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Refined convergence theory for semi-Lagrangian schemes for pure advection

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REFINED CONVERGENCE THEORY FOR SEMI-LAGRANGIAN SCHEMES FOR PURE ADVECTION

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Abstract. We consider generalized linear transient advection problems for differential forms on a bounded domain in $\mathbb{R}^n$. We provide comprehensive a priori convergence estimates for their spatio-temporal discretization by means of a semi-Lagrangian approach combined with a discontinuous Galerkin method. We establish a new asymptotic estimate $O(h^{r+1}\tau^{-\frac{1}{2}})$ for the L^2-norm of the error, where h is the spatial meshwidth, τ denotes the timestep, and r is the polynomial degree of the piecewise polynomial discrete differential forms used as trial functions. Numerical experiments hint that the estimate is sharp for certain trial spaces and may be sub-optimal for others.

1. Introduction

A huge body of numerical analysis literature deals with numerical methods for the transient 2nd-order advection-diffusion problem

$$\begin{array}{rcll} \partial_t u - \operatorname{div}\varepsilon\,\mathbf{grad}\,u + \boldsymbol{\beta}\cdot\mathbf{grad}\,u & = & f & \text{in } \Omega, \\ u & = & g_{\mathrm{D}} & \text{on } \Gamma_0\cup\Gamma_{\mathrm{in}}\subset\partial\Omega, \\ u(\cdot,0) & = & u_0. & \end{array} \tag{1}$$

The non-negative smooth function $\varepsilon = \varepsilon(x)$ is called the diffusion coefficient, $\boldsymbol{\beta} : \overline{\Omega} \mapsto \mathbb{R}^n$ stands for a given Lipschitz continuous stationary velocity field, $\mathbf{n}_\Omega$ is the outward normal and $f \in C^1\left([0,T];L^2(\Omega)\right)$ is a given source function, $T > 0$ the final time. Dirichlet boundary conditions g_D can be imposed on the inflow boundary part Γ_{in} or where the diffusion coefficient ε is positive.

A key issue is the design of numerical methods that are robust in the singular perturbation limit of vanishing diffusion coefficient $\varepsilon \to 0$. This article is devoted to the analysis of so-called semi-Lagrangian methods, which have little difficulty coping with singularly perturbed advection-diffusion problems.

The semi-Lagrangian approach to scalar advection-diffusion problems like (1) has been investigated in a host of research papers, see, e.g., [3–5, 9–11, 21, 23, 26]. A survey of the literature can be found in [12, Section 5]. These works exclusively address the scalar problem (1), whereas, apart from [12–14, 22], little attention has been paid to semi-Lagrangian methods for the non-scalar case, e.g., the so-called magnetic advection-diffusion problem for a vector field $\mathbf{u} : \Omega \to \mathbb{R}^3$ [13], describing the evolution of magnetic fields in conducting media:

$$\begin{array}{rcll} \partial_t\mathbf{u} + \mathbf{curl}\,\varepsilon\,\mathbf{curl}\,\mathbf{u} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} & = & \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} & = & \mathbf{g}_{\mathrm{D}} & \text{on } \Gamma_0\cup\Gamma_{\mathrm{in}}, \\ \mathbf{u}(\cdot,0) & = & \mathbf{u}_0. & \end{array} \tag{2}$$

In this article we present a new convergence theory for fully discrete semi-Lagrangian methods for the limit case $\varepsilon = 0$, where the velocity $\boldsymbol{\beta}$ has vanishing

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normal components everywhere on the boundary of Ω, i.e., as in [13], we focus on a *pure advection problem*. We are not going to delve into a discussion of design and implementation of the methods, but refer to the companion paper [13] instead. The new results improve the asymptotic convergence estimates of order $O(h^{r+1}\tau^{-1})$ for $h, \tau \to 0$ obtained in [13] to $O(h^{r+1}\tau^{-\frac{1}{2}})$, where h is the mesh width of the spatial triangulation, and τ stands for the (uniform) timestep size. The new estimates were inspired by an argument from [19, p. 52]. They are clearly sharper than previous results obtainted in [21] (asymptotic order $O(h^2\tau^{-1})$) and [7] (order $O(\tau + h)$) for (1) and $\varepsilon = 0$. They also generalize the result in [19, page 52] to a larger class of advection problems, to non-constant velocity, and non-zero source terms.

Following a trend in the numerical analysis of PDEs (cf. [1,2,8]), in this article the presentation relies on the calculus of differential forms, a perspective also adopted and elaborated in [13] and [12, 15]. Beside offering concise notations, this has the big benefit that both (1) and (2) can be treated in a unified framework.

The outline of this article is as follows. In the next section we recall the advection problem for differential forms and restate the semi-Lagrangian methods. Afterwards we prove the main convergence theorem Theorem 3.3. Numerous numerical experiments in the final section confirm the relevance of the results also in the preasymptotic range for h and τ.

2. Advection of differential forms and semi-Lagrangian methods

We refer to the books [6], [18] and [25] and the article [1] for a comprehensive introduction to differential forms. Here, we merely list important concepts and notations.

