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N-term Galerkin Wiener chaos approximations of elliptic PDEs with lognormal Gaussian random inputs*

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Abstract

We consider diffusion in a random medium modeled as diffusion equation with lognormal Gaussian diffusion coefficient. Sufficient conditions on the log permeability are provided in order for a weak solution to exist in certain Bochner-Lebesgue spaces with respect to a Gaussian measure. The stochastic problem is reformulated as an equivalent deterministic parametric problem on $\mathbb{R}^{\mathbb{N}}$. It is shown that the weak solution can be represented as Wiener-Itô Polynomial Chaos series of Hermite Polynomials of a countable number of i.i.d standard Gaussian random variables taking values in $\mathbb{R}^1$.

We establish sufficient conditions on the random inputs for weighted sequence of norms of the Wiener-Itô decomposition coefficients of the random solution to be p-summable for some $0 < p < 1$. For random inputs with additional spatial regularity, stronger norms of the weighted coefficient sequence in the random solutions' Wiener-Itô decomposition are shown to be p-summable for the same value of $0 < p < 1$.

We infer rates of nonlinear, best N-term Wiener Polynomial Chaos approximations of the random field, as well as for Finite Element discretizations of these approximations from a dense, nested family $V_0 \subset V_1 \subset V_2 \subset ....V$ of finite element spaces of continuous, piecewise linear Finite Elements.

Key Words: Lognormal Gaussian Random Field, Stochastic Diffusion Equation, Wiener-Itô decomposition, polynomial chaos, random media, best N-term approximation, Hermite Polynomials

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1 Introduction

In recent years, partial differential equations with random inputs have attracted interest due to their relevance for quantifying uncertainty in engineering and in the sciences. Broad classes of numerical methods to estimate statistics of random solutions include sampling techniques such as Monte-Carlo and Quasi-Monte Carlo methods, Stochastic collocation techniques and spectral discretization techniques consisting of Galerkin projection onto (generalized) polynomial chaos bases. Whereas the former are rather general, the latter require careful study of the probability measure and a spectral basis adapted to the probability space of the random inputs. A common feature of the latter class of problems is their parametrization as a deterministic problem on a *parameter space of countably infinite dimension*. A key analytical question in this context is then the approximability of the parametric, deterministic solution in terms of tensorized polynomial systems which are orthonormal with respect to the probability measure. This approach has gained increasing significance in recent years. We mention only the book [10] and the papers [19, 18, 2, 1, 15, 14, 13, 3, 12] and the references there.

In particular, in the context of adaptive solution algorithms, so-called best N-term approximation rates are of interest as a benchmark for the best possible achieveable by adaptive, deterministic approximation methods of stochastic Galerkin or also of stochastic collocation type. One principal aim of this work is to prove that, indeed, adaptive polynomial chaos approximations can afford higher rates of convergence in terms of the number of overall degrees of freedom than the commonly used Monte Carlo sampling methods or even their more efficient variants, the multi-level Monte-Carlo Finite Element Methods, whose complexity was recently analyzed in [4].

In the case of probability measures with compact support such as, e.g. the uniform distribution, best N-term approximation results in terms of tensorized Legendre polynomials (which are the natural orthogonal polynomials for the uniform probability measure) have been obtained in [7, 6]. In many applications, however, countably many, independent and identically normally distributed random inputs are assumed. In this case, the natural polynomial system for the representation of system's random response are well-known to be tensorized Hermite polynomials; this goes back N. Wiener (see, e.g., [17]) and is, therefore, termed *Wiener polynomial chaos*, or WPC, representation.

To obtain best N-term approximation rates for truncated Wiener polynomial chaos expansions, for solutions of elliptic partial differential equations with random inputs and for probability measures with unbounded support, such as lognormal models of permeabilities in subsurface flow models, is one purpose of the present paper.

Its outline is a follows: in the next section, following [11, 16] we specify the lognormal diffusion problem and present its reduction to a parametric, deterministic problem on a subset Γ of the infinite-dimensional parameter space $\mathbb{R}^{\mathbb{N}}$ which we show, however, to be measurable with respect to a parametric family of Gaussian measures on $\mathbb{R}^{\mathbb{N}}$, and to be of full measure. We then establish well-posedness of the parametric, deterministic problem and measurability of the solution of the parametric deterministic problem for all parameter vectors in a subset Γ of $\mathbb{R}^{\mathbb{N}}$ of full (Gaussian) measure. We present a weak formulation of this parametric, deterministic problem and prove its well-posedness. We then show that the parametric solution can be expanded into a polynomial chaos type series with respect to a countable family i.i.d Gaussian random variables. Moreoever, we establish conditions on the p-summability of the Hermite coefficients of the solution, under suitable decay condition of the random coefficients of the problem.

Throughout, we shall use the following notation: $\mathbb{N}$ denotes the set of natural numbers, and we define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $D \subset \mathbb{R}^d$, $d \geq 2$, we denote a bounded domain with Lipschitz boundary ∂D.
By $\mathbb{R}^{\mathbb{N}}$ we denote the set of all sequences of real numbers and observe that $\mathbb{R}^{\mathbb{N}} = \mathbb{R} \times \mathbb{R} \times ... = \mathbb{R}^{\infty}$. By $\mathcal{F}$, we denote the set of "finitely supported" countable multiindices, i.e.

$$\mathcal{F} = \left\{ \nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty \right\} \ .$$

Here, by $|\nu| = \nu_1 + \nu_2 + ...$, we denote the "length" of the multiindex $\nu \in \mathbb{N}_0^{\mathbb{N}}$. Evidently, a multiindex $\nu \in \mathcal{F}$ can have only finitely many nonzero entries ν_j. For $\nu \in \mathcal{F}$, we denote by $\mathfrak{n} \subset \mathbb{N}$ the "support set" of ν, i.e. the (finite) set of all $j \in \mathbb{N}$ such that $\nu_j \neq 0$, *with j repeated $\nu_j \geq 1$ times*. Hence, $|\mathfrak{n}| = |\nu|$. We shall always associate to $\nu \in \mathcal{F}$ the support set $\mathfrak{n}$ and to $\mu \in \mathcal{F}$ the set $\mathfrak{m} \subset \mathbb{N}$.

For $y \in U$, we denote by $\partial_y^{\nu} u(\cdot, y)$ the mixed partial derivative of order ν and likewise $\partial_y^{\mu} u(\cdot, y)$. On occasion, we shall also write $\partial_y^{\mathfrak{n}}$ in place of ∂_y^{ν} and likewise for $\partial_y^{\mathfrak{m}}$.

2 Problem Formulation

Let $D \subset \mathbb{R}^d$ denote a bounded domain with a Lipschitz boundary ∂D and denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. In the domain D, we consider stochastic isotropic elliptic problems

$$-\nabla \cdot (a(x,\omega)\nabla u(x,\omega)) = f(x) \quad \text{in} \quad D \;, \qquad u|_{\partial D} = 0 \;. \tag{2.1}$$

Here, the coefficient $a : D \times \Omega \mapsto \mathbb{R}$ a stochastic diffusion coefficient, and f is the deterministic source term (a stochastic source term that is uncorrelated to a could equally well be considered; to avoid unnecessarily involved exposition and notation, we do not elaborate on this case). By $V = H_0^1(D)$ we denote the closed subspace of the Sobolev space $H^1(D)$ of functions whose boundary values vanish in the sense of trace, with norm

$$\|v\|_V := \left(\int_D |\nabla v(x)|^2 dx\right)^{1/2} \;. \tag{2.2}$$

For given random coefficient $a(x,\omega)$ and any $w, v \in V$, we define the stochastic bilinear form

$$b(\omega; w, v) := \int_D a(x,\omega)\nabla w \cdot \nabla v dx;$$

and we consider the source term f as element of the dual space V' of V. Then, for any $\omega \in \Omega$, the weak formulation of (2.1) reads: find $u(\omega) \in V$ such that

$$b(\omega; u(\omega), v) = \langle f, v\rangle \quad \forall v \in V \;. \tag{2.3}$$

Here, and in what follows, we denote by $\langle\cdot,\cdot\rangle$ the extension by continuity of the $L^2(D)$ innerproduct to the $V' \times V$ duality pairing. To prove well-posedness of (2.3), we use the Lax-Milgram Lemma. To invoke it, we specify conditions which ensure that the diffusion coefficient is positive, and show that, under additional conditions, the collection of pathwise solutions $\{u(\omega) : \omega \in \Omega\}$ is measurable with respect to a suitable probablity measure.

2.1 Model elliptic PDE with lognormal Gaussian Parameters

For the coefficient $a(x,\omega)$ of the problem (2.1), we assume a Karhúnen–Loève type expansion of $\log(a-a_*)$, where a_* is a bounded function on D with $a_*(x) \geq 0$ for all $x \in D$. Thus, we assume a stochastic diffusion coefficient of the form

$$a(x,\omega) = a_*(x) + a_0(x)\exp\left(\sum_{m=1}^{\infty} Y_m(\omega)\psi_m(x)\right) \;, \quad x \in D \;, \tag{2.4}$$

for $y(\omega) = (Y_m(\omega))_{m\in\mathbb{N}} : \Omega \to \mathbb{R}^{\mathbb{N}}$.

To fix the scaling in the Karhúnen–Loève expansion (2.4), *we further assume that the* $Y_m(\omega)$, $m \in \mathbb{N}$ *are independent standard Gaussian random variables in* $\mathbb{R}^1$. This is the case if, for example, $\log(a - a_*)$ is Gaussian and we expand it in its Karhúnen–Loève series, or more generally if $(\psi_m)_{m\in\mathbb{N}}$ are orthonormal in the Cameron–Martin space of the distribution of $\log(a - a_*)$, see [16, Section 2.4].

By the above assumptions, the law of the sequence of random variables $y = (Y_1(\omega), Y_2(\omega), \ldots)$ is defined on the probability space $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \gamma)$, with the Gaussian measure γ given by

$$\gamma = \bigotimes_{m=1}^{\infty} N_1 \tag{2.5}$$

(see, e.g., [5]). In (2.4), we assume that $\psi_m \in L^\infty(D)$ for all $m \in \mathbb{N}_0$, $a_0(x) \geq \check{a}_0 > 0$ for all $x \in D$, $a_*(x) \geq 0$ and

$$\sum_{m=1}^{\infty} \|\psi_m\|_{L^\infty(D)} < \infty \;, \tag{2.6}$$

i.e. we require that the sequence

$$b = (b_m)_{m\geq 1} = (\|\psi_m\|_{L^\infty(D)})_{m\geq 1} \in \ell^1(\mathbb{N}) \;. \tag{2.7}$$

Given a sequence $b \in \ell^1(\mathbb{N})$ we define the set

$$\Gamma_b := \left\{ y \in \mathbb{R}^{\mathbb{N}} \,;\, \sum_{m=1}^{\infty} b_m |y_m| < \infty \right\} . \tag{2.8}$$

For each $y \in \Gamma_b$, we define the *deterministic, parametric* coefficient as

$$a(x,y) = a_*(x) + a_0(x) \exp\left(\sum_{m=1}^{\infty} y_m \psi_m(x) \right) , \quad x \in D. \tag{2.9}$$

The series in (2.9) converges in $L^\infty(D)$ for all $y \in \Gamma_b \subset \mathbb{R}^{\mathbb{N}}$. We observe from (2.9) that as $a_*(x) \geq 0$ for all x, for every $y \in \Gamma_b$ holds

$$\forall \nu \in \mathcal{F} : \quad \left\| \frac{\partial_y^\nu a(\cdot,y)}{a(\cdot,y)} \right\|_{L^\infty(D)} \leq b^\nu . \tag{2.10}$$

Moreover, the set Γ_b of admissible parameter vectors is γ-measurable and of full measure: there holds (see [16, Lemma 2.28])

Lemma 2.1 *For any sequence* $b \in \ell^1(\mathbb{N})$*,*

$$\Gamma_b \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \quad \text{and} \quad \gamma(\Gamma_b) = 1 .$$

In the following, if the dependence of the set Γ_b on the sequence b is clear from the context, we omit it in the notation.

