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# Optimal space-time adaptive wavelet methods for degenerate parabolic PDEs

O. Reichmann

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

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# Optimal Space-Time adaptive Wavelet Methods for degenerate parabolic PDEs

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**Abstract** We analyze parabolic PDEs with certain type of weakly singular or degenerate time-dependent coefficients and prove existence and uniqueness of weak solutions in an appropriate sense. A localization of the PDEs to a bounded spatial domain is justified. For the numerical solution a space-time wavelet discretization is employed. An optimality result for the iterative solution of the arising systems can be obtained. Applications to fractional Brownian motion models in option pricing are presented.

**Keywords** Degenerate parabolic differential equations, wavelets, adaptivity, optimal computational complexity, best N-term approximation, fractional Brownian Motion.

**Mathematics Subject Classification (2000)** 35K65 · 41A25 · 65T60 · 65N22 · 60G22

## 1 Introduction

This work aims at the analysis of certain type of degenerate linear parabolic differential equations and the design of an efficient algorithm for their numerical treatment. The numerical analysis of degenerate parabolic Kolmogorov equations with weakly singular or degenerate coefficients is of independent interest. We present the pricing of European type options under a fractional Brownian Motion (FBM) market model as our main application.

O. Reichmann
Rämistrasse 102 8092 Zürich
Tel.: +41 44 632 5566 0
Fax: +41 1 632 1104
E-mail: oleg.reichmann@sam.math.ethz.ch

The arising PDE reads as follows:

$$\partial_t u - t^\gamma L u = f \text{ on } I \times D \quad (1.1)$$
$$u(0) = g, \quad (1.2)$$

where $L$ denotes a diffusion operator, $g$ the sufficiently smooth initial data, $\gamma$ a constant with $\gamma \in (-1, 1)$, $I = [0, 1]$ and a Lipschitz domain $D \subset \mathbb{R}^d$ for $d \geq 1$. Note that negative exponents $\gamma$ lead to an explosion at $t = 0$, while positive $\gamma$ lead to a degeneracy of the diffusion coefficients. Therefore the initial condition has to be imposed in an appropriate sense.
We consider a weak space-time formulation in the sense of [3,34], as a possible singularity or degeneracy of the diffusion coefficients impedes the application of classical parabolic theory, cf. [2,30]. The use of appropriate wavelet bases in the space-time domain leads to Riesz bases for the ansatz and test spaces, cf. [6,34]. As pricing problems are typically posed on unbounded spatial domains, a localization for the PDE with different boundary conditions and the arising truncation estimates are presented.
The use of Riesz bases in conjunction with the compressibility of the corresponding operator enables us to prove the optimality of the solution process for the arising bi-infinite linear system.
The fractional Brownian motion was introduced by Kolmogorov [25] under the name "Wiener Spiral". The current name is due to the pioneer work of Mandelbrot and Van Neus [26]. Theoretical properties such as stochastic integration with respect to FBM and stochastic differential equations driven by FBM have received a lot of attention, cf. [5,21,20,24] and the monograph [7]. Applications of fractional Brownian motion are not restricted to finance [29], but an extensive amount of literature is devoted to applications in modeling foreign exchange options, weather derivatives and other types of products. For simple contracts such as plain vanilla European options closed form solutions can be derived, for instance [28,5]. In general these are not available and numerical methods have to be employed. Though there exists literature on path simulation for FBM, eg. [1,26,31,36], deterministic solution methods have, to our knowledge, not been analyzed so far.
The remainder of the paper is structured as follows. In the following section we present two uniqueness and existence results for degenerate parabolic PDEs in a weak space-time formulation with different enforcement of the initial conditions. In Section 3 the discretization of the PDEs is presented using space-time wavelets. Section 4 presents an optimality result for the solution of the arising bi-infinite systems using the algorithm of [11] or [12]. Subsequently, the application of the derived theory to the pricing of European options under an FBM market model is described. Finally, we conclude and bring up some open questions.

## 2 Weak formulation

In this section we derive two weak space-time formulations for degenerate parabolic equations such as (1.1)-(1.2) in arbitrary space dimensions. The main difference between the two formulations described lies in the enforcement of the initial condition. Well-posedness results as well as a-priori estimates can be obtained based on eigenfunction expansion of the operator $L$.

### 2.1 Essential initial condition

We consider the following degenerate parabolic problem:

$$\partial_t u - t^\gamma L u = f \text{ on } I \times D, \tag{2.1}$$

$$u(0) = g, \tag{2.2}$$

where $L$ is defined by

$$L := \frac{\gamma+1}{2} \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} a_{j,k}(x) \frac{\partial}{\partial x_k},$$

for $\gamma = 2H - 1$, $H \in (0,1)$, a bounded Lipschitz domain $D \subset \mathbb{R}^d$, finite time interval $I := (0,T)$ and smooth functions $\overline{a} \geq a_{j,k}(x) \geq \underline{a} > 0$, $1 \leq j,k \leq d$. The bilinear form $a(\cdot,\cdot)$ associated with $L$ reads

$$a(u,v) : V \times V \to \mathbb{R}, \qquad a(u,v) = \langle Lu, v\rangle, \quad \forall u,v \in V. \tag{2.3}$$

To state the variational formulation of (2.1)-(2.2) we introduce the following spaces

$$\mathcal{X} := H^1_{t^{-\gamma/2}}(I; V^*) \cap L^2_{t^{\gamma/2}}(I;V) \tag{2.4}$$

$$\cong \left(H^1_{t^{-\gamma/2}}(I) \otimes V^*\right) \cap \left(L^2_{t^{\gamma/2}}(I) \otimes V\right),$$

$$\mathcal{Y} := L^2_{t^{\gamma/2}}(I;V) \cong L^2_{t^{\gamma/2}}(I) \otimes V, \tag{2.5}$$

$$\mathcal{X}_{(0} := \{w \in \mathcal{X} : w(0,\cdot) = 0 \text{ in } V^*\}, \tag{2.6}$$

$$\mathcal{X}_{0)} := \{w \in \mathcal{X} : w(T,\cdot) = 0 \text{ in } V^*\}, \tag{2.7}$$

where $V := H^1_0(D)$, $V^* = H^{-1}(D)$, $L^2_{t^{\gamma/2}}(I) = \overline{C^\infty(0,1)}^{\|\cdot\|_{L^2_{t^{\gamma/2}}(I)}}$ and $H^1_{t^{\gamma/2}}(I) = \overline{C^\infty(0,1)}^{\|\cdot\|_{H^1_{t^{\gamma/2}}(I)}}$. The weighted norms are given by

$$\|u\|^2_{L^2_{t^{\gamma/2}}(I)} := \int_I u^2 t^\gamma \, dt, \qquad \|u\|^2_{H^1_{t^{\gamma/2}}(I)} := \int_I u^2 t^\gamma \, dt + \int_I \dot{u}^2 t^\gamma \, dt.$$

We now show the following result.

**Theorem 2.1** *For every $f \in \mathcal{Y}^*$, $g = 0$ (2.1) admits a unique solution $u \in \mathcal{X}_{(0}$ and there holds the a-priori error estimate*

$$\|u\|_{\mathcal{X}} \leq \sqrt{2}\, \|f\|_{\mathcal{Y}^*}.$$

The proof follows from the inf-sup condition, the surjectivity and the continuity of the corresponding bilinear form using, eg. [4] or [8, III, Theorem 4.3]. These properties will be proved in the following. For the subsequent result we set $b_\lambda : X \times Y \to \mathbb{R}$, with $X := \left\{u \in L^2_{t^{\gamma/2}}(I) \cap H^1_{t^{-\gamma/2}}(I) : u(0) = 0\right\}$ and $Y := L^2_{t^{\gamma/2}}(I)$,

$$b_\lambda(u, v) = \int_I t^{\gamma/2} v \left(\lambda^{-\frac{1}{2}} t^{-\gamma/2} \dot{u} + \lambda^{\frac{1}{2}} t^{\gamma/2} u\right) dt, \quad \lambda > 0.$$

We remark that $H^1_{t^{-\gamma/2}}(I) \subset C_0(I)$ holds, this follows as in Lemma 2.4. For $u$ in $X$ we define the seminorm:

$$\|u\|_{X^\lambda} := \left\|\lambda^{-\frac{1}{2}} t^{-\gamma/2} \dot{u} + \lambda^{\frac{1}{2}} t^{\gamma/2} u\right\|_{L^2(I)}.$$

**Lemma 2.1** *For $\lambda > 0$ and $u \in X$, define the norm $\|u\|_\lambda$ by*

$$\|u\|_\lambda^2 := \lambda^{-1} \left\|t^{-\gamma/2} \dot{u}\right\|^2_{L^2(I)} + \lambda \left\|t^{\gamma/2} u\right\|^2_{L^2(I)}.$$

*Then, for all $u \in X$ holds:*

$$\|u\|_\lambda \le \|u\|_{X^\lambda} \le \sqrt{2}\, \|u\|_\lambda .$$

*Proof* Let $u \in X$, then

$$\|u\|^2_{X^\lambda} = \lambda^{-1} \left\|t^{-\gamma/2} \dot{u}\right\|^2 + \lambda \left\|t^{\gamma/2} u\right\|^2 + 2 \int_I u \dot{u}\, dt = \|u\|_\lambda^2 + |u(T)|^2 \ge \|u\|^2_\lambda .$$

Further,

$$\begin{aligned} 2 \left|\int_I u \dot{u}\, dt\right| &\le 2\lambda^{1/2} \left\|t^{\gamma/2} u\right\|_{L^2(I)} \lambda^{-1/2} \left\|t^{-\gamma/2} \dot{u}\right\|_{L^2(I)} \\ &\le \lambda \left\|t^{\gamma/2} u\right\|^2_{L^2(I)} + \lambda^{-1} \left\|t^{-\gamma/2} \dot{u}\right\|^2_{L^2(I)} \end{aligned}$$

and therefore $\|u\|^2_{X^\lambda} \le 2\, \|u\|^2_\lambda$.

