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Regularity and generalized polynomial chaos approximation of parametric and random 2nd order hyperbolic partial differential equations*

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Regularity and Generalized Polynomial Chaos Approximation of Parametric and Random 2nd Order Hyperbolic Partial Differential Equations *

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Abstract

Initial boundary value problems of linear second order hyperbolic partial differential equations whose coefficients depend on countably many random parameters are reduced to a parametric family of deterministic initial boundary value problems on an infinite dimensional parameter space. This parametric family is approximated by Galerkin projection onto finitely supported polynomial systems in the parameter space. We establish uniform stability with respect to the support of the resulting coupled hyperbolic systems, and provide sufficient smoothness and compatibility conditions on the data for the solution to exhibit analytic respectively Gevrey regularity with respect to the countably many parameters. Sufficient conditions for the p-summability of the generalized polynomial chaos expansion of the parametric solution in terms of the countably many input parameters are obtained and rates of convergence of best N-term polynomial chaos type approximations of the parametric solution are given. In addition, regularity both in space and time for the parametric family of solutions is proved for data satisfying certain compatibility conditions. The results allow obtaining convergence rates and stability of sparse space-time tensor product Galerkin discretizations in the parameter space.

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The original report, titled "Analytic Regularity and Generalized Polynomial Chaos Approximation of Parametric and Random 2nd Order Hyperbolic Partial Differential Equations", contained a mathematical error and is superseded by the current one.

Key Words: Wave Equation, generalized polynomial chaos, random media, best N-term approximation
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1 Introduction

The linear wave equation with random data arises in numerous problems in applied mathematics and scientific computing. We mention only seismic imaging and nondestructive testing (see, e.g., the seminal [7] and the recent papers [2, 1] and the references there). In these applications, particular interest is on the wave equation with random coefficients. Attention in the above references has been on asymptotic analysis techniques for the wave propagation problem in random media. This point of view has mandated, in particular, strong assumptions on the randomness such as stationarity, homogeneity and the like.

In the present paper, we present a representation theorem and regularity results for the solution of linear wave equations with a class of random coefficients which need neither be stationary nor homogeneous in physical space. We show that the law of the random solution can be represented as a deterministic

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function of a countable number of coordinates. For a class of equations with regular right hand side and compatible initial conditions, we also show that this solution is, as a function of the coordinates, *smooth* as a mapping from the parameter domain into suitable Sobolev spaces in which the deterministic wave equation is well-posed. We investigate the smoothness of the parametric solution in terms of analytic respectively Gevrey regularity.

We also show that the solution admits mean square convergent (with respect to the probability measure of the input data) polynomial chaos expansions on the infinite dimensional parameter space. We establish bounds on the size of the polynomial expansion coefficients of the parametric solution and establish, in particular, *sparsity of these polynomial chaos expansions* of the random solution in terms of the input data's fluctuation decay. Our analysis also applies to input data depending on finitely many parameters as well as to data with only few or finitely many compatibility conditions, in which case the parametric solution exhibits only finite regularity.

1.1 Stochastic wave equation

For $0 < T < \infty$, we consider in $I = (0, T)$ the following class of linear, second order hyperbolic equations with random coefficients: let D be a bounded Lipschitz domain in $\mathbb{R}^d$. We define the space-time cylinder $Q_T = I \times D$. In Q_T, we consider the stochastic wave equation

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (a(x,\omega)\nabla u) = g(t,x),\ \ u|_{t=0} = g_1,\ \ u_t|_{t=0} = g_2. \tag{1.1}$$

By H we denote the space $L^2(D)$ and by V a subspace of $H^1(D)$ with the appropriate boundary condition such that $H^1_0(D) \subseteq V \subseteq H^1(D)$. The solution $u(t,\cdot) \in V$ for all $t \in I$. We assume that the coefficient $a(x,\omega)$ is a random field on a probability space (Ω, Σ, P) over $L^\infty(D)$. The forcing g and initial data g_1 and g_2 are assumed to be deterministic.[1] To ensure well-posedness of (1.1), we require:

Assumption 1.1 *There are constants* $0 < a_{\min} \le a_{\max} < \infty$ *so that*

$$\forall\, \omega \in \Omega: \quad 0 < a_{\min} \le \operatorname{essinf}\{a(x,\omega) : x \in D\} \le \| a(\cdot,\omega)\|_{L^\infty(D)} \le a_{\max}\ .$$

To state the weak form of the initial boundary value problem (1.1), we require

$$g \in L^2(I;H), \quad g_1 \in V, \quad g_2 \in H\ . \tag{1.2}$$

Further, we introduce the Bochner spaces

$$\mathcal{X} = L^2(I;V) \cap H^1(I;H) \cap H^2(I;V'), \qquad \mathcal{Y} = L^2(I;V) \times V \times H\ . \tag{1.3}$$

A *weak solution of the hyperbolic initial boundary value problem* (1.1) is any function $u \in \mathcal{X}$ such that

$$\begin{aligned}\int_I \langle \frac{d^2u}{dt^2}(t,\cdot), v_0(t,\cdot)\rangle_H dt + \int_I\int_D a(x,\omega)\nabla u(t,x,\omega)\cdot\nabla v_0(t,x)dxdt + \langle u(0), v_1\rangle_V + \langle u_t(0), v_2\rangle_H \\ = \int_I\int_D g(t,x)v_0(t,x)dxdt + \langle g_1, v_1\rangle_V + \langle g_2, v_2\rangle_H,\ \ \forall v = (v_0,v_1,v_2) \in \mathcal{Y}.\end{aligned} \tag{1.4}$$

Proposition 1.2 *Under Assumption 1.1 and condition* (1.2)*, for every* $\omega \in \Omega$*, the problem (1.4) admits a unique weak solution* $u \in \mathcal{X}$*. The following estimate holds*

$$\|u\|_{\mathcal{X}} \le C(\|g\|_{L^2(I;H)} + \|g_1\|_V + \|g_2\|_H), \tag{1.5}$$

where the constant C *is independent of the coefficient realizations* $\Omega \ni \omega \to a(\cdot,\omega)$*.*

This proposition is a special case of Theorem 29.1 of Wloka ([8]). Inspecting the proof in that reference, it can be inferred that the constant C in the bound (1.5) depends only on T and on $a_{\min}$ and $a_{\max}$ in Assumption 1.1.

[1] We could also assume randomness in these quantities and obtain completely analogous results without any additional mathematical difficulties, merely at the expense of more involved notation.

With a view towards numerical analysis of approximation schemes, we impose further structural assumptions on the coefficient a in (1.1). Specifically, we shall assume throughout this work that the random coefficient a in (1.1) can be characterized by a sequence of infinitely many, scalar random variables $y_j : \Omega \to [-1, 1]$: a is given in the generic form

$$a(x,\omega) = \bar{a}(x) + \sum_{j\ge 1} y_j(\omega)\psi_j(x), \tag{1.6}$$

where ψ_j belong to $L^\infty(D)$. The generic representation (1.6) is highly ambiguous, as y_j and ψ_j could be rescaled. As in our previous work in corresponding elliptic and parabolic problems [5, 4, 6], we require that the coefficient sequence $\{\psi_j\}$ satisfies the following assumption

Assumption 1.3 *The functions $\bar{a}(x)$ and ψ_j in the parametric representation* (1.6) *of the random coefficient $a(x, \omega)$ satisfy*

$$\sum_{j\ge 1} \|\psi_j\|_{L^\infty(D)} \le \frac{\kappa}{1+\kappa}\bar{a}_{\min}$$

with $\bar{a}_{\min} = \mathrm{essinf}_{x\in D}\bar{a}(x) > 0$ and some $\kappa > 0$.

Assumption 1.3 implies in particular that Assumption 1.1 is satisfied by choosing

$$a_{\min} := \bar{a}_{\min} - \frac{\kappa}{1+\kappa}\bar{a}_{\min} = \frac{1}{1+\kappa}\bar{a}_{\min}. \tag{1.7}$$

1.2 Probability Spaces

Under the structural assumption (1.6) on the random coefficient, the law of the random solution u of (1.1) takes the form of a parametric deterministic function of (in general countably many components of) $y \in U$ where $U = [-1, 1]^{\mathbb{N}}$. The variational problem can be cast in the form of a parametric family of deterministic problems for y. In the next sections, we study sparse tensor discretizations of a variational problem for u as a deterministic function of all parameters (t, x, y) in $I \times D \times U$. In order to clarify the relation of the deterministic approximations of $u(t, x, y)$ with the random solution of (1.1), we define probability measures on the parameter domain U. To this end, we introduce the σ-algebra on U by $\Theta = \bigotimes_{j\ge 1} \mathcal{B}^1([-1, 1])$ where $\mathcal{B}^1([-1, 1])$ denotes the Borel σ-algebra on the interval $[-1, 1]$. On the measure space (U, Θ), we define the product measure

$$\rho(dy) := \bigotimes_{j\ge 1} dy_j/2.$$

Since $\frac{1}{2}dy_j$ is a probability measure on $(-1, 1)$, so is $d\rho(y)$ on (U, Θ) and hence (U, Θ, ρ) is a probability space. As y_j are distributed uniformly, for any set of the form $S = \prod_{j=1}^{\infty} S_j$ with $S_j \in \mathcal{B}^1([-1, 1])$, it holds

$$\rho(S) = \prod_{j=1}^{\infty} P\{\omega : y_j(\omega) \in S_j\}.$$

For $0 < p \le \infty$, we denote by $L^p(U, \rho)$ the space of measurable functions $v : U \to \mathbb{R}$ such that $|v|^p$ is ρ-integrable over U. For a generic separable Hilbert space $\mathcal{V}$, we denote analogously by $L^p(U, \rho; \mathcal{V})$ the space of ρ-measurable mappings $v : U \to \mathcal{V}$ for which $\|v\|_{\mathcal{V}}^p$ is ρ-integrable. We introduce Bochner spaces $\underline{\mathcal{X}} = L^2(U, \rho; \mathcal{X})$ and $\underline{\mathcal{Y}} = L^2(U, \rho; \mathcal{Y})$ and note $\underline{\mathcal{X}} \simeq L^2(U, \rho) \otimes \mathcal{X}$, $\underline{\mathcal{Y}} \simeq L^2(U, \rho) \otimes \mathcal{Y}$, where $\otimes$ denotes the tensor product of separable Hilbert spaces.

1.3 Parametric deterministic wave equation

Given a forcing function $g(t, x)$ and initial data $g_1(x)$ and $g_2(x)$ satisfying (1.2), for each $y \in U$ we consider the initial boundary value problem

$$\frac{\partial^2 u(t,x,y)}{\partial t^2} - \nabla\cdot(a(x,y)\nabla u(t,x,y)) = g(t,x) \text{ in } Q_T,\ u|_{t=0} = g_1,\ u_t|_{t=0} = g_2, \tag{1.8}$$

where $u(t,\cdot)\in V$. The coefficient $a(x,y)$ is defined as

$$a(x,y)=\bar{a}(x)+\sum_{j=1}^{\infty} y_j\psi_j(x). \tag{1.9}$$

For each $y\in U$, we define the bilinear map $b:\mathcal{X}\times\mathcal{Y}\to\mathbb{R}$ by

$$b(y;w,(v_0,v_1,v_2))=\int_I\langle\frac{d^2w}{dt^2}(t,\cdot),v_0(t,\cdot)\rangle_H dt+\int_I\int_D a(x,y)\nabla w(t,x)\cdot\nabla v_0(t,x)dxdt+\langle u(0),v_1\rangle_V+\langle u_t(0),v_2\rangle_H. \tag{1.10}$$

We also define the linear form on $\mathcal{Y}$

$$f(v)=\int_I\int_D g(t,x)v_0(t,x)dxdt+\langle g_1,v_1\rangle_V+\langle g_2,v_2\rangle_H.$$

The variational formulation of the problem (1.8) is: Find $u(y)\in\mathcal{X}$ such that

$$b(y;u,v)=f(v)\quad\forall v=(v_0,v_1,v_2)\in\mathcal{Y}\,. \tag{1.11}$$

Proposition 1.4 *Under Assumption 1.1 and conditions* (1.2)*, for every $y\in U$, the problem (1.11) admits a unique weak solution $u(y)\in\mathcal{X}$. The weak solutions $\{u(y):y\in U\}\subset\mathcal{X}$ satisfy the apriori estimate*

$$\forall y\in U:\quad \|u(\cdot,\cdot,y)\|_{\mathcal{X}}\le C(\|g\|_{L^2(I;H)}+\|g_1\|_V+\|g_2\|_H), \tag{1.12}$$

where the constant C is independent of y.