Lemma 2.2 *For all* $y \in \Gamma$*, the diffusion coefficient* (2.9) *is well-defined and satisfies*

$$0 < \check{a}(y) := \operatorname*{ess\,inf}_{x \in D} a(x,y) \leq \operatorname*{ess\,sup}_{x \in D} a(x,y) =: \hat{a}(y) < \infty \tag{2.11}$$

with

$$\hat{a}(y) \leq \|a_*\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \exp\left(\sum_{m=1}^{\infty} b_m |y_m| \right) ,$$
$$\check{a}(y) \geq \operatorname*{ess\,inf}_{x \in D} a_*(x) + \check{a}_0 \exp\left(- \sum_{m=1}^{\infty} b_m |y_m| \right) .$$

Proof: Let $y \in \Gamma$ and $x \in D$ with $|\psi_m(x)| \leq b_m$ for all $m \in \mathbb{N}$. Then

$$\sum_{m=1}^{\infty} |\psi_m(x)||y_m| \leq \sum_{m=1}^{\infty} b_m |y_m| < \infty .$$

By continuity and positivity of $\exp(\cdot)$, for $y \in \Gamma_b$,

$$\exp\left(\sum_{m=1}^{\infty} \psi_m(x) y_m \right) = \prod_{m=1}^{\infty} \exp(\psi_m(x) y_m) \in (0,\infty) . \tag{2.12}$$

Then the claim follows from Kakutani's Theorem (see, e.g. [5]). □

Due to Lemmas 2.1 and 2.2, we consider Γ as the parameter space instead of $\mathbb{R}^{\mathbb{N}}$. Even though Γ is not a product domain, we can define product measures such as γ on Γ by restriction.

Lemma 2.2 shows that the stochastic diffusion coefficient $a(x,\omega)$ in (2.4) is well defined, bounded from above and to admit a positive lower bound for almost all $\omega \in \Omega$. Thus the stochastic diffusion equation (2.1) and, equivalently, the stochastic variational form (2.3) admits a unique solution $u(\omega) \in V$ for almost all $\omega \in \Omega$.

For each $y \in \Gamma$, we consider the *parametric deterministic* elliptic problem

$$\begin{aligned} -\nabla \cdot (a(x,y)\nabla u(x,y)) &= f(x) , \quad x \in D , \\ u(x,y) &= 0 , \quad x \in \partial D \end{aligned} \tag{2.13}$$

with the solution $u(y) \in V$. For $y \in \Gamma$, we define the parametric, deterministic bilinear form

$$b(y; w, v) := \int_D a(x,y) \nabla w(x) \cdot \nabla v(x) dx \ , \quad w, v \in V \ , \tag{2.14}$$

and reinterpret the forcing term f as a map into the dual space V' by

$$f(v) := \int_D f(x) v(x) dx \ , \quad v \in V \ , \tag{2.15}$$

with the integral understood as extension of the $L^2(D)$-innerproduct to the $V' \times V$-duality pairing by continuity.

The *parametric, deterministic variational formulation* of the lognormal diffusion equation (2.13) is given by the linear variational problem of determining, for $y \in \Gamma$, an element $u(y) \in V$ such that

$$b(y; u(y), v) = f(v) \qquad \forall v \in V \ . \tag{2.16}$$

Theorem 2.3 *For all $y \in \Gamma$, (2.16) has a unique solution $u(y) \in V$. It satisfies*

$$\|u(y)\|_V \le \frac{1}{\check{a}(y)} \|f(\cdot)\|_{V'} \qquad \forall y \in \Gamma \ . \tag{2.17}$$

Proof: By Lemma 2.2 and (2.2), the bilinear form $b(y; \cdot, \cdot)$ is continuous and coercive on V with coercivity constant $\check{a}(y)$ for all $y \in \Gamma$. The claim follows by the Lax–Milgram lemma. □

Next, we review solvability of elliptic problems with log-normal coefficients as discussed in [11] and [16] to the extent that we require later.

2.2 Auxiliary Gaussian Measures

For any sequence $\boldsymbol{\sigma} = (\sigma_m)_{m \in \mathbb{N}} \in \exp(\ell^1(\mathbb{N}))$, *i.e.* $\sigma_m = \exp(s_m)$ with $(s_m)_m \in \ell^1(\mathbb{N})$, we define the product measure

$$\gamma_{\boldsymbol{\sigma}} := \bigotimes_{m=1}^{\infty} N_{\sigma_m^2} \tag{2.18}$$

on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$, where $N_{\sigma_m^2}$ denotes the centered Gaussian measure on $\mathbb{R}^1$ with standard deviation $\sigma_m > 0$. We denote the standard Gaussian measure on $\mathbb{R}^{\mathbb{N}}$ by $\gamma = \gamma_{\mathbf{1}}$.

Proposition 2.4 *([11]) For all $\boldsymbol{\sigma} = (\sigma_m)_{m \in \mathbb{N}} \in \exp(\ell^1(\mathbb{N}))$, the measure $\gamma_{\boldsymbol{\sigma}}$ is equivalent to γ. The density of $\gamma_{\boldsymbol{\sigma}}$ with respect to γ is given explicitly by*

$$\zeta_{\boldsymbol{\sigma}}(y) = \left(\prod_{m=1}^{\infty} \frac{1}{\sigma_m} \right) \exp\left(-\frac{1}{2} \sum_{m=1}^{\infty} (\sigma_m^{-2} - 1) y_m^2 \right) \ . \tag{2.19}$$

Proposition 2.4 implies in particular that $\gamma_{\boldsymbol{\sigma}}(\Gamma) = 1$ for any $\boldsymbol{\sigma} \in \exp(\ell^1(\mathbb{N}))$. Therefore, the restriction of $\gamma_{\boldsymbol{\sigma}}$ to Γ is a probability measure.

We consider sequences $\boldsymbol{\sigma}$ that depend exponentially on $b = (b_m)_{m \in \mathbb{N}}$, whose terms are given by

$$\sigma_m(\chi) := \exp(\chi b_m) \ , \quad m \in \mathbb{N} \ , \quad \chi \in \mathbb{R} \ . \tag{2.20}$$

We abbreviate $\gamma_\chi := \gamma_{\boldsymbol{\sigma}(\chi)}$ and $\zeta_\chi := \zeta_{\boldsymbol{\sigma}(\chi)}$. In particular, $\gamma = \gamma_{\mathbf{1}} = \gamma_0$. Then we have

Lemma 2.5 *([16, Lemma 2.32]) Let $\eta < \chi$ and $k \ge 0$. Then*

$$\forall y \in \Gamma : \quad \frac{\zeta_\eta(y)}{\zeta_\chi(y)} \exp\left(k \sum_{m=1}^{\infty} b_m |y_m| \right) \le \exp\left(\left(\frac{k^2 e^{2\chi \|b\|_{\ell^\infty(\mathbb{N})}}}{4(\chi - \eta)} + \chi - \eta \right) \|b\|_{\ell^1(\mathbb{N})} \right) \ . \tag{2.21}$$

If, in particular, $k = 0$ then (2.21) reads

$$\forall y \in \Gamma : \quad \frac{\zeta_\eta(y)}{\zeta_\chi(y)} \le \exp\left((\chi - \eta) \|b\|_{\ell^1(\mathbb{N})} \right) \ . \tag{2.22}$$

We also have

Proposition 2.6 *([16, Proposition 2.33]) Let $0 < p < \infty$ and $\eta < \chi$. Then*

$$L^p(\Gamma, \gamma_\chi) \subset L^p(\Gamma, \gamma_\eta) \tag{2.23}$$

and

$$\|v\|_{L^p(\Gamma,\gamma_\eta)} \le \exp\left(\frac{\chi-\eta}{p}\|b\|_{\ell^1(\mathbb{N})}\right)\|v\|_{L^p(\Gamma,\gamma_\chi)} \qquad \forall v \in L^p(\Gamma, \gamma_\chi) \ . \tag{2.24}$$

Proposition 2.6 also applies to Lebesgue–Bochner spaces of functions taking values in, for example, V or V'. We will use it with $\eta = 0$, such that $\gamma_\eta = \gamma$.

2.3 Integrability of the Solution

We now briefly discuss integrability properties of the solution u of (2.16). Borel measurability of the map $\mathbb{R}^{\mathbb{N}} \supset \Gamma \ni y \mapsto u(y) \in V$ is shown in [11, Lemma 3.4] under the assumption that f is Borel measurable as a map from $\mathbb{R}^{\mathbb{N}}$ to V'. Under stronger assumptions, measurability of u also follows from Theorem 2.17 below.

Proposition 2.7 *Let $0 < p < \infty$ and $\varrho > 0$. The solution u of (2.16) is in $L^p(\Gamma, \gamma; V)$ and satisfies*

$$\|u\|_{L^p(\Gamma,\gamma;V)} \le \bar{c}_{\varrho,p}\|f\|_{V'}$$

with

$$\bar{c}_{\varrho,p} = \min\left\{\frac{\exp\left(\frac{\varrho}{p}\|b\|_{\ell^1(\mathbb{N})}\right)}{\operatorname{ess\,inf}_{x\in D} a_*(x)}, \ \frac{1}{\check{a}_0}\exp\left(\|b\|_{\ell^1(\mathbb{N})}\left(\frac{p\exp(2\varrho\|b\|_{\ell^\infty(\mathbb{N})})}{4\varrho}+\frac{\varrho}{p}\right)\right)\right\} \ .$$

The propsition is a special case of Proposition 2.34 of [16] where this assertion is shown in the more general case that when $f \in L^p(\Gamma, \gamma_\varrho; V')$, it holds

$$\|u\|_{L^p(\Gamma,\gamma;V)} \le \bar{c}_{\varrho,p}\|f\|_{L^p(\Gamma,\gamma_\varrho;V')} \ .$$

We also need integrability of u with respect to the measure γ_ϱ. There holds (see [11, Lemma 3.10]):

Lemma 2.8 *For all $\varrho \ge 0$ and all $0 < r < \infty$,*

$$\exp\left(\sum_{m=1}^{\infty} b_m|y_m|\right) \in L^r(\Gamma, \gamma_\varrho)$$

with

$$\left\|\exp\left(\sum_{m=1}^{\infty} b_m|y_m|\right)\right\|_{L^r(\Gamma,\gamma_\varrho)} \le \exp\left(\frac{r}{2}\|b\|^2_{\ell^2(\mathbb{N})}\exp(2\varrho\|b\|_{\ell^\infty(\mathbb{N})}) + \sqrt{\frac{2}{\pi}}\|b\|_{\ell^1(\mathbb{N})}\exp(\varrho\|b\|_{\ell^\infty(\mathbb{N})})\right)$$

Theorem 2.9 *Let $0 < q < p < \infty$ and $\varrho \ge 0$. The solution u of (2.16) is in $L^q(\Gamma, \gamma_\varrho; V)$ and satisfies*

$$\|u\|_{L^q(\Gamma,\gamma_\varrho;V)} \le \widetilde{c}_{\varrho,q,p}\|f\|_{V'}$$

with

$$\widetilde{c}_{\varrho,q,p} = \frac{1}{\check{a}_0}\exp\left(\frac{qp}{2(p-q)}\|b\|^2_{\ell^2(\mathbb{N})}\exp(2\varrho\|b\|_{\ell^\infty(\mathbb{N})}) + \sqrt{\frac{2}{\pi}}\|b\|_{\ell^1(\mathbb{N})}\exp(\varrho\|b\|_{\ell^\infty(\mathbb{N})})\right) \ ,$$

or, if $\operatorname{ess\,inf}_{y\in\Gamma} a_*(y) > 0$ and $q \le p$, also with

$$\widetilde{c}_{\varrho,q,p} = \frac{1}{\operatorname{ess\,inf}_{x\in D} a_*(x)} \ .$$

This theorem is a special case of Theorem 2.36 in [16]. Indeed, in [16] it is shown that when f depends on y and is in $L^p(\Gamma, \gamma_\varrho; V')$, there holds

$$\|u\|_{L^q(\Gamma,\gamma_\varrho;V)} \leq \widetilde{c}_{\varrho,q,p}\|f\|_{L^p(\Gamma,\gamma_\varrho;V')}$$

In particular, if $f \in L^p(\Gamma, \gamma_\varrho; V')$ with $p > 2$, then $u \in L^2(\Gamma, \gamma_\varrho; V)$ and

$$\|u\|_{L^2(\Gamma,\gamma_\varrho;V)} \leq \widetilde{c}_{\varrho,p}\|f\|_{L^p(\Gamma,\gamma_\varrho;V')} \tag{2.25}$$

with

$$\widetilde{c}_{\varrho,p} = \frac{1}{\check{a}_0} \exp\left(\frac{p}{p-2} \exp(2\varrho\|b\|_{\ell^\infty(\mathbb{N})})\|b\|^2_{\ell^2(\mathbb{N})} + \sqrt{\frac{2}{\pi}} \exp(\varrho\|b\|_{\ell^\infty(\mathbb{N})})\|b\|_{\ell^1(\mathbb{N})}\right) . \tag{2.26}$$

In our case, f is independent of y so the assertion $u \in L^2(\Gamma, \gamma_\varrho; V)$ holds. As $p \to \infty$, we find that

$$\widetilde{c}_{\varrho,\infty} = \frac{1}{\check{a}_0} \exp\left(\exp(2\varrho\|b\|_{\ell^\infty(\mathbb{N})})\|b\|^2_{\ell^2(\mathbb{N})} + \sqrt{\frac{2}{\pi}} \exp(\varrho\|b\|_{\ell^\infty(\mathbb{N})})\|b\|_{\ell^1(\mathbb{N})}\right) < \infty .$$

The space of Gaussian random fields with finite second moments admits a Wiener-Itô decomposition corresponding to expansions of such random fields in terms of Hermite polynomials of Gaussians. The main result of the present paper is to show regularity for the Wiener-Itô decomposition of the solution of the diffusion problem. Specifically, we show that the terms of its Wiener-Itô decomposition are p summable for some power $0 < p < 2$. To this end, we denote by $H_n(t)$ the Hermite polynomial of degree $n \in \mathbb{N}$, normalized so that

$$\|H_n(t)\|_{L^2(\mathbb{R},N_1)} = 1 . \tag{2.27}$$

Note that $H_0 \equiv 1$. For $y \in \Gamma$ and for $\nu \in \mathcal{F}$, we define

$$H_\nu(y) := \prod_{m\geq 1} H_{\nu_m}(y_m) = H_{\nu_1}(y_1)H_{\nu_2}(y_2)\ldots . \tag{2.28}$$

Since $\nu \in \mathcal{F}$, the formally infinite product in (2.28) contains only finitely many nontrivial factors.

