On tensor products of quasi-Banach spaces

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Abstract

Due to applications in approximation theory we are interested in tensor products of quasi-Banach spaces. Though a general abstract theory seems not possible beyond basic topological issues because the dual spaces are possibly trivial, we aim at extending some basic notions like crossnorms, reasonable and uniform norms. In the present paper this is done for quasi-Banach spaces with separating duals, and this condition turns out to be the (in a certain sense) minimal requirement. Moreover, we study extensions of the classical injective and p-nuclear tensor norms to quasi-Banach spaces. In particular, we give a sufficient condition for the p-nuclear quasi-norms to be crossnorms, which particularly applies to the case of weighted ℓ_p-sequence spaces.

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1 Introduction

In recent years tensor product bases as well as tensor product spaces became of interest in numerical analysis. When dealing with high-dimensional problems their properties and relatively easy implementation were extremely useful. To name only some examples we refer to the work of Yserentant [26], who showed that eigenfunctions of the electronic Schrödinger equation belong to certain tensor product Sobolev spaces. Moreover, we refer to [5] (in particular the contribution by Stevenson) and [2] for surveys on adaptive wavelet methods and sparse grid techniques.

The general framework for these topics is often referred to as "high-dimensional problems". This means, one is interested in problems (arising from PDEs or approximation theory) dealing with functions in many variables, or families of functions with parameters coming from a high-dimensional parameter space. In all these variants tensor product functions are of special interest since they provide an easy access to those high-dimensional functions, and they have obvious advantages for computation. Hence one needs to know which functions may be approximated well by tensor product functions, i.e. one is interested in characterizations of tensor product spaces, or rather, which function spaces can be characterized by tensor product bases.

While for tensor products of Hilbert spaces, many of these aspects are well understood due to the special properties of these Hilbert spaces, the theory of tensor products of Banach spaces is far more involved.

Tensor products of Hilbert and Banach spaces and other basic notions were first introduced by von Neumann, Whitney, Schatten and others in the 1940s. An important step in the history of this theory was the work of Grothendieck, in particular his results on nuclear spaces [7] and his famous Résumé [6]. Since the 1970s it became an important tool in the study of Banach spaces, and also in Operator theory due to being closely related to the theory of Operator ideals. We refer to [14] for an introductory survey and to the monograph [3] for a detailed treatment of this theory.

On the other hand this treatment of tensor product spaces had one shortcoming, which in the context of numerical analysis and approximation theory turned out to be a major disadvantage. From a purely functional analytic point of view the treatment of tensor products of Banach spaces was extremely satisfying and produced lots of results giving a deep insight into the (local) structure of Banach spaces. Moreover, the restriction to Banach spaces not only was quite convenient, but it seems quite natural, as the theory makes heavy use of dual spaces and the Hahn-Banach extension theorem. It is well-known that for quasi-Banach spaces this theorem fails to be true, and the dual spaces of Banach spaces may be trivial.

However, in modern approximation theory quasi-Banach spaces became increasingly relevant. In this context they often appear when characterizing so-called approximation spaces, i.e. given a fixed approximation method one is interested in the collection of all functions with a given convergence rate. For example, when studying the problem of nonlinear m-term approximation with respect to wavelet-type bases and measuring the error in some L_p-norm, the corresponding approximation spaces are certain Besov-spaces which are Banach spaces or quasi-Banach spaces, depending on the parameters involved (we refer to [4] for a survey on nonlinear approximation). However, studying the same problem for tensor product bases, one is lead to tensor products of Besov-spaces, in particular also to tensor products of quasi-Banach spaces, see [9, 10, 8].

The aim of the present paper, which is based on earlier work by Nitsche [16] and Sickel/Ullrich [19], is to investigate under which conditions on quasi-Banach spaces one can carry over at least some of the important results for their respective tensor products. As regarding topological issues like existence of Hausdorff topologies on the (algebraic) tensor product of general topological vector spaces this has been done by Turpin and Waelbroeck, see [22, 23, 25], but little seems to be known concerning tensor quasi-norms.

In a first section, we look at different approaches towards tensor products, which are known to coincide for Banach spaces, and give necessary and sufficient conditions for the dual spaces of the respective quasi-Banach spaces. Afterwards we consider the injective tensor norm as one particularly famous example for norms on tensor products of quasi-Banach spaces. As we shall see, there are three different versions of injective quasi-norms. In connection with these quasi-norms we shall demonstrate that also for other basic properties of such tensor norms several non-equivalent extensions to quasi-Banach spaces occur.

The third part of this article is devoted to the study of p-nuclear quasi-norms. For parameters $1 \leq p \leq \infty$ these are a well-known scale of tensor norms. Once more there are different extensions to quasi-Banach spaces. We will prove the basic properties for these extensions as well as relations between them. In particular we will give a sufficient condition for these quasi-norms to be crossnorms. In the last section we apply these considerations first to weighted ℓ_p-sequence spaces and afterwards to function spaces of Sobolev- and Besov-type. These spaces are well-known to allow a characterization in terms of wavelet bases and associated sequence spaces, and we shall study the dependence of the norm of the corresponding isomorphisms on the dimension of the underlying domain.

2 Tensor products of (quasi-)Banach spaces

2.1 Algebraic and analytic definition

In algebra tensor product constructions are known for several different structures. The starting point for one possibility of an explicit construction for vector spaces X and Y (with respect to the same field; here we concentrate on real or complex vector spaces) is the free vector space $F(X,Y)$ on $X \times Y$, i.e. the set

$$
F(X,Y) := \operatorname{span}\Big\{ x \otimes y : x \in X, y \in Y \Big\}
$$
$$
= \bigg\{ \sum_{j=1}^{n} \lambda_j x_j \otimes y_j : x_j \in X, y_j \in Y, \lambda_j \in \mathbb{C}, j = 1, \dots, n, n \in \mathbb{N} \bigg\}.
$$

Afterwards the algebraic tensor product $X \otimes Y$ is defined as the quotient space of $F(X,Y)$ with respect to the subspace

$$
U := \operatorname{span}\Big(\big\{ (x_1 + x_2) \otimes y - x_1 \otimes y - x_2 \otimes y : x_1, x_2 \in X, y \in Y \big\}
$$
$$
\cup \big\{ x \otimes (y_1 + y_2) - x \otimes y_1 - x \otimes y_2 : x_1, x_2 \in X, y \in Y \big\}
$$
$$
\cup \big\{ \lambda(x \otimes y) - (\lambda x) \otimes y, \lambda(x \otimes y) - x \otimes (\lambda y) : x \in X, y \in Y, \lambda \in \mathbb{C} \big\} \Big).
$$

In this way the canonical mapping $(x, y) \longmapsto x \otimes y$ from $X \times Y$ to $X \otimes Y$ becomes bilinear.

The usual functional analytic approach for normed spaces X and Y is slightly different. Once more one starts with $F(X,Y)$, but this times this space is equipped with the following equivalence relation. We say $f = \sum_{j=1}^{n} \lambda_j x_j \otimes y_j \in F(X,Y)$ generates an operator $A_f : X' \longrightarrow Y$ by the determination

$$
A_f \psi := \sum_{j=1}^{n} \lambda_j \psi(x_j) y_j, \qquad \psi \in X'.
$$

Then we define for $f, g \in F(X,Y)$, $f = \sum_{j=1}^{n} \lambda_j^1 x_j^1 \otimes y_j^1$, $g = \sum_{j=1}^{m} \lambda_j^2 x_j^2 \otimes y_j^2$

$$
f \simeq g \iff A_f(\psi) = A_g(\psi) \qquad \text{for all } \psi \in X',
$$

i.e. f and g generate the same operator from the dual space X' of X to Y. Of interest now is the quotient space $X \otimes^A Y = F(X,Y)/\simeq$, which is found to coincide as a vector space with $X \otimes Y$. By this definition the connection with linear mappings from X' to Y is made obvious right from the beginning.

This second (analytic) approach applies to quasi-normed spaces as well, but since the dual space is possibly trivial, this equivalence relation as well as the respective quotient space might become trivial. To avoid this, i.e. to ensure the equivalence of both approaches, we have to impose certain restrictions on the quasi-normed spaces. This situation is clarified by the following theorem.

Theorem 1. Let X and Y be two quasi-normed spaces. Then it holds $X \otimes Y = X \otimes^A Y$ if, and only if, X' separates the points in X, i.e. for every $x \in X \setminus \{0\}$ there exists a functional $\varphi_x \in X'$, such that $\varphi_x(x) \neq 0$.

A quasi-Banach space X with this property is said to have a separating dual.

Proof. In order to show the coincidence of both spaces we have to show that $U = V := \{f \in F(X,Y) : A_f = 0\}$ holds. The inclusion $U \subset V$ is obvious. For the reverse inclusion we remark that the condition on X' is equivalent to $A_{x \otimes y} \neq 0$ for all $x \neq 0$ and $y \neq 0$. To show now $V \subset U$, we show instead, that from $f \notin U$ follows $f \notin V$.

We shall use the fact, that for every $f \notin U$ there exists an (algebraically) equivalent representation $f = \sum_{i=1}^n x_i \otimes y_i$, where $\{x_1, \ldots, x_n\} \subset X$ and $\{y_1, \ldots, y_n\} \subset Y$ are linearly independent (this can be seen analogously to [14, Lemma 1.1]). The linearity of $f \longmapsto A_f$, the linear independency of $\{y_1, \ldots, y_n\}$ and the assumption for X' (applied to the vectors $x_i \neq 0$) now yield $A_f \neq 0$. □

In a similar way, we can also consider operators

$$B_f : Y' \longrightarrow X, \qquad B_f \phi = \sum_{i=1}^n \phi(y_i) x_i, \qquad f = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y.$$

We then observe that

$$B_f = B_g \iff A_f = A_g \iff f \simeq g$$

holds for all $f, g \in X \otimes Y$ if, and only if, X' and Y' are separating. This follows from $F(X,Y) \cong F(Y,X)$ and $X \otimes^A Y \cong X \otimes Y \cong Y \otimes X \cong Y \otimes^A X$, where the isomorphism is provided by the canonical identification $x \otimes y \longmapsto y \otimes x$, $x \in X$, $y \in Y$ (for the algebraic tensor product this is always true, and the assumptions assure that this extends to the respective (functional analytic) equivalence relations). Due to this observation we henceforth always assume that X and Y both have separating duals (without always explicitly mentioning it).

Remark 1. If one is only interested in equipping the algebraic tensor product $X \otimes Y$ of general topological vector spaces with just some topological structure, then one does not need information on the respective dual spaces. For example, if X and Y are equipped with topologies which are Hausdorff, then there exists a topology on $X \otimes Y$ which is Hausdorff as well. For more information on such topological issues, we refer to Turpin [22, 23] and Waelbroeck [25].

2.2   Tensor products of distributions

The calculus of tensor products of (tempered) distributions is based on the following result, see [19, Appendix B] and further references given there. We shall use the notation $\varphi \otimes^f \psi$ for the usual tensor product of functions, i.e. if $\varphi : \Omega_1 \longrightarrow \mathbb{C}$ and $\psi : \Omega_2 \longrightarrow \mathbb{C}$, then $\varphi \otimes^f \psi : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{C}$, $(\varphi \otimes^f \psi)(x,y) = \varphi(x)\psi(y)$.

Proposition 1. Let $S_i \in \mathcal{S}'(\mathbb{R}^{d_1})$ and $T_i \in \mathcal{S}'(\mathbb{R}^{d_2})$, $i = 1, \ldots, n$. Then there exists a uniquely determined distribution $U \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$, such that for all functions $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ and $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$

$$U(\varphi \otimes^f \psi) = \sum_{i=1}^n S_i(\varphi) \cdot T_i(\psi)$$

holds. In case $n = 1$, this distribution $U \equiv S \otimes^D T$ is given explicitly by the formula

$$U(\rho(x,y)) = T_y(S_x(\rho(x,y))) = S_x(T_y(\rho(x,y))), \qquad \rho \in \mathcal{S}(\mathbb{R}^{d_1+d_2}),$$

and in the general case, U can be written as

$$U = \sum_{i=1}^n S_i \otimes^D T_i.$$

Usually this proposition is stated for the case $n = 1$ only, but it extends immediately to finite linear combinations.

This last proposition ensures, that $\sum_{i=1}^{n} S_i \otimes^D T_i$ is again a well-defined tempered distribution. Moreover, if S and T are regular distributions, generated by functions $f : \mathbb{R}^{d_1} \longrightarrow \mathbb{C}$ and $g : \mathbb{R}^{d_2} \longrightarrow \mathbb{C}$, then it can be easily seen, that also $S \otimes^D T$ is a regular distribution, generated by $f \otimes^f g : \mathbb{R}^{d_1 + d_2} \longrightarrow \mathbb{C}$.

At the end we intend to apply the theory to tensor products of Sobolev and Besov spaces. This motivates a closer look on spaces of tempered distributions.

Hence, let X and Y be quasi-Banach spaces of tempered distributions. The first question to be addressed is whether their dual spaces are rich enough to provide meaningful results for tensor products.