This is, again, a special case of Theorem 29.1 of [8].

The previous proposition establishes merely existence of solutions for every selection of the parameter vector $y\in U$. In order to relate these parametric, deterministic solution family to the random solutions, we need to verify measurability of this solution family with respect to measures on the parameter domain.

Proposition 1.5 *The map $u:U\to\mathcal{X}$ is strongly measurable as a Bochner function.*

Proof Choose $\phi\in\mathcal{X}$ arbitrary. The inner product in $\mathcal{X}$ is given by

$$\langle u(y),\phi\rangle_{\mathcal{X}}=\langle u(y),\phi\rangle_{L^2(I,V)}+\langle u(y),\phi\rangle_{H^1(I,H)}+\langle u(y),\phi\rangle_{H^2(I;V')}.$$

We show that $\langle u(y),\phi\rangle_{\mathcal{X}}$ is measurable function from U to $\mathbb{R}$.

Let $\{w_n:\ n\in\mathbb{N}\}$ be a basis of V. We introduce the m-term truncated expansions

$$g_{1m}=\sum_{i=1}^m\xi^1_{im}w_i,\quad g_{2m}=\sum_{i=1}^m\xi^2_{im}w_i, \tag{1.13}$$

with $g_{1m}\to g_1$ in V and $g_{2m}\to g_2$ in H when $m\to\infty$. Let

$$u_m(t,\cdot,y)=\sum_{i=1}^m\zeta_{im}(t,y)w_i(\cdot)$$

satisfy the system

$$\frac{d^2}{dt^2}\langle u_m(t,\cdot,y),w_j\rangle_H+\int_D a(x,y)\nabla u_m(t,x,y)\cdot\nabla w_j(x)dx=\langle g(t,\cdot),w_j\rangle_H,\ \text{for } 1\le j\le m, \tag{1.14}$$
$$u_m(0,\cdot)=g_{1m}(\cdot),\ u'_m(0,\cdot)=g_{2m}(\cdot).$$

Define $\mathbf{A}_m(y)$ to be the $m\times m$ matrix

$$\mathbf{A}_m(y)=\Big(\int_D a(x,y)\nabla w_i(x)\cdot\nabla w_j(x)dx\Big),\quad 1\le i,j\le m.$$

Next, denote by $\mathbf{B}_m = (\langle w_i, w_j \rangle_H),\ 1 \le i,j \le m$ the Gram matrix. By linear independence of the w_i, it has a non zero determinant. Let $G_m(t)$ be the vector $(\langle g(t), w_j \rangle_H)$ and define $\zeta_m(t,y)$ to be the vector $(\zeta_{im}(t,y)),\ 1 \le i \le m$. Then the vector function ζ_m solves the system of differential equations

$$\mathbf{B}_m \frac{d^2\zeta_m(t,y)}{dt^2} + \mathbf{A}_m(y)\zeta_m(t,y) = G_m(t),$$

so

$$\frac{d^2\zeta_m(t,y)}{dt^2} + \mathbf{B}_m^{-1}\mathbf{A}_m(y)\zeta_m(t,y) = \mathbf{B}_m^{-1}G_m(t).$$

Let

$$\zeta_m'(t,y) = \frac{d\zeta_m(t,y)}{dt}.$$

We then define the column vector of length $2m$ by $\bar{\zeta}_m(t,y) = (\zeta_m(t,y), \zeta_m'(t,y))$. We denote by $\mathbf{I}$ the $m \times m$ identity matrix and $\mathbf{O}$ the $m \times m$ matrix whose entries are all zero, and define further

$$\mathbf{C}_m(y) = \begin{bmatrix} \mathbf{O} & -\mathbf{I} \\ \mathbf{B}_m^{-1}\mathbf{A}_m(y) & \mathbf{O} \end{bmatrix}, \quad m = 1, 2, ...$$

Let $\bar{G}_m(t) = (\mathbf{0}, \mathbf{B}_m^{-1}G_m(t))$ where $\mathbf{0}$ is the column vector of length m with zero entries.

Then the first order differential equation for $\bar{\zeta}_m$ reads

$$\frac{d\bar{\zeta}_m(t,y)}{dt} + \mathbf{C}_m(y)\bar{\zeta}_m(t,y) = \bar{G}_m(t), \tag{1.15}$$

with the initial condition $\bar{\zeta}_m(0,y) = \xi_m = (\xi_{1m}^1, \ldots, \xi_{mm}^1, \xi_{1m}^2, \ldots, \xi_{mm}^2)$. The solution $\bar{\zeta}_m$ can be written as

$$\bar{\zeta}_m(t,y) = e^{-t\mathbf{C}_m(y)}\left(\int_0^t e^{\tau\mathbf{C}_m(y)}\bar{G}_m(\tau)d\tau + \xi_m\right). \tag{1.16}$$

We show next that for each value m and for each $\phi \in \mathcal{X}$, the functional $\langle u_m, \phi \rangle_{\mathcal{X}}$ is measurable. We note that for $y, y' \in U$,

$$\begin{aligned}
&|\langle u_m(\cdot,\cdot,y), \phi\rangle_{\mathcal{X}} - \langle u_m(\cdot,\cdot,y'), \phi\rangle_{\mathcal{X}}| \le \|u_m(\cdot,\cdot,y) - u_m(\cdot,\cdot,y')\|_{\mathcal{X}}\|\phi\|_{\mathcal{X}} \\
&\le C(m) \sup_{0\le t\le T}(\|\zeta_m(t,y) - \zeta_m(t,y')\|_{\mathbb{R}^m} + \|\zeta_m'(t,y) - \zeta_m'(t,y')\|_{\mathbb{R}^m} + \|\zeta_m''(t,y) - \zeta_m''(t,y')\|_{\mathbb{R}^m})\|\phi\|_{\mathcal{X}} \\
&\le C(m) \sup_{0\le t\le T}\|\bar{\zeta}_m(t,y) - \bar{\zeta}_m(t,y')\|_{\mathbb{R}^{2m}}\|\phi\|_{\mathcal{X}}\ .
\end{aligned}$$

For every m holds

$$\begin{aligned}
\|\bar{\zeta}_m(t,y) - \bar{\zeta}_m(t,y')\|_{\mathbb{R}^{2m}} \quad &\le \quad \|e^{-t\mathbf{C}_m(y)}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \int_I \|e^{\tau\mathbf{C}_m(y)} - e^{\tau\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}\|\bar{G}_m(\tau)\|_{\mathbb{R}^{2m}}d\tau \\
&+ \quad \|e^{-t\mathbf{C}_m(y)} - e^{-t\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \int_I \|e^{\tau\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}\|\bar{G}_m(\tau)\|_{\mathbb{R}^{2m}}d\tau \\
&+ \quad \|e^{-t\mathbf{C}_m(y)} - e^{-t\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}\|\xi_m\|_{\mathbb{R}^{2m}}.
\end{aligned}$$

We note that for every $0 \le t \le T < \infty$,

$$\|e^{-t\mathbf{C}_m(y)}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \le e^{T\|\mathbf{C}_m(y)\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}}.$$

On the other hand, for every $m \in \mathbb{N}$ and every $y \in U$ it holds

$$\|\mathbf{C}_m(y)\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \le C(m) \max_{1\le i,j\le m}|\mathbf{C}_{mij}(y)| \le C(m)\sup_{x,y}|a(x,y)| \le C(m)\ .$$

Further, we have that

$$\|e^{\tau\mathbf{C}_m(y)} - e^{\tau\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \le Te^{T\|\mathbf{C}_m(y)\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}}e^{T\|\mathbf{C}_m(y)-\mathbf{C}_m(y')\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}}\|\mathbf{C}_m(y) - \mathbf{C}_m(y')\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}},$$

and

$$\|e^{-t\mathbf{C}_m(y)} - e^{-t\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \le Te^{T\|\mathbf{C}_m(y)\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}}e^{T\|\mathbf{C}_m(y)-\mathbf{C}_m(y')\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}}\|\mathbf{C}_m(y) - \mathbf{C}_m(y')\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}.$$

From

$$\|\mathbf{C}_m(y) - \mathbf{C}_m(y')\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}} \le C(m)\max_{i,j}|\mathbf{C}_{mij}(y) - \mathbf{C}_{mij}(y')| \le C(m)\sup_x|a(x,y) - a(x,y')|$$

we conclude that

$$\|\bar{\zeta}_m(t,y) - \bar{\zeta}_m(t,y')\|_{\mathbb{R}^{2m}} \le C(m)\sup_x|a(x,y) - a(x,y')|.$$

This implies

$$|\langle u_m(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}} - \langle u_m(\cdot,\cdot,y'),\phi\rangle_{\mathcal{X}}| \le C(m)\sup_x|a(x,y) - a(x,y')|\ . \tag{1.17}$$

We now show that for every $\alpha \in \mathbb{R}$, the set

$$Y_\alpha = \{y:\ \langle u_m(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}} > \alpha\}$$

is an element of the σ-algebra defined in U.

We consider the set T_i of all $y \in U$ such that $\bar{y} = (y_1, y_2, \ldots, y_i, z_1, z_2, \ldots)$ belongs to Y_α for all $z_j \in (-1,1)$, $j = 1,2,\ldots$. From (1.17) we deduce that for each $y \in Y_\alpha$, if

$$\sup_x|a(x,y) - a(x,y')| < r \tag{1.18}$$

for a sufficiently small constant r, then $y' \in Y_\alpha$. Therefore each vector $y \in Y_\alpha$ belongs to a set T_i for some i. Let $R_i \subset (-1,1)^i$ denote the set of $t = (t_1, t_2, \ldots, t_i)$ such that $(t_1, \ldots, t_i, z_1, z_2, \ldots) \in T_i$ for all $z_j \in (-1,1)$ $(j = 1,2,\ldots)$. From (1.17) and (1.18), R_i is an open set and thus can be represented as a countable union of open cubes. Therefore T_i can be represented as a countable union of cubes, say $\prod_{j\ge1} S_j$ where S_j is an open interval in $(-1,1)$ and $S_j = (-1,1)$ when j is sufficiently large. Thus T_i is measurable and so is Y_a.

Therefore, for every ϕ and every m, the mapping $U \ni y \mapsto \langle u_m(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}}$ is measurable as a mapping from U to $\mathbb{R}$. Next, we define

$$X_\alpha = \{y \in U : \langle u(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}} \ge \alpha\}\ .$$

As $u_m \rightharpoonup u$ in $\mathcal{X}$ (see the proof of [8, Theorem 29.1]), $\langle u_m(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}} \to \langle u(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}}$ as $m \to \infty$. Therefore

$$X_\alpha = \bigcap_{n=1}^\infty \bigcup_{m=1}^\infty \bigcap_{i=m}^\infty \{y:\ \langle u_i(\cdot,\cdot,y),\phi\rangle_{\mathcal{X}} > \alpha - \frac{1}{n}\},$$

which is measurable. We conclude that $u(\cdot,\cdot,y)$ as a map from U to $\mathcal{X}$ is measurable. □

Let $\underline{\mathcal{X}} = L^2(U,\rho;\mathcal{X})$ and $\underline{\mathcal{Y}} = L^2(U,\rho;\mathcal{Y})$. We define the bilinear form $B(\cdot,\cdot) : \underline{\mathcal{X}} \times \underline{\mathcal{Y}} \to \mathbb{R}$ and the linear form $F(\cdot) : \underline{\mathcal{Y}} \to \mathbb{R}$ as

$$B(u,v) = \int_U b(y;u,v)d\rho(y), \quad F(v) = \int_U f(v)d\rho(y).$$

Consider the variational problem: find

$$u \in \underline{\mathcal{X}} \quad \text{such that} \quad B(u,v) = F(v) \qquad \forall v \in \underline{\mathcal{Y}}. \tag{1.19}$$

Proposition 1.6 *Under Assumptions 1.1 and 1.3, problem (1.19) admits a unique solution $u \in \underline{\mathcal{X}}$.*

Proof The solution of (1.11) is uniformly bounded in $\mathcal{X}$ for all $y \in U$. Further, $u(\cdot,\cdot,y)$ is measurable as a map from U to $\mathcal{X}$, so $u(\cdot,\cdot,\cdot) \in \underline{\mathcal{X}}$. The existence part is obvious.

