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Asymptotics for Helmholtz and Maxwell solutions in 3-D open waveguides

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Abstract

We extend Sommerfeld and Silver-Müller radiation conditions to acoustic and electromagnetic fields propagating over three isotropic layers in $\mathbb{R}^3$. In the outer layers, classical conditions only hold for waves propagating in the region $|x_3| > r^\gamma$ with $\gamma \in \left(\frac{1}{4}, \frac{1}{2}\right)$. For $|x_3| < r^\gamma$ and inside the slab, asymptotic behaviors depend on the presence of surface or guided modes given by the discrete spectrum of the associated operator.

1 Introduction

Although layered structures in optics and acoustics have been long studied [19], [10], the persistent interest from both engineering and scientific communities comes from the continuous improvement in manufacturing techniques for optical integrated circuits [24]. Today, layered optical waveguides take part in a plethora of applications ranging from basic light guidance [22], to more complex devices such as strip-geometry semiconductor lasers [23], [2], and photonic crystal structures [21], [15].

In its simplest form, a waveguide is made by three layers of isotropic media. The middle one or *core* possesses a finite thickness and a different dielectric coefficient compared to the surrounding layers, also referred to as *cladding*. We thus speak of *open waveguides* in opposition to *closed waveguides*, in which a metallic enclosure contains the radiation from propagating outside the core. Under certain conditions, these structures are capable of guiding radiation inside the slab while outside energy decays exponentially. These modes do not form a complete eigenfunction set in which the field from an arbitrary source can be expanded. Thus, radiative modes linked to the continuum spectrum must be included to deliver an entire description. However, guided and radiative parts possess different behaviors at infinity [3, 18]. This prevents accurate descriptions by standard numerical methods and theoretically, existence and uniqueness results have for long remained open problems. In recent works, uniqueness of solutions for the Helmholtz equation for 2-D waveguides with small perturbations was achieved [8, 9] and a similar result is obtained in [5] via a generalized Fourier transform when one of the outer layers is replaced by a Dirichlet condition.

In this work, we present rigourous asymptotics for outgoing acoustic and Maxwell waves in time harmonic regime in $\mathbb{R}^3$ using the *limiting absorption principle*. This constitutes a milestone towards a general existence result for open waveguides and uniqueness proofs in the fashion of [11]. On the application side, these precise characterizations allows for the development of new *ad hoc* numerical techniques and improvement of PMLs and similar techniques.

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1.1 Problem Setting

Let $h \in \mathbb{R}_+$ be bounded and and introduce the intervals:

$$I_1 := (h, +\infty) \ , \qquad I_2 := (0, h) \ , \qquad I_3 := (-\infty, h). \tag{1}$$

We consider the following three-layer decomposition of $\mathbb{R}^3$:

$$\Omega_1 := \{\mathbf{x} \in \mathbb{R}^3 \ : \ x_3 \in I_1\} \ , \qquad \Omega_2 := \{\mathbf{x} \in \mathbb{R}^3 \ : \ x_3 \in I_2\} \ , \qquad \Omega_3 := \{\mathbf{x} \in \mathbb{R}^3 \ : \ x_3 \in I_3\},$$

with interfaces $\Gamma_0 = \overline{\Omega}_2 \cap \overline{\Omega}_3$ and $\Gamma_h = \overline{\Omega}_2 \cap \overline{\Omega}_1$, and define for simplicity $\Omega := \bigcup_i \Omega_i$. Each domain $\{\Omega_i\}_{i=1}^3$ is characterised by different parameters according to the physics considered. In the case of linear electromagnetism, permittivity and permeability coefficients are given by values in vacuum, ϵ_0 and μ_0, correspondingly multiplied by relative ones $\epsilon_i, \mu_i \in L^\infty(\Omega_i)$ both positive. Hence, inside Ω_i, the light speed c_i equals $c_0/\sqrt{\epsilon_i \mu_i}$ where $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. In the acoustic case, real positive and bounded constants c_i refer to sound speeds. Parameters $\eta_i \in \mathbb{R}_+$, representing viscosities in the acoustic case or conductivities in the EM one, immediately guarantee the well-posedness of the system, i.e. bounded energy. Nonetheless, we will be mostly interested in the case when they tend to zero and so, we set $\eta_i \equiv \eta$ in all layers.

1.1.1 Time-dependent formulation

Let $\mathcal{U}(\mathbf{x}, t)$ represent either the scalar pressure field, $\mathcal{P}$, or one of the three-dimensional vector fields, $\boldsymbol{\mathcal{E}}$ or $\boldsymbol{\mathcal{H}}$, describing scattered sound or EM waves, respectively. After some rearrangements, the following common time-dependent PDEs must be satisfied:

$$\begin{cases} \left(-c_i^{-2}\partial_t^2 \ - \ \eta\partial_t \ + \ \Delta\right)\mathcal{U}(\mathbf{x}, t) \ = \ \mathsf{F}_\mathcal{U}\mathcal{F}(\mathbf{x}, t) \ , & (\mathbf{x}, t) \in \Omega_i \times \mathbb{R}_+, \quad i = 1, 2, 3, \\ + \text{ physics-dependent transmission conditions}, & (\mathbf{x}, t) \in \Gamma_{0,h} \times \mathbb{R}_+, \\ + \text{ initial conditions}, & (\mathbf{x}, t) \in \Omega_i \times \{0\}, \quad i = 1, 2, 3, \\ + \text{ outgoing behavior}, & \end{cases} \tag{2}$$

where Δ is the Laplacian operator and $\mathcal{F}$ is an excitation according to the physics considered, compactly supported in Ω_2. The partial differential operator $\mathsf{F}_\mathcal{U}$ provides necessary modifications, e.g., in the EM case we set $\boldsymbol{\mathcal{F}}$ as an electric current for which it holds

$$\mathsf{F}_{\boldsymbol{\mathcal{E}}}(\mathbf{x}, t) \ = \ \mu_i\mu_0\partial_t + (\epsilon_i\epsilon_0)^{-1}\,\mathbf{grad}\int_0^t \operatorname{div}(\cdot)d\tau \qquad \text{and} \qquad \mathsf{F}_{\boldsymbol{\mathcal{H}}}(\mathbf{x}, t) \ = \ \mathbf{rot} \tag{3}$$

while for sound scattering $\mathsf{F}_\mathcal{P}$ equals identity.

1.1.2 TE and TM modes. Transmission conditions.

Due to rotational invariance, one decomposes the pair of EM fields $(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{H}})$ with values in $\mathbb{R}^6$ into *transverse electric* (TE), $\mathcal{E}_3 \equiv 0$, and *transverse magnetic* (TM), $\mathcal{H}_3 \equiv 0$, modes which can be entirely characterized by normal components, $\mathcal{H}_3^{TE}$ and $\mathcal{E}_3^{TM}$, respectively. Indeed, for each polarization there are dyadic partial differential operators:

$$\mathsf{E}_\mathcal{U} \ : \ \mathcal{U} \longmapsto \boldsymbol{\mathcal{E}} \qquad \text{and} \qquad \mathsf{H}_\mathcal{U} \ : \ \mathcal{U} \longmapsto \boldsymbol{\mathcal{H}}, \qquad \mathcal{U} = \left\{\mathcal{H}_3^{TE}, \mathcal{E}_3^{TM}\right\}, \tag{4}$$

mapping the corresponding driving normal component into the remaining EM fields components. The dyad form is due to the excitation by vector sources in $\mathbb{R}^3$. Consequently, for each polarization, there are three different scalar sources to be considered, written $\mathsf{F}_U^j\boldsymbol{\mathcal{F}} := (\mathsf{F}_\mathcal{U}\boldsymbol{\mathcal{F}}) \cdot \hat{\mathbf{x}}_j$ with $j = 1, 2, 3$. On the other hand, the normal components $\mathcal{U} = \{\mathcal{H}_3^{TE}, \mathcal{E}_3^{TM}\}$ are solutions of the scalar form of (2) with jump conditions:

$$[\alpha\mathcal{U}] \;=\; 0 \qquad \text{and} \qquad [\partial_3\mathcal{U}] \;=\; 0 \qquad \text{on} \quad \Gamma_{0,h}, \quad \forall t \in \mathbb{R}_+, \tag{5}$$

with α being either μ or ϵ, respectively. In the acoustic case, transmission conditions are given by zero Dirichlet and Neumann jumps, i.e. $\alpha \equiv 1$. Thus, henceforth we focus on the scalar form of (2).

1.1.3 Time harmonic or Helmholtz formulation

By linearity, periodic excitations in time with a pulsation $\omega \in \mathbb{R}_+$, allow solutions of (2) to take the form:

$$\mathcal{U}(\mathbf{x}, t) \;=\; \mathfrak{Re}\left\{U(\mathbf{x})e^{\pm\imath\omega t}\right\} \tag{6}$$

with complex-valued $U(\mathbf{x})$. Once a convention is chosen, and after equating exponential terms out, the time dependence is only portrayed by the sign of the absorption term. Let us choose the minus sign in (6) and accordingly modify (2). Define the real and complex wavenumbers $k_i^2 := (\omega/c_i)^2$ and $k_{i,\eta}^2 := k_i^2 + \imath\omega\eta$.

Hypothesis 1.1. *We will further assume that* $0 < k_3 \leq k_1 < k_2 < +\infty$.

This will ensure the existence of guided modes [14] when η vanishes. We are interested in solving the family of time-harmonic problems for η tending to zero:

$$(P_\eta) \;:=\; \begin{cases} \Delta U_\eta(\mathbf{x}) \;+\; k_{i,\eta}^2 U_\eta(\mathbf{x}) \;=\; 0 & \mathbf{x} \in \Omega_i\,, \quad i = 1, 3 \\ \Delta U_\eta(\mathbf{x}) \;+\; k_{2,\eta}^2 U_\eta \;=\; \mathsf{F}_U(\mathbf{x}, \omega)F(\mathbf{x}) & \mathbf{x} \in \Omega_2 \\ [\alpha U_\eta] \;=\; 0 & \mathbf{x} \in \Gamma_{0,h} \\ [\partial_3 U_\eta] \;=\; 0 & \mathbf{x} \in \Gamma_{0,h}, \\ + \text{outgoing behavior}, \end{cases} \tag{7}$$

where now the field's dependence on η is given by the subscript and $U_\eta \in H^1_{\text{loc}}(\Delta, \Omega)$, the space of local L^2-functions with locally square integrable Laplacians in Ω. Notice that $F(\mathbf{x})$ is complex-valued, compactly supported, and that F_U is either the same operator as before (acoustic) or its projected action along $\hat{\mathbf{x}}_j$ (EM), $j = 1, 2, 3$, with derivatives in t replaced by powers of ω. In general, one can explicitly write

$$\mathsf{F}_U \;=\; \sum_{l,m,n,p=0}^{L,M,N,P} c_{lmnp}^U \omega^p \partial_1^l \partial_2^m \partial_3^n \qquad \text{(introducing indices in EM)} \tag{8}$$

where $c_{lmn}^U \in \mathbb{C}$ are constants associated to the derivatives $\{l, m, n, p\} \in N_0$ of the physics-dependent operator with $\{L, M, N, P\} \in \mathbb{N}_0$ bounded.

As long as $\mathfrak{Im}\{k_{i,\eta}\} > 0$, for $i = 1, 2, 3$, the above problems are well-defined and solutions belong in fact to $H^1(\Delta, \Omega)$. The limit problem $(P_0) := \lim_{\eta\downarrow 0}(P_\eta)$ shows the existence of surface modes and requires radiation conditions to retrieve the outgoing propagation sense in time.