The univariate Hermite polynomials form an orthonormal basis of $L^2(\mathbb{R}^1, \gamma_1)$ (see, e.g. [8, Proposition 9.4] or [5, Lemma 1.3.2 i)]). By [16, Proposition 2.38], for (2.28) the tensorized Hermite polynomials $(H_\nu)_{\nu\in\mathcal{F}}$, form an orthonormal basis of $L^2(\Gamma, \gamma)$. We transform these to an orthonormal basis of $L^2(\Gamma, \gamma_\varrho)$ using the map

$$\tau_\varrho : \mathbb{R}^\mathbb{N} \to \mathbb{R}^\mathbb{N} , \quad (y_m)_{m\in\mathbb{N}} \mapsto (e^{-\varrho b_m} y_m)_{m\in\mathbb{N}} . \tag{2.29}$$

Note that τ_ϱ maps Γ bijectively onto Γ.

Lemma 2.10 *For all $\varrho \in \mathbb{R}$, the map*

$$L^2(\Gamma, \gamma) \to L^2(\Gamma, \gamma_\varrho) , \quad v \mapsto v \circ \tau_\varrho \tag{2.30}$$

is a unitary isomorphism of Hilbert spaces. Furthermore,

$$\int_\Gamma v(y)\gamma(dy) = \int_\Gamma v(\tau_\varrho(y))\gamma_\varrho(dy) \qquad \forall v \in L^2(\Gamma, \gamma) . \tag{2.31}$$

Proof: The standard Gaussian measure γ is the image of γ_ϱ under the map τ_ϱ. i.e. $\gamma(E) = \gamma_\varrho(\tau_\varrho^{-1}(E))$ for all $E \in \mathcal{B}(\Gamma)$. This is easily checked for sets $E = \{y \in \Gamma\, ;\, y_m \leq \lambda\}$ with $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$. Then (2.31) is the transformation theorem. The remaining part of the assertion is a direct consequence. □

The next assertion is closely related to the Wiener-Itô decomposition of $L^2(\mathbb{R}^\mathbb{N}, \gamma)$.

Proposition 2.11 *For all $\varrho \in \mathbb{R}$, $(H_\nu \circ \tau_\varrho)_{\nu\in\mathcal{F}}$ is an orthonormal basis of $L^2(\Gamma, \gamma_\varrho)$.*

Proof: The claim follows from Lemma 2.10 since $(H_\nu)_{\nu\in\mathcal{F}}$ from (2.28) is an orthonormal basis of $L^2(\Gamma, \gamma)$, [8, Theorem 9.7]. □

Corollary 2.12 *Let $\varrho \geq 0$. Then the solution u of (2.16) can be represented in the form*

$$u(y) = \sum_{\nu\in\mathcal{F}} u_\nu H_\nu(\tau_\varrho(y)) \ , \quad y \in \Gamma \ , \tag{2.32}$$

with convergence in $L^2(\Gamma, \gamma_\varrho; V)$, for the coefficients

$$u_\nu = \int_\Gamma u(\tau_\varrho^{-1}(y))H_\nu(y)\gamma(dy) \in V \ , \quad \nu \in \mathcal{F} \ . \tag{2.33}$$

Furthermore, the coefficient vector $\boldsymbol{u} := (u_\nu)_{\nu\in\mathcal{F}} \in \ell^2(\mathcal{F}; V)$ and there holds the isometry

$$\|\boldsymbol{u}\|_{\ell^2(\mathcal{F};V)} = \|u\|_{L^2(\Gamma,\gamma_\varrho;V)} \tag{2.34}$$

and the a-priori bound

$$\|\boldsymbol{u}\|_{\ell^2(\mathcal{F};V)} \leq \widetilde{c}_{\varrho,p}\|f\|_{V'} \tag{2.35}$$

with the constant $\widetilde{c}_{\varrho,p}$ from (2.26).

Proof: By Theorem 2.9 with $q = 2$, the solution u of (2.16) is in $L^2(\Gamma, \gamma_\varrho; V)$. Then (2.32) is the expansion of u in the orthonormal basis from Proposition 2.11, and (2.33) follows from (2.31) since

$$u_\nu = \int_\Gamma u(y)H_\nu(\tau_\varrho(y))\gamma_\varrho(dy) = \int_\Gamma u(\tau_\varrho^{-1}(y))H_\nu(y)\gamma(dy) \ .$$

Equation 2.35 is a consequence of (2.25) and of Parseval's identity. □

2.4 Weak Formulation on a Problem-Dependent Space

Since the diffusion coefficient $a(x, y)$ is not uniformly bounded in $y \in \Gamma$, simply integrating (2.16) over Γ with respect to γ does not lead to a well-posed linear variational problem on $L^2(\Gamma, \gamma; V)$. As shown below, this difficulty can be overcome by considering a variational from with respect to a "stronger" Gaussian measure.

If $a_*(x)$ is not bounded away from zero we integrate (2.16) with respect to a measure that is stronger than γ in the sense of Proposition 2.6, but not by as much as γ_ϱ. For parameters $0 \leq \vartheta < 1$ and $\varrho > 0$, define

$$B_{\vartheta\varrho}(w, v) := \int_\Gamma b(y; w(y), v(y))\gamma_{\vartheta\varrho}(dy) = \int_\Gamma \int_D a(x, y)\nabla w(x, y) \cdot \nabla v(x, y) dx\gamma_{\vartheta\varrho}(dy) \tag{2.36}$$

and

$$F_{\vartheta\varrho}(v) := \int_\Gamma f(v(y))\gamma_{\vartheta\varrho}(dy) = \int_\Gamma \int_D f(x)v(x, y)dx\gamma_{\vartheta\varrho}(dy) \tag{2.37}$$

for suitable w and v. For the variational formulation, we define the space

$$\mathcal{V}_{\vartheta\varrho} := \{v : \Gamma \to V \ : \ \mathcal{B}(\Gamma)\text{-measurable} \, ; \ B_{\vartheta\varrho}(v, v) < \infty\} \ . \tag{2.38}$$

We consider elements of $\mathcal{V}_{\vartheta\varrho}$ as equivalence classes of γ-almost everywhere identical functions.

Proposition 2.13 *The space $\mathcal{V}_{\vartheta\varrho}$ endowed with the inner product $B_{\vartheta\varrho}(\cdot, \cdot)$ is a Hilbert space.*

We refer to [11, Proposition 3.6] for the proof of Proposition 2.13. The argument is analogous to a standard proof that $L^2(\mathbb{R})$ is a Hilbert space.

Lemma 2.14 *For all $w, v \in L^2(\Gamma, \gamma_\varrho; V)$,*

$$|B_{\vartheta\varrho}(w, v)| \leq \hat{c}_{\vartheta\varrho}\|w\|_{L^2(\Gamma,\gamma_\varrho;V)}\|v\|_{L^2(\Gamma,\gamma_\varrho;V)}$$

with

$$\hat{c}_{\vartheta\varrho} = \left(\|a_*\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)} \exp\left(\frac{\exp(2\varrho\|b\|_{\ell^\infty(\mathbb{N})})}{4(1-\vartheta)\varrho}\|b\|_{\ell^1(\mathbb{N})}\right)\right) \exp\left((1-\vartheta)\varrho\|b\|_{\ell^1(\mathbb{N})}\right) \ .$$

Proof: By continuity of $b(y;\cdot,\cdot)$ for $y \in \Gamma$,

$$|B_{\vartheta\varrho}(w,v)| \leq \int_\Gamma \frac{\zeta_{\vartheta\varrho}(y)}{\zeta_\varrho(y)} \hat{a}(y) \|w(y)\|_V \|v(y)\|_V \gamma_\varrho(dy)$$
$$\leq \left\| \frac{\zeta_{\vartheta\varrho}}{\zeta_\varrho} \hat{a} \right\|_{L^\infty(\Gamma,\gamma)} \|w\|_{L^2(\Gamma,\gamma_\varrho;V)} \|v\|_{L^2(\Gamma,\gamma_\varrho;V)}$$

and the claim follows from Lemmas 2.2 and 2.5 with $\eta = \vartheta\varrho$, $\chi = \varrho$ and $k = 1$. □

Lemma 2.15 *For all $v \in L^2(\Gamma,\gamma;V)$ with $B_{\vartheta\varrho}(v,v) < \infty$, the bilinear form $B_{\vartheta\varrho}(\cdot,\cdot)$ is coercive, i.e.*

$$\forall v \in L^2(\Gamma,\gamma;V): \quad B_{\vartheta\varrho}(v,v) \geq \check{c}_{\vartheta\varrho} \|v\|^2_{L^2(\Gamma,\gamma;V)}$$

with coercivity constant $\check{c}_{\vartheta\varrho}$ given by

$$\check{c}_{\vartheta\varrho} = \left(\operatorname*{ess\,inf}_{x\in D} a_*(x) + \check{a}_0 \exp\left(-\frac{e^{2\vartheta\varrho\|b\|_{\ell^\infty(\mathbb{N})}}}{4\vartheta\varrho} \|b\|_{\ell^1(\mathbb{N})} \right) \right) \exp\left(-\vartheta\varrho\|b\|_{\ell^1(\mathbb{N})} \right) .$$

Proof: Using coercivity of $b(y;\cdot,\cdot)$ for $y \in \Gamma$, we obtain

$$B_{\vartheta\varrho}(v,v) \geq \int_\Gamma \zeta_{\vartheta\varrho}(y)\check{a}(y)\|v(y)\|_V^2 \gamma(dy) \geq \operatorname*{ess\,inf}_{y\in\Gamma} \{\zeta_{\vartheta\varrho}(y)\check{a}(y)\} \|v\|^2_{L^2(\Gamma,\gamma;V)}$$

and the claim follows from Lemmas 2.2 and from 2.5 with $\eta = 0$, $\chi = \vartheta\varrho$ and $k = 1$. □

Proposition 2.16 *If $\vartheta > 0$, the Hilbert space $\mathcal{V}_{\vartheta\varrho}$ is related to Lebesgue–Bochner spaces by the continuous embeddings*

$$L^2(\Gamma,\gamma;V) \supset \mathcal{V}_{\vartheta\varrho} \supset L^2(\Gamma,\gamma_\varrho;V) .$$

For $\vartheta = 0$, this still holds if $\operatorname{ess\,inf}_{x\in D} a_*(x) > 0$.

Proof: Lemmas 2.14 and 2.15 imply

$$\check{c}_{\vartheta\varrho}\|v\|^2_{L^2(\Gamma,\gamma;V)} \leq B_{\vartheta\varrho}(v,v) \leq \hat{c}_{\vartheta\varrho}\|v\|^2_{L^2(\Gamma,\gamma_\varrho;V)}$$

for all $v \in L^2(\Gamma,\gamma_\varrho;V)$. □

Also, using (2.22) with $\eta = \vartheta\varrho$ and $\chi = \varrho$, it follows that if $f \in L^2(\Gamma,\gamma_\varrho;V')$, then $F_{\vartheta\varrho}$ is in the dual of $\mathcal{V}_{\vartheta\varrho}$. There holds the following result from [11, Corollary 3.8].