Lemma 1. Let X be a topological vector space, such that we have a continuous embedding $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Then X' separates the points in X.

Proof. We consider the natural injection $\mathcal{J} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}''(\mathbb{R}^n)$, which is defined by $(\mathcal{J}\varphi)(f) = f(\varphi)$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $f \in \mathcal{S}'(\mathbb{R}^n)$. Due to the assumed topological embedding $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$ we immediately find $\mathcal{J}\varphi \in X'$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$.

Now let $f \in X$, $f \neq 0$. Then we also have $f \neq 0$ in the sense of $\mathcal{S}'(\mathbb{R}^{d_1})$. This means there is some function $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$ such that $f(\varphi) \neq 0$. This immediately implies $(\mathcal{J}\varphi)(f) \neq 0$, which yields the desired functional from X'. $\qquad\square$

Thus when dealing with tensor products of spaces of tempered distributions the minimal assumption is a continuous topological embedding into the space $\mathcal{S}'(\mathbb{R}^n)$. In particular, all types of (Fourier analytical) Sobolev and Besov spaces satisfy this condition.

Another interesting aspect arises from Proposition 1. Due to the uniqueness assertion the set

$$X \otimes^D Y = \left\{ \sum_{i=1}^{n} f_i \otimes^D g_i : f_i \in X, g_i \in Y, i = 1, \ldots, n, n \in \mathbb{N} \right\}$$

is a well-defined subspace of $\mathcal{S}'(\mathbb{R}^{d_1 + d_2})$. Moreover, Proposition 1 motivates the following definition for all $h = \sum_{i=1}^{n} \lambda_i f_i \otimes g_i$ and $w = \sum_{j=1}^{m} \mu_j u_j \otimes v_j$ from $F(X, Y)$:

$$\sum_{i=1}^{n} \lambda_i f_i \otimes g_i \cong \sum_{j=1}^{m} \mu_j u_j \otimes v_j$$

$$\Longleftrightarrow \sum_{i=1}^{n} \lambda_i f_i(\varphi) \cdot g_i(\psi) = \sum_{j=1}^{m} \mu_j u_j(\varphi) \cdot v_j(\psi) \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^{d_1}), \psi \in \mathcal{S}(\mathbb{R}^{d_2}).$$

The relation $\cong$ then turns out to be an equivalence relation on $F(X, Y)$, and we have $X \otimes^D Y = F(X, Y)/\cong$ via the obvious identification $f \otimes g \longmapsto f \otimes^D g$. This yields another approach towards tensor product spaces which is applicable also for quasi-Banach spaces.

Remark 2. Note that for $h = \sum_{i=1}^{n} f_i \otimes g_i$ we have

$$\sum_{i=1}^{n} f_i(\varphi) g_i(\cdot) = \sum_{i=1}^{n} (\mathcal{J}\varphi)(f_i) g_i = A_h(\mathcal{J}\varphi) \in Y ,$$

and similarly

$$\sum_{i=1}^{n} g_i(\psi) f_i(\cdot) = \sum_{i=1}^{n} (\mathcal{J}\psi)(g_i) f_i = B_h(\mathcal{J}\psi) \in X \,.$$

Since we do not require a dense embedding $\mathcal{J}(\mathcal{S}(\mathbb{R}^{d_1})) \hookrightarrow X$ or $\mathcal{J}(\mathcal{S}(\mathbb{R}^{d_2})) \hookrightarrow Y$, we need to investigate whether $h \cong w$ implies $A_h = A_w$ or $B_h = B_w$.

Before we come to a comparison of $X \otimes Y$ with $X \otimes^D Y$ we shall prove an auxiliary assertion.

Lemma 2. Let X be a topological vector space, such that we have a continuous embedding $X \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Then $\mathcal{J}(\mathcal{S}(\mathbb{R}^n))$ is $\sigma(X', X)$-dense in X', where $\mathcal{J}$ is once more the natural injection $\mathcal{J} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}''(\mathbb{R}^n)$.

The proof of this lemma requires some preparation. We start by recalling some notions for dual pairs of vector spaces and weak topologies.

A dual pair of vector spaces is any pair of vector spaces X and Y over the same field (here $\mathbb{C}$), together with a bilinear form, which will be denoted by $\langle \cdot, \cdot \rangle$. Moreover, they have to satisfy the compatibility conditions

$$\begin{aligned} \forall x \in X \setminus \{0\} \quad \exists y \in Y : \quad \langle x, y \rangle \neq 0 \,, \\ \forall y \in Y \setminus \{0\} \quad \exists x \in X : \quad \langle x, y \rangle \neq 0 \,. \end{aligned}$$

The weak topology $\sigma(X, Y)$ on X is the locally convex topology generated by the family of seminorms $\{p_y(x) = |\langle x, y \rangle| : y \in Y\}$.
For given sets $A \subset X$ and $B \subset Y$ the *polar* of A is defined as

$$A^0 := \{y \in Y : \Re\langle x, y \rangle \leq 1 \quad \text{for all } x \in A\} \,,$$

and the polar of B is defined as

$$B^0 := \{x \in X : \Re\langle x, y \rangle \leq 1 \quad \text{for all } y \in B\} \,.$$

The bi-polar A^{00} is defined accordingly. If we denote by $\mathrm{co}(C)$ the convex hull of some set C, then the main tool for the proof of Lemma 2 reads as follows.

Proposition 2 (Bi-polar theorem). Let (X, Y) be a dual pair, and let $A \subset X$. Then it holds

$$A^{00} = \overline{\mathrm{co}(A \cup \{0\})} \,,$$

where the closure is taken in the $\sigma(X, Y)$-topology.

Note that neither X nor Y a priori need to be locally convex spaces. Though the proof of this theorem uses a version of the Hahn-Banach theorem for locally convex spaces, this is applied to the locally convex space $(X, \sigma(X, Y))$ only, not to X itself. For further details we refer to [18] (p. 126, Theorem 1.5) or [1] (Chapter IV, p. 53).

Proof of Lemma 2. We remind on the observation $\mathcal{J}\varphi \in X'$ for every $\varphi \in \mathcal{S}(\mathbb{R}^n)$, see Lemma 1. Moreover, in view of Lemma 1 we conclude that (X', X) together with the induced dual pairing is a dual pair.

We now consider the set $A = \mathcal{J}\big(\mathcal{S}(\mathbb{R}^n)\big) \subset X'$. Then we immediately find $A^0 = \{0\}$. This follows indirectly: Let $x \in A^0$, $x \neq 0$. Then we also have $x \neq 0$ in the sense of $\mathcal{S}'(\mathbb{R}^n)$, i.e. there is a function $\varphi \in \mathcal{S}(\mathbb{R}^n)$, such that $x(\varphi) \neq 0$. W.l.o.g. assume $x(\varphi) \in \mathbb{R}$, $x(\varphi) > 0$. Then we find $x(\lambda\varphi) > 0$ for all real numbers $\lambda > 0$. This implies we can find some λ_0, such that $x(\lambda_0\varphi) > 1$. But since $\lambda_0\varphi \in \mathcal{S}(\mathbb{R}^n)$ this contradicts $x \in A^0$.

Moreover, $A^0 = \{0\}$ trivially implies $A^{00} = X'$. Then we find by the Bi-polar theorem

$$X' = A^{00} = \overline{\text{co}(A \cup \{0\})} = \overline{A} = \overline{\mathcal{J}\big(\mathcal{S}(\mathbb{R}^n)\big)} \,,$$

where the closure is taken in the $\sigma(X', X)$-topology. This completes the proof. $\qquad\square$

The main result concerning the comparison of $X \otimes Y$ and $X \otimes^D Y$ is the following theorem.

Theorem 2. Let X be a topological vector space, such that we have a continuous embedding $X \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_1})$. Moreover, let Y be an arbitrary subspace of $\mathcal{S}'(\mathbb{R}^{d_2})$. Then we have $X \otimes Y = X \otimes^A Y$, and it holds $X \otimes^A Y = X \otimes^D Y$ in the sense, that the canonical identification $f \otimes g \longmapsto f \otimes^D g$ is a linear isomorphism mapping the spaces onto each other.

Note that we do not need any topological structure for Y.

Proof. The identification $X \otimes Y = X \otimes^A Y$ follows immediately from Theorem 1 and Lemma 1. For the other assertion, let $h, w \in X \otimes Y$ and $H, W \in X \otimes^D Y$, where

$$h = \sum_{i=1}^{n} f_i \otimes g_i \,, \quad H = \sum_{i=1}^{n} f_i \otimes^D g_i \,, \quad w = \sum_{j=1}^{m} u_j \otimes v_j \,, \quad W = \sum_{j=1}^{m} u_j \otimes^D v_j \,.$$

We shall prove that $h = w$ in the sense of $X \otimes^A Y = X \otimes Y$ if, and only if $H = W$ in the sense of $\mathcal{S}'(\mathbb{R}^{d_1+d_2})$, i.e. the mapping $f \otimes g \longmapsto f \otimes^D g$ can be extended to a well-defined linear isomorphism from $X \otimes^A Y$ onto $X \otimes^D Y$.

Step 1: $h = w$ implies $H = W$.
The assumptions $h = w$ explicitly means

$$\sum_{i=1}^{n} \phi(f_i)g_i = \sum_{j=1}^{m} \phi(u_j)v_j \quad \text{for all} \quad \phi \in X' \,.$$

This particularly applies to $\phi = \mathcal{J}\varphi$, $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$. Hence we find

$$\sum_{i=1}^{n} f_i(\varphi)g_i = \sum_{j=1}^{m} u_j(\varphi)v_j$$

as an equation in $Y \subset \mathcal{S}'(\mathbb{R}^{d_2})$. But this immediately yields

$$\sum_{i=1}^{n} f_i(\varphi)g_i(\psi) = \sum_{j=1}^{m} u_j(\varphi)v_j(\psi) \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^{d_1}), \psi \in \mathcal{S}(\mathbb{R}^{d_2}) \,,$$

which by the uniqueness assertion of Proposition 1 is equivalent to $H = W$.

Step 2: $H = W$ **implies** $h = w$.
We first note that it is sufficient to prove the assertion in the case $W = 0$. Moreover, the assumption $H = 0$ is equivalent to

$$\sum_{i=1}^{n} f_i(\varphi)g_i = 0 \quad \text{for every} \quad \varphi \in \mathcal{S}(\mathbb{R}^{d_1}),$$

which is to be understood as an equation in $\mathcal{S}'(\mathbb{R}^{d_2}) \supset Y$. Using the mapping $\mathcal{J}$ this can be rewritten as

$$\sum_{i=1}^{n} \big(\mathcal{J}\varphi\big)(f_i)g_i = 0 \quad \text{for every} \quad \varphi \in \mathcal{S}(\mathbb{R}^{d_1}).$$

Now it is obvious from the definition of the weak*-topology, that this implies $h = 0$ if, and only if $\mathcal{J}\big(\mathcal{S}(\mathbb{R}^{d_1})\big)$ is weak*-dense in X'. But this is exactly the content of Lemma 2. $\qquad\square$

3 Injective quasi-norms

3.1 Associated norms and Banach envelopes

Tensor products arise quite naturally in the study of bilinear forms and their relation to linear operators. Hence it is no great surprise that also the Hahn-Banach extension theorem is frequently used in proofs, in particular the well-known identity

$$\|x|V\| = \sup\Big\{|\psi(x)| \ : \ \|\psi|V'\| \le 1\Big\} = \sup\Big\{|\psi(x)| \ : \ \|\psi|V'\| = 1\Big\}, \tag{1}$$

which is valid for any (real or complex) normed vector space V.
It is well-known that the Hahn-Banach extension theorem generally fails to be true for quasi-Banach spaces. Hence a natural question when trying to transfer proofs from Banach spaces to quasi-Banach spaces is whether at least the identity (1) remains valid. The answer is given by the following lemma.

Lemma 3. Let X be a quasi-Banach space. Then we define

$$\||x|X\|| := \sup\Big\{|\psi(x)| \ : \ \|\psi|X'\| \le 1\Big\}, \qquad x \in X.$$

(i) The functional $\||\cdot|X\||$ defines a norm on X if, and only if, X' separates the points in X.

(ii) It holds $\|x|X\| = \||x|X\||$ for all $x \in X$ if, and only if, $\|\cdot|X\|$ is a norm on X.

(iii) We have $\||x|X\|| \ge c\,\|x|X\|$ for all $x \in X$ if, and only if, $\||\cdot|X\||$ is an equivalent norm on X.

The proof is obvious. We only mention, that $\||\cdot|X\||$ is called the *associated norm* on X. Its properties are consequences of the fact that the dual spaces of quasi-Banach spaces are always Banach spaces and the observation $\||x|X\|| = \|\mathcal{J}x|X''\|$, where $\mathcal{J} : X \longrightarrow X''$ is the canonical mapping used before.

Moreover, the completion of X with respect to the associated norm is called the *Banach envelope* and will be denoted by X^E. In particular, it follows $X' = (X^E)'$. For further details and references we refer to [12].

