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Abstract

In this paper, using the Newton's formula of Lagrange interpolation, we present a new proof of the anisotropic error bounds for Lagrange interpolation of any order on the triangle, rectangle, tetrahedron and cube in a unified way.

Key words: Lagrange interpolation, Anisotropic error bounds, Newton's interpolation formula.

1 Introduction

It is known that the polynomial interpolations are the foundations of construction the finite elements and the interpolation error estimates play a key role in deriving a-priori error estimates of the finite element methods. The main strategy of the traditional interpolation theory is fairly standard, namely, first deriving the estimate on the reference element and then an application of a coordinate transformation between a general element and the reference element, see [11, 7] and references therein. For the triangular and rectangular elements in two dimension and the tetrahedral and cubic elements in three dimension, the mapping between a general element and the reference element is an affine mapping, so in the following we call these elements affine elements.

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The classical error estimates of the polynomial interpolation on the affine elements need the regular [11] or nondegenerate [7] condition, i.e., the ratio of the diameters of the element and the biggest ball contained in the element is uniformly bounded. This condition restricts the applications of the finite elements. It is found (see e.g., [6, 15]) a long time ago that this condition is not necessary for some interpolation error estimates. We call the element does not satisfy the regular condition the anisotropic element. Recently, the research of the anisotropic elements is rapidly developed, and there are several different methods dealing with them. [3, 4] gave one anisotropic form of the interpolation error on the reference element. They got the anisotropic interpolation error estimates on a general element for some Lagrange and Hermite elements under the maximal angle and coordinate system conditions. The corresponding appeared derivatives are along the coordinate directions. [9, 10] extend this method by presenting a simple anisotropic criterion on the reference element and analyzed some nonconforming elements. [1, 2, 12, 13] got the anisotropic error estimates for low order Lagrange and R-T interpolations by using of the average property of the interpolation and the appeared derivatives under consideration are along the directions of the element boundary. The different forms of the anisotropic error estimate of the linear triangular Lagrange interpolation are obtained by the decomposition of the transformation matrix between a general element and the reference element in [14] and by Taylor's expansion in [8].

In this paper, the anisotropic interpolation error estimates of Lagrange interpolations with any order on the affine elements (triangle, rectangle, cube and tetrahedron) are derived in a unified new way. On the reference element the anisotropic error estimates of the interpolations are proved by Newton's formula of the Lagrange interpolation and a special property of the divided difference, which are different from [4]. The appeared derivatives are along the directions of the element boundary (as in [2, 13]) and independent length scales in different directions are extracted (as in [4]). No geometry condition of the element is needed for rectangular and cubic elements. The sine of the biggest internal angle of the element and the regular vertex property factor [2] appear explicitly in the triangular and the tetrahedral elements, respectively, then standard arguments will lead to the estimates that depend on the biggest internal angle of the element and the regular vertex property factor.

2 Lagrange Interpolation Remainder Term On Reference Elements

2.1 The property of the divided difference

Let $x_0 < x_1 < \cdots < x_m$ be a uniform partition, $d = x_{i+1} - x_i, 0 \leq i \leq m-1$. It is easy to get the following result by inductive method.

Lemma 2.1.

$$\int_{x_1}^{x_2} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} g(t_m)\mathrm{d}t_m = \int_{x_0}^{x_1} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} g(t_m+d)\,\mathrm{d}t_m. \tag{2.1}$$

Let $f[x_0, \cdots, x_m]$ be the usual divided difference (see [5]), then we get the following lemma.

Lemma 2.2. Suppose $f(x)$ is sufficiently smooth, then

$$f[x_0, \cdots, x_m] = \frac{1}{m!d^m} \int_{x_0}^{x_1} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)\,\mathrm{d}t_m. \tag{2.2}$$

Proof. We use the inductive method.

When $m = 1, f[x_0, x_1] = \frac{1}{d}\int_{x_0}^{x_1} f'(t_1)dt_1$, (2.2) is evident.

Suppose (2.2) holds for any $m \geq 1$, then

$$\begin{aligned}
& f[x_0, \cdots, x_{m+1}] \\
= \; & (f[x_1, \cdots, x_{m+1}] - f[x_0, \cdots, x_m])/(x_{m+1} - x_0) \\
= \; & \frac{1}{(m+1)d}[\int_{x_1}^{x_2} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)\,\mathrm{d}t_m \\
& - \int_{x_0}^{x_1} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} f^{(m)}(t_m)\,\mathrm{d}t_m]/(m!d^m) \\
\overset{(2.1)}{=} \; & \frac{1}{(m+1)!d^{m+1}} \int_{x_0}^{x_1} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} [f^{(m)}(t_m + d) - f^{(m)}(t_m)]\,\mathrm{d}t_m \\
= \; & \frac{1}{(m+1)!d^{m+1}} \int_{x_0}^{x_1} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \mathrm{d}t_2 \cdots \int_{t_{m-1}}^{t_{m-1}+d} \mathrm{d}t_m \int_{t_m}^{t_m+d} f^{(m+1)}(t_{m+1})\,\mathrm{d}t_{m+1}.
\end{aligned}$$

This completes the proof.

Remark 1. Lemma 2.2 is similar to Hermite-Gennochi Theorem ([5, Theorem 3.3]).

Using the inductive method again, we can get:

Lemma 2.3. For all $0 \leq l \leq m$, $f[x_0, \cdots, x_m]$ can be expressed by

$$f[x_0, \cdots, x_m] = \sum_{i=0}^{m-l} c_i f[x_i, \cdots, x_{i+l}], \tag{2.3}$$

where c_i $(0 \leq i \leq m - l)$ is only dependent on l and d.