1.2 Main Results: Far-field Asymptotics for Helmholtz and EM solutions

Introduce the following coordinate systems: (1) upper and lower hemispherical (r, θ, ϕ) ones centered at Γ_h for Ω_1 and at Γ_0 for Ω_3, respectively; and, (2) cylindrical ones (ρ, φ, x_3) with $0 < x_3 < h$ in Ω_2. Then, for $\eta = 0$, the following propositions hold

Proposition 1.1. *Assume the existence of M guided modes, with wavenumbers located at circumferences described by $|\boldsymbol{\xi}| = \xi_p^m$, with $\xi_p^m > 0$ for all $m = 1, \ldots M$. Let $\gamma \in (\frac{1}{4}, \frac{1}{2})$. Moreover, let us admit for the limit problem (P_0) the decomposition:*

$$U \;=\; U_g \;+\; U_{rad} \qquad \textit{with} \qquad U_g \;=\; \sum_{m=1}^{M} \alpha_m U_p^m\,, \qquad \alpha_m\,, \tag{9}$$

where U_{rad} and U_g are radiative and guided parts, the latter composed of allowed modes U_p^m. Then, it holds

$$\begin{cases} \left|\frac{\partial U}{\partial r} - \imath k_i U\right| \;=\; \mathcal{O}\left(r^{-\left(2\gamma+\frac{1}{2}\right)}\right) & \textit{for} \quad \mathbf{x}\in\Omega_i,\ i=1,3, \quad |x_3| > r^\gamma, \\ \left|\frac{\partial U}{\partial r} - \imath\sum_{m=1}^M \alpha_m \xi_p^m U_p^m\right| \;=\; \mathcal{O}\left(r^{-\left(\frac{3}{2}-\gamma\right)}\right) & \textit{for} \quad \mathbf{x}\in\Omega_i,\ i=1,3, \quad |x_3| < r^\gamma, \\ \left|\frac{\partial U}{\partial \rho} - \imath\sum_{m=1}^M \alpha_m \xi_p^m U_p^m\right| \;=\; \mathcal{O}\left(\rho^{-\frac{3}{2}}\right) & \textit{for} \quad \mathbf{x}\in\Omega_2. \end{cases} \tag{10}$$

Proposition 1.2 (Silver-Müller-type conditions)**.** *Define the impedances:*

$$\mathsf{z}_{i,r} \;:=\; (\mu_i/\epsilon_i)^{1/2} \qquad \textit{and} \qquad \mathsf{z}_{i,\rho}^m \;:=\; \xi_p^m/(\omega\epsilon_i) \;=\; \mathsf{z}_{i,r}\,\xi_p^m/k_i$$

If excited by an electrical current, outgoing transverse magnetic fields satisfy the following conditions:

$$\begin{cases} \left|\mathbf{H}^i \;+\; \mathsf{z}_{i,r}^{-1}\mathbf{E}\wedge\mathbf{n}\right| \;=\; \mathcal{O}\left(r^{-\left(2\gamma+\frac{1}{2}\right)}\right) & \textit{for} \quad \mathbf{x}\in\Omega_i,\ i=1,3, \quad |x_3| > r^\gamma \\ \left|\mathbf{H} \;+\; \sum_{m=1}^M \alpha_m \left(\mathsf{z}_{i,\rho}^m\right)^{-1}\mathbf{E}_p^m\wedge\mathbf{n}\right| \;=\; \mathcal{O}\left(r^{-\left(\frac{3}{2}-\gamma\right)}\right) & \textit{for} \quad \mathbf{x}\in\Omega_i,\ i=1,3, \quad |x_3| < r^\gamma \\ \left|\mathbf{H} \;+\; \sum_{m=1}^M \alpha_m \left(\mathsf{z}_{2,\rho}^m\right)^{-1}\mathbf{E}_p^m\wedge\mathbf{n}\right| \;=\; \mathcal{O}\left(\rho^{-\frac{3}{2}}\right) & \textit{for} \quad \mathbf{x}\in\Omega_2 \end{cases} \tag{11}$$

where $\mathbf{E}_p^m$ are the associated electric guided modes described in Proposition 1.1, $\mathbf{n} = \mathbf{x}/r$ in $\mathbf{x}\in\Omega_i$, $i = 1, 3$, and $\mathbf{n} = \boldsymbol{\rho}/\rho$ in Ω_2. Similar conditions for transverse electric modes hold by reversing the roles of $\mathbf{H}$ and $\mathbf{E}$.

2 Radiation Conditions Derivation

In order to prove the above results, we study the associated Green's functions, g_η, as one can recover fields U for arbitrary but compactly supported F by convolution, i.e. $U_\eta = g_\eta * (\mathsf{F}_U F)$. Hence, the far-field behavior is indeed the one given by g_η, whose derivation constitutes most of this work. For this, we first obtain explicit surface spectral forms by applying the polar Fourier transform. This yields a system of ordinary differential equations in x_3 as shown in Section 2.1.2 whose solution is given in Proposition 2.1. With this, in Section 2.3, we carry out the asymptotic analysis of the inverse surface Fourier transform when η goes to zero.

2.1 Surface spectral Green's functions and dyads

Let us replace the source $F(\mathbf{x})$ with a scalar (acoustic) or directional (EM) delta Dirac distribution at $\mathbf{x}-\mathbf{y}$, with $\mathbf{x}\in\mathbb{R}^3$ and $\mathbf{y}\in\Omega_2$, to derive the associated spatial Green's functions $g_\eta(\mathbf{x},\mathbf{y})$ or dyads $\mathbf{g}_\eta(\mathbf{x},\mathbf{y})$ to problem (P_η). Since layer parameters α_i are piecewise constant, the functions depend on $(\mathbf{x},\mathbf{y})$ only through their difference [13] and, by translational invariance, we can set $y_1 = x_1$ and $y_2 = x_2$ so that the only source parameter is $y_3 \in I_2$.

2.1.1 Surface Fourier transform

Let $\mathbf{x}' = (x_1, x_2)$ and $\boldsymbol{\xi} = (\xi_1, \xi_2)$. Let $\varphi \in S(\mathbb{R}^2_{\mathbf{x}'}\times\mathbb{R})$ where S denotes the Schwarz space, then its *surface Fourier transform*, $\mathcal{F}$, denoted $\widehat{\varphi}\in S(\mathbb{R}^2_{\boldsymbol{\xi}}\times\mathbb{R})$, is

$$\widehat{\varphi}(\boldsymbol{\xi}, x_3) \;=\; (\mathcal{F}\varphi)(\boldsymbol{\xi}, x_3) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\varphi(\mathbf{x}', x_3)e^{\imath\boldsymbol{\xi}\cdot\mathbf{x}'}\,d\mathbf{x}' \tag{12}$$

with inverse

$$\left(\mathcal{F}^{-1}\widehat{\varphi}\right)(\mathbf{x}', x_3) \;=\; \varphi(\mathbf{x}', x_3) \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2}\widehat{\varphi}(\boldsymbol{\xi}, x_3)e^{-\imath\boldsymbol{\xi}\cdot\mathbf{x}'}\,d\boldsymbol{\xi}. \tag{13}$$

If u lies in $S'(\mathbb{R}^2_{\mathbf{x}'} \times \mathbb{R})$, the space of tempered distributions, its partial Fourier transform, $\widehat{u} \in S'(\mathbb{R}^2_{\boldsymbol{\xi}} \times \mathbb{R})$, is obtained by duality. Now, for an open interval $I \subset \mathbb{R}$, we can define the space of distributions partially tempered over $\mathbb{R}^2_{\mathbf{x}'} \times I$ as follows:

$$S'(\mathbb{R}^2_{\mathbf{x}'} \times I) \ := \ \left\{ u \ \in \ \mathcal{D}'(\mathbb{R}^2_{\mathbf{x}'} \times I) \ : \ \forall \, \psi \ \in \ C_0^\infty(I) \ , \ \psi(x_3) \, u \ \in \ S'(\mathbb{R}^2_{\mathbf{x}'} \times \mathbb{R}) \right\}$$

and all the above definitions apply [6]. The next transforms will be extensively used

$$\text{Dirac delta:} \qquad \widehat{\delta}(\boldsymbol{\xi}, x_3) \ = \ \frac{1}{2\pi} \delta(x_3) \otimes \mathbf{1}_{\boldsymbol{\xi}}; \tag{14}$$

$$\text{derivation:} \qquad \widehat{\partial_1^m \partial_2^n \partial_3^l u}(\boldsymbol{\xi}, x_3) \ = \ (-\imath \xi_1)^m \, (-\imath \xi_2)^n \, \partial_3^l \, \widehat{u}(\boldsymbol{\xi}, x_3). \tag{15}$$

Lastly, it is convenient to express the surface Fourier transform in polar coordinates, defined as $\xi_1 = \xi \cos\phi$, $\xi_2 = \xi \sin\phi$, describing the (ξ_1, ξ_2)-plane for $\xi \in [0, \infty)$ and $\phi \in (0, 2\pi)$. Hence, the inverse transform can be written as

$$f(\mathbf{x}) \ = \ \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty \widehat{f}(\boldsymbol{\xi}, x_3) e^{-\imath \xi \, \mathbf{t}(\phi) \cdot \mathbf{x}'} \, \xi d\xi d\phi \tag{16}$$

where the shorthand $\mathbf{t}(\phi) \ = \ \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$ has been used, i.e. $\boldsymbol{\xi} = \xi \, \mathbf{t}(\phi)$.

2.1.2 Spectral problem formulation

Define restrictions of the Green's function over each layer $g^i_\eta := g_\eta|_{\Omega_i}$. Application of the surface Fourier transform in polar coordinates to (7) leads to the following systems of ODEs in x_3: find $\widehat{g}^i_\eta \in S'(\mathbb{R}^2_{\boldsymbol{\xi}} \times I_i \times I_2)$, for $i = 1, 2, 3$, such that for $\boldsymbol{\xi} \in \mathbb{R}^2$ and $y_3 \in I_3$ it holds

$$(\widehat{P}_\eta) := \begin{cases} \partial_3^2 \widehat{g}^i_\eta \ - \ \chi^2_{i,\eta} \widehat{g}^i_\eta \ = \ 0, & x_3 \ \in \ I_i, \quad i = 1, 3 \\ \partial_3^2 \widehat{g}^2_\eta \ - \ \chi^2_{2,\eta} \widehat{g}^2_\eta \ = \ \widehat{\mathsf{F}_U \delta}, & x_3 \ \in \ I_2, \\ [\alpha \widehat{g}_\eta] \ = \ 0, & x_3 = 0, h, \\ [\partial_3 \widehat{g}_\eta] \ = \ 0, & x_3 = 0, h, \\ + \text{ decay conditions} & |x_3| \ \longrightarrow \ +\infty. \end{cases} \tag{17}$$

where we have set $\chi^2_{i,\eta}(\xi) := \xi^2 - k^2_{i,\eta}$ for which if $\eta = 0$, we will simply write χ_i. It holds

$$\widehat{\mathsf{F}_U \delta}(\boldsymbol{\xi}, x_3, y_3) \ = \ \sum_{n=0}^{N} \widehat{\beta}^U_n(\boldsymbol{\xi}) \delta^{(n)}(x_3 - y_3) \tag{18}$$

wherein

$$\widehat{\beta}^U_n(\boldsymbol{\xi}) \ := \ \frac{1}{2\pi} \sum_{l,m,p=0}^{L,M,P} c^U_{lmnp} \omega^p (-\imath \xi)^{l+m} \, t_1^l(\phi) \, t_2^m(\phi). \tag{19}$$

For the moment we only focus on (17), though we keep in mind the different sources for EM possessing the same form of (18).

2.1.3 Solutions of homogeneous equations

Solutions of the homogeneous ordinary differential equation in $(\widehat{P}_\eta)$ take the form:

$$\widehat{g}^i_\eta(\boldsymbol{\xi}, x_3, y_3) \ = \ K^i_{1,\eta}(\boldsymbol{\xi}, y_3) \, e^{-(x_3 - y_3)\chi_{i,\eta}} \ + \ K^i_{2,\eta}(\boldsymbol{\xi}, y_3) \, e^{(x_3 - y_3)\chi_{i,\eta}}, \qquad i = 1, 3, \tag{20}$$

where the distributions $K^i_{j,\eta} \in S'(\mathbb{R}^2_{\boldsymbol{\xi}} \times I_2)$, $j = 1, 2$, are obtained by imposing boundary and decay conditions, as shown briefly. However, we must establish an interpretation of $\chi_{i,\eta}$ as square-roots in the complex plane.

2.1.4 Square root determination

Let $z \in \mathbb{C}$ such that $\mathfrak{Re}\{z\} = \xi$ and assume $\eta > 0$. We set the square root over the complex plane

$$\chi_{i,\eta} \;:\; z \longmapsto \sqrt{z^2 \,-\, k_{i,\eta}^2}\,, \qquad i = 1,2,3,$$

as the product between $\sqrt{z \,-\, k_{i,\eta}}$ and $\sqrt{z \,+\, k_{i,\eta}}$, defined over over $\mathbb{C}$ minus the non-negative and non-positive imaginary axis, respectively. That is,

$$\arg(z \,-\, k_{i,\eta}) \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right) \qquad \text{and} \qquad \arg(z \,+\, k_{i,\eta}) \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

Remark 2.1. Set $\eta = 0$. Then, if $\mathfrak{Im}\{z\} = 0$ and $\mathfrak{Re}\{z\} = \xi \in \mathbb{R}$, it holds $\arg(\xi - k_i) \in \{-\pi, 0\}$ and $\arg(\xi + k_i) \in \{0, \pi\}$ and thus, χ_i takes either real or purely complex values. The latter occurs if $|\xi| < k_i$ since the term $\xi + k_i$ has an argument equal to zero for all $\xi > -k_i$ and $\xi - k_i = (k_i \,-\, \xi)e^{-\imath\pi}$ so that $\sqrt{\xi - k_i} = -\imath\sqrt{k_i - \xi}$ and, in fact, $\sqrt{\xi^2 - k_i^2} = -\imath\sqrt{k_i^2 - \xi^2}$.