Since ℓ_p, $p < 1$, is known not to be locally convex and hence also not normable, both conditions in (ii) and (iii) would exclude these spaces. Thus they are far too restrictive to be feasible. This means many proofs for tensor products of Banach spaces relying on equation (1) cannot be carried over without changes to the quasi-Banach case, at least without exceptional additional assumptions on the spaces involved.

Before we return to tensor product spaces, we shall recall a last well-known notion for quasi-Banach spaces.

Definition 1. Let $0 < p \leq 1$, and let X be a quasi-Banach space. Then X is called a *p-Banach space* and its quasi norm *p-norm*, respectively, if

$$\left\| \, f + g \mid X \, \right\|^p \leq \left\| \, f \mid X \, \right\|^p + \left\| \, g \mid X \, \right\|^p \qquad \text{for all} \quad f, g \in X \, .$$

It is clear, that every Banach space is a 1-Banach space and every norm is a 1-norm. Furthermore, it can be shown that for every quasi-Banach space $(X, \| \cdot \|)$ there exists a $p \in (0, 1]$ and a p-norm $\| \cdot \|^*$ on X, which is equivalent to $\| \cdot \|$, i.e. $(X, \| \cdot \|^*)$ is a p-Banach space. We refer to [17] for details and further references.

3.2 Injective quasi-norms and their basic properties

When studying properties of quasi-norms and tensor product spaces we shall only be concerned with two particular aspects. The notions we are going to introduce now are immediate extensions of the respective versions for Banach spaces. Thereby, if δ is some quasi-norm on $X \otimes Y$, we denote by $X \otimes_\delta Y$ the completion of $X \otimes Y$ with respect to δ.

Definition 2.

(i) A quasi-norm δ on $X \otimes Y$ satisfying

$$\delta(x \otimes y) = \|x|X\| \cdot \|y|Y\| \qquad \text{for all } x \in X, y \in Y \, ,$$

is called *crossnorm*.

(ii) Let $T_i : X_i \longrightarrow Y_i$, $i = 1, 2$, be bounded linear operators mapping quasi-Banach spaces X_i into quasi-Banach spaces Y_i. We define a linear mapping $T_1 \otimes T_2$ on $F(X_1, X_2)$ (and on $X_1 \otimes X_2$) by the property

$$(T_1 \otimes T_2)(x_1 \otimes x_2) = (T_1 x_1) \otimes (T_2 x_2) \, , \qquad x_1 \in X_1, x_2 \in X_2 \, , \tag{2}$$

and linear extension. Then a quasi-norm δ is called uniform, if

$$\delta\big((T_1 \otimes T_2)h, Y_1, Y_2\big) \leq \big\|T_1\big|\mathcal{L}(X_1, Y_1)\big\| \cdot \big\|T_2\big|\mathcal{L}(X_2, Y_2)\big\|\delta(h, X_1, X_2) \tag{3}$$

holds for all $h \in X_1 \otimes X_2$.

(iii) For functionals $\varphi \in X'$ and $\psi \in Y'$, we can define a functional $\varphi \otimes \psi$ on $F(X, Y)$ (and on $X \otimes Y$ as well) via

$$(\varphi \otimes \psi)(x \otimes y) = \varphi(x) \cdot \psi(y), \qquad x \in X, y \in Y,$$

and linear extension. A quasi-norm δ on $X \otimes Y$ is called reasonable, if $\varphi \otimes \psi$ is bounded on $X \otimes Y$ with respect to δ, and its continuous extension $\varphi \otimes_\delta \psi$ to $X \otimes_\delta Y$ satisfies

$$\left\| \varphi \otimes_\delta \psi \,\middle|\, \left(X \otimes_\delta Y\right)' \right\| = \left\| \varphi \middle| X' \right\| \cdot \left\| \psi \middle| Y' \right\|.$$

Concerning these definitions we only mention, that $\phi \otimes \psi \in (X \otimes Y)'$ is always well-defined due to the observation $(\phi \otimes \psi)(z) = \psi(A_z \phi)$. Moreover, we similarly find from the linearity of T_1 and T_2 that two representations $\sum_{i=1}^n x_i \otimes y_i$ and $\sum_{j=1}^m f_j \otimes g_j$ are equivalent algebraically if, and only if, $\sum_{i=1}^n T_1 x_i \otimes T_2 y_i$ and $\sum_{j=1}^m T_1 f_j \otimes T_2 g_j$ are equivalent, i.e. $T_1 \otimes T_2$ is well-defined on $X_1 \otimes X_2$. Every uniform quasi-norm δ admits for every operator $T_1 \otimes T_2$ a unique quasi-norm-preserving continuous extension T, such that

$$T : X_1 \otimes_\delta X_2 \longrightarrow Y_1 \otimes_\delta Y_2 \qquad \text{and} \qquad T \in \mathcal{L}(X_1 \otimes_\delta X_2, Y_1 \otimes_\delta Y_2).$$

We will denote this extension T by $T_1 \otimes_\delta T_2$.

As a first important example we are now going to study the injective tensor norm, which we will denote by λ. For Banach spaces, it is usually defined as

$$\lambda(z, X, Y) = \left\| A_z \middle| \mathcal{L}(X', Y) \right\| = \sup_{\|\phi|X'\| \le 1} \left\| \sum_{i=1}^n \phi(x_i) y_i \,\middle|\, Y \right\|, \qquad z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y,$$

where the definition is independent of the representation of z by the definition of the underlying equivalence relation in $X \otimes Y$.

Moreover, if we define the functional $\phi \otimes \psi \in X' \otimes Y' \subset (X \otimes Y)'$ as above, we further find for $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$

$$\left\| A_z \middle| \mathcal{L}(X', Y) \right\| = \sup_{\|\phi|X'\| \le 1} \left\| \sum_{i=1}^n \phi(x_i) y_i \,\middle|\, Y \right\| = \sup_{\|\phi|X'\| \le 1} \sup_{\|\psi|Y'\| \le 1} \left| \psi\left(\sum_{i=1}^n \phi(x_i) y_i \right) \right|$$

$$= \sup_{\|\phi|X'\| \le 1} \sup_{\|\psi|Y'\| \le 1} \left| (\phi \otimes \psi)(z) \right| = \sup_{\|\psi|Y'\| \le 1} \sup_{\|\phi|X'\| \le 1} \left| \phi\left(\sum_{i=1}^n \psi(y_i) x_i \right) \right|$$

$$= \sup_{\|\psi|Y'\| \le 1} \left\| \sum_{i=1}^n \psi(y_i) x_i \,\middle|\, X \right\| = \left\| B_z \middle| \mathcal{L}(Y', X) \right\|.$$

For quasi-Banach spaces this chain of equalities no longer needs to be true. Though the operators A_z and B_z still generate the same equivalence relation (at least, if X and Y have separating duals), their quasi-norms are usually incomparable. This can be easily seen for dyads $x \otimes y$, where one immediately verifies

$$\left\| A_{x \otimes y} \middle| \mathcal{L}(X', Y) \right\| = \left\| |x|X\| \right\| \cdot \|y|Y\| \qquad \text{and} \qquad \left\| B_{x \otimes y} \middle| \mathcal{L}(Y', X) \right\| = \|x|X\| \cdot \left\| |y|Y\| \right\|$$

Hence the above calculation gives rise to three (generally different) versions of injective quasi-norms:

Definition 3. Let X and Y be quasi-Banach spaces with separating duals. Then we put

$$\lambda_A(z, X, Y) = \big\| A_z \big| \mathcal{L}(X', Y) \big\|,$$
$$\lambda_B(z, X, Y) = \big\| B_z \big| \mathcal{L}(Y', X) \big\|,$$
$$\lambda_C(z, X, Y) = \sup_{\|\phi|X'\|\leq 1} \sup_{\|\psi|Y'\|\leq 1} \big| (\phi \otimes \psi)(z) \big|.$$

The independence of λ_C of the representation of z follows from the well-definedness of $\phi \otimes \psi$, and we observe

$$\lambda_C(z, X, Y) = \sup_{\|\phi|X'\|\leq 1} \sup_{\|\psi|Y'\|\leq 1} \big| (\phi \otimes \psi)(z) \big| = \sup_{\|\phi|X'\|\leq 1} \big\|\big| A_z \phi \,\big| Y \big\|\big\| = \sup_{\|\psi|Y'\|\leq 1} \big\|\big| B_z \psi \,\big| X \big\|\big\|.$$

The functionals λ_A and λ_B define quasi-norms on $X \otimes Y$ (this follows from the linearity of the mappings $z \mapsto A_z$ and $z \mapsto B_z$ and the properties of the quasi-norms on X and Y). Moreover, λ_A is a p-Norm, if Y is a p-Banach space, and λ_B is a p-Norm, if X is a p-Banach space (the p-triangle inequalities follow from the respective ones on X or Y). Finally, λ_C is always a norm (we remind on the standing assumption, that X' and Y' shall have separating duals; the norm-properties follow by the usual arguments for operator norms such as $\|\| \cdot \,|X\, \|\|$ and $\|\| \cdot \,|Y\, \|\|$).

As we saw above, neither λ_A nor λ_B are crossnorms, and for λ_C we find

$$\lambda_C(x \otimes y, X, Y) = \big\|\big| x \,\big| X \big\|\big\| \cdot \big\|\big| y \,\big| Y \big\|\big\|, \qquad x \in X, \, y \in Y. \tag{4}$$

This follows immediately from the respective definitions. Furthermore, we shall add, that neither of these injective quasi-norms is equivalent to some crossnorm, which is a direct consequence of Lemma 3.

As a corollary of these considerations, we find

Lemma 4. Let X and Y be quasi-Banach spaces with separating duals. Then it holds

$$X \otimes_{\lambda_C} Y = X^E \otimes_{\lambda_C} Y^E = X^E \otimes_\lambda Y^E$$

The first identity follows from (4), the latter one from the mentioned observation $\lambda \equiv \lambda_A = \lambda_B = \lambda_C$ for Banach spaces.

In particular, we obtain from $(\ell_p)^E = \ell_1$ (which holds with equality of norms)

Corollary 1. Let $0 < p, q \leq 1$. Then it holds

$$\ell_p \otimes_{\lambda_C} \ell_q = \ell_1 \otimes_\lambda \ell_1.$$

The main aspect of these considerations is the observation that the above notion of a crossnorm no longer is the only version relevant for the study of quasi-norms, though it remains the most important one, since often the associated norm is no easily accessible.

For the notion of a reasonable norm, the situation is quite different. This stems from the mentioned fact that the dual space of an arbitrary quasi-Banach space is a Banach space (though possibly trivial), and the observation that a tensor norm δ on $X \otimes Y$ is reasonable if, and only if, the induced norm δ^* on $X' \otimes Y' \subset (X \otimes Y)'$ is a crossnorm. Hence even if X and Y are quasi-Banach spaces, this property is concerned only with Banach spaces, so no change is necessary.

Before we return to the injective quasi-norms we add a simple criterion for some quasi-norm δ to be reasonable. Its counterpart for Banach spaces is well-known (see e.g. [14]).

Lemma 5. Let X and Y be quasi-Banach spaces, and let δ be a quasi-norm on $X \otimes Y$. Further we assume

$$\delta(x \otimes y) \leq \|x|X\| \cdot \|y|Y\|, \qquad x \in X, y \in Y, \tag{5}$$
$$|(\phi \otimes \psi)(z)| \leq \|\phi|X'\| \cdot \|\psi|Y'\| \cdot \delta(z), \qquad z \in X \otimes Y, \phi \in X', \psi \in Y'. \tag{6}$$

Then δ is a reasonable quasi-norm.

Note that this lemma only requires the weakest version of a crossnorm-inequality for δ.

Proof. If we denote by δ^* the induced operator quasi-norm on $(X \otimes Y, \delta)'$ the assumption (6) yields

$$\delta^*(\phi \otimes \psi) \leq \|\phi\,|\,X'\| \cdot \|\psi\,|\,Y'\| \tag{7}$$

Moreover, by definition of δ^* we have for all $x \in X$ and $y \in Y$

$$|\phi(x)| \cdot |\psi(y)| = |(\phi \otimes \psi)(x \otimes y)| \leq \delta^*(\phi \otimes \psi, X, Y)\delta(x \otimes y, X, Y) \tag{8}$$

Using the estimate (5) and taking a supremum over the unit balls of X and Y we find

$$\|\phi\,|\,X'\| \cdot \|\psi\,|\,Y'\| \leq \delta^*(\phi \otimes \psi) \tag{9}$$

Combining (7) and (9) proves that δ^* is a crossnorm on $X' \otimes Y'$. As mentioned above this is equivalent to δ being reasonable. $\qquad\square$

For the injective quasi-norms we have the following counterpart of well-known properties of λ.

Lemma 6. Let X and Y be quasi-Banach spaces with separating duals. Then λ_A, λ_B and λ_C are reasonable. Moreover, for every reasonable quasi-norm α on $X \otimes Y$ we have $\alpha(z) \geq \lambda_C(z, X, Y)$ for all $z \in X \otimes Y$.