Let Ω be a smooth, oriented n-dimensional Riemannian manifold. Then, let $\Lambda^k(\Omega)$ denote the space of smooth differential forms. A smooth differential form ω assigns to each point $x \in \Omega$ and k vectors $\mathbf{v}_1, \dots \mathbf{v}_k$ from the tangent space at x a number. For x fixed a smooth differential form ω induces a k-linear alternating mapping ω_x on the tangent space at x (c.f. Table 1).

Based on the inner product on the tangent bundle of Ω we can define an inner product $(\omega, \eta)_\Omega = \int_\omega \langle \omega, \eta \rangle \,\mathrm{vol}$, where $\langle \cdot, \cdot \rangle$ is the inner product of alternating mappings. Completion of $\Lambda^k(\Omega)$ in the norm $\|\omega\|^2_{L^2(\Omega)} := (\omega, \omega)_\Omega$ yields the Hilbert space $L^2\Lambda^k(\Omega)$. Analogously to the Sobolev spaces $H^m(\Omega)$ and $W^{m,p}(\Omega)$ for scalar functions with $m > 0$ derivatives in $L^2(\Omega)$ and $L^p(\Omega)$ [25, Section 1.3] we define Sobolev-spaces $W^{m,p}\Lambda^k(\Omega)$ and $H^m\Lambda^k(\Omega)$ for differential forms by requiring that the map

$$x \mapsto \omega_x(\mathbf{v}_1(x), \dots, \mathbf{v}_k(x)) \tag{3}$$

is in $W^{m,p}(\Omega)$ and $H^m(\Omega)$, where $\mathbf{v}_1(x), \dots, \mathbf{v}_k(x)$ are smooth vector fields. In the following $\|\cdot\|_{W^{m,p}\Lambda^k(\Omega)}$ ($|\cdot|_{W^{m,p}\Lambda^k(\Omega)}$) and $\|\cdot\|_{H^m\Lambda^k(\Omega)}$ ($|\cdot|_{H^m\Lambda^k(\Omega)}$) will denote the corresponding (semi) -norms. We use also the standard notations $\boldsymbol{W}^{m,p}(\Omega)$, $|\boldsymbol{\beta}|_{\boldsymbol{W}^{m,p}(\Omega)}$ and $\|\boldsymbol{\beta}\|_{\boldsymbol{W}^{m,p}(\Omega)}$ to denote Sobolev spaces, Sobolev semi-norms and Sobolev norms of vector valued functions with $m > 0$ derivatives in $L^p(\Omega)$.

The stationary, Lipschitz continuous vector field $\boldsymbol{\beta} : \Omega \to \mathbb{R}^n$ induces a flow $X_\tau(x) = X(\tau, x)$ with $X : \Omega \times \mathbb{R} \mapsto \Omega$, where

$$\frac{\partial}{\partial \tau} X_\tau(x) = \boldsymbol{\beta}(X_\tau(x)), \quad X_0(x) = x\ . \tag{4}$$

Here and below we make the following assumption:

Assumption 2.1. *We assume that $\boldsymbol{\beta} : \Omega \to \mathbb{R}^n$ has vanishing normal components at the boundary of Ω.*

k	*differential form*	*vector proxy*
0	$x \mapsto \omega_x$	$u(x) := \omega_x$
1	$x \mapsto \{\mathbf{v} \mapsto \omega_x(\mathbf{v})\}$	$\mathbf{u}(x) \cdot \mathbf{v} := \omega_x(\mathbf{v})$
2	$x \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega_x(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(x) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \omega_x(\mathbf{v}_1, \mathbf{v}_2)$
3	$x \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega_x(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(x) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \omega_x(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

TABLE 1. In 3D Euclidian space the vector proxies of forms ω are scalar functions u or vector fields $\mathbf{u}$ [16, Table 2.1].

An essential concept for a unifying treatment of linear advection problems is the concept of the pullback map $X_\tau^* : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$, that is induced by the flow map X_τ:

$$\int_\Sigma X_\tau^* \omega := \int_{X_\tau(\Sigma)} \omega, \quad \forall k\text{-dim. submanifolds } \Sigma. \tag{5}$$

The natural advection operator for differential forms is the Lie derivative:

$$\mathsf{L}_{\boldsymbol{\beta}}\, \omega := \frac{\partial}{\partial \tau} X_\tau^* \omega_{|\tau=0}. \tag{6}$$

The Lie derivative $\mathsf{L}_{\boldsymbol{\beta}}\, \omega$ of a k-form measures the rate of change of ω in direction of $\boldsymbol{\beta}$, with respect to the evaluation of ω on any k-dimensional sub-manifold Σ:

$$\int_\Sigma \mathsf{L}_{\boldsymbol{\beta}}\, \omega = \int_\Sigma \frac{\partial}{\partial \tau} \left(X_\tau^* \omega\right)_{|\tau=0} = \frac{\partial}{\partial \tau} \left(\int_\Sigma X_\tau^* \omega \right)_{|\tau=0} = \frac{\partial}{\partial \tau} \left(\int_{X_\tau(\Sigma)} \omega \right)_{|\tau=0}. \tag{7}$$

Then, the linear advection problem for differential forms reads: For given $\omega_0 \in \Lambda^k(\Omega)$ and $\varphi \in \Lambda^k(\Omega)$ find $\omega(t) \in \Lambda^k(\Omega)$, $0 < t < T$, such that

$$\omega(0) = \omega_0, \quad \frac{\partial}{\partial t} \omega(t) + \mathsf{L}_{\boldsymbol{\beta}}\, \omega(t) = \varphi(t),\ 0 \le t \le T\ . \tag{8}$$

The assumption 2.1 implies that $X(\tau, x) \in \Omega$, for all τ and $x \in \Omega$, and we have the following representation formula for the solution:

$$(\omega(t))_x = \left(X_{-t}^* \omega(0)\right)_x + \int_0^t \left(X_{\tau-t}^* \varphi(\tau)\right)_x d\tau, \tag{9}$$

which will be the starting point for the construction of Semi-Lagrangian methods for (8).