2.1.5 Solutions for the inhomogeneous equation in Ω_2

Since $\widehat{\mathsf{F}_U\delta}$ can be written as a sum (18), by linearity, we can set

$$\widehat{g}_\eta^2 \;=\; \sum_{n=0}^{N} \widehat{g}_\eta^{2,n} \tag{21}$$

wherein each $\widehat{g}_\eta^{2,n} \in S'(\mathbb{R}^2_{\boldsymbol{\xi}} \times I_2 \times I_2)$ is a distributional solution of the problem:

$$\partial_3^2 \widehat{g}_\eta^{2,n}(\boldsymbol{\xi}, x_3, y_3) \;-\; \chi_{2,\eta}^2 \widehat{g}_\eta^{2,n}(\boldsymbol{\xi}, x_3, y_3) \;=\; \widehat{\beta}_n^U(\boldsymbol{\xi})\delta^{(n)}(x_3 - y_3) \tag{22}$$

with boundary conditions at $x = \{0, h\}$ as in (17). Due to the punctual support of the exciting terms, we can introduce an artificial layer at $x_3 = y_3$ and split the interval I_2 into $I_{2+} := (y_3, h)$ and $I_{2-} := (0, y_3)$. This induces a decomposition of the spatial domain Ω_2 into

$$\Omega_2^+ \;=\; \{\mathbf{x} \in \Omega_2 \;:\; x_3 \in I_{2+}\} \qquad \text{and} \qquad \Omega_2^- \;=\; \{\mathbf{x} \in \Omega_2 \;:\; x_3 \in I_{2-}\}.$$

Hence, in each $\Omega_2^\pm$ only homogeneous equations must be satisfied and, consequently, the corresponding spectral solutions, denoted $\widehat{g}^{2+,n}$ and $\widehat{g}^{2-,n}$, have the form (20) with coefficients $K_{j,\eta}^{2\pm,n}$. For a given n, we interrelate $K_{j,\eta}^{2\pm,n}$ by imposing jump conditions at $x_3 = y_3$ originated by the source term in (22). For this, let us formally introduce the integral operators:

$$\left(\mathcal{T}_n^\pm u\right)(x_3) \;:=\; \int_{\pm\infty}^{x_3} \cdots \int_{\pm\infty}^{t_3} \int_{\pm\infty}^{t_2} u'(t_1)\, dt_1 dt_2 \cdots dt_n \tag{23}$$

where integration is carried out n times with $\mathcal{T}_1^\pm$ being the identity operator and $\mathcal{T}_0^\pm$ differentiation in x_3. We define the combined operator $\mathcal{T}_n$ acting over $g_\eta^{2\pm,n}$ along x_3:

$$\left(\mathcal{T}_n g_\eta^{2\pm,n}\right)(x_3) \;:=\; K_{1,\eta}^{2\pm,n}\left(\mathcal{T}_n^+ e^{-(\cdot - y_3)\chi_{2,\eta}}\right)(x_3) \;+\; K_{2,\eta}^{2\pm,n}\left(\mathcal{T}_n^- e^{(\cdot - y_3)\chi_{2,\eta}}\right)(x_3) \tag{24}$$

and write jump conditions at $x_3 = y_3$ for all $\boldsymbol{\xi} \in \mathbb{R}^2$:

$$\left[\mathcal{T}_n g_\eta^{2,n}\right]_{x_3 = y_3} \;=\; \left(\mathcal{T}_n g_\eta^{2+,n} - \mathcal{T}_n g_\eta^{2-,n}\right)\Big|_{x_3 = y_3} \;=\; \widehat{\beta}_n^U \qquad \text{and} \qquad \left[\mathcal{T}_{n+1} g_\eta^{2,n}\right]_{x_3 = y_3} = 0.$$

Notice that the classic Neumann condition is retrieved when $n = 0$. Operators $\mathcal{T}_n^{\pm}$ act over exponential terms as

$$\left(\mathcal{T}_n^{\pm} e^{\mp(\cdot - y_3)\chi_{2,\eta}}\right)(x_3) \;=\; (\mp 1)^{n-1}\chi_{2,\eta}^{1-n} e^{\mp(x_3-y_3)\chi_{2,\eta}} \tag{25}$$

so that

$$\left(\mathcal{T}_n g_\eta^{2\pm,n}\right)(x_3) \;=\; \chi_{2,\eta}^{1-n}\left\{K_{1,\eta}^{2\pm,n}(-1)^{n-1}\, e^{-(x_3-y_3)\chi_{2,\eta}} \;+\; K_{2,\eta}^{2\pm,n}\, e^{(x_3-y_3)\chi_{2,\eta}}\right\}. \tag{26}$$

One can now directly compute the jumps:

$$\begin{aligned}
\left[\mathcal{T}_n g_\eta^{2,n}\right]_{x_3=y_3} \;&=\; (-1)^{n-1}\chi_{2,\eta}^{1-n}\left\{K_{1,\eta}^{2+,n} - K_{1,\eta}^{2-,n}\right\} \;+\; \chi_{2,\eta}^{1-n}\left\{K_{2,\eta}^{2+,n} - K_{2,\eta}^{2-,n}\right\},\\
\left[\mathcal{T}_{n+1} g_\eta^{2,n}\right]_{x_3=y_3} \;&=\; (-1)^{n}\chi_{2,\eta}^{-n}\left\{K_{1,\eta}^{2+,n} - K_{1,\eta}^{2-,n}\right\} \;+\; \chi_{2,\eta}^{-n}\left\{K_{2,\eta}^{2+,n} - K_{2,\eta}^{2-,n}\right\},
\end{aligned}$$

and obtain two equations for the four unknowns $K_{j,\eta}^{2\pm,n}$:

$$K_{1,\eta}^{2+,n} - K_{1,\eta}^{2-,n} \;=\; (-1)^{n-1}\frac{1}{2}\chi_{2,\eta}^{n-1}\widehat{\beta}_n^U \qquad \text{and} \qquad K_{2,\eta}^{2+,n} - K_{2,\eta}^{2-,n} \;=\; \frac{1}{2}\chi_{2,\eta}^{n-1}\widehat{\beta}_n^U; \tag{28}$$

the missing relations coming from transmission conditions at $x = \{0, h\}$. One can regroup each individual term $K_{j,\eta}^{2\pm,n}$ into $K_{j,\eta}^{2\pm}$ by adding in n Eqs. (28) as follows

$$K_{1,\eta}^{2+} - K_{1,\eta}^{2-} \;=\; \frac{1}{2}\sum_{n=0}^{N}(-1)^{n-1}\chi_{2,\eta}^{n-1}\widehat{\beta}_n^U \;=:\; \widehat{L}_{1,\eta}^U, \tag{29}$$

$$K_{2,\eta}^{2+} - K_{2,\eta}^{2-} \;=\; \frac{1}{2}\sum_{n=0}^{N}\chi_{2,\eta}^{n-1}\widehat{\beta}_n^U \;=:\; \widehat{L}_{2,\eta}^U, \tag{30}$$

where we have defined the right-hand side source variables $\widehat{L}_{j,\eta}^U$ for convenience. Even and odd components $\widehat{\Upsilon}_{e,\eta}^U$, $\widehat{\Upsilon}_{o,\eta}^U$ can also be introduced:

$$\widehat{\Upsilon}_{o,\eta}^U \;:=\; \sum_{p=0}^{\lfloor\frac{N-1}{2}\rfloor}\chi_{2,\eta}^{2p}\widehat{\beta}_{2p+1}^U, \qquad \text{and} \qquad \widehat{\Upsilon}_{e,\eta}^U \;:=\; \sum_{p=0}^{\lceil\frac{N-1}{2}\rceil}\chi_{2,\eta}^{2p}\widehat{\beta}_{2p}^U. \tag{31a}$$

Observe that the functions $\widehat{\Upsilon}_{o,e}^U$ are polynomial with respect to ξ and $\mathbf{t}(\phi)$ and their relation to $\widehat{L}_{j,\eta}^U$ is as follows

$$\widehat{L}_{1,\eta}^U \;=\; \frac{1}{2}\left(\widehat{\Upsilon}_{o,\eta}^U - \chi_{2,\eta}^{-1}\widehat{\Upsilon}_{e,\eta}^U\right) \qquad \text{and} \qquad \widehat{L}_{2,\eta}^U \;=\; \frac{1}{2}\left(\widehat{\Upsilon}_{o,\eta}^U + \chi_{2,\eta}^{-1}\widehat{\Upsilon}_{e,\eta}^U\right) \tag{32}$$

2.2 Spectral solution

Denote by R_{ij}^η the complex *Fresnel reflection coefficient* for a wave in Ω_i reflected by region Ω_j:

$$R_{ij}^\eta \;:=\; \frac{\alpha_j\chi_{i,\eta} - \alpha_i\chi_{j,\eta}}{\alpha_i\chi_{j,\eta} \,+\, \alpha_j\chi_{i,\eta}} \tag{33}$$

dependent on $\boldsymbol{\xi} \in \mathbb{R}^2$ and η. Observe that $R_{ji}^\eta = -R_{ij}^\eta$. We also define a complex *transmission coefficient*, T_{ij}^η, for the wave transmitted into the jth layer coming from the ith one, defined as

$$T_{ij}^\eta \;:=\; 1 \,+\, R_{ij}^\eta \;=\; \frac{2\alpha_j\chi_{i,\eta}}{\alpha_i\chi_{j,\eta} \,+\, \alpha_j\chi_{i,\eta}} \tag{34}$$

Finally, we introduce the following complex-valued surface spectral function:

$$\mathsf{Det}_\eta \;:=\; R^\eta_{21} R^\eta_{23}\, \exp\left(-2h\chi_{2,\eta}\right) \,-\, 1 \tag{35}$$

which has a physical sense explained later on.

Proposition 2.1. *If* Det_η *is non-zero, the solution to the spectral problem* $(\widehat{P}_\eta)$ *is*

$$\widehat{g}^1_\eta \;=\; \frac{\alpha_2}{\alpha_1}\frac{T^\eta_{21}}{\mathsf{Det}_\eta}\, e^{-(h-y_3)\chi_{2,\eta}} \left\{-\widehat{L}^U_{1,\eta} \,+\, R^\eta_{23}\, e^{-2y_3\chi_{2,\eta}} \widehat{L}^U_{2,\eta}\right]\, e^{-(x_3-h)\chi_{1,\eta}} \qquad x_3 \in I_1, \tag{36a}$$

$$\widehat{g}^{2+}_\eta \;=\; \frac{1}{\mathsf{Det}_\eta} \left[-\widehat{L}^U_{1,\eta} e^{y_3\chi_{2,\eta}} \,+\, R^\eta_{23}\, e^{-y_3\chi_{2,\eta}} \widehat{L}^U_{2,\eta}\right] \left[e^{-x_3\chi_{2,\eta}} \,+\, R^\eta_{21}\, e^{-2h\chi_{2,\eta}}\, e^{x_3\chi_{2,\eta}}\right] \qquad x_3 \in I_{2+}, \tag{36b}$$

$$\widehat{g}^{2-}_\eta \;=\; \frac{1}{\mathsf{Det}_\eta} \left[\widehat{L}^U_{2,\eta}\, e^{-y_3\chi_{2,\eta}} - R^\eta_{21}\, e^{-(2h-y_3)\chi_{2,\eta}} \widehat{L}^U_{1,\eta}\right] \left[R^\eta_{23} e^{-x_3\chi_{2,\eta}} \,+\, e^{x_3\chi_{2,\eta}}\right] \qquad x_3 \in I_{2-}, \tag{36c}$$

$$\widehat{g}^3_\eta \;=\; \frac{\alpha_2}{\alpha_3}\frac{T^\eta_{23}}{\mathsf{Det}_\eta} \left[\widehat{L}^U_{2,\eta}\, e^{-y_3\chi_{2,\eta}} \,-\, R^\eta_{21}\, e^{-(2h-y_3)\chi_{2,\eta}} \widehat{L}^U_{1,\eta}\right]\, e^{x_3\chi_{3,\eta}} \qquad x_3 \in I_3, \tag{36d}$$

where the dependence on $(\boldsymbol{\xi}, y_3) \in \mathbb{R}^2 \times I_2$ *is implied. Coefficients* $\widehat{L}^U_{1,\eta}$ *and* $\widehat{L}^U_{2,\eta}$ *are defined in (29) and (30), respectively.*

Proof. Based on Sections 2.1.3 and 2.1.5, we can write the solutions for $(\widehat{P}_\eta)$ as follows:

$$\begin{aligned}
\widehat{g}^1_\eta \;&=\; \widehat{K}^1_{1,\eta}\, e^{-(x_3-h)\chi_{1,\eta}} && x_3 \,>\, h\\
\widehat{g}^{2+}_\eta \;&=\; \widehat{K}^{2+}_{1,\eta}\, e^{-(x_3-y_3)\chi_{2,\eta}} \,+\, \widehat{K}^{2+}_{2,\eta}\, e^{(x_3-y_3)\chi_{2,\eta}} && y_3 \,<\, x_3 \,<\, h\\
\widehat{g}^{2-}_\eta \;&=\; \widehat{K}^{2-}_{1,\eta}\, e^{-(x_3-y_3)\chi_{2,\eta}} \,+\, \widehat{K}^{2-}_{2,\eta}\, e^{(x_3-y_3)\chi_{2,\eta}} && 0 \,<\, x_3 \,<\, y_3\\
\widehat{g}^3_\eta \;&=\; \widehat{K}^3_{2,\eta}\, e^{x_3\chi_{3,\eta}} && x_3 \,<\, 0
\end{aligned}$$

as they decay at infinity when $\eta > 0$. The limiting case when η goes to zero and $|\xi|^2 < k_i^2$ will be discussed further below. Imposing jump conditions at $x = \{0, h\}$ and by definition of reflection and transmission coefficients, it holds

$$\widehat{K}^{2+}_{2,\eta} \;=\; R^\eta_{21}\, e^{-2(h-y_3)\chi_{2,\eta}} \widehat{K}^{2+}_{1,\eta} \qquad \text{and} \qquad \widehat{K}^{2-}_{1,\eta} \;=\; R^\eta_{23}\, e^{-2y_3\chi_{2,\eta}} \widehat{K}^{2-}_{2,\eta}$$

