'\big)\Big)',\\ \mathbf{V} &:= L_2((0,T);\mathbf{H}^1(\Omega)) \cap H^1_{0,\{T\}}((0,T);\mathbf{H}^1(\Omega)'),\\ Q &:= L_2((0,T);L_{2,0}(\Omega)), \end{aligned}$$

the mapping $\mathbf{L} : (\mathbf{u},p)\mapsto(\mathbf{f},g)$ *as in* (2.1) *with bilinear forms from* (3.2) *defines a boundedly invertible linear mapping* $\mathbf{U}\times P\to\mathbf{V}'\times Q'$. *(Here, as usual, dual spaces should be interpreted with respect to the identifications* $L_2(\Omega)'\simeq L_2(\Omega)$, $L_{2,0}(\Omega)'\simeq L_{2,0}(\Omega)$, *or* $L_2((0,T);L_{2,0}(\Omega))'\simeq L_2((0,T);L_{2,0}(\Omega))$, *respectively.)*

To prove this theorem, in the following, we will verify the conditions of the abstract existence and uniqueness result, Theorem 2.1.

The bilinear forms $a : \mathbf{U}\times\mathbf{V}\to\mathbb{R}$, $b : P\times\mathbf{V}\to\mathbb{R}$, and $c : \mathbf{U}\times Q\to\mathbb{R}$ are bounded. For b, this follows from $\operatorname{div}\in\mathcal{L}(\mathbf{H}^1(\Omega),L_{2,0}(\Omega))$ and $\operatorname{div}\in\mathcal{L}(\mathbf{H}^1(\Omega)',\tilde{H}^2(\Omega)')$, the latter, because of the density of $\mathcal{D}(\Omega)$ in $\mathbf{H}^1(\Omega)'$, being equivalent to $\nabla\in\mathcal{L}(\tilde{H}^2(\Omega),\mathbf{H}^1(\Omega))$. We conclude that $I\otimes\operatorname{div}_{\mathbf{x}}\in\mathcal{L}(\mathbf{V},P')$, being equivalent to $b:\mathbf{V}\times P\to\mathbb{R}$ is bounded.

For $\mathbf{u}\in\mathbf{U}$, $q\in Q$, one has $c(\mathbf{u},q) = -\int_0^T\int_\Omega \nabla_{\mathbf{x}} q\cdot\mathbf{u}\,d\mathbf{x}dt$. Since Ω is a *bounded Lipschitz domain*,

$$\nabla\in\mathcal{L}(L_{2,0}(\Omega),(H^1_0(\Omega)^n)') \text{ is a homeomorphism onto its range} \tag{3.3}$$

([Neč67], cf. [Tem79, Ch.1, Remark 1.4(ii)]). By an application of Lemma 2.3, this means that $\inf_{0\neq q\in L_{2,0}(\Omega)}\sup_{0\neq \mathbf{u}\in H_0^1(\Omega)}\frac{\int_\Omega q \operatorname{div}\mathbf{u} d\mathbf{x}}{\|q\|_{L_{2,0}(\Omega)}\|\mathbf{u}\|_{H^1(\Omega)^n}}>0$, and so also that $\inf_{0\neq q\in L_{2,0}(\Omega)}\sup_{0\neq \mathbf{u}\in \mathbf{H}^1(\Omega)}\frac{\int_\Omega q \operatorname{div}\mathbf{u}\, d\mathbf{x}}{\|q\|_{L_{2,0}(\Omega)}\|\mathbf{u}\|_{H^1(\Omega)^n}}>0$. Since, additionally, $(\mathbf{u},q)\mapsto \int_\Omega q \operatorname{div}\mathbf{u}\, d\mathbf{x}$ is bounded on $\mathbf{H}^1(\Omega)\times L_{2,0}(\Omega)$, one has that $\nabla\in\mathcal{L}(L_{2,0}(\Omega),\mathbf{H}^1(\Omega)')$, and so $I\otimes\nabla_{\mathbf{x}}\in\mathcal{L}(Q,\mathbf{U}')$ are a homeomorphisms onto their ranges by Lemma 2.3. Knowing the boundedness of $c:\mathbf{U}\times Q\to\mathbb{R}$, the latter is equivalent to Condition (ii) of Theorem 2.1.

To show Condition (i) of Theorem 2.1, we set

$$\boldsymbol{\mathcal{H}}^1(\Omega):=\{\mathbf{u}\in\mathbf{H}^1(\Omega):\operatorname{div}\mathbf{u}=0\},$$

define $\boldsymbol{\mathcal{H}}^0(\Omega)$ as the closure of $\boldsymbol{\mathcal{H}}^1(\Omega)$ in $L_2(\Omega)^n$, and set

$$\boldsymbol{\mathcal{H}}^{-1}(\Omega):=\boldsymbol{\mathcal{H}}^1(\Omega)'$$

(to be interpreted with respect to the identification $\boldsymbol{\mathcal{H}}^0(\Omega)'\simeq\boldsymbol{\mathcal{H}}^0(\Omega)$.)

Since $\operatorname{div}\mathbf{H}^1(\Omega)\subset L_{2,0}(\Omega)$, it holds that

$$\ker(\operatorname{div}\in\mathcal{L}(\mathbf{U},Q'))=L_2((0,T);\boldsymbol{\mathcal{H}}^1(\Omega)). \tag{3.4}$$

Lemma 3.2. *With $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$ denoting the closure of $\boldsymbol{\mathcal{H}}^1(\Omega))$ in $\mathbf{H}^1(\Omega)'$, it holds that*

$$\ker(I\otimes\operatorname{div}\in\mathcal{L}(\mathbf{V},P'))=L_2((0,T);\boldsymbol{\mathcal{H}}^1(\Omega))\cap H^1_{0,\{T\}}((0,T);\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)).$$

Proof. In view of (3.4) and the definitions of $\mathbf{V}$ and P', it suffices to show that $\boldsymbol{\mathcal{N}}:=\ker(\operatorname{div}\in\mathcal{L}(\mathbf{H}^1(\Omega)',\tilde{H}^2(\Omega)'))=\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$. By $\operatorname{div}\in\mathcal{L}(\mathbf{H}^1(\Omega)',\tilde{H}^2(\Omega)')$, $\boldsymbol{\mathcal{N}}$ contains $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$.

To prove that $\boldsymbol{\mathcal{N}}\subset\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$, it suffices to show the reversed inclusion for their polar sets

$$\begin{aligned}&\{\mathbf{u}\in\mathbf{H}^1(\Omega):\langle\mathbf{u},\mathbf{w}\rangle_{L_2(\Omega)}=0,\ \mathbf{w}\in\boldsymbol{\mathcal{N}}\}\\ &\qquad\supset\{\mathbf{u}\in\mathbf{H}^1(\Omega):\langle\mathbf{u},\mathbf{w}\rangle_{L_2(\Omega)}=0,\ \mathbf{w}\in\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)\}.\end{aligned} \tag{3.5}$$

The set on the right is contained in $\{\mathbf{u}\in\mathbf{H}^1(\Omega):\langle\mathbf{u},\mathbf{w}\rangle_{L_2(\Omega)}=0,\ \mathbf{w}\in\mathcal{D}(\Omega),\ \operatorname{div}\mathbf{w}=0\}$. As shown by De Rham ([dR84], cf. [Tem79, Ch. 1, Prop. 1.1]), a distribution $\mathbf{u}$ that vanishes on all divergence-free test functions is a gradient of another distribution. If, additionally $\mathbf{u}\in\mathbf{H}^1(\Omega)$, then necessarily $\mathbf{u}\in\nabla\tilde{H}^2(\Omega)$.

The adjoint of $\operatorname{div}\in\mathcal{L}(\mathbf{H}^1(\Omega)',\tilde{H}^2(\Omega)')$ is $-\nabla\in\mathcal{L}(\tilde{H}^2(\Omega),\mathbf{H}^1(\Omega))$. The latter operator is bounded and so closed. The *closed range theorem* now tells us that the space on the left in (3.5) is equal to $\nabla\tilde{H}^2(\Omega)$, which completes the proof. □

Lemma 3.3. *If the $L_2(\Omega)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}^0(\Omega)$ is bounded on $\mathbf{H}^1(\Omega)$, then $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)=\boldsymbol{\mathcal{H}}^{-1}(\Omega)$.*

Proof. As shown in, e.g., [Tem79, Ch.1, Th. 1.4], the closure of the divergence-free test functions in $L_2(\Omega)^n$ is $\{\mathbf{u}\in L_2(\Omega)^n:\operatorname{div}\mathbf{u}=0,\ \mathbf{u}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}$, and so this space is contained in $\boldsymbol{\mathcal{H}}^0(\Omega)$. On the other hand, if for $(\mathbf{u}_k)_k\subset\boldsymbol{\mathcal{H}}^1(\Omega)$, $\mathbf{u}_k\to\mathbf{u}$ in $L_2(\Omega)^n$, and so in $\mathcal{D}(\Omega)'$, then $\operatorname{div}\mathbf{u}=0$, and so $\mathbf{u}_k\to\mathbf{u}$ in $H(\operatorname{div};\Omega)$, in particular meaning that $\mathbf{u}\cdot\mathbf{n}=\lim_{k\to\infty}\mathbf{u}_k\cdot\mathbf{n}=0$ on $\partial\Omega$. We conclude that

$$\boldsymbol{\mathcal{H}}^0(\Omega)=\{\mathbf{u}\in L_2(\Omega)^n:\operatorname{div}\mathbf{u}=0,\ \mathbf{u}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\}.$$