Remark 2.2. *Functions and vector fields can provide models for differential forms. The association is by no means unique. A popular identification is provided by the so-called* Euclidean vector proxies *depicted in Table 1. This isomorphism establishes a link between exterior calculus and vector analysis, see Table 2 for an overview of vector proxy representation of pullbacks for differential forms in* $\mathbb{R}^3$ *and the corresponding Lie derivatives. This reveals that the transport parts of* (1) *and* (2) *are in fact incarnations of* (8) *for* $k = 0$ *and* $k = 1$*, respectively.*

The generic semi-Lagrangian approach boils down to a finite element Galerkin discretization of (9) plus timestepping. Let $\mathcal{T}$ be a triangulation of Ω and $\Lambda_h^k(\mathcal{T}) \subset L^2\Lambda^k(\Omega)$ denote some piecewise polynomial approximation space for differential k-forms in Ω on the triangulation $\mathcal{T}$. The subscript h plays the dual role of labeling

k	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega$	$\phi^*\omega$
0	$\boldsymbol{\beta}\cdot\mathbf{grad}\,u$	$u(\phi(x))$
1	$\mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u})+\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}$	$D\phi(x)^T\mathbf{u}(\phi(x))$
2	$\mathbf{curl}(\mathbf{u}\times\boldsymbol{\beta})+\boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u}$	$\det D\phi(x)D\phi(x)^{-1}\mathbf{u}(\phi(x))$
3	$\mathrm{div}(u\boldsymbol{\beta})$	$\det D\phi(x)u(\phi(x))$

TABLE 2. Correspondences of operations on forms ω with operations on scalar functions u or vectorial functions (vector proxies) $\mathbf{u}$ in 3D Euclidean space. ϕ is a diffeomorphism, e.g. X_τ. [1,16]

discrete entities and designating the mesh width of $\mathcal{T}$. We assume that for suitable $r,s\in\mathbb{N}_0$ there are constants $K=K(r,s)$ independent of h such that

$$\inf_{\omega_h\in\Lambda^k_h(\mathcal{T})}\|\omega_h-\omega\|_{L^2\Lambda^k(\Omega)}\le Kh^{\min(r+1,s)}\|\omega\|_{H^s\Lambda^k(\Omega)}\quad\forall\omega\in H^s\Lambda^k(\Omega)\;. \tag{10}$$

Obviously, r will be related to the polynomial degree of the approximation space.

To formulate a semi-Lagrangian method, we consider a partitioning of the time interval of the form $[0,T]=\bigcup_{n=0}^{N-1}[t^n,t^{n+1}]$ with $t^n=\tau n$ and $\tau=\frac{T}{N}$. Then the semi-Lagrangian Galerkin timestepping scheme for the advection problem (8) constructs sequences $(\omega_h^n)_{n=0}^N$, $\omega_h^n\in\Lambda_h(\mathcal{T})$, approximating $(\omega(t^n))_{n=0}^N$ by solving the following *discrete variational problem*: Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n\in\Lambda_h^k(\mathcal{T})$, such that for all $\eta_h\in\Lambda_h^k(\mathcal{T})$:

$$\begin{aligned}\left(\omega_h^0,\eta_h\right)_\Omega&=(\omega_0,\eta_h)_\Omega\,,\\ \frac{1}{\tau}\left(\omega_h^{n+1},\eta_h\right)_\Omega-\frac{1}{\tau}\left(X_{-\tau}^*\omega_h^n,\eta_h\right)_\Omega&=\int_{t^n}^{t^{n+1}}\left(X^*_{t^{n+1}-s}\varphi(s),\eta_h\right)_\Omega ds\end{aligned} \tag{11}$$

3. A PRIORI ERROR ESTIMATES

For the error analysis of the semi-Lagrangian discretization (11) we have to gauge the effect of transport with the flow. This is done in the following two auxiliary lemmas.

Lemma 3.1 (Proposition A.1 in [12]). *Let $\boldsymbol{\beta}\in\boldsymbol{W}^{1,\infty}(\Omega)$ satisfy Assumption 2.1. Then, for $\omega,\eta\in L^2\Lambda^k(\Omega)$, we have the expansion*

$$\left(X_{-\tau}^*\omega,X_{-\tau}^*\eta\right)_{X_\tau(\Omega)}=\,(\omega,\eta)_\Omega+R(\boldsymbol{\beta},\tau)\,(\omega,\eta)_\Omega\,, \tag{12}$$

with $|R(\boldsymbol{\beta},\tau)|\le C(\boldsymbol{\beta})\tau$ independent of ω and η.