Theorem 2.17 *The solution u of* (2.16) *is the unique solution in $\mathcal{V}_{\vartheta\varrho}$ of the linear variational problem*

$$B_{\vartheta\varrho}(u,v) = F_{\vartheta\varrho}(v) \qquad \forall v \in \mathcal{V}_{\vartheta\varrho} . \tag{2.39}$$

2.5 Stochastic Galerkin Approximation

Using the variational formulation (2.39) of (2.16), we can define Galerkin projections of u onto suitable spaces. Let $\mathcal{V}_N \subset L^2(\Gamma,\gamma_\varrho;V) \subset \mathcal{V}_{\vartheta\varrho}$ be finite dimensional. Then the Galerkin projection of u onto $\mathcal{V}_N$ is the unique element $u_N \in \mathcal{V}_N$ satisfying

$$B_{\vartheta\varrho}(u_N,v_N) = F_{\vartheta\varrho}(v_N) \qquad \forall v_N \in \mathcal{V}_N . \tag{2.40}$$

This u_N is well-defined since, being finite dimensional, $\mathcal{V}_N$ is a closed subspace of $\mathcal{V}_{\vartheta\varrho}$, and thus also a Hilbert space when endowed with the inner product $B_{\vartheta\varrho}(\cdot,\cdot)$.

Theorem 2.18 *The Galerkin projection u_N satisfies*

$$\|u - u_N\|_{L^2(\Gamma,\gamma;V)} \leq \sqrt{\frac{\hat{c}_{\vartheta\varrho}}{\check{c}_{\vartheta\varrho}}} \inf_{v_N\in\mathcal{V}_N} \|u - v_N\|_{L^2(\Gamma,\gamma_\varrho;V)} . \tag{2.41}$$

Proof: Theorem 2.9 implies that $u \in L^2(\Gamma, \gamma_\varrho; V)$. By definition, u_N is the orthogonal projection of u onto $\mathcal{V}_N$ with respect to the inner product $B_{\vartheta\varrho}(\cdot,\cdot)$. Therefore, it minimizes the projection error in the norm induced by $B_{\vartheta\varrho}(\cdot,\cdot)$. Using Lemmas 2.14 and 2.15, we have

$$\begin{aligned}\check{c}_{\vartheta\varrho}\|u-u_N\|^2_{L^2(\Gamma,\gamma;V)} &\leq B_{\vartheta\varrho}(u-u_N, u-u_N)\\ &= \inf_{v_N\in\mathcal{V}_N} B_{\vartheta\varrho}(u-v_N, u-v_N)\\ &\leq \hat{c}_{\vartheta\varrho} \inf_{v_N\in\mathcal{V}_N}\|u-v_N\|^2_{L^2(\Gamma,\gamma_\varrho;V)} ,\end{aligned}$$

and the claim follows. □

Remark 2.19 *The errors on the two sides of the estimate* (2.41) *are measured in different norms. Therefore, Theorem 2.18 states that the Galerkin projection is almost quasi-optimal. Inserting the values of $\hat{c}_{\vartheta\varrho}$ and $\check{c}_{\vartheta\varrho}$ from Lemmas 2.14 and 2.15, we see that the constant in* (2.41) *is*

$$\sqrt{\frac{\hat{c}_{\vartheta\varrho}}{\check{c}_{\vartheta\varrho}}} = \sqrt{\frac{\|a_*\|_{L^\infty(D)} + \|a_0\|_{L^\infty(D)}\exp\left(\frac{e^{2\varrho\|b\|_{\ell^\infty}}}{4(1-\vartheta)\varrho}\|b\|_{\ell^1}\right)}{\operatorname{ess\,inf}_{x\in D} a_*(x) + \check{a}_0\exp\left(-\frac{e^{2\vartheta\varrho\|b\|_{\ell^\infty}}}{4\vartheta\varrho}\|b\|_{\ell^1}\right)}\exp\left(\frac{\varrho}{2}\|b\|_{\ell^1}\right)} .$$

In particular, it tends to ∞ as ϱ approaches 0 *or* ∞*, or if ϑ approaches* 1*. If a_* is not bounded away from* 0*, then the constant also tends to ∞ as ϑ approaches* 0*.*

Motivated by Corollary 2.12, we consider in particular spaces $\mathcal{V}_N$ of the form

$$\mathcal{V}_N := \left\{v \in L^2(\Gamma,\gamma_\varrho; V)\,;\; v_\nu \in V_{N,\nu}\ \forall \nu \in \mathcal{F}\right\} , \tag{2.42}$$

where $V_{N,\nu} \subset V$ is a finite dimensional subspace for all $\nu \in \mathcal{F}$, and $V_{N,\nu} = \{0\}$ for all but finitely many $\nu \in \mathcal{F}$. In (2.42), $(v_\nu)_{\nu\in\mathcal{F}}$ are the Hermite coefficients of $v \in L^2(\Gamma,\gamma_\varrho; V)$ with respect to the scaled Hermite polynomials $(H_\nu \circ \tau_\varrho)_{\nu\in\mathcal{F}}$ from Proposition 2.11, *i.e.*

$$v_\nu = \int_\Gamma v(\tau_\varrho^{-1}(y))H_\nu(y)\gamma(dy) , \quad \nu \in \mathcal{F} . \tag{2.43}$$

Then $\mathcal{V}_N$ is a finite dimensional subspace of $L^2(\Gamma,\gamma_\varrho; V)$, and its dimension is the sum of the dimensions of $V_{N,\nu}$ over $\nu \in \mathcal{F}$.

Corollary 2.20 *For $\mathcal{V}_N$ be of the form* (2.42) *the Galerkin projection u_N satisfies*

$$\|u-u_N\|_{L^2(\Gamma,\gamma;V)} \leq \sqrt{\frac{\hat{c}_{\vartheta\varrho}}{\check{c}_{\vartheta\varrho}}}\left(\sum_{\nu\in\mathcal{F}}\inf_{v_\nu\in V_{N,\nu}}\|u_\nu - v_\nu\|^2_V\right)^{1/2} . \tag{2.44}$$

Proof: The claim follows from Theorem 2.18 and from Parseval's identity since $(H_\nu \circ \tau_\varrho)_{\nu\in\mathcal{F}}$ is an orthonormal basis of $L^2(\Gamma,\gamma_\varrho; V)$. □

3 Regularity of the parametric solution

For a given parameter vector $y \in \Gamma$, we consider the parametric, deterministic problem (2.13) with the parametric variational formulation (2.16). We are interested in bounding partial derivatives $\partial_y^\nu u(\cdot,y)$ for any $\nu \in \mathcal{F}$. To this end, we assume

$$\exists F(\cdot) \in L^2(D)^d \quad \text{s.t.} \quad f(\cdot) = -\nabla\cdot F(\cdot) \quad \text{in} \quad V' . \tag{3.1}$$

We use the positivity of $a(\cdot,y)$ for $y \in \Gamma$ and (3.1) to rewrite the parametric deterministic problem (2.16) as follows: find $u(\cdot,y) \in V$ such that

$$u(\cdot,y)\in V \quad b(y;u(\cdot,y),v) = -\int_D a^{-1/2}(x,y)F(x)\cdot a^{1/2}(x,y)\nabla v dx \quad \forall v \in V . \tag{3.2}$$

Inserting into (3.2) the test function $v = u(\cdot, y)$, we find

$$\int_D a(x,y)|\nabla u(x,y)|^2 dx = -\int_D F(x)\cdot\nabla u(x,y)dx \le \|a^{-1/2}F(\cdot)\|_{L^2(D)}\|a^{1/2}\nabla u(\cdot,y)\|_{L^2(D)} \ .$$

For $y \in \Gamma$ we define the a-dependent norms

$$\|v\|_a := \left(\int_D a(x,y)|\nabla v|^2 dx\right)^{1/2}$$

and, for $f \in V'$ with $F \in L^2(D)^d$ as in (3.1),

$$\|f\|_{a^{-1}} := \left(\int_D a^{-1}(x,y)|F(x)|^2 dx\right)^{1/2} .$$

With these notations in hand, applying the Cauchy-Schwarz inequality to (3.2), we find for every $y \in \Gamma$ that

$$\|u(\cdot,y)\|_a^2 = |b(y;u(\cdot,y),u(\cdot,y))| \le \| f(\cdot)\|_{a^{-1}}\|u(\cdot,y)\|_a$$

so that we obtain the a-priori estimate

$$\forall y \in \Gamma: \quad \|u(\cdot,y)\|_a \le \|f(\cdot)\|_{a^{-1}} \ . \tag{3.3}$$

Next, we prove estimates for $\partial_y^\nu u(\cdot,y)$ for $\nu \in \mathcal{F}$.

Theorem 3.1 *Under the assumption* (2.7)*, for $f \in V'$ which is independent of y, for every $y \in \Gamma$ such that there holds*

$$\|\partial_y^\nu u(\cdot,y)\|_a \le |\nu|!\bar{b}^\nu\|f(\cdot)\|_{a^{-1}} \ , \tag{3.4}$$

where the sequence $\bar{b}$ is defined by $\bar{b} := b/\log_e 2$ with the sequence b as defined in (2.7).

Proof: For $\nu = 0 \in \mathcal{F}$, (3.4) reduces to the a-priori estimate (3.3). For $|\nu| > 0$, we proceed (3.4) by induction with respect to $|\nu|$.

The induction will be accomplished by differentiation of the parametric weak formulation (2.16) with respect to y. We shall require the Leibnitz rule: given any two smooth functions f, g of $y \in U$, for any $\nu \in \mathcal{F}$ with associated support set $\mathfrak{n} \subset \mathbb{N}$ holds

$$\partial_y^{\mathfrak{n}}(fg) = \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})} (\partial_y^{\mathfrak{m}} f)(\partial_y^{\mathfrak{n}\setminus\mathfrak{m}} g) \ . \tag{3.5}$$

Here, for a finite subset $\mathfrak{m}$ of $\mathbb{N}$, $\mathfrak{P}(\mathfrak{m})$ denotes the power set of $\mathfrak{m}$.

Applying for $\nu \in \mathcal{F}$ with support set $\mathfrak{n}$ the partial derivative ∂_y^ν to (2.13), the y-independence of f implies

$$\begin{aligned}\forall v \in V: \quad \int_D a(x,y)\nabla(\partial_y^\nu u)\cdot\nabla v dx &= -\sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} \int_D (\partial_y^{\mathfrak{n}\setminus\mathfrak{m}} a)(x,y)\nabla(\partial_y^{\mathfrak{m}} u)\cdot\nabla v dx \\ &= -\sum_{0\prec\mu\preceq\nu} \frac{\nu!}{\mu!(\nu-\mu)!}\int_D \partial_y^\mu a \nabla\partial^{\nu-\mu} u\cdot\nabla v dx \ ,\end{aligned} \tag{3.6}$$

where $\mu \prec \nu$ means that $\mu_i \le \nu_i \ \forall i$ with $\mu_i < \nu_i$ for at least one index i, and $\mu \preceq \nu$ means that $\forall i$, $\mu_i \le \nu_i$; 0 denotes the member of $\mathcal{F}$ whose all components are zero. We refer to the Appendix for a more detailed derivation of this identity.