The interpolation polynomial $\mathrm{I}f(x)$ of $f(x)$ satisfying $\mathrm{I}f(x_i) = f(x_i)(0 \leq i \leq m)$ can be expressed in the following two forms, where (2.4) is called Lagrange's formula and (2.5) is called Newton's formula (see [5]):

$$\mathrm{I}f(x) = \sum_{i=0}^{m} f(x_i)p_i(x), \tag{2.4}$$

where $p_i(x)$ $(0 \leq i \leq m)$ $\in P_m$ (the polynomial space of degree less or equal to m) and $p_i(x_j) = \delta_{ij},\ 0 \leq i, j \leq m$.

$$\mathrm{I}f(x) = \sum_{i=0}^{m} f[x_0, \cdots, x_i] \prod_{j=0}^{i-1} (x - x_j). \tag{2.5}$$

2.2 Rectangular Elements

Let the reference element $\hat{K} = [0,1]^2,\ d = 1/k,\ k$ is a positive integer, $\hat{x}_i = \hat{y}_i = id,\ i = 0, \cdots, k$. Suppose $\hat{u}(\hat{x}, \hat{y}) \in C(\hat{K})$, then bi-$k$-interpolation polynomial $\hat{\mathrm{I}}\hat{u}$ of $\hat{u}$ satisfying $\hat{\mathrm{I}}\hat{u}(\hat{x}_i, \hat{y}_j) = \hat{u}(\hat{x}_i, \hat{y}_j)(0 \leq i, j \leq k)$ has the following expression (cf. [11])

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k} \sum_{j=0}^{k} \hat{u}(\hat{x}_i, \hat{y}_j) \hat{p}_i(\hat{x}) \hat{p}_j(\hat{y}),$$

where $\hat{p}_i(t) \in P_k(\hat{K}),\ \hat{p}_i(\hat{x}_l) = \hat{p}_i(\hat{y}_l) = \delta_{il}, 0 \leq i, l \leq k$. Obviously

$$\hat{\mathrm{I}}\hat{u} = \hat{u},\ \forall \hat{u} \in Q_k, \tag{2.6}$$

where Q_k is the polynomial space of the degree $\leq k$ with respect to each variable.

Similar as one dimension case, $\hat{\mathrm{I}}\hat{u}$ can be expressed as the following Newton's formula,

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k} \sum_{r=0}^{k} \hat{u}[\hat{x}_0, \cdots, \hat{x}_i; \hat{y}_0, \cdots, \hat{y}_r] \prod_{j=0}^{i-1} (\hat{x} - \hat{x}_j) \prod_{s=0}^{r-1} (\hat{y} - \hat{y}_s), \tag{2.7}$$

where $\hat{u}[\hat{x}_0, \cdots, \hat{x}_i; \hat{y}_0, \cdots, \hat{y}_r]$ is the i-order divided difference with respect to $\hat{x}$ and r-order divided difference with respect to $\hat{y}$ of $\hat{u}(\hat{x}, \hat{y})$, $\prod\limits_{j=0}^{i-1} (\hat{x} - \hat{x}_j) = 1$ for $i = 0$ and $\prod\limits_{s=0}^{r-1} (\hat{y} - \hat{y}_s) = 1$ for $r = 0$.

Let us consider a simple example before treating the general case. Taking $k = 1$ and $\frac{\partial \hat{\mathrm{I}}\hat{u}}{\partial \hat{x}}$ as an example, then it is easy to check that the interpolation function can be written as

$$\begin{aligned} \hat{\mathrm{I}}\hat{u} &= \hat{u}[\hat{x}_0; \hat{y}_0] + \hat{u}[\hat{x}_0; \hat{y}_0, \hat{y}_1](\hat{y} - \hat{y}_0) + \hat{u}[\hat{x}_0, \hat{x}_1; \hat{y}_0](\hat{x} - \hat{x}_0) \\ &\quad + \hat{u}[\hat{x}_0, \hat{x}_1; \hat{y}_0, \hat{y}_1](\hat{x} - \hat{x}_0)(\hat{y} - \hat{y}_0). \end{aligned}$$

So

$$\frac{\partial \hat{\mathrm{I}}\hat{u}}{\partial \hat{x}} = \hat{u}[\hat{x}_0, \hat{x}_1; \hat{y}_0] + \hat{u}[\hat{x}_0, \hat{x}_1; \hat{y}_0, \hat{y}_1](\hat{y} - \hat{y}_0).$$

Then it can be checked easily with (2.3) and (2.2) that

$$\hat{u}[\hat{x_0},\hat{x_1};\hat{y_0}] = \int_{\hat{x_0}}^{\hat{x_1}} \frac{\partial \hat{u}[\hat{x},\hat{y_0}]}{\partial \hat{x}} d\hat{x}$$

and

$$\hat{u}[\hat{x_0},\hat{x_1};\hat{y_0},\hat{y_1}] = \int_{\hat{x_0}}^{\hat{x_1}} \frac{\partial \hat{u}[\hat{x};\hat{y_0},\hat{y_1}]}{\partial \hat{x}} d\hat{x}.$$

Let us consider the general case and set $\alpha = (\alpha_1, \alpha_2), \alpha_1$ and α_2 are non-negative integers, $|\alpha| = \alpha_1 + \alpha_2$, then

$$\hat{D}^\alpha \hat{\mathrm{I}}\hat{u} = \sum_{i=\alpha_1}^{k} \sum_{r=\alpha_2}^{k} \hat{u}[\hat{x_0},\cdots,\hat{x_i};\hat{y_0},\cdots,\hat{y_r}] G_i(\hat{x}) R_r(\hat{y}), \tag{2.8}$$

where

$$G_i(\hat{x}) = \frac{\mathrm{d}^{\alpha_1}}{\mathrm{d}\hat{x}^{\alpha_1}}(\prod_{j=0}^{i-1}(\hat{x}-\hat{x_j})),\ R_r(\hat{y}) = \frac{\mathrm{d}^{\alpha_2}}{\mathrm{d}\hat{y}^{\alpha_2}}(\prod_{s=0}^{r-1}(\hat{y}-\hat{y_s})). \tag{2.9}$$