Let Π denote the $L_2(\Omega)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}^0(\Omega)$. From $\boldsymbol{\mathcal{H}}^1(\Omega) \subset \boldsymbol{\mathcal{H}}^0(\Omega) \cap \mathbf{H}^1(\Omega)$, we have $\boldsymbol{\mathcal{H}}^1(\Omega) \subset \Im\Pi|_{\mathbf{H}^1(\Omega)}$. On the other hand, if Π is bounded on $\mathbf{H}^1(\Omega)$, then $\Im\Pi|_{\mathbf{H}^1(\Omega)} \subset \boldsymbol{\mathcal{H}}^0(\Omega) \cap \mathbf{H}^1(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \operatorname{div}\mathbf{u} = 0\} = \boldsymbol{\mathcal{H}}^1(\Omega)$, i.e.,

$$\boldsymbol{\mathcal{H}}^1(\Omega) = \Im\Pi|_{\mathbf{H}^1(\Omega)}. \tag{3.6}$$

If, for some $(f_n)_n \subset \boldsymbol{\mathcal{H}}^1(\Omega)$, $f_n \to f$ in $\mathbf{H}^1(\Omega)'$, then, viewed as functionals on $\boldsymbol{\mathcal{H}}^1(\Omega)$, $f_n \to f$ in $\boldsymbol{\mathcal{H}}^{-1}(\Omega)$, i.e., $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \subset \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.

Conversely, let $f \in \boldsymbol{\mathcal{H}}^{-1}(\Omega)$. Then there exists a $(f_n)_n \subset \boldsymbol{\mathcal{H}}^1(\Omega)$ with $f_n \to f$ in $\boldsymbol{\mathcal{H}}^{-1}(\Omega)$. For any $\mathbf{u} \in \mathbf{H}^1(\Omega)$, $f_n((I-\Pi)\mathbf{u}) = \langle f_n, (I-\Pi)\mathbf{u}\rangle_{L_2(\Omega)^n} = \langle (I-\Pi)f_n, \mathbf{u}\rangle_{L_2(\Omega)^n} = 0$. So, after trivially extending f to a functional on $\mathbf{H}^1(\Omega)$ by means of $f(\Im(I-\Pi)) = 0$, by the boundedness of Π on $\mathbf{H}^1(\Omega)$ and (3.6) we have $\|f-f_n\|_{\mathbf{H}^1(\Omega)'} = \sup_{0\neq\mathbf{u}\in\mathbf{H}^1(\Omega)} \frac{|(f-f_n)(\Pi u)|}{\|\mathbf{u}\|_{H^1(\Omega)^n}} \lesssim \sup_{0\neq\mathbf{u}\in\mathbf{H}^1(\Omega)} \frac{|(f-f_n)(\Pi\mathbf{u})|}{\|\Pi\mathbf{u}\|_{H^1(\Omega)^n}} = \|f-f_n\|_{\boldsymbol{\mathcal{H}}^{-1}(\Omega)}$, or $\boldsymbol{\mathcal{H}}^{-1}(\Omega) \subset \tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$. ☐

Theorem 3.4. *If Ω is convex or has a C^2 boundary, then $\tilde{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) = \boldsymbol{\mathcal{H}}^{-1}(\Omega)$.*

Proof. As shown in, e.g., [Tem79, Ch.1, Th. 1.4], one has the following *Helmholtz decomposition*

$$L_2(\Omega)^n = \boldsymbol{\mathcal{H}}^0(\Omega) \oplus^\perp \nabla(H^1(\Omega)/\mathbb{R}). \tag{3.7}$$

The $L_2(\Omega)^n$-orthogonal projector Π onto $\boldsymbol{\mathcal{H}}^0(\Omega)$ is known as the Leray projector. Given $\mathbf{u} \in L_2(\Omega)^n$, $\nabla z = (I-\Pi)\mathbf{u}$ is the solution of $\langle \mathbf{u} - \nabla z, \nabla w\rangle_{L_2(\Omega)^n} = 0$ ($w \in H^1(\Omega)/\mathbb{R}$).

When $\mathbf{u} \in \mathbf{H}^1(\Omega)$, this z solves the *Poisson problem* with *Neumann boundary conditions*

$$\begin{cases} -\Delta z = \operatorname{div}\mathbf{u} & \text{on } \Omega, \\ \frac{\partial z}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega, \\ \int_\Omega z\,dx = 0. & \end{cases}$$

By the condition on Ω, this Poisson problem is $H^2(\Omega)$-*regular*, and so

$$\|\nabla z\|_{H^1(\Omega)^n} \lesssim \|z\|_{H^2(\Omega)} \lesssim \|\operatorname{div}\mathbf{u}\|_{L_2(\Omega)^n} \lesssim \|\mathbf{u}\|_{H^1(\Omega)^n},$$

i.e., $I-\Pi$ and thus Π is bounded on $\mathbf{H}^1(\Omega)$. Now the result follows from Lemma 3.3. ☐

Using that on $\boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$, $(\mathbf{w}, \mathbf{v}) \mapsto \int_\Omega \nu\nabla\,\mathbf{w} : \nabla\,\mathbf{v}\,d\mathbf{x}$ is *bounded* and satisfies a *Gårding inequality*, we have the following result about the well-posedness of the variational formulation of the parabolic problem that results from the reduction of the instationary Stokes problem, with the homogeneous constraint $\operatorname{div}_{\mathbf{x}}\mathbf{u} = 0$, to a system of equations for the divergence-free velocities only:

Theorem 3.5. *With*

$$\boldsymbol{\mathcal{X}} := L_2((0,T), \boldsymbol{\mathcal{H}}^1(\Omega)), \quad \boldsymbol{\mathcal{Y}} := L_2((0,T), \boldsymbol{\mathcal{H}}^1(\Omega)) \cap H^1_{0,\{T\}}((0,T); \boldsymbol{\mathcal{H}}^{-1}(\Omega)),$$

$\mathbf{A} := \mathbf{u} \mapsto (\mathbf{v} \mapsto a(\mathbf{u}, \mathbf{v}))$ *is a boundedly invertible linear mapping from $\boldsymbol{\mathcal{X}}$ to $\boldsymbol{\mathcal{Y}}'$.*

Proof. The statement is equivalent to $\mathbf{A}'$ being boundedly invertible from $\boldsymbol{\mathcal{Y}}$ to $\boldsymbol{\mathcal{X}}'$, which in turn, by making the change of variable $T-t$ to t, is equivalent to the statement that

$$\mathbf{u} \mapsto (\mathbf{v} \mapsto \int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v}\, d\mathbf{x} dt + \int_0^T \int_\Omega \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v}\, d\mathbf{x} dt),$$

from $L_2((0,T), \boldsymbol{\mathcal{H}}^1(\Omega)) \cap H^1_{0,\{0\}}((0,T); \boldsymbol{\mathcal{H}}^{-1}(\Omega))$ to $L_2((0,T), \boldsymbol{\mathcal{H}}^{-1}(\Omega))$ is boundedly invertible. The boundedness of this mapping follows easily. The mapping corresponds to a variational formulation of a parabolic problem with homogeneous initial datum in the space of divergence-free velocities. The boundedness of the inverse is a consequence of $(\mathbf{u}, \mathbf{v}) \mapsto \int_\Omega \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v}\, d\mathbf{x}$ being bounded and coercive on $\boldsymbol{\mathcal{H}}^1(\Omega) \times \boldsymbol{\mathcal{H}}^1(\Omega)$, and it is shown, e.g., as a special case of [SS09, Thm. 4.1], where a possible inhomogeneous initial condition is imposed weakly. ☐

The characterizations of the kernels given by (3.4), Lemma 3.2, and Theorem 3.4, together with Theorem 3.5 imply Condition (i) of Theorem 2.1.

Condition (iii) of Theorem 2.1 is equivalent to (iii)$'$ which, by Lemma 2.3, is equivalent to

$$I \otimes \operatorname{div}_{\mathbf{x}} : \mathcal{L}\Big(L_2((0,T), \mathbf{H}^1(\Omega)) \cap H^1_{0,\{T\}}((0,T), \mathbf{H}^1(\Omega)'), \\ L_2((0,T); L_{2,0}(\Omega)) \cap H^1_{0,\{T\}}((0,T), \tilde{H}^2(\Omega)')\Big) \text{ is surjective.} \tag{3.8}$$

Note that since $I \otimes \operatorname{div}_{\mathbf{x}}$ is not injective, to prove (3.8) it is generally *not* sufficient to show that $I \otimes \operatorname{div}_{\mathbf{x}}$ is surjective both as mapping in $\mathcal{L}(L_2((0,T), \mathbf{H}^1(\Omega)), L_2((0,T); L_{2,0}(\Omega)))$ and as mapping in $\mathcal{L}(H^1_{0,\{T\}}((0,T), \mathbf{H}^1(\Omega)'), H^1_{0,\{T\}}((0,T), \tilde{H}^2(\Omega)'))$.