Proof. (provided for the sake of completeness) By density of $\Lambda^k(\Omega)$ in $L^2\Lambda^k(\Omega)$ it is enough to prove the assertions for smooth $\eta,\omega\in\Lambda^k(\Omega)$. In what follows $S(k,n)$ is the set of permutations σ of numbers $\{1,2,\dots n\}$, such that $\sigma(1)<\dots<\sigma(k)$ and $\sigma(k+1)<\dots<\sigma(n)$. By multi-linearity we have for orthonormal vector fields $\mathbf{e}_1,\dots\mathbf{e}_n$ and $\sigma\in S(j,n)$, $\gamma\in\Lambda^j(\Omega)$ and $x\in\Omega$ [24, Page 610]:

$$\begin{aligned}&(X_\tau^*\gamma)_x(\mathbf{e}_{\sigma(1)},\dots,\mathbf{e}_{\sigma(j)})\\ &\qquad=\sum_{\sigma'\in S(j,n)}\det\Big((DX_\tau(x))_{\sigma',\sigma}\Big)\,\gamma_{\,X_\tau(x)}\,(\mathbf{e}_{\sigma'(1)},\dots,\mathbf{e}_{\sigma'(j)}),\end{aligned} \tag{13}$$

where the quantities $\det\Big((DX_\tau(x))_{\sigma',\sigma}\Big)$ are known as the j-minors of the differential $DX_\tau(x)$ with respect to $\mathbf{e}_1,\dots,\mathbf{e}_n$, i.e. the determinants of those submatrices of $DX_\tau(x)$, that contain the rows σ' and columns σ. By the definition of the inner product of differential forms we have

$$\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)} = \int_{X_\tau(\Omega)} \langle X^*_{-\tau}\omega, X^*_{-\tau}\eta\rangle \,\mathrm{vol},$$

where $\langle\cdot,\cdot\rangle : \Lambda^k(\Omega) \times \Lambda^k(\Omega) \to \Lambda^0(\Omega) = C^\infty(\Omega)$ [25, Definition 1.2.2 b)] is the scalar product of alternating linear forms, which reads

$$\langle\omega,\eta\rangle := \sum_{\sigma\in S(k,n)} \omega(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)})\eta(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)}) \,.$$

Hence, (13) and the change of variable formula yield

$$\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)} = (\det(DX_\tau)\mathbf{M}_k(DX_\tau)\omega, \mathbf{M}_k(DX_\tau)\eta)_\Omega \ , \tag{14}$$

with

$$\begin{aligned}(\mathbf{M}_j(DX_\tau)\gamma)_x(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(j)}) :=& \\ &\sum_{\sigma'\in S(j,n)} \det\Big(\left(DX_\tau(x)\right)_{\sigma',\sigma}\Big)\,\gamma_x(\mathbf{e}_{\sigma'(1)},\ldots,\mathbf{e}_{\sigma'(j)}).\end{aligned} \tag{15}$$

Now, Taylor expansion of $\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)}$ in τ around $\tau = 0$ boils down to Taylor expansion of the multiplication coefficients $\det\Big(\left(DX_\tau(x)\right)_{\sigma',\sigma}\Big)$. From the Taylor expansion of $DX_\tau(x)$, $DX_\tau(x) = I_n + \int_0^\tau D\boldsymbol{\beta}(X_s(x))DX_s(x)ds$ and the Taylor expansion of det(), $\det(\mathbf{A}(\tau)) = \det(\mathbf{A}(0)) + \int_0^\tau \mathrm{tr}(\mathsf{Adj}(\mathbf{A}(s))\frac{d\mathbf{A}}{ds})ds$ and (15) we infer

$$\begin{aligned}(\mathbf{M}_j(DX_\tau)\gamma)_x(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(j)}) &= \sum_{\sigma'\in S(j,n)} \det((I_n)_{\sigma',\sigma})\gamma_x(\mathbf{e}_{\sigma'(1)},\ldots,\mathbf{e}_{\sigma'(j)}), \\ &+ \sum_{\sigma'\in S(j,n)} R_{\sigma',\sigma}(D\boldsymbol{\beta},\tau)\gamma_x(\mathbf{e}_{\sigma'(1)},\ldots,\mathbf{e}_{\sigma'(j)}) \ ,\end{aligned} \tag{16}$$

with Adj and tr the adjugate and trace operator for matrices, the unit matrix $I_n \in \mathbb{R}^{n\times n}$, and $|R_{\sigma',\sigma}(D\boldsymbol{\beta},\tau)| \le \tau C_{\sigma',\sigma}(D\boldsymbol{\beta})$, $C_{\sigma',\sigma}$ independent of τ. Combining (14) and (16) yields the assertion. ☐

Lemma 3.2. *If $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ satisfies Assumption 2.1, we can bound*

$$\left\|X^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k(\Omega)} \le (1+c\tau)\left\|\omega\right\|^2_{L^2\Lambda^k(\Omega)} \quad \forall\omega\in L^2\Lambda^k(\Omega) \ , \tag{17}$$

with $c = c(\|\boldsymbol{\beta}\|_{\boldsymbol{W}^{1,\infty}(\Omega)}) > 0$ independent of τ.