Proof. The proof follows along the lines of the one for Banach spaces. In particular, we have for $\phi \in X'$ and $\psi \in Y'$

$$|(\phi \otimes \psi)(z)| = \|\phi|X'\| \cdot \|\psi|Y'\| \cdot \left|\left(\frac{\phi}{\|\phi|X'\|} \otimes \frac{\psi}{\|\psi|Y'\|}\right)\left(\sum_{i=1}^{n} x_i \otimes y_i\right)\right|$$
$$\leq \|\phi|X'\| \cdot \|\psi|Y'\| \cdot \lambda_C(z, X, Y).$$

Now the reasonability of λ_C follows from Lemma 5 (the assumption (5) is a consequence of (4)). The respective assertions for λ_A and λ_B follow from $\lambda_C(z) \leq \lambda_A(z)$ and $\lambda_C(z) \leq \lambda_B(z)$ for all $z \in X \otimes Y$. Now let α be any reasonable tensor quasi-norm. Then we have

$$|(\phi \otimes \psi)(z)| \leq \|\phi \otimes \psi|(X \otimes Y)'\| \cdot \alpha(z) = \|\phi|X'\| \cdot \|\psi|Y'\| \cdot \alpha(z).$$

Taking the supremum over the unit balls of X' and Y' yields $\lambda_C(z) \leq \alpha(z)$. $\qquad\square$

3.3 Uniform quasi-norms

Next we try to determine whether the notion of a uniform quasi-norm needs to be modified as well. The answer is two-fold. On the one hand, the important inequality (3) makes sense for quasi-Banach spaces, and even the consequence about the extendability of $T_1 \otimes T_2$ remains valid. However, when trying to determine whether the equation

$$\|T_1 \otimes_\delta T_2|\mathcal{L}(X_1 \otimes_\delta X_2, Y_1 \otimes_\delta Y_2)\| = \|T_1|\mathcal{L}(X_1, Y_1)\| \cdot \|T_2|\mathcal{L}(X_2, Y_2)\|$$

holds true, i.e. whether the operator quasi-norm is a crossnorm, its relation to the modified notions of crossnorms becomes clear.

To begin with we slightly modify the classical situation of uniform crossnorms, and demonstrate how the above identity for the operator norms can be derived in this situation.

Proposition 3. Let α be a quasi-norm on $F \otimes G$, and let β be a quasi-norm on $X \otimes Y$, where F, G, X, Y are quasi-Banach spaces. Moreover, we assume

$$\alpha(f \otimes g) \le \|f|F\| \cdot \|g|G\| \qquad \text{for all } f \in F, g \in G, \tag{10}$$

$$\beta(x \otimes y) = \|x|X\| \cdot \|y|Y\| \qquad \text{for all } x \in X, y \in Y, \tag{11}$$

$$\beta\big((S \otimes T)h\big) \le \big\|S|\mathcal{L}(F, X)\big\| \cdot \big\|T|\mathcal{L}(G, Y)\big\| \cdot \alpha(h). \tag{12}$$

Then it holds

$$\big\|S \otimes_\alpha T|\mathcal{L}(F \otimes_\alpha G, X \otimes_\beta Y)\big\| = \big\|S|\mathcal{L}(F, X)\big\| \cdot \big\|T|\mathcal{L}(G, Y)\big\|.$$

Proof. Immediately from the definition of the operator quasi-norm we find

$$\begin{aligned}
\big\|S \otimes_\alpha T|\mathcal{L}(F \otimes_\alpha G, X \otimes_\beta Y)\big\| &= \sup_{\alpha(h) \le 1} \beta\big((S \otimes T)(h)\big) \\
&\ge \sup_{f \in F, g \in G: \alpha(f \otimes g) \le 1} \beta\big((S \otimes T)(f \otimes g)\big) \\
&\ge \sup_{\|f|F\| \le 1, \|g|G\| \le 1} \beta\big((Sf) \otimes (Tg)\big),
\end{aligned}$$

where in the last estimate we used assumption (10). Using property (11) we further find

$$\big\|S \otimes_\alpha T|\mathcal{L}(F \otimes_\alpha G, X \otimes_\beta Y)\big\| \ge \sup_{\|f|F\| \le 1} \sup_{\|g|G\| \le 1} \|Sf|X\| \cdot \|Tg|Y\| = \big\|S|\mathcal{L}(F, X)\big\| \cdot \big\|T|\mathcal{L}(G, Y)\big\|.$$

Together with (12) this proves the claim. $\qquad \square$

While the result is a final one and clearly gives a satisfying answer in the classical situation, the assumed crossnorm-property generally no longer holds true for quasi-Banach spaces and quasi-norms as we have seen before. Moreover, also the assumption (10) (or (3)) turn out to not sufficient when dealing with other types of crossnorm-properties. Exemplary we will treat the injective quasi-norms in detail.

Proposition 4. Let X_1, X_2, Y_1, Y_2 be quasi-Banach spaces with separating duals, and let $0 < p \le 1$. Moreover, let $T_1 \in \mathcal{L}(X_1, Y_1)$ and $T_2 \in \mathcal{L}(X_2, Y_2)$. Then it holds

$$\big\|T_1 \otimes_{\lambda_A} T_2|\mathcal{L}(X_1 \otimes_{\lambda_A} X_2, Y_1 \otimes_{\lambda_A} Y_2)\big\| = \big\|T_1|\mathcal{L}(X_1, Y_1^E)\big\| \cdot \big\|T_2|\mathcal{L}(X_2, Y_2)\big\|, \tag{13}$$

$$\big\|T_1 \otimes_{\lambda_B} T_2|\mathcal{L}(X_1 \otimes_{\lambda_B} X_2, Y_1 \otimes_{\lambda_B} Y_2)\big\| = \big\|T_1|\mathcal{L}(X_1, Y_1)\big\| \cdot \big\|T_2|\mathcal{L}(X_2, Y_2^E)\big\|, \tag{14}$$

$$\big\|T_1 \otimes_{\lambda_C} T_2|\mathcal{L}(X_1 \otimes_{\lambda_C} X_2, Y_1 \otimes_{\lambda_C} Y_2)\big\| = \big\|T_1|\mathcal{L}(X_1, Y_1^E)\big\| \cdot \big\|T_2|\mathcal{L}(X_2, Y_2^E)\big\|. \tag{15}$$

As a first step, we shall determine the operator quasi-norm of the usual dual operator T' of some given operator $T \in \mathcal{L}(X, Y)$. For Banach spaces $\|T'|\mathcal{L}(Y', X')\| = \|T|\mathcal{L}(X, Y)\|$ is well-known, but clearly this no longer needs to be true for quasi-Banach spaces, in particular since the dual spaces might be trivial. However, for quasi-Banach spaces with separating duals, a precise answer is possible.

Lemma 7. Let X and Y be quasi-Banach spaces with separating duals, and let $T \in \mathcal{L}(X, Y)$. Then it holds

$$\big\| T' \big| \mathcal{L}(Y', X') \big\| = \big\| T \big| \mathcal{L}(X, Y^E) \big\| = \big\| T \big| \mathcal{L}(X^E, Y^E) \big\|,$$

where T is extended by continuity to the whole of X^E.

Proof. By straightforward calculations, inserting the relevant definitions, we find

$$
\begin{aligned}
\big\| T' \big| \mathcal{L}(Y', X') \big\| &= \sup_{\|\psi|Y'\| \leq 1} \|T'\psi|X'\| = \sup_{\|\psi|Y'\| \leq 1} \sup_{\|x|X\| \leq 1} |(T'\psi)(x)| \\
&= \sup_{\|x|X\| \leq 1} \sup_{\|\psi|Y'\| \leq 1} |\psi(Tx)| = \sup_{\|x|X\| \leq 1} |\|Tx|Y\|| \equiv \sup_{\|x|X\| \leq 1} \|Tx|Y^E\| \\
&= \big\| T \big| \mathcal{L}(X, Y^E) \big\|.
\end{aligned}
$$

This proves the claim. $\qquad\square$

Proof of Proposition 4.

Step 1: Treatment of λ_C.
Let $z \in X_1 \otimes X_2$ be given by $z = \sum_{i=1}^n x_i \otimes y_i$. Then we find

$$
\begin{aligned}
\lambda_C\big((T_1 \otimes T_2)(z), Y_1, Y_2\big) &= \sup_{\|\phi|Y_1'\| \leq 1} \sup_{\|\psi|Y_2'\| \leq 1} \big|(\phi \otimes \psi)\big((T_1 \otimes T_2)(z)\big)\big| \\
&= \sup_{\|\phi|Y_1'\| \leq 1} \sup_{\|\psi|Y_2'\| \leq 1} \left| \sum_{i=1}^n (T_1'\phi)(x_i)(T_2'\psi)(y_i) \right| \\
&\leq \big\|T_1'|\mathcal{L}(Y_1', X_1')\big\| \cdot \big\|T_2'|\mathcal{L}(Y_2', X_2')\big\| \sup_{\|\widetilde\phi|X_1'\| \leq 1} \sup_{\|\widetilde\psi|X_2'\| \leq 1} \left| \sum_{i=1}^n \widetilde\phi(x_i)\widetilde\psi(y_i) \right| \\
&= \big\|T_1'|\mathcal{L}(Y_1', X_1')\big\| \cdot \big\|T_2'|\mathcal{L}(Y_2', X_2')\big\| \cdot \lambda_C\big(z, X_1, X_2\big).
\end{aligned}
$$

This implies the first inequality needed to prove (15). The reverse inequality follows in the same way as in the proof of Proposition 3, using property (4) in the second part of the calculation. We end up with

$$\big\|T_1 \otimes_{\lambda_C} T_2 \big| \mathcal{L}(X_1 \otimes_\alpha X_2, Y_1 \otimes_{\lambda_C} Y_2)\big\| \geq \big\|T_1 \big| \mathcal{L}(X_1, Y_1^E)\big\| \cdot \big\|T_2 \big| \mathcal{L}(X_2, Y_2^E)\big\|.$$

In view of Lemma 7 this proves the claim.

Step 2: Treatment of λ_A and λ_B.

Directly from the respective definitions we find for $z \in X_1 \otimes X_2$

$$\lambda_A((T_1 \otimes T_2)(z), Y_1, Y_2) = \sup_{\|\psi|Y_1'\| \leq 1} \left\|\sum_{j=1}^n \psi(T_1 x_j) T_2 y_j \,\Big|\, Y_2\right\| = \sup_{\|\psi|Y_1'\| \leq 1} \left\|T_2\left(\sum_{j=1}^n \psi(T_1 x_j) y_j\right) \Big|\, Y_2\right\|$$

$$\leq \|T_2|\mathcal{L}(X_2, Y_2)\| \sup_{\|\psi|Y_1'\| \leq 1} \left\|\sum_{j=1}^n (T_1'\psi)(x_j) y_j \,\Big|\, Y_2\right\|$$

$$\leq \|T_2|\mathcal{L}(X_2, Y_2)\| \cdot \|T_1'|\mathcal{L}(Y_1', X_1')\| \sup_{\|\phi|X_1'\| \leq 1} \left\|\sum_{j=1}^n \phi(x_j) y_j \,\Big|\, Y_2\right\|$$

$$= \|T_2|\mathcal{L}(X_2, Y_2)\| \cdot \|T_1'|\mathcal{L}(Y_1', X_1')\| \lambda_A(z, X_1, X_2),$$

which immediately yields

$$\left\|T_1 \otimes_{\lambda_A} T_2 \,\big|\, \mathcal{L}(X_1 \otimes_{\lambda_A} X_2, Y_1 \otimes_{\lambda_A} Y_2)\right\| \leq \left\|T_1' \,\big|\, \mathcal{L}(Y_1', X_1')\right\| \cdot \left\|T_2 \,\big|\, \mathcal{L}(X_2, Y_2)\right\|.$$

The reverse estimate once more follows as in the proof of Proposition 3. In particular we find

$$\left\|T_1 \otimes_{\lambda_A} T_2 \,\big|\, \mathcal{L}(X_1 \otimes_{\lambda_A} X_2, Y_1 \otimes_{\lambda_A} Y_2)\right\| \geq \sup_{\|x|X_1\| \leq 1, \|y|X_2\| \leq 1} \lambda_A\big((T_1 x) \otimes (T_2 y), Y_1, Y_2\big)$$

$$\geq \sup_{\|x|X_1\| \leq 1} \sup_{\|y|X_2\| \leq 1} \||T_1 x|Y_1\|| \cdot \|T_2 y|Y\|$$

$$= \left\|T_1 \,\big|\, \mathcal{L}(X_1, Y_1^E)\right\| \cdot \left\|T_2 \,\big|\, \mathcal{L}(X_2, Y_2)\right\|.$$

In view of Lemma 7 this shows the claim. The proof for λ_B is completely analogous. $\qquad\square$

Remark 3. Note that these results correspond to the assertion in Lemma 4 and the respective counterparts

$$X \otimes_{\lambda_A} Y = X^E \otimes_{\lambda_A} Y \qquad \text{and} \qquad X \otimes_{\lambda_B} Y = X \otimes_{\lambda_B} Y^E.$$

Starting with these identifications, Proposition 4 can be obtained either from Proposition 3 or in the same way as for λ in the case of Banach spaces.