Therefore, $\widehat{K}^1_{1,\eta}$ and $\widehat{K}^3_{2,\eta}$ are given in terms of $\widehat{K}^{2+}_{1,\eta}$ and $\widehat{K}^{2-}_{2,\eta}$, respectively, as

$$\widehat{K}^1_{1,\eta} \;=\; \frac{\alpha_2}{\alpha_1} T^\eta_{21}\, e^{-(h-y_3)\chi_{2,\eta}} \widehat{K}^{2+}_{1,\eta} \qquad \text{and} \qquad \widehat{K}^3_{2,\eta} \;=\; \frac{\alpha_2}{\alpha_3} T^\eta_{23}\, e^{-y_3\chi_{2,\eta}} \widehat{K}^{2-}_{2,\eta}$$

where factors α_i/α_j are equal to one in acoustics and thus only show up for EM normal fields –in contrast to tangential ones studied in [7]. We relate coefficients inside the waveguide via (29) and (30) and obtain the linear system:

$$\begin{pmatrix} R^\eta_{21}\, e^{-2(h-y_3)\chi_{2,\eta}} & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & R^\eta_{23}\, e^{-2y_3\chi_{2,\eta}} \end{pmatrix} \begin{pmatrix} \widehat{K}^{2+}_{1,\eta} \\ \widehat{K}^{2+}_{2,\eta} \\ \widehat{K}^{2-}_{1,\eta} \\ \widehat{K}^{2-}_{2,\eta} \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ \widehat{L}^U_{1,\eta} \\ \widehat{L}^U_{2,\eta} \\ 0 \end{pmatrix}$$

whose determinant is equal to Det_η as introduced in (35). Lastly, if Det_η is different from zero, Cramer's rule delivers the stated result. □

Remark 2.2. For the physical cases considered, the reader can easily verify that associated source terms $\widehat{L}^U_{j,\eta}$, for $j = 1, 2$, are equal in modulus, i.e. $|\widehat{L}^U_{j,\eta}| \equiv |\widehat{L}^U_\eta|$. Consequently, one can further simplify (36a)-(36d) into the general form:

$$\widehat{g}^i_\eta(\boldsymbol{\xi}, x_3, y_3) \;=\; \widehat{L}^U_\eta(\boldsymbol{\xi})\frac{\Xi^i_\eta(\xi, y_3)}{\mathsf{Det}_\eta(\xi)} \times \text{ exponential terms in } x_3 \tag{38}$$

where $\xi = |\boldsymbol{\xi}|$. The functions $\Xi^i_\eta(\xi, y_3)$ are built from terms R^η_{ij}, T^η_{ij} and $\chi_{i,\eta}$ which depend solely on the radial spectral coordinate.

2.2.1 Spectral EM dyad transversal terms

The spectral form of the remaining elements in the EM Green's dyads can be readily be found by applying the surface Fourier transform over operators E_U and H_U [14]. We state them without terms in $\delta(x_3 - y_3)$ as these do not contribute to the far-field. Let ε_{ijk} the Levi-Civita tensor [17]. In the case of TM modes, we have an electric field dyad normal component $\widehat{\mathbf{g}}^E_{e,3}$ composed of three scalars corresponding to sources along $j = 1, 2, 3$:

$$\widehat{\mathbf{g}}^E_{h,T} \;=\; \varepsilon_{T3T''}\frac{t_{T''}(\phi)}{\xi}\omega\epsilon\widehat{\mathbf{g}}^E_{e,3} \tag{39a}$$

$$\widehat{\mathbf{g}}^E_{e,T} \;=\; -\frac{1}{\imath\omega\epsilon}\varepsilon_{T3T'}\,\partial_3\widehat{\mathbf{g}}^E_{h,T'} \tag{39b}$$

where $T = 1, 2$ and $T', T'' = 1, 2, 3$. For TE modes, the normal magnetic field spectral component is $\widehat{\mathbf{g}}^H_{h,3}$:

$$\widehat{\mathbf{g}}^H_{e,T} \;=\; -\varepsilon_{T3T''}\frac{\omega\mu}{\xi}t_{T''}(\phi)\widehat{\mathbf{g}}^H_{h,3} \tag{40a}$$

$$\widehat{\mathbf{g}}^H_{h,T} \;=\; \frac{1}{\imath\omega\mu}\varepsilon_{T3T'}\,\partial_3\widehat{\mathbf{g}}^H_{e,T'} \tag{40b}$$

2.3 Asymptotic analysis for vanishing absorption

We now present asymptotics of the inverse Fourier transforms of the surface spectral Green's functions obtained in Section 2.1 when η goes to zero:

$$g^i_\eta(\mathbf{x}', x_3, y_3) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \widehat{g}^i_\eta(\boldsymbol{\xi}, x_3, y_3)e^{-\imath\xi\mathbf{t}(\phi)\cdot\mathbf{x}'}\xi d\xi d\phi \tag{41}$$

with $\mathbf{x}' = (x_1, x_2)$ for $\mathbf{y} = (0, 0, y_3)$ and where $\widehat{g}^i_\eta$ has the form (38). For this, we rewrite the integrals (41) in the standard form:

$$g^i_\eta(\lambda, \cdot, \cdot, y_3) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty \Psi^i_\eta(\xi, \phi, y_3)e^{\lambda\Phi^i_\eta(\xi,\phi,\cdot,\cdot)}\xi d\xi d\phi \tag{42}$$

where one considers λ as the large parameter. This last one depends on the choice of coordinate system. The term Φ^i_η denotes the associated phase, obtained by multiplying exponential terms coming from $\widehat{g}_\eta$ and the Fourier transform exponential. The remaining terms form part of the amplitude function Ψ^i_η. As mentioned before, the subscripts η disappear when considering $\eta \downarrow 0$. Notice that, when $\eta = 0$, the terms χ_i are even in ξ, the functions Ξ^i also are.

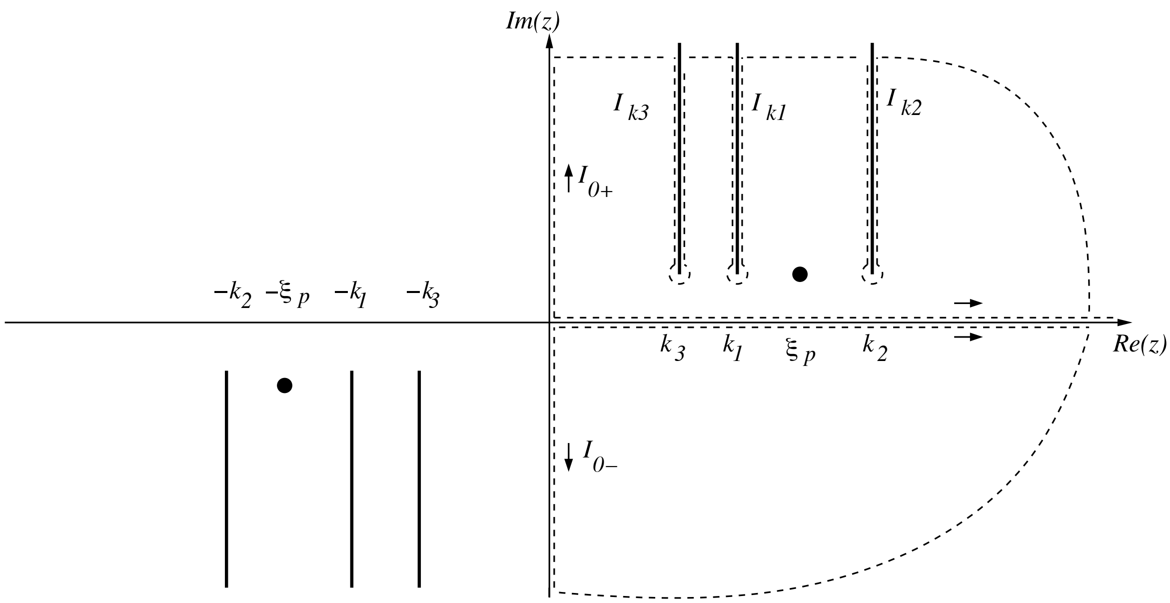

Figure 1: Complex paths for integrals in z at a fixed spectral angle ϕ (Initial form).

2.3.1 General procedure

Recall Hypothesis 1.1 and choose a slab height h that allows only two conjugate poles to exist, denoted $\pm\xi_p \in \mathbb{R}$. In order to apply the limiting absorption principle set $\eta > 0$. By doing so, the real poles $|\xi_p|$ are displaced in the imaginary axis as $\xi_{p,\eta} := \pm(\xi_p + \imath\eta)$ [12]. Hence, the positive (negative) pole lies on the upper (lower) half-plane of the complex plane $\mathbb{C}_+$ ($\mathbb{C}_-$). Then, letting η go to zero, asymptotics are obtained as a sum of contributions coming from:

1. Stationary phase points $\boldsymbol{\xi}_s^i$ in Φ^i, with behaviors denoted $\mathbb{I}^i_{\boldsymbol{\xi}_s^i}$ and derived via the *stationary phase method* [20];

2. for fixed integration angle ϕ, we regard

$$J^i(\lambda,\cdot,\cdot,y_3,\phi) \sim \int_0^\infty \Psi^i_\eta(\xi,\phi,y_3)e^{\lambda\Phi^i_\eta(\xi,\phi,\cdot,\cdot)}\xi d\xi, \quad \lambda\to+\infty, \quad \eta\downarrow 0. \tag{43}$$

   by replacing ξ with the complex variable z and using the *residue theorem* [1] for the complex contours shown in Fig. 1 for $\mathfrak{Re}\{z\} \geq 0$. Thus, we define analytic continuations for Ψ^i_η, Φ^i_η which contain the square-root terms $\chi_{i,\eta}$ as defined in Section 2.1.4. Following the *steepest descent method* [4], we list all possible critical complex (real) points z_c (ξ_c) associated to the integral in z:

   - surface mode or pole contributions located at $z_c = \pm\xi_{p,\eta}$, given by the complex residue;
   - branch points located at $z_c = \pm k_{i,\eta}$ for $i = 1,2,3$;
   - the integration end-points at $z_c = 0 \pm \imath\eta$.

   After taking the limit $\eta\downarrow 0$, these last results are finally integrated with respect to ϕ, and added up to obtain

$$\mathbb{I}^i = \mathbb{I}^i_{\boldsymbol{\xi}^s} + \sum_{\xi_c}\mathbb{I}^i_{\xi_c} \qquad \text{with} \qquad \mathbb{I}^i_{\xi_c}(\lambda,\cdot,\cdot,y_3) \sim \frac{1}{2\pi}\int_0^{2\pi} J^i_{\xi_c}(\lambda,\cdot,\cdot,y_3,\phi)d\phi \tag{44}$$

2.4 Forms in Ω_i for $i = 1, 3$

Introduce hemi-spherical coordinates with origin at $(0, 0, h)$ for Ω_1 and at $(0, 0, 0)$ for Ω_3. This is, for $r > 0$, $\varphi \in (0, 2\pi)$, and $\theta \in (0, \frac{\pi}{2})$ in Ω_1 or $\theta \in (\frac{\pi}{2}, \pi)$ in Ω_3, we have the equivalences: $x_1 = r \sin\theta \cos\varphi$, $x_2 = r \sin\theta \sin\varphi$, and $x_3 - h = \cos\theta$ for Ω_1 or $x_3 = \cos\theta$ for Ω_3. Then, the amplitude and phase in (42) are given by

$$\Psi^i_\eta(\xi, \phi, y_3) \;:=\; \widehat{L}^U_\eta(\xi, \phi) \frac{\Xi^i_\eta(\xi, y_3)}{\mathsf{Det}_\eta(\xi)}, \tag{45a}$$

$$\Phi^i_\eta(\xi, \phi, \theta, \varphi) \;:=\; -\left|\cos\theta\right| \chi_{i,\eta}(\xi) - \imath\xi \sin\theta \cos(\phi - \varphi)\,. \tag{45b}$$

By the form of the phase it is clear that both stationary points and branch points occur.

