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CONVERGENCE OF LOWEST ORDER SEMI-LAGRANGIAN SCHEMES

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Abstract. We consider generalized linear transient advection-diffusion problems for differential forms on a bounded domain in $\mathbb{R}^n$. We provide comprehensive a priori convergence estimates for their spatio-temporal discretization by means of a semi-Lagrangian approach combined with a discontinuous Galerkin method. Under rather weak assumptions on the velocity underlying the advection we establish an asymptotic L^2-estimate $O(\tau + h^r + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}})$, where h is the spatial meshwidth, τ denotes the timestep, and r the polynomial degree of the forms used as trial functions. This estimate can even be improved considerably in a variety of special settings.

1. Introduction

A huge amount of research has been directed at numerical methods for transient 2nd-order advection-diffusion problems for an unknown scalar function $u = u(x,t)$ on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{aligned} \partial_t u - \operatorname{div} \varepsilon \,\mathbf{grad}\, u + \boldsymbol{\beta} \cdot \mathbf{grad}\, u &= f && \text{in } \Omega, \\ u &= g_{\mathrm{D}} && \text{on } \Gamma_0 \cup \Gamma_{\mathrm{in}}, \\ u(\cdot, 0) &= u_0. \end{aligned} \tag{1}$$

The non-negative smooth function $\varepsilon = \varepsilon(x)$ is called the diffusion coefficient, $\boldsymbol{\beta} : \overline{\Omega} \mapsto \mathbb{R}^n$ stands for a given Lipschitz continuous velocity field, $\mathbf{n}_\Omega$ is the outward normal and $f \in C^1\left([0,T]; L^2(\Omega)\right)$ is a given source function, $T > 0$ the final time.

The boundary splits into two disjoint parts $\Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}} = \partial\Omega$, the inflow and outflow boundary, with

$$\Gamma_{\mathrm{in}} = \{x \in \partial\Omega, \boldsymbol{\beta} \cdot \mathbf{n}_\Omega < 0\} \quad \text{and} \quad \Gamma_{\mathrm{out}} = \{x \in \partial\Omega, \boldsymbol{\beta} \cdot \mathbf{n}_\Omega \geq 0\}. \tag{2}$$

Further, the part of the boundary $\partial\Omega$ where the diffusion parameter ε is positive, e.g.

$$\Gamma_0 = \{x \in \partial\Omega, \varepsilon(x) > 0\} \tag{3}$$

is called elliptic boundary. We have to impose Dirichlet or Neumann boundary conditions on $\Gamma_0 \cup \Gamma_{\mathrm{in}}$, cf. the Dirichlet data g_{D} in (1).

Another important advection-diffusion problem is the so-called magnetic advection-diffusion problem for a vectorfield $\mathbf{u} : \Omega \to \mathbb{R}^3$ [28], describing the evolution of magnetic fields in conducting media:

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{curl}\, \varepsilon\, \mathbf{curl}\, \mathbf{u} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) + \mathbf{curl}\, \mathbf{u} \times \boldsymbol{\beta} &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}_{\mathrm{D}} && \text{on } \Gamma_0 \cup \Gamma_{\mathrm{in}}, \\ \mathbf{u}(\cdot, 0) &= \mathbf{u}_0. \end{aligned} \tag{4}$$

The most widely used numerical methods for (1) and (4) are Eulerian schemes that perform spatial discretization on a fixed mesh and then introduce timestepping

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in the spirit of the so-called method of lines. Stability of the spatial discretization for dominant advection is a key issue; it is well known that straightforward Galerkin finite element discretization incurs severe pollution by spurious oscillation for $0 < \varepsilon \ll 1$, unless excessively fine meshes are employed. A wide array of stable spatial discretization methods has been devised for the scalar problem (1); examples are the Discontinuous Galerkin methods [33, 39, 51], Galerkin/Least-Squares methods [34] or subgrid viscosity techniques [22]. We refer to the monograph [54] for a detailed discussion of such methods. Sophisticated estimates confirm that these methods, when applied to stationary scalar advection-diffusion problems, are immune to pollution outside boundary layers or internal layers. We refer to [24] for such estimates for Discontinuous Galerkin methods and to [43] and [54, Theorem 3.41] for a Galerkin/Least-Squares method.

For Eulerian methods, in particular, if ε is large (locally), implicit timestepping is advisable for stability reasons. This entails solving a discrete stationary advection diffusion problem in each timestep, that is, a large sparse linear system of equations with non-symmetric system matrix. Such systems are notoriously challenging for iterative solvers.

A "solver-friendly" [65] alternative that, in addition, manages to circumvent all stability problems, is provided by the class of semi-Lagrangian methods, whose analysis is the focus of this article. Like Eulerian methods they rely on a *single fixed mesh* for spatial discretization. However, their derivation starts from combining the temporal derivative in (1) and (4) and the advection part of the spatial differential operator into a so-called material derivative, which is approximated by a difference quotient. This implies tracking trajectories of the velocity field $\boldsymbol{\beta}$, which is typical of Lagrangian discretization schemes for transport problems. The semi-Lagrangian idea has been introduced for scalar advection-diffusion problems like (1) in a host of research papers, see, e.g., [9–11, 21, 23, 25, 48, 55, 60]. A survey of the literature can be found in Section 5. Theses works exclusively address the scalar case, whereas, apart from [28–30, 52], little attention has been paid to semi-Lagrangian methods for (28).

In order to treat (1) and (4) in a common framework we adopt the perspective of differential forms throughout this article. Doing this, both turn out to be members of a much larger family of advection-diffusion problems. This is elaborated in Section 2. The use of differential forms also helps reveal fundamental structural properties shared by all advection-diffusion problems.

We point out, that the benefits of using the calculus of continuous and discrete differential forms in the derivation and numerical analysis of discretizations of second-order boundary value problems has become well established by now [5, 6, 20] and has proved to be a very fruitful idea. The reader may judge whether this is again confirmed by our work.

The rest of the paper is organized as follows. In Section 3 we present a well-posedness results for transient advection-diffusion of differential forms. Then, in Section 4 we introduce the semi-Lagrangian Galerkin method for such problems and formulate, in Section 5, the main result. The proof of this result, in Section 7, is based on the analysis of an auxillary Galerkin method for the stationary advection-diffusion problem in Section 6.

2. Differential Forms and Vector Proxies

Let Ω be a smooth, oriented n-dimensional manifold with boundary. The sets of smooth differential k-forms $\Lambda^k(\Omega)$ are the smooth sections of alternating k-linear forms defined on the tangent spaces $T_x\Omega$ of Ω with $x \in \Omega$ [57, p. 19]. We refer to the books [16], [35] and [57] for a comprehensive introduction to differential forms and recall here only the basic algebraic operations. In what follows $S(k,n)$ is the set of permutations σ of numbers $\{1,2,\dots n\}$, such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. We use $\mathrm{sign}(\sigma)$ to denote the sign of a permutation σ.

Definition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, oriented n-dimensional Riemannian manifold with volume form $\mu \in \Lambda^n(\Omega)$. In the following definitions $\mathbf{v}_1,\dots,\mathbf{v}_n$ are arbitrary smooth vector fields on Ω while $\mathbf{e}_1,\dots,\mathbf{e}_n$ are orthonormal vector fields on Ω. We define*

- *the exterior product $\wedge : \Lambda^j(\Omega) \times \Lambda^k(\Omega) \to \Lambda^{j+k}(\Omega)$ [57, Definition 1.2.2 a)]:*

$$(\omega \wedge \eta)(\mathbf{v}_1,\dots,\mathbf{v}_{j+k}) := \sum_{\sigma \in S(k,j+k)} \mathrm{sign}(\sigma)\omega(\mathbf{v}_{\sigma(1)}\dots,\mathbf{v}_{\sigma(j)})\eta(\mathbf{v}_{\sigma(j+1)},\dots,\mathbf{v}_{\sigma(j+k)}).$$

- *the scalar product $(\cdot,\cdot) : \Lambda^k(\Omega)\times\Lambda^k(\Omega) \to \Lambda^0(\Omega) = C^\infty(\Omega)$ [57, Definition 1.2.2 b)]:*

$$(\omega,\eta) := \sum_{\sigma \in S(k,n)} \omega(\mathbf{e}_{\sigma(1)},\dots,\mathbf{e}_{\sigma(n)})\eta(\mathbf{e}_{\sigma(1)},\dots,\mathbf{e}_{\sigma(n)}).$$

- *the Hodge operator $\star : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ [57, Definition 1.2.2 c)]:*

$$\eta \wedge \star\omega := (\eta,\omega)\,\mu, \quad \forall \eta \in \Lambda^k(\Omega).$$

- *the exterior derivative $\mathsf{d} : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ at $x \in \Omega$ [38, Proposition 3.2]:*

$$(\mathsf{d}\,\omega)_x(\mathbf{v}_1(x),\dots,\mathbf{v}_{k+1}(x)) = \sum_{j=1}^{k+1}(-1)^{j+1}\partial_{\mathbf{v}_j(x)}\omega(x)(\mathbf{v}_1(x),\dots,\hat{\mathbf{v}}_j(x),\dots,\mathbf{v}_{k+1}(x)),$$

  where $\partial_{\mathbf{v}_j}$ denotes the partial derivative in direction $\mathbf{v}_j$ and $\hat{\mathbf{v}}_j$ indicates a suppressed argument.
- *the contraction $\mathsf{i}_{\boldsymbol{\beta}} : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ for a vector field $\boldsymbol{\beta}$ [57, Definition 1.2.2 d)]:*

$$(\mathsf{i}_{\boldsymbol{\beta}}\,\omega)(\mathbf{v}_1,\dots\mathbf{v}_k) = \omega(\boldsymbol{\beta},\mathbf{v}_1,\dots,\mathbf{v}_k).$$

- *the Lie derivative $\mathsf{L}_{\boldsymbol{\beta}} : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ [57, page 21]:*

$$\mathsf{L}_{\boldsymbol{\beta}}\,\omega = \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega + \mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega. \tag{5}$$

- *the pullback $\Phi^* : \Lambda^k(\Omega') \to \Lambda^k(\Omega)$ for a smooth map $\Phi : \Omega \to \Omega'$ from Ω to a manifold Ω' [57, page 22]:*

$$(\Phi^*\omega)_x(\mathbf{v}_1,\dots,\mathbf{v}_k) = \omega_{\Phi(x)}(D\Phi_x\mathbf{v}_1,\dots,D\Phi_x\mathbf{v}_k),$$

  where ω_x denotes the evaluation of $\omega \in \Lambda^k(\Omega)$ at x and $D\Phi_x$ is the differential of Φ at x.
- *the trace $\mathrm{tr} : \Lambda^k(\Omega) \to \Lambda^k(\partial\Omega)$ is the pullback of the inclusion map $\imath : \partial\Omega \to \Omega$ [5, page 16].*

The stationary, Lipschitz continuous vector field $\boldsymbol{\beta} : \Omega \to T_x\Omega$ induces a flow $X_\tau(x) = X(\tau,x)$ with $X : \Omega \times \mathbb{R} \mapsto \Omega$, where

$$\frac{\partial}{\partial\tau}X_\tau(x) = \boldsymbol{\beta}(X_\tau(x)), \quad X_0(x) = x. \tag{6}$$

It is an important result due to Cartan that [38, p. 142, prop. 5.3]

$$\mathsf{L}_{\boldsymbol{\beta}}\,\omega = \frac{\partial}{\partial\tau}X_\tau^*\omega|_{\tau=0}, \quad \omega \in \Lambda^k(\Omega). \tag{7}$$

Recall that $\mu \in \Lambda^n(\Omega)$ is the volume form on Ω. Completion of $\Lambda^k(\Omega)$ in the norm $\|\omega\|^2_{L^2\Lambda^k(\Omega)} := (\omega,\omega)_\Omega := \int_\Omega (\omega,\omega)\,\mu$ yields the Hilbert space $L^2\Lambda^k(\Omega)$. Analogously to the Sobolev spaces $H^m(\Omega)$ and $W^{m,p}(\Omega)$ for scalar functions with $m > 0$ derivatives in $L^2(\Omega)$ and $L^p(\Omega)$ [57, Section 1.3] we define Sobolev-spaces $W^{m,p}\Lambda^k(\Omega)$ and $H^m\Lambda^k(\Omega)$ for differential forms by requiring that the map

$$x \mapsto \omega_x(\mathbf{v}_1(x),\ldots,\mathbf{v}_k(x)) \tag{8}$$

is in $W^{m,p}(\Omega)$ and $H^m(\Omega)$. In the following $\|\cdot\|_{W^{m,p}\Lambda^k(\Omega)}$ $(|\cdot|_{W^{m,p}\Lambda^k(\Omega)})$ and $\|\cdot\|_{H^m\Lambda^k(\Omega)}$ $(|\cdot|_{H^m\Lambda^k(\Omega)})$ will denote the corresponding (semi)-norms. We use also the standard notations $\boldsymbol{W}^{m,p}(\Omega)$, $|\boldsymbol{\beta}|_{\boldsymbol{W}^{m,p}(\Omega)}$ and $\|\boldsymbol{\beta}\|_{\boldsymbol{W}^{m,p}(\Omega)}$ to denote Sobolev spaces, Sobolev semi-norms and Sobolev norms of vector valued functions with $m > 0$ derivatives in $L^p(\Omega)$.

For the analysis of transport problems of differential forms in a Hilbert space setting it is useful to introduce also the formal L^2-adjoints of the exterior derivative, contraction and Lie derivative [31, Page 8].