Choosing in identity (3.6) the test function $v = \partial_y^{\mathfrak{n}} u = \partial_y^\nu u$, we find for every $y \in \Gamma$

$$\begin{aligned}\|(\partial_y^{\mathfrak{n}} u(\cdot,y)\|_a^2 &= -\sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} \int_D (\partial_y^{\mathfrak{n}\setminus\mathfrak{m}} a)(x,y)\nabla(\partial_y^{\mathfrak{m}} u)\cdot\nabla(\partial_y^{\mathfrak{n}} u)dx \\ &\le \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\setminus\{\mathfrak{n}\}} \left\|\frac{\partial_y^{\mathfrak{n}\setminus\mathfrak{m}} a(\cdot,y)}{a(\cdot,y)}\right\|_{L^\infty(D)} \|\partial_y^{\mathfrak{m}} u(\cdot,y)\|_a\|\partial_y^{\mathfrak{n}} u(\cdot,y)\|_a\end{aligned}$$

which implies with (2.10) that

$$
\begin{aligned}
\|\partial_y^{\mathfrak{n}} u(\cdot,y)\|_a &\le \sum_{i=0}^{|\mathfrak{n}|-1} \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n}):|\mathfrak{m}|=i} \left\| \frac{\partial_y^{\mathfrak{n}\setminus\mathfrak{m}} a(\cdot,y)}{a(\cdot,y)} \right\|_{L^\infty(D)} \|\partial_y^{\mathfrak{m}} u(\cdot,y)\|_a \\
&\le \sum_{i=0}^{|\mathfrak{n}|-1} \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n}):|\mathfrak{m}|=i} b^{\mathfrak{n}\setminus\mathfrak{m}} \|\partial_y^{\mathfrak{m}} u(\cdot,y)\|_a \ .
\end{aligned}
\tag{3.7}
$$

We next note that

$$
\#\{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n}) : |\mathfrak{m}| = i\} = \binom{|\mathfrak{n}|}{i} \ .
$$

We define the sequence $d = (d_n)_{n\ge 0}$ by the recursion

$$
d_0 := 1, \quad \forall j \ge 1: \quad d_j := \sum_{i=0}^{j-1} \binom{j}{i} d_i \ . \tag{3.8}
$$

We now claim that for all $\nu \in \mathcal{F}$ with support set $\mathfrak{n} \subset \mathbb{N}$, we have

$$
\|\partial_y^{\mathfrak{n}} u(\cdot,y)\|_a \le d_{|\mathfrak{n}|} b^{\mathfrak{n}} \|f\|_{a^{-1}} \ . \tag{3.9}
$$

For $|\nu| = 0$, (3.9) is just the bound (3.3). For $|\nu| > 0$, we assume that (3.9) is already proved for all $\mu \in \mathcal{F}$ such that $|\mu| \le n-1$ for some $n \ge 1$. Next, for $\nu \in \mathcal{F}$ such that $|\nu| = n$ with associated support set $\mathfrak{n}$, we find from (3.7) that

$$
\begin{aligned}
\|\partial_y^{\mathfrak{n}} u(\cdot,y)\|_a &\le \sum_{i=0}^{|\mathfrak{n}|-1} \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n}):|\mathfrak{m}|=i} b^{\mathfrak{n}\setminus\mathfrak{m}} \|\partial_y^{\mathfrak{m}} u(\cdot,y)\|_a \\
&\le \sum_{i=0}^{|\mathfrak{n}|-1} \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n}):|\mathfrak{m}|=i} b^{\mathfrak{n}\setminus\mathfrak{m}} d_{|\mathfrak{m}|} b^{\mathfrak{m}} \|f\|_{a^{-1}} \\
&= \left(\sum_{i=0}^{|\mathfrak{n}|-1} \binom{|\mathfrak{n}|}{i} d_i \right) b^{\mathfrak{n}} \|f\|_{a^{-1}} \\
&= d_{|\mathfrak{n}|} b^{\mathfrak{n}} \|f\|_{a^{-1}} \ .
\end{aligned}
$$

This completes the induction step and hence the proof of (3.9). The assertion (3.4) now follows from the bound

$$
d_n \le \frac{n!}{(\log_e 2)^n} \qquad \forall n \in \mathbb{N}_0
$$

which is proved, by referring to generating functions, for example in [3]. □

4 Best N term approximation

For best N-term approximation rates, we study the summability of the sequence of coefficients $(u_\nu)_{\nu\in\mathcal{F}}$ in (2.33). In particular, we will show that the sequence $(\|u_\nu\|_V)_\nu$ belongs to a space $\ell^p(\mathcal{F})$ under certain summability conditions for the coefficients ψ_m of the expansion (2.9).

4.1 p-Summability of $\|u_\nu\|_V$

The summability property of $(\|u_\nu\|_V)_\nu$ depends on the summability of the coefficients of the expansion (2.9). We will work under the following assumption on the summability of the input's coefficients ψ_k.

Assumption 4.1 *There exists $0 < p \le 1$ such that the sequence $(b_k)_k$ defined in (2.7) satisfies*

$$
(kb_k)_k \in \ell^p(\mathbb{N}) \ .
$$

We will first provide an elementary estimate on Gaussians which will be used repeatedly.

Lemma 4.2 *For all $t > 0$,*

$$\int_{-\infty}^{\infty} \exp(-z^2/(2\sigma^2) + |z|t)\frac{dt}{\sigma\sqrt{2\pi}} \leq \exp(\sigma^2 t^2/2 + \sigma t\sqrt{2/\pi}).$$

Proof We calculate

$$\begin{aligned}
\int_{-\infty}^{\infty} \exp(-z^2/(2\sigma^2) + |z|t)\frac{dz}{\sigma\sqrt{2\pi}} &= \exp(\sigma^2 t^2/2)\int_{-\infty}^{\infty} \exp(-(|z| - \sigma^2 t)^2/(2\sigma^2))\frac{dz}{\sigma\sqrt{2\pi}} \\
&= \exp(\sigma^2 t^2/2)\Big(\int_{-\infty}^{0} \exp(-(z + \sigma^2 t)^2/(2\sigma^2))\frac{dz}{\sigma\sqrt{2\pi}} \\
&\qquad + \int_{0}^{\infty} \exp(-(z - \sigma^2 t)^2/2\sigma^2)\frac{dz}{\sigma\sqrt{2\pi}}\Big) \\
&= \exp(\sigma^2 t^2/2)\Big(\int_{-\infty}^{\sigma^2 t} \exp(-z^2/(2\sigma^2))\frac{dz}{\sigma\sqrt{2\pi}} \\
&\qquad + \int_{-\sigma^2 t}^{\infty} \exp(-z^2/(2\sigma^2)\frac{dz}{\sigma\sqrt{2\pi}}\Big) \\
&= \exp(\sigma^2 t^2/2)\Big(\int_{-\infty}^{\infty} \exp(-z^2/(2\sigma^2))\frac{dz}{\sigma\sqrt{2\pi}} \\
&\qquad + \int_{-\sigma^2 t}^{\sigma^2 t} \exp(-z^2/(2\sigma))\frac{dz}{\sigma\sqrt{2\pi}}\Big) \\
&= \exp(\sigma^2 t^2/2)\left(1 + \int_{-\sigma^2 t}^{\sigma^2 t} \exp(-z^2/(2\sigma^2))\frac{dz}{\sigma\sqrt{2\pi}}\right) \\
&\leq \exp(\sigma^2 t^2/2)(1 + \sigma t\sqrt{2/\pi}) \\
&\leq \exp(\sigma^2 t^2/2)\exp(\sigma t\sqrt{2/\pi})\ .
\end{aligned}$$

□

To estimate the norm $\|u_\nu\|_V$, we use the following result.

Lemma 4.3 *For $s_j \in \{1, 2, \ldots, t\}$ ($j = 1, \ldots, m$),*

$$(s_1 + \ldots + s_m)! \leq t^{tm} 1^{s_1} 2^{s_2} \ldots m^{s_m}.$$

Proof We prove by induction. When $m = 1$ there holds $s_1! \leq t! < t^t$. Assume that the assertion holds for all orders up to some value $m > 1$. We have

$$(s_1 + \ldots + s_m + 1) \ldots (s_1 + \ldots + s_m + s_{m+1}) \leq (tm+1) \ldots (tm + s_{m+1})) \leq t^{s_{m+1}}(m+1)^{s_{m+1}} \leq t^t (m+1)^{s_{m+1}}\ .$$

Therefore

$$(s_1 + \ldots + s_{m+1})! \leq t^{tm} 1^{s_1} 2^{s_2} \ldots m^{s_m} t^t (m+1)^{s_{m+1}} = t^{t(m+1)} 1^{s_1} 2^{s_2} \ldots (m+1)^{s_{m+1}}\ .$$

□

Based on this estimate, we can show the following summability property for the coefficients $u_\nu \in V$ of the expansion (2.32).

Proposition 4.4 *Under Assumption 4.1, the coefficients $(u_\nu)_\nu$ of the expansion* (2.32) *satisfy* $(\|u_\nu\|_V)_\nu \in \ell^p(\mathcal{F})$.

Proof Let $S = (i_1, \ldots, i_m) \subset \mathbb{N}$ be any subset of $\mathbb{N}$, and denote by $\bar{S} := \mathbb{N}\backslash S$ its complement. With the index set S, we associate the product Hermite differential operator

$$\mathcal{L}_S = (-1)^m \prod_{j=1}^{m} \left(\frac{d^2}{dy_{i_j}^2} - \frac{1}{\sigma_{i_j}^2} y_{i_j} \frac{d}{dy_{i_j}}\right).$$

We note that

$$\left(\prod_{j=1}^{m} e^{-y_{i_j}^2/(2\sigma_{i_j}^2)}\right)\mathcal{L}_S=\prod_{j=1}^{m}\frac{d}{dy_{i_j}}\left(e^{-y_{i_j}^2/(2\sigma_{i_j}^2)}\frac{d}{dy_{i_j}}\right)$$

is self-adjoint over the space of m-variate, continuously differentiable functions g where g and the first derivatives of g grow at most exponentially at infinity. Next, we observe that the Hermite polynomials $H_n(t/\sigma)$ satisfy the eigenproblems

$$-\left(\frac{d^2}{dt^2}-\frac{t}{\sigma^2}\frac{d}{dt}\right)H_n\left(\frac{t}{\sigma}\right)=n\sigma^{-2}H_n\left(\frac{t}{\sigma}\right)\ .$$

For $j\in\mathbb{N}$, let Γ_j be a copy of $\mathbb{R}$ and $y_j\in\Gamma_j$. We denote by $\Gamma_S=\otimes_{j=1}^{m}\Gamma_{i_j}$ and by $y_S=(y_{i_1},\ldots,y_{i_m})$ a point in Γ_S. For such S and for any $\nu\in\mathcal{F}$, we define

$$\lambda_S(\nu)=\prod_{j=1}^{m}\nu_{i_j}\sigma_{i_j}^{-2}\ .$$

Let $\Gamma_{\bar S}=\{\bar y=(y_j)_{j\notin S}:\ \sum_{j\notin S}y_jb_j<\infty\}$. Then $\Gamma=\Gamma_{\bar S}\times\Gamma_S$. Fixing y_j for $j\notin S$, we have

$$\begin{aligned}&\left(\prod_{j=1}^{m}\frac{1}{\sigma_{i_j}\sqrt{2\pi}}\right)\int_{\Gamma_S}u\exp\left(-\sum_{j=1}^{m}y_{i_j}^2/(2\sigma_{i_j}^2)\right)\lambda_S(\nu)H_\nu(\tau_\varrho(y))dy_S\\
&=\left(\prod_{j=1}^{m}\frac{1}{\sigma_{i_j}\sqrt{2\pi}}\right)\int_{\Gamma_S}u\exp\left(-\sum_{j=1}^{m}y_{i_j}^2/(2\sigma_{i_j}^2)\right)\mathcal{L}_S(H_\nu(\tau_\varrho(y))dy_S\\
&=\left(\prod_{j=1}^{m}\frac{1}{\sigma_{i_j}\sqrt{2\pi}}\right)\int_{\Gamma_S}\exp\left(-\sum_{j=1}^{m}y_{i_j}^2/(2\sigma_{i_j}^2)\right)\mathcal{L}_S(u)H_\nu(\tau_\varrho(y))dy_S\ .\end{aligned}$$

Therefore

$$\int_\Gamma u\lambda_S(\nu)H_\nu(\tau_\varrho(y))d\gamma_\varrho(y)=\int_\Gamma\mathcal{L}_S(u)H_\nu(\tau_\varrho(y))d\gamma_\varrho(y)\ .$$

This shows that

$$\sum_{\nu\in\mathcal{F}}u_\nu\lambda_S(\nu)H_\nu(\tau_\varrho(y))=\mathcal{L}_S(u)\ .$$

Applying the operator $\mathcal{L}_S$ r times, we find

$$\sum_{\nu\in\mathcal{F}}u_\nu\lambda_S^r(\nu)H_\nu(\tau_\varrho(y))=\mathcal{L}_S^r(u).$$

From this, we obtain

$$\sum_{\nu\in\mathcal{F}}\|u_\nu\|_V^2\lambda_S^{2r}(\nu)=\int_\Gamma\|\mathcal{L}_S^r(u)\|_V^2d\gamma_\varrho(y)\ . \tag{4.1}$$

We note that there are polynomials $q_j(t)$ $(j=1,\ldots,2r)$ of degrees at most r such that

$$\left(\frac{d^2}{dt^2}-\frac{t}{\sigma^2}\frac{d}{dt}\right)^r=\sum_{j=1}^{2r}q_j(t)\frac{d^j}{dt^j},$$