Below, we will construct a mapping div^+ with $\operatorname{div} \circ \operatorname{div}^+ = I$, such that

$$\operatorname{div}^+ \in \mathcal{L}(L_{2,0}(\Omega), \mathbf{H}^1(\Omega)), \quad \operatorname{div}^+ \in \mathcal{L}(\tilde{H}^2(\Omega)', \mathbf{H}^1(\Omega)'). \tag{3.9}$$

Since, consequently, $I \otimes \operatorname{div}^+_{\mathbf{x}}$ is a right-inverse for the mapping from (3.8), this will imply the surjectivity of the latter mapping.

We define $\operatorname{div}^+ : g \mapsto \mathbf{u}$ by

$$\begin{cases} \mathbf{u} + \nabla p &= \mathbf{f} & \text{on } \Omega, \\ \operatorname{div} \mathbf{u} &= g & \text{on } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{f} = 0$, or, more precisely, by its variational formulation to find $(\mathbf{u}, p) \in L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}$ such that

$$\int_\Omega \mathbf{u} \cdot \mathbf{v} + \int_\Omega \nabla p \cdot \mathbf{v} + \int_\Omega \nabla q \cdot \mathbf{u} = \mathbf{f}(\mathbf{v}) + g(q) \quad ((\mathbf{v}, q) \in L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}). \tag{3.10}$$

From the fact that $\nabla \in \mathcal{L}(H^1(\Omega)/\mathbb{R}, L_2(\Omega)^n)$ is a homeomorphism onto its range, it immediately follows that this variational problem, for general $\mathbf{f} \in L_2(\Omega)^n$, defines a boundedly invertible operator from $L_2(\Omega)^n \times H^1(\Omega)/\mathbb{R}$ to its dual.

Under the condition of Theorem 3.4, for $\mathbf{f} \in \mathbf{H}^1(\Omega)$ and $g \in L_{2,0}(\Omega)$, the solution p of

$$\begin{cases} -\Delta p &= g - \operatorname{div} \mathbf{f} & \text{on } \Omega, \\ \frac{\partial p}{\partial n} &= 0 & \text{on } \partial\Omega, \end{cases}$$

is in $\tilde{H}^2(\Omega)$, and $\mathbf{u} := \mathbf{f} - \nabla p \in \mathbf{H}^1(\Omega)$. We infer that under this condition, (3.10) defines a boundedly invertible mapping from $\mathbf{H}^1(\Omega) \times \tilde{H}^2(\Omega)$ to $\mathbf{H}^1(\Omega) \times L_{2,0}(\Omega)$, and so by symmetry of the operator, also from $\mathbf{H}^1(\Omega)' \times L_{2,0}(\Omega)$ to $\mathbf{H}^1(\Omega)' \times \tilde{H}^2(\Omega)'$. We conclude that (3.9) and thus Condition (iii) of Theorem 2.1 are valid.

Having verified all conditions of Theorem 2.1, the proof of Theorem 3.1 is completed.

The variational formulation (3.2) of the Stokes problem (3.1) was derived by applying integration by parts over time. This has the advantage that the initial condition $\mathbf{u}(0,\cdot) = \mathbf{u}_0$ enters the variational formulation as a natural boundary condition, i.e., in the right-hand side. In any case for a *homogeneous initial condition*, i.e., $\mathbf{u}(0,\cdot) = \mathbf{0}$, an alternative variational formulation is obtained by not applying integration by parts over time. It reads as a variational formulation of the form (2.1), where

$$
\begin{cases}
a(\mathbf{u},\mathbf{v}) &= \int_0^T\int_\Omega \frac{\partial \mathbf{u}}{\partial t}\cdot \mathbf{v}\, d\mathbf{x}dt + \int_0^T\int_\Omega \nu\nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\mathbf{v}\, d\mathbf{x}dt, \\
b(p,\mathbf{v}) &= -\int_0^T\int_\Omega p \operatorname{div}\mathbf{v}\, d\mathbf{x}dt, \\
c(\mathbf{u},q) &= \int_0^T\int_\Omega q \operatorname{div}\mathbf{u}\, d\mathbf{x}dt, \\
\mathbf{f}(\mathbf{v}) &= \int_0^T\int_\Omega \tilde{\mathbf{f}}\cdot\mathbf{v}\, d\mathbf{x}dt + \int_0^T\int_{\partial\Omega}\sum_{i=1}^{n-1}(\mathbf{v}\cdot\boldsymbol{\tau}_i)g_i d\mathbf{x}dt \\
g(q) &= \int_0^T\int_\Omega g\, q\, d\mathbf{x}dt.
\end{cases}
\tag{3.11}
$$

Theorem 3.6. *With*

$$
\begin{aligned}
&\mathbf{U} := L_2((0,T);\mathbf{H}^1(\Omega)) \cap H^1_{0,\{0\}}((0,T);\mathbf{H}^1(\Omega)'),\\
&P := L_2((0,T);L_{2,0}(\Omega)),\\
&\mathbf{V} := L_2((0,T);\mathbf{H}^1(\Omega)),\\
&Q := \Big(L_2((0,T);L_{2,0}(\Omega)) \cap H^1_{0,\{0\}}\big((0,T);\tilde{H}^2(\Omega)'\big)\Big)',
\end{aligned}
$$

the mapping $\mathbf{L} : (\mathbf{u},p) \mapsto (\mathbf{f},g)$ *as in* (2.1) *with bilinear forms from* (3.11) *defines a boundedly invertible linear mapping* $\mathbf{U}\times P \to \mathbf{V}'\times Q'$.

Proof. Denoting the spaces $\mathbf{U}$, $\mathbf{V}$, P, and Q, and operator $\mathbf{L}$ from Theorem 3.1 here as $\bar{\mathbf{U}}$, $\bar{\mathbf{V}}$, $\bar{P}$, $\bar{Q}$, and $\bar{\mathbf{L}}$, and defining $(Rw)(t,x) = w(T-t,x)$, we have

$$(\mathbf{L}(\mathbf{u},p))(\mathbf{v},q) = (\bar{\mathbf{L}}(R\mathbf{v},-Rq))(R\mathbf{u},-Rp) = (\bar{\mathbf{L}}'(R\mathbf{u},-Rp))(R\mathbf{v},-Rq).$$

From $\bar{\mathbf{L}}' \in \mathcal{L}(\bar{\mathbf{V}}\times\bar{Q}, \bar{\mathbf{U}}'\times\bar{P}')$ being a boundedly invertible, and $R\mathbf{U} = \bar{\mathbf{V}}$, $RP = \bar{Q}$, $R\mathbf{V} = \bar{\mathbf{U}}$, and $RQ = \bar{P}$, the proof is completed. □

4. The instationary Stokes problem, with no-slip boundary conditions, as a well-posed operator equation

Let $\Omega \subset \mathbb{R}^n$ be a domain. Later in this section, we will have to add the condition that Ω is such that the stationary Stokes problem is $H^2(\Omega)^n \times H^1(\Omega)$-regular.

Given vector fields $\tilde{\mathbf{f}}$ on $[0,T]\times\Omega$ and $\mathbf{u}_0$ on Ω, and a function g on $[0,T]\times\Omega$, we consider the instationary inhomogeneous Stokes problem with *no-slip boundary* *conditions* to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}}\, p = \tilde{\mathbf{f}} & \text{on } [0,T]\times\Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } [0,T]\times\Omega, \\ \mathbf{u} = 0 & \text{on } [0,T]\times\partial\Omega, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 & \text{on } \Omega. \end{cases}$$

By taking the canonical scalar product of the first equation with smooth test functions $\mathbf{v}$, that as function of $\mathbf{x}$ vanish at $\partial\Omega$, and that as function of t vanish at $t = T$, and by applying integration by parts in space and time, and by multiplying the second equation by smooth functions q, and by applying integration by parts, we arrive at a variational problem of the form (2.1), where

$$\begin{cases} a(\mathbf{u},\mathbf{v}) &=& -\int_0^T\int_\Omega \mathbf{u}\cdot \frac{\partial \mathbf{v}}{\partial t}\, d\mathbf{x}dt + \int_0^T\int_\Omega \nu\nabla_{\mathbf{x}}\mathbf{u} : \nabla_{\mathbf{x}}\mathbf{v}\, d\mathbf{x}dt, \\ b(p,\mathbf{v}) &=& \int_0^T\int_\Omega \mathbf{v}\cdot\nabla p\, d\mathbf{x}dt, \\ c(\mathbf{u},q) &=& -\int_0^T\int_\Omega \mathbf{u}\cdot\nabla q\, d\mathbf{x}dt, \\ \mathbf{f}(\mathbf{v}) &=& \int_0^T\int_\Omega \tilde{\mathbf{f}}\cdot\mathbf{v}\, d\mathbf{x}dt + \int_\Omega \mathbf{u}_0\cdot\mathbf{v}(0,\cdot)\, d\mathbf{x}, \\ g(q) &=& \int_0^T\int_\Omega g\, q\, d\mathbf{x}dt. \end{cases} \tag{4.1}$$