Now we show uniqueness. Let $\phi(t,x,y) = \psi(t,x)w(y)$ where $\psi(t,x) \in \mathcal{X}$ and $w(y) \in L^2(U,d\rho)$. We then have from (1.19):

$$\int_U\Big(\int_I\langle\frac{d^2u}{dt^2}(t,\cdot,y),\psi(t,\cdot)\rangle_H dt + \int_I\int_D a(x,y)\nabla u(t,x,y)\cdot\nabla\psi(t,x)dxdt\Big)w(y)d\rho(y)$$
$$= \int_U\Big(\int_I\int_D g(t,x)\psi(t,x)dxdt\Big)w(y)d\rho(y).$$

As this holds for all $w(y) \in L^2(U, d\rho(y))$, we find that

$$\int_I \langle \frac{d^2 u}{dt^2}, \psi \rangle_H + \int_I \int_D a(x,y) \nabla u(t,x,y) \cdot \nabla \psi(t,x,y) dx dt = \int_I \int_D g(t,x) \psi(t,x) dx dt$$

for almost all $y \in U$. This together with the initial condition shows that $u(t,x,y)$ is unique. □

2 Semidiscrete Galerkin Approximation

2.1 Polynomial spaces in U

Let $(L_n)_{n \geq 0}$ denote the univariate Legendre polynomials normalized according to

$$\int_{-1}^{1} |L_n(t)|^2 \frac{dt}{2} = 1. \tag{2.1}$$

Note that in this normalization, $L_0(t) \equiv 1$. We shall use tensor products of Legendre polynomials of multi-degrees taking values in the set $\mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \|\nu\|_1 < \infty\}$, i.e. the set of all sequences $\nu = (\nu_j)_{j \geq 1}$ of nonnegative integers such that only a finite number of ν_j are non zero. For such ν, we define the tensorized Legendre polynomials

$$L_\nu(y) = \prod_{j \geq 1} L_{\nu_j}(y_j), \qquad \nu \in \mathcal{F}.$$

The family L_ν forms a countable orthonormal basis of $L^2(U, \rho)$. Therefore, each function $u \in \underline{\mathcal{X}}$ can be written as

$$u = \sum_{\nu \in \mathcal{F}} u_\nu L_\nu, \qquad u_\nu \in \mathcal{X} \tag{2.2}$$

and an analogous representation is valid for $v \in \underline{\mathcal{Y}}$.

2.2 Spectral semidiscretization in y

For any set $\Lambda \subset \mathcal{F}$ of finite cardinality we define the following subspaces of $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$

$$\underline{\mathcal{X}}_\Lambda = \{u_\Lambda(t,x,y) = \sum_{\nu \in \Lambda} u_\nu(t,x) L_\nu(y) : \ u_\nu \in \mathcal{X}\} \subset \underline{\mathcal{X}},$$

and

$$\underline{\mathcal{Y}}_\Lambda = \{v_\Lambda(t,x,y) = \sum_{\nu \in \Lambda} v_\nu(t,x) L_\nu(y) : \ v_\nu \in \mathcal{Y}\} \subset \underline{\mathcal{Y}}.$$

Denoting $v_\nu = (v_{0\nu}, v_{1\nu}, v_{2\nu})$, we may write

$$v_{0\Lambda}(t,x,y) = \sum_{\nu \in \Lambda} v_{0\nu}(t,x) L_\nu(y), \ \ v_{1\Lambda}(x,y) = \sum_{\nu \in \Lambda} v_{1\nu}(x) L_\nu(y) \text{ and } v_{2\Lambda}(x,y) = \sum_{\nu \in \Lambda} v_{2\nu}(x) L_\nu(y).$$

We consider the following semidiscrete Galerkin projection of u onto $\underline{\mathcal{X}}$: find

$$u_\Lambda \in \underline{\mathcal{X}}_\Lambda \quad \text{such that} \quad B(u_\Lambda, v_\Lambda) = F(v_\Lambda) \quad \forall v_\Lambda \in \underline{\mathcal{Y}}_\Lambda. \tag{2.3}$$

Theorem 2.1 *Under Assumptions 1.1 and 1.3, for every subset $\Lambda \subset \mathcal{F}$ of finite cardinality there exists a unique solution $u_\Lambda \in \underline{\mathcal{X}}_\Lambda$ to the Galerkin equations (2.3).*

Proof Let $u_\Lambda = \sum_{\nu \in \Lambda} u_\nu L_\nu$ and $v_\Lambda = \sum_{\mu \in \Lambda} v_\mu L_\mu$. Problem (2.3) can be written as a coupled system of wave equations for the coefficient functions $u_\nu(t,x)$ for every $\nu \in \Lambda \subset \mathcal{F}$ (with implied summation over repeated indices $\mu, \nu \in \Lambda$)

$$\begin{aligned} &\int_I \langle \frac{d^2 u_\nu}{dt^2}, v_{0\nu} \rangle_H dt + \int_I \int_D A^{\nu\mu}(x) \nabla u_\mu \cdot \nabla v_{0\nu} dx dt + \langle u_\nu(0), v_{1\nu} \rangle_V + \langle \frac{du_\nu}{dt}(0), v_{2\nu} \rangle_H \\ = \ &\int_I \int_D g(t,x) v_{0\nu}(t,x) dx dt \delta_{0\nu} + \langle g_1, v_{1\nu} \rangle_V \delta_{0\nu} + \langle g_2, v_{2\nu} \rangle_H \delta_{0\nu} \ , \qquad \nu \in \mathcal{F} \end{aligned} \tag{2.4}$$

where $\delta_{0\nu} = 1$ if all entries of ν are zero and $\delta_{0\nu} = 0$ otherwise (the analysis will be more involved when g, g_1 and g_2 are random). In (2.4), the coefficients $\{A^{\nu\mu} : \mu, \nu \in \Lambda\}$ are defined as

$$A^{\nu\mu}(x) = \int_U a(x,y) L_\nu(y) L_\mu(y) d\rho(y), \qquad \nu, \mu \in \mathcal{F} \ . \tag{2.5}$$

For each $\nu \in \Lambda$, we consider a vector $\xi^\nu = (\xi^\nu_i)_{i=1}^d \in \mathbb{R}^d$. Then (with implied summation over repeated indices $\mu, \nu \in \Lambda$) we obtain from Assumption 1.1

$$A^{\nu\mu}(x)\xi^\nu_i \xi^\mu_i = \int_U a(x,y)(L_\nu(y)\xi^\nu_i)(L_\mu(y)\xi^\mu_i) d\rho(y) \geq a_{\min} \sum_{i=1}^d \sum_{\nu \in \Lambda} (\xi^\nu_i)^2. \tag{2.6}$$

This shows that the matrix $(A^{\nu\mu}_{ij}) := (A^{\nu\mu}\delta_{ij})$ for $\nu, \mu \in \Lambda$ and $i, j = 1, \ldots, d$ is positive definite. For every $\Lambda \subseteq \mathcal{F}$, problem (2.3) thus has a unique solution. □

In the argument that follows we shall use point values of u and of the first time derivatives of u; to this end, we introduce the space

$$\mathcal{Z} := H^1(I;V) \cap H^2(I;H) \subset C^0(\overline{I};V) \cap C^1(\overline{I};H) \ . \tag{2.7}$$

Note that $\mathcal{Z} \subset \mathcal{X}$. The following quasi-optimality like error estimate for semidiscrete approximations of parametric solutions in $\mathcal{Z}$ holds.

Proposition 2.2 *Assume that $u \in L^2(U, \rho; \mathcal{Z})$. Then for all $\nu \in \mathcal{F}$ the coefficient u_ν in (2.2) belongs to $\mathcal{Z}$. Assume further that for a subset $\Lambda \subset \mathcal{F}$, $u_\Lambda \in L^2(U, \rho; \mathcal{Z})$. Then holds the error bound*

$$\|u - u_\Lambda\|_{L^2(U,\rho;\mathcal{X})} \leq c \| \sum_{\nu \in \mathcal{F}\setminus\Lambda} u_\nu L_\nu \|_{L^2(U,\rho;\mathcal{Z})} = c \left(\sum_{\nu \in \mathcal{F}\setminus\Lambda} \|u_\nu\|^2_{\mathcal{Z}} \right)^{1/2} . \tag{2.8}$$

Here, the constant $c > 0$ depends only on the coefficient bounds a_{min} and a_{max} in Assumption 1.1.

Proof In (2.2), we write $u = \bar{u}_\Lambda + \bar{\bar{u}}_\Lambda$ where

$$\bar{u}_\Lambda = \sum_{\nu \in \Lambda} u_\nu L_\nu, \quad \text{and} \quad \bar{\bar{u}}_\Lambda = \sum_{\nu \in \mathcal{F}\setminus\Lambda} u_\nu L_\nu.$$

From (1.19) and (2.3) we have for all $v_\Lambda \in \mathcal{Y}$,

$$\begin{aligned}
&\int_U \int_I \left\langle \frac{d^2(\bar{u}_\Lambda - u_\Lambda)}{dt^2}, v_{0\Lambda} \right\rangle_H dt d\rho(y) + \int_U \int_I \int_D a(x,y) \nabla(\bar{u}_\Lambda - u_\Lambda) \cdot \nabla v_{0\Lambda} dx dt d\rho(y) + \\
&\langle \bar{u}_\Lambda(0,\cdot,\cdot) - u_\Lambda(0,\cdot,\cdot), v_{1\Lambda} \rangle_V + \langle (\bar{u}_\Lambda)_t(0,\cdot,\cdot) - (u_\Lambda)_t(0,\cdot,\cdot), v_{2\Lambda} \rangle_H \\
= \ & -\int_U \int_I \left\langle \frac{d^2 \bar{\bar{u}}_\Lambda}{dt^2}, v_{0\Lambda} \right\rangle_H dt d\rho(y) - \int_U \int_I \int_D a(x,y) \nabla \bar{\bar{u}}_\Lambda \cdot \nabla v_{0\Lambda} dx dt d\rho(y) - \\
&\langle \bar{\bar{u}}_\Lambda(0,\cdot,\cdot), v_{1\Lambda} \rangle_V - \langle (\bar{\bar{u}}_\Lambda)_t(0,\cdot,\cdot), v_{2\Lambda} \rangle_H .
\end{aligned} \tag{2.9}$$

Inserting in (2.9) the test functions $v_{0\Lambda} = w\phi$ where $\phi \in L^2(I)$ and $w \in L^2(U, \rho; V)$, $v_{1\Lambda} = 0$ and $v_{2\Lambda} = 0$, we get

$$\begin{aligned}
&\int_I \int_U \left\langle \frac{d^2(\bar{u}_\Lambda - u_\Lambda)}{dt^2}, w \right\rangle_H \phi(t) d\rho(y) dt + \int_I \int_U \int_D a(x,y) \nabla(\bar{u}_\Lambda - u_\Lambda) \cdot \nabla w \phi(t) dx d\rho(y) dt \\
= \ & -\int_I \int_U \left\langle \frac{d^2 \bar{\bar{u}}_\Lambda}{dt^2}, w \right\rangle_H \phi(t) d\rho(y) dt - \int_I \int_U \int_D a(x,y) \nabla \bar{\bar{u}}_\Lambda \cdot \nabla w \phi(t) dx d\rho(y) dt.
\end{aligned} \tag{2.10}$$

As this holds for all $\phi \in L^2(I)$, we get for all $w \in L^2(U,\rho;V)$ and for almost all $t \in I$

$$\begin{aligned} &\int_U \left\langle \frac{d^2(\bar{u}_\Lambda - u_\Lambda)}{dt^2}(t,\cdot,y), w(\cdot,y)\right\rangle_H d\rho(y) + \int_U\int_D a(x,y)\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y)\cdot\nabla w(x,y)dxd\rho(y) \\ = \; & -\int_U \left\langle \frac{d^2\bar{\bar{u}}_\Lambda}{dt^2}(t,\cdot,y), w(\cdot,y)\right\rangle_H d\rho(y) - \int_U\int_D a(x,y)\nabla\bar{\bar{u}}_\Lambda(t,x,y)\cdot\nabla w(x,y)dxd\rho(y)\ . \end{aligned} \tag{2.11}$$