Obviously $G_i(\hat{x}), R_r(\hat{y}) \in \hat{Q}_{k-\alpha_1,k-\alpha_2}$, here $\hat{Q}_{m,n}$ is a polynomial space of the degrees of $\hat{x}$ and $\hat{y}$ less or equal to m and n, respectively.
By (2.3) and (2.2) we have

$$\begin{aligned}\hat{u}[\hat{x_0},\cdots,\hat{x_i};\hat{y_0},\cdots,\hat{y_r}] &= \sum_{j=0}^{i-\alpha_1}\sum_{s=0}^{r-\alpha_2} c_{js}\hat{u}[\hat{x_j},\cdots,\hat{x}_{j+\alpha_1};\hat{y_s},\cdots,\hat{y}_{s+\alpha_2}] \\ &= \tfrac{k^{\alpha_1+\alpha_2}}{\alpha_1!\alpha_2!}\sum_{j=0}^{i-\alpha_1}\sum_{s=0}^{r-\alpha_2} c_{js}\int_{\hat{x_j}}^{\hat{x}_{j+1}}\mathrm{d}t_1\int_{t_1}^{t_1+d}\cdots\mathrm{d}t_{\alpha_1-1}\int_{t_{\alpha_1-1}}^{t_{\alpha_1-1}+d} \\ &\quad [\int_{\hat{y_s}}^{\hat{y}_{s+1}}\mathrm{d}s_1\int_{s_1}^{s_1+d}\cdots\mathrm{d}s_{\alpha_2-1}\int_{s_{\alpha_2-1}}^{s_{\alpha_2-1}+d}\tfrac{\partial^{\alpha_1+\alpha_2}\hat{u}(\hat{x},\hat{y})}{\partial\hat{x}^{\alpha_1}\partial\hat{y}^{\alpha_2}}\,\mathrm{d}\hat{y}]\,\mathrm{d}\hat{x} \\ &\triangleq \hat{L}(\hat{D}^\alpha\hat{u}),\end{aligned} \tag{2.10}$$

where $\hat{L}(\hat{w}) = \frac{k^{\alpha_1+\alpha_2}}{\alpha_1!\alpha_2!}\sum_{j=0}^{i-\alpha_1}\sum_{s=0}^{r-\alpha_2} c_{js}\int_{\hat{x_j}}^{\hat{x}_{j+1}}\mathrm{d}t_1\int_{t_1}^{t_1+d}\cdots\mathrm{d}t_{\alpha_1-1}\int_{t_{\alpha_1-1}}^{t_{\alpha_1-1}+d}[\int_{\hat{y_s}}^{\hat{y}_{s+1}}\mathrm{d}s_1\int_{s_1}^{s_1+d}$ $\cdots\mathrm{d}s_{\alpha_2-1}\int_{s_{\alpha_2-1}}^{s_{\alpha_2-1}+d}\hat{w}\,\mathrm{d}\hat{y}]\,\mathrm{d}\hat{x}$.
It is easy to see that $[\hat{x_j},\hat{x_{j+1}}],\ [t_l,t_{l+d}]\ (1\le l\le \alpha_1-1)$, and $[\hat{y_s},\hat{y_{s+1}}],\ [s_l,s_{l+d}]$ $(1\le l\le\alpha_2-1)$ are all in $[0,1]$, the above integration is on $\hat{K}$ or a side of $\hat{K}$ or a fixed line lies in $\hat{K}$, hence by the Sobolev trace theorems we can obtain

$$|\hat{L}(\hat{\omega})| \le \hat{c}\|\hat{\omega}\|_{W^{1,p}(\hat{K})},\quad 2\le p\le\infty.$$

Substituting (2.10) into (2.8) results

$$\hat{D}^\alpha\hat{\mathrm{I}}\hat{u} = \hat{T}(\hat{D}^\alpha\hat{u}), \tag{2.11}$$

where

$$\hat{T}(\hat{\omega}) = \sum_{i=\alpha_1}^{k}\sum_{r=\alpha_2}^{k}\hat{L}(\hat{\omega})G_i(\hat{x})R_r(\hat{y}). \tag{2.12}$$

Obviously,

$$\begin{cases} \|\hat{T}(\hat{\omega})\|_{W^{m,q}(\hat{K})} \leq \hat{c}\|\hat{\omega}\|_{W^{1,p}(\hat{K})}, \forall\, m \geq 0, \quad 1 \leq q \leq \infty, \quad 2 \leq p \leq \infty, \\ \hat{T}(\hat{\omega}) = \hat{\omega}, \ \forall\, \hat{\omega} \in Q_{k-\alpha_1, k-\alpha_2}. \end{cases} \tag{2.13}$$

In fact $\forall\, \hat{\omega} \in Q_{k-\alpha_1,k-\alpha_2}$, $\exists\, \hat{v} \in Q_k$ such that $\hat{D}^\alpha \hat{v} = \hat{\omega}$, then

$$\hat{T}(\hat{\omega}) = \hat{T}(\hat{D}^\alpha \hat{v}) \overset{(2.11)}{=} \hat{D}^\alpha \hat{\mathrm{I}}\hat{u} \overset{(2.6)}{=} \hat{D}^\alpha \hat{u} = \hat{\omega}$$

2.3 Cubic Elements

We can extend the results from rectangular elements to cubic elements in a straightforward way. let $\hat{K} = [0,1]^3,\ d = 1/k,\ \hat{x}_i = \hat{y}_i = \hat{z}_j = id,\ 0 \leq i \leq k$, then the interpolation polynomial $\hat{\mathrm{I}}\hat{u}$ of $\hat{u}(\hat{x},\hat{y},\hat{z})$ satisfying $\hat{\mathrm{I}}\hat{u}(\hat{x}_i,\hat{y}_r,\hat{z}_l) = \hat{u}(\hat{x}_i,\hat{y}_r,\hat{z}_l),\ 1 \leq i,r,l \leq k+1$, has the following expression