Remark 4.1. With $\hat{H}^2(\Omega) := \{p \in H^2(\Omega) : \nabla p = 0 \text{ on } \partial\Omega\}/\mathbb{R}$, following the exposition in Sect. 3, an obvious choice for the spaces $\mathbf{U}, P$ and $\mathbf{V}, Q$ for the variables $\mathbf{u}, p$ and $\mathbf{v}, q$, would be

$$L_2((0,T); H_0^1(\Omega)^n), \quad \Big(L_2((0,T); L_{2,0}(\Omega)) \cap H^1_{0,\{T\}}\big((0,T); \hat{H}^2(\Omega)'\big)\Big)',$$
$$L_2((0,T); H_0^1(\Omega)^n) \cap H^1_{0,\{T\}}((0,T); H^{-1}(\Omega)^n), \quad L_2((0,T); L_{2,0}(\Omega)),$$

where $H^{-1}(\Omega) = H_0^1(\Omega)'$ with respect to the identification $L_2(\Omega)' \simeq L_2(\Omega)$. With this choice, the resulting space $\boldsymbol{\mathcal{H}}^1(\Omega)$ of divergence free spatial functions would read as $\{\mathbf{u} \in H_0^1(\Omega)^n : \operatorname{div}\mathbf{u} = 0\}$, with, as in Sect. 3, its closure $\boldsymbol{\mathcal{H}}^0(\Omega)$ in $L_2(\Omega)^n$ being $\{\mathbf{u} \in L_2(\Omega)^n : \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$. Now when following the analysis from Sect. 3, the problem is that the $L_2(\Omega)^n$-orthogonal projector onto $\boldsymbol{\mathcal{H}}^0(\Omega)$, i.e., the Leray projector, does not preserve no-slip boundary conditions, and therefore is not bounded on $H_0^1(\Omega)^n$.

In view of Remark 4.1, we will define trial- and testspaces by making a shift in smoothness indices for the spatial variables.

Theorem 4.2. *With*

$$\mathbf{U} := L_2((0,T); L_2(\Omega)^n),$$
$$P := \Big(L_2((0,T); H^1(\Omega)/\mathbb{R}) \cap H^1_{0,\{T\}}\big((0,T); (H^1(\Omega)/\mathbb{R})'\big)\Big)',$$
$$\mathbf{V} := L_2((0,T); (H_0^1(\Omega) \cap H^2(\Omega))^n) \cap H^1_{0,\{T\}}((0,T); L_2(\Omega)^n),$$
$$Q := L_2((0,T); H^1(\Omega)/\mathbb{R}),$$

the mapping $\mathbf{L} : (\mathbf{u}, p) \mapsto (\mathbf{f}, g)$ *as in* (2.1) *with bilinear forms from* (4.1) *defines a boundedly invertible linear mapping* $\mathbf{U} \times P \to \mathbf{V}' \times Q'$*. (Here, dual spaces should be* *interpreted with respect to the identifications* $L_{2,0}(\Omega)' \simeq L_{2,0}(\Omega)$*, or Condition* (i) $L_2((0,T);L_{2,0}(\Omega))' \simeq L_2((0,T);L_{2,0}(\Omega))$*, respectively.*)

To prove this theorem, in the following, we will verify the conditions of Theorem 2.1.

In Sect. 3, for the verification of the Conditions (i) and (iii), we used $H^2(\Omega)$-regularity of the Poisson problem on Ω with homogeneous Neumann boundary conditions. In this section, instead we will use the $H^2(\Omega)^n \times H^1(\Omega)$-regularity of the *stationary Stokes problem* on Ω with Dirichlet boundary conditions. In variational form, this problem reads as finding, for given $\mathbf{f} \in H^{-1}(\Omega)^n$, $g \in (H^1(\Omega)/\mathbb{R})'$, the solution $(\mathbf{u},p) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)$ of

(4.2)
$$\int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v}\, d\mathbf{x} + \int_\Omega \mathbf{v}\cdot\nabla p\, d\mathbf{x} - \int_\Omega \mathbf{u}\cdot\nabla q\, d\mathbf{x} = \mathbf{f}(\mathbf{v}) + g(q)\ ((\mathbf{v},q) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)).$$

From (3.3), and the general theory from Sect. 2, one verifies the well-known fact that this problem defines a boundedly invertible operator from $H_0^1(\Omega)^n \times L_{2,0}(\Omega)$ to its dual. Under additional conditions on Ω, the problem can be shown to be $H^2(\Omega)^n \times H^1(\Omega)$-regular:

Theorem 4.3. *For domains* Ω *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *that either have a* C^2 *boundary or that are convex with a piecewise smooth boundary,* (4.2) *defines a boundedly invertible mapping from* $(H^2(\Omega) \cap H_0^1(\Omega))^n \times H^1(\Omega)/\mathbb{R}$ *to* $L_2(\Omega) \times H^1(\Omega)/\mathbb{R}$*, and from* $L_2(\Omega) \times (H^1(\Omega)/\mathbb{R})'$ *to* $((H^2(\Omega) \cap H_0^1(\Omega))^n)' \times (H^1(\Omega)/\mathbb{R})'$*.*

A proof can be found in [KO76, Dau89] in two - or three-dimensions, respectively. The last mentioned result follows from symmetry of the variational problem by taking adjoints.

Lemma 4.4. *It holds that*

$$\begin{aligned}
&\ker(\nabla' \in \mathcal{L}(L_2(\Omega)^n, (H^1(\Omega)/\mathbb{R})'))\\
&\qquad = \{\mathbf{u} \in L_2(\Omega)^n : \operatorname{div}\mathbf{u} = 0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\} := \breve{\boldsymbol{\mathcal{H}}}^0(\Omega),\\
&\ker(\nabla' \in \mathcal{L}(H_0^1(\Omega), L_{2,0}(\Omega))) = \{\mathbf{u} \in H_0^1(\Omega)^n : \operatorname{div}\mathbf{u} = 0\} := \breve{\boldsymbol{\mathcal{H}}}^1(\Omega),\\
&\ker(\nabla' \in \mathcal{L}((H^2(\Omega) \cap H_0^1(\Omega)^1)^n, H^1(\Omega)/\mathbb{R}))\\
&\qquad = \{\mathbf{u} \in (H^2(\Omega) \cap H_0^1(\Omega))^n : \operatorname{div}\mathbf{u} = 0\} := \breve{\boldsymbol{\mathcal{H}}}^2(\Omega).
\end{aligned}$$

Proof. Since in the last two cases $\nabla = -\operatorname{div}'$ by definition, we only have to verify the first statement, i.e., that

$$\boldsymbol{\mathcal{N}} := \ker(\nabla' \in \mathcal{L}(L_2(\Omega)^n, (H^1(\Omega)/\mathbb{R})')) = \breve{\boldsymbol{\mathcal{H}}}^0(\Omega).$$

For $\mathbf{u} \in \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$, $p \in H^1(\Omega)/\mathbb{R}$, one has $\int_\Omega \nabla p \cdot \mathbf{u}\, d\mathbf{x} = 0$, i.e., $\breve{\boldsymbol{\mathcal{H}}}^0(\Omega) \subset \boldsymbol{\mathcal{N}}$. To prove the reversed inclusion, we have to show that

(4.3)
$$\begin{aligned}
&\{\mathbf{u} \in L_2(\Omega)^n : \langle \mathbf{u},\mathbf{w}\rangle_{L_2(\Omega)} = 0,\ \mathbf{w} \in \boldsymbol{\mathcal{N}}\}\\
&\qquad \supset \big\{\mathbf{u} \in L_2(\Omega)^n : \langle \mathbf{u},\mathbf{w}\rangle_{L_2(\Omega)} = 0,\ \mathbf{w} \in \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)\big\}.
\end{aligned}$$

The set on the right is part of $\{\mathbf{u} \in L_2(\Omega)^n : \langle \mathbf{u},\mathbf{w}\rangle_{L_2(\Omega)} = 0,\ \mathbf{w} \in \mathcal{D}(\Omega),\ \operatorname{div}\mathbf{w} = 0\}$. As shown by De Rham ([dR84], cf. [Tem79, Ch. 1, Prop. 1.1]), a distribution $\mathbf{u}$ that vanishes on all divergence-free test functions is a gradient of another distribution. If, additionally $\mathbf{u} \in L_2(\Omega)^n$, then necessarily $\mathbf{u} \in \nabla(H^1(\Omega)/\mathbb{R})$.