Let $w = \frac{d}{dt}(\bar{u}_\Lambda - u_\Lambda)$. Then $w \in L^2(U,\rho;L^2(I;V))$ as both $\bar{u}_\Lambda$ and u_Λ are in $L^2(U,\rho;\mathcal{Z})$ by assumption, and we have

$$\begin{aligned} &\int_U \left\langle \frac{d^2(\bar{u}_\Lambda - u_\Lambda)}{dt^2}(t,\cdot,y), \frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(t;\cdot,y)\right\rangle_H d\rho(y) \\ &+\int_U\int_D a(x,y)\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y)\cdot\nabla\frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(t,x,y)dxd\rho(y) \\ = \; & -\int_U \left\langle \frac{d^2\bar{\bar{u}}_\Lambda}{dt^2}(t,\cdot,y), \frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(t,\cdot,y)\right\rangle_H d\rho(y) \\ &-\int_U\int_D a(x,y)\nabla\bar{\bar{u}}_\Lambda(t,x,y)\cdot\nabla\frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(t,x,y)dxd\rho(y) \\ = \; & -\int_U \left\langle \frac{d^2\bar{\bar{u}}_\Lambda}{dt^2}(t,\cdot,y), \frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(t,\cdot,y)\right\rangle_H d\rho(y) \\ &-\int_U\int_D a(x,y)\frac{d}{dt}(\nabla\bar{\bar{u}}_\Lambda(t,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y))dxd\rho(y) \\ &+\int_U\int_D a(x,y)\nabla\frac{d\bar{\bar{u}}_\Lambda}{dt}(t,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y)dxd\rho(y). \end{aligned}$$

From this we deduce using once more with the assumption $u \in L^2(U,\rho;\mathcal{Z})$ and the embedding $\mathcal{Z} \subset C^0(\overline{I};V)\cap C^1(\overline{I};H)$ that for all $t \in I$

$$\begin{aligned} &\frac{1}{2}\left\|\frac{d}{dt}(\bar{u}_\Lambda - u_\Lambda)(t,\cdot,\cdot)\right\|^2_{L^2(U,\rho;H)} \\ &+\frac{1}{2}\int_U\int_D a(x,y)\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y)dxd\rho(y) \\ = \; & -\int_0^t\int_U \left\langle \frac{d^2\bar{\bar{u}}_\Lambda}{dt^2}(\tau,\cdot,y), \frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(\tau,\cdot,y)\right\rangle_H d\tau d\rho(y) \\ &-\int_U\int_D a(x,y)\nabla\bar{\bar{u}}_\Lambda(t,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(t,x,y)dxd\rho(y) \\ &+\int_U\int_D a(x,y)\nabla\bar{\bar{u}}_\Lambda(0,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(0,x,y)dxd\rho(y) \\ &+\int_0^t\int_U\int_D a(x,y)\nabla\frac{d\bar{\bar{u}}_\Lambda}{dt}(\tau,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(\tau,x,y)dxd\rho(y)d\tau \\ &+\frac{1}{2}\left\|\frac{d}{dt}(\bar{u}_\Lambda - u_\Lambda)(0,\cdot,\cdot)\right\|^2_{L^2(U,\rho;H)} \\ &+\frac{1}{2}\int_U\int_D a(x,y)\nabla(\bar{u}_\Lambda - u_\Lambda)(0,x,y)\cdot\nabla(\bar{u}_\Lambda - u_\Lambda)(0,x,y)dxd\rho(y)\ . \end{aligned}$$

Inserting $v_{1\Lambda} = \bar{u}_\Lambda(0,\cdot,\cdot) - u_\Lambda(0,\cdot,\cdot)$, $v_{0\Lambda} = 0$ and $v_{2\Lambda} = 0$ into (2.9), we infer that

$$\|\bar{u}_\Lambda(0,\cdot,\cdot) - u_\Lambda(0,\cdot,\cdot)\|_V \le \|\bar{\bar{u}}_\Lambda(0,\cdot,\cdot)\|_V.$$

Inserting

$$v_{2\Lambda} = \frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(0,\cdot,\cdot), \quad v_{0\Lambda} = 0, \quad \text{and} \quad v_{1\Lambda} = 0$$

into (2.9), we find that

$$\left\|\frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}(0,\cdot,\cdot)\right\|_H \le \left\|\frac{d\bar{\bar{u}}_\Lambda}{dt}(0,\cdot,\cdot)\right\|_H.$$

We note that there is a constant c that depends on T such that

$$\|\bar{\bar{u}}_\Lambda(0,\cdot,\cdot)\|_{L^2(U,\rho;V)} \le c\|\bar{\bar{u}}_\Lambda(\cdot,\cdot,\cdot)\|_{L^2(U,\rho;\mathcal{Z})}, \quad \|\frac{d\bar{\bar{u}}_\Lambda}{dt}(0,\cdot,\cdot)\|_{L^2(U,\rho;H)} \le c\|\bar{\bar{u}}_\Lambda(\cdot,\cdot,\cdot)\|_{L^2(U,\rho;\mathcal{Z})}.$$

From these bounds we deduce that for each $0 < t < T$ that

$$\begin{aligned}
&\left\|\frac{d}{dt}(\bar{u}_\Lambda - u_\Lambda)(t,\cdot,\cdot)\right\|^2_{L^2(U,\rho;H)} + \|(\bar{u}_\Lambda - u_\Lambda)(t,\cdot,\cdot)\|^2_{L^2(U,\rho;V)} \\
&\le c\left\|\frac{d^2\bar{\bar{u}}_\Lambda}{dt^2}\right\|_{L^2(I;L^2(U,\rho;H))}\left\|\frac{d(\bar{u}_\Lambda - u_\Lambda)}{dt}\right\|_{L^2(I;L^2(U,\rho:H))} \\
&+ c\|\bar{\bar{u}}_\Lambda(t,\cdot,\cdot)\|_{L^2(U,\rho;V)}\|(\bar{u}_\Lambda - u_\Lambda)(t,\cdot,\cdot)\|_{L^2(U,\rho;V)} \\
&+ c\|\bar{\bar{u}}_\Lambda\|_{L^2(U,\rho;\mathcal{Z})}\|\bar{u}_\Lambda - u_\Lambda\|_{L^2(U,\rho;L^2(I;V))} + c\|\bar{\bar{u}}_\Lambda\|^2_{L^2(U,\rho;\mathcal{Z})}.
\end{aligned}$$

Integrating both sides of this inequality from $t = 0$ to $t = T$, we obtain

$$\left\|\frac{d}{dt}(\bar{u}_\Lambda - u_\Lambda)\right\|_{L^2(I;L^2(U,\rho;H))} + \|\bar{u}_\Lambda - u_\Lambda\|_{L^2(I;L^2(U,\rho;V))} \le c\|\bar{\bar{u}}_\Lambda\|_{L^2(U,\rho;\mathcal{Z})}.$$

From this and (2.9), we deduce that

$$\left\|\frac{d^2}{dt^2}(\bar{u}_\Lambda - u_\Lambda)\right\|_{L^2(U,\rho;L^2(I,V'))} \le c\|\bar{\bar{u}}_\Lambda\|_{L^2(U,\rho;\mathcal{Z})}.$$

Thus

$$\|\bar{u}_\Lambda - u_\Lambda\|_{L^2(U,\rho;\mathcal{X})} \le c\|\bar{\bar{u}}_\Lambda\|_{L^2(U,\rho;\mathcal{Z})},$$

which implies the first part of the assertion (2.8), i.e.

$$\|u - u_\Lambda\|_{L^2(U,\rho;\mathcal{X})} \le c\|\sum_{\nu\in\mathcal{F}\setminus\Lambda} u_\nu L_\nu\|_{L^2(U,\rho;\mathcal{Z})}.$$

The second part follows then from the normalization (2.1) and Parseval's equality. □

Proposition 2.2 implies, in effect, *quasioptimality* of the $L^2(U,\rho;\mathcal{X})$ projection $u_\Lambda \in \underline{\mathcal{X}}_\Lambda$ defined in (2.3). We note, however, that in its proof, the extra regularity $u \in L^2(U,\rho;\mathcal{Z})$ was required.

2.3 Regularity with respect to t

We now establish a regularity result for u and u_Λ which ensures the validity of the regularity $u, u_\Lambda \in L^2(U,\mathcal{Z},d\rho)$ and, hence, implies the semidiscrete error bound (2.8). To this end, we define the smoothness space $W \subset V$ as the space of all solutions to the Dirichlet problem

$$-\Delta u = f \quad \text{in} \quad D, \qquad u|_{\partial D} = 0, \tag{2.12}$$

with $f \in L^2(D)$, i.e.

$$W = \{v \in V : \Delta v \in L^2(D)\}. \tag{2.13}$$

We define the W-(semi) norm and the W-norm by

$$|v|_W = \|\Delta v\|_{L^2(D)}, \qquad \|v\|_W := \|v\|_V + |v|_W. \tag{2.14}$$

It is well-known $W = H^2(D) \cap V$ for convex $D \subset \mathbb{R}^d$. For the following result we define the function space

$$\mathcal{W} = L^2(I; W) \cap H^1(I; V) \cap H^2(I; H) \ , \tag{2.15}$$

where W is defined in (2.13). Note that for $\mathcal{Z}$ defined in (2.7) holds $\mathcal{W} \subset \mathcal{Z}$.

Proposition 2.3 *Under Assumptions 1.1 and 1.3, and if, in addition, $a(\cdot,\cdot) \in L^\infty(U, W^{1,\infty}(D))$, $g \in H^1(I; H)$, $g_1 \in W$ and $g_2 \in V$, then for every subset $\Lambda \subset \mathcal{F}$ of finite cardinality holds*

$$u_\Lambda \in L^2(U, \rho; \mathcal{W}) \subset L^2(U, \rho; \mathcal{Z}).$$

Proof We proceed in two steps:
i): The coefficients and initial condition in the hyperbolic system (2.4) satisfy the compatibility condition

$$\frac{d}{dt}u_\nu(0) \in V, \ \frac{d^2 u_\nu}{dt^2}(0) = g(0)\delta_{0\nu} + \nabla A^{\nu\mu}(\cdot) \cdot \nabla g_1(\cdot)\delta_{0\mu} + A^{\nu\mu}(\cdot)\Delta g_1(\cdot)\delta_{0\mu} \in H, \quad g_t \in L^2(I; H). \tag{2.16}$$

A standard bootstrap argument following, for example, the proof of [8, Theorem 30.1] for regularity of a nonparametric, scalar second order wave equation shows that $u_\nu \in H^1(I; V) \cap H^2(I; H)$ for all $\nu \in \Lambda$. Since Λ is a finite set, it follows $u_\Lambda \in L^2(U, \rho; \mathcal{Z})$.
ii): Next, we observe that the weak equation (2.4) is equivalent to the coupled system of wave equations in $(t, x) \in I \times D$ (with implied summation over repeated indices):

$$\frac{d^2 u_\nu(t,x)}{dt^2} - \nabla \cdot (A^{\nu\mu}(x)\nabla u_\mu(t,x)) = g_\nu(t,x) \ , \tag{2.17}$$

where $g_\nu(t,x) = g(t,x)\delta_{0\nu}$. Using that $a(\cdot,\cdot) \in L^\infty(U, W^{1,\infty}(D))$, we find from (2.5) that $A^{\nu\mu}(\cdot) \in W^{1,\infty}(D)$ for every $\nu, \mu \in \Lambda$. We next observe that from (2.17) it follows (summation over repeated indices)

$$\frac{d^2 u_\nu(t,x)}{dt^2} - A^{\nu\mu}(x)\Delta u_\mu(t,x) - \nabla A^{\nu\mu}(x) \cdot \nabla u_\mu(t,x) = g_\nu(t,x) \in C^0(\overline{I}; L^2(D)) \quad \forall \nu \in \Lambda.$$

From part i) of the proof and from the embedding in (2.7) it follows that (with summation over repeated indices)

$$A^{\nu\mu}(x)\Delta u_\mu(t,x) = \frac{d^2 u_\nu(t,x)}{dt^2} - \nabla A^{\nu\mu}(x) \cdot \nabla u_\mu(t,x) - g_\nu(t,x) \in C^0(\overline{I}; L^2(D)) \subset L^2(I; H) \quad \forall \nu \in \Lambda.$$

$$\forall \nu \in \Lambda : \quad \Delta u_\nu(t,x) \in L^2(I; H)$$

which implies by the definition of the space W that

$$\forall \nu \in \Lambda : \quad u_\nu(t,x) \in L^2(I; W).$$

With part i) of the proof and the definition of $\mathcal{W}$ the assertion follows. □

Next, we establish the regularity $u \in L^2(U, \rho; \mathcal{Z})$. Under some additional conditions on the coefficients and the initial data of (1.1), we show that $u(\cdot,\cdot,y)$ is bounded uniformly in the norm of $\mathcal{Z}$ defined in (2.7) for all $y \in U$. We make the following assumption on the coefficient $a(x, y)$.