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k}\sum_{r=0}^{k}\sum_{l=0}^{k} \hat{u}[\hat{x}_0,\cdots,\hat{x}_i;\hat{y}_0,\cdots,\hat{y}_r;\hat{z}_0,\cdots,\hat{z}_l]\prod_{j=0}^{i-1}(\hat{x}-\hat{x}_j)\prod_{s=0}^{r-1}(\hat{y}-\hat{y}_s)\prod_{t=0}^{l-1}(\hat{z}-\hat{z}_t). \tag{2.14}$$

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then

$$\hat{D}^\alpha \hat{\mathrm{I}}\hat{u} = \hat{T}(\hat{D}^\alpha \hat{u}), \tag{2.15}$$

here

$$\hat{T}(\hat{\omega}) = \sum_{i=\alpha_1}^{k}\sum_{r=\alpha_2}^{k}\sum_{l=\alpha_3}^{k} \hat{L_{irl}}(\hat{\omega}) G_i(\hat{x}) R_r(\hat{y}) H_l(\hat{z}) \tag{2.16}$$

with $G_i(\hat{x}),\ R_r(\hat{y})$ is as (2.9) and

$$H_l(\hat{x}) = \frac{\mathrm{d}^{\alpha_3}}{\mathrm{d}\hat{x}^{\alpha_3}}\Big(\prod_{t=0}^{l-1}(\hat{z}-\hat{z}_t)\Big), \tag{2.17}$$

$$\hat{L}_{irl}(\hat{\omega}) = \frac{k^{\alpha_1+\alpha_2+\alpha_3}}{\alpha_1!\alpha_2!\alpha_3!}\sum_{j=0}^{i-\alpha_1}\sum_{s=0}^{r-\alpha_2}\sum_{t=0}^{l-\alpha_3} c_{jst} \underset{\frown}{\iiint} \hat{\omega}\,\mathrm{d}\hat{X}, \tag{2.18}$$

where

$$\begin{aligned}\underset{\frown}{\iiint} \hat{\omega}\,\mathrm{d}\hat{X} = &\int_{\hat{x}_j}^{\hat{x}_{j+1}} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \cdots \mathrm{d}t_{\alpha_1-1} \int_{t_{\alpha_1-1}}^{t_{\alpha_1-1}+d} \\ &\Big[\int_{\hat{y}_s}^{\hat{y}_{s+1}} \mathrm{d}s_1 \int_{s_1}^{s_1+d} \cdots \mathrm{d}s_{\alpha_2-1} \int_{s_{\alpha_2-1}}^{s_{\alpha_2-1}+d} \\ &\Big(\int_{\hat{z}_t}^{\hat{z}_{t+1}} \mathrm{d}q_1 \int_{q_1}^{q_1+d} \cdots \mathrm{d}q_{\alpha_3-1} \int_{q_{\alpha_3-1}}^{q_{\alpha_3-1}+d} \hat{\omega}\,\mathrm{d}\hat{z}\Big)\,\mathrm{d}\hat{y}\Big]\,\mathrm{d}\hat{x}.\end{aligned}$$

The above integration is on $\hat{K}$ or on a face (or a fixed internal face) of $\hat{K}$ or on an edge (or a fixed internal edge) of $\hat{K}$, similar to the rectangular case, by trace theorem and Cauchy-Schwarz inequality, we have

$$\begin{cases} \|\hat{T}(\hat{\omega})\|_{W^{m,q}(\hat{K})} \leq \hat{c}\|\hat{\omega}\|_{W^{1,p}(\hat{K})}, \forall\, m \geq 0, 3 \leq p \leq \infty, 1 \leq q \leq \infty, \\ \hat{T}(\hat{\omega}) = \hat{\omega},\ \forall\, \hat{\omega} \in Q_{k-\alpha_1,k-\alpha_2,k-\alpha_3}. \end{cases} \tag{2.19}$$

2.4 Triangular Elements

Let $\hat{K} = \{(\hat{x},\hat{y});\ \hat{x} \geq 0,\ \hat{y} \geq 0,\ \hat{x}+\hat{y} \leq 1\},\ d = 1/k,\ \hat{x}_i = id,\ \hat{y}_r = rd,\ 0 \leq i+r \leq k$, then the Lagrange interpolation polynomial $\hat{\mathrm{I}}\hat{u}$ of degree k of $\hat{u}(\hat{x},\hat{y})$, satisfying $\hat{\mathrm{I}}\hat{u}(\hat{x}_i,\hat{y}_r) = \hat{u}(\hat{x}_i,\hat{y}_r),\ 0 \leq i+r \leq k$ can be expressed by[10]

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k}\sum_{r=0}^{k-i} \hat{u}(\hat{x}_i,\hat{y}_r)\hat{p}_i(\hat{x})\hat{q}_r(\hat{y}),$$

where $\hat{p}_i(\hat{x}) \in P_i(\hat{K}),\ 0 \leq i \leq k,\ \hat{p}_i(\hat{x_j}) = \delta_{ij},\ \hat{q}_r(\hat{y}) \in P_{k-i}(\hat{K}),\ 0 \leq r \leq k-i,\ \hat{q}_r(\hat{y_s}) = \delta_{rs}$.
The following expression is the Newton's formula of $\hat{\mathrm{I}}\hat{u}$,

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k}\sum_{r=0}^{k-i} \hat{u}[\hat{x_0},\cdots,\hat{x_i};\hat{y_0},\cdots,\hat{y_r}] \prod_{j=0}^{i-1}(\hat{x}-\hat{x_j}) \prod_{s=0}^{r-1}(\hat{y}-\hat{y_s}). \tag{2.20}$$