Definition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a smooth, oriented n-dimensional manifold with volume form $\mu \in \Lambda^n(\Omega)$. We define*

- *the exterior co-derivative:*

$$\star\,\delta\,\omega := (-1)^k\,\mathsf{d}\star\omega, \quad \omega \in \Lambda^k(\Omega), \tag{9}$$

- *the co-contraction:*

$$\star\,\mathsf{j}_{\boldsymbol{\beta}}\,\omega := (-1)^k\,\mathsf{i}_{\boldsymbol{\beta}}\star\omega, \quad \omega \in \Lambda^k(\Omega), \tag{10}$$

- *the Lie co-derivative:*

$$\star\,\mathcal{L}_{\boldsymbol{\beta}}\,\omega = -\,\mathsf{L}_{\boldsymbol{\beta}}\star\omega, \quad \omega \in \Lambda^k(\Omega). \tag{11}$$

With these definitions we derive the following product rules from the usual product rules for exterior derivative [38, Proposition 3.3], contraction [38, Page 139] and Lie derivative [38, Proposition 5.3] for $\omega \in \Lambda^j(\Omega)$ and $\eta \in \Lambda^k(\Omega)$:

$$\mathsf{d}(\omega\wedge\star\eta) = \mathsf{d}\,\omega\wedge\star\eta + (-1)^{j+k}\omega\wedge\star\,\delta\,\eta, \tag{12}$$

$$\mathsf{i}_{\boldsymbol{\beta}}(\omega\wedge\star\eta) = \mathsf{i}_{\boldsymbol{\beta}}\,\omega\wedge\star\eta + (-1)^{j+k}\omega\wedge\star\,\mathsf{j}_{\boldsymbol{\beta}}\,\eta, \tag{13}$$

$$\mathsf{L}_{\boldsymbol{\beta}}(\omega\wedge\star\eta) = \mathsf{L}_{\boldsymbol{\beta}}\,\omega\wedge\star\eta - \omega\wedge\star\,\mathcal{L}_{\boldsymbol{\beta}}\,\eta. \tag{14}$$

These formulas are valid for $j + n - k > n$, by the convention that $\mathsf{d}\,\omega$ and $\mathsf{i}_{\boldsymbol{\beta}}\,\omega$ are zero whenever $\omega \in \Lambda^j(\Omega)$ with $j > n$.

With these notations at our disposal we formulate the non-stationary transport problem for time-dependent differential forms $\omega(t) \in \Lambda^k(\Omega)$:

$$\begin{aligned}
\partial_t\omega(t) + \delta\,\varepsilon\,\mathsf{d}\,\omega(t) + \mathsf{L}_{\boldsymbol{\beta}}\,\omega(t) &= \varphi, && \text{in } \Omega,\\
\operatorname{tr}\omega(t) &= \operatorname{tr}\psi_{\mathrm{D}}(t), && \text{on } \Gamma_{\mathrm{in}},\\
\operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\,\omega(t)) &= \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}}\,\psi_{\mathrm{D}}(t)), && \text{on } \Gamma_{\mathrm{in}},\\
\operatorname{tr}\omega(t) &= \operatorname{tr}\psi_{\mathrm{D}}(t), && \text{on } \Gamma_0,\\
\omega(0) &= \omega_0.
\end{aligned} \tag{15}$$

with boundary condition on the inflow boundary Γ_{in} and elliptic boundary Γ_0, see (2) and (3).

From Cartan's formula (7) it is clear, why (15) is called transport problem for differential forms if $\varepsilon = 0$. Moreover, in this case we find the formal solution for this problem:

(16)

$$(\omega(t))_x = \begin{cases} \left(X^*_{-t}\omega(0)\right)_x + \int_0^t \left(X^*_{\tau-t}\varphi(\tau)\right)_x d\tau, & X_{\tau-t}(x) \notin \partial\Omega\,\forall\tau\in[0,t],\\ \left(X^*_{t(x)-t}\psi_{\mathrm{D}}(t(x))\right)_x + \int_{t(x)}^t \left(X^*_{\tau-t}\varphi(\tau)\right)_x d\tau, & X_{t(x)-t}(x) \in \partial\Omega. \end{cases}$$

k	differential form	vector proxy
0	$x \mapsto \omega(x)$	$u(x) := \omega(x)$
1	$x \mapsto \{\mathbf{v} \mapsto \omega(x)(\mathbf{v})\}$	$\mathbf{u}(x) \cdot \mathbf{v} := \omega(x)(\mathbf{v})$
2	$x \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(x)(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(x) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \omega(x)(\mathbf{v}_1, \mathbf{v}_2)$
3	$x \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(x)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(x) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \omega(x)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

TABLE 1. In 3D Euclidian space the vector proxies of forms ω are scalar functions u or vectorial functions $\mathbf{u}$ [32, Table 2.1].

k	$\mathsf{d}\,\omega$	$\mathsf{i}_{\boldsymbol{\beta}}\,\omega$	$\delta\,\omega$	$\mathsf{j}_{\boldsymbol{\beta}}\,\omega$	tr	ϕ^*
0	$\mathbf{grad}\, u$			$u\boldsymbol{\beta}$	$u(x)$	$u(\phi(x))$
1	$\mathbf{curl}\,\mathbf{u}$	$\boldsymbol{\beta} \cdot \mathbf{u}$	$-\operatorname{div} \mathbf{u}$	$-\mathbf{u} \times \boldsymbol{\beta}$	$\mathbf{n}_\Omega(x) \times \mathbf{u}(x)$	$D\phi(x)^T \mathbf{u}(\phi(x))$
2	$\operatorname{div} \mathbf{u}$	$\mathbf{u} \times \boldsymbol{\beta}$	$\mathbf{curl}\,\mathbf{u}$	$\boldsymbol{\beta} \cdot \mathbf{u}$	$\mathbf{u}(x) \cdot \mathbf{n}_\Omega(x)$	$\det D\phi(x) D\phi(x)^{-1} \mathbf{u}(\phi(x))$
3		$u\boldsymbol{\beta}$	$-\mathbf{grad}\, u$			$\det D\phi(x) u(\phi(x))$

k	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega$	$\mathcal{L}_{\boldsymbol{\beta}}\,\omega$	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega + \mathcal{L}_{\boldsymbol{\beta}}\,\omega$
0	$\boldsymbol{\beta} \cdot \mathbf{grad}\, u$	$-\operatorname{div}(u\boldsymbol{\beta})$	$-u \operatorname{div} \boldsymbol{\beta}$
1	$\mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) + \mathbf{curl}\,\mathbf{u} \times \boldsymbol{\beta}$	$\mathbf{curl}(\boldsymbol{\beta} \times \mathbf{u}) - \boldsymbol{\beta} \operatorname{div} \mathbf{u}$	$D\boldsymbol{\beta}\mathbf{u} + (D\boldsymbol{\beta})^T \mathbf{u} - \mathbf{u} \operatorname{div} \boldsymbol{\beta}$
2	$\mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta}) + \boldsymbol{\beta} \operatorname{div} \mathbf{u}$	$\boldsymbol{\beta} \times \mathbf{curl}\,\mathbf{u} - \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u})$	$\mathbf{u} \operatorname{div} \boldsymbol{\beta} - D\boldsymbol{\beta}\mathbf{u} - (D\boldsymbol{\beta})^T \mathbf{u}$
3	$\operatorname{div}(u\boldsymbol{\beta})$	$-\boldsymbol{\beta} \cdot \mathbf{grad}\, u$	$u \operatorname{div} \boldsymbol{\beta}$

TABLE 2. Correspondences of operations on forms ω with operations on scalar functions u or vectorial functions (vector proxies) $\mathbf{u}$ in 3D Euclidean space. ϕ is a diffeomorphism and $D\boldsymbol{\beta}$ is the Jacobi matrix. The vector proxies of $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ follow from standard vector calculus identities. [5, 32]

If we have a non-vanishing inflow boundary, for $x \in \Omega$ there might exist a value $t(x) \in \mathbb{R}$, with $0 < t(x) < t$ such that $X_{t(x)-t}(x) \in \partial\Omega$ and the solution depends on prescribed boundary data. The representation formula (16) will be key to the derivation and analysis of semi-Lagrangian methods for (15).

Remark 2.3. *Based on coordinate charts differential forms allow a description by means of functions and vector fields ("vector proxies"). For 3D Euclidian space the usual correspondencies are given in Tables 1 and 2. This reveals that* (1) *and* (4) *incarnate* (15) *for $k = 0$ and $k = 1$, respectivley in 3D.*

3. Well-posedness of transient advection-diffusion problems

We use the Hille-Yosida Theorem [22, Theorem 6.52] to show existence and uniqueness of solutions of (15). Let L be a separable Hilbert space with inner product $(\cdot,\cdot)_L$. Let $A : W \subset L \mapsto L$ be a linear, maximal and monotone operator, i.e.

$$\forall f \in L, \exists v \in W, v + Av = f \tag{17}$$

and

$$\forall v \in W,\ (Av, v)_L \geq 0. \tag{18}$$

It can be shown that in this case the space W, equipped with the scalar product $(u, v)_L + (Au, Av)_L$ is a Hilbert space. We define a bilinear form a as $\mathsf{a}(u, v) = (Au, v)_L$ for all $u \in W$ and $v \in L$ and consider the following model problem:

For $f \in C^1([0, T]; L)$ and $u_0 \in W$, find $u \in C^1([0, T]; L) \cap C^0([0, T]; W)$ such that

$$\begin{aligned} (d_t u, v)_L + \mathsf{a}(u, v) &= (f, v)_L\,, \quad \forall v \in L, \forall t > 0, \\ (u(0), v)_L &= (u_0, v)_L\,, \quad \forall v \in L. \end{aligned} \tag{19}$$

The Hille-Yosida Theorem [22, Theorem 6.52] gives existence and uniqueness of solutions:

Theorem 3.1 (Hille-Yosida). *Let L be a separable Hilbert space with inner product $(\cdot, \cdot)_L$. Let $A : W \subset L \mapsto L$ be a linear, maximal and monotone operator and $\mathsf{a}(u, v) = (Au, v)_L$ for all $u \in W$ and $v \in L$. For all $f \in C^1([0, T]; L)$ and $u_0 \in W$ the problem* (19) *has a unique solution.*

Here, we have $A = \delta\, \varepsilon\, \mathsf{d} + \mathsf{L}_{\boldsymbol{\beta}}$ and set

$$W = \{\omega \in L^2\Lambda^k(\Omega), A\omega \in L^2\Lambda^k(\Omega), \mathrm{tr}_{|\Gamma_{\mathrm{in}}}\omega = 0, \mathrm{tr}_{|\Gamma_{\mathrm{in}}} \mathsf{i}_{\boldsymbol{\beta}}\, \omega = 0, \mathrm{tr}_{|\Gamma_0}\omega = 0\}$$

and $L = L^2\Lambda^k(\Omega)$. We equip the space W with the norm

$$\|\omega\|_W^2 := \|\omega\|_L^2 + \|(\delta\, \varepsilon\, \mathsf{d} + \mathsf{L}_{\boldsymbol{\beta}})\omega\|_L^2\,.$$

To ensure monotonicity (18) we need to assume that the operator $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ is non-negative.

Assumption 3.2. *We assume that $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}} : L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega)$ is non-negative.*

Remark 3.3. *By part b) of Proposition A.1 we know that Assumption 3.2 is a condition on the velocity field $\boldsymbol{\beta}$. This assumption is not very restrictive since we can always introduce a change of variables $\omega' = e^{\alpha t}\omega$ with $\alpha \in \mathbb{R}$, $\alpha > 0$ and rescale time such that the arising operator $\alpha\mathsf{id} + \mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ is positive.*

The following lemma establishes then the crucial step for proving well-posedness.

Lemma 3.4. *Under Assumption 3.2 the operator $A = \delta\, \varepsilon\, \mathsf{d} + \mathsf{L}_{\boldsymbol{\beta}} : W \subset L \to L$ is a maximal and monotone operator.*

Proof. Proving maximality (17) is equivalent to existence and uniqueness of solutions of the following variational formulation:

For $f \in L$ find $\omega \in W$ such that

$$(\omega, \eta)_L + (\delta\, \varepsilon\, \mathsf{d}\, \omega, \eta)_L + (\mathsf{L}_{\boldsymbol{\beta}}\, \omega, \eta)_L = (f, \eta)_L\,, \quad \forall \eta \in L.$$

To prove existence and uniqueness we verify the assumptions of the Banach-Neĉas-Babûska Theorem [22, p. 85]: The bilinear form

$$\mathsf{a}(\omega, \eta) := (\omega, \eta)_L + (\delta\, \varepsilon\, \mathsf{d}\, \omega, \eta)_L + (\mathsf{L}_{\boldsymbol{\beta}}\, \omega, \eta)_L$$

is continuous on $W \times L$ and we show that it satisfies an inf-sup-condition. First the non-negativity assumption and the product rule for Lie derivatives (14) imply stability in L:

$$\begin{aligned}\mathsf{a}(\omega,\omega) &= (\omega,\omega)_L + (\delta\,\varepsilon\,\mathsf{d}\,\omega,\omega)_L + (\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\omega)_L \\ &= (\omega,\omega)_L + (\varepsilon\,\mathsf{d}\,\omega,\mathsf{d}\,\omega)_L - \int_{\partial\Omega} \varepsilon\,\mathrm{tr}\,\omega\wedge\mathrm{tr}\star\mathsf{d}\,\omega) \\ &\quad + \frac{1}{2}\left((\mathsf{L}_{\boldsymbol{\beta}}+\mathcal{L}_{\boldsymbol{\beta}})\omega,\omega\right)_L + \frac{1}{2}\int_{\partial\Omega}\mathrm{tr}\,\mathsf{i}_{\boldsymbol{\beta}}(\omega\wedge\star\omega) \\ &\geq \|\omega\|_L^2\,.\end{aligned}$$

The last inequality follows from $\int_{\partial\Omega}\mathrm{tr}\,\mathsf{i}_{\boldsymbol{\beta}}\,\mu = \int_{\partial\Omega}\boldsymbol{\beta}\cdot\mathbf{n}_\Omega\,\mathsf{i}_{\mathbf{n}_\Omega}\,\mu$ and $(\boldsymbol{\beta}\cdot\mathbf{n}_\Omega)_{|\partial\Omega\setminus\Gamma_{\mathrm{in}}} \geq 0$ and the imposed homogeneous boundary conditions, since

$$\int_{\Gamma_{\mathrm{in}}}\mathrm{tr}\,\mathsf{i}_{\boldsymbol{\beta}}(\omega\wedge\star\omega) = \int_{\Gamma_{\mathrm{in}}}\underbrace{\mathrm{tr}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega}_{=0}\wedge\mathrm{tr}\star\omega + \int_{\Gamma_{\mathrm{in}}}\underbrace{\mathrm{tr}\,\omega}_{=0}\wedge\mathrm{tr}\star\mathsf{j}_{\boldsymbol{\beta}}\,\omega = 0, \tag{20}$$

by (13). The L-stability implies

$$\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L} \geq \frac{\mathsf{a}(\omega,\omega)}{\|\omega\|_L} \geq \|\omega\|_L\,,$$

and we deduce

$$\begin{aligned}\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L} &= \sup_{\eta\in L}\frac{(\omega,\eta)_L + (\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\eta)_L + (\delta\,\varepsilon\,\mathsf{d}\,\omega,\eta)_L}{\|\eta\|_L} \\ &\geq \sup_{\eta\in L}\frac{(\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\eta)_L + (\delta\,\varepsilon\,\mathsf{d}\,\omega,\eta)_L}{\|\eta\|_L} - \sup_{\eta\in L}\frac{(\omega,\eta)_L}{\|\eta\|_L} \\ &= \sup_{\eta\in L}\frac{(\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\eta)_L + (\delta\,\varepsilon\,\mathsf{d}\,\omega,\eta)_L}{\|\eta\|_L} - \|\omega\|_L \\ &\geq \|(\delta\,\varepsilon\,\mathsf{d}+\mathsf{L}_{\boldsymbol{\beta}})\omega\|_L - \sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L}.\end{aligned}$$