Assumption 2.4 *We assume in (1.9) that $\bar{a} \in W^{1,\infty}(D)$ and $\psi_j \in W^{1,\infty}(D)$ such that*

$$\sum_{j=1}^{\infty} \|\psi_j\|_{W^{1,\infty}(D)} < \infty.$$

Proposition 2.5 *If Assumption 2.4 holds and if moreover the compatibility condition*

$$g \in H^1(I;H),\ g_1 \in W,\ g_2 \in V \tag{2.18}$$

holds, then for every $y \in U$ it holds $u(\cdot,\cdot,y) \in \mathcal{Z}$ and its $\mathcal{Z}$ norm is bounded uniformly for all $y \in U$.

Proof We proceed along the lines of proof for the nonparametric problem as outlined, for example, in [8]: we consider the parametric hyperbolic problem

$$\frac{d^2v}{dt^2} - \nabla\cdot(a(x,y)\nabla v(x,y)) = \frac{dg}{dt},\ v(0) = g_2,\ \frac{dv}{dt}(0) = g(0) + \nabla\cdot(a(x,y)\nabla g_1). \tag{2.19}$$

As $g \in H^1(I;H)$ so $g(0) \in H$. From Assumption 2.4 and from the condition that $g_1 \in W$, we infer $g(0) + \nabla\cdot(a(x,y)\nabla g_1) \in H$. From [8, Theorem 29.1], the initial boundary value problem (2.19) admits a unique solution $v \in \mathcal{X}$; and the norm $\|v\|_{\mathcal{X}}$ has an upper bound that depends only on $a_{\min}$, $a_{\max}$, g_t and the initial conditions. It can be shown that (see the proof of [8, Theorem 30.1]),

$$\frac{du}{dt}(\cdot,\cdot,y) = v. \tag{2.20}$$

Therefore $u(\cdot,\cdot,y) \in \mathcal{Z}$ and, by the apriori estimate (1.12) (applied to v) also its norm in $\mathcal{Z}$ is bounded uniformly for all $y \in U$. □

Proposition 2.6 *With Assumption 2.4 and conditions (2.18), the mapping $u : U \to \mathcal{Z}$ is measurable.*

Proof Let $\phi \in L^2(I;V) \cap H^1(I;H)$. We will show that for fixed $\phi \in L^2(I;V) \cap H^1(I;H)$, the mapping

$$U \ni y \mapsto \langle v, \phi\rangle_{L^2(I;V)} + \langle v, \phi\rangle_{H^1(I;H)}$$

is a measurable map from U to $\mathbb{R}$; here $v = du/dt$ is the solution of Problem (2.19). The proof proceeds along the lines of the proof of Proposition 1.5. For (2.19) we again consider the differential equation (1.15) with the initial condition $\bar\zeta_m(0,y) = \xi_m = (\xi^1_{1m},\dots,\xi^1_{mm},\xi^2_{1m},\dots,\xi^2_{mm})$ where

$$\sum_{j=1}^m \xi^1_{jm} w_j \to g_2 \text{ in } V \text{ when } m \to \infty$$

and, as $m \to \infty$,

$$\sum_{j=1}^m \xi^2_{jm} w_j \to g(0) + \nabla a(x,y)\cdot\nabla g_1 + a(x,y)\Delta g_1 \text{ in } H \text{ when } m \to \infty;$$

here the coefficients ξ^2_{jm} depend on y. The solution $\bar\zeta_m$ of (1.15) (for problem (2.19)) is written by (1.16) for this new initial condition ξ_m which depends on y. We claim that for each m there exists $c(m) > 0$ such that for every x, y, t there holds

$$|e^{-t\mathbf{C}_m(y)}\xi_m(y) - e^{-t\mathbf{C}_m(y')}\xi_m(y')| \le c(m)(\sup_x |a(x,y) - a(x,y')| + \sup_x |\nabla a(x,y) - \nabla a(x,y')|)\ .$$

To prove the claim, we write

$$|e^{-t\mathbf{C}_m(y)}\xi_m(y) - e^{-t\mathbf{C}_m(y')}\xi_m(y')| \le \|e^{-t\mathbf{C}_m(y)} - e^{-t\mathbf{C}_m(y')}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}|\xi_m(y)| + \|e^{-t\mathbf{C}_m(y)}\|_{\mathbb{R}^{2m},\mathbb{R}^{2m}}|\xi_m(y) - \xi_m(y')|.$$

Choosing $\sum_{j=1}^m \xi^2_{jm} w_j$ as the orthogonal projection in H of $g(0) + \nabla a(x,y)\cdot\nabla g_1 + a(x,y)\Delta g_1$ onto the linear hull of $\{w_1,\dots,w_m\}$, there is a constant $c(m)$ which does not depend on y such that $|\xi_m(y)| < c(m)$ for all $y \in U$. Furthermore,

$$|\xi_m(y) - \xi_m(y')| \le c(m)(\sup_x |\nabla_x(a(x,y) - a(x,y'))| + \sup_x |a(x,y) - a(x,y')|).$$

Estimate (1.17) then becomes

$$\sup_{0\le t\le T} \|\bar\zeta_m(y) - \bar\zeta_m(y')\|_{\mathbb{R}^{2m}} \le c(m)(\sup_x |\nabla_x(a(x,y) - a(x,y'))| + \sup_x |a(x,y) - a(x,y')|).$$

With Assumption 2.4, by a similar argument, it can be shown that for the solution v_m of the discrete problem (1.14) (applied for (2.19)) and for every ϕ, the mapping $U \ni y \mapsto \langle v_m, \phi\rangle_{L^2(I;V)} + \langle v_m, \phi\rangle_{H^1(I;H)}$ is measurable from U to $\mathbb{R}$. The assertion follows. □

From Propositions 2.5 and 2.6 we deduce $u \in L^2(U,\rho;\mathcal{Z})$, so $u_\nu \in \mathcal{Z}$ for all $\nu \in \mathcal{F}$.

3 Best N term approximation

As the solution v of the problem (2.19) satisfies $v \in L^2(U,\rho;\mathcal{X})$, and due to (2.20), its coefficients v_ν in the expansion $v = \sum_{\nu\in\mathcal{F}} v_\nu L_\nu$ belong to $\mathcal{X}$. For $|\nu|>0$, $u_\nu(0,\cdot)=0$. By the Poincaré in $H^1(I)$, there exists $c>0$ such that

$$\forall \nu \in \mathcal{F}: \quad \|u_\nu\|_{L^2(I;V)} \le c\left\|\frac{du_\nu}{dt}\right\|_{L^2(I;V)} \quad \text{and} \quad \|u_\nu\|_{L^2(I;H)} \le c\left\|\frac{du_\nu}{dt}\right\|_{L^2(I;H)} .$$

Therefore, $\|u_\nu\|_{\mathcal{Z}} \le c\|v_\nu\|_{\mathcal{X}}$. The approximation (2.8) can be written as

$$\|u-u_\Lambda\|_{L^2(U,\rho;\mathcal{X})} \le c\Big(\sum_{\nu\in\mathcal{F}\setminus\Lambda}\|v_\nu\|^2_{\mathcal{X}}\Big)^{1/2}.$$

To obtain an explicit rate of convergence for this approximation in terms of the cardinality N of Λ, we need summability of $\|v_\nu\|_{\mathcal{X}}$. We establish this for the case where v as a map from U to $\mathcal{X}$ is infinitely differentiable, by recursive differentiation of the partial differential equation as in [5] for elliptic problems. We emphasize that analytic continuation and complex variables techniques as employed in [4] do not apply here, as the usual existence and uniqueness theory for second order hyperbolic equations (as presented, e.g., in [8]) do not apply when the elliptic spatial operator is not selfadjoint, as is the case for the operator obtained by analytic continuation of $a(x,z)$.

3.1 Differentiability of v with respect to y

To show differentiability of the solution v of (2.19), with respect to parameter y_k, we require additional regularity of v with respect to x and to t as shown in the following lemma.

Lemma 3.1 *For $y\in U$, consider the hyperbolic problem*

$$\frac{d^2w}{dt^2} - \nabla\cdot(a(x,y)\nabla w) = f(t,x,y),\ w(0,x,y)=f_1(x,y),\ w_t(0,x,y)=f_2(x,y)\ . \tag{3.1}$$

Assume that $w(\cdot,\cdot,y)$ is uniformly bounded in $L^2(I;W)$ for all $y\in U$ and is continuous as a map from U to $L^2(I;W)$ so that

$$\lim_{y'\to y}\|\Delta(w(\cdot,\cdot,y')-w(\cdot,\cdot,y))\|_{L^2(I;H)}=0, \tag{3.2}$$

f is differentiable as a map from U to $L^2(I;H)$, f_1 is differentiable as a map from U to V, and f_2 is differentiable as a map from U to H. Then, under Assumption 2.4, for all $k\in\mathbb{N}$ w is differentiable with respect to y_k, and the derivative $\partial_{y_k}w(t,x,y)\in V$ is the unique solution of the problems

$$\frac{d^2}{dt^2}(\partial_{y_k}w) - \nabla\cdot(a(x,y)\nabla\partial_{y_k}w) = \nabla\psi_k\cdot\nabla w + \psi_k\Delta w + \partial_{y_k}f(y)$$

with the initial conditions

$$\partial_{y_k}v(0,\cdot,y)=\partial_{y_k}f_1(y), \quad \textit{and} \quad \frac{d}{dt}\partial_{y_k}v(0,\cdot,y)=\partial_{y_k}f_2(y)\ .$$

Proof We prove the (strong) differentiability by analyzing suitable difference quotients. To this end, for $\delta\neq 0$, let $y'\in U$ be such that y'_l differs from y_l only when $l=k$, and $y'_k - y_k = \delta$. Assume that the coefficients ψ_k satisfy Assumption 2.4 and $\Delta w(\cdot,\cdot,y')\in L^2(D)$. Then $w(\cdot,\cdot,y)-w(\cdot,\cdot,y')$ satisfies the equation

$$\begin{aligned}\frac{d^2}{dt^2}&(w(\cdot,\cdot,y')-w(\cdot,\cdot,y)) - \nabla\cdot(a(x,y)\nabla(w(\cdot,\cdot,y')-w(\cdot,\cdot,y)))\\ &= \nabla\cdot((a(x,y')-a(x,y))\nabla w(\cdot,\cdot,y')) + f(y')-f(y)\ ,\end{aligned} \tag{3.3}$$

with the initial condition

$$w(0,\cdot,y')-w(0,\cdot,y)=f_1(y')-f_1(y) \ \text{ and } \ \frac{d}{dt}(w(0,\cdot,y')-w(0,\cdot,y))=f_2(y')-f_2(y)\ .$$

We deduce from this identity that

$$\|w(\cdot,\cdot,y')-w(\cdot,\cdot,y)\|_{\mathcal{X}} \le c\delta\left(\|\nabla\psi_k\|_{L^\infty(D)}\|\nabla w(\cdot,\cdot,y')\|_V + \|\psi_k\|_{L^\infty(D)}\|\Delta w(\cdot,\cdot,y')\|_{L^2(D)}\right)$$
$$+c\left(\|f(y')-f(y)\|_{L^2(I;H)} + \|f_1(y')-f_1(y)\|_V + \|f_2(y')-f_2(y)\|_H\right).$$

For y, y' as above, let

$$\bar{w} = \frac{1}{\delta}(w(\cdot,\cdot,y')-w(\cdot,\cdot,y))$$

and let $\eta(t,\cdot)\in V$ denote the solution of the problem

$$\frac{d^2}{dt^2}\eta - \nabla\cdot(a(x,y)\nabla\eta) = \nabla\cdot(\psi_k\nabla w(\cdot,\cdot,y)) + \partial_{y_k}f(y), \tag{3.4}$$

with the initial condition

$$\eta(0,\cdot,y) = \partial_{y_k}f_1(y) \ \text{ and } \ \frac{d}{dt}\eta(0,\cdot,y) = \partial_{y_k}f_2(y).$$