Let $\alpha = (\alpha_1,\alpha_2)$, then in the same way as in the rectangular case,

$$\hat{D}^{\alpha}\hat{\mathrm{I}}\hat{u} = \sum_{i=\alpha_1}^{k}\sum_{r=\alpha_2}^{k-i} \hat{u}[\hat{x_0},\cdots,\hat{x_i};\hat{y_0},\cdots,\hat{y_r}]G_i(\hat{x})R_r(\hat{y}) = \hat{T}(\hat{D}^{\alpha}\hat{u}), \tag{2.21}$$

where

$$\hat{T}(\hat{\omega}) = \sum_{i=\alpha_1}^{k}\sum_{r=\alpha_2}^{k-i} \hat{L}_{ir}(\hat{\omega})G_i(\hat{x})R_r(\hat{y}),$$

$$\hat{L}_{ir}(\hat{D}^{\alpha}\hat{u}) = \hat{u}[\hat{x_0},\cdots,\hat{x_i};\hat{y_0},\cdots,\hat{y_r}] = \sum_{j=0}^{i-\alpha_1}\sum_{s=0}^{r-\alpha_2} \hat{c_{js}} \iint_{\backsim} \hat{D}^{\alpha}\hat{u}\,\mathrm{d}\hat{x}\,\mathrm{d}\hat{y}$$

and

$$\iint\limits_{\smile} \mathrm{d}\hat{x}\,\mathrm{d}\hat{y} = \int_{\hat{x}_j}^{\hat{x}_{j+1}} \mathrm{d}t_1 \int_{t_1}^{t_1+d} \cdots \mathrm{d}t_{\alpha_1-1} \int_{t_{\alpha_1-1}}^{t_{\alpha_1-1}+d}$$
$$[\int_{\hat{y}_s}^{\hat{y}_{s+1}} \mathrm{d}s_1 \int_{s_1}^{s_1+d} \cdots \mathrm{d}s_{\alpha_2-1} \int_{s_{\alpha_2-1}}^{s_{\alpha_2-1}+d} \mathrm{d}\hat{y}]\,\mathrm{d}\hat{x}.$$

Proceeding as before, we have

$$\begin{cases} \|\hat{T}(\hat{\omega})\|_{W^{m,q}(\hat{K})} \le \hat{c}\|\hat{\omega}\|_{W^{1,p}(\hat{K})}, \forall\, m \ge 0, \ \ 1 \le q \le \infty, \ \ 2 \le p \le \infty, \\ \hat{T}(\hat{\omega}) = \hat{\omega}, \ \forall\, \hat{\omega} \in P_{k-|\alpha|}. \end{cases} \tag{2.22}$$

2.5 Tetrahedral Element

Let $\hat{K} = \{(\hat{x},\hat{y},\hat{z});\ \hat{x} \ge 0,\ \hat{y} \ge 0, \hat{z} \ge 0,\ \hat{x}+\hat{y}+\hat{z} \le 1\},\ d = 1/k,\ \hat{x}_i = id,\ \hat{y}_r = rd, \hat{z}_l = ld,\ 0 \le i+r+l \le k$, then the interpolation polynomial $\hat{\mathrm{I}}\hat{u}$ of degree k of $\hat{u}(\hat{x},\hat{y},\hat{z})$ satisfying $\hat{\mathrm{I}}\hat{u}(\hat{x}_i,\hat{y}_r,\hat{z}_l) = \hat{u}(\hat{x}_i,\hat{y}_r,\hat{z}_l),\ 0 \le i+r+l \le k$ has the following form,

$$\hat{\mathrm{I}}\hat{u} = \sum_{i=0}^{k}\sum_{r=0}^{k-i}\sum_{l=0}^{k-i-r} \hat{u}[\hat{x}_0,\cdots,\hat{x}_i;\hat{y}_0,\cdots,\hat{y}_r;\hat{z}_0,\cdots,\hat{z}_l]\prod_{j=0}^{i-1}(\hat{x}-\hat{x}_j)\prod_{s=0}^{r-1}(\hat{y}-\hat{y}_s)\prod_{t=0}^{l-1}(\hat{z}-\hat{z}_t).$$

Let $\alpha = (\alpha_1,\alpha_2,\alpha_3)$, it is easy to see that $\hat{D}^\alpha \mathrm{I}\hat{u}$ can be expressed as (2.21), here

$$\hat{T}(\hat{\omega}) = \sum_{i=\alpha_1}^{k}\sum_{r=\alpha_2}^{k-i}\sum_{l=\alpha_3}^{k-i-r} \hat{L_{irl}}(\hat{\omega}) G_i(\hat{x}) R_r(\hat{y}) H_l(\hat{z}), \tag{2.23}$$

where $\hat{L_{irl}}(\hat{\omega})$ is as (2.18) and $G_i(\hat{x}),\ R_r(\hat{y}),\ H_l(\hat{z})$ are as (2.9) and (2.17).

Similar to (2.19), we have

$$\begin{cases} \|\hat{T}(\hat{\omega})\|_{W^{m,q}(\hat{K})} \le \hat{c}\|\hat{\omega}\|_{W^{1,p}(\hat{K})}, \ \forall\, m \ge 0, 3 \le p \le \infty, 1 \le q \le \infty, \\ \hat{T}(\hat{\omega}) = \hat{\omega}, \ \forall\, \hat{\omega} \in Q_{k-\alpha_1,k-\alpha_2,k-\alpha_3}. \end{cases} \tag{2.24}$$

2.6 Interpolation Remainders

Let $\hat{K}$ be the reference element ($\hat{K} = [0,1]^n,\ n = 2$ for rectangular element and $n = 3$ for cubic element; and $\hat{K} = \{(\hat{x},\hat{y});\ \hat{x} \ge 0,\ \hat{y} \ge 0,\ \hat{x}+\hat{y} \le 1\}$ for triangular element and $\hat{K} = \{(\hat{x},\hat{y},\hat{z});\ \hat{x} \ge 0,\ \hat{y} \ge 0, \hat{z} \ge 0,\ \hat{x}+\hat{y}+\hat{z} \le 1\}$ for tetrahedron element) $\hat{\mathrm{I}} : C^0(\hat{K}) \to \hat{P}$ is the above Lagrange interpolation operator, here $\hat{P} = Q_k$ for rectangular and cubic element, $\hat{P} = P_k$ for triangular and tetrahedral element. Then we have