**Lemma 2.2** *We have*

$$\inf_{0 \ne u \in \mathcal{X}_{(0}} \sup_{0 \ne v \in \mathcal{Y}} \frac{B(u, v)}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} \ge \frac{1}{\sqrt{2}}, \tag{2.8}$$

$$\forall\, 0 \ne v \in \mathcal{Y} : \sup_{u \in \mathcal{X}_{(0}} B(u, v) > 0 \tag{2.9}$$

*and*

$$\sup_{0 \ne u \in \mathcal{X}_{(0}, 0 \ne v \in \mathcal{Y}} \frac{|B(u, v)|}{\|u\|_{\mathcal{X}} \|v\|_{\mathcal{Y}}} < \infty, \tag{2.10}$$

*where*

$$B(u, v) := \int_0^T \left(\langle v, \dot{u}\rangle + t^\gamma a(u, v)\right) dt, \tag{2.11}$$

*for $u \in \mathcal{X}_{(0}$ and $v \in \mathcal{Y}$.*

*Proof* Let $u \in \mathcal{X}$. Then $u = \sum_{\lambda\in\sigma} u_\lambda \phi_\lambda$, $v \in \mathcal{Y}$, $v = \sum_{\lambda\in\sigma} v_\lambda \phi_\lambda$, where $\phi_\lambda$ are the eigenfunctions of $L$ and $\sigma \subset \mathbb{R}_+$ denotes the countable family of eigenvalues of $L$, where the $(\phi_\lambda)_{\lambda\in\sigma}$ are assumed to form an orthonormal basis of $L^2(D)$, then

$$\begin{aligned} B(u,v) &= \int_0^T \left(\langle v, \dot{u}\rangle + t^\gamma a(u,v)\right) dt \\ &= \sum_{\lambda\in\sigma} \int_0^T \lambda^{1/2} v_\lambda t^{\gamma/2} \left(\lambda^{-1/2} t^{-\gamma/2} \dot{u}_\lambda + \lambda^{1/2} t^{\gamma/2} u_\lambda\right) dt. \end{aligned}$$

Therefore,

$$\begin{aligned} |B(u,v)| &\le \left(\sum_{\lambda\in\sigma} \lambda \int_0^T t^\gamma |v_\lambda|^2 \, dt\right)^{1/2} \\ &\quad \times \left(\sum_{\lambda\in\sigma} \int_0^T \left|\lambda^{-1/2} t^{-\gamma/2} \dot{u}_\lambda + \lambda^{1/2} t^{\gamma/2} u_\lambda\right|^2 dt\right)^{1/2} \\ &= \|v\|_{L^2_{t^{\gamma/2}}(I;V)} \left(\sum_{\lambda\in\sigma} \|u_\lambda\|^2_{X^\lambda}\right)^{1/2} \\ &\le \|v\|_{L^2_{t^{\gamma/2}}(I;V)} \sqrt{2} \left(\sum_{\lambda\in\sigma} \|u_\lambda\|^2_\lambda\right)^{1/2} = \sqrt{2}\, \|u\|_{\mathcal{X}} \, \|v\|_{\mathcal{Y}} . \end{aligned}$$

This implies (2.10). Next given $u \in \mathcal{X}_{(0}$, we define $v_u = \sum_{\lambda\in\sigma} \phi_\lambda v_\lambda$ by

$$v_\lambda = \lambda^{-1} t^{-\gamma} \dot{u}_\lambda + u_\lambda,$$

then

$$\begin{aligned} \|v_u\|^2_{\mathcal{Y}} &= \sum_{\lambda\in\sigma} \lambda \int_0^T t^\gamma \left(\lambda^{-1} t^{-\gamma} \dot{u}_\lambda + u_\lambda\right)^2 \, dt \\ &= \sum_{\lambda\in\sigma} \int_0^T \left(\lambda^{-1/2} \dot{u}_\lambda t^{-\gamma/2} + \lambda^{1/2} u_\lambda t^{\gamma/2}\right)^2 \, dt \\ &= \sum_{\lambda\in\sigma} \|u_\lambda\|^2_{X^\lambda} \le 2\, \|u\|^2_{\mathcal{X}} . \end{aligned} \tag{2.12}$$

$$\begin{aligned} B(u,v_u) &= \int_0^T \langle v_u, \dot{u}\rangle + t^\gamma a(u, v_u) \, dt \\ &= \sum_{\lambda\in\sigma} \int_0^T \langle \lambda^{-1} t^{-\gamma} \dot{u}_\lambda + u_\lambda, \dot{u}_\lambda\rangle + \lambda t^\gamma \left(\lambda^{-1} t^{-\gamma} \dot{u}_\lambda + u_\lambda, u_\lambda\right) \, dt \\ &= \sum_{\lambda\in\sigma} \int_0^T \left(\lambda^{-1} t^{-\gamma} |\dot{u}_\lambda|^2 + \frac{d}{dt} |u_\lambda|^2 + \lambda t^\gamma |u_\lambda|^2 \, dt\right) \\ &= \|u\|^2_{\mathcal{X}} + \|u(T)\|^2_{L^2(D)} - \|u(0)\|^2_{L^2(D)} . \end{aligned}$$

This implies (2.8) using (2.12). Let now $v = \sum_{\lambda\in\sigma} v_\lambda \phi_\lambda$ be given, we define $u_v = \sum_\lambda u_\lambda \phi_\lambda$, where $(u_\lambda)_{\lambda\in\sigma}$ is given as solutions of the following sequence of initial value problems.

$$\lambda^{-1} t^{-\gamma} \dot{u}_\lambda + u_\lambda = v_\lambda \text{ for } t \in (0,T), \quad u_\lambda(0) = 0.$$

In the following it will be shown that $v \in \mathcal{Y}$ implies $u_v \in \mathcal{X}$. We have

$$\begin{aligned}
\|v\|_{\mathcal{Y}}^2 &= \sum_{\lambda\in\sigma} \int_0^T t^\gamma \lambda \left|v_\lambda\right|^2 \, dt \\
&= \sum_{\lambda\in\sigma} \int_0^T \lambda \left| \lambda^{-1/2} t^{-\gamma} \dot{u}_\lambda + \lambda^{1/2} u_\lambda \right|^2 \\
&= \sum_{\lambda\in\sigma} \|u_\lambda\|_{X^\lambda}^2 \geq \sum_{\lambda\in\sigma} \|u_\lambda\|_\lambda^2 = \|u_v\|_{\mathcal{X}}^2 .
\end{aligned}$$

We are now able to prove statement (2.9).

$$\begin{aligned}
B(u_v, v) &= \int_0^T \langle v, \dot{u}_v \rangle + t^\gamma a(u_v, v) \, dt \\
&= \sum_{\lambda\in\sigma} \int_0^T v_\lambda \dot{u}_\lambda + \lambda u_\lambda v_\lambda t^\gamma \, dt \\
&= \sum_{\lambda\in\sigma} \int_0^T \lambda t^\gamma \left|v_\lambda\right|^2 \, dt = \|v\|_{\mathcal{Y}}^2 > 0.
\end{aligned}$$

**Theorem 2.2** *For every $f \in \mathcal{Y}^*$ the problem* (2.1)-(2.2) *with $g = 0$ admits a unique solution $u \in \mathcal{X}_{(0}$ satisfying*

$$u \in \mathcal{X}_{(0} : \quad B(u, v) = \langle f, v \rangle, \quad \forall v \in \mathcal{Y}.$$

*With $\mathcal{X}$ and $\mathcal{Y}$ as in* (2.4)-(2.5) *and $B(\cdot,\cdot)$ as in Lemma 2.2, we have the a-priori estimate*

$$\|u\|_{\mathcal{X}}^2 \leq 2 \|f\|_{\mathcal{Y}}^2 .$$

*The existence of a unique weak solution for non-homogeneous initial data follows via the following change of variable $\widetilde{v}(t,x) = v(t,x) - g$, for $g \in V$. The function $\widetilde{v}(t,x)$ satisfies the same PDE as $v(t,x)$ with homogeneous initial conditions and a different right hand side.*

## 2.2 Natural initial condition

As we assume non-homogeneous initial conditions, we can either transform the problem into a homogeneous setting as described in Section 2.1 or impose natural conditions as follows:

$$\int_0^T (v, \dot{u}) dt = -\int_0^T (\dot{v}, u) dt + (u, v)|_0^T \text{ for } v, u \in C^\infty(I).$$

For $u(0) \neq 0$ we impose homogeneous Dirichlet conditions on $v$, i.e. we require $v(T) = 0$. The variational formulation with weak enforcement of the initial conditions then reads: given $f \in \mathcal{X}^*_{0)}$, $g \in V$:

$$u \in \mathcal{Y}: \quad B^*(u,v) = \langle v, f\rangle + \langle v(0), g\rangle, \quad \forall v \in \mathcal{X}_{0)}, \tag{2.13}$$

where $B^*(\cdot,\cdot)$ is given by

$$B^*(u,v) = \int_0^T (-\langle \dot{v}, u\rangle + a(u,v))\,dt, \text{ for } u \in \mathcal{Y},\, v \in \mathcal{X}_{0)}, \tag{2.14}$$

with $a(\cdot,\cdot)$ given in (2.3). We define the functional $l^*(v)$ on $\mathcal{X}$ as follows:

$$l^*(v) := \langle v, f\rangle + \langle v(0), g\rangle.$$

**Lemma 2.3** *For $f \in \mathcal{X}^*_{0)}$ and for $g \in V$, $l^*$ is a continuous, linear functional on $\mathcal{X}_{0)}$, i.e., there exists a $C > 0$ s.t.*

$$\forall v \in \mathcal{X}_{0)}: \quad |l^*(v)| \leq C \left( \|f\|_{\mathcal{X}^*_{0)}} + \|g\|_V \right) \|v\|_{\mathcal{X}_{0)}}.$$

*Proof* For $f \in \mathcal{X}^*_{0)}$ we have:

$$|\langle v, f\rangle| \leq \|v\|_{\mathcal{X}_{0)}} \|f\|_{\mathcal{X}^*_{0)}}.$$

By the embedding given in (2.15) we obtain for $v \in \mathcal{X}_{0)}$

$$\|v(0)\|_{V^*} \leq \|v\|_{C^0(\overline{I},V^*)} \leq C\,\|v\|_{\mathcal{X}},$$

which implies,

$$|\langle v(0), g\rangle| \leq \|g\|_V \|v(0)\|_{V^*} \leq C\,\|g\|_V \|v\|_{\mathcal{X}}.$$

This implies the claimed result.