Proof. This lemma is a special case of Lemma 3.1. ☐

The following main theorem generalizes the result of [19, page 52] to advection problems for differential forms with non-constant velocity and non-homogeneous right hand side.

Theorem 3.3. *Let ω, $(\omega_h^n)_{n=0}^N$ be the solutions of (8) and (11), respectively. If $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$, $\omega \in L^\infty([0,T], H^s\Lambda^k(\Omega))$, Assumption (2.1) holds, and, additionally, $\Lambda_h^k(\mathcal{T})$ furnishes the approximation property (10), then we get*

$$\max_{0\le n\le N}\|\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)} \le Ch^{\min(r+1,s)}\tau^{-\frac{1}{2}} \max_{0\le n\le N}\|\omega(t^m)\|_{H^s\Lambda^k(\Omega)}, \tag{18}$$

where $C > 0$ depends only on K from (10) and $\boldsymbol{\beta}$.

Proof. Let P_h denote the L^2-projection onto $\Lambda_h^k(\mathcal{T})$. Then the solution formula (9), the definition (11) of ω_h^{n+1} and the Pythagorean theorem yield the following *recursive estimate* for the norm of the error $e^{n+1} := \omega(t^{n+1}) - \omega_h^{n+1}$:

$$\begin{aligned}\left\|e^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} &= \left\|\omega(t^{n+1}) - P_h\omega(t^{n+1}) + P_h\omega(t^{n+1}) - \omega_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} \\ &= \left\|\omega(t^{n+1}) - P_h\omega(t^{n+1}) + P_h X^*_{-\tau}\left(\omega(t^n) - \omega_h^n\right)\right\|^2_{L^2\Lambda^k(\Omega)} \\ &= \left\|\omega(t^{n+1}) - P_h\omega(t^{n+1})\right\|^2_{L^2\Lambda^k(\Omega)} + \left\|P_h X^*_{-\tau}\left(\omega(t^n) - \omega_h^n\right)\right\|^2_{L^2\Lambda^k(\Omega)} .\end{aligned}$$

Since we assume vanishing normal component of $\boldsymbol{\beta}$ on the boundary of Ω (Assumption 2.1), we can use Lemma 3.2 together with the L^2-stability of the orthogonal projection P_h to obtain

$$\left\|e^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} \leq \xi^{n+1} + (1 + c\tau)\left\|e^n\right\|^2_{L^2\Lambda^k(\Omega)} ,$$

with the spatial projection errors $\xi^{k+1} := \left\|\omega(t^{n+1}) - P_h\omega(t^{n+1})\right\|^2_{L^2\Lambda^k(\Omega)}$. Summation, combined with $1 + x \leq \exp(x)$ gives

$$\begin{aligned}\left\|e^n\right\|^2_{L^2\Lambda^k(\Omega)} &\leq \sum_{i=0}^{n}(1 + c\tau)^{n-i}\xi^i \leq \frac{\exp(c\tau(n+1)) - 1}{\exp(c\tau) - 1}\max_{0\leq i\leq n}\xi^i \\ &\leq \frac{\exp(cT) - 1}{\exp(c\tau) - 1}\max_{0\leq i\leq n}\xi^i \leq \frac{\exp(cT)}{c\tau}\max_{0\leq i\leq n}\xi^i.\end{aligned}$$

Finally we use the approximation property (10) of $\Lambda_h^k(\mathcal{T})$ to control the ξ^i and obtain the assertion. ☐

The scheme (11) is impractical for most applications, since it assumes that the bilinear form $\left(X^*_{-\tau}u_h, v_h\right)_\Omega$ can be computed exactly. Fully discrete schemes use approximate pullbacks.

Remark 3.4. *As in [14] we can introduce fully discrete schemes by means of approximated pullbacks and quadrature formulas for the right hand sides. A similar convergence analysis as the one in [14, Theorem 2] gives estimates that are explicit in the additional approximation parameters. To keep this article focused, we merely refer to [14, Section 5] for a detailed explanation of the various approximation steps towards fully discrete semi-Lagrangian schemes.*

4. Numerical experiments

Now we study the behaviour of the semi-Lagrangian scheme for the rotating hump problem, c.f. [5, 11, 17]. We consider an advection problem (8) for 0-forms on the unit disc $\Omega := \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ with vanishing source term and the velocity field

$$\boldsymbol{\beta}(\mathbf{x}) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} , \quad \mathbf{x} = (x_1, x_2) \in \Omega .$$

In vector proxy notation, *cf.* Sect 2 and Table 1, the analytical solution reads

$$u(t, \mathbf{x}) = u_0(\mathbf{R}(t)\mathbf{x}), \quad \mathbf{R}(t) := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \ t \in \mathbb{R} ,$$

where $u_0 = u_0(\mathbf{x})$ is the initial data.