Theorem 2.4. Suppose that $n \le p \le \infty, 1 \le q \le \infty, 0 \le m \le k,\ W^{k+1,p}(\hat{K}) \hookrightarrow$

$W^{m,q}(\hat{K}), W^{k+1,p}(\hat{K}) \hookrightarrow C^0(\hat{K}),$ α is an index, $|\alpha| = m$, then there exists a constant $\hat{c} > 0$ such that

$$\|\hat{D}^\alpha(\hat{u} - \hat{\mathrm{I}}\hat{u})\|_{L^q(\hat{K})} \le \hat{c}|\hat{D}^\alpha\hat{u}|_{W^{k+1-m,p}(\hat{K})}. \tag{2.25}$$

Proof. By (2.11), (2.15) and (2.20) we have

$$\hat{D}^\alpha(\hat{u} - \hat{\mathrm{I}}\hat{u}) = \hat{D}^\alpha\hat{u} - \hat{T}(\hat{D}^\alpha\hat{u}).$$

Then (2.24) is followed by (2.13), (2.19), (2.22) and (2.24).
Remark 2. Apel has proved (2.25) in [4], but our method is different from Apel's. Our main arguments are using the Newton's formula of Lagrange interpolation and the special property of the divided difference, which admit us to prove (2.25) for rectangular, cubic, triangular and tetrahedral elements in a unified way .

3 Lagrange Interpolation Errors In General Elements

3.1 Rectangular and Cubic Elements

We denote a general rectangular element by $K = [x_0, x_0+h_1] \times [y_0, y_0+h_2]$ and a general cubic element by $[x_0, x_0+h_1] \times [y_0, y_0+h_2] \times [z_0, z_0+h_3]$, here $a_0(x_0, y_0)$ or $a_0(x_0, y_0, z_0)$ is a vertex of K and h_1, h_2 or h_1, h_2, h_3 are edge lengths of K. Let $X = F_K(\hat{X})$ be the affine mapping from $\hat{K}$ to K, then

$$X = B\hat{X} + a_0, \tag{3.1}$$

where $B = diag(h_1, h_2)$ for the rectangular element and $B = diag(h_1, h_2, h_3)$ for the cubic element . Obviously

$$D^\alpha = h^{-\alpha}\hat{D}^\alpha,\ \hat{D}^\alpha = h^\alpha D^\alpha, \tag{3.2}$$

where $h^\alpha = h_1^{\alpha_1}h_2^{\alpha_2}$ or $h^\alpha = h_1^{\alpha_1}h_2^{\alpha_2}h_3^{\alpha_3}$
Lemma 3.1. Suppose $a_i \ge 0,\ 1 \le i \le N,\ q \ge 1,\ 1/q + 1/q' = 1$, then

$$N^{-\frac{1}{q'}}\sum_{i=1}^N a_i \le (\sum_{i=1}^N a_i^q)^{\frac{1}{q}} \le N^{\frac{1}{q}}\sum_{i=1}^N a_i. \tag{3.3}$$

Let $\mathrm{I}u = \hat{\mathrm{I}}\hat{u} \circ F_K^{-1}(X),$ $\hat{\mathrm{I}}$ is defined by (2.7) or (2.14), then I is affine equivalent [11]. Furthermore, we have the following interpolation error estimate.
Theorem 3.2. Under the same assumptions as Theorem 2.4, then for rectangular and cubic elements, we have

$$|u - \mathrm{I}u|_{W^{m,q}(K)} \le \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}(\sum_{|\beta|=k+1-m} h^{\beta p}|D^\beta u|^p_{W^{k+1,p}(K)})^{\frac{1}{p}}. \tag{3.4}$$

where $\hat{c}$ is independent of K, $detB$ is the Jacobian of B.
Proof.

$$
\begin{aligned}
|u-\mathrm{I}u|_{W^{m,q}(K)} &= \left(\sum_{|\alpha|=m}\|D^\alpha(u-\mathrm{I}u)\|^q_{L^q(K)}\right)^{\frac{1}{q}} \\
&\stackrel{(3.2)}{=} \left(\sum_{|\alpha|=m} h^{-\alpha q} detB \|\hat{D}^\alpha(\hat{u}-\hat{\mathrm{I}}\hat{u})\|^q_{L^q(\hat{K})}\right)^{\frac{1}{q}} \\
&\stackrel{(2.25)}{\leq} \left(\sum_{|\alpha|=m} h^{-\alpha q} detB\cdot \hat{c}|\hat{D}^\alpha \hat{u}|^q_{W^{k+1-m,p}(\hat{K})}\right)^{\frac{1}{q}} \\
&= \hat{c}(detB)^{\frac{1}{q}}\left[\sum_{|\alpha|=m} h^{-\alpha q}(\sum_{|\beta|=k+1-m}\|\hat{D}^{\alpha+\beta}\hat{u}\|^p_{L^p(\hat{K})})^{\frac{q}{p}}\right]^{\frac{1}{q}} \\
&\stackrel{(3.2)}{=} \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}[\sum_{|\alpha|=m}(\sum_{|\beta|=k+1-m} h^{\beta p}\|D^{\alpha+\beta}u\|^p_{L^p(K)})^{\frac{q}{p}}]^{\frac{1}{q}} \\
&\stackrel{(3.3)}{\leq} \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}\sum_{|\alpha|=m}(\sum_{|\beta|=k+1-m} h^{\beta p}\|D^{\alpha+\beta}u\|^p_{L^p(K)})^{\frac{1}{p}} \\
&\leq \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}(\sum_{|\alpha|=m}\sum_{|\beta|=k+1-m} h^{\beta p}\|D^{\alpha+\beta}u\|^p_{L^p(K)})^{\frac{1}{p}} \\
&= \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}(\sum_{|\beta|=k+1-m} h^{\beta p}|D^\beta u|^p_{W^{m,p}(K)})^{\frac{1}{p}} \quad \#
\end{aligned}
$$