By superposition,

$$\frac{d^2}{dt^2}(\bar{w}-\eta) - \nabla\cdot(a(x,y)\nabla(\bar{w}-\eta)) = \nabla\cdot(\psi_k\nabla(w(\cdot,\cdot,y')-w(\cdot,\cdot,y))) + \frac{1}{\delta}(f(y')-f(y)) - \partial_{y_k}f(y), \tag{3.5}$$

with the initial conditions

$$(\bar{w}-\eta)(0,\cdot) = \frac{1}{\delta}(f_1(y')-f_1(y)) - \partial_{y_k}f_1(y), \quad (\bar{w}_t-\eta_t)(0,\cdot) = \frac{1}{\delta}(f_2(y')-f_2(y)) - \partial_{y_k}f_2(y).$$

From Assumption (3.2), when $\delta\to 0$,

$$\|\Delta(w(\cdot,\cdot,y')-w(\cdot,\cdot,y))\|_{L^2(I;H)} \to 0.$$

Together with

$$\left\|\frac{f(\cdot,\cdot,y')-f(\cdot,\cdot,y)}{\delta} - \partial_{y_k}f(\cdot,y)\right\|_{L^2(I;H)} \to 0,$$

$$\left\|\frac{f_1(\cdot,y')-f_1(\cdot,y)}{\delta} - \partial_{y_k}f_1(\cdot,y)\right\|_V \to 0,$$

and

$$\left\|\frac{f_2(\cdot,y')-f_2(\cdot,y)}{\delta} - \partial_{y_k}f_2(\cdot,y)\right\|_H \to 0,$$

when $\delta\to 0$, we get

$$\lim_{\delta\to 0}\|w-\eta\|_{\mathcal{X}} = 0, \text{ i.e } \eta = \partial_{y_k}w.$$

□

Remark 3.2 *The spatial regularity which we assumed in Lemma 3.1 can not be essentially weakened, as is easily seen from the Cauchy problem of the wave equation* (1.8) *in* $D=\mathbb{R}$*, with positive coefficient* $a(y)$ *that is independent of of* x*. Then, for each* $y\in U$*, for* $g=g_2=0$ *in* (1.8)*, by d'Alembert's formula it holds that*

$$u(t,x,y) = \frac{1}{2}[g_1(x-c(y)t) + g_1(x+c(y)t)]$$

where $c(y)=\sqrt{a(y)}>0$ *denotes the (constant) signal propagation speed. Evidently, regularity of* $g_1(x)$ and *of* $a(y)$ *is necessary for smooth dependence of* $u(\cdot,\cdot,y)$ *on* y*. Similar arguments show that also smoothness of* g_2 *and of* g *in* (1.8) *is necessary for smooth parameter dependence of* u *on* $y\in U$*.*

Remark 3.3 *In order for* (3.2) *to hold, we need regularity for* (3.3)*. This requires compatibility conditions for the initial conditions.*

Consider the particular case of equation (2.19)*, this requires further regularity of* g_1*, and for non constant* g_1 *further regularity for* $a(x,y)$ *than Assumption 2.4. In order to be able to differentiate further, we need similar regularity for* $\partial_{y_k}v$*. Therefore, we restrict our consideration to a subclass of problems as specified in the following assumption.*

Assumption 3.4 *We assume that the initial conditions g_1 and g_2 are constant ($g_1 = g_2 = 0$ in the case of Dirichlet problems) and $g \in C^\infty([0,T]; H)$, i.e. g is infinitely differentiable as a map from I to H. Furthermore, $\frac{d^l g}{dt^l}(0,\cdot) \in V$ is independent of x for all $l \in \mathbb{N}$.*

We use Assumption 3.4 to prove that $\partial_y^\nu v$ exists for all $\nu \in \mathcal{F}$. To this end, we first establish the regularity of v with respect to x and t.

Lemma 3.5 *For all integers l, $v \in H^l(I;V)$ and $\Delta v \in H^l(I;H)$. Further $\frac{d^l v}{dt^l}$ satisfies the problem*

$$\frac{d^2}{dt^2}\frac{d^l v}{dt^l} - \nabla\cdot(a(x,y)\nabla\frac{d^l v}{dt^l}) = \frac{d^{l+1}g}{dt^{l+1}}, \quad \frac{d^l v}{dt^l}(0,\cdot) = \frac{d^{l-1}g}{dt^{l-1}}(0,\cdot), \quad \textit{and} \quad \frac{d^{l+1}v}{dt^{l+1}}(0,\cdot) = \frac{d^l g}{dt^l}(0,\cdot)\,. \tag{3.6}$$

Proof We show this by induction. The function v satisfies the problem

$$\frac{d^2 v}{dt^2} - \nabla\cdot(a(x,y)\nabla v) = \frac{dg}{dt}, \quad v(0,\cdot) = g_2, \quad \frac{dv}{dt}(0,\cdot) = g(0,\cdot)\,.$$

Therefore, since the initial conditions are compatible,

$$\frac{d^2}{dt^2}\frac{dv}{dt} - \nabla\cdot(a(x,y)\nabla\frac{dv}{dt}) = \frac{d^2 g}{dt^2}, \quad \frac{dv}{dt}(0,\cdot) = g(0,\cdot), \quad \frac{d^2 v}{dt^2}(0,\cdot) = \frac{dg}{dt}(0,\cdot)\,.$$

The conclusion therefore holds for $l = 1$. Assume that the conclusion holds for l. We note that the compatibility conditions of the wave equations for $d^{l+1}v/dt^{l+1}$ hold as $d^l g/dt^l(0,\cdot)$ is in V, and

$$\nabla\cdot\left(a(x,y)\nabla\frac{d^{l-1}g}{dt^{l-1}}(0,\cdot)\right) + \frac{d^{l+1}g}{dt^{l+1}}(0,\cdot) = \frac{d^{l+1}g}{dt^{l+1}}(0,\cdot)$$

is independent of x and is in H. Therefore $d^l v/dt^l \in \mathcal{X}$ is the unique solution of the problem (3.6) for all l. As $d^{l+2}v/dt^{l+2} \in H$ for all l,

$$-\Delta\frac{d^l v}{dt^l} = \frac{1}{a(x,y)}\Big(\frac{d^{l+1}g}{dt^{l+1}} - \frac{d^{l+2}v}{dt^{l+2}} + \nabla a(x,y)\cdot\nabla\frac{d^l v}{dt^l}\Big),$$

is in H as $a(x,y) \geq a_{\min}$. The Lemma is thus proved. □

Proposition 3.6 *For all $\nu \in \mathcal{F}$, $\partial_y^\nu v$ exists as a derivative of the map v from U to $\mathcal{X} \cap L^2(I;W)$. Further for all integers l, the functions $\partial_y^\nu v \in H^l(I;V)$, $\Delta\partial_y^\nu v \in H^l(I;H)$ and $\partial_y^\nu v$ are the unique solution of the problem*

$$\frac{d^2}{dt^2}\frac{d^l}{dt^l}\partial_y^\nu v - \nabla\cdot(a(x,y)\nabla\frac{d^l}{dt^l}\partial_y^\nu v) = \sum_j \nu_j\left[\nabla\psi_j\nabla\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v + \psi_j\Delta\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v\right], \tag{3.7}$$

with homogeneous initial conditions.

Proof We prove this proposition by induction. First, we consider the case $|\nu| = 1$. Choose $w = v$ in (3.1). We have for any $y, y' \in U$

$$\frac{d^2}{dt^2}(v(\cdot,\cdot,y') - v(\cdot,\cdot,y)) - \nabla\cdot(a(x,y)\nabla(v(\cdot,\cdot,y') - v(\cdot,\cdot,y)) = \nabla\cdot((a(x,y') - a(x,y))\nabla v(\cdot,\cdot,y'))\,, \tag{3.8}$$

with homogeneous initial conditions. As $v(\cdot,\cdot,y') \in H^1(I;W)$, the right hand side is in $H^1(I;H)$. The compatibility conditions therefore hold. Differentiating both sides with respect to t, we get

$$\frac{d^2}{dt^2}\frac{d}{dt}(v(\cdot,\cdot,y') - v(\cdot,\cdot,y)) - \nabla\cdot(a(x,y)\nabla\frac{d}{dt}(v(\cdot,\cdot,y') - v(\cdot,\cdot,y)) = \nabla\cdot((a(x,y') - a(x,y))\nabla\frac{d}{dt}v(\cdot,\cdot,y'))\,, \tag{3.9}$$

with zero initial conditions. We therefore conclude that

$$\left\|\frac{d^2}{dt^2}(v(\cdot,\cdot,y')-v(\cdot,\cdot,y))\right\|_{L^2(I;H)} \le c\left(\|a(\cdot,y')-a(\cdot,y)\|_{L^\infty(D)}+\|\nabla a(\cdot,y')-\nabla a(\cdot,y)\|_{L^\infty(D)}\right),$$

which converges to 0 when $\delta \to 0$. From this and (3.8), we infer that condition (3.2) holds. Therefore we obtain that the partial derivative $\partial_{y_k} v$ exists and is the solution of the initial boundary value problem

$$\frac{d^2\partial_{y_k}v}{dt^2}-\nabla\cdot(a(x,y)\nabla\partial_{y_k}v)=\nabla\psi_k\nabla v+\psi_k\Delta v \tag{3.10}$$

with zero initial conditions.

As the right hand side is in $H^l(I;H)$ for all $l\in\mathbb{N}$, equation (3.7) for $|\nu|=1$ is shown from (3.10) in the same manner as in the proof of Lemma 3.5. This equation can also be derived by inserting $w=\dfrac{d^lu}{dt^l}$ in (3.1). Similarly, by differentiating (3.9) with respect to t, we have

$$\left\|\frac{d^3}{dt^3}(v(\cdot,\cdot,y')-v(\cdot,\cdot,y))\right\|_{L^2(I;H)}\to 0 \text{ as } \delta\to 0,$$

so

$$\left\|\Delta\frac{d}{dt}(v(\cdot,\cdot,y')-v(\cdot,\cdot,y))\right\|_{L^2(I;H)}\to 0 \text{ as } \delta\to 0.$$

As the initial conditions of (3.5) are homogeneous, differentiating both sides of (3.5) with respect to t (where $w=v$), we show that

$$\left\|\frac{d^2}{dt^2}(\bar{w}-\eta)\right\|_{L^2(I;H)}\to 0 \text{ as } \delta\to 0.$$

From this we deduce $\Delta(\bar{w}-\eta)\to 0$ in $L^2(I;H)$ as $\delta\to 0$. Therefore we obtain differentiability of v also when it is regarded as a function from U to $L^2(I;W)$.

For $|\nu|>1$, we prove the assertion by induction with respect to $|\nu|$. To this end, let $e_k\in\mathcal{F}$ denote the multi-index with all components being zero, except the kth component which equals 1. The induction hypothesis gives

$$\frac{d^2}{dt^2}\partial_y^{\nu-e_k}v-\nabla\cdot(a(x,y)\nabla\partial_y^{\nu-e_k}v)=\sum_j(\nu-e_k)_j\left[\nabla\psi_j\nabla\partial_y^{\nu-e_k-e_j}v+\psi_j\Delta\partial_y^{\nu-e_k-e_j}v\right]$$

with homogeneous initial conditions. We consider this equation in the place of (3.1) where $w=\partial_y^{\nu-e_k}v$ As by assumption the right hand side is differentiable as a map from U to $H^1(I;H)$, differentiating both sides of (3.3) with respect to t, we deduce that

$$\left\|\frac{d^2}{dt^2}(\partial_y^{\nu-e_k}v(\cdot,\cdot,y')-\partial_y^{\nu-e_k}v(\cdot,\cdot,y))\right\|_{L^2(I;H)}\to 0$$

when $\delta\to 0$. Therefore the condition (3.2) for $w=\partial_y^{\nu-e_k}v$ holds. We can thus differentiate $\partial_y^{\nu-e_k}v$ with respect to y_k and conclude as in the case $|\nu|=1$. □

3.2 p-Summability of $\|v_\nu\|_{\mathcal{X}}$

To establish $\ell^p(\mathcal{F})$ summability of $\|v_\nu\|_{\mathcal{X}}$, we consider smooth and compatible data g with quantitative bounds on the smoothness as follows.