We need the following embedding result.

**Lemma 2.4** *For $\mathcal{X} := H^1_{t^{-\gamma/2}}(I;V^*) \cap L^2_{t^{\gamma/2}}(I;V)$ the following continuous embedding holds:*

$$\mathcal{X} \subset C^0(\overline{I}, D(\Lambda^{\frac{1}{2}-\frac{|\gamma|}{2}})), \tag{2.15}$$

*where $\Lambda$ denotes the operator $\Lambda = L^{1/2}$, as defined in [14, Chapter VIII, §3, Definition 8]. The operator $\Lambda^\theta$ denotes the holomorphic interpolant between $V$ and $V^*$.*

*Proof* Consider first $\gamma \in (-1,0)$, then $L^2_{t^{\gamma/2}}(I,V) \subset L^2_{t^{-\gamma/2}}(I,V)$. For the space $H^1_{t^{-\gamma/2}}(I,V^*) \cap L^2_{t^{-\gamma/2}}(I,V)$, the claimed result follows from [15, Chapter XVIII, §1, Remark 6]. Let now $\gamma \in (0,1)$. Then $H^1_{t^{-\gamma/2}}(I,V^*) \subset H^1_{t^{\gamma/2}}(I,V^*)$, therefore we can again apply [15, Chapter XVIII, §1, Remark 6] and conclude.

*Remark 2.1* (i) The space $H^1_{t^{-\gamma/2}}(I, V^*) \cap L^2_{t^{-\gamma/2}}(I, V)$, for $\gamma \in (0,1)$, is continuously embedded in
$C^0(\overline{I}, D(\Lambda^{\frac{1}{2}+\frac{\gamma}{2}}))$, cf.[15, Chapter XVIII, §1, Remark 6].

(ii) The elementary embedding of $\mathcal{X}$ in $C^0(\overline{I}, V^*)$ can be shown as follows, cf. [23, Proposition 1.1],

$$\int_0^T \|v(t)\|_{V^*}\, dt \leq \left( \int_0^T \|v(t)\|^2_{V^*}\, t^{-\gamma} dt \right)^{1/2} \left( \int_0^T t^{\gamma} dt \right)^{1/2}.$$

Therefore the mapping $K : u \to u'$, $K : \mathcal{X} \to L^1_{\mathrm{loc}}(I, V^*)$ is continuous. This implies that $v$ is absolutely continuous on $\overline{I}$ with values in $V^*$. Note that this does not imply the continuity of the embedding.

(iii) We obtain an analogous result for the weight function $(T-t)^{\gamma}$ instead of $t^{\gamma}$.

(iv) To our knowledge, it is not known if the embedding given in Lemma 2.4 is sharp.

**Theorem 2.3** *Let $B^*(\cdot,\cdot)$ be given as in* (2.14) *and $\mathcal{X}$, $\mathcal{Y}$ as in* (2.4)-(2.5). *Then the following estimates hold*

$$\begin{aligned}
&\inf_{0 \neq u \in \mathcal{Y}} \sup_{0 \neq v \in \mathcal{X}_{0)}} \frac{B^*(u,v)}{\|u\|_{\mathcal{Y}} \|v\|_{\mathcal{X}_{0)}}} \geq \frac{1}{\sqrt{2}}, \\
&\forall 0 \neq v \in \mathcal{X}_{0)} : \quad \sup_{u \in \mathcal{Y}} B^*(u,v) > 0, \\
&\sup_{0 \neq v \in \mathcal{X}_{0)}, 0 \neq u \in \mathcal{Y}} \frac{|B(u,v)|}{\|u_{\mathcal{Y}}\| \, \|v\|_{\mathcal{X}}} < \infty.
\end{aligned}$$

*Proof* The proof is analogous to the proof of Lemma 2.2.

**Corollary 2.1** *For every $g \in V$ and $f \in \mathcal{X}^*_{0)}$, there exists a unique weak solution $u \in \mathcal{Y}$ in the sense that $u$ satisfies* (2.13).

*Remark 2.2* Note that for this formulation smoothness of the initial data is required, i.e. $g \in V$. This is stronger than in the standard parabolic setting, as in this situation $g \in L^2(D)$ is sufficient in order to prove well-posedness of the corresponding weak formulation. This stronger condition stems from the fact that in the setup only the continuous embedding $\mathcal{X} \subset C^0(\overline{I}, \Lambda^{\frac{1}{2}-\frac{|\gamma|}{2}})$ can be proved, while in the standard parabolic case $\left(L^2(I,V) \cap H^1(I,V^*)\right) \subset C^0(\overline{I}, L^2(D))$ holds.

*Remark 2.3* Alternatively the following formulation with natural initial conditions could also be considered. Find $w \in \mathcal{X}$ such that

$$\begin{aligned}
B^{\dagger}(w,v) &= f^{\dagger}(v), \text{ for all } v := (v_1, v_2) \in \mathcal{Y} \times V, \text{ where} \qquad (2.16)\\
B^{\dagger}(w,v) &= \int_0^T (\langle \dot{w}, v_1 \rangle + a(w, v_1))\, dt + \langle w(0), v_2 \rangle, \\
f^{\dagger}(v) &= \langle v_1, f \rangle + (g, v_2).
\end{aligned}$$

The well-posedness of (2.16) follows as in Lemma 2.2. The advantage of formulation (2.16) is the absence of any boundary conditions in the temporal domain, therefore the bases presented in the next section can be used for the discretization without any additional considerations.

## 3 Discretization

For the space-time discretization of the degenerate parabolic PDE, given by (2.1), we follow [34] and [3]. A crucial role for the efficient discretization is the use of tensor product Riesz bases on the space-time domain. We construct appropriate bases in the following and prove the necessary norm equivalences.

### 3.1 Wavelets

To present the space-time discretization, we briefly recapitulate basic definitions and results on wavelets from, e.g., [10] and the references therein. For specific spline wavelet constructions on a bounded interval $I$, we refer to e.g. [16], [32] and [35]. Our use of compactly supported, piecewise polynomial multiresolution systems (rather than the more commonly employed B-spline finite element spaces) for the Galerkin discretization of corresponding equations is motivated by the following key properties of these spline wavelet systems: a) the approximation properties of the multiresolution sytems equal those of the B-spline systems, b) the spline wavelet systems form *Riesz bases* on the corresponding spaces allowing for simple and efficient *preconditioning of the arising matrices*, c) the spline wavelet systems can be designed to have a large number of *vanishing moments*. We recapitulate the basic definitions from, e.g., [10,35] to which we also refer for further references and additional details, such as the construction of higher order wavelets.
Our wavelet systems are two-parameter systems $\{\psi_{l,k}\}_{l=-1,\dots,\infty,k\in\nabla_l}$ of compactly supported functions $\psi_{l,k}$, where $\nabla_l$ denotes the set of wavelet indices on level $l$. Here the first index, $l$, denotes "level" of refinement resp. resolution: wavelet functions $\psi_{l,k}$ with large values of the level index are well-localized in the sense that $\mathrm{diam}(\mathrm{supp}\psi_{l,k}) = O(2^{-l})$. The second index, $k \in \nabla_l$, measures the *localization* of wavelet $\psi_{l,k}$ within the interval $I$ at scale $l$ and ranges in the index set $\nabla_l$. In order to achieve maximal flexibility in the construction of wavelet systems (which can be used to satisfy other requirements, such as minimizing their support size or to minimize the size of constants in norm equivalences), we consider wavelet systems for the spatial discretization which are *biorthogonal* in $L^2(I)$, consisting of a primal wavelet system $\{\psi_{l,k}\}_{l=-1,\dots,\infty,k\in\nabla_l}$ which is a Riesz basis of $L^2(I)$ and a corresponding dual wavelet system $\{\widetilde{\psi}_{l,k}\}_{l=-1,\dots,\infty,k\in\nabla_l}$ (which will never be used explicitly in our algorithms).

The primal wavelet bases $\psi_{l,k}$ span finite dimensional spaces

$$\mathcal{W}^l := \text{span}\,\{\psi_{l,k} : k \in \nabla_l\}\ , \quad \mathcal{V}^L := \bigoplus_{l=-1}^{L-1} \mathcal{W}^l \quad l = -1, 0, 1 \ldots .$$

The dual spaces are defined analogously in terms of the dual wavelets $\widetilde{\psi}_{l,k}$ by

$$\widetilde{\mathcal{W}}^l := \text{span}\,\{\widetilde{\psi}_{l,k} : k \in \nabla_l\}\ , \quad \widetilde{\mathcal{V}}^L := \bigoplus_{l=-1}^{L-1} \widetilde{\mathcal{W}}^l \quad l = -1, 0, 1 \ldots .$$

In the sequel we require the following properties of the wavelet functions to be used on our Galerkin discretization schemes, we assume without loss of generality $I = (0,1)$ for the time interval and $D = (0,1)^d$ for the physical domain. The use of a hypercube as the spatial domain enables us to construct the basis functions for the discretization of the physical space as tensor products of univariate basis functions. Besides, we could also use sparse tensor products to overcome the curse of dimension, cf. [17] for the elliptic case. Domains of this form arise naturally in the discretization of pricing equations due to localization, cf. Section 5. We now state the requirements for the temporal wavelet basis $\Theta = \{\theta_\lambda : \lambda \in \nabla_\Theta\}$, where $\nabla_\Theta$ denotes the set of all wavelet indices.