4 p-nuclear-norms for quasi-Banach spaces

For Banach spaces X and Y, the p-nuclear tensor norm $\alpha_p(\cdot, X, Y)$ is well-known, where $1 < p < \infty$, see e.g. [14]. Its definition can be extended to values $p \leq 1$ in different ways, which are seen to coincide for Banach spaces (for completeness we shall repeat those arguments below). However, they are generally different for quasi-Banach spaces.

Definition 4. Let X and Y be quasi-Banach spaces, and let $f \in X \otimes Y$.

(i) Let $0 < p \leq 1$. Then we define the p-nuclear norm γ_p by

$$\gamma_p(f, X, Y) := \inf\left\{\left(\sum_{j=1}^n \|x_j|X\|^p \cdot \|y_j|Y\|^p\right)^{1/p} : f = \sum_{j=1}^n x_j \otimes y_j\right\}.$$

(ii) Now let $0 < p \leq \infty$ and $\frac{1}{q} = \left(1 - \frac{1}{p}\right)_+$. Then we define

$$\alpha_p(f, X, Y) := \inf\left\{\left(\sum_{j=1}^n \|x_j|X\|^p\right)^{1/p} \sup_{\|\psi|Y'\| \leq 1} \left(\sum_{j=1}^n |\psi(y_j)|^q\right)^{1/q} : f = \sum_{j=1}^n x_j \otimes y_j\right\},$$

with the usual modification in case $q = \infty$ (i.e. $p \leq 1$).

(iii) Finally, let again $0 < p \leq \infty$. Then we define

$$\beta_p(f, X, Y) := \inf\left\{\left(\sum_{j=1}^n \|x_j|X\|^p\right)^{1/p} \sup_{\|\lambda|\ell_p^n\| \leq 1} \left\|\sum_{j=1}^n \lambda_j y_j \,\Big|\, Y\right\| : f = \sum_{j=1}^n x_j \otimes y_j\right\},$$

where ℓ_p^n is the vector space $\mathbb{C}^n$, equipped with the usual (quasi-)norm $\|\lambda|\ell_p^n\| = \left(\sum_{j=1}^n |\lambda_j|^p\right)^{1/p}$.

The version (i) was used already by Grothendieck in [6]. It can be shown that for Banach spaces the projective norm $\gamma_1 = \gamma$ is always equal to the 1-nuclear norm α_1 as defined in [14] (which justifies the above notion in case $p = 1$). On the other hand, (ii) and (iii) are more immediate extensions of the usual formulations of the p-nuclear tensor norm.

4.1 Properties of γ_p

In this subsection, we are concerned with some basic properties of the quasi-norms γ_p defined above (among others that they are indeed p-norms).

Lemma 8. Let X be a quasi-Banach space with separating dual, and let Y be a p-Banach space. Then $\gamma_p(\cdot, X, Y)$, $0 < p \leq 1$, is a reasonable p-norm on $X \otimes Y$.
Moreover, let X_1 and X_2 be quasi-Banach spaces with separating duals, and let Y_1 and Y_2 be further quasi-Banach spaces. Then it holds (3) for every $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$ and $h \in X_1 \otimes X_2$, i.e. γ_p is uniform.

Proof. We shall follow closely the corresponding proofs for $\gamma_1 \equiv \gamma$ in [14].
Due to the assumptions Theorem 1 is applicable, hence the functional analytic tensor product coincides with the algebraic one. In other words, given $z \in X \otimes Y$, $z \neq 0$, there is a functional $\phi \in X'$, $\|\phi|X'\| \leq 1$, such that we have $A_z\phi \neq 0$, where $A_z : X' \longrightarrow Y$ is the operator associated to z. Then we find

$$0 < \|A_z\phi \,|\, Y\| \leq \left\|\sum_{j=1}^n \phi(x_i)y_i \,\Big|\, Y\right\|$$

$$\leq \left(\sum_{i=1}^n |\phi(x_i)|^p \cdot \|y_i \,|\, Y\|^p\right)^{1/p} \leq \left(\sum_{i=1}^n \|x_i \,|\, X\|^p \cdot \|y_i \,|\, Y\|^p\right)^{1/p}.$$

Taking the infimum over all representations of z yields $\gamma_p(z, X, Y) > 0$. The other p-norm properties for γ_p are obvious.
For the proof of the reasonability, we first observe

$$\gamma_p(x \otimes y, X, Y) \leq \|x|X\| \cdot \|y|Y\|, \qquad x \in X, y \in Y. \tag{16}$$

Now let $\phi \in X'$, $\psi \in Y'$ and $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. Then we find

$$|(\phi \otimes \psi)(z)| \leq \sum_{i=1}^{n} |(\phi \otimes \psi)(x_i \otimes y_i)| = \sum_{i=1}^{n} |\phi(x_i)| \cdot |\psi(y_i)|$$

$$\leq \| \phi \,|\, X' \| \cdot \| \psi \,|\, Y' \| \sum_{i=1}^{n} \| x_i \,|\, X \| \cdot \| y_i \,|\, Y \|$$

Taking the infimum over all representations of z we obtain

$$|(\phi \otimes \psi)(z)| \leq \| \phi \,|\, X' \| \cdot \| \psi \,|\, Y' \| \gamma_1(z, X, Y) \leq \| \phi \,|\, X' \| \cdot \| \psi \,|\, Y' \| \gamma_p(z, X, Y)\,,$$

where we additionally used the monotonicity of the ℓ_p-quasi-norms. Now Lemma 5 shows that γ_p is reasonable.

Finally, let $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$ and $z = \sum_{i=1}^{n} u_i \otimes v_i \in X_1 \otimes X_2$. Then it holds

$$\gamma_p\left(\sum_{i=1}^{n} T_1 u_i \otimes T_2 v_i \right) \leq \left(\sum_{i=1}^{n} \| T_1 u_i \,|\, Y_1 \|^p \cdot \| T_2 v_i \,|\, Y_2 \|^p \right)^{1/p}$$

$$\leq \| T_1 |\mathcal{L}(X_1, Y_1) \| \cdot \| T_2 |\mathcal{L}(X_2, Y_2) \| \left(\sum_{i=1}^{n} \| u_i |X_1 \|^p \| v_i |X_2 \|^p \right)^{1/p}$$

Taking the infimum over all representations of z yields (3). $\qquad\square$

Remark 4. Note, that the assumption of Y being a p-Banach space can be slightly weakened to Y being p-normable, i.e. there has to be some p-norm on Y equivalent to $\| \cdot \,|Y\|$, since this condition was used only to show that $\gamma_p(z, X, Y) > 0$ for $z \neq 0$.

The following corollary is a consequence of the uniformity. Its proof is analogous to the one for the normed case in [19, Appendix B].

Corollary 2. Let X_1, X_2, Y_1, Y_2 be quasi-Banach spaces, which fulfil the assumptions of Lemma 8, and let $0 < p < 1$. Suppose that $T_1 \in \mathcal{L}(X_1, Y_1)$ and $T_2 \in \mathcal{L}(X_2, Y_2)$ are isomorphisms. Then also $T_1 \otimes_{\gamma_p} T_2$ is an isomorphism from $X_1 \otimes_{\gamma_p} X_2$ onto $Y_1 \otimes_{\gamma_p} Y_2$.

A particular application for this corollary will be discussed in the last section.

4.2 Properties of α_p and β_p

Lemma 9. Let X and Y be quasi-Banach spaces with separating duals. Furthermore, let X be a p-Banach space, where $0 < p \leq 1$. Then $\alpha_p(\cdot, X, Y)$ is a p-norm.

Proof. The homogenity is obvious. We remind on the operators B_z and the observation, that the quotient spaces with respect to the corresponding equivalence relation coincide with $X \otimes Y$. Now let $z \in X \otimes Y$, $z = \sum_{i=1}^{n} x_i \otimes y_i \neq 0$. Then due to the assumption on Y' we find some $\phi_z \in Y'$,

$\|\phi_z|Y'\| \le 1$, such that $B_z\phi_z \ne 0$, and hence

$$0 < \|B_z\phi_z|X\|^p = \left\| \sum_{i=1}^{n} \phi_z(y_i)x_i \,\bigg|\, X \right\|^p \le \sum_{i=1}^{n} |\phi_z(y_i)|^p \|x_i|X\|^p$$

$$\le \left(\sum_{i=1}^{n} \|x_i|X\|^p \right) \max_{j=1,\dots,n} |\phi_z(y_j)|^p \le \left(\sum_{i=1}^{n} \|x_i|X\|^p \right) \sup_{\|\psi|Y'\|\le 1} \max_{j=1,\dots,n} |\psi(y_j)|^p \,.$$

Taking the infimum over all representations of z and afterwards the supremum over the unit ball of Y' yields

$$0 < \big\|B_z \big| \mathcal{L}(Y', X)\big\| \le \alpha_p(z, X, Y) \,.$$

We finally prove the p-triangle inequality. Let $\varepsilon > 0$, and choose a representation of $z \in X \otimes Y$, such that

$$\sum_{j=1}^{n} \big\|x_j\big|X\big\|^p \sup_{\|\psi|Y'\|\le 1} \max_{j=1,\dots,n} |\psi(y_j)|^p \le \alpha_p(z, X, Y)^p + \varepsilon \,,$$

where we additionally may assume

$$\sum_{j=1}^{n} \big\|x_j\big|X\big\|^p \le \alpha_p^p(z, X, Y) + \varepsilon \,, \qquad \sup_{\|\psi|Y'\|\le 1} \max_{j=1,\dots,n} |\psi(y_j)| \le 1 \,.$$

Moreover, we choose a representation of $w \in X \otimes Y$,

$$w = \sum_{i=n+1}^{m} x_i \otimes y_i \,,$$

with analogous properties. Then we have in particular

$$|\psi(y_j)| \le 1 \,, \qquad j = 1, \dots, m \,, \qquad \psi \in Y' \,, \quad \|\psi|Y'\| \le 1 \,,$$

and hence

$$\sup_{\|\psi|Y'\|\le 1} \max_{j=1,\dots,m} |\psi(y_j)| \le 1 \,.$$

Moreover, we easily find

$$\sum_{j=1}^{m} \big\|x_j\big|X\big\|^p \le \alpha_p(z, X, Y)^p + \alpha_p(w, X, Y)^p + 2\varepsilon \,.$$

Combinig the last two estimates yields (upon taking the infimum over all representations of $z + w$)

$$\alpha_p(z + w, X, Y)^p \le \sum_{j=1}^{m} \big\|x_j\big|X\big\|^p \sup_{\|\psi|Y'\|\le 1} \max_{j=1,\dots,m} |\psi(y_j)| \le \alpha_p(z, X, Y)^p + \alpha_p(w, X, Y)^p + 2\varepsilon \,.$$

For $\varepsilon \to 0$ we obtain the desired inequality. $\qquad\qquad\square$

Remark 5. Note, that the assumption of X being a p-Banach space is only used to show

$$\left\|B_z\big|\mathcal{L}(Y',X)\right\| \leq \alpha_p(z,X,Y)\,,$$

in particular, the p-triangle inequality did not need any additional assumption. Hence, this assumption once more can be weakened to X being p-normable.

Lemma 10. Let X be a quasi-Banach space with separating dual. Furthermore, let Y be a p-Banach space, where $0 < p \leq 1$. Then $\beta_p(\cdot,X,Y)$ is a p-norm.

Proof. The homogenity is obvious. For the p-triangle inequality we refer to Nitsche [16].
Now let $z \in X \otimes Y$, $z = \sum_{i=1}^{n} x_i \otimes y_i \neq 0$. Moreover, let $\phi_z \in X'$, $\|\phi_z|X'\| \leq 1$, such that $A_z\phi_z \neq 0$ (such a linear functional exists due to the assumption). Then it follows

$$0 < \|A_z\phi_z|Y\|^p = \left\|\sum_{i=1}^{n} \phi_z(x_i)y_i \,\bigg|\, Y\right\|^p \leq \left(\sum_{i=1}^{n} |\phi_z(x_i)|^p\right) \sup_{\|\lambda|\ell_p^n\|\leq 1}\left\|\sum_{j=1}^{n} \lambda_j y_j \,\bigg|\, Y\right\|^p.$$

This is an immediate consequence of the homogenity of the quasi-norm on Y. Hence we find (as before taking the infimum over all representations of z and afterwards the supremum over the unit ball of X')

$$0 < \left\|A_z\big|\mathcal{L}(X',Y)\right\| \leq \beta_p(z,X,Y)\,. \qquad \square$$

Lemma 11. Let $0 < p \leq 1$. Then α_p is reasonable and uniform.