Assumption 3.7 *There are constants $c_0>0$, $\mathfrak{d}\ge 1$ and $\delta>0$ such that*

$$\forall l\in\mathbb{N}: \quad \left\|\frac{d^lg}{dt^l}\right\|_{L^2([0,T];H)}+\left\|\frac{d^lg}{dt^l}(0)\right\|_H\le c_0\mathfrak{d}^{l-1}((l-1)!)^\delta.$$

We observe that the case $\delta = 1$ corresponds to function g which are analytic functions of t whereas Assumption 3.7 with $\delta > 1$ corresponds to g belonging to the so-called Gevrey class $\mathcal{G}^\delta$.

We will first establish energy bounds for the solution of the hyperbolic problem (1.1)

Proposition 3.8 *Consider the equation* (1.1) *on a time interval* $I = [0, T]$. *If* $u \in H^1(I; V)$, *then there is a positive contant* α_0 *such that*

$$\left\| \frac{du}{dt} \right\|^2_{L^2(I;H)} + \|u\|^2_{L^2(I;V)} \le \alpha_0^2 (1 + T^2)^2 (\|g\|^2_{L^2(I;H)} + \|g_1\|^2_V + \|g_2\|^2_H). \tag{3.11}$$

Assume further that for $\frac{dg}{dt} \in L^2(I; H)$ *and*

$$\frac{d^2}{dt^2}\frac{du}{dt} - \nabla\cdot(a(x,y)\nabla\frac{du}{dt}) = \frac{dg}{dt}$$

with compatible initial conditions

$$\frac{du}{dt}(0,\cdot) = g_2 \in V \quad \text{and} \quad \frac{d^2u}{dt^2}(0,\cdot) = g(0,\cdot) + \nabla\cdot(a(\cdot,y)\nabla g_1) \in H.$$

Then there exists a constant $\beta_0 > 0$ *such that*

$$\|\Delta u\|_{L^2(I;H)} \le \frac{1}{a_{\min}}\|g\|_{L^2(I;H)} + \beta_0(1+T^2)\Bigg(\|g\|_{L^2(I;H)} + \left\|\frac{dg}{dt}\right\|_{L^2(I;H)} + \|g_1\|_V + \|g_2\|_H + \left\|\frac{du}{dt}(0)\right\|_V + \left\|\frac{d^2u}{dt^2}(0)\right\|_H \Bigg). \tag{3.12}$$

We will prove this proposition in the Appendix.

We now establish estimates for the solution v of problem (2.19). Let

$$\alpha = \alpha_0(1+T^2)(1 + 1/\mathfrak{d} + 1/\mathfrak{d}^2), \tag{3.13}$$

and

$$\beta = \frac{1}{a_{\min}} + \beta_0(1+T^2)(2 + \mathfrak{d} + 2/\mathfrak{d} + 1/\mathfrak{d}^2), \tag{3.14}$$

where the constant $\mathfrak{d}$ is as in Assumption 3.7.

Lemma 3.9 *With Assumption 3.7, we have for* $\delta \ge 1$

$$\left\|\frac{d^l v}{dt^l}\right\|_{\mathcal{X}} \le c_0\alpha\mathfrak{d}^l(l!)^\delta, \quad \text{and} \quad \left\|\Delta\frac{d^l v}{dt^l}\right\|_{L^2(I;H)} \le c_0\beta\mathfrak{d}^l((l+1)!)^\delta.$$

Proof We deduce from (3.6) and (3.11) that

$$\left\|\frac{d^l v}{dt^l}\right\|_{\mathcal{X}} \le \alpha_0(1+T^2)\Bigg(\left\|\frac{d^{l+1}g}{dt^{l+1}}\right\|_{L^2(I;H)} + \left\|\frac{d^{l-1}g}{dt^{l-1}}(\cdot,0)\right\|_V + \left\|\frac{d^l g}{dt^l}(\cdot,0)\right\|_H\Bigg),$$

which implies

$$\left\|\frac{d^l v}{dt^l}\right\|_{\mathcal{X}} \le c_0\alpha_0(1+T^2)(\mathfrak{d}^l + \mathfrak{d}^{l-1} + \mathfrak{d}^{l-2})(l!)^\delta = c_0\alpha\mathfrak{d}^l(l!)^\delta.$$

We also have from (3.12)

$$\left\|\Delta\frac{d^l v}{dt^l}\right\|_{L^2(I;H)} \le \frac{1}{a_{\min}}\left\|\frac{d^{l+1}g}{dt^{l+1}}\right\|_{L^2(I;H)} + \beta_0(1+T^2)\Bigg(\left\|\frac{d^{l+1}g}{dt^{l+1}}\right\|_{L^2(I;H)} + \left\|\frac{d^{l+2}g}{dt^{l+2}}\right\|_{L^2(I;H)} + \left\|\frac{d^l v}{dt^l}(0)\right\|_V + \left\|\frac{d^{l+1}v}{dt^{l+1}}(0)\right\|_H + \left\|\frac{d^{l+1}v}{dt^{l+1}}(0)\right\|_V + \left\|\frac{d^{l+2}v}{dt^{l+2}}(0)\right\|_H\Bigg).$$

Thus

$$\left\|\Delta\frac{d^l v}{dt^l}\right\|_{L^2(I;H)} \leq \left(\frac{1}{a_{\min}}c_0\mathfrak{d}^l + c_0\beta_0(1+T^2)\Big(\mathfrak{d}^l+\mathfrak{d}^{l+1}+\mathfrak{d}^{l-2}+\mathfrak{d}^{l-1}+\mathfrak{d}^{l-1}+\mathfrak{d}^l\Big)\right)((l+1)!)^\delta$$
$$= c_0\beta\mathfrak{d}^l((l+1)!)^\delta.$$

□

We now establish bounds for $\partial_y^\nu v$ inductively. For α and β as in (3.13), (3.14), let

$$b_k = \|\nabla\psi_k\|_{L^\infty(D)}\alpha + \|\psi_k\|_{L^\infty(D)}\beta. \tag{3.15}$$

We have the following result

Proposition 3.10 *Under Assumption 3.7, for every $\nu\in\mathcal{F}$ and every $l\in\mathbb{N}$,*

$$\left\|\frac{d^l}{dt^l}\partial_y^\nu v\right\|_{\mathcal{X}} \leq c_0\alpha|\nu|!b^\nu\mathfrak{d}^l((l+|\nu|)!)^\delta,\ \textit{and}\ \left\|\Delta\frac{d^l}{dt^l}\partial_y^\nu u\right\|_{L^2(I;H)} \leq c_0\beta|\nu|!b^\nu\mathfrak{d}^l((l+|\nu|+1)!)^\delta\ . \tag{3.16}$$

Proof We proceed by induction. When $\nu=0$, from Lemma 3.9, the assertion holds. For $|\nu|>0$, using

$$\left\|\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v\right\|_{\mathcal{X}} \leq c_0\alpha(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l((l+|\nu|-1)!)^\delta,$$

and

$$\left\|\Delta\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v\right\|_{L^2(I;H)} \leq c_0\beta(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l((l+|\nu|)!)^\delta,$$

we have from (3.7) and (3.11), using (3.15) that

$$\left\|\frac{d^l}{dt^l}\partial_y^\nu v\right\|_{\mathcal{X}} \leq c_0\alpha_0(1+T^2)\sum_j\nu_j\Big(\|\nabla\psi_j\|_{L^\infty(D)}\alpha(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l +$$
$$\|\psi_j\|_{L^\infty(D)}\beta(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l\Big)((l+|\nu|)!)^\delta$$
$$< c_0\alpha\sum_j\nu_j(\|\nabla\psi_j\|_{L^\infty(D)}\alpha+\|\psi_j\|_{L^\infty(D)}\beta)(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l((l+|\nu|)!)^\delta$$
$$= c_0\alpha|\nu|!b^\nu\mathfrak{d}^l((l+|\nu|)!)^\delta.$$

Furthermore, from (3.7) and (3.12)

$$\left\|\Delta\frac{d^l}{dt^l}\partial_y^\nu v\right\|_{L^2(I;H)} \leq \frac{1}{a_{\min}}\sum_j\nu_j\left\|\nabla\psi_j\nabla\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v+\psi_j\Delta\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v\right\|_{L^2(I;H)}$$
$$+\beta_0(1+T^2)\left(\sum_j\nu_j\left\|\nabla\psi_j\nabla\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v+\psi_j\Delta\frac{d^l}{dt^l}\partial_y^{\nu-e_j}v\right\|_{L^2(I;H)}\right.$$
$$\left.+\sum_j\nu_j\left\|\nabla\psi_j\nabla\frac{d^{l+1}}{dt^{l+1}}\partial_y^{\nu-e_j}v+\psi_j\Delta\frac{d^{l+1}}{dt^{l+1}}\partial_y^{\nu-e_j}v\right\|_{L^2(I;H)}\right)$$
$$\leq \frac{1}{a_{\min}}c_0\sum_j\nu_j(\|\nabla\psi_j\|_{L^\infty(D)}\alpha+\|\psi_j\|_{L^\infty(D)}\beta)(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l((l+|\nu|)!)^\delta$$
$$+c_0\beta_0(1+T^2)\left(\sum_j\nu_j(\|\nabla\psi_j\|_{L^\infty(D)}\alpha+\|\psi_j\|_{L^\infty(D)}\beta)(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^l((l+|\nu|)!)^\delta\right.$$
$$\left.+\sum_j\nu_j(\|\nabla\psi_j\|_{L^\infty(D)}\alpha+\|\psi_j\|_{L^\infty(D)}\beta)(|\nu|-1)!b^{\nu-e_j}\mathfrak{d}^{l+1}((l+|\nu|+1)!)^\delta\right)$$
$$= c_0\frac{1}{a_{\min}}|\nu|!b^\nu\mathfrak{d}^l((l+|\nu|)!)^\delta + c_0\beta_0(1+T^2)|\nu|!b^\nu\mathfrak{d}^l(1+\mathfrak{d})((l+|\nu|+1)!)^\delta$$
$$< c_0\beta|\nu|!b^\nu\mathfrak{d}^l((l+|\nu|+1)!)^\delta.$$

□

When $l = 0$, we get

$$\|\partial_y^\nu v\|_{\mathcal{X}} \le c_0 \alpha (|\nu|!)^{1+\delta} b^\nu.$$

We now consider the Legendre expansion of v:

$$v = \sum_{\nu \in \mathcal{F}} v_\nu L_\nu(y).$$

To establish p-summability of $\|v_\nu\|_{\mathcal{X}}$, we employ the following result.

Lemma 3.11 *For $s_j \in \{1, 2, \ldots, t\}$ ($j = 1, \ldots, m$),*

$$(s_1 + \ldots + s_m)! \le t^{tm} 1^{s_1} 2^{s_2} \ldots m^{s_m}.$$

Proof We prove by induction. When $m = 1$: $s_1! \le t! < t^t$. Assume that the assertion holds for m. We have

$$(s_1+\ldots+s_m+1)\ldots(s_1+\ldots+s_m+s_{m+1}) \le (tm+1)\ldots(tm+s_{m+1})) \le t^{s_{m+1}}(m+1)^{s_{m+1}} \le t^t(m+1)^{s_{m+1}}.$$

Therefore

$$(s_1 + \ldots + s_{m+1})! \le t^{tm} 1^{s_1} 2^{s_2} \ldots m^{s_m} t^t (m+1)^{s_{m+1}} = t^{t(m+1)} 1^{s_1} 2^{s_2} \ldots (m+1)^{s_{m+1}} .$$

□

To establish p-summability for $\|v_\nu\|_{\mathcal{X}}$, we make the following assumption.

Assumption 3.12 *We assume that for a value $p \in (0, 1)$*

$$\sum_{k=1}^\infty k^{p(\delta+1)} (\|\psi_k\|^p_{L^\infty(D)} + \|\nabla \psi_k\|^p_{L^\infty(D)}) < \infty.$$

We then have the following summability result for $\|v_\nu\|_{\mathcal{X}}$.

Proposition 3.13 *Under Assumptions 3.7 and 3.12,* $\Big(\|v_\nu\|_{\mathcal{X}}\Big)_{\nu \in \mathcal{F}} \in \ell^p(\mathcal{F})$.