(t1) Biorthogonality: the basis functions $\theta_{l,k}$, $\widetilde{\theta}_{l,k}$ satisfy

$$\langle \theta_{l,k}, \widetilde{\theta}_{l',k'} \rangle = \delta_{l,l'}\delta_{k,k'} . \tag{3.1}$$

(t2) Local support: the diameter of the support is proportional to the meshsize $2^{-l}$,

$$\text{diam supp}\ \theta_{l,k} \sim 2^{-l} \quad , \quad \text{diam supp}\ \widetilde{\theta}_{l,k} \sim 2^{-l} . \tag{3.2}$$

(t3) Piecewise polynomial of order $p_t$, where piecewise means that the singular support consists of a uniformly bounded number of points over all levels.

(t4) Vanishing moments: The primal basis functions $\theta_{l,k}$ are assumed to satisfy vanishing moment conditions up to order $p_t > 1$

$$\langle \theta_{l,k}, x^\alpha \rangle = 0\ ,\ \alpha = 0, \ldots, d = p_t,\ l \geq 0. \tag{3.3}$$

The dual wavelets are assumed to satisfy

$$\langle \widetilde{\theta}_{l,k}, x^\alpha \rangle = 0\ ,\ \alpha = 0, \ldots, \widetilde{d},\ l \geq 0, \tag{3.4}$$

for $\widetilde{d} \geq d$.

(t5) We assume the following norm equivalences, for all $0 \leq s \leq \kappa$ and a $\kappa \geq 1$

$$\|u\|_s^2 \sim \sum_{l=-1}^{\infty} \sum_{k \in \nabla_l} 2^{2ls} \left| u_k^l \right|^2 , \quad u_k^l = \langle \widetilde{\theta}_{k,l}, u \rangle,$$

where $\|\cdot\|_s$ denotes the $H^s(0,1)$-norm.

Further we require that the wavelets and the dual wavelets for the time domain belong to $W^{1,\infty}(0,1)$ and the boundary wavelets for the time discretization satisfy:

$$\begin{aligned}
\left|\theta_k^l(x)\right| &\leq C_\theta 2^{l/2}(2^l x)^{\beta},\\
\left|(\theta_k^l)'(x)\right| &\leq C_\theta 2^{3l/2}(2^l x)^{\beta-1},\ x \in [0,2^{-l}],\ \beta \in \mathbb{N}_0,\ k \in \nabla_l^L,\\
\left|\widetilde{\theta}_k^l(x)\right| &\leq C_\theta C_\theta 2^{l/2}(2^l x)^{\widetilde{\beta}},\\
\left|(\widetilde{\theta}_k^l)'(x)\right| &\leq C_\theta 2^{3l/2}(2^l x)^{\widetilde{\beta}-1},\ x \in [0,2^{-l}],\ \widetilde{\beta} \in \mathbb{N}_0,\ k \in \widetilde{\nabla}_l^L,
\end{aligned}$$

where $\gamma/2+\beta > -\frac{1}{2}$ and $-\gamma/2+\widetilde{\beta} > -\frac{1}{2}$ with $\gamma$ as in (2.1). The sets $\nabla_l^L$ and $\widetilde{\nabla_l^L}$ are given as follows, $\nabla_l^L := \{k \in \nabla_l : 0 \in \text{supp}\theta_k^l\}$ and $\widetilde{\nabla}_l^L := \{k \in \widetilde{\nabla}_l : 0 \in \text{supp}\widetilde{\theta}_k^l\}$. We refer to [13] for explicit constructions.
The spatial basis is constructed as follows: we define the subspace $V_L$ of $H_0^1(D)$, for $D=[0,1]^d$, as the full tensor product of $d$ univariate approximation spaces, i.e. $V_L := \bigotimes_{1\leq i\leq d} \mathcal{V}^{l_i}$, which can be written as

$$V_L = \{\sigma_{\mathbf{l},\mathbf{k}} : -1 \leq l_i \leq L-1, k_i \in \nabla_{l_i}, i=1,\ldots,d\},$$

with basis functions $\sigma_{\mathbf{l},\mathbf{k}} = \sigma_{l_1,k_1}\cdots\sigma_{l_d,k_d}$, $-1 \leq l_i \leq L-1$, $k_i \in \nabla_{l_i}$, $i = 1,\ldots,d$, where $\nabla_{l_i}$ denotes the set of wavelet coefficients in the $i$-th coordinate on level $l_i$. We can write $V_L$ in terms of increment spaces

$$V_L = \bigoplus_{-1\leq l_i\leq L-1} \mathcal{W}^{l_1}\otimes\ldots\otimes\mathcal{W}^{l_d}.$$

We denote by $\Sigma = \{\sigma_\mu : \mu \in \nabla_\Sigma\} = \bigotimes_{i=1}^d \Sigma_i$, $\Sigma_i = \{\sigma_{\mu_i} : \mu_i \in \nabla_{\Sigma_i}\}$. The tensor product spatial basis satisfies the following assumptions, where $\nabla_\Sigma$ is the set of all wavelet multi-indices and $\nabla_{\Sigma_i}$ denotes the set of all wavelet indices in the $i$-th coordinate.

(s1) Local support: the diameter of the support is proportional to the meshsize $2^{-l}$,

$$\text{diam supp } \sigma_{l,k} \sim 2^{-l}. \tag{3.5}$$

(s2) Continuity: the primal basis function are assumed to be elements in $C^{r_x}(0,1)$, with $r_x \leq p_x - 2$.
(s3) Piecewise polynomial of order $p_x$, where piecewise means that the singular support consists of a uniformly bounded number of points.
(s4) Vanishing moments: The primal basis functions $\sigma_{l,k}$ are assumed to satisfy vanishing moment conditions up to order for $p_x > 1$

$$\langle \sigma_{l,k}, x^\alpha\rangle = 0\ ,\ \alpha = 0,\ldots,d = p_x,\ l \geq 0. \tag{3.6}$$

(s5) Orthonormality in $L^2(0,1)$.
(s6) Riesz basis property in $L^2(0,1)$ and renormalized in $H_0^1(0,1)$ and $H^{-1}(0,1)$.

We refer to [18] and [19] for explicit constructions.

### 3.2 Time Discretization

Using the wavelet constructions of the previous section we are now able to obtain Riesz bases for the spaces $L^2_{t^{\gamma/2}}(0,1)$ and $H^1_{t^{\gamma/2}}(0,1)$

**Theorem 3.1** *The norm* $|||\cdot|||_{L^2_{t^{\gamma/2}}(0,1)}$ *is given as*

$$|||u|||^2_{L^2_{t^{\gamma/2}}(0,1)} := \sum_{l=-1}^{\infty} \sum_{k\in\nabla_l} (2^{-l}k)^{\gamma} \left|u^l_k\right|^2 , \tag{3.7}$$

*where* $u \in L^2_{t^{\gamma/2}}(0,1)$ *admits the unique representation*

$$u = \sum_{l=-1}^{\infty} \sum_{k\in\nabla_l} u^l_k \theta^l_k, u^l_k = \langle \widetilde{\theta}_{k,l}, u\rangle.$$

*Then the following norm equivalence holds for all functions* $u \in L^2_{t^{\gamma/2}}(0,1)$*:*

$$\|u\|^2_{L^2_{t^{\gamma/2}}(0,1)} \sim |||u\,|||^2_{L^2_{t^{\gamma/2}}(0,1)} . \tag{3.8}$$

*Proof* The result follows from [6, Theorem 3.3] setting $\omega = t^{\gamma/2}$ and checking Assumption 3.1 and 3.2 in [6].

A similar result can be obtained for $H^1_{t^{\gamma/2}}(0,1)$ using the following theorem:

**Theorem 3.2** *Let* $\Theta$ *be as above and let* $u \in H^1_{t^{\gamma/2}}(0,1)$*, then*

$$\|u'\|^2_{L^2_{t^{\gamma/2}}(0,1)} \sim \sum_l 2^{2l} \sum_k (2^{-l}k)^{\gamma} \left|u^l_k\right|^2 .$$

*Proof* See [6, Theorem 5.1].

Therefore $\Theta$ forms after diagonal scaling a Riesz basis of $H^1_{t^{\gamma/2}}(0,1)$.

*Remark 3.1* Note that analogous results can be obtained for the weight function $w(t) = \prod_{j=1}^{k}(t_k - t)^{\gamma_k}$.

### 3.3 Space-time discretization

We are now able to construct a Riesz basis for the spaces $\mathcal{X}$ and $\mathcal{Y}$ in the case of a bounded spatial domain. The spaces have the following tensor product structure:

$$\mathcal{X} = (L^2_{t^{\gamma/2}}(I) \otimes V) \cap (H^1_{t^{-\gamma/2}}(I) \otimes V^*) \text{ and } \mathcal{Y} = L^2_{t^{\gamma/2}} \otimes V,$$

where $V = H^1_0(D)$. Let $\Sigma$ and $\Theta$ be given as above, then we obtain from [22, Proposition 1 and 2] that the collection $\Theta \otimes \Sigma$ normalized in $\mathcal{X}$, i.e.,

$$\left\{ (t,x) \to \frac{\theta_\lambda(t)\sigma_\mu(x)}{\sqrt{\|\sigma_\mu\|^2_V + \|\theta_\lambda\|^2_{H^1_{t^{-\gamma/2}}(I)} \|\sigma_\mu\|^2_{V^*}}} : (\lambda,\mu) \in \nabla_{\mathcal{X}} := \nabla_\Theta \times \nabla_\Sigma \right\}$$

is a Riesz basis for $\mathcal{X}$ and that $\Theta \otimes \Sigma$ normalized in $\mathcal{Y}$, i.e.,

$$\left\{(t,x) \to \frac{\theta_\lambda(t)\sigma_\mu(x)}{\|\sigma_\mu\|_V} : (\lambda,\mu) \in \nabla_\mathcal{X}\right\}$$

is a Riesz basis for $\mathcal{Y}$.