2.4.1 Stationary point contribution

We multiply the integrand by a cut-off function $\vartheta \in \mathcal{D}(\mathbb{R}^2)$ such that ϑ is equal to one on a neighborhood of the stationary point $\boldsymbol{\xi}^i_s$ and zero elsewhere. This leaves only the contribution from the stationary point. Let $B_k(\boldsymbol{\xi}^i_s) \subset \mathbb{R}^2$ denote the ball centered at the saddle point of radius $k \in \mathbb{R}_+$. We change variables and calculate, for $\eta \equiv 0$,

$$\mathbb{I}^i_{\boldsymbol{\xi}^i_s}(r, \theta, \varphi, y_3) \;\sim\; \frac{1}{2\pi} \int_{B_k(\boldsymbol{\xi}_s)} \Psi^i(\xi_1, \xi_2, y_3)\, e^{\imath r \tilde{\Phi}^i(\xi_1, \xi_2, \theta, \varphi)} d\xi_1 d\xi_2$$

wherein we have followed Remark 2.1 to modify the phase (45b) by defining

$$\tilde{\Phi}^i(\xi_1, \xi_2, \theta, \varphi) \;:=\; |\cos\theta| \sqrt{k_i^2 - \xi^2} - \xi_1 \sin\theta\cos\varphi \;-\; \xi_2 \sin\theta \sin\varphi.$$

The only stationary point is

$$\boldsymbol{\xi}^i_s \;=\; (\xi^i_s, \xi^i_s) \;=\; (-k_i \sin\theta\cos\varphi, -k_i \sin\theta \sin\varphi), \qquad i = 1, 3,$$

which lies in the ball $B_{k_i}(\mathbf{0}) = \{\boldsymbol{\xi} \,:\, |\boldsymbol{\xi}| \,\leq\, k_i\}$ with Hessian matrix given by

$$\mathrm{H}_{\tilde{\Phi}}(\boldsymbol{\xi}^i_s) \;=\; -\frac{1}{k_i} \begin{pmatrix} 1 + \tan^2\theta\cos^2\varphi & \tan^2\theta\cos\varphi\sin\varphi \\ \tan^2\theta\cos\varphi\sin\varphi & 1 + \tan^2\theta\sin^2\varphi \end{pmatrix}$$

from where $\det \mathrm{H}_{\tilde{\Phi}}(\boldsymbol{\xi}^i_s) = \sec^2\theta / k_i^2$. Moreover, the matrix has eigenvalues of opposite signs and consequently $\operatorname{sign} \mathrm{H}_{\tilde{\Phi}}(\boldsymbol{\xi}^i_s) = 0$, the stationary point thus being a saddle point. Application of the stationary phase method yields

$$\mathbb{I}^i_{\boldsymbol{\xi}^i_s}(r, \theta, \varphi, y_3) \;=\; k_i \left|\cos\theta\right| \Psi^i(\boldsymbol{\xi}_s^i, y_3) \frac{e^{\imath k_i r}}{r} + \mathcal{O}(r^{-2}) \qquad i = 1, 3. \tag{46}$$

Remark 2.3. If $\sin\theta = 0$, the stationary point is also a critical point for J^i [see (43)], i.e. the end-point at $z = 0$, and the above result is divided by two [20].

2.4.2 Surface mode or pole contribution

We now consider asymptotic contributions along a fixed angle (43). In order to do so independently from r, we choose the complex paths so as to eliminate the integral contributions for large z and apply Jordan's lemma [16], i.e.

$$J^i_{\xi_p} \;=\; \imath 2\pi \lim_{\eta \downarrow 0} \mathrm{Res}_{z = \xi_{p,\eta}} \left(z \Psi^i_\eta e^{r \Phi^i_\eta} \right) \tag{47}$$

Hence, by looking at our square root definitions (see Section 2.1.4), we write $z = Re^{\imath\tau}$ with $R \in \mathbb{R}$, $\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and study the integrand behavior. First, we analyze the ubiquitous $\chi_{i,\eta}$:

$$\lim_{|R|\to+\infty} \chi_{i,\eta} = \lim_{|R|\to+\infty} \sqrt{(Re^{\imath\tau})^2 - k_{i,\eta}^2} = \lim_{|R|\to+\infty} |R|e^{\imath\tau}, \quad i = 1,2,3, \tag{48}$$

which show that all terms T_{ij}^{η}, R_{ij}^{η} are bounded. Second, we observe that exponential terms of the form $\exp(\pm s\chi_{i,\eta})$, with $s > 0$ and real, behave as

$$\lim_{|R|\to+\infty} \exp(\pm s\chi_{i,\eta}) = \lim_{|R|\to+\infty} \exp\left(\pm s|R|e^{\imath\tau}\right) = \lim_{|R|\to+\infty} \exp\left[\pm s|R|\cos\tau\right]$$

which, in view of the chosen domain for τ, converge to zero only for the negative sign. Hence, from (35),

$$\lim_{|R|\to+\infty} \mathsf{Det}_\eta(Re^{\imath\tau}) = 1$$

Since the particular expressions for Ξ_η^i in (38) are well-defined and bounded, Ψ_η^i grows at most polynomially for large z due to the term $\widehat{L}_\eta^U$. Finally, we look at the real part of the exponential term:

$$\mathfrak{Re}\left\{e^{-r|\cos\theta||R|\exp(\imath\tau)}e^{-\imath|R|\exp(\imath\tau)r\sin\theta\cos(\phi-\varphi)}\right\} = e^{-r|R|(|\cos\theta|\cos\tau-\sin\tau\sin\theta\cos(\phi-\varphi))}$$

For both Ω_1 and Ω_3, the elevation angle θ lies in $(0,\pi)$, and therefore $\sin\theta$ and $|\cos\theta|$ are positive. Thus, one can define the integration contours in relation exclusively to the sign of $\cos(\phi-\varphi)$ so that integrals over paths at a fixed distance R vanish as R goes to infinity.

Case $\cos(\phi-\varphi) \geq 0$

Path integrals lie on the lower half-plane following the sense shown in Fig. 1. Hence, poles are not included and the only potential contribution comes from the integral departing from $z = 0$:

$$J_{0^-}^i(r,\theta,\varphi,y_3,\phi) \sim \int_{0^+-\imath 0^+}^{0^+-\imath\infty} \Psi_\eta^i(z,\phi,y_3)\, e^{r\Phi_\eta^i(z,\phi,\theta,\varphi)} z dz \quad \lambda\to+\infty, \quad \eta\downarrow 0, \quad i = 1,3,$$

lying on the fourth quadrant of the complex plane. The contribution is calculated in Section 2.4.9.

Case $\cos(\phi-\varphi) \leq 0$

Integrate over $\mathbb{C}_+$ and encircle the pole located at $\xi_{p,\eta}$. Its residue for vanishing η is

$$\begin{aligned}\lim_{\eta\downarrow 0}\mathrm{Res}_{z=\xi_{p,\eta}}\left(z\Psi_\eta^i e^{r\Phi_\eta^i}\right) &= \lim_{\eta\downarrow 0}\lim_{z\to\xi_{p,\eta}}(z-\xi_{p,\eta})z\Psi_\eta^i(z,\phi,y_3)\, e^{r\Phi^i(z,\phi,\theta,\varphi)} \\ &= \xi_p\,\widehat{L}^U(\xi_p,\phi)\Xi^i(\xi_p,y_3)\, e^{r\Phi^i(\xi_p,\theta,\varphi,\phi)}\lim_{z\to\xi_p}\frac{z-\xi_p}{\mathsf{Det}(z)}\end{aligned} \tag{49}$$

wherein we have exchange limits by analyticity over the cut complex plane and functions $\widehat{L}^U$, Ξ^i and Φ^i [see (45a)] are well-defined at ξ_p. Since the determinant is null when $\eta \equiv 0$, we take the last limits using l'Hôpital's rule:

$$\lim_{z\to\xi_p}\frac{z-\xi_p}{\mathsf{Det}(z)} = \lim_{z\to\xi_p}\left[\mathsf{Det}'(z)\right]^{-1}.$$

The derivative of the determinant can be found as follows: let $f(z) = \mathsf{Det}(z) + 1$ and take the natural logarithm:

$$\log f(z) = \log\left(R_{21}R_{23}e^{-2h\chi_2}\right).$$

Derivation of the above yields,

$$\frac{f'}{f} \;=\; \frac{(R_{21})'}{R_{21}} \;+\; \frac{(R_{23})'}{R_{23}} \;-\; 2h\chi_2'$$

At $z = \xi_p$, we have, $f(\xi_p) = 1$, and consequently,

$$\mathsf{Det}'(\xi_p) \;=\; \frac{R_{21}'(\xi_p)}{R_{21}(\xi_p)} \;+\; \frac{R_{23}'(\xi_p)}{R_{23}(\xi_p)} \;-\; 2h\chi_2'(\xi_p).$$

The derivative of $\chi_i' = \xi/\chi_i$, and therefore,

$$\frac{R_{ij}'}{R_{ij}} \;=\; \frac{1}{R_{ij}}\left(\frac{\alpha_j\frac{\xi}{\chi_i} - \alpha_i\frac{\xi}{\chi_j}}{\alpha_i\chi_j \,+\, \alpha_j\chi_i} \;-\; R_{ij}\frac{\alpha_j\frac{\xi}{\chi_i} + \alpha_i\frac{\xi}{\chi_j}}{\alpha_i\chi_j \,+\, \alpha_j\chi_i}\right) \;=\; \frac{2\alpha_j\alpha_i\xi}{\chi_i\chi_j}\left(\frac{k_i^2 \,-\, k_j^2}{\alpha_j^2\chi_i^2 \,-\, \alpha_i^2\chi_j^2}\right).$$

Thus,

$$\mathsf{Det}'(\xi_p) \;=\; \frac{2\xi_p}{\chi_2(\xi_p)}\widetilde{\mathsf{Det}}'(\xi_p) \tag{50}$$

where we have defined

$$\widetilde{\mathsf{Det}}'(\xi_p) \;:=\; \frac{\alpha_1\alpha_2}{\chi_1}\left(\frac{k_2^2 \,-\, k_1^2}{\alpha_1^2\chi_2^2 \,-\, \alpha_2^2\chi_1^2}\right) \;+\; \frac{\alpha_3\alpha_2}{\chi_3}\left(\frac{k_2^2 \,-\, k_3^2}{\alpha_3^2\chi_2^2 \,-\, \alpha_2^2\chi_3^2}\right) \;-\; h \tag{51}$$

Since the $k_{1,3} \neq k_2$, and $h > 0$, the above quantity is well-defined and we can safely conclude

$$J^i_{\xi_p}(r,\theta,\varphi,y_3,\phi) \;=\; \imath 2\pi\chi_2(\xi_p)\widehat{L}^U(\xi_p,\phi)\frac{\Xi^i(\xi_p,y_3)}{2\widetilde{\mathsf{Det}}'(\xi_p)}\, e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}-\imath r\xi_p\sin\theta\cos(\phi-\varphi)}$$

for $i = 1, 3$.

2.4.3 Angular integration

The entire contribution coming from the pole is now obtained by integrating over ϕ. Since the residue is zero for $\cos(\phi-\varphi) > 0$, we use the indicator function $\mathbf{1}_A(\varphi)$ equal to one when $\varphi \in A$ and zero elsewhere to write

$$\begin{aligned}\mathbb{I}^i_{\xi_p}(r,\theta,\varphi,y_3) \;&\sim\; \frac{1}{2\pi}\int_0^{2\pi} \mathbf{1}_{\{\phi\,:\,\cos(\phi-\varphi)<0\}}(\phi)J^i_{\xi_p}(r,\theta,\varphi,y_3,\phi)d\phi \\ &\sim\; \imath\chi_2(\xi_p)\frac{\Xi^i(\xi_p,y_3)}{2\widetilde{\mathsf{Det}}'(\xi_p)}e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}}\, W^i_{\xi_p}(r\sin\theta,\varphi),\end{aligned}$$

for $i = 1, 3$, where the last term equals

$$W^i_{\xi_p}(\rho,\varphi) \;:=\; \int_0^{2\pi} \mathbf{1}_{\{\phi\,:\,\cos(\phi-\varphi)<0\}}(\phi)\widehat{L}^U(\xi_p,\phi)\, e^{-\imath\xi_p\rho\cos(\phi-\varphi)}\, d\phi$$

with ρ being the projection of r over the equatorial plane, i.e. $\rho = r\sin\theta$. Application of the stationary phase method, for the phase rewritten as $w(\phi) := -\xi_p\cos(\phi-\varphi)$, yields stationary points at $\sin(\phi-\varphi) = 0$, i.e. $\phi^s = m\pi + \varphi$, with $m = 0, 1, 2$ at most. This gives,

$$w(\phi^s) = -\xi_p(-1)^m\,, \qquad \partial_\phi^2\, w(\phi^s) = \xi_p(-1)^m.$$

However, $\mathbf{1}_{\{\phi\,:\,\cos(\phi-\varphi)<0\}}$ is nonzero only for $m=1$. Thus, bearing in mind that both ϕ and φ belong to the interval $(0,2\pi)$, the method yields

$$W^i_{\xi_p}(\rho,\varphi) \;=\; \widehat{L}^U(\xi_p,\pi+\varphi)\left(\frac{2\pi}{\rho\xi_p}\right)^{1/2} e^{\imath\rho\xi_p\,-\,\imath\pi/4} \;+\; \mathcal{O}\left(\rho^{-3/2}\right). \tag{52}$$

Summarizing results for $i=1,3$ we obtain

$$\mathbb{I}^i_{\xi_p} \;=\; \chi_2(\xi_p)\widehat{L}^U(\xi_p,\pi+\varphi)\frac{\Xi^i(\xi_p,y_3)}{2\,\widetilde{\mathsf{Det}}'(\xi_p)}e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}}\left(\frac{2\pi}{\rho\xi_p}\right)^{1/2} e^{\imath\rho\xi_p\,+\,\imath\pi/4} \;+\; \mathcal{O}\left(\rho^{-3/2}\right)$$

where the phase $-\imath\pi/4$ is changed due to the $\imath$ factor coming from the residue theorem.