This yields

$$\left((1+1)^2+1\right)\left(\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L}\right)^2 \geq \|(\delta\,\varepsilon\,\mathsf{d}+\mathsf{L}_{\boldsymbol{\beta}})\omega\|_L^2 + \|\omega\|_L^2\,,$$

i.e. the inf-sup-inequality

$$\inf_{\omega\in W}\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\omega\|_W\,\|\eta\|_L} \geq 5^{-\frac{1}{2}}.$$

Next we establish the injectivity condition in the Banach-Neĉas-Babûska Theorem [22, p. 85]. Let $\eta\in L$ such that $\mathsf{a}(\omega,\eta)=0$ for all $\omega\in W$. A density argument gives $\eta+\delta\,\varepsilon\,\mathsf{d}\,\eta+\mathcal{L}_{\boldsymbol{\beta}}\,\eta=0$, which implies $\|\eta\|_W\leq\infty$. Testing with $\omega\in\Lambda^k(\Omega)\cap W$ we find $\mathrm{tr}\star\eta=0$ and $\mathrm{tr}\,\mathsf{i}_{\boldsymbol{\beta}}\star\eta=0$ at $\partial\Omega\setminus\Gamma_{\mathrm{in}}$, $\mathrm{tr}\star\mathsf{d}\,\omega=0$ at Γ_0 and deduce

$$\begin{aligned}0 &= (\eta,\eta)_L + (\eta,\delta\,\varepsilon\,\mathsf{d}\,\eta)_L + (\eta,\mathcal{L}_{\boldsymbol{\beta}}\,\eta)_L \\ &= (\eta,\eta)_L + (\mathsf{d}\,\eta,\varepsilon\,\mathsf{d}\,\eta) - \int_{\partial\Omega}\varepsilon\,\mathrm{tr}(\eta\wedge\star\,\mathsf{d}\,\eta) \\ &\quad + \frac{1}{2}\left((\eta,\mathcal{L}_{\boldsymbol{\beta}}\,\eta)_L + (\eta,\mathsf{L}_{\boldsymbol{\beta}}\,\eta)_L\right) - \frac{1}{2}\int_{\partial\Omega}\mathrm{tr}\,\mathsf{i}_{\boldsymbol{\beta}}(\eta\wedge\star\eta) \\ &\geq \|\eta\|_L^2\,,\end{aligned}$$

i.e. $\eta=0$.

Summing up, Assumption 3.2 and the boundary conditions in the definition of W ensure that the Lie derivative is a maximal and monotone operator. $\square$

Remark 3.5. *The identity* (20) *shows that we could impose other inflow boundary conditions. For the following four conditions we can show well-posedness:*

(1) $\operatorname{tr} \mathsf{i}_{\boldsymbol\beta}\, \omega = \operatorname{tr} \mathsf{i}_{\boldsymbol\beta}\, \psi_{\mathrm{D}}$ *and* $\operatorname{tr} \omega = \operatorname{tr} \psi_D$ *on* Γ_{in}*;*
(2) $\operatorname{tr} \mathsf{i}_{\boldsymbol\beta}\, \omega = \operatorname{tr} \mathsf{i}_{\boldsymbol\beta}\, \psi_D$ *and* $\operatorname{tr} \mathsf{i}_{\boldsymbol\beta} \star\omega = \operatorname{tr} \mathsf{i}_{\boldsymbol\beta} \star\psi_D$ *on* Γ_{in}*;*
(3) $\operatorname{tr} \star\omega = \operatorname{tr} \star\psi_D$ *and* $\operatorname{tr} \omega = \operatorname{tr} \psi_D$ *on* Γ_{in}*;*
(4) $\operatorname{tr} \star\omega = \operatorname{tr} \star\psi_D$ *and* $\operatorname{tr} \mathsf{i}_{\boldsymbol\beta} \star\omega = \operatorname{tr} \mathsf{i}_{\boldsymbol\beta} \star\psi_D$ *on* Γ_{in}*;*

If the normal component of $\boldsymbol\beta$ *vanishes everywhere on* $\partial\Omega$*, we do not need to impose any inflow boundary conditions, since* Γ_{in} *is empty.*

4. Semi-Lagrangian Methods

We review the construction of a Semi-Lagrangian discretization of (15). More details for pure advection ($\varepsilon = 0$) can be found in [28,30]. Let $\mathcal{T}$ be a triangulation of Ω and $\omega \in L^2\Lambda^k(\Omega)$ piecewise smooth on $\mathcal{T}$. Let $\mathcal{F}$ be the set of all $n-1$-dimensional open faces of all elements $T \in \mathcal{T}$ and assume an arbitrary orientation of faces $f \in \mathcal{F}$, i.e. the faces have a distinguished normal $\mathbf{n}_f$. If a face f is contained in the boundary of some element T then either $\mathbf{n}_f = \mathbf{n}_{T|_f}$ or $\mathbf{n}_f = -\mathbf{n}_{T|_f}$. Then ω^+ and ω^- denote the two different restrictions of $\omega \in \Lambda^k(\Omega)$ to f, e.g. $\omega_x^+ := \lim_{s\to 0^+} \omega_{x+s\mathbf{n}_f}$ for $x \in f$. With these restriction we define the jump $[\omega]_f = \omega^- - \omega^+$ and the average $\{\omega\}_f = \frac{1}{2}(\omega^- + \omega^+)$. For $f \subset \partial\Omega$ we assume f to be oriented such that $\mathbf{n}_f$ points outward. Let $\mathcal{F}^\circ$ and $\mathcal{F}^\partial$ be the set of interior and boundary facets, respectively; $\mathcal{F}^\partial_-, \mathcal{F}^\partial_+, \mathcal{F}^\partial_0 \subset \mathcal{F}^\partial$ is the set of facets on the inflow boundary Γ_-, the outflow boundary Γ_+ and the elliptic boundary Γ_0. Let $(\cdot,\cdot)_\Omega$ be the L^2-inner product on any $\Lambda^k(\Omega)$. In the following, $\Lambda^k_h(\mathcal{T})$ denotes some piecewise polynomial approximation space on the triangulation $\mathcal{T}$ for k-forms in Ω. The discretization of the diffusion operator $\delta\,\varepsilon\,\mathsf{d}$ is based on the bilinear form

$$
\begin{aligned}
\mathsf{b}(\omega_h,\eta_h) := {} & \sum_{T\in\mathcal{T}} (\varepsilon\,\mathsf{d}\,\omega_h, \mathsf{d}\,\eta_h)_T \\
& + \sum_{f\in\mathcal{F}^\circ} \left(\int_f \varepsilon \operatorname{tr}[\omega_h]_f \wedge \operatorname{tr}\star\{\mathsf{d}\,\eta_h\}_f - \int_f \varepsilon \operatorname{tr}[\eta_h]_f \wedge \operatorname{tr}\star\{\mathsf{d}\,\omega_h\}_f \right) \\
& + \sum_{f\in\mathcal{F}^\partial_0} \left(\int_f \varepsilon \operatorname{tr}\omega_h \wedge \operatorname{tr}\star\mathsf{d}\,\eta_h - \int_f \varepsilon \operatorname{tr}\eta_h \wedge \operatorname{tr}\star\mathsf{d}\,\omega_h \right) \\
& + \sum_{f\in\mathcal{F}^\partial_0} \int_f s_f \varepsilon \operatorname{tr} \mathsf{i}_{\mathbf{n}_f}(\omega_h \wedge \star\eta_h) + \sum_{f\in\mathcal{F}^\circ} s_f \varepsilon \operatorname{tr} \mathsf{i}_{\mathbf{n}_f}([\omega_h]_f \wedge \star[\eta_h]_f),
\end{aligned} \tag{21}
$$

for $\omega_h, \eta_h \in \Lambda^k_h(\mathcal{T})$, $s_f > 0$, and another bilinear form

$$
\ell(\psi,\eta_h) = \sum_{f\in\mathcal{F}^\partial_0} \int_f \varepsilon \operatorname{tr}\psi \wedge \operatorname{tr}\star\mathsf{d}\,\eta_h + \int_f s_f \varepsilon \operatorname{tr} \mathsf{i}_{\mathbf{n}_f}(\psi \wedge \star\eta_h), \tag{22}
$$

for $\psi \in \Lambda^k(\Omega)$ and $\eta_h \in \Lambda^k_h(\mathcal{T})$. These are related to the discontinuous Galerkin (DG) non-symmetric interior penalty discretization [33, 46, 53]. The penalty parameter s_f is inversely proportional to the local mesh size.

Recall that X_τ is the flow of the velocity field $\boldsymbol\beta$. Here and in the following, we assume that $\boldsymbol\beta$ is defined on an open neighbourhood of Ω. For fixed small τ the map $X_{-\tau}$ induces the decomposition $\Omega = \Omega_{\mathrm{in}} \cup \Omega_0$, with $X_{-\tau}(\Omega_{\mathrm{in}}) \cap \Omega = \{\}$ and $X_{-\tau}(\Omega_0) \subset \Omega$. Further we have $X_\tau(\Omega) = \Omega_0 \cup \Omega_{\mathrm{out}}$ with $\Omega_{\mathrm{out}} = X_\tau(\Omega) \setminus \Omega_0$; see Figure 1.

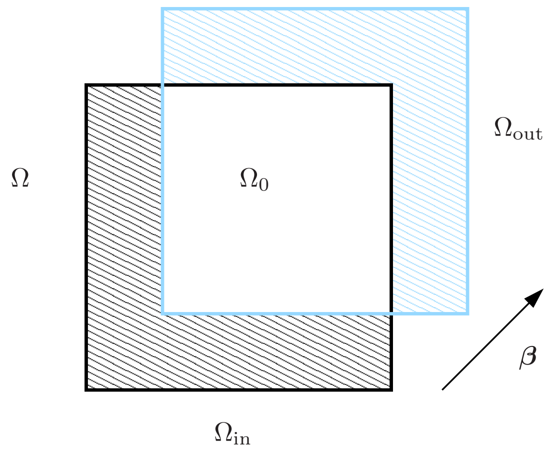

FIGURE 1. Illustration of the definition of the domains Ω_0, Ω_{in} and Ω_{out} for $\boldsymbol{\beta}=$ const: the black lines and the light blue lines bound Ω and $X_\tau(\Omega)$, respectively. The black shaded area is Ω_{in} and the light blue shaded area is Ω_{out}.

For the advection operator we introduce the 'weak discrete material derivative' [28, page 1477] [29, page 8]

$$\mathsf{a}_\tau\left(\omega_h,\eta_h\right):=\frac{1}{\tau}\left(\omega_h,\eta_h\right)_\Omega-\frac{1}{\tau}\left(X^*_{-\tau}\omega_h,\eta_h\right)_{\Omega_0},\quad \omega_h,\eta_h\in\Lambda^k_h\left(\mathcal{T}\right) \tag{23}$$

and

$$\mathsf{g}_\tau\left(\psi,\eta_h\right):=\frac{1}{\tau}\left(\widetilde{\psi},\eta_h\right)_{\Omega_{\text{in}}},\quad \psi\in\Lambda^k\left(\Omega\right),\eta_h\in\Lambda^k_h\left(\mathcal{T}\right), \tag{24}$$

where $\widetilde{\psi}$ is an extension of ψ into Ω_{in} that is constant along the characteristic lines of $\boldsymbol{\beta}$. More precisely, if we define the time $t(x)$ for $x\in\Omega_{\text{in}}$ such that $X_{-t(x)}(x)\in\Gamma_{\text{in}}$ we set

$$\widetilde{\psi}_x=\left(X^*_{-t(x)}\psi\right)_x. \tag{25}$$

To formulate a semi-Lagrangian method, we consider a partitioning of the time interval of the form $[0,T]=\bigcup_{n=0}^{N-1}[t^n,t^{n+1}]$ with $t^n=\tau n$ and $\tau=\frac{T}{N}$. Then the semi-Lagrangian Galerkin timestepping scheme for the advection-diffusion problem (15) constructs sequences $(\omega_h^n)_{n=0}^N$, $\omega_h^n\in\Lambda^k_h\left(\mathcal{T}\right)$, approximating $(\omega(t^n))_{n=0}^N$ according to:

- Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n\in\Lambda^k_h\left(\mathcal{T}\right)$, such that for all $\eta_h\in\Lambda^k_h\left(\mathcal{T}\right)$:

$$\left(\omega_h^0,\eta_h\right)_\Omega=\left(\omega_0,\eta_h\right)_\Omega,$$

$$\begin{aligned}\mathsf{b}\left(\omega_h^{n+1},\eta\right)+\frac{1}{\tau}\left(\omega_h^{n+1},\eta_h\right)_\Omega-\frac{1}{\tau}\left(\omega_h^n,\eta_h\right)_\Omega&\\ +\mathsf{a}_\tau\left(\omega_h^n,\eta_h\right)=\left(\varphi(t^{n+1}),\eta_h\right)_\Omega+\ell\left(\psi_{\mathrm{D}}(t^n),\eta_h\right)+\mathsf{g}_\tau\left(\widetilde{\psi^n_{\mathrm{D}}},\eta\right).&\end{aligned} \tag{26}$$

$\widetilde{\psi^n_{\mathrm{D}}}$ is the extension $\left(X^*_{-t(x)}\psi_{\mathrm{D}}(t^n)\right)_x$ for $x\in\Omega_{\text{in}}$ introduced in (25).