Proof. Let X and Y be arbitrary quasi-Banach spaces. Furthermore, let $\phi \in X'$, $\psi \in Y'$ and $f = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. Then it holds

$$|(\phi \otimes \psi)(f)| = \left|\sum_{i=1}^{n} \phi(x_i)\psi(y_i)\right| \leq \|\phi|X'\|\left(\sum_{i=1}^{n} \|x_i|X\|\right)\max_{j=1,\dots,n}|\psi(y_j)|$$

$$\leq \|\phi|X'\|\left(\sum_{i=1}^{n} \|x_i|X\|^p\right)^{1/p}\|\psi|Y'\|\sup_{\|\eta|Y'\|\leq 1}\max_{j=1,\dots,n}|\eta(y_j)|\,,$$

where we used the monotonicity of the ℓ_p^n-quasi-norms. Taking the infimum over all representations of f we obtain

$$|(\phi \otimes \psi)(f)| \leq\| \phi|X'\| \cdot \|\psi|Y'\| \cdot \alpha_p(f,X,Y)\,.$$

Since the inequality (5) is obvious, from Lemma 5 now follows that $\alpha_p(\cdot,X,Y)$ is reasonable. Concerning the uniformity, let X_1, X_2, Y_1 and Y_2 be quasi-Banach spaces, and let $T_i \in \mathcal{L}(X_i,Y_i)$,

$i = 1, 2$. Then we find for $z = \sum_{j=1}^{n} f_j \otimes g_j \in X_1 \otimes X_2$

$$\alpha_p\big((T_1 \otimes T_2)z, Y_1, Y_2\big)$$

$$\leq \left(\sum_{j=1}^{n} \|T_1 f_j | Y_1\|^p\right)^{1/p} \sup_{\|\psi | Y_2'\| \leq 1} \max_{j=1,\dots,n} |\psi(T_2 g_j)|$$

$$\leq \|T_1 | \mathcal{L}(X_1, Y_1)\| \left(\sum_{j=1}^{n} \|f_j | X_1\|^p\right)^{1/p} \sup_{\|\psi | Y_2'\| \leq 1} \max_{j=1,\dots,n} |(T_2^* \psi)(g_j)|$$

$$\leq \|T_1 | \mathcal{L}(X_1, Y_1)\| \cdot \|T_2' | \mathcal{L}(Y_2', X_2')\| \left(\sum_{j=1}^{n} \|f_j | X_1\|^p\right)^{1/p} \sup_{\|\psi | Y_2'\| \leq 1} \max_{j=1,\dots,n} \left|\left(\frac{T_2' \psi}{\|T_2' | \mathcal{L}(Y_2', X_2')\|}\right)(g_j)\right|$$

$$\leq \|T_1 | \mathcal{L}(X_1, Y_1)\| \cdot \|T_2 | \mathcal{L}(X_2, Y_2)\| \left(\sum_{j=1}^{n} \|f_j | X_1\|^p\right)^{1/p} \sup_{\|\phi | X_2'\| \leq 1} \max_{j=1,\dots,n} \left|\phi(g_j)\right|.$$

Concerning the dual operator T_2' we only need the inequality $\|T_2' | \mathcal{L}(Y_2', X_2')\| \leq \|T_2 | \mathcal{L}(X_2, Y_2)\|$ at this point. Taking the infimum over all representations of z, we end up with the inequality (3). $\square$

4.3 Equivalence of these norms for Banach spaces

For Banach spaces and $1 \leq p \leq \infty$ it is well-known that we have $\alpha_p = \beta_p$. Moreover, we find $\alpha_1 = \gamma_1 \equiv \gamma$, the projective tensor norm. For a proof of the former assertion, we refer to [14], the latter one is covered by the following lemma.

Lemma 12. Let X and Y be Banach spaces and $0 < p \leq 1$. Then $\alpha_p = \gamma_p$. Moreover, these norms are crossnorms, i.e. $\gamma_p(x \otimes y, X, Y) = \alpha_p(x \otimes y, X, Y) = \|x|X\| \cdot \|y|Y\|$ for all $x \in X$ and $y \in Y$.

Proof. Since every Banach space is also a p-Banach space for every value $p < 1$, we immediately obtain from Lemmas 9 and 14 (the required inequality for this lemma is obvious) the estimate $\alpha_p \leq \gamma_p$. Hence it remains to prove the reverse relation.
Let $z = \sum_{i=1}^{n} x_i \otimes y_i$, then we find

$$\gamma_p(z, X, Y)^p \leq \sum_{i=1}^{n} \|x_i|X\|^p \cdot \|y_i|Y\|^p = \sum_{i=1}^{n} \|x_i|X\|^p \cdot \sup_{\|\psi|Y'\| \leq 1} |\psi(y_i)|^p$$

$$\leq \left(\sum_{i=1}^{n} \|x_i|X\|^p\right) \sup_{\|\psi|Y'\| \leq 1} \max_{j=1,\dots,n} |\psi(y_i)|^p.$$

Taking the infimum over all representations of z yields the desired estimate.
For the crossnorm-assertion we first find for all $x \in X$, $y \in Y$, $\phi \in X'$, $\psi \in Y'$

$$|\phi(x)| \cdot |\psi(y)| = |(\phi \otimes \psi)(x \otimes y)| \leq \gamma_p^*(\phi \otimes \psi)\gamma_p(x \otimes y) = \|\phi|X'\| \cdot \|\psi|Y'\| \cdot \gamma_p(x \otimes y),$$

where we used Lemma 8 (γ_p is reasonable). Now using (1) we find by taking the supremum over the unit balls of X' and Y'

$$\gamma_p(x \otimes y, X, Y) \geq \|x|X\| \cdot \|y|Y\|.$$

Since the reverse inequality is obvious, this proves the crossnorm property for γ_p and hence also for α_p (this could be proved directly as well, using analogous arguments). $\square$

Lemma 13. Let X and Y be Banach spaces and $0 < p \leq 1$. Then $\alpha_p = \beta_p$.

Proof. Using the usual identifications $\left(\ell_1^n\right)' = \ell_\infty^n$ and $\left(\ell_\infty^n\right)' = \ell_1^n$ we observe first

$$
\sup_{\|\psi|Y'\| \leq 1} \max_{j=1,\ldots,n} |\psi(y_j)| \equiv \sup_{\|\psi|Y'\| \leq 1} \left\| \overline{\psi} \, |\ell_\infty^n \right\| = \sup_{\|\psi|Y'\| \leq 1} \sup_{\|\mu|(\ell_\infty^n)'\| \leq 1} \left| \mu(\overline{\psi}) \right|
$$

$$
= \sup_{\|\psi|Y'\| \leq 1} \sup_{\|\lambda|\ell_1^n\| \leq 1} \left| \sum_{j=1}^n \lambda_j \psi(y_j) \right| = \sup_{\|\lambda|\ell_1^n\| \leq 1} \sup_{\|\psi|Y'\| \leq 1} \left| \psi \left(\sum_{j=1}^n \lambda_j y_j \right) \right|
$$

$$
= \sup_{\|\lambda|\ell_1^n\| \leq 1} \left\| \sum_{j=1}^n \lambda_j y_j \, \Big| Y \right\| \geq \sup_{\|\lambda|\ell_p^n\| \leq 1} \left\| \sum_{j=1}^n \lambda_j y_j \, \Big| Y \right\|,
$$

where the last step is a consequence of the ℓ_p-monotonicity (for decreasing values of p the unit balls of ℓ_p^n become smaller). Inserting this into the respective definitions for the tensor norms yields $\alpha_p(z, X, Y) \geq \beta_p(z, X, Y)$.
The reverse inequality is a special case of Lemma 16 below. $\qquad \square$

Remark 6. Note that neither Lemma 12 nor Lemma 13 use any information on the space X, i.e. the assertions and the corresponding proofs remain valid in case X is only a quasi-Banach space.

4.4 Relations in case of Quasi-Banach spaces

The first relation is based on the p-norm property of all the variants of the p-nuclear quasi-norm.

Lemma 14. Let $0 < p \leq 1$, and let X and Y be two quasi-Banach spaces. If α is some p-norm satisfying

$$
\alpha(x \otimes y) \leq \|x|X\| \cdot \|y|Y\| \quad \text{for all} \quad x \in X, y \in Y,
$$

then $\alpha(z) \leq \gamma_p(z, X, Y)$ for all $z \in X \otimes Y$.

Proof. The proof follows along the lines of the one for the projective tensor norm $\gamma_1 \equiv \gamma$. Let $z = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, then

$$
\alpha(z)^p \leq \sum_{i=1}^n \alpha(x_i \otimes y_i)^p \leq \sum_{i=1}^n \|x_i|X\|^p \cdot \|y_i|Y\|^p.
$$

Taking the infimum over all representations of z yields the desired estimate. $\qquad \square$

In particular, we find $\alpha_p \leq \gamma_p$ and $\beta_p \leq \gamma_p$ for all $0 < p \leq 1$.

Lemma 15. Let X be an arbitrary quasi-Banach space, and let Y be a p-Banach space, $0 < p \leq 1$. Then it holds $\beta_p(\cdot, X, Y) = \gamma_p(\cdot, X, Y)$.

Proof. Since we have $\|e^j|\ell_p^n\| = 1$, $j = 1, \ldots, n$, for the canonical basis vectors, it follows

$$
\|y_j|Y\| \leq \sup_{\|\lambda|\ell_p^n\| \leq 1} \left\| \sum_{i=1}^n \lambda_i y_i \, \Big| Y \right\|, \qquad j = 1, \ldots, n,
$$

for all $y_1, \ldots, y_n \in Y$. This immediately yields

$$\left(\sum_{j=1}^n \|x_j|X\|^p \cdot \|y_j|Y\|^p \right)^{1/p} \leq \left(\sum_{j=1}^n \|x_j|X\|^p \right)^{1/p} \sup_{\|\lambda|\ell_p^n\| \leq 1} \left\| \sum_{j=1}^n \lambda_j y_j \,\Big|\, Y \right\|$$

for all $x_1, \ldots, x_n \in X$ and $y_1, \ldots, y_n \in Y$. Taking the infimum over all representations of $z \in X \otimes Y$ we obtain $\beta_p(z, X, Y) \geq \gamma_p(z, X, Y)$ for all $z \in X \otimes Y$, which in view of the remark above completes the proof. $\qquad\square$

Remark 7. If Y is a p-Banach space, one can verify

$$\max_{j=1,\ldots,n} \|y_j|Y\| = \sup_{\|\lambda|\ell_p^n\| \leq 1} \left\| \sum_{i=1}^n \lambda_i y_i \,\Big|\, Y \right\|.$$

To see this, one uses the same estimate as in the proof above, and subsequently one applies the p-triangle inequality.

Since the definition of α_p involves duality more directly, its comparisons to the other quasi-norms are more subtle.

Lemma 16. Let X and Y be quasi-Banach spaces, and let $0 < p \leq 1$. Then it holds

$$\alpha_p(f, X, Y) \leq \beta_p(f, X, Y), \qquad f \in X \otimes Y\,.$$

Proof. The observation $\||y|Y\|| \leq \|y|Y\|$ yields for arbitrary $y_1, \ldots, y_n \in Y$

$$\sup_{\|\lambda|\ell_p^n\| \leq 1} \left\| \sum_{i=1}^n \lambda_i y_i \,\Big|\, Y \right\| \geq \sup_{\|\lambda|\ell_p^n\| \leq 1} \sup_{\|\psi|Y'\| \leq 1} \left| \psi\left(\sum_{i=1}^n \lambda_i y_i \right) \right| = \sup_{\|\psi|Y'\| \leq 1} \sup_{\|\lambda|\ell_p^n\| \leq 1} \left| \sum_{i=1}^n \lambda_i \psi(y_i) \right|.$$

Now we choose an index $j \in \{1, \ldots, n\}$, such that $|\psi(y_j)| = \max_{i=1,\ldots,n} |\psi(y_i)|$. We further choose $\mu = e^j \in \ell_p^n$, the jth canonical unit vector. Then we have $\|\mu|\ell_p^n\| = 1$ and hence

$$\sup_{\|\psi|Y'\| \leq 1} \sup_{\|\lambda|\ell_p^n\| \leq 1} \left| \sum_{i=1}^n \lambda_i \psi(y_i) \right| \geq \sup_{\|\psi|Y'\| \leq 1} \left| \sum_{i=1}^n \mu_i \psi(y_i) \right| = \sup_{\|\psi|Y'\| \leq 1} \max_{i=1,\ldots,n} |\psi(y_i)|.$$

Inserting this into the respective definitions finally yields the assertion. $\qquad\square$

Corollary 3. Let $0 < p \leq 1$. Then β_p is reasonable and uniform.