Since $\nabla \in \mathcal{L}(H^1(\Omega)/\mathbb{R}, L_2(\Omega)^n)$ and so closed, the closed range theorem tells us that the space on the left in (4.3) is equal to $\nabla(H^1(\Omega)/\mathbb{R})$, which completes the proof. □

It holds that $\breve{\boldsymbol{\mathcal{H}}}^2(\Omega) \hookrightarrow \breve{\boldsymbol{\mathcal{H}}}^1(\Omega) \hookrightarrow \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$ with dense embeddings. For $i \in \{1,2\}$, we set $\breve{\boldsymbol{\mathcal{H}}}^{-i}(\Omega) := (\breve{\boldsymbol{\mathcal{H}}}^i(\Omega))'$, where this dual space should be interpreted with respect to the identification $(\breve{\boldsymbol{\mathcal{H}}}^0(\Omega))' \simeq \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$.

The stationary Stokes problem (4.2) with $g = 0$ can be reduced to a problem involving divergence-free velocities only. It reads as finding, for given $\mathbf{f} \in \breve{\boldsymbol{\mathcal{H}}}^{-1}(\Omega) \supset H^{-1}(\Omega)^n$, $\mathbf{u} \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)$ that solves

$$\int_\Omega \nu \nabla \mathbf{u} : \nabla \mathbf{v}\, d\mathbf{x} = \mathbf{f}(\mathbf{v}) \quad (\mathbf{v} \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)). \tag{4.4}$$

As follows from Theorem 2.1, together with the characterizations of the kernels from Lemma 4.4, and Theorem 4.3, we have:

Corollary 4.5. *The variational problem* (4.4) *defines a boundedly invertible linear mapping from* $\breve{\boldsymbol{\mathcal{H}}}^1(\Omega)$ *to* $\breve{\boldsymbol{\mathcal{H}}}^{-1}(\Omega)$*, and, under the conditions on* Ω *from Theorem 4.2, from* $\breve{\boldsymbol{\mathcal{H}}}^2(\Omega)$ *to* $\breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$*, and from* $\breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$ *to* $\breve{\boldsymbol{\mathcal{H}}}^{-2}(\Omega)$*.*

After these preparations dealing with the stationary Stokes problem, we are ready to prove Theorem 4.2 by verifying the conditions of Theorem 2.1.

Similarly as in Sect. 3, one shows that the bilinear forms $a : \mathbf{U} \times \mathbf{V} \to \mathbb{R}$, $b : P \times \mathbf{V} \to \mathbb{R}$, and $c : \mathbf{U} \times Q \to \mathbb{R}$ are bounded.

The operator $I \otimes \nabla_{\mathbf{x}} \in \mathcal{L}(Q, \mathbf{U}')$ is a homeomorphism onto its range, which by Lemma 2.3 shows Condition (ii) of Theorem 2.1.

As an obvious consequence of Lemma 4.4, we have

$$\ker(I \otimes \nabla' \in \mathcal{L}(\mathbf{U}, Q')) = L_2((0,T); \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)) \tag{4.5}$$

$$\ker(I \otimes \nabla' \in \mathcal{L}(\mathbf{V}, P')) = L_2((0,T); \breve{\boldsymbol{\mathcal{H}}}^2(\Omega)) \cap H^1_{0,\{T\}}((0,T); \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)) \tag{4.6}$$

Theorem 4.6. *With*

$$\boldsymbol{\mathcal{X}}_1 := L_2((0,T); \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)),\ \boldsymbol{\mathcal{Y}}_1 := L_2((0,T); \breve{\boldsymbol{\mathcal{H}}}^2(\Omega)) \cap H^1_{0,\{T\}}((0,T); \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)),$$

under the condition on Ω *from Theorem 4.3,* $\mathbf{A} : \mathbf{u} \mapsto (\mathbf{v} \mapsto a(\mathbf{u}, \mathbf{v}))$ *defines a boundedly invertible linear mapping from* $\boldsymbol{\mathcal{X}}_1$ *to* $\boldsymbol{\mathcal{Y}}_1'$*.*

Proof. We follow [Ste11b, proof of Thm. 4.2]. The boundedness of $\mathbf{A}$ follows easily.

The boundedness of $\mathbf{A}^{-1}$ is equivalent to $(\mathbf{A}')^{-1} \in \mathcal{L}(\boldsymbol{\mathcal{X}}_1', \boldsymbol{\mathcal{Y}}_1)$. To demonstrate the latter, we have to show that for any $\mathbf{f} \in \boldsymbol{\mathcal{X}}_1 \simeq \boldsymbol{\mathcal{X}}_1'$, the variational problem of finding $\mathbf{z}$ such that

$$\int_0^T \int_\Omega -\mathbf{w} \cdot \frac{\partial \mathbf{z}}{\partial t}\, d\mathbf{x}dt + \int_0^T \int_\Omega \nabla \nu \mathbf{w} : \nabla \mathbf{z}\, d\mathbf{x}dt = \int_0^T \int_\Omega \mathbf{f} \cdot \mathbf{w}\, d\mathbf{x}dt \quad (\mathbf{w} \in \boldsymbol{\mathcal{X}}_1), \tag{4.7}$$

has a unique solution $\mathbf{z} \in \boldsymbol{\mathcal{Y}}_1$ with $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$.

Although this result may follow from the theory of analytic semigroups, we give a more elementary derivation. With

$$\boldsymbol{\mathcal{X}}_0 := L_2((0,T); \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)),\ \boldsymbol{\mathcal{Y}}_0 := L_2((0,T); \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)) \cap H^1_{0,\{T\}}((0,T); \breve{\boldsymbol{\mathcal{H}}}^{-1}),$$

similar to Theorem 3.5, we have that for $\mathbf{f} \in \boldsymbol{\mathcal{X}}_0' \supset \boldsymbol{\mathcal{X}}_1'$, (4.7), with test space $\boldsymbol{\mathcal{X}}_0$, has a unique solution $\mathbf{z} \in \boldsymbol{\mathcal{Y}}_0$. Below, we will show that for a subspace of sufficiently smooth $\mathbf{f}$, this solution is in $\boldsymbol{\mathcal{Y}}_1$, and thus that (4.7) holds for all $\mathbf{w} \in \boldsymbol{\mathcal{X}}_1$, and moreover that $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$. Since the subspace of these smooth $\mathbf{f}$ will be dense in $\boldsymbol{\mathcal{X}}_1$, this will complete the proof.

Equation (4.7) is the variational formulation of the problem of finding, for $t \in [0,T]$, $\mathbf{z}(t,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)$ that satisfies

(4.8)
$$\begin{cases} \int_\Omega -\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\cdot \mathbf{w}\, d\mathbf{x} + \int_\Omega \nu \nabla \mathbf{w} : \nabla \mathbf{z}(t,\cdot)\, d\mathbf{x} = \int_\Omega \mathbf{f}(t,\cdot)\cdot \mathbf{w}\, d\mathbf{x} \quad (\mathbf{w} \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)), \\ \mathbf{z}(T,\cdot) = 0. \end{cases}$$

Note that as function of $\tilde{t} = T - t$, $\mathbf{z}$ satisfies a standard parabolic *initial* value problem. As shown in [Wlo82, Ch.IV,§27], if $\mathbf{f} \in H^2((0,T);\breve{\boldsymbol{\mathcal{H}}}^{-1}(\Omega))$ with $\mathbf{f}(T,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^2(\Omega)$ and $\frac{\partial \mathbf{f}}{\partial t}(T,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$, then its solution $\mathbf{z} \in H^2((0,T);\breve{\boldsymbol{\mathcal{H}}}^1(\Omega))$.

By substituting $\mathbf{w} = -\frac{\partial \mathbf{z}}{\partial t}(t,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega)$ in (4.8), we obtain

$$\|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\Omega)^n} - \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \nu \nabla \mathbf{z}(t,\cdot) : \nabla \mathbf{z}(t,\cdot)\, d\mathbf{x} = -\int_\Omega \mathbf{f}(t,\cdot)\cdot \frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\, d\mathbf{x}.$$

By integrating this equality over time, applying $\mathbf{z}(T,\cdot) = 0$ and Cauchy-Schwarz' inequality, and by additionally assuming that $\mathbf{f} \in L_2((0,T);\breve{\boldsymbol{\mathcal{H}}}^0(\Omega))$, we arrive at

$$\int_0^T \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\Omega)^n} dt \leq \frac{1}{2}\int_0^T \|\mathbf{f}(t,\cdot)\|^2_{L_2(\Omega)^n} dt + \frac{1}{2}\int_0^T \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\Omega)^n} dt,$$

or

$$\int_0^T \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|^2_{L_2(\Omega)^n} dt \leq \int_0^T \|\mathbf{f}(t,\cdot)\|^2_{L_2(\Omega)^n} dt. \tag{4.9}$$

By additionally assuming that $\mathbf{f}(t,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$, from $\frac{\partial \mathbf{z}}{\partial t}(t,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^1(\Omega) \subset \breve{\boldsymbol{\mathcal{H}}}^0(\Omega)$ and the $\boldsymbol{\mathcal{H}}^2(\Omega)$-regularity result from Corollary 4.5, (4.8) shows that $\mathbf{z}(t,\cdot) \in \breve{\boldsymbol{\mathcal{H}}}^2(\Omega)$ with $\|\mathbf{z}(t,\cdot)\|_{H^2(\Omega)^n} \lesssim \|\mathbf{f}(t,\cdot)\|_{L_2(\Omega)^n} + \|\frac{\partial \mathbf{z}}{\partial t}(t,\cdot)\|_{L_2(\Omega)^n}$. By integrating this inequality over time and applying (4.9), we obtain that

$$\|\mathbf{z}\|_{L_2((0,T);\breve{\boldsymbol{\mathcal{H}}}^2(\Omega))} \lesssim \|\mathbf{f}\|_{L_2((0,T);L_2(\Omega)^n)} \tag{4.10}$$

By combining (4.9) and (4.10), we have $\|\mathbf{z}\|_{\boldsymbol{\mathcal{Y}}_1} \lesssim \|\mathbf{f}\|_{\boldsymbol{\mathcal{X}}_1}$ and the proof is completed. □

The characterizations of the kernels (4.5) and (4.6) together with Theorem 4.6 imply Condition (i) of Theorem 2.1.