The polynomials $q_j(t)$ are of the form

$$q_j(t)=\sum_{k=1}^{r}\left(\sum_{l=1}^{r}\frac{1}{\sigma^{2l}}q_{jkl}\right)t^k,$$

where q_{jkl} only depends on j, k, l and r. As $\sigma\geq1$, there is a constant $C_1(r)$ so that for all j and t

$$|q_j(t)|\leq C_1(r)(1+|t|)^r\ .$$

Thus

$$\begin{aligned}\|\mathcal{L}_S^r(u)(\cdot,y)\|_V &= \left\|\prod_{j=1}^m \Big(\frac{d^2}{dy_{i_j}^2} - \frac{y_{i_j}}{\sigma_{i_j}^2}\frac{d}{dy_{i_j}}\Big)^r u\right\|_V \\ &\le C_1(r)^m \prod_{j=1}^m (1+|y_{i_j}|)^r \Big(\sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} \left\| \frac{d^{s_1}}{dy_{i_1}^{s_1}} \cdots \frac{d^{s_m}}{dy_{i_m}^{s_m}} u \right\|_V \Big)\end{aligned}$$

we deduce

$$\|\mathcal{L}_S^r(u)(\cdot,y)\|_V^2 \le C_1(r)^{2m}(2r)^m \prod_{j=1}^m (1+|y_{i_j}|)^{2r} \Big(\sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} \left\| \frac{d^{s_1}}{dy_{i_1}^{s_1}} \cdots \frac{d^{s_m}}{dy_{i_m}^{s_m}} u \right\|_V^2 \Big) .$$

Using estimate (3.4), we find

$$\begin{aligned}\|\mathcal{L}_S^r(u)(\cdot,y)\|_V^2 &\le C_1(r)^{2m}(2r)^m \prod_{j=1}^m (1+|y_{i_j}|)^{2r} \left(\sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} ((s_1+\ldots+s_m)!)^2 \bar{b}_{i_1}^{2s_1}\ldots \bar{b}_{i_m}^{2s_m} \right) \\ &\qquad . \sup_x (a(x,y)^{-1}) \|f(\cdot)\|_{a^{-1}}^2 \\ &\le C_1(r)^{2m}(2r)^m \prod_{j=1}^m (1+|y_{i_j}|)^{2r} (2r)^{4rm} \left(\sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} (1\bar{b}_{i_1})^{2s_1}\ldots (m\bar{b}_{i_m})^{2s_m} \right) \\ &\qquad \cdot \frac{1}{(\inf_x a_0(x))^2} \exp\Big(2\sum_{j\ge 1} |y_j| \|\psi_j\|_{L^\infty(D)}\Big) \|F(x)\|_{L^2(D)}^2 .\end{aligned} \tag{4.2}$$

Let κ be a positive constant such that

$$0 < \kappa \le \frac{1}{4}\exp(-2\varrho \max_j \|\psi_j\|_{L^\infty(D)}) \le \frac{1}{4\sigma_j^2} \tag{4.3}$$

for all j with the choice of σ_j in (2.20) where $\chi = \varrho$. Let further $C_2(r)$ denote a positive constant so that

$$\forall t > 0: \quad (1+t)^{2r} \le C_2(r) e^{t^2\kappa} .$$

With the constants chosen in this way, we estimate

$$\begin{aligned}&\int_\Gamma \prod_{j=1}^m (1+|y_{i_j}|)^{2r} \exp\Big(2\sum_{j\ge 1}|y_j|\|\psi_j\|_{L^\infty(D)}\Big) d\gamma_\varrho(y) \\ &\le (C_2(r))^m \prod_{j\in S} \int_{-\infty}^{\infty} \exp\Big(-y_j^2(1/(2\sigma_j^2)-\kappa) + 2|y_j|\|\psi_j\|_{L^\infty(D)}\Big) \frac{dy_j}{\sigma_j\sqrt{2\pi}} \\ &\quad \times \prod_{j\notin S} \int_{-\infty}^{\infty} \exp\Big(-y_j^2/(2\sigma_j^2) + 2|y_j|\|\psi_j\|_{L^\infty(D)}\Big) \frac{dy_j}{\sigma_j\sqrt{2\pi}} .\end{aligned} \tag{4.4}$$

From Lemma 4.2, for $t > 0$ we obtain

$$\begin{aligned}&\int_{-\infty}^{\infty} \exp\Big(-z^2(1/(2\sigma_j^2)-\kappa) + |z|t\Big) \frac{dz}{\sigma_j\sqrt{2\pi}} \\ &\le \int_{-\infty}^{\infty} \exp\Big(-z^2/(4\sigma_j^2) + |z|t\Big) \frac{dz}{\sigma_j\sqrt{2\pi}} \\ &\le \sqrt{2}\exp\Big(\sigma_j^2 t^2 + \sigma_j t 2/\sqrt{\pi}\Big)\end{aligned}$$

where we have used inequality (4.3). Therefore

$$\begin{aligned}
&\int_\Gamma \prod_{j=1}^m (1+|y_{i_j}|)^{2r} \exp\left(\sum_{j\ge 1} 2|y_j|\|\psi_j\|_{L^\infty(D)}\right) d\gamma_\varrho(y)\\
&\le (C_2(r))^m 2^{m/2} \exp\left(\sum_{j\in S} 4\sigma_j^2\|\psi_j\|^2_{L^\infty(D)} + 4\sigma_j\|\psi_j\|_{L^\infty(D)}/\sqrt{\pi}\right)\\
&\qquad\qquad \times \exp\left(\sum_{j\notin S} 2\sigma_j^2\|\psi_j\|^2_{L^\infty(D)} + 2\sigma_j\|\psi_j\|_{L^\infty(D)}\sqrt{2/\pi}\right)\\
&\le c(C_2(r))^m 2^{m/2},
\end{aligned}$$

where the last inequality is deduced from the fact that $1 \le \sigma_j \le \exp(\varrho \max_j \|\psi_j\|_{L^\infty(D)})$. From (4.2), we then obtain the bound

$$\int_\Gamma \|\mathcal{L}_S^r u(\cdot,y)\|_V^2 d\gamma_\varrho(y) \le K^{2m} \sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} (i_1\bar{b}_{i_1})^{2s_1}\dots(i_m\bar{b}_{i_m})^{2s_m},$$

for a sufficiently large constant K which depends on $r\in\mathbb{N}$.

We deduce from (4.1) that for $\nu\in\mathcal{F}$ with $\mathrm{supp}(\nu)=S\subset\mathbb{N}$,

$$\|u_\nu\|_V \le K^m \sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} (i_1\bar{b}_{i_1})^{s_1}\dots(i_m\bar{b}_{i_m})^{s_m} \frac{1}{\nu_{i_1}^r\dots\nu_{i_m}^r}\sigma_{i_1}^{2r}\dots\sigma_{i_m}^{2r}\ .$$

When $r>1/p$, let $M=\sum_{k\ge 1} k^{-rp}$. We have,

$$\begin{aligned}
\sum_{\mathrm{supp}(\nu)=S} \|u_\nu\|_V^p &\le \exp(2prm\varrho\max_j\|\psi_j\|_{L^\infty(D)})K^{mp}M^m \left(\sum_{\substack{s_j=1,\dots,2r\\ j=1,\dots,m}} (i_1\bar{b}_{i_1})^{s_1}\dots(i_m\bar{b}_{i_m})^{s_m}\right)^p\\
&= L^m \prod_{j=1}^m \left(\sum_{s=1}^{2r}(i_j\bar{b}_{i_j})^s\right)^p,
\end{aligned}$$

where $L:=\exp(2rp\varrho\max_j\|\psi_j\|_{L^\infty(D)})K^pM$. Thus

$$\sum_{\nu\in\mathcal{F}}\|u_\nu\|_V^p = \sum_{i_1,\dots,i_m=1}^\infty \prod_{j=1}^m L\left(\sum_{s=1}^{2r}(i_j\bar{b}_{i_j})^s\right)^p \le \prod_{k=1}^\infty\left(1+L\Big(\sum_{s=1}^{2r}(k\bar{b}_k)^s\Big)^p\right) \le \exp\left(L\sum_{k=1}^\infty\Big(\sum_{s=1}^{2r}(k\bar{b}_k)^s\Big)^p\right),$$

which is finite when $(kb_k)_k\in\ell^p(\mathbb{N})$. □

4.2 Best N-term convergence rate

For a subset $\Lambda\subset\mathcal{F}$ of finite cardinality N, we define by

$$\mathcal{V}_{\varrho,\Lambda} = \{v=\sum_{\nu\in\Lambda} v_\nu H_\nu(\tau_\varrho(y)) : v_\nu\in V\}\subset L^2(U,\gamma_\varrho;V)\subset\mathcal{V}_{\vartheta_\varrho}$$

the set of N-term truncated Hermite expansions with "active" coefficients indexed by $\nu\in\Lambda$. We consider the *stochastic Galerkin approximation* (2.40) *for* $\mathcal{V}_N=\mathcal{V}_{\varrho,\Lambda}$:
Find $u_\Lambda\in\mathcal{V}_{\varrho,\Lambda}$ such that

$$B_{\vartheta_\varrho}(u_\Lambda,v_\Lambda) = F_{\vartheta_\varrho}(v_\Lambda)\quad \forall\, v_\Lambda\in\mathcal{V}_{\varrho,\Lambda}. \tag{4.5}$$

By Lemma 2.15, for any set $\Lambda\subset\mathcal{F}$ this problem admits a unique solution u_Λ, the Galerkin projection of the solution u onto $\mathcal{V}_N=\mathcal{V}_{\varrho,\Lambda}$. The following result shows that Assumption 4.1 implies convergence rates of these Galerkin approximations, *provide* the sets $\Lambda_N\subset\mathcal{F}$ of "active" components in the Wiener-Ito decomposition of the random field u are judiciously chosen.

Proposition 4.5 *Under Assumption 4.1, for every $N \in \mathbb{N}$ there exists an index set $\Lambda_N \subset \mathcal{F}$ of cardinality not exceeding N such that the parametric, weak solution u of equation* (2.13) *and the stochastic Galerkin approximation u_{Λ_N} of* (4.5) *satisfies*

$$\|u - u_{\Lambda_N}\|_{L^2(U,\gamma;V)} \leq c(\vartheta, \varrho) N^{-(1/p-1/2)} .$$

Proof Let $\Lambda \subset \mathcal{F}$ be a subset of finite cardinality, and define the partial sum of the Wiener-Itô decomposition of u in (2.32) over Λ by

$$v(x,y) = \sum_{\nu \in \Lambda} u_\nu(x) H_\nu(\tau_\varrho(y)) .$$

From (2.35), it follows that

$$\|u - u_\Lambda\|_{L^2(U,\gamma;V)} \leq c(\vartheta, \varrho)\|u - v\|_{L^2(U,\gamma_\varrho;V)} \leq c(\vartheta, \varrho) \left(\sum_{\nu \notin \Lambda} \|u_\nu\|_V^2 \right)^{1/2} .$$

Assumption 4.1 implies, by Proposition 4.4, that $(\|u_\nu\|_V)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$. Choosing $\Lambda = \Lambda_N$ as the set corresponding to the set of N coefficients u_ν which are largest in norm $\|u_\nu\|_V$, we deduce from Stechkin's lemma (see, e.g. [7]) that

$$\|u - u_\Lambda\|_{L^2(U,\gamma;V)} \leq c(\vartheta, \varrho) N^{-(1/p-1/2)} .$$

□

5 Spatial Regularity and Finite Element Approximation

So far, we considered the *semidiscrete* stochastic Galerkin approximation of the parametric, deterministic solution. In practice, however, the Wiener-Itô coefficients of the stochastic Galerkin approximation u_Λ are not explicitly available and must be approximated from a suitable Finite Element subspace of V, introducing an additional discretization error. In order to obtain convergence rates for this Finite Element approximation, we require additional regularity of the Wiener-Itô coefficients. In principle, regularity for diffusion problems is a standard matter; in the present setting, however, we require regularity of the *parametric diffusion problem* with uniform control of the constants' dependence on the parameter vector $y \in \Gamma$.

As in [7, 6], in the analysis of the spatial regularity we only aim at bounds for the second weak derivatives of the parametric solution $u(x, y)$ which are required for convergence rate estimates of continuous, piecewise linear Finite Element Methods, and exploit moreover that the stochastic coefficient $a(x, y)$ is isotropic.