Remark 4.1. *The semi-Lagrangian scheme* (26) *for the pure advection problem, that is problem* (15) *with* $\varepsilon=0$*, boils down to the Galerkin projection of the formal solution* (16)*, where we choose a low order quadrature for the evaluation of the right* *hand side*

$$\varphi^{n+1} := \varphi(t^{n+1}) \approx \tau \int_{t^n}^{t^{n+1}} X^*_{t-t^{n+1}} \varphi(t) dt.$$

To see this, notice

$$\frac{1}{\tau}\left(\omega_h^{n+1}, \eta_h\right)_\Omega - \frac{1}{\tau}\left(\omega_h^{n}, \eta_h\right)_\Omega + \mathsf{a}_\tau\left(\omega_h^{n}, \eta_h\right) = \frac{1}{\tau}\left(\omega_h^{n+1}, \eta_h\right)_\Omega - \frac{1}{\tau}\left(X^*_{-\tau}\omega_h^{n}, \eta_h\right)_{\Omega_0}.$$

Candidate spaces for $\Lambda_h^k(\mathcal{T})$ are the spaces of discrete differential forms, introduced in [5] and [6], that are subspaces of appropriate Sobolev-spaces of differential forms. In $\mathbb{R}^3$, these spaces correspond to the standard Lagrangian finite element spaces ($k = 0$), to the $\boldsymbol{H}(\mathbf{curl}, \Omega)$ ($k = 1$) and $\boldsymbol{H}(\mathrm{div}, \Omega)$ ($k = 2$) conforming finite element spaces of Nédélec's first [44] and second family [45] and to spaces of discontinuous piecewise polynomial functions ($k = 3$), respectively. Then in the definitions of $\mathsf{b}(\cdot, \cdot)$ and $\ell(\cdot, \cdot)$ all the integral terms over interior faces $f \in \mathcal{F}^\circ$ vanish, because $\mathrm{tr}([\omega_h]_f) = 0$. While these spaces feature a lot of interesting mathematical structure, in our derivation we will need only certain optimal approximation properties like $\inf_{\eta \in \Lambda_h^k(\mathcal{T})} \|\omega - \eta\|_{L^2\Lambda^k(\Omega)} = O(h^{r+1})$ for ω sufficiently smooth, where r is the degree of the piecewise polynomials that are contained in $\Lambda_h^k(\mathcal{T})$ and h the local mesh size. Therefore, other possible choices for $\Lambda_h^k(\mathcal{T})$ include the usual globally discontinuous approximation spaces used in discontinuous Galerkin methods.

Remark 4.2. *To elucidate the relationship of* (26) *with methods proposed in the literature (usually stated in terms of vector proxies there), we provide the vector proxy incarnation of* (1) *and* (4) *for homogeneous boundary conditions. Let* V_h *and* $\mathbf{V}_h$ *denote some scalar and vectorial finite dimensional approximation spaces. Then the semi-Lagrangian Galerkin schemes for the two boundary value problems are: Given* $u_h^0 \in V_h$ *find* $u_h^n \in V_h, n = 1, 2, \ldots, N$ *such that for all* $v_h \in V_h$

$$\begin{aligned}\tau \mathsf{b}\left(u_h^{n+1}, v_h\right) + \int_\Omega \left(u_h^{n+1}(x) - u_h^n\left(X_{-\tau}(x)\right)\right) v_h(x) dx \\ = \tau \int_\Omega \varphi^{n+1}(x) v_h(x) dx;\end{aligned} \tag{27}$$

and: Given $\mathbf{u}_h^0 \in \mathbf{V}_h$ *find* $\mathbf{u}_h^n \in \mathbf{V}_h, n = 1, 2, \ldots, N$ *such that for all* $\mathbf{v}_h \in \mathbf{V}_h$

$$\begin{aligned}\tau \mathsf{b}\left(\mathbf{u}_h^{n+1}, \mathbf{v}_h\right) + \int_\Omega \left(\mathbf{u}_h^{n+1}(x) - DX_{-\tau}^T(x)\mathbf{u}_h^n\left(X_{-\tau}(x)\right)\right) \mathbf{v}_h(x) dx \\ = \tau \int_\Omega \varphi^{n+1}(x) \mathbf{v}_h(x) dx.\end{aligned} \tag{28}$$

For $\varepsilon = 0$ *the scheme* (27) *agrees with the so-called "exactly integrated semi-Semi-Lagrange Galerkin scheme" in [50]. Actual implementations, e.g. in [21, 48, 59, 61, 63], of this method require further approximation steps, e.g. approximation of trajectories or the evaluation of the inner products* $\int_\Omega u_h(X_{-\tau}(x)) v_h(x) dx$ *(see Figure 2). A flawed treatment of these additional approximations in* (27) *of* (28) *can lead to unconditionally unstable [41] or non-convergent semi-Lagrangian timestepping schemes [29].*

In introducing the change of variables $y := X_{-\tau}(x)$ *we could replace the products* $u_h^n(X_{-\tau}(x)) v_h(x)$ *in* (27) *and* (28) *with* $u_h^n(y) v_h(X_\tau(y))$*. Sometimes such representations are referred to as the weak Lagrange-Galerkin method [42] or (localized) adjoint Lagrange-Galerkin method [17, 26, 27] (LAM, ELLAM). Also the treatment of inflow boundary conditions can be found in the literature on LAM and ELLAM [17, 26, 27].*

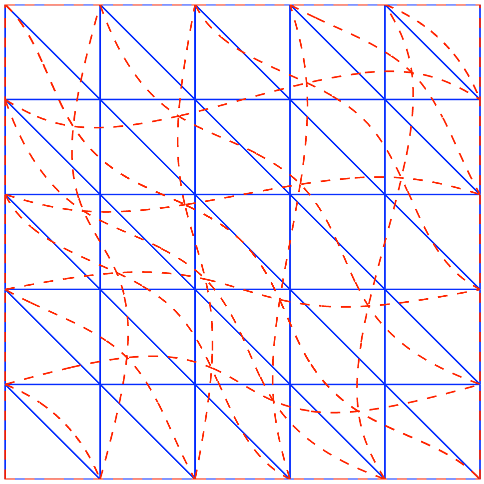

FIGURE 2. A mesh $\mathcal{T}$ (blue, solid lines) on $\Omega = [0,1]^2$ and its image mesh $X_\tau(\mathcal{T})$ under the flow induced by $\boldsymbol{\beta} = \left(\frac{1}{16}\sin(2\pi x)\sin(2\pi y), \frac{1}{16}\sin(2\pi x)\sin(2\pi y)\right)$. The inner product $\int_\Omega u_h(X_{-\tau}(x))v_h(x)dx$ in (27) and (28) is an inner product of two functions that are piecewise smooth on two different meshes: $v_h(x)$ is piecewise smooth on $\mathcal{T}$ and $u_h(X_{-\tau}(x))$ is piecewise smooth on $X_\tau(\mathcal{T})$.

5. Previous Work on Semi-Lagrangian Galerkin Methods and Main Result

Ample convergence theory is available for variants of the semi-Lagrangian discretization (27) of the scalar advection-diffusion boundary value problem (1). It has yielded three types of a priori convergence results that we are going to discuss below. The first class of results accepts the dependence of constants in the estimates on the diffusion coefficient ε and their blowing up when $\varepsilon \to 0$. The second class of results provides estimates that are uniformly in ε and the third class concentrates on estimates for the limit case $\varepsilon = 0$.

Even though convergence theory for piecewise polynomial trial spaces of higher degree is available, we only review the results for lowest-order approximation spaces, i.e., if not stated otherwise, V_h is the space of piecewise linear $H^1(\Omega)$-conforming finite elements. It turned out to be surprisingly difficult to establish any convergence result for fully discrete semi-Lagrangian methods in the general setting of a bounded domain and non-vanishing normal component of $\boldsymbol{\beta}$ at parts of the boundary, when τ and h are roughly linearly proportional.

Results that are non-uniform in the diffusion coefficient can be found in [13,21, 48,59,61,63].

The early work of Pironneau [48] proved the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\varepsilon)(\tau + h + h^2\tau^{-1})$, where it is assumed that $\operatorname{div}\boldsymbol{\beta} = 0$ and $\boldsymbol{\beta}$ has vanishing normal component on the boundary of Ω. The exact flow $X(\tau)$ is approximated by a flow $X_h(\tau)$ corresponding to a piecewise constant approximation of the velocity $\boldsymbol{\beta}$. Then all the integrals occurring in (27), and in particular $\int_\Omega u_h(X_{h,-\tau}(x))v_h(x)dx$, can be computed exactly [48, Page 314.]. Subsequently, Douglas and Russel in [21] proved the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\varepsilon)(\tau + h^2)$ in one dimension for $\Omega = \mathbb{R}$. Their result accounts for characteristics that are approximated by an explicit Euler step but assumes that $\int_\Omega u_h(x - \tau\boldsymbol{\beta}(x))v_h(x)dx$ can be computed exactly. For general $\boldsymbol{\beta}$

this assumption is impractical (see Figure 2). Later Süli [59] extended the so-called area-weighting technique for advection problems [41] to advection-diffusion problems in unbounded domains. For a fully discrete semi-Lagrangian scheme he proved the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\varepsilon)(\tau + h)$. Bermejo used a similar technique to formulate fully discrete semi-Lagrangian schemes on rectangular meshes [12] and showed $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\varepsilon)(\tau + \max(h^2\tau^{-1}, \tau))$. Wang, Ewing, and Russell [61] considered the ELLAM variants of semi-Lagrangian schemes, *cf.* [17,26,27], for the scalar advection-diffusion problem with constant velocity on intervals $\Omega \subset \mathbb{R}$. Their analysis can accommodate various types of boundary conditions and yields the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\varepsilon)(\tau + h^2)$. Recently, Wang and Wang [63] generalized this to the case of non-constant $\boldsymbol{\beta}$ and $\Omega \subset \mathbb{R}^n$ and proved the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\varepsilon)(\min(\tau, h) + \tau + h^2$ for a fully discrete ELLAM method. Both these last works assume that the integrals $\int_\Omega u_h(X_{-\tau}(x))v_h(x)dx$ are computed exactly.

Available estimates that are uniform in the diffusion coefficient assume either vanishing normal components of $\boldsymbol{\beta}$ at the boundary of Ω [8] or impose periodic boundary conditions [62]. Both these settings avoid the critical case of steep boundary layers: in the former case Bause und Knabner [8] gave the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\tau + h^2 + \min(h^2, h^2/\tau))$, while in the latter case Wang and Wang [62] proved the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq c(\sqrt{\varepsilon}\tau + \tau + h)$. Both results hinge on the exact evaluation of the integrals $\int_\Omega u_h(X_{-\tau}(x))v_h(x)dx$.

Error bounds that tackle the limit case $\varepsilon = 0$ can be found in [2, 19, 37, 40, 48]: Pironneau [48] gave the estimate $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq C_1(\tau, h) + c(h^2\tau^{-1})$, where $C_1(\tau, h) = O(h^{m_1}) + O(\tau^{m_2})$ reflects an error due to the approximation of the trajectories. It is assumed that $\operatorname{div}\boldsymbol{\beta} = 0$ and $\boldsymbol{\beta}$ has vanishing normal component on the boundary of Ω. Later, for a variant of the scheme from [21] with periodic boundary conditions, in [19] Dawson and co-workers showed the estimate $\|u(t^n) - u_h^n\|_{l^2(\mathbb{Z})} \leq C(\tau + h)$. The result of Johnson [37], $\|u(t_n) - u_h^n\|_{L^2(\Omega)} \leq ch^2\tau^{-\frac{1}{2}}$, assumed that $\boldsymbol{\beta} = const$ and $\Omega = \mathbb{R}^n$. Another important result is due to Lucier [40] and Arbogast and Wang [2]: The semi-Lagrangian scheme for scalar conservation laws, $k = n$ in (26), and V_h being the space of piecewise constant finite elements agrees with Godunov's method, for which $\|u(t_n) - u_h^n\|_{L^1(\Omega)} \leq C_1(\tau, h) + c(h + h\tau^{-\frac{1}{2}})$ is shown [2,40], where $C_1(\tau) = O(\tau^{m_2})$ is due to the approximation of the trajectories.

In contrast to the scalar problem, there are almost no results addressing semi-Lagrangian methods for the non-scalar problem (4). We would like to mention the Arbitrary Lagrangian-Eulerian (ALE) method from [52]. Yet this approach relies on a series of distorted meshes, while the semi-Lagrangian methods work on a single fixed mesh.

Besides the semi-Lagrangian Galerkin method considered here, there is also a different kind of semi-Lagrangian schemes that use interpolation operators instead of L^2-projection to map the quantities $u_h(X_{-\tau}(x))$ onto the approximation space. We refer to the literature, e.g. [29, 49, 58], for a discussion and theoretical results on such methods.

If we neglect for a moment complications introduced by the treatment of boundary conditions and the evaluation of the non-standard inner product, we find a discrepancy between the theoretical results for vanishing and non-vanishing diffusion: For $\tau = O(h)$ and piecewise linear approximation spaces the ε-uniform estimates of [8, 62] yield an error of order $O(h)$, while the best result for vanishing diffusion gives an error of order $h^{1+\frac{1}{2}}$ [37]. However, the proof of this estimate (see [37, p. 52]) seems to be confined to the Cauchy problem for linear advection.

Apparently it is not possible to establish similar estimates for both non-vanishing diffusion and inflow boundary with these techniques.

In this article we present a new kind of analysis for semi-Lagrangian Galerkin methods that is based on an auxiliary discretization of the stationary advection-diffusion problem. Thanks to the use of differential forms our analysis covers not only scalar but also non-scalar advection-diffusion problems, like the magnetic advection-diffusion problem (4) from magnetohydrodynamics. We prove an L^2-estimate of order $O(\|\varepsilon\|_{L^\infty(\Omega)} h^r + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} + \tau)$ for unstructured simplicial meshes and approximation spaces with local approximation order $O(h^{r+1})$ in $L^2(\Omega)$. This estimate holds for conforming and non-conforming approximation spaces alike, and it includes the case of non-vanishing inflow boundary data, extending the results in [29]. In the case of vanishing inflow boundary data we get an L^2-estimate of order $O(\|\varepsilon\|_{L^\infty(\Omega)} h^r + h^{r+1}\tau^{-\frac{1}{2}} + \tau)$, that for $\varepsilon \ll h$ agrees with the stronger results for the Cauchy problem with vanishing diffusion [37]. We point out that all these estimates hinge on (strict) positivity assumptions for an expression depending on derivatives of $\boldsymbol{\beta}$. More precisely we make the following assumption, closely related to Assumption 3.2.