## 4 Optimality

We are interested in optimality of the approximation of the solution process of the bi-infinite linear system, which arises from the discretization of (2.1) using the bases as described in the previous section. We derive estimates for the work requiered to solve of the arising linear systems, under the assumption that the best $N$-term approximation of the solution vector $\mathbf{u}$ converges with a certain rate $s$. This class of elements in $l^2(\nabla_\mathcal{X})$ is formalized in the following definition.

**Definition 4.1** For $s > 0$ the approximation class $\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))$ is defined as follows:

$$\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X})) := \{\mathbf{v} \in l^2(\nabla_\mathcal{X}) : \|\mathbf{v}\|_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))} < \infty\},$$

where $\|\mathbf{v}\|_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))} := \sup_{\varepsilon>0}\left(\varepsilon \times [\min\{N \in \mathbb{N}_0 : \|\mathbf{v}-\mathbf{v}_N\|_{l^2(\nabla_\mathcal{X})} \le \varepsilon\}]^s\right)$ and $\mathbf{v}_N$ denotes the best $N$-term approximation of $\mathbf{v}$.

Let $s > 0$ be such that $\mathbf{u} \in \mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))$, in order to be able to bound the complexity of an iterative solution method for the bi-infinite system $\mathbf{Bu} = \mathbf{f}$, with appropriate $\mathbf{B}$ and $\mathbf{f}$, one needs a suitable bound on the complexity of an approximate matrix-vector product in terms of the prescribed tolerance. We formalize this in the notion of $s^*$-admissibility.

**Definition 4.2** $\mathbf{B} \in \mathcal{L}(l^2(\nabla_\mathcal{X}), l^2(\nabla_\mathcal{Y}))$ is $s^*$-admissible if there exists a routine which yields, for any $\varepsilon > 0$ and any finitely supported $\mathbf{w} \in l^2(\nabla_\mathcal{X})$, a finitely supported $\mathbf{z} \in l^2(\nabla_\mathcal{Y})$ with $\|\mathbf{Bw}-\mathbf{z}\| < \varepsilon$. For any $\overline{s} \in (0, s^*)$, there exists an admissibility constant $a_{\mathbf{B},\overline{s}}$ such that

$$\#\mathrm{supp}\mathbf{z} \le a_{\mathbf{B},\overline{s}}\varepsilon^{-1/\overline{s}}\|\mathbf{w}\|^{1/\overline{s}}_{\mathcal{A}^{1/\overline{s}}_\infty(l^2(\nabla\mathcal{X}))}$$

and the number of arithmetic operations and storage locations used by the call of the routine is bounded by some absolute multiple of

$$a_{\mathbf{B},\overline{s}}\varepsilon^{-1/\overline{s}}\|\mathbf{w}\|^{1/\overline{s}}_{\mathcal{A}^{1/\overline{s}}_\infty(l^2(\nabla\mathcal{X}))} + \#\mathrm{supp}\mathbf{w} + 1.$$

Next we introduce the concept of $s^*$-computability.

**Definition 4.3** The mapping $\mathbf{B} \in \mathcal{L}(l^2(\nabla_\mathcal{X}), l^2(\nabla_\mathcal{Y}))$ is $s^*$-computable if, for each $N \in \mathbb{N}$ there exists a $\mathbf{B}_N \in \mathcal{L}(l^2(\nabla_\mathcal{X}), l^2(\nabla_\mathcal{Y}))$ having in each column at most N nonzero entries whose joint computation takes an absolute multiple of $N$ operations, such that the computability constants

$$c_{\mathbf{B},\overline{s}} := \sup_{N \in \mathbb{N}} \|\mathbf{B} - \mathbf{B}_N\|^{1/\overline{s}}_{l^2(\nabla_\mathcal{X}) \to l^2(\nabla_\mathcal{Y})}$$

are finite for any $\overline{s} \in (0, s^*)$.

In the following we assume that for $f \in \mathcal{Y}$ and any $\varepsilon > 0$ we can compute $\mathbf{f}_\varepsilon \in l^2(\nabla_\mathcal{Y})$ with

$$\|\mathbf{f} - \mathbf{f}_\varepsilon\|_{l^2(\nabla_\mathcal{Y})} \leq \varepsilon \text{ and } \#\text{supp } \mathbf{f}_\varepsilon \lesssim \min\{N : \|\mathbf{f} - \mathbf{f}_N\| \leq \varepsilon\},$$

with the number of arithmetic operations and storage locations used by the computation of $\mathbf{f}_\varepsilon$ bounded by some absolute multiple of $\#\text{supp}\mathbf{f}_\varepsilon + 1$. The following theorem links the two concepts of $s^*$-admissibility and $s^*$-computability, cf. [34, Theorem 4.10].

**Theorem 4.1** *An $s^*$-computable $\mathbf{B}$ is $s^*$-admissible.*

We use the following result from [34, Corollary 4.6].

**Corollary 4.1** *If $\mathbf{B} \in \mathcal{L}(l^2(\nabla_\mathcal{X}), l^2(\nabla_\mathcal{Y}))$ and $\mathbf{C} \in \mathcal{L}(l^2(\nabla_\mathcal{Y}), l^2(\nabla_\mathcal{Z}))$, then so is $\mathbf{CB} \in \mathcal{L}(l^2(\nabla_\mathcal{X}), l^2(\nabla_\mathcal{Z}))$*

The adaptive wavelet methods from [11] and [12] can be shown to be optimal for $s^*$-admissible $\mathbf{B}$ and $\mathbf{u} \in \mathcal{A}^{1/\overline{s}}_\infty(l^2(\nabla\mathcal{X}))$.

**Theorem 4.2** *Consider the bi-infinite system $\mathbf{Bu} = \mathbf{f}$ and let $\mathbf{B}$ be $s^*$-admissible, then for any $\varepsilon > 0$, both adaptive wavelet methods from [11, 12] produce an approximation $\mathbf{u}_\varepsilon$ to $\mathbf{u}$ with $\|\mathbf{u} - \mathbf{u}_\varepsilon\|_{l^2(\nabla_\mathcal{X})} \leq \varepsilon$. If $\mathbf{u} \in \mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))$, then $\#\text{supp}\mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s} \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))}$ and if, moreover, $s < s^*$, then the number of arithmetic operations and storage locations required by a call of either of these adaptive wavelet solvers with tolerance $\varepsilon$ is bounded by some multiple of*

$$\varepsilon^{-1/s}(1 + a_{\mathbf{B},s}) \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))} + 1.$$

*The multiples depend only on $s$ when it tends to 0 or $\infty$, and on $\|\mathbf{B}\|$ and $\|\mathbf{B}^{-1}\|$ when they tend to infinity.*

The following proposition is very useful, as the coefficients in the PDE (2.1) separate, i.e., using appropriate bases for the discretization leads to linear systems that possess a tensor product structure, cf. [34, Proposition 8.1].

**Proposition 4.1** *For some $s^* > 0$, let $\mathbf{C}$, $\mathbf{D}$ be $s^*$-computable. Then*

*(a)* $\mathbf{C} \otimes \mathbf{D}$ *is $s^*$-computable with computability constant satisfying, for $0 < \overline{s} < \tilde{s} < s^*$, $c_{\mathbf{C}\otimes\mathbf{D},\overline{s}} \lesssim (c_{\mathbf{C},\tilde{s}} c_{\mathbf{D},\tilde{s}})^{\tilde{s}/\overline{s}}$ and*

*(b) for any* $\varepsilon \in (0, s^*)$, $\mathbf{C} \otimes \mathbf{D}$ *is* $(s^* - \varepsilon)$*-computable, with computability constant* $c_{\mathbf{C}\otimes\mathbf{D},\overline{s}}$ *satisfying, for* $0 < \overline{s} < s^* - \varepsilon < \tilde{s} < s^*$, $c_{\mathbf{C}\otimes\mathbf{D},\overline{s}} \lesssim \max(c_{\mathbf{C},\tilde{s}}) \max(c_{\mathbf{D},\tilde{s}})$.