Remark 2.4. The function decreases exponentially in the vertical direction, whereas the decrease is as $\rho^{-1/2}$ as we approach the $x_3=\{0,h\}$ planes. If the function Det possesses many zeros, denoted ξ_p^m, then we must add the contributions coming from each residue. These represent all the possible guided modes in the slab.

2.4.4 Branch point contributions

Relevant branch points are $k_{j,\eta}$, $j=1,2,3$, located on $\mathbb{C}_+$ (see Fig 1), thereby defining three contributions $J^i_{k_j}$, $i=1,3$, when η vanishes. At each branch cut, the original contour follows a loop-hole. First, we show that the integrals are well-defined at these points and therefore integral paths can be as close as desired to the branch cut.

2.4.5 Hole integrals

Indeed, at $k_{j,\eta}$, we calculate the limits:

$$\lim_{\nu\downarrow 0}\left|\Psi^i_\eta(k_{j,\eta}+\nu e^{\imath\tau},\phi,y_3)\,e^{r\Phi^i_\eta(k_{j,\eta}+\nu e^{\imath\tau},\phi,\theta,\varphi)}\right|,\quad j=1,2,3,\quad i=1,3.$$

Clearly, for $j\neq i$, $\lim_{\nu\downarrow 0}\chi_{i,\eta}(k_{j,\eta}+\nu e^{\imath\tau})=\sqrt{k^2_{j,\eta}-k^2_{i,\eta}}$ is well-defined. When $i=j$, we have

$$\lim_{\nu\downarrow 0}\sqrt{2k_{i,\eta}+\nu e^{\imath\tau}}\sqrt{\nu e^{\imath\tau}} \;=\; \sqrt{2k_{i,\eta}}e^{\imath\tau/2}\lim_{\nu\downarrow 0}\sqrt{\nu} \tag{53}$$

and, consequently, coefficients R^η_{ij} and T^η_{ij} have well-defined limits. Thus, functions Ξ^i_η and Det_η are also well behaved at the points $k_{j,\eta}$ for all $j=1,2,3$. Now, the source $\widehat{L}^U_\eta$ may contain terms of the form $\chi^{-1}_{2,\eta}$–when $\widehat{\Upsilon}^U_{e,\eta}$ is nonzero (32)– which are singular as $\nu^{-1/2}$ when for $z\to k_{2,\eta}$ [see (53)]. Since the Jacobian is equal to ν around $k_{j,\eta}$, for all cases, integrals

$$\lim_{\nu\downarrow 0}\int_{-3\pi/2}^{\pi/2}\Psi^i_\eta(k_{j,\eta}+\nu e^{\imath\tau},\phi,y_3)\,e^{r\Phi^i_\eta(k_{i,\eta}+\nu e^{\imath\tau},\phi,\theta,\varphi)}(k_{i,\eta}+\nu e^{\imath\tau})\nu e^{\imath\tau}\,d\tau$$

vanish. Hence, one is left with vertical integrals at each side of the branch cuts shown in Fig. 1.

2.4.6 Integrals parallel to the branch cut

For the moment, let us neglect angular variables and introduce $z_{j,\eta}:=\imath s+k_{j,\eta}$ and $z^\pm_{j,\eta,\nu}:=z_{j,\eta}\pm\nu$ with $s\in\mathbb{R}_+$ as the new integration variable. We must compute

$$J^i_{k_j} \;=\; \lim_{\eta,\nu\downarrow 0}\left(\int_{+\infty}^{0}\Psi^i_\eta(z^-_{j,\eta,\nu})e^{r\Phi^i_\eta(z^-_{j,\eta,\nu})}z^-_{j,\eta,\nu}dz^-_{j,\eta,\nu}+\int_0^{+\infty}\Psi^i_\eta(z^+_{j,\eta,\nu})e^{r\Phi^i_\eta(z^+_{j,\eta,\nu})}z^+_{j,\eta,\nu}dz^+_{j,\eta,\nu}\right). \tag{54}$$

For simplicity, let us also define local polar coordinates:

$$\begin{pmatrix} \rho^j_\pm(z) \\ \tau^j_\pm(z) \end{pmatrix} := \begin{pmatrix} |z \mp k_{j,\eta}| \\ \arg(z \mp k_{j,\eta}) \end{pmatrix}, \quad \rho^j_\pm \in \mathbb{R}_+, \quad \tau^j_+ \in \left(-\frac{3\pi}{2}, \frac{\pi}{2}\right), \quad \tau^j_- \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right), \tag{55}$$

for $z \in \mathbb{C}$ and $j = 1, 2, 3$. When ν goes to zero, the angle τ^j_+ takes the values

$$\tau^j_+ \left(z^+_{j,\eta,0}\right) = \frac{\pi}{2} \quad \text{and} \quad \tau^j_+ \left(z^-_{j,\eta,0}\right) = -\frac{3\pi}{2}, \qquad s \in \mathbb{R}_+$$

while τ^j_- does not vary. Notice that coordinates $(\rho^j_\pm, \tau^j_\pm)$ remain unchanged if defined with respect to $z^\pm_{i,\eta,0}$ with $i \neq j$. Based on the representation

$$\chi_{i,\eta}(z) \; = \; \sqrt{\rho^i_+ \rho^i_-} \exp\left[\imath \left(\frac{\tau_+ + \tau_-}{2}\right)\right]$$

we can state the following relations:

$$\chi_i(z^+_{i,\eta,0}) = \sqrt{\rho^i_+ \rho^i_-} e^{\imath\pi/4} e^{\imath\tau^i_-/2}, \quad \chi_i(z^-_{i,\eta,0}) = -\chi_i(z^+_{i,\eta,0}), \quad \chi_j(z^+_{i,\eta,0}) = \chi_j(z^-_{i,\eta,0}) \quad i \neq j \tag{56}$$

From these, one can deduce

$$\begin{aligned} R^\eta_{ij}(z^-_{i,\eta,0}) \; &= \; \frac{-\alpha_j\chi_i(z^+_{i,\eta,0}) - \alpha_i\chi_j(z^+_{i,\eta,0})}{-\alpha_j\chi_i(z^+_{i,\eta,0}) + \alpha_i\chi_j(z^+_{i,\eta,0})} \; = \; \left[R^\eta_{ij}(z^+_{i,\eta,0})\right]^{-1}, \\ T^\eta_{ij}(z^-_{i,\eta,0}) \; &= \; \frac{-2\alpha_j\chi_i(z^+_{i,\eta,0})}{-\alpha_j\chi_i(z^+_{i,\eta,0}) + \alpha_i\chi_j(z^+_{i,\eta,0})} \; = \; T^\eta_{ij}(z^+_{i,\eta,0}) \left[R^\eta_{ij}(z^+_{i,\eta,0})\right]^{-1}, \end{aligned}$$

and consequently,

$$\begin{aligned} \mathsf{Det}(z^-_{i,\eta,0}) &= \left[R_{21}(z^+_{i,\eta,0}) R_{23}(z^+_{i,\eta,0}) e^{-2h\chi_2(z^+_{i,\eta,0})}\right]^{-1} - 1 \\ &= -\mathsf{Det}(z^+_{i,\eta,0}) \left[R_{21}(z^+_{i,\eta,0}) R_{23}(z^+_{i,\eta,0}) e^{-2h\chi_2(z^+_{i,\eta,0})}\right]^{-1}. \end{aligned}$$

With the above, the reader can verify the helpful result:

$$\Psi^i_\eta(z^-_{j,\eta,0}) \; = \; \Psi^i_\eta(z^+_{j,\eta,0}) \; = \; \Psi^i_\eta(z_{j,\eta}) \qquad j = 1, 2, 3, \quad i = 1, 3,$$

and one can write integrals (54) as

$$J^i_{k_j} \; = \; \lim_{\eta \downarrow 0} \lim_{R \to \infty} \int_0^R \Psi^i_\eta(z_{j,\eta}) \left[e^{r\Phi^i_\eta(z^+_{j,\eta,0})} \; - \; e^{r\Phi^i_\eta(z^-_{j,\eta,0})}\right] z_{j,\eta}(s) dz_{j,\eta}(s) \tag{58}$$

and consider solely the behavior of Φ^i_η. Clearly, the phase Φ^i_η does not change at either side of the branch cuts located at $k_{j,\eta}$ for $j \neq i$ [see (45b)] due to property (56). Thus, the square brackets term in (58) is equal to zero, and

$$\mathbb{I}^i_{k_j}(r, \theta, \varphi) \; = \; 0 \qquad j \neq i, \quad i = 1, 3.$$

2.4.7 Contribution when $i = j$

On the other hand, Φ^i_η does change when crossing the branch cut located in $k_{i,\eta}$ as it passes through the Riemann sheets of $\chi_{i,\eta}$. Replacing Φ^i_η in (58), yields

$$J^i_{k_i} \;=\; \lim_{\substack{\eta\downarrow 0\\ R\to\infty}} \int_0^R \Psi^i_\eta(z_{i,\eta}) e^{\imath r z_{i,\eta}\sin\theta|\cos(\phi-\varphi)|} \left[e^{-r|\cos\theta|\chi_{i,\eta}(z^+_{i,\eta,0})} \;-\; e^{r|\cos\theta|\chi_{i,\eta}(z^+_{i,\eta,0})}\right] z_{i,\eta}(s)\, dz_{i,\eta}(s)$$

We now deform the original contour to that given by the steepest descent direction and take the limit in η. For $0 \le \theta < \pi/2$ and $\cos(\phi-\varphi) \,<\, 0$, we regard the phase when z is close to $k_{i,\eta}$:

$$\Phi^i(z) \;\sim\; \imath k_i \sin\theta\,|\cos(\phi-\varphi)| \;-\; |\cos\theta|\,\sqrt{2k_i}\,(z-k_i)^{1/2}$$

By identifying the above with (77), we obtain $a = \cos\theta\sqrt{2k_i}$, $\alpha = \pi$ and $n = 1/2$. From (74), the angle $\Theta_p = 0$ and therefore, it follows the real axis. Thus, we modify our original contour so that the integral now goes along $\mathfrak{Re}\{z\} = 0$. We then regard the integral

$$J^i_{k_i} \;=\; \int_{C_{k_i}} \Psi^i(z) e^{\imath r z\sin\theta|\cos(\phi-\varphi)|} e^{-r\cos\theta\chi(z)}\, z\, dz$$

where C_{k_i} is the steepest descent path for which the imaginary part of the phase is kept constant, i.e.,

$$\mathfrak{Im}\left\{\Phi^i(z) \;-\; \Phi^i(k_i)\right\} \;=\; 0 \tag{59}$$

Using the coordinates defined in (55) which satisfy

$$\rho^i_+\sin\tau^i_+ = \rho^i_-\sin\tau^i_-\,, \qquad \rho^i_+\cos\tau^i_+ \,+\, 2k_i = \rho^i_-\cos\tau^i_- \quad \text{if} \quad 0 \,\le\, \tau^i_- \,<\, \tau^i_+ \,\le\, \pi/2$$

we write condition (59) as

$$\mathfrak{Im}\left\{\imath\rho^i_+ e^{\imath\tau^i_+}\sin\theta|\cos(\phi-\varphi)| \;-\; \cos\theta\sqrt{\rho^i_+\rho^i_-}\,e^{\imath(\tau_+ + \tau_-)/2}\right\} \;\sim\; 0$$

$$\rho^i_+\cos\tau^i_+\sin\theta|\cos(\phi-\varphi)| \;-\; \cos\theta\sqrt{\rho^i_+\rho^i_-}\,\sin\left(\frac{\tau_+ \,+\, \tau_-}{2}\right) \;\sim\; 0$$

In the first quadrant, for very large $|z|$ it holds $\rho_- \,\sim\, \rho_+$ and $\tau_+ \sim \tau_-$. Thus,

$$\tan\theta|\cos(\phi-\varphi)| \;\sim\; \tan\tau_+$$

Although the steepest descent path depends upon $\tan\theta|\cos(\phi-\varphi)|$, it is always located on the first quadrant of the complex plane as $\theta \,\in\, (0,\pi/2)$

$$0 \;\le\; \tan\theta|\cos(\phi-\varphi)| \;\le\; \tan\theta$$

If $\theta = 0$, τ_+ vanishes. This is consistent with a steepest descent path following the real axis when there is no oscillatory term in Φ^i. Thus, asymptotically, the path followed is that of a line with slope $\tan\tau_+$ whose main contribution is given by (80)

$$J^i_{k_i}(r,\theta,\phi,\varphi) \;\sim\; \Psi^i(k_i,\phi,y_3)\frac{1}{r^2\cos^2\theta}\, e^{\imath r k_i\sin\theta|\cos(\phi-\varphi)|} + \mathcal{O}(r^{-3}) \qquad \theta \,\in\, (0,\pi/2)$$

as $\widehat{L}^U(z,\phi)$ is well-defined at k_i for $i \neq 2$.