Assumption 5.1. *We assume that* $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}} : L^2\Lambda^k(\Omega) \to L^2\Lambda^k(\Omega)$ *is* strictly *positive, i.e. there exists a constant* $\alpha_0 > 0$ *such that*

$$((\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\omega, \omega)_\Omega \geq \alpha_0 (\omega, \omega)_\Omega, \quad \forall \omega \in \Lambda^k(\Omega). \tag{29}$$

Again, for the transient problem this assumption is not very restrictive, due to the rescaling argument from Remark 3.3.

Our main result is the following theorem.

Theorem 5.2. *Let* ω, $(\omega_h^n)_{n=0}^N$ *be the solutions of* (15) *and* (26), *respectively. If* $\boldsymbol{\beta} \in \boldsymbol{W}^{2,\infty}(\Omega)$, *Assumption* (5.1) *holds, and, additionally,* $\Lambda_h^k(\mathcal{T})$ *furnishes the approximation property for* $s > 0, r \geq s$:

$$\inf_{\eta \in \Lambda_h^k(T)} |\omega - \eta|_{H^s\Lambda^k(T)} \leq K h^{r+1-s} \|\omega\|_{H^{r+1}\Lambda^k(T)}, \quad \omega \in H^{r+1}\Lambda^k(T), T \in \mathcal{T},$$

with $K > 0$ *independent of* h, *then, for sufficiently small* τ, *we get*

$$\max_{0 \leq n \leq N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq C \left(\|\varepsilon\|_{L^\infty(\Omega)} h^r + h^{r+1} + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} + \tau \right), \tag{30}$$

where $C > 0$ *depends on* K, $\|\partial_t \omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\|\omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\left\|\partial_t^2 \omega(t)\right\|_{L^2\Lambda^k(\Omega)}$, $\|\partial_t \varphi(t)\|_{L^2\Lambda^k(\Omega)}$, $|\partial_t \omega(t)|_\varepsilon$) *and* $\phi(t)$, *but is independent of* τ *and* h.[1]

We give a proof of this theorem in Section 7.

Remark 5.3. *The proof of Theorem 5.2 shows that the term* $\tau^{\frac{1}{2}}$ *in our error estimate* (30) *is due to non-vanishing inflow data. In the case of vanishing inflow data we obtain the estimate:*

$$\max_{0 \leq n \leq N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq C \left(\|\varepsilon\|_{L^\infty(\Omega)} h^r + h^{r+1} + h^{r+1}\tau^{-\frac{1}{2}} + \tau \right), \tag{31}$$

instead of (30). *Further, the implicit dependence of* C *on the diffusion coefficient* ε *due to the higher order norms of the solution* ω *can be removed for sufficiently smooth data, cf. [62, Theorem 5.2].*

---

[1] By the phrase that a constant is *independent of* h we mean that it may only depend on the shape-regularity of the mesh cells, but not on their size.

6. Auxiliary Method of Characteristics

In this section we present a Galerkin method for the *s*tationary advection-diffusion problem

$$
\begin{aligned}
\delta\,\varepsilon\,\mathsf{d}\,\omega + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega,\\
\operatorname{tr}\omega &= \operatorname{tr}\psi_D, && \text{on } \Gamma_{\text{in}},\\
\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega &= \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\psi_D, && \text{on } \Gamma_{\text{in}}\\
\operatorname{tr}\omega &= \operatorname{tr}\psi_D, && \text{on } \Gamma_0.
\end{aligned}
\tag{32}
$$

We use the Cartan formula (7) to introduce a so-called characteristic method for (32). Characteristic methods for the scalar stationary advection problems have been introduced in [14], convergence for the scalar advection problem in $\mathbb{R}^2$ was proved in [7]. Although we do prove convergence for our characteristic method for differential k-forms in $\mathbb{R}^n$ we merely use it as a technical tool; the characteristic methods will be instrumental for the analysis of semi-Lagrangian method for the non-stationary advection-diffusion problem (15).

Fixing $\tau > 0$, we use (21), (23), (24) and (25) and define the characteristic Galerkin scheme for the advection-diffusion problem (32): Find $\omega_h \in \Lambda_h^k(\mathcal{T})$ such that:

$$
\mathsf{b}\left(\omega_h,\eta_h\right) + \mathsf{a}_\tau\left(\omega_h,\eta_h\right) = \left(\varphi,\eta_h\right) + \ell\left(\psi_{\mathrm{D}},\eta_h\right) + \mathsf{g}_\tau\left(\widetilde{\psi}_{\mathrm{D}},\eta_h\right), \quad \forall \eta_h \in \Lambda_h^k(\mathcal{T}).
\tag{33}
$$

The technique to prove convergence for the characteristic methods resembles the analysis of discontinuous Galerkin methods for the scalar problem, see, e.g. [3, 15]. The idea is to prove convergence in some mesh dependent norm.

First, we collect some important results for the advection operator. We define a norm

$$
\begin{aligned}
\|\omega\|_{h,\tau}^2 := \|\omega\|_{L^2\Lambda^k(\Omega)}^2 &+ \frac{1}{2\tau}\left\|\omega - X_{-\tau}^*\omega\right\|_{L^2\Lambda^k(\Omega_0)}^2\\
&+ \frac{1}{2\tau}\|\omega\|_{L^2\Lambda^k(\Omega_{\text{in}})}^2 + \frac{1}{2\tau}\left\|X_{-\tau}^*\omega\right\|_{L^2\Lambda^k(\Omega_{\text{out}})}^2
\end{aligned}
\tag{34}
$$

parametrized by τ and prove stability of $\mathsf{a}(\tau,\cdot)\cdot$ (cf. (23)) in this norm for sufficiently small τ under the Assumption 3.2. By Part b) of Proposition A.1 we know that this condition is an assumption on the velocity field $\boldsymbol{\beta}$.

Lemma 6.1. *Under Assumption* (5.1) *and for sufficiently small $\tau > 0$ we have for all $\omega \in \Lambda_h^k(\mathcal{T}) \cup \Lambda^k(\Omega)$:*

$$
\mathsf{a}_\tau(\omega,\omega) \geq \min(\alpha_0, 1)\,\|\omega\|_{h,\tau}^2,
$$

where α_0 is the constant in Assumption 5.1.

Proof. We find

$$
\begin{aligned}
\mathsf{a}_\tau(\omega,\omega) =& \frac{1}{\tau}(\omega,\omega)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega,\omega\right)_{\Omega_0}\\
=& \frac{1}{2\tau}(\omega,\omega)_{\Omega_0} - \frac{1}{2\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_0}\\
&+ \frac{1}{2\tau}\left(\omega - X_{-\tau}^*\omega, \omega - X_{-\tau}^*\omega\right)_{\Omega_0} + \frac{1}{\tau}(\omega,\omega)_{\Omega_{\text{in}}}\\
=& \frac{1}{2\tau}(\omega,\omega)_\Omega - \frac{1}{2\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_0\cup\Omega_{\text{out}}} + \frac{1}{2\tau}(\omega,\omega)_{\Omega_{\text{in}}}\\
&+ \frac{1}{2\tau}\left(\omega - X_{-\tau}^*\omega, \omega - X_{-\tau}^*\omega\right)_{\Omega_0} + \frac{1}{2\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_{\text{out}}}\\
\geq& \min(\alpha_0, 1)\,\|\omega\|_{h,\tau}^2,
\end{aligned}
$$

where the last estimate follows from assumption (29) by the following identity:

$$(\omega,\omega)_\Omega - \left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_0\cup\Omega_{\mathrm{out}}} = \int_\Omega \omega\wedge\star\omega - \int_\Omega \omega\wedge X_\tau^*\star X_{-\tau}^*\omega. \tag{35}$$

□

The next Lemma gives a continuity estimate for $\mathsf{a}_\tau(\omega,\eta)$.

Lemma 6.2. *For τ sufficiently small we have*

$$\mathsf{a}_\tau(\omega,\eta_h) \le C(\frac{1}{\sqrt{\tau}}+1+\tau)\|\omega\|_{L^2\Lambda^k(\Omega)}\|\eta_h\|_{h,\tau}, \quad \omega\in L^2\Lambda^k(\Omega), \eta_h\in\Lambda_h^k(\mathcal{T}),$$

with $C = C(\boldsymbol{\beta}) \ge 0$ independent of τ and the mesh size h.

Proof. First we rewrite a_τ:

$$\begin{aligned}\mathsf{a}_\tau(\omega,\eta_h) =& \frac{1}{\tau}(\omega,\eta_h)_\Omega + \frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h - \eta_h\right)_{\Omega_0} - \frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h\right)_{\Omega_0}\\ =& \frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h - \eta_h\right)_{\Omega_0} + \frac{1}{\tau}(\omega,\eta_h)_\Omega\\ &- \frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h\right)_{\Omega_0\cup\Omega_{\mathrm{out}}} + \frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h\right)_{\Omega_{\mathrm{out}}}\end{aligned}$$

and then estimate the individual terms in the last sum:

$$\begin{aligned}\left|\frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h-\eta_h\right)_{\Omega_0}\right| &\le \sqrt{\frac{1+C\tau}{\tau}}\|\omega\|_{L^2\Lambda^k(\Omega)}\frac{1}{\sqrt{\tau}}\left\|\eta_h - X_{-\tau}^*\eta_h\right\|_{L^2\Lambda^k(\Omega_0)},\\ \left|\frac{1}{\tau}(\omega,\eta_h)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h\right)_{\Omega_0\cup\Omega_{\mathrm{out}}}\right| &\le C(\boldsymbol{\beta})(1+\tau)\|\omega\|_{L^2\Lambda^k(\Omega)}\|\eta_h\|_{L^2\Lambda^k(\Omega)},\\ \left|\frac{1}{\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta_h\right)_{\Omega_{\mathrm{out}}}\right| &\le \sqrt{\frac{1+C\tau}{\tau}}\|\omega\|_{L^2\Lambda^k(\Omega)}\frac{1}{\sqrt{\tau}}\left\|X_{-\tau}^*\eta_h\right\|_{L^2\Lambda^k(\Omega_{\mathrm{out}})}.\end{aligned}$$

The second estimate is based on the expansion (47) and the bound (46). The first and third estimate use boundedness of the pullback for sufficiently small τ (see Part a) of Proposition A.1):

$$\left\|X_{-\tau}^*\omega\right\|_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})} \le \sqrt{1+C\tau}\,\|\omega\|_{L^2\Lambda^k(\Omega)}.$$

Recalling definition (34) of the norm $\|\cdot\|_{h,\tau}$ we deduce the assertion. □

In contrast to discontinuous Galerkin methods obviously the characteristic method is not consistent. But we can control the consistency error and prove convergence in an energy norm $|||\cdot|||^2 := |\cdot|_\varepsilon^2 + \|\cdot\|^2_{L^2\Lambda^k(\Omega)}$, where $|\cdot|_\varepsilon$ is the semi-norm associated to the bilinear form $\mathsf{b}(\cdot,\cdot)$ defined in (21), i.e.

$$\begin{aligned}&|\omega|_\varepsilon^2 := \mathsf{b}(\omega,\omega) \qquad (36)\\ &= \sum_{T\in\mathcal{T}}(\varepsilon\,\mathsf{d}\,\omega,\mathsf{d}\,\omega)_T + \sum_{f\in\mathcal{F}_0^\partial}\int_f s_f\varepsilon\,\mathrm{tr}\,\mathbf{i}_{\mathbf{n}_f}(\omega\wedge\star\omega) + \sum_{f\in\mathcal{F}^\circ} s_f\varepsilon\,\mathrm{tr}\,\mathbf{i}_{\mathbf{n}_f}([\omega]_f\wedge\star[\omega]_f).\end{aligned}$$

Theorem 6.3. *For $r\in\mathbb{N}$ let $\omega\in H^{\max(2,r+1)}\Lambda^k(\Omega)$ and $\omega_h\in\Lambda_h^k(\mathcal{T})$ be the solutions of the advection-diffusion problem (32) and its characteristic Galerkin discretization (33), respectively. If $\boldsymbol{\beta}\in\boldsymbol{W}^{2,\infty}(\Omega)$, Assumption (5.1) holds and, additionally, for $0\le s\le r$ and $K>0$ independent of ω and h, the approximation space $\Lambda_h^k(\mathcal{T})$ furnishes the approximation property*

$$\inf_{\eta_h\in\Lambda_h^k(T)}|\omega-\eta_h|_{H^s\Lambda^k(T)} \le Kh^{r+1-s}\|\omega\|_{H^{r+1}\Lambda^k(T)}, \quad \omega\in H^{r+1}\Lambda^k(T), T\in\mathcal{T},$$

then, for sufficiently small τ and with $C > 0$ independent of the mesh size $h := \max_T(h_T)$, timestep size τ and diffusion coefficient ε, we get:

$$|||\omega - \omega_h||| \leq C \left(\left\| \sqrt{\varepsilon} \right\|_{L^\infty(\Omega)} h^r + h^{r+1} + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} \right) \|\omega\|_{H^{\max(2,r+1)}\Lambda^k(\Omega)}.$$

Proof. Let $\bar{\omega}_h$ denote the L^2-projection of ω onto $\Lambda_h^k(\mathcal{T})$, then:

$$|||\omega - \omega_h||| \leq |||\omega - \bar{\omega}_h||| + |||\bar{\omega}_h - \omega_h||| \,. \tag{37}$$

Clearly, for the first term on the right hand side the approximation property gives:

$$|||\omega - \bar{\omega}_h||| \leq C \left(\left\| \sqrt{\varepsilon} \right\|_{L^\infty(\Omega)} h^r + h^{r+1} \right) \|\omega\|_{H^{\max(2,r+1)}\Lambda^k(\Omega)}.$$