Proof. The assumption (6) of Lemma 5 now follows immediately from the one for α_p, and the inequality $\beta_p(x \otimes y, X, Y) \leq \|x|X\| \cdot \|y|Y\|$ is obvious.
Furthermore, by linearity we always have $T_2(\sum_{i=1}^n \lambda_i y_i) = \sum_{i=1}^n \lambda_i T_2 y_i$, hence the estimate (3) follows directly by applying the definition of β_p to the representation $Tz = \sum_{i=1}^n (T_1 x_i) \otimes (T_2 y_i)$. $\qquad\square$

In order to establish $\alpha_p = \beta_p = \gamma_p$ it now would be sufficient to show $\alpha_p \geq \gamma_p$. However, even for the prominent (and particularly simple) example $X = Y = \ell_p$ this fails to be true. In [19] it was shown $\ell_p(I) \otimes_{\gamma_p} \ell_p(I) = \ell_p(I^2)$, $0 < p \leq 1$, which holds with equality of quasi-norms. In particular, this implies $\gamma_p(x \otimes y) = \|x|\ell_p(I)\| \cdot \| y|\ell_p(I)\|$ for all $x, y \in \ell_p(I)$. Since $\left(\ell_p(I)\right)^E = \ell_1(I)$, we have $\alpha_p(x \otimes y) = \|x|\ell_p(I)\| \cdot \|y|\ell_1(I)\|$ (see Lemma 17 below), which yields $\alpha_p(x \otimes y) < \gamma_p(x \otimes y)$ already for all those dyads, where y is different from multiples of the unit vectors e^i, $i \in I$.
Hence, in general the quasi-norms α_p and γ_p are different, and even non-equivalent. More precisely, they are equivalent if, and only if, Y is normable, see Lemma 3.

4.5 Crossnorm-properties

In this section we shall investigate the p-nuclear tensor norms from Definition 4 regarding possible crossnorm-properties. To begin with, we observe that α_p generally is neither a crossnorm, nor equivalent to one. Instead, we have the following assertion.

Lemma 17. Let X and Y be quasi-Banach spaces, and let $0 < p \leq 1$. Then it holds

$$\alpha_p(z, X, Y) \geq \lambda_B(z, X, Y), \qquad z \in X \otimes Y,$$

and furthermore

$$\alpha_p(x \otimes y, X, Y) = \|x|X\| \cdot \|\|y|Y\|\|, \qquad x \in X, y \in Y. \tag{17}$$

Proof. The first assertion can already be found in the proof of Lemma 9, where we have shown

$$\alpha_p(z, X, Y) \geq \big\|B_z\big|\mathcal{L}(Y', X)\big\| = \lambda_B(z, X, Y).$$

Applying this to $z = x \otimes y$, we find

$$\alpha_p(x \otimes y, X, Y) \geq \big\|B_{x \otimes y}\big|\mathcal{L}(Y', X)\big\|$$
$$= \sup_{\|\psi|Y'\| \leq 1} \|\psi(y)x|X\| = \|x|X\| \sup_{\|\psi|Y'\| \leq 1} |\psi(y)| = \|x|X\| \cdot \|\|y|Y\|\|.$$

Since the reverse estimate simply follows by inserting $x \otimes y$ into the definition of α_p, the proof is complete. $\qquad \square$

From (17) we immediately obtain results for $X \otimes_{\alpha_p} Y$ similarly as in Lemma 4.

Lemma 18. Let X be a p-Banach space, $0 < p \leq 1$, and Y a further quasi-Banach space, both with separating duals. Then it holds

$$X \otimes_{\alpha_p} Y = X \otimes_{\alpha_p} Y^E = X \otimes_{\beta_p} Y^E = X \otimes_{\gamma_p} Y^E.$$

The latter identities follow from the observation $\alpha_p(\cdot, F, G) = \beta_p(\cdot, F, G) = \gamma_p(\cdot, F, G)$, whenever G is a Banach space (see Remark 6).

For γ_p we obtain no final result for general spaces X and Y.

Lemma 19. Let X and Y be p-Banach spaces, where $0 < p \leq 1$. Then it holds

$$\gamma_p(z, X, Y) \geq \max\Big(\lambda_A(z, X, Y), \lambda_B(z, X, Y)\Big), \qquad z \in X \otimes Y.$$

In particular, we find

$$\max\Big(\|x|X\| \cdot \|\|y|Y\|\|, \|\|x|X\|\| \cdot \|y|Y\|\Big) \leq \gamma_p(x \otimes y, X, Y) \leq \|x|X\| \cdot \|y|Y\| \tag{18}$$

for all $x \in X$ and $y \in Y$.

Proof. In the proof of Lemma 8 we have shown

$$\left\| A_z \phi \,\middle|\, Y \right\| \le \gamma_p(z, X, Y)\,, \qquad z \in X \otimes Y\,, \phi \in X'\,.$$

This yields $\gamma_p(z, X, Y) \ge \lambda_A(z, X, Y)$ for all $z \in X \otimes Y$. The proof of $\gamma_p(z, X, Y) \ge \lambda_B(z, X, Y)$ follows by analogous arguments. This yields the first assertion. Applying this result to $z = x \otimes y$ one immediately obtains the second one. $\qquad\square$

Also the estimate (18) can be reformulated in terms of Banach envelopes.

Corollary 4. Let $0 < p \le 1$, and let X and Y be p-Banach spaces with separating duals. Then it holds

$$X \otimes_{\gamma_p} Y \hookrightarrow X^E \otimes_{\gamma_p} Y \;\cap\; X \otimes_{\gamma_p} Y^E \hookrightarrow X^E \otimes_{\gamma_p} Y^E\,.$$

Remark 8. Without any additional assumption on the quasi-norms of X and Y we still get

$$\left\| x \,|\, X \right\| \cdot \left\| y \,|\, Y \right\| \le \gamma_p(x \otimes y, X, Y) \le \|x|X\| \cdot \|y|Y\|\,,$$

based on the reasonability of γ_p (see Lemmas 8 and 17). This corresponds to the trivial embedding $X \otimes_{\gamma_p} Y \hookrightarrow X^E \otimes_{\gamma_p} Y^E$.

However, sharper estimates for $\gamma_p(x \otimes y, X, Y)$ or $\beta_p(x \otimes y, X, Y)$ hold true, at the cost of stronger assumptions on the space X. Then we can prove that β_p or γ_p are (almost) crossnorms.

Theorem 3. Let X be a quasi-Banach space with separating dual, and let Y be another quasi-Banach space. Assume that for arbitrary $x, x_1, \ldots, x_n \in X$ and $y, y_1, \ldots, y_n \in Y$, such that $x \otimes y \ne 0$ and

$$x \otimes y = \sum_{i=1}^n x^i \otimes y^i\,,$$

the following condition is fulfilled:

$$\sup_{\varphi \in X^* \setminus \{0\}} \frac{\sum_{i=1}^n \|x_i|X\|^p}{\sum_{i=1}^n |\varphi(x_i)|^p} \cdot \frac{|\varphi(x)|^p}{\|x|X\|^p} \ge C_1 > 0\,, \tag{19}$$

where X^* denotes the *algebraic* dual X^* of X, and C_1 does not depend on $x, x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. Then it holds

$$\beta_p(x \otimes y, X, Y) \ge C_1^{1/p} \|x|X\| \cdot \|y|Y\| \qquad \text{for all } x \in X \text{ and } y \in Y.$$

Note that the restriction on $x, x_1, \ldots, x_n \in X$ implies that $\sum_{i=1}^n |\varphi(x_i)|^p$ is non-vanishing for all functionals $\varphi \in X^* \setminus \{0\}$.
Clearly, we always have $C_1 \le 1$ (take $n = 1$ and $x = x_1$). The proof is based on an argument by Nitsche [16]. We further remark that for Banach spaces and ℓ_p-spaces we have $C_1 = 1$ (even upon replacing X^* by X').

Proof. Let $x, x_1, \ldots, x_n \in X$ and $y, y_1, \ldots, y_n \in Y$ be given as required. Moreover, consider an arbitrary functional $\varphi \in X^* \setminus \{0\}$. We put $\eta^i = \varphi(x^i)/M$. Then it follows

$$
\left(\sum_{i=1}^n \|x^i|X\|^p \right)^{1/p} \sup_{\|\mu|\ell_p^n\| \leq 1} \left\| \sum_{i=1}^n \mu_i y^i \middle| Y \right\| \geq \left(\sum_{i=1}^n \|x^i|X\|^p \right)^{1/p} \left\| \sum_{i=1}^n \eta_i y^i \middle| Y \right\|
$$

$$
= \frac{1}{M} \left(\sum_{i=1}^n \|x^i|X\|^p \right)^{1/p} \left\| \sum_{i=1}^n \varphi(x^i) y^i \middle| Y \right\|
$$

$$
= \frac{1}{M} \left(\sum_{i=1}^n \|x^i|X\|^p \right)^{1/p} \|\varphi(x)y|Y\|
$$

$$
= \left(\frac{\left(\sum_{i=1}^n \|x^i|X\|^p \right)^{1/p}}{\left(\sum_{i=1}^n |\varphi(x^i)|^p \right)^{1/p}} \cdot \frac{|\varphi(x)|}{\|x|X\|} \right) \|x|X\| \cdot \|y|Y\|.
$$

Taking the supremum over all such functionals and afterwards taking the infimum over all equivalent representations of $x \otimes y$ yields the claimed result. $\qquad\square$

To finish this subsection we adapt another result of Nitsche, giving an example of spaces satisfying condition (19). We suppose

- X is a p-Banach space,

- X contains a Schauder basis $B = (b_i)_{i \in I}$, where I is a countable index set,

- the mapping $J : f \longmapsto \left(\lambda_i(f) \|b_i|X\| \right)_{i \in I}$ is bounded from X to $\ell_p(I)$.

Theorem 4. Under these assumptions it holds

$$
\beta_p(x \otimes y, X, Y) \geq C_0 \|x|X\| \cdot \|y|Y\| \qquad \text{for all } x \in X \text{ and } y \in Y,
$$

where the constant C_0 is given by $C_0 = \|J|\mathcal{L}(X, \ell_p(I))\|^{-1}$.

Proof. We shall show that every space X with a basis as assumed satisfies the condition (19) with $C_1 = C_0^p$. Throughout the proof $(\lambda_i)_{i \in I} \subset X'$ denotes the system of the correponding coefficient functionals. Now let $x, x_1, \ldots, x_n \in X$, be given as in the last theorem, which particularly implies $x \neq 0$ and

$$
M_i = \frac{\left(\sum_{j=1}^n |\lambda_i(x^j)|^p \right)^{1/p}}{\left(\sum_{j=1}^n \|x^j|X\|^p \right)^{1/p}} > 0 \qquad \text{for all } i \in I.
$$

The assumption on the mapping J then yields

$$
\sum_{i \in I} M_i^p \|b_i|X\|^p \leq \|J\|^p.
$$

Now consider the set

$$
\Phi := \left\{ \varphi_i = M_i^{-1} \operatorname{sgn}(\lambda_i(x)) \lambda_i : i \in I \right\} \subset X'.
$$

By construction we find for all $\varphi \in \Phi$

$$\sum_{j=1}^{n} |\varphi(x^j)|^p = \sum_{j=1}^{n} \|x^j|X\|^p \,.$$

Now assume that there is no $\varphi \in \Phi$, such that $\varphi(x) \geq C_0\|x|X\|$ (observe that $\varphi(x)$ is real for all $\varphi \in \Phi$). This implies $\varphi_i(x) < C_0\|x|X\|$ for all $i \in I$ and hence (summing up over $i \in I$)

$$\sum_{i \in I} M_i^p \|b_i|X\|^p = \sum_{i \in I} \left(\frac{|\lambda_i(x)|}{\varphi_i(x)} \right)^p \|b_i|X\|^p > \sum_{i \in I} \frac{|\lambda_i(x)|^p}{C_0^p \|x|X\|^p} \|b_i|X\|^p \,.$$

Moreover we find from the p-triangle inequality

$$\|x|X\|^p = \left\| \sum_{i \in I} \lambda_i(x) b_i \Big| X \right\|^p \leq \sum_{i \in I} \|\lambda_i(x) b_i|X\|^p = \sum_{i \in I} |\lambda_i(x)|^p \|b_i|X\|^p \,.$$

Altogether we find

$$\|J\|^p \geq \sum_{i \in I} M_i^p \|b_i|X\|^p > C_0^{-p} \,.$$

Hence in case $C_0 = \|J|\mathcal{L}(X, \ell_p(I))\|^{-1}$ this is a contradiction, thus a functional $\varphi \in \Phi$ with $\varphi(x) \geq C_0\|x|X\|$ does exist. This functional then yields condition (19) with $C_1 = C_0^p$, what proves the claim. $\qquad \square$

5 Applications

As a first application of the results of the last section we will have a closer look at ℓ_p-type sequence spaces. In particular, we consider weighted spaces $\ell_p(w, I)$, defined via the norm

$$\|a|\ell_p(w, I)\| := \left(\sum_{i \in I} |a_i|^p w_i \right)^{1/p}, \qquad 0 < p < \infty \,,$$

where $w = (w_i)_{i \in I}$ is an arbitrary sequence of positive real numbers.