2.4.8 Angular integration

We now compute the complete contribution $\mathbb{I}^i_{k_i}$ by integrating over ϕ. In the special case $\theta = \pi/2$, the term $\mathbb{I}^i_{k_i}$ vanishes. If $\theta \in (0, \pi/2)$, we apply the stationary phase method by using the same results provided in Section 2.4.2, i.e.

$$\mathbb{I}^i_{k_i}(r,\theta,\varphi) \;=\; \frac{1}{2\pi}\frac{1}{r^2\cos^2\theta}\frac{\Xi^i(k_i,y_3)}{\mathsf{Det}(k_i)}\widehat{L}^U(k_i,\pi+\varphi)\left(\frac{2\pi}{\rho k_i}\right)^{1/2} e^{\imath\rho k_i\,-\,\imath\pi/4} \;+\; \mathcal{O}\left(\rho^{-3/2}\right) \tag{60}$$

valid for $\theta \in (0, \pi/2)$.

2.4.9 End point contributions

Consider the integrals departing from $z = 0$ towards $\pm\imath\infty$ shown in Fig 1:

$$J^i_{0^\pm} \;=\; \lim_{\eta,\nu\downarrow 0}\int_0^\infty \Psi^i_\eta(z^\pm_{0,\nu},\phi,y_3)\, e^{r\Phi^i_\eta(\theta,\varphi,z^\pm_{0,\nu},\phi)} z^\pm_{0,\nu}(s)dz^\pm_{0,\nu}(s)$$

where $z^\pm_{0,\nu} := \pm\imath s + \nu$ with $s, \nu \in \mathbb{R}_+$. We study the phase at $s = 0$ for the integral in z using the derivative

$$\partial_z\Phi^i_\eta = -\left|\cos\theta\right| z\chi^{-1}_{i,\eta} \;-\; \imath\sin\theta\cos(\phi-\varphi) \tag{61}$$

If $\theta > 0$ or $\cos(\phi - \varphi) \neq 0$, the end point is neither a stationary point nor a branch point, and we can set $n = 1$ and use formula (79) from the steepest descent method. Taking the limit in η, from (61), $\alpha = \mp\pi/2$ depending on the sign of $\cos(\phi - \varphi)$, $\Theta_1 = \pi - \alpha$ and $|\partial_z\Phi^i(0)| = \sin\theta|\cos(\phi - \varphi)|$. Therefore, $\beta = 2$ in (79) and the integrals in z for both signs of the cosine are asymptotically equal to

$$\begin{aligned} J^i_{0^\mp}(r,\theta,\varphi,y_3,\phi) \;&=\; \widehat{L}^U(0,\phi)\frac{\Xi^i(0,y_3)}{\mathsf{Det}(0)}\frac{1}{\left(r\sin\theta|\cos(\phi-\varphi)|\right)^2}\, e^{\imath rk_i\cos\theta\pm\imath\pi/2} \;+\; \mathcal{O}\left(\rho^{-4}\right) \\ &=\; \pm\widehat{L}^U(0,\phi)\frac{\Xi^i(0,y_3)}{\mathsf{Det}(0)}\frac{1}{\rho^2|\cos(\phi-\varphi)|^2}\, e^{\imath rk_1\cos\theta\,+\,\imath\pi/2} \;+\; \mathcal{O}\left(\rho^{-4}\right) \end{aligned}$$

the plus and minus signs coming from the phase in $\pi/2$.

2.4.10 Angular integration

Integration over ϕ yields,

$$\mathbb{I}^i_0(r,\theta,\varphi) \;=\; \frac{1}{2\pi}\int_0^{2\pi}\left(\mathbf{1}_{\{\phi:\cos(\phi-\varphi)<0\}}J^i_{0^+}(r,\phi,\varphi) + \mathbf{1}_{\{\phi:\cos(\phi-\varphi)>0\}}J^i_{0^-}(r,\phi,\varphi)\right)\,d\phi = 0$$

since $\theta > 0$ and $\cos(\phi - \varphi) \neq 0$, the denominators never vanish and the above integrals are bounded. Moreover, regardless of the sign of $\cos(\phi - \varphi)$ they have the same result due to the form of the term $\widehat{L}^U(0, \phi)$ – either t_j or constant – and therefore the contributions of order ρ^{-2} cancel each other.

Remark 2.5. Now, if $\theta = 0$ or $\cos(\phi - \varphi) = 0$, the above is no longer valid, and $z = 0$ turns to be a stationary point, for which $n = 2$

$$\partial^2_z\Phi^i_\eta = -\frac{|\cos\theta|}{\chi_{i,\eta}}\left(1 \;-\; \frac{z^2}{\chi^2_{i,\eta}}\right)$$

from where, if η goes to zero, $\partial^2_z\Phi^i = -\imath\cos\theta/k_i$, $\alpha = -\pi/2$, $a = \cos\theta/k_i$ and $\theta_\mp = 3\pi/4, -\pi/4$ and using formula (76), we obtain

$$J^i_{0^\mp}(r,\theta,\varphi) \;\sim\; \frac{\widehat{L}^U(0,\phi)}{2}\frac{\Xi^i(0,y_3)}{\mathsf{Det}(0)}\left[\frac{2k_i}{r\,|\cos\theta|}\right]^{3/2}\Gamma\left(\frac{3}{2}\right)e^{\imath rk_i\cos\theta\,+\,\imath 3\Theta_\mp}$$

and by integrating in ϕ the total contribution is equal to zero by the same arguments as before.

2.5 Forms in Ω_2

In this case, the normal direction x_3 is bounded, and hence asymptotics are obtained along horizontal directions. We use the cylindrical coordinates:

$$x_1 = \rho\cos\varphi\ , \quad x_2 = \rho\sin\varphi\ , \quad x_3^+ \in (y_3,\, h)\ , \quad x_3^- \in (0,\, y_3)$$

with $\rho > 0$ and $\varphi \in (0, 2\pi)$, so that

$$g^{2\pm}(\rho,\varphi,x_3^\pm,y_3) \;=\; \lim_{\eta\downarrow 0}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty}\Psi_\eta^{2\pm}(\xi,\varphi,y_3)e^{\rho\Phi^2(\boldsymbol{\xi},\phi,\varphi)}\,\xi d\xi d\phi \tag{62}$$

where now

$$\begin{aligned}
\Psi_\eta^{2\pm}(\boldsymbol{\xi},x_3^\pm,y_3) &= \widehat{L}_\eta^U(\boldsymbol{\xi})\,\frac{\Xi_\eta^{2\pm}(\xi,y_3)}{\mathsf{Det}_\eta(\xi)}X_\eta^{2\pm}(\xi,x_3^\pm)\\
\Phi^2(\boldsymbol{\xi},\phi,\varphi) &= -\imath\xi\cos(\phi-\varphi)\\
X_\eta^{2+}(\xi,x_3^+) &= e^{-x_3^+\chi_{2,\eta}} \;+\; R_{21}\,e^{-2h\chi_{2,\eta}}\,e^{x_3^+\chi_{2,\eta}}\\
X_\eta^{2-}(\xi,x_3^-) &= R_{23}e^{-x_3^-\chi_{2,\eta}} \;+\; e^{x_3^-\chi_{2,\eta}}
\end{aligned}$$

where functions $X_\eta^{2\pm}$, $\Xi_\eta^{2\pm}$, are well-defined in ξ. Given the form of Φ^2, it is clear that no saddle points occur for the integral in ξ and the asymptotic behavior of the Green's function G^2 is given by the pole contribution:

$$g^{2\pm} \;=\; \chi_2(\xi_p)\psi^{2\pm}(\xi_p,\pi+\varphi)\frac{\Xi^{2\pm}(\xi_p,y_3)}{2\,\widetilde{\mathsf{Det}}'(\xi_p)}X^{2\pm}(\xi_p,x_3)\left(\frac{2\pi}{\rho\xi_p}\right)^{1/2}e^{\imath\rho\xi_p+\imath\pi/4} \;+\; \mathcal{O}\left(\rho^{-3/2}\right)$$

2.5.1 Results for scalar Helmholtz and EM normal components

Proposition 2.2. *Consider the coordinate sets describing for each Ω_i defined before. Assume the existence of a single surface mode ξ_p and let $\gamma \in (\frac{1}{4},\frac{1}{2})$. Then, the far-field of the scalar or vectorial Green's functions g^i when η vanishes is given by*

- *For Ω_i, $i = 1,3$ and $r|\cos\theta| > r^\gamma$,*

$$g^i(r,\theta,\varphi,y_3) \;=\; \Lambda_r^{U,i}(\theta,\varphi,y_3)\,\frac{e^{\imath k_i r}}{r} \;+\; \mathcal{O}\left(r^{-\left(2\gamma+\frac{1}{2}\right)}\right)$$

  and, for $0 < r|\cos\theta| < r^\gamma$:

$$g^i(r,\theta,\varphi,y_3) \;=\; \Lambda_\rho^{U,i}(\varphi,y_3)e^{-r\cos\theta\sqrt{\xi_p^2-k_i^2}}\,\frac{e^{\imath\rho\xi_p+\imath\pi/4}}{\rho^{1/2}} \;+\; \mathcal{O}\left(r^{-\left(\frac{3}{2}-\gamma\right)}\right)$$

- *On the other hand, for $\Omega_{2\pm}$, we have*

$$g^2(\rho,\varphi,x_3,y_3) \;=\; \Lambda_\rho^{U,2\pm}(\varphi,y_3)X^{2\pm}(\xi_p,x_3)\frac{e^{\imath\rho\xi_p+\imath\pi/4}}{\rho^{1/2}} \;+\; \mathcal{O}\left(\rho^{-3/2}\right) \tag{64}$$

with according scalar or vector terms depending on the precise field U described

$$\begin{aligned}
\Lambda_r^{U,i}(\theta,\varphi,y_3) &:= k_i\,|\cos\theta|\,\widehat{L}^U(-k_i\sin\theta,\varphi)\,\frac{\Xi^i(k_i\sin\theta,y_3)}{\mathsf{Det}(k_i\sin\theta)} && i = 1,3\\
\Lambda_\rho^{U,i}(\varphi,y_3) &:= -\chi_2(\xi_p)\widehat{L}^U(\xi_p,\varphi)\frac{\Xi^i(\xi_p,y_3)}{2\,\widetilde{\mathsf{Det}}'(\xi_p)}\left(\frac{2\pi}{\xi_p}\right)^{1/2} && i = 1,2\pm,3
\end{aligned}$$

where $\widetilde{\mathsf{Det}}'$ is obtained by taking the limit of Det *at the guided mode wavenumber and is well defined.*

Remark 2.6. This result is consistent with [11]. In the definition of Λ_ρ^i, we have use the fact that $t_j(\pi+\varphi) = -t_j(\varphi)$ by the definition of $\mathbf{t}$. For EM, we have neglected the dependence on the vectorial sources.

Remark 2.7. In the case of scalar Helmholtz, we can already prove Proposition 1.1 by using the above asymptotics as the far-field of the solution $U = g_\eta * F$ with different values ξ_p^m.

2.6 Asymptotics for transversal EM fields components

With the above information, we can easily compute the asymptotic behavior for the transversal fields for each polarization. Recall (39) and (40) and observe that $\widehat{\mathbf{g}}^E_{e,Tj}$ and $\widehat{\mathbf{g}}^H_{h,Tj}$ are deduced only by deriving in x_3 the transversal fields. Hence, they do not need to be calculated from their spectral form, and we focus only on the field components $\widehat{\mathbf{g}}^P_{q,T}$, with $(q,P) = \{(h,E),(e,H)\}$, $T = 1,2$. These last terms have the general form

$$\widehat{\mathbf{g}}^{P,i}_{q,T} \ = \ \pm\varepsilon_{T3T''}\frac{\omega\alpha_i}{\xi}\, t_{T''}(\phi)\widehat{\mathbf{g}}^{P,i}_{p,3} \quad , \qquad p \neq q \tag{66}$$

where the positive and negative signs correspond to TM and TE modes, respectively. We take asymptotics for their inverse Fourier transform on each component $j = 1,2,3$:

$$g^i_{q,Tj}(\mathbf{x},y_3) \ = \ \pm\varepsilon_{T3T''}\frac{\omega\alpha_i}{2\pi}\lim_{\eta\downarrow 0}\int_0^{2\pi}\int_0^\infty \frac{t_{T''}(\phi)}{\xi}\,\Psi^i_{j,\eta}(\xi,\phi,x_3,y_3)e^{r\Phi^i_\eta(\xi,\phi,\varphi,x_3)}\xi d\xi d\phi$$

where the dependence on j lies in $\Psi^i_{j,\eta}$. Clearly, the integral critical points do not change, and hence, we must only carry out minor adjustments to the previous calculations. Thence, we state directly the modifications for the stationary points, poles and branch points for each domain.