For this problem, we can compute the inner products $\left(X^*_{-\tau}\omega_h, \eta_h\right)_\Omega$ up to very high accuracy. Hence, we avoid the difficulties connected to the approximation of X_τ, e.g. instabilities [20] or inconsistencies [14]: First, we can use exact tracking of the characteristics and second, the functions $X^*_{-\tau}\omega_h$, $\omega_h \in \Lambda_h^k(\mathcal{T})$, are again finite element functions on a mesh $\mathcal{T}'$ with straight edges. To compute $\left(X^*_{-\tau}\omega_h, \eta_h\right)_\Omega$ we split the integral over Ω into a sum of integrals over all intersections of elements of

$\mathcal{T}$ and $\mathcal{T}'$. Both $X^*_{-\tau}\omega_h$ and η_h are smooth on these intersections, hence we can employ high order quadrature rules. Since in general the intersections are convex polygons we obtain these quadrature rules via auxiliary triangulations.

In the sequel we monitor the error $\|u(T) - u_h^n\|_{L^2(\Omega)}$, $T = \pi$ for the following initial data, c.f. [17]:

(19)
$$u_{0,i} = \begin{cases} \cos\left(\frac{3}{2}\pi\sqrt{x_1^2 + (x_2 - 0.5)^2}\right)^i & x_1^2 + (x_2 - 0.5)^2 \leq \frac{1}{9} \\ 0 & \text{else} \end{cases}, \quad i = 1, 2, 3, 4, 5.$$

The larger the index i the smoother is $u_{0,i}$. We consider triangulations with straight edges and boundary vertices that lie on the boundary of the original domain Ω, i.e. in each refinement step the boundary nodes of the mesh are placed onto the boundary of the domain, c.f. Figure 1. Note that the support of $u(t, \mathbf{x})$ never comes close to $\partial\Omega$. Therefore, we do not expect the polygonal approximation of $\partial\Omega$ to affect the results.

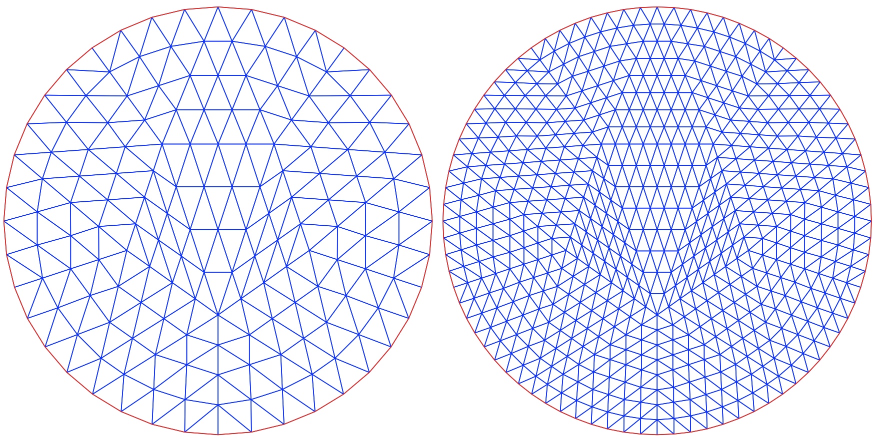

FIGURE 1. Two triangulations of the domain Ω.

Galerkin discretization is based on *continuous* linear Lagrangian elements (**Experiment 1**), continuous quadratic Lagrangian elements (**Experiment 2**), continuous hierarchic elements of polynomial order 1, 2, 3 and 4 (**Experiment 3**) and spaces of *discontinuous* piecewise polynomials of degree 0, 1, 2, 3 and 4 (**Experiment 4**), respectively.

If we link the timestep size τ to the mesh size h by $\tau = \frac{\delta}{\sqrt{2}}h$, we get *higher rates of convergence* (see Figures 2, 5, 8, 9), than those predicted by Theorem 3.3. Interestingly, in the case of continuous approximation spaces, we observe a remarkable difference between odd and even polynomial degree r. While in the former case we see clearly convergence of order $O(h^{r+1})$ (Figures 2 and 8), in the latter case the convergence seem to be between $O(h^{r+\frac{1}{2}})$ and $O(h^{r+1})$ (Figures 5 and 8). For discontinuous elements we observe convergence of order $O(h^{r+1})$ throughout.

If we keep the timestep size τ fixed we observe exactly the rates $O(h^{r+1})$, that follow by Theorem 3.3 (see Figures 3 and 6). Also the impact of the smoothness of the solution on the rate of convergence, is reflected by the results of the numerical experiments (see Figures 4 and 7).

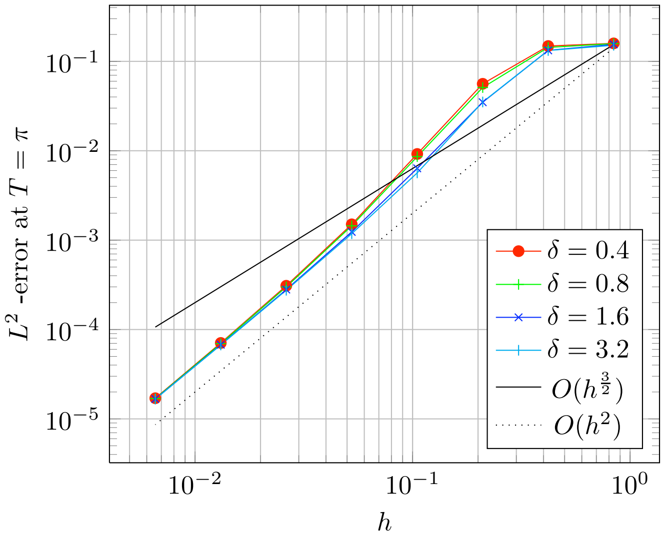

FIGURE 2. Experiment 1 a): linear elements, locally supported bump $u_{0,3} \in H^3(\Omega)$ as initial data, $\tau = \frac{\delta}{\sqrt{2}}h$.