5.1 Spatial Regularity

To quantify the spatial regularity of the Hermite coefficients as well as for the ensuing Finite Element convergence analysis, it will be convenient to define the space

$$W = \{u \in V : \ \Delta u \in L^2(D)\}, \tag{5.1}$$

equipped with the norm

$$\|u\|_W = \|u\|_V + \|\Delta u\|_{L^2(D)} .$$

The space W is a closed subspace of V, which is known to coincide for convex domains D with $H^2(D) \cap H_0^1(D)$. We denote by

$$v_{\mathtt{m}} = \nabla(a \nabla \partial_y^{\mathtt{m}} u) = a \Delta \partial_y^{\mathtt{m}} u + \nabla a \cdot \nabla \partial_y^{\mathtt{m}} u . \tag{5.2}$$

We then have

$$a^{1/2} \Delta \partial_y^m u = a^{-1/2} v_{\mathtt{m}} - a^{-1/2} \nabla a \cdot \nabla \partial_y^{\mathtt{m}} u . \tag{5.3}$$

The gradient ∇a in equations (5.2) and (5.3) is only formal, as it is not well defined for all $y \in \Gamma_b$. We thus consider the parametric, deterministic problem (2.13) for parameter vectors y from a subset $\Gamma_{\hat{b}} \subset \mathbb{R}^{\mathbb{N}}$ of full measure, for which $\nabla a(\cdot, y)$ is well defined. To define this set, denote by $\hat{b} = (\hat{b}_k)_{k \geq 1}$ the sequence

$$\hat{b}_k := \|\psi_k\|_{L^\infty(D)} + \|\nabla \psi_k\|_{L^\infty(D)} \quad k = 1, 2, ... \tag{5.4}$$

We now impose the additional assumption

Assumption 5.1 *The coefficients* $a_*, a_0 \in W^{1,\infty}(D)$ *and*

$$\hat{b} = (\|\psi_k\|_{L^\infty(D)} + \|\nabla\psi_k\|_{L^\infty(D)})_k \in \ell^1(\mathbb{N}) .$$

Under Assumption 5.1, we may define the set $\Gamma_{\hat{b}} \subset \mathbb{R}^{\mathbb{N}}$ as the set Γ_b in (2.8), with $\hat{b}_m$ in place of b_m. Then $\Gamma_{\hat{b}} \subset \Gamma_b$ and, by Lemma 2.1, the set $\Gamma_{\hat{b}}$ has full (Gaussian) measure in $\mathbb{R}^{\mathbb{N}}$. Then for all $y \in \Gamma_{\hat{b}}$,

$$\nabla a(x,y) \quad = \quad \nabla a_*(x) + \nabla a_0(x)\exp\left(\sum_{k=1}^\infty y_k\psi_m(x)\right) + a_0(x)\exp\left(\sum_{k=1}^\infty y_k\psi_m(x)\right)\sum_{k=1}^\infty y_k\nabla\psi_k(x) .$$

We observe that due to $\hat{b}_k \geq b_k$, it holds that $\Gamma_b \supset \Gamma_{\hat{b}}$. Therefore we have, under Assumption 5.1, for every $y \in \Gamma_{\hat{b}}$

$$\|a^{1/2}\Delta\partial_y^{\mathfrak{m}}u\|_{L^2(D)} \leq \|a^{-1/2}v_{\mathfrak{m}}\|_{L^2(D)} + \left(\|a^{-1}\nabla a_*\|_{L^\infty(D)} + \left\|\frac{\nabla a_0}{a_0}\right\|_{L^\infty(D)} + \sum_{k=1}^\infty |y_k|\|\nabla\psi_k\|_{L^\infty(D)}\right)\|a^{1/2}\nabla\partial_y^{\mathfrak{m}}u\|_{L^2(D)} . \qquad (5.5)$$

From (3.6), we get

$$v_{\mathfrak{n}} = -\sum_{m\in\mathfrak{P}(\mathfrak{n})\backslash\{\mathfrak{n}\}} \nabla(\partial_y^{\mathfrak{n}\backslash\mathfrak{m}}a(x,y))\cdot\nabla\partial_y^{\mathfrak{m}}u + \partial_y^{\mathfrak{n}\backslash\mathfrak{m}}a(x,y)\Delta\partial_y^{\mathfrak{m}}u .$$

We have

$$\begin{aligned}\nabla(\partial_y^{\mathfrak{n}\backslash\mathfrak{m}}a(x,y)) \quad = \quad & \Bigg[\nabla a_0(x)\exp\left(\sum_{k=1}^\infty y_k\psi_k(x)\right) \\ & +a_0(x)\exp\left(\sum_{k=1}^\infty y_k\psi_k(x)\right)\left(\sum_{k=1}^\infty y_k\nabla\psi_k(x)\right)\Bigg]\psi_1(x)^{\nu_1-\mu_1}\psi_2(x)^{\nu_2-\mu_2}\ldots \\ & +a_0(x)\exp\left(\sum_{k=1}^\infty y_k\psi_k(x)\right)\nabla\left(\psi_1(x)^{\nu_1-\mu_1}\psi_2(x)^{\nu_2-\mu_2}\ldots\right) .\end{aligned}$$

From this we obtain

$$\begin{aligned}&\|a^{-1}\nabla(\partial_y^{\mathfrak{n}\backslash\mathfrak{m}}a(\cdot,y))\|_{L^2(D)} \\ &\leq \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\backslash\{\mathfrak{n}\}}\left[\left\|\frac{\nabla a_0}{a_0}\right\|_{L^\infty(D)} + \sum_{k=1}^\infty y_k\|\nabla\psi_k\|_{L^\infty(D)}\right]\|\psi_1\|_{L^\infty(D)}^{\nu_1-\mu_1}\|\psi_2\|_{L^\infty(D)}^{\nu_2-\mu_2}\cdots \\ &\quad +\|\psi_1\|_{L^\infty(D)}^{\nu_1-\mu_1}\|\psi_2\|_{L^\infty(D)}^{\nu_2-\mu_2}\cdots\sum_{k=1}^\infty\frac{(\nu_k-\mu_k)\|\nabla\psi_k\|_{L^\infty(D)}}{\|\psi_k\|_{L^\infty(D)}} .\end{aligned}$$

Under Assumption 5.1, we have the estimate

$$\|\psi_k\|_{L^\infty(D)}^{\nu_k-\mu_k} + (\nu_k-\mu_k)\|\psi_k\|_{L^\infty(D)}^{\nu_k-\mu_k-1}\|\nabla\psi_k\|_{L^\infty(D)} \leq \hat{b}_k^{\nu_k-\mu_k},$$

and we deduce that

$$\|a^{-1}\nabla(\partial_y^{\mathfrak{n}\backslash\mathfrak{m}}a(\cdot,y))\|_{L^2(D)} \leq \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\backslash\{\mathfrak{n}\}}\left(\left\|\frac{\nabla a_0}{a_0}\right\|_{L^\infty(D)} + \sum_{k=1}^\infty y_k\|\nabla\psi_k\|_{L^\infty(D)} + 1\right)\hat{b}^{\mathfrak{n}\backslash\mathfrak{m}} .$$

Therefore

$$\begin{aligned}\|a^{-1/2}(\cdot,y)v_{\mathfrak{n}}\|_{L^2(D)} \quad \leq \quad & \sum_{\mathfrak{m}\in\mathfrak{P}(\mathfrak{n})\backslash\{\mathfrak{n}\}}\left(\left\|\frac{\nabla a_0}{a_0}\right\|_{L^\infty(D)} + \sum_{k=1}^\infty y_k\|\nabla\psi_k\|_{L^\infty(D)} + 1\right)\hat{b}^{\mathfrak{n}\backslash\mathfrak{m}}\|a^{1/2}\nabla\partial_y^{\mathfrak{m}}u\|_{L^2(D)} \\ & +\hat{b}^{\mathfrak{n}\backslash\mathfrak{m}}\|a^{1/2}\Delta\partial_y^{\mathfrak{m}}u\|_{L^2(D)} .\end{aligned}$$

From this and (5.5), we have for all $y \in \Gamma_{\hat{b}}$

$$\|a^{-1/2} v_{\mathfrak{n}}\|_{L^2(D)} \leq \sum_{\mathfrak{m} \in \mathfrak{P}(\mathfrak{n}) \setminus \{\mathfrak{n}\}} A(y) \hat{b}^{\mathfrak{n} \setminus \mathfrak{m}} \|a^{1/2} \nabla \partial_y^{\mathfrak{m}} u\|_{L^2(D)} + \hat{b}^{\mathfrak{n} \setminus \mathfrak{m}} \|a^{-1/2} v_{\mathfrak{m}}\|_{L^2(D)}$$

where the constant $A(y)$ is, for $y \in \Gamma_{\hat{b}}$, defined by

$$A(y) = \|a^{-1}(\cdot, y) \nabla a_*(\cdot)\|_{L^\infty(D)} + 2\|a_0(\cdot)^{-1} \nabla a_0(\cdot)\|_{L^\infty(D)} + 2 \sum_{k=1}^{\infty} |y_k| \|\nabla \psi_k\|_{L^\infty(D)} + 1 \,. \tag{5.6}$$

From (3.7), we have for $y \in \Gamma_{\hat{b}}$ that

$$A(y)^{-1} \|a^{-1/2} v_{\mathfrak{n}}\|_{L^2(D)} + \|\partial_y^{\mathfrak{n}} u(\cdot, y)\|_a \leq \sum_{\mathfrak{m} \in \mathfrak{P}(\mathfrak{n}) \setminus \{\mathfrak{n}\}} 2 \hat{b}^{\mathfrak{n} \setminus \mathfrak{m}} (A(y)^{-1} \|a^{-1/2} v_{\mathfrak{m}}\|_{L^2(D)} + \|\partial_y^{\mathfrak{m}} u\|_a) \,.$$

We therefore have

Theorem 5.2 *Under Assumption 5.1 and for $f \in L^2(D)$, we have for $y \in \Gamma_{\hat{b}}$, with $A(y)$ as in* (5.6)*,*

$$A(y)^{-1} \|a^{-1/2} v_\nu\|_{L^2(D)} + \|\partial_y^\nu u(\cdot, y)\|_a \leq (A(y)^{-1} \|a^{-1/2} f\|_{L^2(D)} + \|f\|_{a^{-1}}) |\nu|! \bar{\hat{b}}^\nu,$$

where the sequence $\bar{\hat{b}}$ is defined by $\bar{\hat{b}}_k := 2\hat{b}_k / \log 2$ with $\hat{b}_k$ as in Assumption 5.1.

Proof: The proof is essentially the same as that for Theorem 3.1. When $\nu = 0$, $\|u(\cdot, y)\|_a \leq \|f\|_{a^{-1}}$ and $v_0 = f$. □

It then follows that

$$\|a^{-1/2} v_\nu\|_{L^2(D)} \leq (\|a^{-1/2} f\|_{L^2(D)} + A(y) \|f\|_{a^{-1}}) |\nu|! \bar{\hat{b}}^\nu \,.$$

From (5.5) and Theorem 3.1 we have

$$\forall y \in \Gamma_{\hat{b}} : \quad \|a^{1/2} \Delta \partial_y^\nu u\|_{L^2(D)} \leq (\|a^{-1/2} f\|_{L^2(D)} + 2A(y) \|f\|_{a^{-1}}) |\nu|! \bar{\hat{b}}^\nu \,.$$

To study the regularity of the coefficients u_ν of the expansion (2.33), we will work under the following assumption.

Assumption 5.3 *The coefficients $a_*, a_0 \in W^{1,\infty}(D)$ and there exists $0 < p < 1$ such that*

$$(k \|\nabla \psi_k\|_{L^\infty(D)})_k \in \ell^p(\mathbb{N}) \,.$$

Note that Assumption 5.3 implies Assumption 5.1. We then have the following result.