Condition (iii) of Theorem 2.1 is equivalent to (iii)$'$, which by Lemma 2.3, is equivalent to

(4.11)
$$\begin{aligned} I \otimes \operatorname{div} \in \mathcal{L}\Big(& L_2((0,T);(H^1_0(\Omega)\cap H^2(\Omega))^n) \cap H^1_{0,\{T\}}((0,T);L_2(\Omega)^n), \\ & L_2((0,T);H^1(\Omega)/\mathbb{R}) \cap H^1_{0,\{T\}}\big((0,T);(H^1(\Omega)/\mathbb{R})'\big)\Big) \text{ is surjective.} \end{aligned}$$

Below, we will construct a mapping div^+ with $\operatorname{div} \circ \operatorname{div}^+ = I$, such that

$$\operatorname{div}^+ \in \mathcal{L}(H^1(\Omega)/\mathbb{R}, (H^1_0(\Omega)\cap H^2(\Omega))^n),\ \operatorname{div}^+ \in \mathcal{L}((H^1(\Omega)/\mathbb{R})', L_2(\Omega)^n). \tag{4.12}$$

Since, consequently, $I \otimes \operatorname{div}_{\mathbf{x}}^{+}$ is a right-inverse for the mapping from (4.11), this will imply the surjectivity of the latter mapping.

We define $\operatorname{div}^{+} : g \mapsto \mathbf{u}$ by

$$\begin{cases} -\Delta \mathbf{u} + \nabla p &= 0 \quad \text{on } \Omega \\ \operatorname{div} \mathbf{u} &= g \quad \text{on } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega \end{cases}$$

or, more precisely, by its variational formulation to find $(\mathbf{u}, p) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)$ such that

$$\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} + \int_\Omega p \operatorname{div} \mathbf{v} + \int_\Omega q \operatorname{div} \mathbf{u} = g(q) \quad ((\mathbf{u}, p) \in H_0^1(\Omega)^n \times L_{2,0}(\Omega)). \tag{4.13}$$

For domains Ω that satisfy the condition from Theorem 4.3, an application of this theorem shows that div^{+} satisfies (4.12). We conclude that (4.11) and thus that Condition (iii) of Theorem 2.1 are valid.

Having verified all conditions of Theorem 2.1, the proof of Theorem 4.2 is completed.

Similar to Theorem 3.6 for the free-slip boundary conditions case, for the instationary Stokes problem with no-slip boundary conditions and a *homogeneous initial condition*, a variational formulation of the form (2.1) can be derived without applying integration by parts. The bilinear forms and right-hand side read as

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) &= \int_0^T \int_\Omega \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{v} \, d\mathbf{x}dt + \int_0^T \int_\Omega \nu \nabla_{\mathbf{x}} \mathbf{u} : \nabla_{\mathbf{x}} \mathbf{v} \, d\mathbf{x}dt, \\ b(p, \mathbf{v}) &= \int_0^T \int_\Omega \mathbf{v} \cdot \nabla p \, d\mathbf{x}dt, \\ c(\mathbf{u}, q) &= -\int_0^T \int_\Omega \mathbf{u} \cdot \nabla q \, d\mathbf{x}dt, \\ \mathbf{f}(\mathbf{v}) &= \int_0^T \int_\Omega \tilde{\mathbf{f}} \cdot \mathbf{v} \, d\mathbf{x}dt, \\ g(q) &= \int_0^T \int_\Omega g \, q \, d\mathbf{x}dt. \end{cases} \tag{4.14}$$

and we have the following result:

Theorem 4.7. *With*

$$\mathbf{U} := L_2((0,T); (H_0^1(\Omega) \cap H^2(\Omega))^n) \cap H^1_{0,\{0\}}((0,T); L_2(\Omega)^n),$$
$$P := L_2((0,T); H^1(\Omega)/\mathbb{R}),$$
$$\mathbf{V} := L_2((0,T); L_2(\Omega)^n),$$
$$Q := \Big(L_2((0,T); H^1(\Omega)/\mathbb{R}) \cap H^1_{0,\{0\}}\big((0,T); (H^1(\Omega)/\mathbb{R})'\big)\Big)',$$

the mapping $\mathbf{L} : (\mathbf{u}, p) \mapsto (\mathbf{f}, g)$ as in (2.1) *with bilinear forms from* (4.14) *defines a boundedly invertible linear mapping $\mathbf{U} \times P \to \mathbf{V}' \times Q'$.*

5. The instationary Navier-Stokes problem with homogeneous initial condition

With the spaces $\mathbf{U}$, P, $\mathbf{V}$, Q from either Theorem 4.7 (no slip boundary conditions) and $n \in \{2, 3\}$, or those from Theorem 3.6 (free-slip boundary conditions) and $n = 2$, we will show that for sufficiently small data $(\mathbf{f}, g) \in \mathbf{V}' \times Q'$, the corresponding *Navier*-Stokes problem has locally a unique solution in $\mathbf{U} \times P$.

Lemma 5.1. *For Banach spaces X and Y, let $B = L + N : X \to Y'$ where $L \in \mathcal{L}(X, Y')$ is boundedly invertible, and $N(0) = 0$.*

For some $R > 0$ and $\alpha < \|L^{-1}\|^{-1}_{\mathcal{L}(Y',X)}$, let

$$\|N(x_1) - N(x_2)\|_{Y'} \leq \alpha \|x_1 - x_2\|_X \quad (x_1, x_2 \in B(0; R) := \{x \in X : \|x\|_X \leq R\}).$$

Then for any $h \in Y'$ with $\|h\|_{Y'} \leq R(\|L^{-1}\|^{-1}_{\mathcal{L}(Y',X)} - \alpha)$, there exists a unique $x \in B(0; R)$ with $B(x) = h$.

Proof. $B(x) = h$ is equivalent to $x = T(x) := L^{-1}(h - N(x))$. For $\|h\|_{Y'} \leq R(\|L^{-1}\|^{-1}_{\mathcal{L}(Y',X)} - \alpha)$ and $x \in B(0; R)$, $\|T(x)\|_X \leq \|L^{-1}\|_{\mathcal{L}(Y',X)}(\|h\|_Y + \alpha \|x\|_X) \leq R$, and, for $x_1, x_2 \in B(0; R)$, $\|T(x_1) - T(x_2)\|_X \leq \|L^{-1}\|_{\mathcal{L}(Y',X)} \alpha \|x_1 - x_2\|$. The proof is completed by an application of *Banach's fixed point theorem.* □

5.1. **No-slip boundary conditions.** Let Ω be a domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ that either has a C^2 boundary, or that is convex, satisfies the cone condition, and has a piecewise smooth boundary.

Given a vector field $\tilde{\mathbf{f}}$ on $[0, T] \times \Omega$, and a function g on $[0, T] \times \Omega$, we consider the instationary Navier–Stokes problem to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}} \mathbf{u} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} p = \tilde{\mathbf{f}} & \text{on } [0, T] \times \Omega, \\ \operatorname{div}_{\mathbf{x}} \mathbf{u} = g & \text{on } [0, T] \times \Omega, \\ \mathbf{u} = 0 & \text{on } [0, T] \times \partial\Omega, \\ \mathbf{u}(0, \cdot) = 0 & \text{on } \Omega. \end{cases} \tag{5.1}$$

It gives rise to a variational problem of the form (2.1) with an extra trilinear term $n(\cdot, \cdot, \cdot)$, that reads as finding $\mathbf{u} \in \mathbf{U}$, $p \in P$ such that

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + c(\mathbf{u}, q) = f(\mathbf{v}) + g(q) - n(\mathbf{u}, \mathbf{u}, \mathbf{v}) \quad (\mathbf{v} \in V, q \in Q), \tag{5.2}$$

where the spaces $\mathbf{U}$, P, $\mathbf{V}$, Q, right-hand side functionals f and g, and bilinear forms a, b, c are as in Theorem 4.7 or (4.14), and

$$n(\mathbf{y}, \mathbf{z}, \mathbf{v}) := \int_0^T \int_\Omega \mathbf{y} \cdot \nabla_{\mathbf{x}} \mathbf{z} \cdot \mathbf{v} \, d\mathbf{x} dt. \tag{5.3}$$

Theorem 5.2. *For sufficiently small $f \in \mathbf{V}'$ and $g \in Q'$,* (5.2) *has a unique solution $(\mathbf{u}, p)$ in some ball in $\mathbf{U} \times P$ around the origin.*

Proof. By Theorem 4.7 and Lemma 5.1, it suffices to show that with $N(\mathbf{u})(\mathbf{v}) := n(\mathbf{u}, \mathbf{u}, \mathbf{v})$, it holds that $N : \mathbf{U} \to \mathbf{V}'$ with

$$\|N(\mathbf{u}) - N(\mathbf{w})\|_{\mathbf{V}'} \leq \zeta(\|\mathbf{u}\|_{\mathbf{U}}, \|\mathbf{w}\|_{\mathbf{U}}) \|\mathbf{u} - \mathbf{w}\|_{\mathbf{U}}$$

for some $\zeta : [0, \infty)^2 \to [0, \infty)$ with $\zeta(\alpha) \to 0$ if $\alpha \to 0$.