Proof First, we consider the case where $1/2 < p < 1$. Let $S = \{i_1, \ldots, i_m\} \subset \mathbb{N}$. We consider the differential operator

$$\mathcal{L}_S = (-1)^m \frac{\partial}{\partial y_{i_1}} \left((1 - y_{i_1}^2) \frac{\partial}{\partial y_{i_1}} \right) \ldots \frac{\partial}{\partial y_{i_m}} \left((1 - y_{i_m}^2) \frac{\partial}{\partial y_{i_m}} \right).$$

We have

$$\|\mathcal{L}_S v\|_{\mathcal{X}} \le 2^m \sum_{s_j = 1,2} \|\partial^{s_1}_{y_{i_1}} \ldots \partial^{s_m}_{y_{i_m}} v\|_{\mathcal{X}} \le 2^m c_0 \alpha \sum_{s_j=1,2} ((|s_1 + \ldots + s_m|)!)^{\delta+1} b^{s_1}_{i_1} \ldots b^{s_m}_{i_m}.$$

We then deduce from Lemma 3.11

$$\begin{aligned}
\|\mathcal{L}_S v\|_{\mathcal{X}} &\le 2^{m(2\delta+3)} c_0 \alpha \sum_{s_j=1,2} 1^{s_1(\delta+1)} \ldots m^{s_m(\delta+1)} b^{s_1}_{i_1} \ldots b^{s_m}_{i_m} \\
&\le 2^{m(2\delta+3)} c_0 \alpha \prod_{j=1}^m (j^{\delta+1} b_{i_j} + j^{2(\delta+1)} b^2_{i_j}) \\
&\le 2^{m(2\delta+3)} c_0 \alpha \prod_{j=1}^m (i_j^{\delta+1} b_{i_j} + i_j^{2(\delta+1)} b^2_{i_j}).
\end{aligned}$$

We note that

$$\mathcal{L}_S L_\nu(y) = \left(\prod_{j=1}^m \nu_{i_j} (\nu_{i_j} + 1) \right) L_\nu(y).$$

Because $\mathcal{L}_S$ is selfadjoint, we have

$$\left(\prod_{j=1}^{m} \nu_{i_j}(\nu_{i_j}+1)\right) v_\nu = \int_U v\mathcal{L}_S L_\nu(y)d\rho(y) = \int_U \mathcal{L}_S v L_\nu(y)d\rho(y).$$

We therefore deduce that

$$\sum_{\nu\in\mathcal{F}} \left(\prod_{j=1}^{m} \nu_{i_j}(\nu_{i_j}+1)\right)^2 \|v_\nu\|_{\mathcal{X}}^2 \le \left(2^{m(2\delta+3)} c_0\alpha \prod_{j=1}^{m} (i_j^{\delta+1} b_{i_j} + i_j^{2(\delta+1)} b_{i_j}^2)\right)^2.$$

From this, we deduce that:

$$\|v_\nu\|_{\mathcal{X}} \le \left(2^{m(2\delta+3)} c_0\alpha \prod_{j=1}^{m} (i_j^{\delta+1} b_{i_j} + i_j^{2(\delta+1)} b_{i_j}^2)\right) \frac{1}{\nu_{i_1}^2 \ldots \nu_{i_m}^2}.$$

Therefore, for $p > 1/2$:

$$\begin{aligned}
\sum_{\mathrm{supp}(\nu)=S} \|v_\nu\|_{\mathcal{X}}^p &\le \left((c_0\alpha)^p 2^{m(2\delta+3)p} \prod_{j=1}^{m} (i_j^{\delta+1} b_{i_j} + i_j^{2(\delta+1)} b_{i_j}^2)^p\right) \sum_{\mathrm{supp}(\nu)=S} \frac{1}{\nu_{i_1}^{2p} \ldots \nu_{i_m}^{2p}} \\
&= (c_0\alpha)^p 2^{m(2\delta+3)p} M^m \prod_{j=1}^{m} (i_j^{\delta+1} b_{i_j} + i_j^{2(\delta+1)} b_{i_j}^2)^p,
\end{aligned}$$

where $M = \sum_{k=1}^\infty k^{-2p} < \infty$ since $p > 1/2$. Thus

$$\begin{aligned}
\sum_{\nu\in\mathcal{F}} \|v_\nu\|_{\mathcal{X}}^p &\le (c_0\alpha)^p \sum_{i_1,\ldots,i_m=1}^{\infty} \left(2^{m(2\delta+3)p} M^m \prod_{j=1}^{m} (i_j^{\delta+1} b_{i_j} + i_j^{2(\delta+1)} b_{i_j}^2)^p\right) \\
&= (c_0\alpha)^p \prod_{k=1}^{\infty} \left(1 + 2^{(2\delta+3)p} M (k^{\delta+1} b_k + k^{2(\delta+1)} b_k^2)^p\right) \\
&\le (c_0\alpha)^p \exp\left(\sum_{k=1}^{\infty} 2^{(2\delta+3)p} M (k^{\delta+1} b_k + k^{2(\delta+1)} b_k^2)^p\right),
\end{aligned}$$

which is finite when $\left(k^{\delta+1} b_k\right)_k \in \ell^p(\mathbb{N})$.

For $p \le 1/2$, we get the same conclusion by applying the operator $\mathcal{L}_S^r$ where r is the smallest integer greater than $1/(2p)$. □

Example 3.14 *Consider the forcing function $g(t) = e^{-1/t^q}$ where $q > 0$ which belongs to $\mathcal{G}^\delta$ for all $\delta \ge 1 + 1/q$ (see [3]). In this case, Assumption 3.12 holds, e.g., when $\|\psi_k\|_{L^\infty(D)}$ and $\|\nabla\psi_k\|_{L^\infty(D)}$ decay faster than $k^{-(2+1/p+1/q)}$.*

3.3 Best N-term convergence rate

To deduce the rate of convergence for the best N terms u_ν in the expansion (2.2), we need the following lemma

Lemma 3.15 *(Stechkin) Let $\alpha = (\alpha_\nu)_{\nu\in\mathcal{F}}$ be a sequence in $\ell^p(\mathcal{F})$. Let $q \ge p > 0$. If $\Lambda_N \in \mathcal{F}$ is a set of indices corresponding to a set of N largest $|\alpha_\nu|$, we have the estimate*

$$(\sum_{\nu\notin\Lambda_N} |\alpha_\nu|^q)^{1/q} \le \|\alpha\|_{\ell^p(\mathcal{F})} N^{-\sigma}, \quad \textit{where} \;\; \sigma = \frac{1}{p} - \frac{1}{q}.$$

We can now state and prove our main result.

Theorem 3.16 *If Assumptions 1.1, 1.3, 2.4, 3.4 and 3.7 hold, there exists a sequence* $(\Lambda_N) \subset \mathcal{F}$ *of index sets with cardinality not exceeding* N *such that the solutions* u_{Λ_N} *of the Galerkin semidiscretized problem (2.3) satisfy*

$$\|u - u_{\Lambda_N}\|_{\underline{\mathcal{X}}} \le CN^{-\sigma}, \quad \sigma = \frac{1}{p} - \frac{1}{2}.$$

Proof As Assumptions 1.1 and 3.4 hold, from Proposition 3.13 we obtain that $u \in L^2(U, \rho; \mathcal{Z})$. This implies that the semidiscrete Galerkin projection onto $\underline{\mathcal{X}}_\Lambda$ is quasioptimal for any finite index set $\Lambda \subset \mathcal{F}$ by Proposition 2.2. Let Λ_N be the subset of $\mathcal{F}$ corresponding to the largest N terms u_ν in the expansion (2.2) according to their $\mathcal{Z}$ norm. The conclusion follows from Lemma 3.15. □

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Appendix

We prove Proposition 3.8. Assuming that $u \in H^1(I; V)$ and letting the test function in the variational form be du/dt, we have

$$\int_D \left\langle \frac{d^2u}{dt^2}, \frac{du}{dt} \right\rangle_H dx + \int_D a(x,y)\nabla u \cdot \nabla \frac{du}{dt} dx = \int_D g \frac{du}{dt},$$

so

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{du}{dt}(t)\right\|_H^2 + \frac{1}{2}\frac{d}{dt}\int_D a(x,y)|\nabla u(t,x,y)|^2 dx \le \|g(t)\|_H \left\|\frac{du}{dt}(t)\right\|_H$$

for all $t \in (0,T)$. Thus

$$\left\|\frac{du}{dt}(t)\right\|_H^2 + a_{\min}\|\nabla u(t)\|_H^2 \le 2\|g\|_{L^2(I;H)} \left\|\frac{du}{dt}\right\|_{L^2(I;H)} + \|g_2\|_H^2 + a_{\max}\|g_1\|_V^2. \tag{3.17}$$

Therefore:

$$\left\|\frac{du}{dt}\right\|^2_{L^2(I;H)} + a_{\min}\|\nabla u\|^2_{L^2(I;H)} \le 2T\|g\|_{L^2(I;H)}\left\|\frac{du}{dt}\right\|_{L^2(I;H)} + T\|g_2\|^2_H + Ta_{\max}\|g_1\|^2_V$$
$$\le 4T^2\|g\|^2_{L^2(I;H)} + \frac{1}{4}\left\|\frac{du}{dt}\right\|^2_{L^2(I;H)} + T\|g_2\|^2_H + Ta_{\max}\|g_1\|^2_V.$$

We note that

$$\|u(t)\|^2_H = \left\|\int_0^t \frac{du}{dt}(\tau)d\tau + g_1\right\|^2_H \le 2\left(\int_0^T \left\|\frac{du}{dt}(t)\right\|_H dt\right)^2 + 2\|g_1\|^2_H$$
$$\le 2T\int_0^T \left\|\frac{du}{dt}(t)\right\|^2_H dt + 2\|g_1\|^2_H \le 2T\left\|\frac{du}{dt}\right\|^2_{L^2(I;H)} + 2\|g_1\|^2_H.$$

Thus

$$\|u\|^2_{L^2(I;H)} \le 2T^2\left\|\frac{du}{dt}\right\|^2_{L^2(I;H)} + 2T\|g_1\|^2_H.$$

Therefore, there is a constant $\alpha_0 > 0$ such that

$$\left\|\frac{du}{dt}\right\|^2_{L^2(I;H)} + \|u\|^2_{L^2(I;V)} \le \alpha_0^2(1+T^2)^2(\|g\|^2_{L^2(I;H)} + \|g_2\|^2_H + \|g_1\|^2_V). \tag{3.18}$$

Assuming that $dg/dt \in L^2(I;H)$ and du/dt satisfies

$$\frac{d^2}{dt^2}\frac{du}{dt} - \nabla(a(x,y)\nabla\frac{du}{dt}) = \frac{dg}{dt},$$

with compatible initial conditions

$$\frac{du}{dt}(0,\cdot) = g_2, \quad \text{and} \quad \frac{d^2u}{dt^2}(0) = g(0,\cdot) - \nabla\cdot(a(\cdot,y)\nabla g_1).$$

For all t, we have

$$-a(x,y)\Delta u = g - \frac{d^2u}{dt^2} + \nabla a\cdot\nabla u.$$

Therefore for all $t \in I$:

$$a_{\min}\|\Delta u(t)\|_{L^2(I;H)} \le \|g(t)\|_{L^2(I;H)} + \left\|\frac{d^2u}{dt^2}(t)\right\|_{L^2(I;H)} + \|\nabla a\|_{L^\infty(D)}\|u\|_{L^2(I;V)}$$
$$\le \|g(t)\|_{L^2(I;H)} + \alpha_0(1+T^2)\left(\left\|\frac{dg}{dt}\right\|_{L^2(I;H)} + \left\|\frac{du}{dt}(0)\right\|_V + \left\|\frac{d^2u}{dt^2}(0)\right\|_H\right) +$$
$$\|\nabla a\|_{L^\infty(D)}\alpha_0(1+T^2)\Big(\|g\|_{L^2(I;H)} + \|g_1\|_V + \|g_2\|_H).$$

Thus there is a positive constant β_0 such that

$$\|\Delta u\|_{L^2(I;H)} \le \frac{1}{a_{\min}}\|g\|_{L^2(I;H)} + \beta_0(1+T^2)\Bigg(\|g\|_{L^2(I;H)} + \left\|\frac{dg}{dt}\right\|_{L^2(I;H)} +$$
$$\|g_1\|_V + \|g_2\|_H + \left\|\frac{du}{dt}(0)\right\|_V + \left\|\frac{d^2u}{dt^2}(0)\right\|_H\Bigg). \tag{3.19}$$

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