3.2 Triangular and Tetrahedral Elements

Let K be a triangle (a tetrahedron) with the vertexes P_0,P_1,P_2 (P_0,P_1,P_2,P_3), $\underset{\sim}{v_1}$,$\underset{\sim}{v_2}$ $(\underset{\sim}{v_1},\underset{\sim}{v_2},\underset{\sim}{v_3})$ be the unit vectors along edges P_0P_1, P_0P_2 (P_0P_1, P_0P_2, P_0P_3) with $l_i=\|P_0P_i\|$, $\angle P_0$ be the maximum angle of the triangle K.
The affine mapping $F:\hat{K}\to K$ is

$$X = F(\hat{X}) = B\hat{X} + P_0 \tag{3.5}$$

where

$$B = B_0\Lambda, \tag{3.6}$$

$B_0=(\underset{\sim}{v_1},\underset{\sim}{v_2})$, $\Lambda=diag(l_1,l_2)$ for the triangular and $B_0=(\underset{\sim}{v_1},\underset{\sim}{v_2},\underset{\sim}{v_3})$, $\Lambda=diag(l_1,l_2,l_3)$ for the tetrahedron. Let $\hat{\nabla}=(\frac{\partial}{\partial\hat{x}},\frac{\partial}{\partial\hat{y}})^T$, $(\hat{\nabla}=(\frac{\partial}{\partial\hat{x}},\frac{\partial}{\partial\hat{y}},\frac{\partial}{\partial\hat{z}})^T)$, $\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y})^T$, $(\nabla=(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z})^T)$, $\nabla_l=(\frac{\partial}{\partial v_1},\frac{\partial}{\partial v_2})^T$, $(\nabla_l=(\frac{\partial}{\partial v_1},\frac{\partial}{\partial v_2},\frac{\partial}{\partial v_3})^T)$, by simple computations we have

$$\nabla_l = B_0^T\nabla,\ \ \hat{\nabla}=\Lambda B_0^T\nabla,\ \ \hat{\nabla}=\Lambda\nabla_l. \tag{3.7}$$

Let

$$\mathrm{I}u = \hat{\mathrm{I}}\hat{u} \circ F_K^{-1}(X), \tag{3.8}$$

where $\hat{\mathrm{I}}$ is defined by (2.20) or (2.23). It is well known that (cf. [11]) I is affine equivalent.

Theorem 3.3. Under the same assumptions as Theorem 2.4, then for triangular and the tetrahedral elements,

$$|u - \mathrm{I}u|_{W_l^{m,q}(K)} \le \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}\Big(\sum_{|\beta|=k+1-m} l^{\beta p}|D_l^{\beta} u|^p_{W_l^{k+1,p}(K)}\Big)^{\frac{1}{p}}, \tag{3.9}$$

where $\hat{c}$ is independent of K, $D_l^{\alpha} = \frac{\partial^{|\alpha|}}{\partial v_1^{\alpha_1}\partial v_2^{\alpha_2}}$ $\big(D_l^{\alpha} = \frac{\partial^{|\alpha|}}{\partial v_1^{\alpha_1}\partial v_2^{\alpha_2}\partial v_3^{\alpha_3}}\big)$, $|v|_{W_l^{m,q}(K)} = \big(\sum_{|\alpha|=m} \|D_l^{\alpha} v\|^q_{L^q(K)}\big)^{\frac{1}{q}}$, $l^{\alpha} = l_1^{\alpha_1} l_2^{\alpha_2} (l^{\alpha} = l_1^{\alpha_1} l_2^{\alpha_2} l_3^{\alpha_3})$.

Proof.

$$\begin{aligned}
|u - \mathrm{I}u|_{W_l^{m,q}(K)} &= \Big(\sum_{|\alpha|=m} \|D_l^{\alpha}(u - \mathrm{I}u)\|^q_{L^q(K)}\Big)^{\frac{1}{q}} \\
&\overset{(3.7)}{=} (detB)^{\frac{1}{q}}\Big(\sum_{|\alpha|=m} l^{-\alpha q}\|\hat{D}^{\alpha}(\hat{u} - \hat{\mathrm{I}}\hat{u})\|^q_{L^q(\hat{K})}\Big)^{\frac{1}{q}} \\
&\overset{(2.25)}{\le} \hat{c}(detB)^{\frac{1}{q}}\Big(\sum_{|\alpha|=m} l^{-\alpha q}|\hat{D}^{\alpha}\hat{u}|^q_{W^{k+1-m,p}(\hat{K})}\Big)^{\frac{1}{q}} \\
&\overset{(3.7)}{=} \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}\Big[\sum_{|\alpha|=m} l^{-\alpha q}\Big(\sum_{|\beta|=k+1-m} l^{(\alpha+\beta)p}\|D_l^{\alpha+\beta} u\|^p_{L^p(K)}\Big)^{\frac{q}{p}}\Big]^{\frac{1}{q}} \\
&\overset{(3.3)}{\le} \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}\sum_{|\alpha|=m}\Big(\sum_{|\beta|=k+1-m} l^{\beta p}\|D_l^{\alpha+\beta} u\|^p_{L^p(K)}\Big)^{\frac{1}{p}} \\
&\le \hat{c}(detB)^{\frac{1}{q}-\frac{1}{p}}\Big(\sum_{|\beta|=k+1-m} l^{\beta p}|D_l^{\beta} u|^p_{W_l^{m,p}(K)}\Big)^{\frac{1}{p}}.
\end{aligned}$$

From (3.7) we can get

Lemma 3.4.

$$|v|_{W^{m,q}(K)} \le \|B_0^{-T}\|^m\, |v|_{W_l^{m,q}(K)}, \tag{3.10}$$

where $\|B_0^{-T}\|$ is the matrix norm.