The rest of the proof targets the second term in (37). The stability estimate of Lemma 6.1 yields

$$\begin{aligned} \min(\alpha_0, 1) \left(|\bar{\omega}_h - \omega_h|_\varepsilon^2 + \|\bar{\omega}_h - \omega_h\|_{h,\tau}^2 \right) \leq \\ \mathsf{b}\left(\bar{\omega}_h - \omega, \bar{\omega}_h - \omega_h\right) + \mathsf{a}_\tau\left(\bar{\omega}_h - \omega, \bar{\omega}_h - \omega_h\right) \\ + \mathsf{b}\left(\omega - \omega_h, \bar{\omega}_h - \omega_h\right) + \mathsf{a}_\tau\left(\omega - \omega_h, \bar{\omega}_h - \omega_h\right). \end{aligned} \tag{38}$$

We find for the consistency error $\mathsf{b}\left(\omega - \omega_h, \eta_h\right) + \mathsf{a}_\tau\left(\omega - \omega_h, \eta_h\right)$, $\eta_h \in \Lambda_h^k(\mathcal{T})$ by the definition of $\mathsf{b}(\cdot,\cdot)$, $\mathsf{a}_\tau(\cdot,\cdot)$, $\mathsf{g}_\tau(\cdot,\cdot)$, $\ell(\cdot,\cdot)$ and $\delta\,\varepsilon\,\mathsf{d}\,\omega + \mathsf{L}_{\boldsymbol{\beta}}\,\omega = \varphi$:

$$\begin{aligned} \left|\mathsf{b}\left(\omega - \omega_h, \eta_h\right) + \mathsf{a}_\tau\left(\omega - \omega_h, \eta_h\right)\right| &= \left|\mathsf{a}_\tau\left(\omega, \eta_h\right) - \mathsf{g}_\tau\left(\widetilde{\psi}_{\mathrm{D}}, \eta_h\right) - \left(\mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h\right)_\Omega\right| \\ &= \left| \frac{1}{\tau}\left(\omega, \eta_h\right)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega, \eta_h\right)_{\Omega_0} - \frac{1}{\tau}\left(\widetilde{\psi}_{\mathrm{D}}, \eta_h\right)_{\Omega_{\mathrm{in}}} - \left(\mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h\right)_\Omega \right| \\ &= \left| \left(\frac{1}{\tau}\left(\omega - X_{-\tau}^*\omega\right) - \mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h \right)_{\Omega_0} + \left(\frac{1}{\tau}\left(\omega - \widetilde{\psi}_{\mathrm{D}}\right) - \mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h \right)_{\Omega_{\mathrm{in}}} \right|. \end{aligned} \tag{39}$$

A bound for the first term in the last inequality follows from (7) and Taylor expansion

$$\frac{1}{\tau}\left(\omega - X_{-\tau}^*\omega\right) - \mathsf{L}_{\boldsymbol{\beta}}\,\omega = \frac{1}{\tau}\int_0^\tau (-s)\frac{\partial^2 X_t^*\omega}{\partial t^2}_{|t=s}\mathrm{d}s = \frac{1}{\tau}\int_0^\tau (-s) X_s^*\,\mathsf{L}_{\boldsymbol{\beta}}^2\,\omega \mathrm{d}s,$$

and we find

$$\left| \left(\frac{1}{\tau}\left(\omega - X_{-\tau}^*\omega\right) - \mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h \right)_{\Omega_0} \right| \leq C\tau\, \|\boldsymbol{\beta}\|_{\boldsymbol{W}^{2,\infty}(\Omega)} \|\omega\|_{H^2\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)}$$

with C independent of h and τ. Recall that $\widetilde{\psi}_{\mathrm{D}}(x) = \left(X_{-t(x)}^*\psi_{\mathrm{D}}\right)_x$ with $X_{-t(x)}(x) \in \Gamma_{\mathrm{in}}$ and $\psi_{\mathrm{D}} = \omega$ on Γ_{in}. For the second term Taylor expansion yields

$$\begin{aligned} \left| \left(\frac{1}{\tau}\left(\omega - \widetilde{\psi}_{\mathrm{D}}\right) - \mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta_h \right)_{\Omega_{\mathrm{in}}} \right| &\leq C \left\|\mathsf{L}_{\boldsymbol{\beta}}\,\omega\right\|_{L^2\Lambda^k(\Omega_{\mathrm{in}})} \|\eta_h\|_{L^2\Lambda^k(\Omega_{\mathrm{in}})} \\ &\leq C\tau^{\frac{1}{2}} \|\boldsymbol{\beta}\|_{\boldsymbol{W}^{1,\infty}(\Omega)} \|\omega\|_{H^1\Lambda^k(\Omega)} \|\eta_h\|_{h,\tau}\,. \end{aligned}$$

The last inequality follow from the definition (34) of the norm $\|\cdot\|_{h,\tau}$. This means that we have the following bound for the consistency error:

$$\begin{aligned} &\left|\mathsf{b}\left(\omega - \omega_h, \bar{\omega}_h - \omega_h\right) + \mathsf{a}_\tau\left(\omega - \omega_h, \bar{\omega}_h - \omega_h\right)\right| \\ &\qquad \leq C\tau^{\frac{1}{2}} \|\boldsymbol{\beta}\|_{\boldsymbol{W}^{2,\infty}(\Omega)} \|\omega\|_{H^2\Lambda^k(\Omega)} \|\bar{\omega}_h - \omega_h\|_{h,\tau}\,. \end{aligned} \tag{40}$$

The continuity estimate in Lemma 6.2 and the approximation property of $\Lambda_h^k(\mathcal{T})$ give:

$$\mathsf{a}_\tau\left(\bar{\omega}_h - \omega, \bar{\omega}_h - \omega_h\right) \leq C\tau^{-\frac{1}{2}} h^{r+1} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)} \|\bar{\omega}_h - \omega_h\|_{h,\tau}\,. \tag{41}$$

Since the bilinear form $\mathsf{b}(\cdot,\cdot)$ arises from the non-symmetric interior penalty method for differential k-forms, we can use inverse inequalities and multiplicative trace inequalities (Proposition A.3) to establish the following estimate completely analoguous to the case $k=0$ (see e.g. [33, Theorem 4.5] or [4, Section 5]).

$$\mathsf{b}(\bar{\omega}_h-\omega,\bar{\omega}_h-\omega_h) \le C\left\|\sqrt{\varepsilon}\right\|_{L^\infty(\Omega)} h^r \|\omega\|_{H^{r+1}\Lambda^k(\Omega)} |\bar{\omega}_h-\omega_h|_\varepsilon\,. \tag{42}$$

Combining the estimates (40), (41) and (42) with (38) yields:

$$|||\bar{\omega}_h-\omega_h||| \le C\left(\left\|\sqrt{\varepsilon}\right\|_{L^\infty(\Omega)} h^r + h^{r+1}\right)\|\omega\|_{H^{\max(2,r+1)}\Lambda^k(\Omega)},$$

which, in turn proves together with (37) the assertion. □

The L^2-estimate that follows from Theorem 6.3 is suboptimal for $\varepsilon>0$. But if we restrict ourselves to a certain class of conforming approximation spaces $\Lambda_h^k(\mathcal{T})\subset H\Lambda^k(\Omega)$, we can improve this result. For instance we may choose $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$, where the spaces $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})\subset H\Lambda^k(\Omega)$, $0<k<n$ and $r\ge 0$, are defined in [6, section 5]. Then there exist projection operators $I_h^k:\Lambda^k(\Omega)\mapsto\Lambda_h^k(\mathcal{T})$ such that

$$\begin{aligned} \mathsf{d}\, I_h^k &= I_h^{k+1}\,\mathsf{d}\,; \\ \left\|\omega - I_h^k\omega\right\|_{L^2\Lambda^k(\Omega)} &\le Kh^{r+1}\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}, \quad \omega\in H^{r+1}\Lambda^k(\Omega)\,; \\ \left\|\mathsf{d}\,\omega - \mathsf{d}\, I_h^k\omega\right\|_{L^2\Lambda^k(\Omega)} &\le Kh^{r+1}\|\mathsf{d}\,\omega\|_{H^{r+1}\Lambda^k(\Omega)}, \quad \omega\in H^{r+1}\Lambda^k(\Omega)\,, \end{aligned} \tag{43}$$

for $K>0$ independent of ω and h (see [6, Theorem 5.9] and [18] for the case of essential boundary conditions). In $\mathbb{R}^3$ the space $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ corresponds to Nédélec's first family [44] of $\boldsymbol{H}(\mathbf{curl},\Omega)$ $(k=1)$ and $\boldsymbol{H}(\mathrm{div},\Omega)$ $(k=2)$ conforming spaces.

Theorem 6.4. *Let $0<k<n$, and for $r\in\mathbb{N}$ let $\omega\in H^{\max(2,r+1)}\Lambda^k(\Omega)$ and $\omega_h\in\Lambda_h^k(\mathcal{T})$, with $\operatorname{tr}\omega_{h|\Gamma_0} = \operatorname{tr}\psi_{\mathrm{D}|\Gamma_0}$, be the solutions to the advection-diffusion problem* (32) *and its discrete variational formulation* (33)*, respectively. If $\boldsymbol{\beta}\in\boldsymbol{W}^{2,\infty}(\Omega)$, Assumption* (5.1) *holds and, additionally, the approximation space $\Lambda_h^k(\mathcal{T})$ has the properties* (43)*, for sufficiently small τ we get:*

$$\begin{aligned} |||\omega-\omega_h||| \le C\left(h^{r+1}+h^{r+1}\tau^{-\frac{1}{2}}+\tau^{\frac{1}{2}}\right)\|\omega\|_{H^{\max(2,r+1)}\Lambda^k(\Omega)} \\ + C\left\|\sqrt{\varepsilon}\right\|_{L^\infty(\Omega)} h^{r+1}\|\mathsf{d}\,\omega\|_{H^{r+1}\Lambda^k(\Omega)}, \end{aligned}$$

with $C>0$ independent of mesh size $h:=\max_T(h_T)$, timestep size τ and diffusion coefficient ε,

Proof. The proof follows the lines of the proof of Theorem 6.3 with $\bar{\omega}_h := I_h^k\omega$, where I_h^k is the projection operator onto $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ with the properties (43). Since $\operatorname{tr}[\omega_h]_f = 0$ for $\omega_h\in\Lambda_h^k(\mathcal{T})\subset H\Lambda^k(\Omega)$ the assertion follows from the approximation properties of I_h^k. □

7. Semi-Lagrangian Galerkin Scheme: Convergence

We would like to stress that the Semi-Lagrangian Galerkin schemes (26) resembles an explicit Eulerian scheme, if $\varepsilon=0$. In light of this similarity it is very likely that this scheme converges at least for sufficiently small timesteps τ also for lowest order spatial approximations. Moreover, we can even prove convergence for sufficiently small timesteps under the assumption that $\mathsf{a}_\tau(\cdot,\cdot)$ allows for a Galerkin projector. Now we prove our main result, Theorem 5.2.

Proof of Theorem 5.2. By Theorem 6.3 we have a Ritz-Galerkin projection $P_h\omega(t^n) \in \Lambda_h^k(\mathcal{T})$ with

$$\mathsf{b}\left(P_h\omega(t^n),\eta_h\right)+\mathsf{a}_\tau\left(P_h\omega(t^n),\eta_h\right) = \left(\varphi(t^n),\eta_h\right)+\ell\left(\psi_{\mathrm{D}}(t^n),\eta_h\right)+\mathsf{g}_\tau\left(\widetilde{\psi}_{\mathrm{D}}^n,\eta_h\right),$$

for all $\eta_h \in \Lambda_h^k(\mathcal{T})$ that fulfills the estimate

$$|||\omega(t^n)-P_h\omega(t^n)||| \leq \\ C_1\left(\left\|\sqrt{\varepsilon}\right\|_{L^\infty(\Omega)}h^r+h^{r+1}+h^{r+1}\tau^{-\frac{1}{2}}+\tau^{\frac{1}{2}}\right)\left\|\omega(t^n)\right\|_{H^m\Lambda^k(\Omega)}$$

for $m=\max(2,r+1)$ and $C_1>0$ independent of h. Let $\bar{\omega}_h^n := P_h\omega(t^n)$, then

$$\mathsf{b}\left(\bar{\omega}_h^{n+1},\eta_h\right)+\frac{1}{\tau}\left(\bar{\omega}_h^{n+1}-\bar{\omega}_h^n,\eta_h\right)_\Omega+\mathsf{a}_\tau\left(\bar{\omega}_h^n,\eta_h\right) = \\ \left(\varphi(t^{n+1}),\eta_h\right)+\ell\left(\psi_{\mathrm{D}}(t^n),\eta_h\right)+\mathsf{g}_\tau\left(\widetilde{\psi}_{\mathrm{D}}^n,\eta_h\right)+\left(R^{n+1},\eta_h\right)_\Omega+\mathsf{b}\left(\bar{\omega}_h^{n+1}-\bar{\omega}_h^n,\eta_h\right),$$

with

$$\left(R^{n+1},\eta_h\right)_\Omega = \left(\frac{1}{\tau}(\bar{\omega}_h^{n+1}-\bar{\omega}_h^n)-\partial_t\omega(t^n),\eta_h\right)_\Omega+\left(\varphi(t^n)-\varphi(t^{n+1}),\eta_h\right)_\Omega.$$

We define $\gamma_h^n := \bar{\omega}_h^n-\omega_h^n$ and find:

$$\mathsf{b}\left(\gamma_h^{n+1},\eta_h\right)+\frac{1}{\tau}\left(\gamma_h^{n+1}-\gamma_h^n,\eta_h\right)_\Omega+\mathsf{a}_\tau\left(\gamma_h^n,\eta_h\right) = \left(R^{n+1},\eta_h\right)_\Omega+\mathsf{b}\left(\bar{\omega}_h^{n+1}-\bar{\omega}_h^n,\eta_h\right),$$

for all $\eta_h \in \Lambda_h^k(\mathcal{T})$, or, equivalently

$$\mathsf{b}\left(\gamma_h^{n+1},\eta_h\right)+\frac{1}{\tau}\left(\gamma_h^{n+1}-X_{-\tau}^*\gamma_h^n,\eta_h\right)_\Omega = \left(R^{n+1},\eta_h\right)_\Omega+\mathsf{b}\left(\bar{\omega}_h^{n+1}-\bar{\omega}_h^n,\eta_h\right).$$