Theorem 5. Let $X = \ell_p(w^1, I)$, $Y = \ell_p(w^2, J)$, $0 < p \leq 1$, where I and J are arbitrary countable index sets. Then condition (19) holds with $C_1 = 1$, and the canonical basis satisfies all conditions in Theorem 4 with $C_0 = 1$. In particular, $\beta_p(\cdot, \ell_p(w^1, I), \ell_p(w^2, J)) = \gamma_p(\cdot, \ell_p(w^1, I), \ell_p(w^2, J))$ is a crossnorm.

On the one hand this theorem extends the main results of Nitsche to the case of weighted sequence spaces, on the other hand this corroborates the findings in [19], where it was proven

$$\ell_p(w^1, I) \otimes_{\delta_p} \ell_p(w^2, J) = \ell_p(w^1 \otimes w^2, I \times J), \qquad 0 < p < \infty \,,$$

with equality of quasi-norms, which implies the crossnorm-property for γ_p. Here $\delta_p = \gamma_p$ if $0 < p \leq 1$ and $\delta_p = \alpha_p$ if $1 < p < \infty$.

One particular application for these tensor product techniques is given by wavelet transformations with respect to tensor product wavelet systems. More precisely, if X and Y are spaces of tempered distributions with wavelet bases $\Phi = \{\phi_i\}_{i \in I}$ and $\Psi = \{\psi_j\}_{j \in J}$, respectively, and if $T_X : X \longrightarrow \mathcal{X}$ and $T_Y : Y \longrightarrow \mathcal{Y}$ are isomorphisms mapping the distribution spaces onto associated sequence spaces, then the system $\Phi \otimes \Psi = \{\phi_i \otimes \psi_j\}_{(i,j) \in I \times J}$ is a basis of $X \otimes_{\gamma_p} Y$ (which itself is a space of tempered distributions as well), and $T_X \otimes_{\gamma_p} T_Y$ maps $X \otimes_{\gamma_p} Y$ onto the sequence space $\mathcal{X} \otimes_{\gamma_p} \mathcal{Y}$. By Corollary 2 this mapping is again an isomorphism.

In this way, the knowledge about tensor products of sequence spaces can immediately be transferred to function spaces. Particularly, this applies to Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ and Sobolev spaces $H_p^s(\mathbb{R}^n)$ and variants thereof, which can be characterized (at least for a certain range of parameters) using Meyer or Daubechies wavelets (to name only some possible bases). For details regarding to the wavelet characterization we refer to the literature, see e.g. [15, 11, 21].

Further applications of these facts are due to the following results, which provide a characterization of tensor products of some prominent function spaces. To this purpose, we define the Sobolev space $W_p^m(\mathbb{R}^n)$, $m \in \mathbb{N}$, $1 < p < \infty$, as the collection of all functions $f \in L_p(\mathbb{R}^n)$, such that

$$\big\|f\big|W_p^m(\mathbb{R}^n)\big\| := \sum_{|\alpha| \leq m} \big\|D^\alpha f\big|L_p(\mathbb{R}^n)\big\|$$

is finite. Here we used the usual multiindex notation, and $D^\alpha f$ denotes the weak derivative of f of order α. These spaces can be generalized using Fourier analytic means. Defining spaces $H_p^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$, as the collection of all tempered distributions satisfying

$$\big\|f\big|H_p^s(\mathbb{R}^n)\big\| := \big\|\mathcal{F}^{-1}(1 + |\xi|^2)^{s/2}\mathcal{F}f\big|L_p(\mathbb{R}^n)\big\| < \infty \,,$$

it holds $W_p^m(\mathbb{R}^n) = H_p^m(\mathbb{R}^n)$, $m \in \mathbb{N}$, $1 < p < \infty$, in the sense of equivalent norms.

For tensor products of these spaces it has been shown (see [19, 8])

$$H_p^{s_1}(\mathbb{R}^n) \otimes_{\alpha_p} H_p^{s_2}(\mathbb{R}^m) = S_p^{(s_1, s_2)}H(\mathbb{R}^n \times \mathbb{R}^m) \,, \qquad 1 < p < \infty \,, \ s_1, s_2 \in \mathbb{R},$$

which generalizes the well-known Hilbert space identity $H^s(\mathbb{R}) \otimes H^s(\mathbb{R}) = H_{\text{mix}}^s(\mathbb{R}^2)$. Moreover, this can be extended to more than two factors in the sense of iterated tensor products,

$$
\begin{aligned}
H_p^{s_1}(\mathbb{R}^{d_1}) \otimes_{\alpha_p} \cdots \otimes_{\alpha_p} H_p^{s_N}(\mathbb{R}^{d_N}) &= H_p^{s_1}(\mathbb{R}^{d_1}) \otimes_{\alpha_p} \left(H_p^{s_2}(\mathbb{R}^{d_2}) \otimes_{\alpha_p} \cdots \otimes_{\alpha_p} H_p^{s_N}(\mathbb{R}^{d_N}) \right) \\
&= S_p^{(s_1, \ldots, s_N)}H(\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_N}) \,, \qquad 1 < p < \infty \,, \ s_1, \ldots, s_N \in \mathbb{R},
\end{aligned}
$$

and it has been shown, that the outcome is independent of the order of iteration. The spaces on the right hand side of these identities are called (fractional) Sobolev spaces of dominating mixed smoothness. These are again defined Fourier analytically via the norm

$$\big\|f\big|S_p^{(s_1, s_2)}H(\mathbb{R}^n \times \mathbb{R}^m)\big\| := \big\|\mathcal{F}^{-1}(1 + |\xi|^2)^{s_1/2}(1 + |\eta|^2)^{s_2/2}\mathcal{F}f\big|L_p(\mathbb{R}^{n+m})\big\| < \infty \,, \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Particular cases of this scale of spaces can once more be described using weak derivatives of functions from L_p. It holds $S_p^{(k_1, k_2)}H(\mathbb{R}^n \times \mathbb{R}^m) = S_p^{(k_1, k_2)}W(\mathbb{R}^n \times \mathbb{R}^m)$, $(k_1, k_2) \in \mathbb{N}^2$, where

$$\big\|f\big|S_p^{(k_1, k_2)}W(\mathbb{R}^n \times \mathbb{R}^m)\big\| := \sum_{|\alpha| \leq k_1} \sum_{|\beta| \leq k_2} \big\|D_x^\alpha D_y^\beta f(x, y)\big|L_p(\mathbb{R}^{n+m})\big\| \,.$$

In particular, this yields the identity

$$W_p^{k_1}(\mathbb{R}^n) \otimes_{\alpha_p} W_p^{k_2}(\mathbb{R}^m) = S_p^{(k_1, k_2)}W(\mathbb{R}^n \times \mathbb{R}^m) \,, \qquad 1 < p < \infty \,, \ k_1, k_2 \in \mathbb{N},$$

which holds with equivalence of norms. For details we refer to the cited literature. Moreover, counterparts of these results are known for spaces on $[0,1]$ and $[0,1]^2$, respectively, see [20].

As a last example we shall discuss Besov spaces $B_{p,p}^s(\mathbb{R}^n)$. Instead of giving their definition we state that they can be characterized using Meyer wavelets or Daubechies wavelets of sufficiently high order (to name only some possible choices). If Ψ is such a wavelet system (in particular we assume it to be an orthonormal basis of $L_2(\mathbb{R}^n)$), then the corresponding wavelet transformation $\mathcal{J}$, i.e. the mapping associating to a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the sequence of its wavelet coefficients $(\lambda_{j,\nu}(f))_{j\in\mathbb{N}_0,\nu\in\nabla}$, is an isomorphism mapping $B_{p,p}^s(\mathbb{R}^n)$ onto the sequence space $b_{p,p}^s$, defined by

$$\big\| a \big| b_{p,p}^s \big\| := \left(\sum_{j=0}^{\infty} 2^{j(s+d/2-d/p)p} \sum_{\nu\in\nabla} |a_{j,\nu}|^p \right)^{1/p}, \qquad s \in \mathbb{R}, \ 0 < p < \infty .$$

We note that these sequence spaces fit in the framework of the weighted ℓ_p-spaces discussed above. In this way we can identify their tensor product $b_{p,p}^{s_1} \otimes_{\delta_p} b_{p,p}^{s_2}$ with the space $s_{p,p}^{(s_1,s_2)} b$, which is given for parameters $s_1, s_2 \in \mathbb{R}$ and $0 < p < \infty$ by the norm

$$\big\| a \big| s_{p,p}^{(s_1,s_2)} b \big\| := \left(\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} 2^{j_1(s_1+d_1/2-d_1/p)p} 2^{j_2(s_2+d_2/2-d_2/p)p} \sum_{\nu\in\widetilde{\nabla}} |a_{(j_1,j_2),\nu}|^p \right)^{1/p} .$$

On the other hand this kind of sequence space is known to arise in characterizations of function spaces of dominating mixed smoothness using tensor product wavelets. In particular, let Ψ_1 be a wavelet basis for $B_{p,p}^{s_1}(\mathbb{R}^{d_1})$ as described above, and let Ψ_2 be a wavelet basis of $B_{p,p}^{s_2}(\mathbb{R}^{d_2})$. Then the tensor product basis $\Psi_1 \otimes \Psi_2$, consisting of all pairwise tensor products of basis vectors, turns out to be a basis for the Besov space of dominating mixed smoothness $S_{p,p}^{(s_1,s_2)} B(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$, and the mapping $\mathcal{I}$ associating to a distribution $f \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$ the sequence of its wavelet coefficients with respect to $\Psi_1 \otimes \Psi_2$ is an isomorphism from $S_{p,p}^{(s_1,s_2)} B(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$ onto $s_{p,p}^{(s_1,s_2)} b$. For further details we refer to [24, 8].

The outcome of these considerations is the identity

$$B_{p,p}^{s_1}(\mathbb{R}^{d_1}) \otimes_{\delta_p} B_{p,p}^{s_2}(\mathbb{R}^{d_2}) = S_{p,p}^{(s_1,s_2)} B(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}), \qquad 0 < p < \infty, \ s_1, s_2 \in \mathbb{R},$$

which now is an immediate consequence of Corollary 2, see also [19]. Moreover, we find for the wavelet isomorphisms the following result.

Theorem 6. Let Ψ be a wavelet basis for $B_{p,p}^s(\mathbb{R})$, where $s \in \mathbb{R}$ and $0 < p < \infty$. Denote by J the corresponding isomorphism mapping $B_{p,p}^s(\mathbb{R})$ onto the associated sequence space $b_{p,p}^s$. Moreover, let Ψ^d be the d-fold tensor product of Ψ, and denote by J^d the corresponding isomorphism mapping $S_{p,p}^{(s,\dots,s)} B(\mathbb{R}^d)$ onto $s_{p,p}^{(s,\dots,s)} b$

- Let $0 < p < \infty$. Then it holds $J^d = J \otimes_{\delta_p} \cdots \otimes_{\delta_p} J$, and

$$\big\| J^d : S_{p,p}^{(s,\dots,s)} B(\mathbb{R}^d) \longrightarrow s_{p,p}^{(s,\dots,s)} b \big\| = \big\| J : B_{p,p}^s(\mathbb{R}) \longrightarrow b_{p,p}^s \big\|^d .$$

- Let $1 \leq p < \infty$. Then it holds $(J^d)^{-1} = J^{-1} \otimes_{\alpha_p} \cdots \otimes_{\alpha_p} J^{-1}$, and

$$\big\| (J^d)^{-1} : s_{p,p}^{(s,\dots,s)} b \longrightarrow S_{p,p}^{(s,\dots,s)} B(\mathbb{R}^d) \big\| = \big\| J^{-1} : b_{p,p}^s \longrightarrow B_{p,p}^s(\mathbb{R}) \big\|^d .$$

This means whenever the quasi-norm of the "one-dimensional" mappings is greater than one, the quasi-norm of the tensor mapping grows exponentially in the number of factors (variables), and hence suffers from the curse of dimension. That observation has some immediate consequences for estimates of certain approximation quantities like approximation, Kolmogorov and entropy numbers or m-term approximation. More precisely, one is interested in their asymptotic behaviour. To this purpose one transfers these problems from function spaces to the corresponding sequence spaces, and afterwards solves the task for the sequence spaces. Afterwards, the results are transferred back to function spaces. The key point in this discretization technique is that the asymptotic order of the estimates is preserved. However, for practical implementation also the constants involved in these estimates became increasingly relevant, in particular their dependence on the dimension of the underlying domains. Here the norm of the above isomorphism comes into play, as in the resulting estimates for the function spaces the constants are the product of the constants for the sequence spaces and the norm of the isomorphism.

In other words: Even if the dependence on the dimension on sequence space level is moderate (i.e. polynomial dependence or even independent of the dimension), this positive behaviour gets lost on function space level.

The proof of Theorem 6 follows immediately from Proposition 3, keeping in mind that for $p \geq 1$ the spaces $B_{p,p}^s(\mathbb{R})$ and $b_{p,p}^s$ are Banach spaces and α_p is a uniform crossnorm, and for $p \leq 1$ we still know that γ_p is uniform and a crossnorm on $b_{p,p}^s \otimes b_{p,p}^s$.

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