From Theorem 3.3 and Lemma 3.4, we get

Theorem 3.5. Under the same assumptions as Theorem 2.4, then for triangular and the tetrahedral elements,

$$|u - \mathrm{I}u|_{W^{m,q}(K)} \le \hat{c}\|B_0^{-T}\|^m(detB)^{\frac{1}{q}-\frac{1}{p}}\Big(\sum_{|\beta|=k+1-m} l^{\beta p}|D_l^{\beta} u|^p_{W_l^{m,p}(K)}\Big)^{\frac{1}{p}}, \tag{3.11}$$

where $\hat{c}$ is independent of K.

Now we estimate $\|B_0^{-T}\|^m$.

1)Triangular element

Let $\underset{\sim}{v_i} = (\cos\varphi_i, \sin\varphi_i)^T$, φ_i be the angle between $\underset{\sim}{v_i}$ and x-axis, $1 \le i \le 2$, and θ be the angle between $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$ then

$$detB_0 = \sin(\varphi_2 - \varphi_1) = \sin\theta,$$

$$\|B_0^{-T}\| = \|(\underset{\sim}{v_1}, \underset{\sim}{v_2})^{-T}\| = \frac{1}{|detB_0|}\left\| \begin{pmatrix} \sin\varphi_2 & -\sin\varphi_1 \\ -\cos\varphi_2 & \cos\varphi_1 \end{pmatrix} \right\| \le \frac{2}{\sin\theta}. \tag{3.12}$$

2) Tetrahedral element
Similarly we have

$$\|B_0^{-T}\| \le \frac{6}{|detB_0|}. \tag{3.13}$$

Let $\mathcal{J} = \{\mathcal{J}_h\}_{h\to 0}$ be a family of decompositions of Ω into tetrahedra. There are three geometry conditions for tetrahedron elements.

1. Regular condition (cf. [11]).

$\mathcal{J}$ is said to be regular if there exists a constant $c_0 > 0$ such that for any $\mathcal{J}_h \in \mathcal{J}$ and any $K \in \mathcal{J}_h$ we have

$$c_0 h_K \le \rho_K,$$

where $h_K = diamK$ and $\rho_K = diamS_K$, here S_K is the biggest ball contained in K.

2. Regular vertex property (cf. [2]).

$\mathcal{J}$ is said to have the regular vertex property if there exists a constant $c_1 > 0$ such that for any $\mathcal{J}_h \in \mathcal{J}$ and any $K \in \mathcal{J}_h$, K has a vertex P_0 such that

$$|detB_0| \ge c_1,$$

where B_0 see (3.6).

3. Maximum angle condition (cf. [17]).

$\mathcal{J}$ is said to satisfy the maximum angle condition if there exists a constant $0 < \varphi_0 < \pi$ such that for any $\mathcal{J}_h \in \mathcal{J}$ and any $K \in \mathcal{J}_h$, the angles inside the faces and the angles between faces are all bounded above by φ_0.

It is well known that (see [17]) the regular vertex property is stronger than the maximum angle condition and weaker than the regular condition. Obviously $detB_0$ in (3.11) is corresponding to the regular vertex property which can degenerate to flat or needle meshes.

Meanwhile $detB_0$ can be expressed by the angles of K at P_0. In fact let $T = P_0Q_1Q_2Q_3$ be tetrahedron generated by the unit vectors $\underset{\sim}{v_i}$, $1 \le i \le 3$ (see Fig.1), $|T|$ be the volume of T. Obviously the angles inside the faces and interfacial angles of T are also K's ones at P_0.

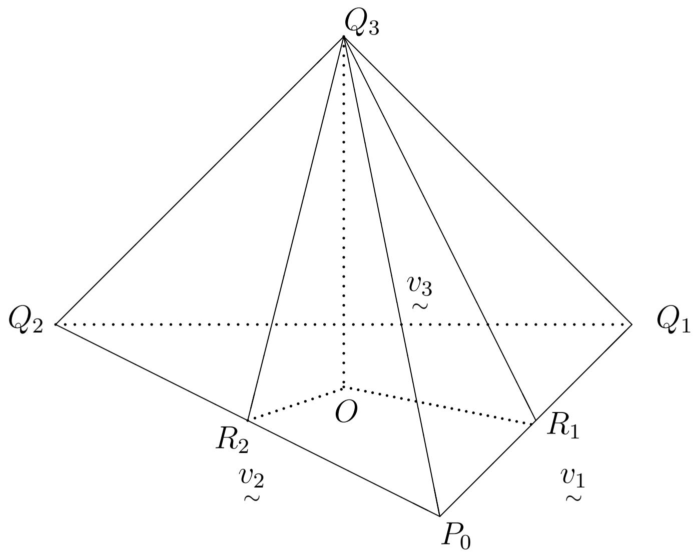

Fig. 1

Denote by $H = |Q_3O|$ the length of spatial altitude perpendicular to $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$. Denote by $r_1 = |Q_3R_1|$ and $r_2 = |Q_3R_2|$ the altitudes perpendicular to $\underset{\sim}{v_1}$ and $\underset{\sim}{v_2}$, respectively. Then $\varphi_1 = \angle Q_3R_1O$, $\varphi_2 = \angle Q_3R_2O$ are the interfacial angles between faces $P_0Q_3Q_1$ and $P_0Q_2Q_1$, $P_0Q_2Q_3$ and $P_0Q_2Q_1$, respectively. Let $\alpha_0 = \angle Q_1P_0Q_2$, $\alpha_i = \angle Q_iP_0Q_3$, $i = 1, 2$. Then

$$\begin{aligned} |detB_0| &= 6|T| = 2|\triangle P_0Q_1Q_2| \cdot H = |\underset{\sim}{v_1}||\underset{\sim}{v_2}| sin\alpha_0 \cdot r_i sin\varphi_i \\ &= sin\alpha_0 \cdot |\underset{\sim}{v_3}| sin\alpha_i \cdot sin\varphi_i = sin\alpha_0 sin\alpha_i sin\varphi_i, \quad i = 1, 2. \end{aligned}$$

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