Let $B(\cdot,\cdot)$ be as in Lemma 2.2, then the corresponding bi-infinite matrix reads, where $[\Theta \otimes \Sigma]_{\mathcal{X}}$ and $[\Theta \otimes \Sigma]_{\mathcal{Y}}$ are the Riesz bases of $\mathcal{X}$ and $\mathcal{Y}$,

$$\begin{aligned}
\mathbf{B} &= B([\Theta \otimes \Sigma]_{\mathcal{X}}, [\Theta \otimes \Sigma]_{\mathcal{Y}}) \\
&= \left[\langle \Theta', \Theta\rangle \otimes (\Sigma, \Sigma) + \int_I t^\gamma a(\Theta \otimes \Sigma, \Theta \otimes \Sigma)\, dt\right] \\
&\quad \times \left(Id_t \otimes \|\Sigma\|_V^{-1}\right) \|\Theta \otimes \Sigma\|_{\mathcal{X}}^{-1} \qquad (4.1)\\
&= \left[\langle [\Theta']_{H^1_{t^{-\gamma/2}}}, \Theta\rangle \otimes (\Sigma, \Sigma)\right] \left(\|\Theta\|_{H^1_{t^{-\gamma/2}}(I)} \otimes \|\Sigma\|_V\right) \qquad (4.2)\\
&\quad \times \|\Theta \otimes \Sigma\|_{\mathcal{X}}^{-1} + \int_I t^\gamma a(\Theta \otimes [\Sigma]_V, \Theta \otimes [\Sigma]_V)\, dt \left(Id_t \otimes \|\Sigma\|_V\right) \|\Theta \otimes \Sigma\|_{\mathcal{X}}^{-1}.
\end{aligned}$$

The load vector reads:

$$\mathbf{f} = \int_I \langle f, \Theta \otimes [\Sigma]_V\rangle\, dt. \qquad (4.3)$$

We remark that the solution algorithms of [11] and [12] are only applicable to symmetric system matrices $\mathbf{B}$, we therefore consider the normal equations

$$\mathbf{B}^*\mathbf{B}\mathbf{u} = \mathbf{B}^*\mathbf{f} \qquad (4.4)$$

instead, cf. [34, Section 4].
We now show the $s^*$-computability of $\mathbf{B}$ and $\mathbf{B}^*$. First consider the term $\langle [\Theta]'_{H^1_{t^{-\gamma/2}}(I)}, \Theta\rangle$. The $\infty$-computability of the bi-infinite matrix and its adjoint follows as in [34, Section 8.2] using the properties of the temporal basis. Next we consider $\langle [\Sigma]_{V^*}, [\Sigma]_V\rangle$. The $\infty$-computability follows from [34, Section 8.3]. We now consider the $s^*$-computability of $\int_I t^\gamma a(\Theta \otimes [\Sigma]_V, \Theta \otimes [\Sigma]_V)$. Due to the properties of the bilinear form, we get:

$$\int_I t^\gamma a(\Theta \otimes [\Sigma]_V, \Theta \otimes [\Sigma]_V) = (\Theta, \Theta)_{L^2_{t^{\gamma/2}}(D)} \otimes a([\Sigma]_V, [\Sigma]_V).$$

Therefore is suffices to investigate the $s^*$-computability of both factors. The $\infty$-computability of $(\Theta, \Theta)_{L^2_{t^{\gamma/2}}(D)}$ follows from [6, Theorem 3.1] as in [34, Section 8.3]. For $a([\Sigma]_V, [\Sigma]_V)$ we can deduce from [33] that it is $s^*$-computable with $s^* = p_x + 1$. We arrive at the following theorem.

**Theorem 4.3** *Consider the weak form of the parabolic problem* (2.1) *on* $\mathcal{X}$, $\mathcal{Y}$ *as in* (2.4)-(2.5) *with bilinear form* $B(\cdot,\cdot)$ *as in* (2.11) *and the right hand side* $\int_I \langle f, \cdot\rangle$ *with* $f$ *as* (2.1). *Its representation using space-time wavelets as in Section 3.3 with appropriate boundary conditions reads* $\mathbf{Bu} = \mathbf{f}$ *with* $\mathbf{B}$ *as in* (4.2) *and* $\mathbf{f}$ *as in* (4.3). *Then for any* $\varepsilon > 0$, *the adaptive wavelet methods from*

*[11] and [12] applied to the normal equations* (4.4) *produce an approximation* $\mathbf{u}_\varepsilon$ *with*

$$\|\mathbf{u} - \mathbf{u}_\varepsilon\| \leq \varepsilon.$$

*If for some* $s > 0$, $\mathbf{u} \in \mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))$, *then supp* $\mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s} \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))}$. *The constant only depends on* $s$ *when it tends to* 0 *or* $\infty$. *If for arbitrary* $s^* > 0$ *it holds that* $s < s^*$, *then the number of operations and storage locations required by one call of the space-time adaptive algorithm with tolerance* $\varepsilon > 0$ *is bounded by some multiple of*

$$\varepsilon^{-1/s} d^2 \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))} + 1,$$

*where this multiple is uniformly bounded in* $d$ *and depends only on* $s \downarrow 0$ *and* $s \to \infty$.

## 5 Application

We describe the application of the results obtained in the Section 2 and Section 4 to PDEs arising in the context of option pricing under FBM market models.

### 5.1 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space supporting a real-valued fractional Brownian motion (FBM) $B_H(t)$ with Hurst parameter $H \in (0,1)$ and let $\mathcal{F}^H_t$ be the $\sigma$-algebra generated by $B_H(s)$, $s \leq t$.

**Definition 5.1** For $H \in (0,1)$, a fractional Brownian motion $B_H$ is a Gaussian process with mean zero, i.e.,

$$\mathbb{E}[B_H(t)] = 0$$

for all $t$ and covariance:

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\},$$

for all $s, t \geq 0$. We assume $B_H(0) = 0$. For $H = \frac{1}{2}$ we obtain a standard Brownian motion.

Our market model reads as follows. If $S(t)$ denotes the spot price of the risky asset, then its dynamics under the real world measure $\mathbb{P}$ is given as:

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t), \quad t \geq 0. \tag{5.1}$$

For the notion of a stochastic integral with respect to a fractional Brownian motion $B_H(t)$ we refer to [24] and [21]. Besides we assume the existence of a risk free bank account $P(t)$ with risk free interest rate $r > 0$. With the Girsanov theorem for FBM, cf. [5, Theorem 2.8] or [24, Theorem 3.18], we obtain the risk adjusted dynamics of the stock $S(t)$ under the equivalent measure $\mathbb{Q}$:

$$dS(t) = rS(t)dt + \sigma S(t)d\widetilde{B}_H(t), \quad t \geq 0,$$

where $\widetilde{B}_H(t)$ is a fractional Brownian motion under $\mathbb{Q}$ and the discounted stock is a quasi-martingale under $\mathbb{Q}$, see [5, Definition 2.3] for the definition of quasi-conditional expectation and quasi-martingales. Note that $\mathbb{Q}$ is not a martingale measure as the stock is not a martingale under $\mathbb{Q}$. Let $G(S)$ be the payoff of a European type contingent claim $V$, for sufficiently smooth $G$. Its value at time $t$ before maturity is given as the discounted quasi-conditional expectation:

$$V(t) = e^{-r(T-t)}\widetilde{\mathbb{E}}_{\mathbb{Q}}[G(S_T)|\mathcal{F}_t^H], \tag{5.2}$$

cf. [5, Theorem 4.2] and [20, Proposition 1]. The option price $V(t)$ admits a PDE representation.

**Theorem 5.1** *Let $v \in C^{1,2}([0,T],\mathbb{R})$ such that $v : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy the following PDE:*

$$\partial_t v(t,S) + rSv(t,S) + H\sigma^2 t^{2H-1} v_{SS}(t,S) - rv(t,S) = 0 \tag{5.3}$$

*with terminal condition $v(T,S) = G(S)$, then*

$$v(t,S) = V(t,S) \quad \textit{for all } t \in [0,T],\ S \in \mathbb{R}_+.$$

*Proof* The result follows from [20, Proposition 2] and [5, Proposition 6.1].

## 5.2 Weak formulation

### *5.2.1 Essential initial conditions*

Consider the following backward Kolmogorov equation arising in option pricing in the context of fractional Brownian motion models, i.e.,

$$\begin{aligned} \partial_t u(t,S) + rS\partial_S u(t,S) + H\sigma^2 t^\gamma S^2 \partial_{SS} u(t,S) &= 0 \text{ on } (0,T) \times \mathbb{R}_+ \\ u(T,S) &= g(S) \text{ on } \mathbb{R}_+, \end{aligned}$$

with $r > 0$, $\sigma > 0$ and $H \in (0,1)$. This setup can be reduced to the setting in Lemma 2.2. Transforming to log-price coordinates and time-to-maturity we obtain the following strong formulation for $\widetilde{v}(\tau,x) = u(T-\tau,e^x)$:

$$\begin{aligned} 0 &= \partial_\tau \widetilde{v}(\tau,x) - \alpha(\gamma)\partial_x \widetilde{v}(\tau,x) - \beta(\gamma)\partial_{xx}\widetilde{v}(\tau,x) \text{ on } (0,T) \times \mathbb{R} \\ \widetilde{v}(0,x) &= g(e^x) \text{ on } \mathbb{R}, \end{aligned}$$

where $\alpha(\gamma) = (r - H\sigma^2(T-\tau)^\gamma)$ and $\beta(\gamma) = H\sigma^2(T-\tau)^\gamma$. After localization, removal of the drift and transformation to excess to payoff the formulation reads as follows $v(\tau,y) = \widetilde{v}\big(\tau, \underbrace{y - \tau r + H\frac{\sigma^2}{\gamma+1}(T-\tau)^{\gamma+1}}_{z(\tau,y)}\big) - \underbrace{g(e^{z(\tau,y)})}_{\widetilde{g}(\tau,y)}$:

$$\partial_\tau(v(\tau,y) + \widetilde{g}(\tau,y)) - \beta(\gamma)\partial_{yy}(v(\tau,y) + \widetilde{g}(\tau,y)) = 0 \text{ on } (0,T)\times D \quad (5.4)$$

$$v(0,y) = 0 \text{ on } D. \quad (5.5)$$

The localization to the bounded interval $D = (-R, R)$ will be justified in Section 5.4. The weak formulation reads: find $v \in \mathcal{X}_{(0}$ such that for all $w \in \mathcal{Y}$

$$B(v,w) = f(v), \quad (5.6)$$

where

$$\begin{aligned}
B(v,w) &= \int_0^T (\langle w, \dot{v}\rangle + a(\tau, v, w))\, d\tau \\
a(\tau,v,w) &= H\frac{\sigma^2}{2}(T-\tau)^\gamma \left(\partial_y v(\tau,y), \partial_y w(\tau,y)\right), \\
f(v) &= -B(\widetilde{g}, v) \\
\mathcal{X} &:= H^1_{(T-\tau)^{-\gamma/2}}(I; V_D^*) \cap L^2_{(T-\tau)^{\gamma/2}}(I; V_D), \\
\mathcal{Y} &:= L^2_{(T-\tau)^{\gamma/2}}(I; V_D), \\
V_D &:= H^1_0(D).
\end{aligned}$$