We take $\eta_h = 2\tau\gamma_h^{n+1}$, use $2p(p-q)=p^2+(p-q)^2-q^2$ and the definition of the semi-norm $|\cdot|_\varepsilon$ and boundedness of $\mathsf{b}(\cdot,\cdot)$ in that semi-norm (see [4, Section 4] or [3, Lemma 2.2]):

$$\begin{aligned}
2\tau\left|\gamma_h^{n+1}\right|_\varepsilon^2&+\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2+\left\|\gamma_h^{n+1}-X_{-\tau}^*\gamma_h^n\right\|_{L^2\Lambda^k(\Omega)}^2\\
&\leq \left\|X_{-\tau}^*\gamma_h^n\right\|_{L^2\Lambda^k(\Omega)}^2+2\tau\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)}\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}\\
&\qquad+C_b2\tau\left|\bar{\omega}_h^{n+1}-\bar{\omega}_h^n\right|_\varepsilon\left|\gamma_h^{n+1}\right|_\varepsilon\\
&\leq \left\|X_{-\tau}^*\gamma_h^n\right\|_{L^2\Lambda^k(\Omega)}^2+\frac{\tau}{\kappa}\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2+\kappa\tau\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2\\
&\qquad+C_b^2\frac{\tau}{2}\left|\bar{\omega}_h^{n+1}-\bar{\omega}_h^n\right|_\varepsilon^2+2\tau\left|\gamma_h^{n+1}\right|_\varepsilon^2,
\end{aligned}$$

for $\kappa>0$. By part a) of Proposition A.1, $\tau \leq \frac{1}{\kappa}$, we deduce

$$\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2 \leq \frac{1+C_2\tau}{1-\kappa\tau}\left\|\gamma_h^n\right\|_{L^2\Lambda^k(\Omega)}^2 \\ +\frac{\tau}{(1-\kappa\tau)\kappa}\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2+\frac{\tau C_b^2}{2(1-\kappa\tau)}\left|\bar{\omega}_h^{n+1}-\bar{\omega}_h^n\right|_\varepsilon^2.$$

From the definitions of R^n and $\bar{\omega}_h^n$ we infer

$$\begin{aligned}
\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)} \leq{}& C_3\tau\max_{t\in[0,T]}\left(\left\|\partial_t^2\omega(t)\right\|_{L^2\Lambda^k(\Omega)}+\left\|\partial_t\varphi(t)\right\|_{L^2\Lambda^k(\Omega)}\right)\\
&+C_4\left(\left\|\sqrt{\varepsilon}\right\|_{L^\infty(\Omega)}h^r+h^{r+1}+h^{r+1}\tau^{-\frac{1}{2}}+\tau^{\frac{1}{2}}\right)\max_{t\in[0,T]}\left\|\partial_t\omega(t)\right\|_{H^m\Lambda^k(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}\left|\bar{\omega}_h^{n+1} - \bar{\omega}_h^n\right|_\varepsilon &\le \left|\bar{\omega}_h^{n+1} - \omega(t^{n+1})\right|_\varepsilon + \left|\bar{\omega}_h^n - \omega(t^n)\right|_\varepsilon + \left|\omega(t^{n+1}) - \omega(t^n)\right|_\varepsilon \\ &\le C_5 \left(\left\|\sqrt{\varepsilon}\right\|_{L^\infty(\Omega)} h^r + h^{r+1} + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} \right) \max_{t\in[0,T]} \|\omega(t^n)\|_{H^m\Lambda^k(\Omega)} \\ &\quad + C_6\tau \max_{t\in[0,T]} |\partial_t \omega(t)|_\varepsilon ,\end{aligned}$$

hence, the assertion (30) follows from triangle inequality, Theorem 6.3 and a discrete Gronwall-like inequality: If a sequence of non-negative numbers satisfies

$$\begin{aligned} b_0 &= a_0 \\ b_{n+1} &\le a_{n+1} + (1 + C\tau) b_n, \quad C > 0 \end{aligned}$$

then we can infer

$$b_N \le \frac{e^{CN\tau} - 1}{C\tau} \max_{1\le i\le N} a_i + e^{CN\tau} b_0.$$

We can apply this with $b_n = \|\gamma_h^n\|_{L^2\Lambda^k(\Omega)}$ and note $n\tau \le T$. □

For the case of the conforming approximation spaces $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ we obtain a similar result.

Theorem 7.1. *Let* $0 < k < n$. *Let* ω, $(\omega_h^n)_{n=0}^N$ *be the solutions of* (15) *and* (26), *respectively. If* $\boldsymbol{\beta} \in \boldsymbol{W}^{2,\infty}(\Omega)$, *Assumption* (5.1) *holds and additionally* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ *[6, section 5], we get for sufficiently small* τ

$$\max_{0\le n\le N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \le C \left(\|\varepsilon\|_{L^\infty(\Omega)} h^{r+1} + h^{r+1} + h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} + \tau \right),$$

where C *depends on* $\|\partial_t\omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\|\partial_t \,\mathsf{d}\, \omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\|\omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\|\mathsf{d}\,\omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\|\omega(t)\|_{H^m\Lambda^k(\Omega)}$, $\left\|\partial_t^2\omega(t)\right\|_{L^2\Lambda^k(\Omega)}$, $\|\partial_t\varphi(t)\|_{L^2\Lambda^k(\Omega)}$, $|\partial_t\omega(t)|_\varepsilon$, *for* $m = \max(2, r+1)$, *but is independent of* τ *and* h.

Proof. The proof is analogous to the proof of the previous theorem taking into account the approximation result of Theorem 6.4. □

Remark 7.2. *Theorems 5.2 and 7.1 give convergence for* $\tau = O(h)$, *lowest order spatial approximation spaces and* $\varepsilon = 0$. *Assumption* (5.1) *can always be taken for granted due to the rescaling argument of Remark 3.3. Proofs of convergence for the semi-Lagrangian scheme for the rescaled variables follow the same lines as the proof of Theorem 5.2, and, in particular, we can establish convergence for* $\tau = O(h)$ *and lowest order approximation spaces.*

Remark 7.3. *The assumption that* $\left(X_{-\tau}^*\omega_h, \eta_h\right)_{\Omega_0}$, $\omega_h, \eta_h \in \Lambda_h^k(\mathcal{T})$, *can be computed exactly is not crucial in our analysis. As in the standard, but non-optimal, analysis of fully discrete semi-Lagrangian methods [29] we can replace the exact flow* X_τ *with a consistent approximation* $\bar{X}_\tau$, *such that*

$$\left\|X_\tau - \bar{X}_\tau\right\|_{\boldsymbol{L}^\infty(\Omega)} \le O(h^{l_1+1}\tau + \tau^{l_2}) \quad \text{and} \quad \left\|X_\tau - \bar{X}_\tau\right\|_{\boldsymbol{W}^{1,\infty}(\Omega)} \le O(h^{l_1}\tau + \tau^{l_2}).$$

for $l_1, l_2 \ge 1$ *and* $h \to 0$ *and* $\tau \to 0$. *In this case we use* $\bar{X}_\tau$ *instead of* X_τ *to define the bilinear form* $\mathsf{a}_\tau(\cdot,\cdot)$ *in* (23) *and the norm* $|\cdot|_{h,\tau}$ *in* (34). *Then the analysis for the convergence estimate for the stationary problem need to be modified. The bound* (39) *of the consistency error* (38) *contains then the additional term* $\left((X_{-\tau}^* - \bar{X}_{-\tau}^*)\omega, \eta_h\right)_\Omega$. *This additional term can be bounded by*

$$\left((X_{-\tau}^* - \bar{X}_{-\tau}^*)\omega, \eta_h\right)_\Omega \le C_l(h^{l_1}\tau + \tau^{l_2})\|\omega\|_{H^1\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)}$$

since a similar argument as in the proof of Proposition A.1 (see also [29, Lemma 5.1]) gives:

$$\left\|X_{-\tau}^*\omega - \bar{X}_{-\tau}^*\omega\right\|_{L^2\Lambda^k(\Omega)}^2 \leq$$
$$C\left|X_{-\tau} - \bar{X}_{-\tau}\right|_{\boldsymbol{W}^{1,\infty}(\Omega)}^2 \|\omega\|_{L^2\Lambda^k(\Omega)}^2 + C\left\|X_{-\tau} - \bar{X}_{-\tau}\right\|_{\boldsymbol{L}^\infty(\Omega)}^2 |\omega|_{H^1\Lambda^k(\Omega)}^2.$$

Details on the construction of such approximate flow maps can be found in [29, Section 5]. The construction relies on the nodal basis functions spanning the space of continuous piecewise polynomial Lagrangian finite element functions and approximations of the trajectories of the degrees of freedoms corresponding to the basis functions.

Remark 7.4. *For non-vanishing diffusion coefficient ε our estimates* (30) *and* (31) *are sub-optimal when compared to the theoretical results in [19, Theorem 3.3], [61] and [63, Theorem 4.1]. And even though these estimates assume the exact calculation of the integrals of type $\int_\Omega u(X_{-\tau}(x))v(x)dx$, there are numerical experiments with fully discrete semi-Lagrangian Galerkin schemes, see e.g. [61, Section 9.1], [64, Section 6.1] or [63, Section 5.1], that suggest that these results hold also for the perturbed methods. Nevertheless, for fully discrete semi-Lagrangian methods a rigorous proof of an estimate of order $O(\tau + h^2)$ in L^2 remains elusive.*

Though in the case of vanishing diffusion coefficient ε many numerical experiments (see [29, Sections 6.1, 6.3], [64, Section 6.2]) hint at a higher order of convergence than our estimates (30) *and* (31)*, we believe that at least in this case our results are sharp on simplicial meshes. Probably the superconvergence behaviour observed in these experiments is closely related to superconvergence of stabilized Galerkin methods for scalar advection. Peterson [47] and Zhou [66] showed that the usual error estimates of order $O(h^{r+\frac{1}{2}})$ in L^2 for the stabilized discontinuous Galerkin method [15, 36, 51] and the SUPG/SDFEM method [22, 34, 43] are sharp. This result carries over to Eulerian discretizations [31, Section 5.1], that can be seen as perturbations of our semi-Lagrangian Galerkin schemes (see [31, Section 4.2] or [28, Lemma 4.2]).*

Appendix A. Auxiliary Estimates

We exploit the close relationship of the operator $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ and the bilinear form $\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta\right)_{X_\tau(\Omega)}$. In Section 3 we use the result of this section for $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ to prove well-posedness of (15). Also the analysis of the auxiliary method of characteristics introduced in Section 6 is based on the following results for $\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta\right)_{X_\tau(\Omega)}$.

Proposition A.1. *Let $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$ and $\omega, \eta \in L^2\Lambda^k(\Omega)$, then*

a) *we have the estimate*

$$\left|\left(X_{-\tau}^*\omega, X_{-\tau}^*\eta\right)_{X_\tau(\Omega)}\right| \leq C(DX_{-\tau}) \|\omega\|_{L^2\Lambda^k(\Omega)} \|\eta\|_{L^2\Lambda^k(\Omega)}, \tag{44}$$

with $C(DX_{-\tau}) = 1 + \tau C(D\boldsymbol{\beta})$ for τ sufficiently small;

b) *the operator $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ is symmetric:*

$$(\omega, (\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\eta)_\Omega = ((\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\omega, \eta)_\Omega, \quad \omega, \eta \in L^2\Lambda^k(\Omega). \tag{45}$$

and we have the estimate

$$\left|(\omega, (\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\eta)_\Omega\right| \leq C\,|\boldsymbol{\beta}|_{\boldsymbol{W}^{1,\infty}(\Omega)} \|\omega\|_{L^2\Lambda^k(\Omega)} \|\eta\|_{L^2\Lambda^k(\Omega)}. \tag{46}$$

If $\boldsymbol{\beta} \in \boldsymbol{W}^{2,\infty}(\Omega)$ and $\omega, \eta \in L^2\Lambda^k(\Omega)$

c) *we have the expansion*

$$\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)} = (\omega,\eta)_\Omega - \tau\,(\omega, (\mathsf{L}_{\boldsymbol\beta} + \mathcal{L}_{\boldsymbol\beta})\eta)_\Omega + R(\boldsymbol\beta,\tau)\,(\omega,\eta)_\Omega\,, \tag{47}$$

with $\|R(\boldsymbol\beta,\tau)\| \leq C(\boldsymbol\beta)\tau^2$ *independent of* ω *and* η.

Proof. 1.) We first examine the special case of Ω beeing a domain in $\mathbb{R}^3$:
The results follow directly from the corresponding vector proxy representations from Table 2: While the assertion b) is obvious in $\mathbb{R}^3$, we recall that

$$\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)} = \int_\Omega \omega \wedge X^*_\tau \star X^*_{-\tau}\eta$$

and hence, we find for differential forms ω in $\mathbb{R}^3$ with vector correspondences u or $\mathbf{u}$:

$$\begin{aligned}
k=0:&\quad (X^*_\tau \star X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))u(x),\\
k=1:&\quad (X^*_\tau \star X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))DX_\tau^{-1}(x)DX_\tau^{-T}(x)\mathbf{u}(x),\\
k=2:&\quad (X^*_\tau \star X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))^{-1}DX_\tau^{T}(x)DX_\tau(x)\mathbf{u}(x),\\
k=3:&\quad (X^*_\tau \star X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))^{-1}u(x),
\end{aligned}$$

which yields the assertion a). Taylor expansion of $\omega \wedge X^*_\tau \star X^*_{-\tau}\omega$ in τ finally proves assertion c).