Recall that $\mathbf{U} = L_2((0, T); (H^1_0(\Omega) \cap H^2(\Omega))^n) \cap H^1_{0,\{0\}}((0, T); L_2(\Omega)^n)$ and $\mathbf{V} = L_2((0, T); L_2(\Omega)^n)$. Using twice a Hölder inequality, twice that $H^1(\Omega) \hookrightarrow L_6(\Omega)$ when $n \leq 3$ (here the cone condition is used), and also twice that $\mathbf{U} \hookrightarrow$

$C([0,T];H^1_0(\Omega)^n)$ ([DL92, Ch. XVIII,§1.3]), being a consequence of $[L_2(\Omega),H^1_0(\Omega)\cap H^2(\Omega)]_{1/2}=H^1_0(\Omega)$, for $\mathbf{y},\mathbf{z}\in\mathbf{U}$ we find

$$\begin{aligned}
&\left(\sup_{0\neq\mathbf{v}\in\mathbf{V}}\frac{|n(\mathbf{y},\mathbf{z},\mathbf{v})|}{\|\mathbf{v}\|_{\mathbf{V}}}\right)^2=\int_0^T\|\mathbf{y}(t,\cdot)\cdot\nabla_{\mathbf{x}}\mathbf{z}(t,\cdot)\|^2_{L_2(\Omega)^n}\,dt\\
&=\int_0^T\sum_{i=1}^n\int_\Omega|\mathbf{y}\cdot\nabla_{\mathbf{x}}z_i|^2\,d\mathbf{x}\,dt\leq\sum_{i=1}^n\int_0^T\int_\Omega|\mathbf{y}|^2|\nabla_{\mathbf{x}}z_i|^2\,d\mathbf{x}\,dt\\
&\leq\sum_{i=1}^n\int_0^T\|\mathbf{y}(t,\cdot)\|^2_{L_6(\Omega)^n}\|\nabla z_i(t,\cdot)\|^2_{L_3(\Omega)^n}\,dt\\
&\leq\sup_{t\in[0,T]}\|\mathbf{y}(t,\cdot)\|^2_{L_6(\Omega)^n}\sum_{i=1}^n\int_0^T\Big(\int_\Omega|\nabla_{\mathbf{x}}z_i|^{\frac32}|\nabla_{\mathbf{x}}z_i|^{\frac32}\Big)^{\frac23}dt\\
&\leq\sup_{t\in[0,T]}\|\mathbf{y}(t,\cdot)\|^2_{L_6(\Omega)^n}\sum_{i=1}^n\int_0^T\|\nabla_{\mathbf{x}}z_i(t,\cdot)\|_{L_2(\Omega)^n}\|\nabla_{\mathbf{x}}z_i(t,\cdot)\|_{L_6(\Omega)^n}dt\\
&\lesssim\sup_{t\in[0,T]}\|\mathbf{y}(t,\cdot)\|^2_{H^1(\Omega)^n}\sup_{t\in[0,T]}\|\mathbf{z}(t,\cdot)\|_{H^1(\Omega)^n}\sqrt{T}\|\mathbf{z}\|_{L_2(0,T);H^2(\Omega)^n)}\lesssim\|\mathbf{y}\|^2_{\mathbf{U}}\|\mathbf{z}\|^2_{\mathbf{U}}.
\end{aligned}$$

As a first consequence, we have $\|N(\mathbf{u})\|^2_{\mathbf{V}'}\lesssim\|\mathbf{u}\|^4_{\mathbf{U}}$, and so in particular, $N:\mathbf{U}\to\mathbf{V}'$.

Secondly, from $n(\mathbf{u},\mathbf{u},\cdot)-n(\mathbf{w},\mathbf{w},\cdot)=n(\mathbf{u}-\mathbf{w},\mathbf{u},\cdot)+n(\mathbf{w},\mathbf{u}-\mathbf{w},\cdot)$, we find

$$\|N(\mathbf{u})-N(\mathbf{w})\|^2_{\mathbf{V}'}\lesssim(\|\mathbf{u}\|^2_{\mathbf{U}}+\|\mathbf{w}\|^2_{\mathbf{U}})\|\mathbf{u}-\mathbf{w}\|^2_{\mathbf{U}},$$

which completes the proof. □

Besides existence and local uniqueness for sufficiently small data, we also have local uniqueness of any solution:

Theorem 5.3. *Let $(\mathbf{u},p)$ be a solution of* (5.2)*, then for sufficiently small $\delta f\in\mathbf{V}'$, $\delta g\in Q'$,* (5.2) *with (f,g) reading as $(f+\delta f,g+\delta g)$ has a unique solution in some ball in $\mathbf{U}\times P$ around $(\mathbf{u},p)$.*

Proof. Writing the solution with perturbed data as $(\mathbf{u}+\delta\mathbf{u},p+\delta p)$, we find that $(\delta\mathbf{u},\delta p)\in\mathbf{U}\times V$ solves

$$a_{\mathbf{u}}(\delta\mathbf{u},\mathbf{v})+b(\delta p,\mathbf{v})+c(\delta\mathbf{u},q)=\delta f(\mathbf{v})+\delta g(q)-n(\delta\mathbf{u},\delta\mathbf{u},\mathbf{v})\quad(\mathbf{v}\in\mathbf{V},\,q\in Q),$$

where

$$a_{\mathbf{u}}(\delta\mathbf{u},\mathbf{v}):=a(\delta\mathbf{u},\mathbf{v})+n(\mathbf{u},\delta\mathbf{u},\mathbf{v})+n(\delta\mathbf{u},\mathbf{u},\mathbf{v}).$$

The bilinear form $a_{\mathbf{u}}$ corresponds to the partial differential operator $\mathbf{w}\mapsto-\nu\boldsymbol{\Delta}_{\mathbf{x}}\mathbf{w}+\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{w}+\mathbf{w}\cdot\nabla_{\mathbf{x}}\mathbf{u}$. Since the perturbations are of lower order, any result that we have proven for the Stokes equations is also valid for the modified Stokes equations with $-\nu\boldsymbol{\Delta}_{\mathbf{x}}\mathbf{w}$ reading as $-\nu\boldsymbol{\Delta}_{\mathbf{x}}\mathbf{w}+\mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{w}+\mathbf{w}\cdot\nabla_{\mathbf{x}}\mathbf{u}$ (not uniformly in $\mathbf{u}$ though). We conclude that the statement is proven similarly to Theorem 5.2. □

Remark 5.4. The point of the above proof is that with

$$\mathbf{B}:\mathbf{U}\times P\to\mathbf{V}'\times Q':(\mathbf{u},p)\mapsto\big((\mathbf{v},q)\mapsto a(\mathbf{u},\mathbf{v})+b(p,\mathbf{v})+c(\mathbf{u},q)+n(\mathbf{u},\mathbf{u},\mathbf{v})\big),$$

the Fréchet derivative

$$D\mathbf{B}(\mathbf{u},p):(\delta\mathbf{u},\delta p)\mapsto\big((\mathbf{v},q)\mapsto a_{\mathbf{u}}(\delta\mathbf{u},\mathbf{v})+b(\delta p,\mathbf{v})+c(\delta\mathbf{u},q)\big)\in\mathcal{L}(\mathbf{U}\times P,\mathbf{V}'\times Q')$$

is boundedly invertible, which is a crucial property for any method for *solving* the Navier-Stokes equations.

5.2. **Free-slip boundary conditions.** Let Ω be a bounded domain in $\mathbb{R}^2$ that either has a C^2 boundary or that is convex with a Lipschitz boundary.

Given a vector field $\tilde{\mathbf{f}}$ on $[0,T]\times\Omega$, we consider the instationary Navier–Stokes problem to find the velocities $\mathbf{u}$ and pressure p that satisfy

$$(5.4)\qquad \begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \boldsymbol{\Delta}_{\mathbf{x}}\mathbf{u} + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + \nabla_{\mathbf{x}} p = \tilde{\mathbf{f}} & \text{on } [0,T]\times\Omega,\\ \operatorname{div}_{\mathbf{x}}\mathbf{u} = 0 & \text{on } [0,T]\times\Omega,\\ \mathbf{u}\cdot\mathbf{n} = 0 & \text{on } [0,T]\times\partial\Omega,\\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\cdot\boldsymbol{\tau}_i = g_i & \text{on } [0,T]\times\partial\Omega,\ 1\le i\le n-1,\\ \mathbf{u}(0,\cdot) = \mathbf{u}_0 & \text{on } \Omega, \end{cases}$$

where $\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_{n-1}$ is an orthonormal set of tangent vectors.