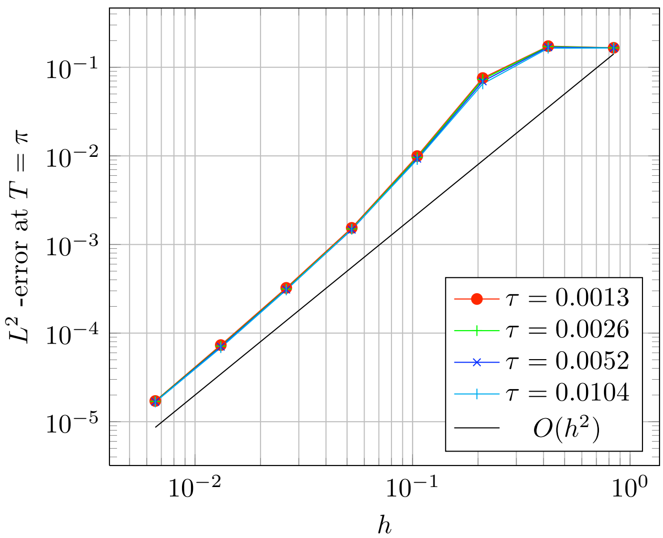

FIGURE 3. Experiment 1 b): linear elements, locally supported bump $u_{0,3} \in H^3(\Omega)$ as initial data, τ fixed.

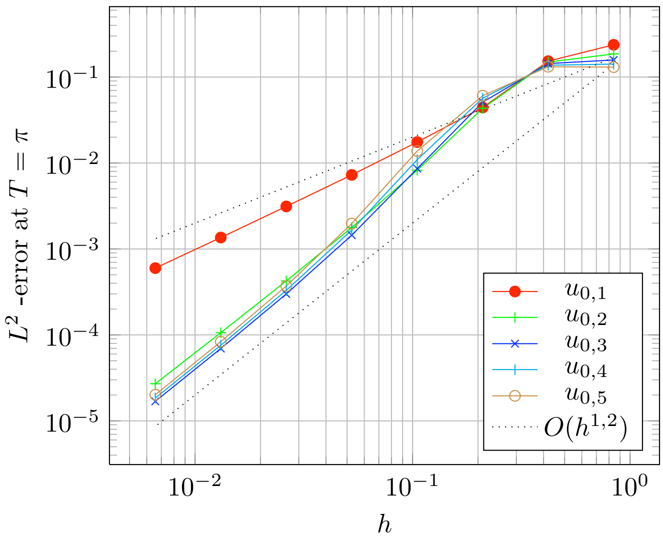

FIGURE 4. Experiment 1 c): linear elements, locally supported bumps of different global regularity as initial data, $\tau = \frac{0.8}{\sqrt{2}}h$.

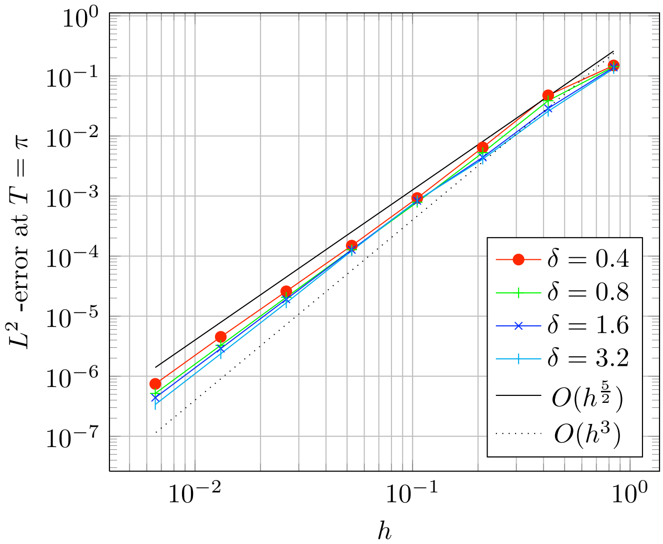

FIGURE 5. Experiment 2 a): quadratic Lagrangian elements, locally supported bump $u_{0,3} \in H^3(\Omega)$ as initial data, $\tau = \frac{\delta}{\sqrt{2}}h$.