2.) General case (see also [29, Lemma 4.1]:
The proof for the general case is very similar, but involves certain technical notations from tensor calculus, if one aims at explicit formulas for the operator $\mathsf{L}_{\boldsymbol\beta} + \mathcal{L}_{\boldsymbol\beta}$ and the constants. By density of $\Lambda^k(\Omega)$ in $L^2\Lambda^k(\Omega)$ it is enough to prove the assertions for smooth $\eta, \omega \in \Lambda^k(\Omega)$.
a) By multi-linearity we have for orthonormal vector fields $\mathbf{e}_1, \dots \mathbf{e}_n$ and $\sigma \in S(j,n)$, $\gamma \in \Lambda^j(\Omega)$ and $x \in \Omega$ [56, Page 610]:

$$\begin{aligned}
&(X^*_\tau\gamma)_x(\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(j)})\\
&\qquad = \sum_{\sigma' \in S(j,n)} \det\left((DX_\tau(x))_{\sigma',\sigma}\right)\gamma_{X_\tau(x)}(\mathbf{e}_{\sigma'(1)}, \dots, \mathbf{e}_{\sigma'(j)}),
\end{aligned} \tag{48}$$

where the quantities $\det\left((DX_\tau(x))_{\sigma',\sigma}\right)$ are known as the j-minors of the differential $DX_\tau(x)$ with respect to $\mathbf{e}_1, \dots, \mathbf{e}_n$, i.e. the determinants of those submatrices of $DX_\tau(x)$, that contain the rows σ' and columns σ. By the definition of the inner product of differential forms we have

$$\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)} = \int_{X_\tau(\Omega)} X^*_{-\tau}\omega \wedge \star X^*_{-\tau}\eta = \int_{X_\tau(\Omega)} \left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)\mu.$$

Hence, by the definition of the inner product of alternating forms and (48), we find

$$\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{X_\tau(\Omega)} = (\det(DX_\tau)\mathbf{M}_k(DX_\tau)\omega, \mathbf{M}_k(DX_\tau)\eta)_\Omega \tag{49}$$

with

$$(\mathbf{M}_j(DX_\tau)\gamma)_x(\mathbf{e}_{\sigma(1)}, \dots, \mathbf{e}_{\sigma(j)}) := \sum_{\sigma' \in S(j,n)} \det\left((DX_\tau(x))_{\sigma',\sigma}\right)\gamma_x(\mathbf{e}_{\sigma'(1)}, \dots, \mathbf{e}_{\sigma'(j)}).$$

This proves the assertion.
b.) We consider $\bar\eta$ and $\bar\omega$ to be extensions of η and ω to $\Lambda^k(\mathbb{R}^n)$ and assume $\boldsymbol\beta \in C^\infty(\Omega)$. First, Cartan's formula (7) and compatibility of exterior product and

pullback yield:

$$\begin{aligned}\bar{\eta}\wedge\star(\mathsf{L}_{\boldsymbol{\beta}}+\mathcal{L}_{\boldsymbol{\beta}})\bar{\omega} &= \lim_{\tau\to 0}\frac{1}{\tau}\left(\bar{\eta}\wedge(\star X_\tau^*\bar{\omega}-\star\bar{\omega})-\bar{\eta}\wedge(X_\tau^*\star\bar{\omega}-\star\bar{\omega})\right)\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\bar{\eta}\wedge(\star X_\tau^*\bar{\omega}-X_\tau^*\star\bar{\omega})\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\left(\bar{\eta}\wedge\star X_\tau^*\bar{\omega}-X_\tau^*(X_{-\tau}^*\bar{\eta}\wedge\star\bar{\omega})\right)\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\left(\,(\bar{\eta},X_\tau^*\bar{\omega})\,\mu - X_\tau^*\left(X_{-\tau}^*\bar{\eta},\bar{\omega}\right)\mu\right).\end{aligned}$$

From the Taylor expansion $DX_\tau(x)=\mathsf{id}+\tau D\boldsymbol{\beta}(x)+O(\tau^2)$ of $DX_\tau(x)$ around $\tau=0$ and the Taylor expansion of det(), $\det(\mathbf{A}+\varepsilon\mathbf{B})=\det(\mathbf{A})+\varepsilon\,\mathrm{tr}(\mathsf{Adj}(\mathbf{A})\mathbf{B})+O(\varepsilon^2)$ and (48) we infer

$$\begin{aligned}(X_\tau^*\gamma)_x(\mathbf{e}_{\sigma(1)},\dots,\mathbf{e}_{\sigma(j)}) &= \sum_{\sigma'\in S(j,n)}\det((I_n)_{\sigma',\sigma})\gamma_{X_\tau(x)}(\mathbf{e}_{\sigma'(1)},\dots,\mathbf{e}_{\sigma'(j)}),\\ &+\tau\sum_{\sigma'\in S(j,n)}\mathrm{tr}\left(\mathsf{Adj}((I_n)_{\sigma',\sigma})(D\boldsymbol{\beta}_x)_{\sigma',\sigma}\right)\gamma_{X_\tau(x)}(\mathbf{e}_{\sigma'(1)},\dots,\mathbf{e}_{\sigma'(j)})+O(\tau^2),\end{aligned}\tag{50}$$

with Adj and tr the adjugate and trace operator for matrices, and the unit matrix $I_n\in\mathbb{R}^{n\times n}$. Introducing the abbreviation

$$\begin{aligned}&\left(\mathbf{M}'_j(X_\tau)\gamma\right)_x(\mathbf{e}_{\sigma(1)},\dots,\mathbf{e}_{\sigma(j)}) =\\ &\qquad\sum_{\sigma'}\mathrm{tr}\left(\mathsf{Adj}((I_n)_{\sigma',\sigma})(D\boldsymbol{\beta}_x)_{\sigma',\sigma}\right)\gamma_{X_\tau(x)}(\mathbf{e}_{\sigma'(1)},\dots,\mathbf{e}_{\sigma'(j)}),\end{aligned}\tag{51}$$

we find:

$$\begin{aligned}&(\bar{\eta}\wedge\star(\mathsf{L}_{\boldsymbol{\beta}}+\mathcal{L}_{\boldsymbol{\beta}})\bar{\omega})_x\\ &\quad=\lim_{\tau\to 0}\left(\,(\bar{\eta}_x,(\mathbf{M}'_k(X_\tau)\bar{\omega})_x)\,\mu+X_\tau^*\,((\mathbf{M}'_k(X_{-\tau})\bar{\eta})_x,\bar{\omega}_x)\,\mu\right)\\ &\qquad+\lim_{\tau\to 0}\frac{1}{\tau}\left(\left(\bar{\eta}_x,\bar{\omega}_{X_\tau(x)}\right)\mu-X_\tau^*\left(\bar{\eta}_{X_{-\tau}(x)},\bar{\omega}_x\right)\mu\right)\\ &\quad=\ (\bar{\eta}_x,(\mathbf{M}'_k(X_0)\bar{\omega})_x)\,\mu+\,((\mathbf{M}'_k(X_0)\bar{\eta})_x,\bar{\omega}_x)\,\mu\\ &\qquad+\lim_{\tau\to 0}\frac{1}{\tau}\left(\left(\bar{\eta}_x,\bar{\omega}_{X_\tau(x)}\right)(\mu-X_\tau^*\mu)\right)\\ &\quad=\ (\bar{\eta}_x,(\mathbf{M}'_k(X_0)\bar{\omega})_x)\,\mu+\,((\mathbf{M}'_k(X_0)\bar{\eta})_x,\bar{\omega}_x)\,\mu-\,(\bar{\eta}_x,\bar{\omega}_x)\,\mathbf{M}'_n(X_0)\mu.\end{aligned}\tag{52}$$

This result holds for any extension of ω and η and the assertion follows by density of $\Lambda^k(\Omega)$ in $L^2\Lambda^k(\Omega)$, since $\mathbf{M}'_k(\cdot)$ depends only on the Jacobian of $\boldsymbol{\beta}$. Thus we see that $\boldsymbol{\beta}\in\boldsymbol{W}^{1,\infty}(\Omega)$ is the minimal smoothness assumption for $\boldsymbol{\beta}$.

c) First, we see that:

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(X_\tau^*\omega,X_\tau^*\eta\right)_{X_{-\tau}(\Omega)|_{\tau=0}} &= \lim_{\tau\to 0}\frac{1}{\tau}\left((\omega,\eta)_\Omega-\left(X_{-\tau}^*\omega,X_{-\tau}^*\eta\right)_{X_\tau(\Omega)}\right)\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\left((\omega,\eta)_\Omega-\left(X_{-\tau}^*\omega,\eta\right)_{X_\tau(\Omega)}\right)\\ &\quad+\lim_{\tau\to 0}\frac{1}{\tau}\left(\left(X_{-\tau}^*\omega,\eta\right)_{X_\tau^*(\Omega)}-\left(X_{-\tau}^*\omega,X_{-\tau}^*\eta\right)_{X_\tau(\Omega)}\right)\\ &= \lim_{\tau\to 0}\frac{1}{\tau}\int_\Omega\omega\wedge(\star\eta-X_\tau^*\star\eta)\\ &\quad+\lim_{\tau\to 0}\frac{1}{\tau}\int_\Omega\omega\wedge X_\tau^*\star\left(\eta-X_{-\tau}^*\eta\right)\\ &= (\omega,\mathcal{L}_{\boldsymbol{\beta}}\,\eta)_\Omega+\,(\omega,\mathsf{L}_{\boldsymbol{\beta}}\,\eta)_\Omega\,.\end{aligned}$$

Then the assertion follows by the Taylor expansion of (49). □

While the previous result is important in the treatment of the advection terms, the treatment of the diffusion requires certain multiplicative trace inequalities.

Recall that (12) implies an integration by parts formula for $\omega \in \Lambda^k(\Omega)$ and $\eta \in \Lambda^{k+1}(\Omega)$ on a bounded domain Ω:

$$\int_{\partial\Omega} \operatorname{tr}\omega \wedge \operatorname{tr}\star\eta = (\mathsf{d}\,\omega, \eta)_\Omega - (\omega, \mathsf{d}\,\eta)_\Omega .$$

Observe that the right hand side is not a semidefinite bilinear form. Nevertheless we have a Cauchy-Schwarz type inequality:

Proposition A.2. *Let Ω be a bounded domain with outward normal $\mathbf{n}_\Omega$. Then, we define a semi-norm for $\omega \in \Lambda^k(\Omega)$ by*

$$|\omega|^2_{\partial\Omega,\mathrm{tr}} := \int_{\partial\Omega} \operatorname{tr} \mathsf{i}_{\mathbf{n}_\Omega}(\omega \wedge \star\omega), \tag{53}$$

and have

$$\left|\int_{\partial\Omega} \operatorname{tr}\omega \wedge \operatorname{tr}\star\eta\right| \le |\omega|_{\partial\Omega,\mathrm{tr}}\, |\eta|_{\partial\Omega,\mathrm{tr}}\,, \quad \omega \in \Lambda^k(\Omega)\,, \eta \in \Lambda^{k+1}(\Omega) \tag{54}$$

for $\omega \in \Lambda^k(\Omega)$ and $\eta \in \Lambda^{k+1}(\Omega)$.

Proof. According to [57, proposition 1.2.6] we have:

$$\operatorname{tr}\omega \wedge \operatorname{tr}\star\eta = (\omega, \mathsf{i}_{\mathbf{n}_\Omega}\,\eta)\, \mathsf{i}_{\mathbf{n}_\Omega}\,\mu,$$

where μ is the volume form of Ω. Hence, the assertion follows from the standard Cauchy-Schwarz inequality for the scalar product of alternating k-forms and

$$(\omega, \omega)\, \mathsf{i}_{\mathbf{n}_\Omega}\,\mu = \mathsf{i}_{\mathbf{n}_\Omega}(\omega, \omega)\mu = \mathsf{i}_{\mathbf{n}_\Omega}(\omega \wedge \star\omega),$$

because, certainly, $(\mathsf{i}_{\mathbf{n}_\Omega}\,\eta, \mathsf{i}_{\mathbf{n}_\Omega}\,\eta) \le (\omega, \omega)$. $\square$

The next proposition states a multiplicative trace inequality (cf. [1, Theorem 3.10]) for the semi-norm $|\cdot|_{\partial\Omega,\mathrm{tr}}$ for a convex polygonal domain Ω.

Proposition A.3. *Assume that Ω is a convex polygonal domain. Let h_Ω be the radius of the smallest n-dimensional ball that contains Ω and ρ_Ω the radius of the largest n-dimensional ball that is contained in Ω. Then we have:*

$$|\omega|^2_{\partial\Omega,\mathrm{tr}} \le 2\frac{h_\Omega}{\rho_\Omega}\, \|\omega\|_{L^2\Lambda^k(\Omega)}\, |\omega|_{H^1\Lambda^k(\Omega)} + \frac{n}{\rho_\Omega}\, \|\omega\|^2_{L^2\Lambda^k(\Omega)}\,. \tag{55}$$

Proof. Without loss of generality, we suppose that the center $\bar{x}$ of the largest inscribed ball is the origin of the coordinate system. We start from the following relation:

$$\int_{\partial\Omega} \operatorname{tr} \mathsf{i}_{\mathbf{x}}(\omega \wedge \star\omega) = \int_\Omega \mathsf{d}\, \mathsf{i}_{\mathbf{x}}(\omega \wedge \star\omega).$$

On the one hand we have the lower bound:

$$\int_{\partial\Omega} \operatorname{tr} \mathsf{i}_{\mathbf{x}}(\omega \wedge \star\omega) \ge \min_{\mathbf{x}\in\partial\Omega}(\mathbf{x}\cdot\mathbf{n}_\Omega(\mathbf{x})) \int_{\partial\Omega} \operatorname{tr} \mathsf{i}_{\mathbf{n}_\Omega}(\omega \wedge \star\omega) = \rho_\Omega\, |\omega|^2_{\partial\Omega,\mathrm{tr}}\,, \tag{56}$$

because $\int_{\partial\Omega} \operatorname{tr} \mathsf{i}_{\mathbf{x}}\,\mu = \int_{\partial\Omega} \mathbf{x}\cdot\mathbf{n}_\Omega\, \mathsf{i}_{\mathbf{n}_\Omega}\,\mu$. Moreover

$$\int_\Omega \mathsf{d}\, \mathsf{i}_{\mathbf{x}}(\omega \wedge \star\omega) = \int_\Omega \mathsf{L}_{\mathbf{x}}(\omega \wedge \star\omega) = \int_\Omega (\mathsf{L}_{\mathbf{x}} + \mathcal{L}_{\mathbf{x}})(\omega \wedge \star\omega) - \int_\Omega \mathsf{j}_{\mathbf{x}}\,\delta(\omega \wedge \star\omega). \tag{57}$$

With the Cauchy inequality the second term on the right hand side is estimated as

$$\left|\int_\Omega \mathsf{j}_{\mathbf{x}}\,\delta(\omega \wedge \star\omega)\right| \le 2\sup_{\mathbf{x}\in\Omega}|\mathbf{x}|\, \|\omega\|_{L^2\Lambda^k(\Omega)}\, |\omega|_{H^1\Lambda^k(\Omega)}. \tag{58}$$

Since further $(\mathsf{L}_{\mathbf{x}} + \mathcal{L}_{\mathbf{x}})\mu = (\operatorname{div}\mathbf{x})\mu$ (see (52) and (51)), the lower bound (56) together with (57) and (58) proves assertion (55). $\square$

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