The well-posedness of this formulation follows analogously to Lemma 2.2. Instead of localization of the problem to a bounded domain we can also consider the equation in exponentially weighted Sobolev spaces, cf. [27, Section 2.2.]

$$\begin{aligned}
L^2_\nu(\mathbb{R}) &:= \left\{ v \in L^1_{\text{loc}}(\mathbb{R}) : ve^{\nu|x|} \in L^2(\mathbb{R}) \right\}, \\
H^1_\nu(\mathbb{R}) &:= \left\{ v \in L^1_{\text{loc}}(\mathbb{R}) : ve^{\nu|x|}, v'e^{\nu|x|} \in L^2(\mathbb{R}) \right\}.
\end{aligned}$$

To obtain a variational in this setup formulation we consider the pricing equation before localization:

$$\partial_\tau(v_s(\tau,y) + \widetilde{g}(\tau,y)) - \beta(\gamma)(\partial_{yy}v_s(\tau,y) + \widetilde{g}(\tau,y)) = 0 \text{ on } (0,T)\times\mathbb{R} \quad (5.7)$$

$$v_s(0,y) = 0 \text{ on } \mathbb{R}. \quad (5.8)$$

We multiply (5.7) by $e^{\nu|y|}$ and test with $we^{\nu|y|}$, $w \in C_0^\infty((0,T)\times\mathbb{R})$:

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_\tau v(\tau,y) w(\tau,y) e^{2\nu|y|}\, dy - H\frac{\sigma^2}{2}(T-\tau)^\gamma \times \\
&\int_{\mathbb{R}} \left[ e^{\nu|y|}\partial_y\left(v(\tau,y)\right)\partial_y\left(e^{\nu|y|}w(\tau,y)\right) - (\partial_y e^{\nu|y|})(\partial_y v(\tau,y))e^{\nu|y|}w(\tau,y)\right] dy \\
&= \langle \partial_\tau v(\tau,y) w(\tau,y)\rangle_\nu - H\frac{\sigma^2}{2}(T-\tau)^\gamma(\partial_y v, \partial_y w)_\nu = -B^\nu(\widetilde{g}, w).
\end{aligned}$$

We obtain existence of a unique solution for the following problem as in (5.6): Find $v \in \mathcal{X}$ such that for all $w \in \mathcal{Y}$

$$B^{\nu}(v,w) = f_{\nu}(w), \tag{5.9}$$

where

$$\begin{aligned}
B^{\nu}(v,w) &= \int_0^T \left(\langle w, \dot{v}\rangle_{\nu} + a^{\nu}(\tau,v,w)\right)\, d\tau \\
a^{\nu}(\tau,v,w) &= H\frac{\sigma^2}{2}(T-\tau)^{\gamma}\left[(\partial_y v, \partial_y w)_{\nu}\right], \quad f_{\nu}(v) = -B^{\nu}(g,v), \\
\mathcal{X}_{\nu} &:= H^1_{(T-\tau)^{-\gamma/2}}(I;V_{\nu}^*) \cap L^2_{(T-\tau)^{\gamma/2}}(I;V_{\nu}), \quad \mathcal{Y}_{\nu} := L^2_{(T-\tau)^{\gamma/2}}(I;V_{\nu}), \\
V_{\nu} &:= H^1_{\nu}(\mathbb{R}),
\end{aligned}$$

for

$$B^{\nu}(g,\cdot) \in H^1_{\nu}(\mathbb{R})^*. \tag{5.10}$$

Note that (5.10) holds for standard options such as European calls and puts, for arbitrary $\nu$. For more exotic options, such as digital contracts or barrier options, with discontinuous payoffs an appropriate smooth approximation of the payoff has to be employed in order for (5.10) to hold.

*Remark 5.1* The well-posedness of the pricing equation for European calls and puts on weighted spaces $V_{\nu}$ for arbitrary positive $\nu$ implies a fast decay of the excess-to-payoff function at infinity. This property will be used to obtain a localization estimate for the equation in Section 5.4.

### *5.2.2 Natural initial conditions*

Instead of the enforcement of essential initial conditions, we now pose the problem with natural initial data, cf. Section 2.2. For the backward Kolmogorov equation (5.7)-(5.8) the formulation reads as follows: given $f^D \in \left(\mathcal{X}^D_{0)}\right)^*$, $f^{-\nu} \in \left(\mathcal{X}^{-\nu}_{0)}\right)^*$, $g^D \in V_D$, $g^{-\nu} \in V_{-\nu}$:

$$u \in \mathcal{Y}_D: \quad B^*_D(u,v) = \langle v, f^D\rangle + (v(0), g^D), \quad \forall v \in \mathcal{X}^D_{0)}, \tag{5.11}$$

$$u \in \mathcal{Y}_{-\nu}: \quad B^*_{-\nu}(u,v) = \langle v, f^{\nu}\rangle + (v(0), g^{\nu}), \quad \forall v \in \mathcal{X}^{-\nu}_{0)}, \tag{5.12}$$

where

$$\begin{aligned}
B_D^*(v,w) &:= \int_0^T \big(-\langle \dot{w}, v\rangle + a^D(\tau,v,w)\big)\, d\tau,\\
B_{-\nu}^*(v,w) &:= \int_0^T \big(-\langle \dot{w}, v\rangle + a^{-\nu}(\tau,v,w)\big)\, d\tau,\\
a^D(\tau,v,w) &:= H\frac{\sigma^2}{2}(T-\tau)^\gamma \left(\partial_y v(\tau,y), \partial_y w(\tau,y)\right),\\
a^{-\nu}(\tau,v,w) &:= H\frac{\sigma^2}{2}(T-\tau)^\gamma \left[(\partial_y v, \partial_y w)_{-\nu}\right],\\
\mathcal{X}_{0)}^D &:= H^1_{(T-\tau)^{-\gamma/2},0)}(I;V_D^*) \cap L^2_{(T-\tau)^{\gamma/2}}(I;V_D),\\
\mathcal{X}_{0)}^{-\nu} &:= H^1_{(T-\tau)^{-\gamma/2},0)}(I;V_{-\nu}^*) \cap L^2_{(T-\tau)^{\gamma/2}}(I;V_{-\nu}),\\
\mathcal{Y}_D &:= L^2_{(T-\tau)^{\gamma/2}}(I;V_D),\\
\mathcal{Y}_{-\nu} &:= L^2_{(T-\tau)^{\gamma/2}}(I;V_{-\nu}).
\end{aligned}$$

The well-posedness of (5.11) and (5.12) can be shown as in Lemma 2.2.

*Remark 5.2* Note that the condition $g^{-\nu} \in V_{-\nu}$ is stronger than (5.10). The stronger condition is only satisfied for standard payoffs such as European calls and puts for $\nu > 1$. A localization of the payoff has to be employed for $\nu \le 1$.

## 5.3 Optimality

We apply the results of Section 4 to the derived formulations.

**Theorem 5.2** *Consider the weak formulation* (5.6) *on* $\mathcal{X}$, $\mathcal{Y}$ *as above. Its representation using space-time wavelets as in Section 3.3 with appropriate boundary conditions reads* $\mathbf{Bu} = \mathbf{f}$ *with* $\mathbf{B}$ *as in* (4.2) *and* $\mathbf{f}$ *as in* (4.3). *Then for any* $\varepsilon > 0$, *the adaptive solution algorithm from [11] and [12] applied to the normal equations* (4.4) *produces an approximation* $\mathbf{u}_\varepsilon$ *with*

$$\|\mathbf{u} - \mathbf{u}_\varepsilon\| \le \varepsilon.$$

*If for some* $s > 0$, $\mathbf{u} \in \mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))$, *then supp* $\mathbf{u}_\varepsilon \lesssim \varepsilon^{-1/s}\, \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))}$. *The constant only depends on* $s$ *when it tends to* $0$ *or* $\infty$. *If for arbitrary* $s^* > 0$ *it holds that* $s < s^*$, *then the number of operations and storage locations required by one call of the space-time adaptive algorithm with tolerance* $\varepsilon > 0$ *is bounded by some multiple of*

$$\varepsilon^{-1/s} d^2\, \|\mathbf{u}\|^{1/s}_{\mathcal{A}^s_\infty(l^2(\nabla_\mathcal{X}))} + 1,$$

*where this multiple is uniformly bounded in* $d$ *and depends only on* $s \downarrow 0$ *and* $s \to \infty$.

*Remark 5.3* An analogous result can be obtained for (5.11) and (2.16). The derivation of such results for the global weighted formulations (5.9) and (5.12) is more involved, as the construction of a Riesz basis for the dual of $V_\nu$ is grueling.

## 5.4 Localization

In the following we describe two localization methods which lead to a formulation of the pricing problem on a bounded domain. The localization error is quantified using probabilistic techniques.

### *5.4.1 Homogeneous Dirichlet Boundary Condition*

The localization to a bounded domain and the use of homogeneous Dirichlet boundary conditions is justified in the following. We follow [27, Section 4.2].