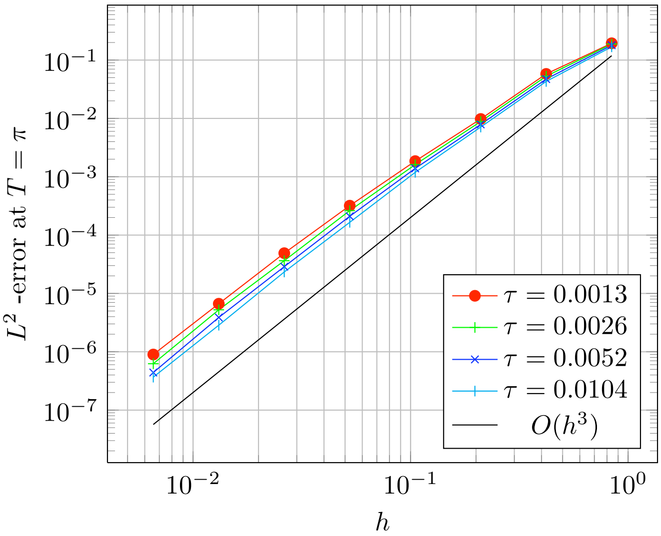

FIGURE 6. Experiment 2 b): quadratic Lagrangian elements, locally supported bump $u_{0,3} \in H^3(\Omega)$ as initial data, τ fixed. Experimental result agrees with Theorem 3.3.

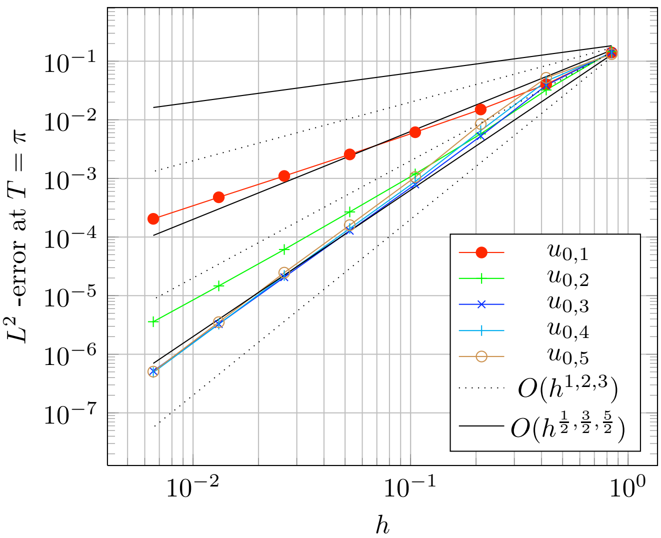

FIGURE 7. Experiment 2 c): quadratic elements, locally supported bumps of different global regularity as initial data, $\tau = \frac{0.8}{\sqrt{2}} h$.

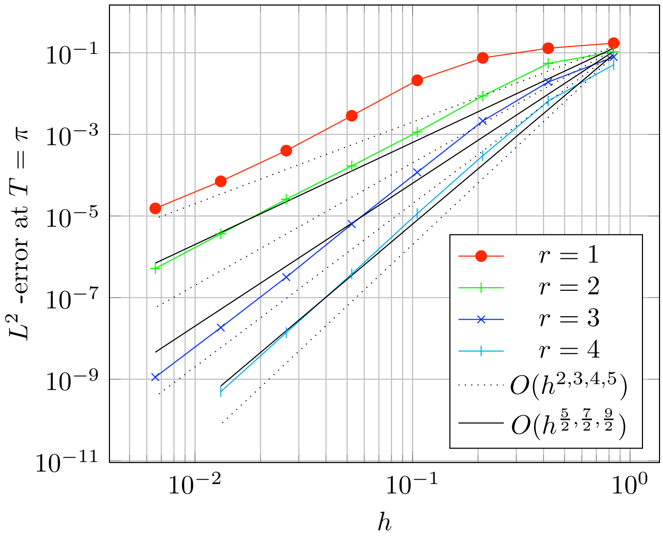

FIGURE 8. Experiment 3: $H^1(\Omega)$ conforming hierarchic elements of different order, locally supported bump $u_{0,5} \in H^5(\Omega)$ as initial data, $\tau = \frac{0.8}{\sqrt{2}}h$.

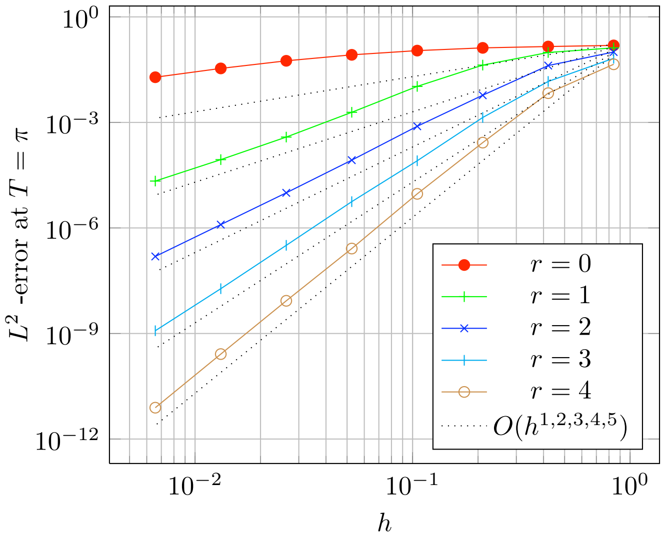

FIGURE 9. Experiment 4: discontinuous elements of different order, locally supported bump $u_{0,5} \in H^5(\Omega)$ as initial data, $\tau = \frac{0.8}{\sqrt{2}}h$.

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