It gives rise to a variational problem of the form (2.1) with an extra nonlinear term, that reads as finding $\mathbf{u}\in\mathbf{U}$, $p\in P$ such that

$$(5.5)\qquad a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) + c(\mathbf{u},q) = f(\mathbf{v}) + n(\mathbf{u},\mathbf{v},\mathbf{u}) \quad (\mathbf{v}\in V,\ q\in Q),$$

where the spaces $\mathbf{U}$, P, $\mathbf{V}$, Q, right-hand side functional f, and bilinear forms a, b, c are as in Theorem 3.6 or (3.11), and the form n is as in (5.3).

We arrived at this variational formulation with $n(\mathbf{u},\mathbf{v},\mathbf{u})$, instead of the expected term $-n(\mathbf{u},\mathbf{u},\mathbf{v})$, by using that for smooth vector fields $\mathbf{u}$ on Ω that have vanishing normals at $\partial\Omega$ and that are divergence-free, and for smooth vector fields $\mathbf{v}$ on Ω,

$$\begin{aligned} n(\mathbf{u},\mathbf{v},\mathbf{v}) = \sum_{i,j=1}^{2}\int_\Omega u_i(\partial_i v_j)v_j\,d\mathbf{x} &= \tfrac12\sum_{i,j=1}^{2}\int_\Omega u_i\partial_i v_j^2\,d\mathbf{x}\\ &= \tfrac12\Big[\sum_j\int_\Omega -\operatorname{div}\mathbf{u}\,v_j^2\,d\mathbf{x} + \int_{\partial\Omega} v_j^2\,\mathbf{u}\cdot\mathbf{n}\,ds\Big] = 0 \end{aligned}$$

Expanding $n(\mathbf{u},\mathbf{v}+\mathbf{w},\mathbf{v}+\mathbf{w})$ for smooth vector fields $\mathbf{v},\mathbf{w}$ on Ω, we arrive at $n(\mathbf{u},\mathbf{v},\mathbf{w}) = -n(\mathbf{u},\mathbf{w},\mathbf{v})$. Note that it is essential that in (5.4) we have imposed $\mathbf{u}\cdot\mathbf{n}=0$ on $[0,T]\times\partial\Omega$, instead of $\mathbf{u}\cdot\mathbf{n}=g$ on $[0,T]\times\partial\Omega$ for some general function.

Theorem 5.5. *For sufficiently small $f\in\mathbf{V}'$,* (5.5) *has a unique solution* $(\mathbf{u},p)$ *in some ball in* $\mathbf{U}\times P$ *around the origin.*

Proof. As shown in [Tem79, Ch.III, §3, Lemma 3.3], for $v\in H^1(\mathbb{R}^2)$ it holds that

$$\|v\|_{L_4(\mathbb{R}^2)} \le 2^{1/4}\|v\|_{L_2(\mathbb{R}^2)}^{\frac12}|v|_{H^1(\mathbb{R}^2)}^{\frac12}.$$

Since $\Omega\subset\mathbb{R}^2$ is a bounded Lipschitz domain, there exists an operator E that extends functions on Ω to functions on $\mathbb{R}^2$ with $E\in\mathcal{L}(L_2(\Omega),L_2(\mathbb{R}^2))$, $E\in\mathcal{L}(H^1(\Omega),H^1(\mathbb{R}^2))$ ([Ste70, Ch.VI,§3, Thm.5]). We conclude that for $v\in H^1(\Omega)$,

$$\|v\|_{L_4(\Omega)} \le \|Ev\|_{L_4(\mathbb{R}^2)} \le 2^{1/4}\|Ev\|_{L_2(\mathbb{R}^2)}^{\frac12}|Ev|_{H^1(\mathbb{R}^2)}^{\frac12} \lesssim \|v\|_{L_2(\Omega)}^{\frac12}\|v\|_{H^1(\Omega)}^{\frac12}.$$

Using this result, and by a few applications of Cauchy-Schwarz inequality, for $\mathbf{y}, \mathbf{v}, \mathbf{z} \in H^1(\Omega)$ we have

$$
\begin{aligned}
&|\int_\Omega \mathbf{y}\cdot\nabla_{\mathbf{x}}\mathbf{v}\cdot\mathbf{z}\,d\mathbf{x}| = |\int_\Omega \sum_{i,j=1}^{2} y_i(\partial_i v_j)z_j\,d\mathbf{x}| \\
&\le \sum_{i,j} \|\partial_i v_j\|_{L_2(\Omega)}\|y_i\|_{L_4(\Omega)}\|z_j\|_{L_4(\Omega)} \\
&\le \sqrt{\sum_{i,j}\|\partial_i v_j\|^2_{L_4(\Omega)}}\sqrt{\sum_i \|y_i\|^2_{L_4(\Omega)}\sum_j \|z_j\|^2_{L_4(\Omega)}} \\
&\lesssim \|\mathbf{v}\|_{H^1(\Omega)^2}\|\mathbf{y}\|^{\frac12}_{L_2(\Omega)^2}\|\mathbf{y}\|^{\frac12}_{H^1(\Omega)^2}\|\mathbf{z}\|^{\frac12}_{L_2(\Omega)^2}\|\mathbf{z}\|^{\frac12}_{H^1(\Omega)^2}.
\end{aligned}
$$

Recalling that $\mathbf{U} = L_2((0,T);\mathbf{H}^1(\Omega))\cap H^1_{0,\{0\}}((0,T);\mathbf{H}^1(\Omega)')$ and $\mathbf{V} = L_2((0,T);\mathbf{H}^1(\Omega))$, for $\mathbf{y}, \mathbf{z} \in \mathbf{U}$, $\mathbf{v} \in \mathbf{V}$, from $\mathbf{U} \hookrightarrow C([0,T];L_2(\Omega))$ we obtain

$$
\begin{aligned}
&|n(\mathbf{y},\mathbf{v},\mathbf{z})| \lesssim \\
&\int_0^T \|\mathbf{v}(t,\cdot)\|_{H^1(\Omega)^2}\|\mathbf{y}(t,\cdot)\|^{\frac12}_{L_2(\Omega)^2}\|\mathbf{y}(t,\cdot)\|^{\frac12}_{H^1(\Omega)^2}\|\mathbf{z}(t,\cdot)\|^{\frac12}_{L_2(\Omega)^2}\|\mathbf{z}(t,\cdot)\|^{\frac12}_{H^1(\Omega)^2}\,dt \\
&\le \sup_{t\in[0,t]}\|\mathbf{y}(t,\cdot)\|^{\frac12}_{L_2(\Omega)}\sup_{t\in[0,t]}\|\mathbf{z}(t,\cdot)\|^{\frac12}_{L_2(\Omega)} \\
&\qquad\qquad \times \int_0^T \|\mathbf{v}(t,\cdot)\|_{H^1(\Omega)^2}\|\mathbf{y}(t,\cdot)\|^{\frac12}_{H^1(\Omega)^2}\|\mathbf{z}(t,\cdot)\|^{\frac12}_{H^1(\Omega)^2}\,dt \\
&\lesssim \|\mathbf{y}\|^{\frac12}_{\mathbf{U}}\|\mathbf{z}\|^{\frac12}_{\mathbf{U}}\|\mathbf{v}\|_{\mathbf{V}}\Big(\int_0^T \|\mathbf{y}(t,\cdot)\|^2_{H^1(\Omega)^2}\,dt\Big)^{\frac14}\Big(\int_0^T \|\mathbf{z}(t,\cdot)\|^2_{H^1(\Omega)^2}\,dt\Big)^{\frac14} \\
&\lesssim \|\mathbf{y}\|_{\mathbf{U}}\|\mathbf{z}\|_{\mathbf{U}}\|\mathbf{v}\|_{\mathbf{V}},
\end{aligned}
$$

or $\sup_{0\neq\mathbf{v}\in\mathbf{V}} \frac{|n(\mathbf{y},\mathbf{v},\mathbf{z})|}{\|\mathbf{v}\|_{\mathbf{V}}} \lesssim \|\mathbf{y}\|_{\mathbf{U}}\|\mathbf{z}\|_{\mathbf{U}}$. Similar to the proof of Theorem 5.2, the latter result together with Theorem 3.6 and Lemma 5.1 completes the proof. □

Similar to the no-slip case, besides existence and local uniqueness for sufficiently small data, we also have local uniqueness of any solution:

Theorem 5.6. *Let* $(\mathbf{u}, p)$ *be a solution of* (5.5)*, then for sufficiently small* $\delta f \in \mathbf{V}'$, (5.2) *with* f *reading as* $f + \delta f$ *has a unique solution in some ball in* $\mathbf{U} \times P$ *around* $(\mathbf{u}, p)$.

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