Proposition 5.4 *Under Assumptions 4.1 and 5.3, the coefficient sequence $(u_\nu)_{\nu \in \mathcal{F}}$ of the Wiener-Itô chaos expansion* (2.32) *satisfies*

$$\sum_{\nu \in \mathcal{F}} \|u_\nu\|_W^p < \infty \,.$$

Proof: The proof of this theorem is analogous to that for Proposition 4.4. We have

$$\begin{aligned}
\|\Delta \mathcal{L}_S^r(u)(\cdot, y)\|_{L^2(D)}^2 &\leq C_1(r)^{2m} (2r)^m \prod_{j=1}^m (1 + |y_{i_j}|)^{2r} \left(\sum_{\substack{s_j = 1, \ldots, 2r \\ j = 1, \ldots, m}} ((s_1 + \ldots + s_m)!)^2 \bar{\hat{b}}_{i_1}^{2s_1} \ldots \bar{\hat{b}}_{i_m}^{2s_m} \right) \\
&\quad . \sup_x (a(x, y)^{-1}) (\|a^{-1/2} f\|_{L^2(D)} + 2A(y) \|f\|_{a^{-1}})^2 \\
&\leq C_1(r)^{2m} (2r)^m \prod_{j=1}^m (1 + |y_{i_j}|)^{2r} \left(\sum_{\substack{s_j = 1, \ldots, 2r \\ j = 1, \ldots, m}} ((s_1 + \ldots + s_m)!)^2 \bar{\hat{b}}_{i_1}^{2s_1} \ldots \bar{\hat{b}}_{i_m}^{2s_m} \right) \\
&\quad . \sup_x (a(x, y)^{-2}) (\|f\|_{L^2(D)} + 2A(y) \|F\|_{L^2(D)})^2.
\end{aligned}$$

We note that for all $y \in \Gamma_{\hat{b}}$ (the value of the constant $c > 0$ in the estimates which follow may depend on f, F and may change from one line to the next)

$$\begin{aligned} |A(y)| &\leq \|\nabla a_*\|_{L^\infty(D)} \sup_x (a(x,y))^{-1} + 2\|a_0^{-1}\nabla a_0\|_{L^\infty(D)} + 2\sum_{k=1}^{\infty} |y_k| \|\nabla\psi_k\|_{L^\infty(D)} + 1 \\ &\leq c\left(1 + \sup_x (a(x,y))^{-1} + \sum_{k=1}^{\infty} |y_k| \|\nabla\psi_k\|_{L^\infty(D)}\right). \end{aligned}$$

Therefore

$$\begin{aligned} &(\|f\|_{L^2(D)} + 2|A(y)|\|F\|_{L^2(D)})^2 \\ &\leq c\left(1 + \sup_x (a(x,y))^{-1} + \sum_{k=1}^{\infty} |y_k| \|\nabla\psi_k\|_{L^\infty(D)}\right)^2 \\ &\leq c\left(\exp\Big(\sum_{k=1}^{\infty} |y_k| \|\psi_k\|_{L^\infty(D)}\Big) + \exp\Big(\sum_{k=1}^{\infty} |y_k| \|\nabla\psi_k\|_{L^\infty(D)}\Big)\right)^2 \\ &\leq c\exp\Big(2\sum_{k=1}^{\infty} |y_k| (\|\psi_k\|_{L^\infty(D)} + \|\nabla\psi_k\|_{L^\infty(D)})\Big). \end{aligned}$$

Thus

$$\begin{aligned} \|\Delta\mathcal{L}_S^r(u)(\cdot,y)\|_{L^2(D)}^2 &\leq C_1(r)^{2m}(2r)^m \prod_{j=1}^{m} (1+|y_{i_j}|)^{2r} \left(\sum_{\substack{s_j=1,\dots,2r \\ j=1,\dots,m}} ((s_1+\ldots+s_m)!)^2 \bar{\hat{b}}_{i_1}^{2s_1} \ldots \bar{\hat{b}}_{i_m}^{2s_m}\right) \\ &\qquad \times \exp\Big(4\sum_{k=1}^{\infty} |y_k| (\|\psi_k\|_{L^\infty(D)} + \|\nabla\psi_k\|_{L^\infty(D)})\Big). \end{aligned}$$

The remaining part of the proof then follows the lines of the argument in the proof of Proposition 4.4. □

5.2 Finite Element Approximation

Let $V_h \subset V$ be a one-parameter family of finite-dimensional spaces of continuous, piecewise linear functions associated to a family of shape regular, quasi uniform partitions of the domain D into simplices with meshwidth $O(h)$. We also denote V_h by V_M where $M(h)$ denotes the finite dimension of the finite element space V_h. The quasiuniformity of the partitions of D implies that $M(h) = O(h^{-1/d})$. We recall the definition (5.1) of the space W, and assume the following approximation property of the family V_h.

Assumption 5.5 *For all functions $v \in W$,*

$$\inf_{v_h \in V_h} \|v - v_h\|_V \leq cM^{-s}\|v\|_W,$$

for some positive constants $c, s > 0$ which are independent of h.

For $\Lambda \subset \mathcal{F}$, let $\mathcal{M} = (M_\nu)_{\nu\in\Lambda}$ be a sequence of positive integers. We denote by

$$\mathcal{V}_{\varrho,\Lambda,\mathcal{M}} = \{v_{\Lambda,\mathcal{M}} \in L^2(U,\gamma_\varrho;V) : \; v_{\Lambda,\mathcal{M}} = \sum_{\nu\in\Lambda} v_{\Lambda,\mathcal{M},\nu}(\cdot) H_\nu(\tau_\varrho(\cdot)), \; v_{\Lambda,\mathcal{M},\nu} \in V_{M_\nu}\}.$$

We then consider the approximating problem for (2.39):
Find $u_{\Lambda,\mathcal{M}} = \sum_{\nu\in\Lambda} u_{\Lambda,\mathcal{M},\nu} H_\nu(\tau_\varrho(\cdot)) \in \mathcal{V}_{\varrho,\Lambda,\mathcal{M}}$ such that

$$B_{\vartheta_\varrho}(u_{\Lambda,\mathcal{M}}, v_{\Lambda,\mathcal{M}}) = F_{\vartheta,\varrho}(v_{\Lambda,\mathcal{M}}), \qquad \forall\, v_{\Lambda,\mathcal{M}} \in \mathcal{V}_{\varrho,\Lambda,\mathcal{M}}. \tag{5.7}$$

From Theorem 2.18, we have

$$\|u - u_{\Lambda,\mathcal{M}}\|_{L^2(\Gamma,\gamma;V)} \leq c(\vartheta,\varrho) \inf_{v_{\Lambda,\mathcal{M}} \in \mathcal{V}_{\varrho,\Lambda,\mathcal{M}}} \|u - v_{\Lambda,\mathcal{M}}\|_{L^2(\Gamma,\gamma_\varrho;V)}.$$

Denoting by $\Lambda \subset \mathcal{F}$ the set of indices corresponding to the coefficients u_ν with the largest V norm, we have for all $v_{\Lambda,\mathcal{M},\nu} \in V$

$$\|u - u_{\Lambda,\mathcal{M}}\|_{L^2(\Gamma,\gamma;V)} \leq c(\vartheta,\varrho) \left(\sum_{\nu \notin \Lambda} \|u_\nu\|_V^2 + \sum_{\nu \in \Lambda} \|u_\nu - v_{\Lambda,\mathcal{M},\nu}\|_V^2 \right)^{1/2}$$
$$\leq c(\vartheta,\varrho) \left(N^{-2r} + \sum_{\nu \in \Lambda} \|u_\nu - v_{\Lambda,\mathcal{M},\nu}\|_V^2 \right)^{1/2}$$

where we defined $r = 1/p - 1/2$ and where we have used Proposition 4.5. Thus

$$\|u - u_{\Lambda,\mathcal{M}}\|_{L^2(\Gamma,\gamma;V)} \leq c(\vartheta,\varrho) \left(N^{-2r} + \sum_{\nu \in \Lambda} M_\nu^{-2s} \|u_\nu\|_W^2 \right)^{1/2} . \tag{5.8}$$

We then choose M_ν with the total number of degrees of freedom

$$N_{dof} = \sum_{\nu \in \Lambda} M_\nu,$$

such that both contributions in the estimate (5.8) are of equal order. This yields the following result.

Theorem 5.6 *Assume that the constant p in Assumption 5.3 satisfies $p \leq 2/(1+2s)$. There is a choice for the dimensions M_ν of the finite element approximating spaces V_ν such that*

$$\|u - u_{\Lambda,\mathcal{M}}\|_{L^2(\Gamma,\gamma;V)} \leq c(\vartheta,\varrho) N_{dof}^{-s}.$$

Proof This theorem is proved as the corresponding result for the Legendre chaos expansion in Cohen et al. [6], using Proposition 5.4, where M_ν are chosen as the solution of the minimizing problem:

$$\min \left\{ \sum_{\nu \in \Lambda} M_\nu : \sum_{\nu \in \Lambda} M_\nu^{-2s} \|u_\nu\|_W^2 \leq N^{-2r} \right\}.$$

□

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Appendix

We now justify the equation (3.6). We proceed by induction. For $|\nu| = 1$, we let $y' \in \Gamma_b$ be such that $y'_m = y_m$ when $m \neq k$ and $y'_k = y_k + \delta$. We then have

$$\int_D a(y)\nabla(u(y') - u(y)) \cdot \nabla v dx = -\int_D (a(y') - a(y))\nabla u(y') \cdot \nabla v.$$

From this, we deduce that

$$\|u(y') - u(y)\|_V \leq \frac{1}{\check{a}(y)} \|a(y') - a(y)\|_{L^\infty(D)} \nabla u(y'),$$

which converges to 0 when $\delta \to 0$. Let $w \in V$ be the solution of the problem

$$\int_D a(y)\nabla w \cdot \nabla v dx = -\int_D \partial_{y_k} a(y)\nabla u(y) \cdot \nabla v dx, \quad \forall\, v \in V.$$

We then have

$$\int_D a(y)\nabla\left(\frac{u(y')-u(y)}{\delta}-w\right)\cdot\nabla v dx = -\int_D\left(\frac{a(y')-a(y)}{\delta}-\partial_{y_k}a(y)\right)\nabla u(y')\cdot\nabla v$$
$$-\int_D \partial_{y_k}a(y)\nabla(u(y')-u(y))\cdot\nabla v,\quad \forall\, v\in V.$$

We then deduce that

$$\left\|\frac{u(y')-u(y)}{\delta}-w\right\|\le\frac{1}{\check{a}(y)}\left(\left\|\frac{a(y')-a(y)}{\delta}-\partial_{y_k}a(y)\right\|_{L^\infty(D)}\|\nabla u(y')\|_{L^2(D)}\right.$$
$$\left.+\|\partial_y a(y)\|_{L^\infty(D)}\|\nabla(u(y')-u(y))\|_{L^2(D)}\right).$$

which converges to 0 when $\delta\to 0$. This shows that

$$\partial_{y_k}u = w.$$

Assume that (3.6) holds for $\nu-e_k$, i.e.

$$\int_D a(x,y)\nabla(\partial_y^{\nu-e_k}u)\cdot\nabla v dx = -\sum_{0\prec\mu\preceq\nu-e_k}\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}\int_D\partial_y^\mu a\nabla\partial_y^{\nu-e_k-\mu}u\cdot\nabla v dx,\quad\forall\, v\in V.$$

By the same argument, we show that for all $v\in V$

$$\int_D a(x,y)\nabla(\partial_y^\nu u)\cdot\nabla v dx = -\sum_{0\prec\mu\preceq\nu-e_k}\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}\left(\int_D\partial_y^{\mu+e_k}a\nabla\partial_y^{\nu-e_k-\mu}u\cdot\nabla v dx\right.$$
$$\left.+\int_D\partial_y^\mu a\nabla\partial_y^{\nu-\mu}u\cdot\nabla v dx\right)-\int_D\partial_{y_k}a\nabla(\partial_y^{\nu-e_k}u)\cdot\nabla v dx$$
$$=-\sum_{e_k\prec\mu\preceq\nu-e_k}\left(\frac{(\nu-e_k)!}{(\mu-e_k)!(\nu-\mu)!}+\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}\right)\int_D\partial_y^\mu a\nabla\partial_y^{\nu-\mu}u\cdot\nabla v dx$$
$$-\sum_{\substack{0\prec\mu\preceq\nu-e_k\\ \mu_k=0}}\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}\int_D\partial_y^\mu a\nabla\partial_y^{\nu-\mu}u\cdot\nabla v dx-\sum_{\substack{0\prec\mu\preceq\nu-e_k\\ \mu_k=\nu_k-1}}\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}\int_D\partial_y^{\mu+e_k}a\nabla\partial_y^{\nu-e_k-\mu}u\cdot\nabla v dx$$
$$-(\nu_k-1)\int_D\partial_{y_k}a\nabla\partial_y^{\nu-e_k}u\cdot\nabla v dx-\int_D\partial_{y_k}a\nabla\partial_y^{\nu-e_k}u\cdot\nabla v dx.$$

We note that

$$\frac{(\nu-e_k)!}{(\mu-e_k)!(\nu-\mu)!}+\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}=\frac{(\nu-e_k)!(\mu_k+\nu_k-\mu_k)}{\mu!(\nu-\mu)!}=\frac{\nu!}{\mu!(\nu-\mu)!};$$

and when $\mu_k=0$,

$$\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}=\frac{\nu!}{\mu!(\nu-\mu)!};$$

in the case $\mu_k=\nu_k-1\ge 0$, it holds

$$\frac{(\nu-e_k)!}{\mu!(\nu-e_k-\mu)!}=\frac{\nu!}{(\mu+e_k)!(\nu-(\mu+e_k))!}.$$

Therefore

$$\int_D a(x,y)\nabla(\partial_y^\nu u)\cdot\nabla v dx = -\sum_{0\prec\mu\preceq\nu}\frac{\nu!}{\mu!(\nu-\mu)!}\int_D\partial_y^\mu a\nabla\partial_y^{\nu-\mu}u\cdot\nabla v dx,\quad\forall\, v\in V.$$

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