**Theorem 5.3** *Let $u_D$ be the sufficiently smooth solution of* (5.4)-(5.5) *and $u$ the sufficiently smooth solution of* (5.7)-(5.8) *with $g(S) = \max\{(S-K), 0\}$ for some $K > 0$, further let $e_D = u_D - u$. Then $e_D$ satisfies the following error bound:*

$$\|e_D(T)\|^2_{L^2(D)} + \|e_D\|^2_{L^2_{(T-t)^\gamma}(H^1([-R/2,R/2]))} \leq e^{-\alpha R},$$

*for some positive constants $\alpha$ and $R$.*

*Proof* Note that $e_R$ satisfies the following equation:

$$\int_0^T (\frac{d}{d\tau} e_D(\tau), v) + a(\tau, e_D(\tau), v)\, d\tau = 0 \qquad \forall v \in H^1_0(D), \tag{5.13}$$

with $a(\cdot,\cdot)$ given as in (5.6). Denote by $\phi$ a cut-off function with the following properties: $\phi \in C_0^\infty(D)$, $\phi \equiv 1$ on $[-R/2, R/2]$ and $\|\phi'\|_{L^\infty(D)} < C$ for some constant $C > 0$ independent of $R$. Inserting $v = \phi^2(x)e_D(\tau, x)$ into (5.13) leads to:

$$\|\phi e_D(T)\|_{L^2(\mathbb{R})} + \int_0^T a(\tau, \phi e_D(\tau), \phi e_D(\tau))\, d\tau = \int_0^T \rho(\tau)\, d\tau,$$

where the residual $\rho(\tau) = a(\tau, \phi e_D, \phi e_D) - a(e_D, \phi^2 e_D)$, with $a(\tau,\cdot,\cdot)$ is given in (5.10). The residual admits the following estimate:

$$\begin{aligned}\int_0^T \rho(\tau)\, d\tau &\leq \int_0^T \int_{\mathbb{R}} H\frac{\sigma^2}{2}(T-\tau)^\gamma(\phi')^2 e_D^2(\tau) e^{\nu|x|} e^{-\nu|x|}\, dx d\tau \\ &\leq e_D^{-\alpha R} \|e(\tau)\|_{L^2_{(T-t)^\gamma}(L^2_\nu(\mathbb{R}))},\end{aligned}$$

for some positive constant $\alpha$ and arbitrary $\nu \in \mathbb{R}$.

*Remark 5.4* Theorem 5.3 gives a rigorous justification for the approximation of the option price (5.2) by the solution of a degenerate parabolic PDE on a bounded domain. Choosing the computational domain sufficiently large with respect to the domain of interest yields an negligible truncation error. In contrast to the subsequent section the argument is purely deterministic. We do not rely on the representation of the option price as a quasi-conditional expectation (5.1).

*5.4.2 Homogeneous Robin Boundary Condition*

We make use of a probabilistic argument to approximate the pricing equation by a local problem with Robin boundary conditions. First the ideas for the case where the price process is driven by a Brownian motion will be presented and then extended to the case of a price process driven by a fractional Brownian motion. The argumentation relies on the following idea. The price process (5.1) is approximated by a process that behaves similar to (5.1) inside the computational domain, but does not leave the computational domain $D$. The behaviour of the approximating process at the boundary of the computational domain will be modeled using local times.
**Brownian Motion:**
The reflected process can be characterized as follows, cf. [9, Theorem 2.1]:

**Theorem 5.4** *Let $(-R, R) = D \subset \mathbb{R}$ be a bounded open interval and $B$ a Brownian motion in $\mathbb{R}$, then there exists a unique pair of continuous stochastic processes $(\widetilde{X}, L)$ adapted to the natural filtration of $B$ such that*

*(i)* $\widetilde{X}(t) \in \overline{D}$ *for all* $t \in [0, T)$ *with* $\widetilde{X} = 0$,
*(ii)* $L$ *is a nondecreasing process such that* $t \to L(t)$ *only increases when the process* $X(t)$ *is on the boundary,*
*(iii)* $\widetilde{X}(t) = B_t + \int_0^t n(\widetilde{X}(r))dL(r)$, *where* $-n(x)$ *is the exterior unit normal vector on* $D$.

The process $L(t)$ is called local time of $X = B$. An intuitive characterization of the local time is given in the following theorem. The result naturally generalizes when $X(0) \neq 0$.

**Theorem 5.5** *Let the assumptions of Theorem 5.4 be satisfied, then*

$$L(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{D_\varepsilon}(\widetilde{X}_r)\, dr, \tag{5.14}$$

*where* $D_\varepsilon = \{x \in D | d(x, \partial D) < \varepsilon\}$ *and* $d(x, \partial D)$ *denotes the Euclidean distance of* $x$ *to the boundary of* $D$. *Besides the following estimate holds:*

$$\mathbb{E}[L(t)] \leq C(t) e^{-\alpha |R|^2}.$$

*Proof* The proof is given in [9, Theorem 2.6].

With these estimates available the localization estimate can now easily be obtained.

**Theorem 5.6** *Let $g$ be globally Lipschitz with Lipschitz constant $K$ and let the assumptions of Theorem 5.4 be satisfied. Then*

$$\left| \mathbb{E}[g(X(t)) - g(\widetilde{X}(t))] \right| \leq K \mathbb{E}[L(t)] \leq K C(t) e^{-\alpha |R|^2}, \tag{5.15}$$

*with $\widetilde{X}$ and $X$ as in Theorem 5.4.*

This justifies the approximation of $\mathbb{E}[g(X(T))]$ by $\mathbb{E}[g(\widetilde{X}(T))]$, for sufficiently large domains of interest. The Kolmogorov equation for $\widehat{v}(t,x) = \mathbb{E}[g(\widetilde{X}(T))|X(t) = x]$ with $X(t) = x + \mu t + \sigma B(t)$ reads

$$\frac{\partial \widehat{v}(t,x)}{\partial t} + \mu \frac{\partial \widehat{v}(t,x)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 \widehat{v}(t,x)}{\partial^2 x^2} = 0 \text{ for } x \in D$$

$$\lim_{x \uparrow R} \left[ \frac{\sigma^2}{2} \frac{\partial \widehat{v}(t,x)}{\partial x} + \mu \widehat{v}(t,x) \right] = 0 \tag{5.16}$$

$$\lim_{x \downarrow -R} \left[ \frac{\sigma^2}{2} \frac{\partial \widehat{v}(t,x)}{\partial x} + \mu \widehat{v}(t,x) \right] = 0. \tag{5.17}$$

The Robin-type boundary conditions (5.16)-(5.17) account for the fact that no probability mass can leave the domain $D$.

**Fractional Brownian motion:**

We proceed as in the Brownian case to approximate the pricing problem on an unbounded domain by the formulation on a bounded domain. Let $\sigma B_H(t)$ denote a fractional Brownian motion and let $\widetilde{X}_t := \sigma B_H(t) + \int_0^t n(\widetilde{X}(r))dL(r)$ denote the reflected fractional Brownian motion on $D$, where $L(t)$ is given by the following definition analogous to (5.14).

**Definition 5.2** Let $t > 0$ and $x \in \mathbb{R}$. The local time of $\sigma B_H$ up to time $t$ on $D$ is given by

$$L(t) = \lim_{\varepsilon \to 0} \int_0^t \frac{1}{2\varepsilon} \mathbb{1}_{D_\varepsilon}(\sigma B_H(r))\, dr, \tag{5.18}$$

with $D_\varepsilon$ is as in Theorem 5.5.

We have the following estimate due to [7, Corollary 10.1.12]

$$\left| \widetilde{\mathbb{E}}[L(t)] \right| \leq C(t) e^{-\alpha(t) R^2},$$

for some positive time-dependent constants $C(t)$ and $\alpha(t)$. Therefore we have the following estimate for sufficiently smooth payoffs $g$.

**Theorem 5.7** *Let $g : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz with constant $K$ and let $v(t,x)$ and $\widehat{v}(t,x)$ be given as:*

$$v(t,x) = \widetilde{\mathbb{E}}[g(\sigma W_H(T))|\mathcal{F}_t^H] \text{ and } \widehat{v}(t,x) = \widetilde{\mathbb{E}}[g(\widetilde{X}_T)|\mathcal{F}_t^H].$$

*Then the following estimate holds:*

$$|v(t,x) - \widehat{v}(t,x)| \leq \left| \widetilde{\mathbb{E}}[KL(T)|\mathcal{F}_t^H] \right| \leq KC(T) e^{-\alpha(T) R^2}.$$

*Remark 5.5* Theorem 5.7 naturally generalizes to processes driven by FBM with (non constant) drift and non-homogeneous initial conditions.

The Kolmogorov equation for $\widehat{v}(t,x) = \widetilde{\mathbb{E}}[g(\widetilde{X}(T))|\mathcal{F}_t^H]$ with $X(t) = x + \int_0^t \mu(t)\,dt + \sigma B_H(t)$ reads:

$$\frac{\partial \widehat{v}(t,x)}{\partial t} + \mu(t)\frac{\partial \widehat{v}(t,x)}{\partial x} + H\sigma^2 \frac{\partial^2 \widehat{v}(t,x)}{\partial^2 x^2} = 0 \text{ for } x \in D$$

$$\lim_{x\uparrow R}\left[H\sigma^2 \frac{\partial \widehat{v}(t,x)}{\partial x} + \mu(t)\widehat{v}(t,x)\right] = 0 \tag{5.19}$$

$$\lim_{x\downarrow -R}\left[H\sigma^2 \frac{\partial \widehat{v}(t,x)}{\partial x} + \mu(t)\widehat{v}(t,x)\right] = 0, \tag{5.20}$$

with final condition $\widehat{v}(T,x) = g(x)$. This justifies the use of Robin boundary conditions for the localization of the pricing equation. The choice of the appropriate boundary conditions is strongly related to the behaviour of the process. Although both localization using Robin and Dirichlet boundary conditions lead to an exponential decay of the truncation error the constants depend on the nature of the process, therefore an a priori choice of the boundary condition, i.e., before the market model is determined, is not meaningful.

## 6 Conclusion

The aim of this work is to contribute to the analysis of linear degenerate parabolic equations. For certain types of equations well-posedness results for weak space-time formulations could be obtained. The space-time domain was discretized using appropriate wavelets bases. This enabled us to obtain Riezs bases of the ansatz and test spaces which led in conjunction with the compressibility of the arising operators to an optimality result for a space-time adaptive solution algorithm of the resulting equivalent bi-infinite linear system. An application of the theory to option pricing problems under fractional Brownain motion market models was presented. For an option pricing problem in the context of FBM well-posedness results for different formulations could be obtained and localization of the pricing problem was justified rigorously.

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