2.6.1 Asymptotics for $\mathbf{g}^P_{q,T}$

Proposition 2.3. *Let $\gamma \in (\frac{1}{4},\frac{1}{2})$. For the corresponding coordinates describing Ω_i, it holds*

- *For Ω_i, $i = 1,3$ and $r|\cos\theta| > r^\gamma$,*

$$\mathbf{g}^{P,i}_{q,T}(\mathbf{x},y_3) \ = \ \mp\varepsilon_{T3T''}\,\omega\alpha_i\frac{t_{T''}(\varphi)}{k_i\sin\theta}\mathbf{\Lambda}^{U,i}_r(\theta,\varphi,y_3)\,\frac{e^{\imath k_i r}}{r} \ + \ \mathcal{O}\left(\frac{1}{r^{2\gamma+\frac{1}{2}}}\right)$$

- *whereas for $0 < r|\cos\theta| < r^\gamma$,*

$$\mathbf{g}^{P,i}_{q,T}(\mathbf{x},y_3) \ = \ \mp\varepsilon_{T3T''}\,\omega\alpha_i\frac{t_{T''}(\varphi)}{\xi_p}\mathbf{\Lambda}^{U,i}_\rho(\varphi,y_3)e^{-r\cos\theta\sqrt{\xi_p^2-k_i^2}}\,\frac{e^{\imath\rho\xi_p+\imath\pi/4}}{\rho^{1/2}} \ + \ \mathcal{O}\left(\frac{1}{r^{\frac{3}{2}-\gamma}}\right)$$

- *For $\Omega_{2\pm}$, it holds*

$$\mathbf{g}^{P,2\pm}_{q,T}(\mathbf{x},y_3) \ = \ \mp\varepsilon_{T3T''}\,\omega\alpha_2\frac{t_{T''}(\varphi)}{\xi_p}\mathbf{\Lambda}^{U,2\pm}_\rho(\varphi,y_3)X^{2\pm}(\xi_p,x_3)\frac{e^{\imath\rho\xi_p+\imath\pi/4}}{\rho^{1/2}} \ + \ \mathcal{O}\left(\frac{1}{\rho^{3/2}}\right)$$

with coefficients $\mathbf{\Lambda}^{U,i}_r$ and $\mathbf{\Lambda}^{U,i}_\rho$ defined as in Proposition 2.2, and where positive (negative) sign and $\alpha_i = \mu$ (or ϵ) correspond to TE (TM) modes.

Hence, by combining Propositions 2.2 and 2.3, we can rewrite the above as follows:

Corollary 2.1. *Asymptotically, the dyad terms $\mathbf{g}^{P,i}_{q,T}$ behave as the normal components $\mathbf{g}^{P,i}_{p,3}$ in the following form*

$$\mathbf{g}^{P,i}_{q,T} \ \sim \ \mp\varepsilon_{T3T''}\,\omega\alpha_i\frac{t_{T''}(\varphi)}{k_{i\|}}\mathbf{g}^{P,i}_{p,3} \qquad q \neq p$$

where $k_{i\|}$ is the projection of the wave number over the slab, i.e., the tangential wavenumber in Ω_i given by

$$k_{i\|} \;=\; \begin{cases} k_i \sin\theta & \textit{for} \quad \mathbf{x} \in \Omega_i, \quad i = 1,3\,, \quad r|\cos\theta| \;>\; r^\gamma \\ \xi_p & \textit{for} \quad \mathbf{x} \in \Omega_i, \quad i = 1,3\,, \quad r|\cos\theta| \;<\; r^\gamma \\ \xi_p & \textit{for} \quad \mathbf{x} \in \Omega_{2\pm} \end{cases} \tag{67}$$

2.6.2 Asymptotics for $\mathbf{g}^{P,i}_{p,T}$

From (39) and (40), we can retrieve the transversal field Green's functions, $\mathbf{g}^E_{e,T}$ and $\mathbf{g}^H_{h,T}$ from the above by using the general form

$$\mathbf{g}^{P,i}_{p,T} \;=\; \pm\frac{\imath}{\omega\alpha_i}\partial_3 \varepsilon_{T3T'}\mathbf{g}^{P,i}_{q,T'} \quad , \qquad q \;\neq\; p \tag{68}$$

where the positive and negative signs also correspond to TM and TE polarizations. Thus,

Proposition 2.4. *The asymptotic form for the dyad components $\mathbf{g}^{P,i}_{p,T}$ is*

$$\mathbf{g}^{P,i}_{p,T} \;\sim\; \imath\,\frac{k_{i\perp}}{k_{i\|}} t_T(\varphi)\mathbf{g}^{P,i}_{p,3}$$

where $k_{i\|}$ and $k_{i\perp}$ are the projections of the wave number over the parallel and perpendicular directions with respect to the slab, satisfying

$$k^2_{i\|} \;+\; k^2_{i\perp} \;=\; k^2_i$$

Proof. Expression (68) together with Proposition 2.1 and the following formula for the multiplication of Levi-Civita tensors:

$$\varepsilon_{T3T'}\varepsilon_{T'3T''} \;=\; -\delta_{TT''}$$

yields the desired result. In detail, we see that:

- for corresponding ranges of θ associated to Ω_i, with $i = 1, 3$, and $r|\cos\theta| \;>\; r^\gamma$,

$$\mathbf{g}^{P,i}_{p,T}(r,\theta,\varphi,y_3) \;=\; \imath\frac{-k_i\cos\theta}{k_i\sin\theta}t_T(\varphi)\mathbf{\Lambda}^{P,i}_r(\theta,\varphi,y_3)\,\frac{e^{\imath k_i r}}{r} \;+\; \mathcal{O}\left(\frac{1}{r^{2\gamma+\frac{1}{2}}}\right)$$

- and for $0 <\; r|\cos\theta| < r^\gamma$,

$$\mathbf{g}^{P,i}_{p,T}(r,\theta,\varphi,y_3) \;=\; \imath\,\frac{-\sqrt{\xi_p^2 - k_i^2}}{\xi_p}t_T(\varphi)\mathbf{\Lambda}^{P,i}_\rho(\varphi,y_3)e^{-r\cos\theta\sqrt{\xi_p^2-k_i^2}}\,\frac{e^{\imath\rho\xi_p \,+\, \imath\pi/4}}{\rho^{1/2}} \;+\; \mathcal{O}\left(\frac{1}{r^{\frac{3}{2}-\gamma}}\right)$$

- For $\Omega_2\pm$, it holds

$$\begin{aligned} \mathbf{g}^{P,2\pm}_{p,T}(\rho,\varphi,y_3) \;&=\; \imath\,\frac{t_T(\varphi)}{\xi_p}\mathbf{\Lambda}^{P,i}_r(\varphi,y_3)\partial_3 X^{2\pm}(\xi_p,x_3)\frac{e^{\imath\rho\xi_p \,+\, \imath\pi/4}}{\rho^{1/2}} \;+\; \mathcal{O}\left(\frac{1}{\rho^{3/2}}\right) \\ &=\; \imath\,\frac{-\imath\sqrt{k_2^2 - \xi_p^2}}{\xi_p}t_T(\varphi)\mathbf{\Lambda}^{P,i}_\rho(\varphi,y_3)\widetilde{X}^{2\pm}(\xi_p,x_3)\frac{e^{\imath\rho\xi_p \,+\, \imath\pi/4}}{\rho^{1/2}} \;+\; \mathcal{O}\left(\frac{1}{\rho^{3/2}}\right) \end{aligned}$$

where the term $\widetilde{X}^{2\pm}$ is the derivative in x_3 of $X^{2\pm}$ divided by $\chi_2(\xi_p)$. □

2.7 Radiation condition proofs sketch

Proof of Proposition 1.1. The fields U are built by convoluting $\mathsf{F}_U F$ with the derived Green's functions. Since F has compact support, fields behave as g^i, $i = 1, 2, 3$. Straightforward derivation of the asymptotic results provided in propositions 2.2, 2.1 and 2.4, along r and ρ for each component yields the stated conditions. □

Proof of Proposition 1.2. As in the previous proof, the fields convey the asymptotic behavior revealed by Green's dyads by construction. These are stated in propositions 2.2, 2.1 and 2.4 which directly show the conditions. We demonstrate the above inequalities for TM-modes assuming a single surface mode, the case of TE polarization being reciprocal.

2.7.1 For $r|\cos\theta| > r^\gamma$

Using $\mathbf{n} = (\sin\theta t_1(\varphi), \sin\theta t_2(\varphi), \cos\theta)$, classical Silver-Müller conditions are retrieved for large r. Indeed, after replacing according to $\mathbf{E}$ and $\mathbf{H}$ the previous asymptotics, for each excitation along $j = 1, 2, 3$, we have the component-wise results:

$$H_1^i + \mathsf{z}_{i,r}^{-1}\left(E_2^i\cos\theta - \sin\theta E_3^i t_2(\varphi)\right) \sim \left\{\frac{\omega\epsilon_i}{k_i\sin\theta}t_2(\varphi) - \mathsf{z}_{i,r}^{-1}\left(\frac{k_i\cos^2\theta}{k_i\sin\theta} + \sin\theta\right)t_2(\varphi)\right\}\Lambda^{E,i}_{\epsilon,j,r}\frac{e^{\imath k_i r}}{r}$$

$$H_2^i - \mathsf{z}_{i,r}^{-1}\left(E_2^i\cos\theta - \sin\theta E_3^i t_2(\varphi)\right) \sim \left\{\frac{-\omega\epsilon_i}{k_i\sin\theta}t_1(\varphi) + \mathsf{z}_{i,r}^{-1}\left(\frac{k_i\cos^2\theta}{k_i\sin\theta} + \sin\theta\right)t_1(\varphi)\right\}\Lambda^{E,i}_{\epsilon,j,r}\frac{e^{\imath k_i r}}{r}$$

and

$$\mathsf{z}_{i,r}^{-1}\sin\theta\left[E_1^i t_2(\varphi) - E_2^i t_1(\varphi)\right] \sim \mathsf{z}_{i,r}^{-1}\frac{-k_i\cos\theta}{k_i\sin\theta}\left[t_1(\varphi)t_2(\varphi) - t_2(\varphi)t_1(\varphi)\right]\Lambda^{E,i}_{\epsilon,j,r}\frac{e^{\imath k_i r}}{r} \tag{70}$$

after factorization and expressing the precise parameters intervening in Λ terms. By definition, $\mathsf{z}_{i,r}\omega\epsilon_i = k_i$ which only leaves terms as $\mathcal{O}\left(r^{-(2\gamma+\frac{1}{2})}\right)$.

2.7.2 For $r|\cos\theta| < r^\gamma$

For clarity, exclude for the moment common terms

$$\Lambda^{E,i}_{\epsilon,j,\rho}(\varphi, y_3)e^{-r|\cos\theta|\sqrt{\xi_p^2-k_i^2}}\, e^{\imath\rho\xi_p + \imath\frac{\pi}{4}}\rho^{-1/2}$$

Then,

$$H_1^i + \mathsf{z}_{i,\rho}^{-1}\left(\cos\theta E_2^i - E_3^i\sin\theta t_2(\varphi)\right) \sim t_2(\varphi)\mathsf{z}_{i,\rho}^{-1}\left[1 - \left(\imath\cos\theta\frac{(\xi_p^2-k_i^2)^{1/2}}{\xi_p} + \sin\theta\right)\right]$$

$$H_2^i - \mathsf{z}_{i,\rho}^{-1}\left(\cos\theta E_1^i - E_3^i\sin\theta t_1(\varphi)\right) \sim -t_1(\varphi)\mathsf{z}_{i,\rho}^{-1}\left[1 - \left(\imath\cos\theta\frac{(\xi_p^2-k_i^2)^{1/2}}{\xi_p} + \sin\theta\right)\right]$$

$$\mathsf{z}_{i,\rho}^{-1}\sin\theta\left[E_1^i t_2(\varphi) - E_2^i t_1(\varphi)\right] = \mathcal{O}\left(r^{-(\frac{3}{2}-\gamma)}\right)$$

the last result obtained from (70). By hypothesis, $|\cos\theta| < r^{\gamma-1}$, and consequently, the imaginary terms are bounded as desired by taking into account the factor $\rho^{-1/2}$. For the real term, the exponential factor $\exp(-r|\cos\theta|)$ decreases faster than any polynomial for θ sufficiently far from $\pi/2$, and hence, the bound is achieved. Finally, as $\left|\theta - \frac{\pi}{2}\right|$ tends to zero, the term $1 - \sin\theta$ vanishes as $\left|\theta - \frac{\pi}{2}\right|^2$ and the statement follows.

2.7.3 For $\Omega_{2\pm}$

We take the normal equal to $\rho = (t_1(\varphi), t_2(\varphi), 0)$ and expand

$$H_1^{2\pm} - \mathsf{z}_{2,\rho}^{-1} E_3^{2\pm} t_2(\varphi) \sim \left[\frac{\omega\epsilon_2}{\xi_p} t_2(\varphi) - \mathsf{z}_{2,\rho}^{-1} t_2(\varphi)\right] \Lambda_{\epsilon,j,r}^{E,2\pm} X_\epsilon^{2\pm} \frac{e^{\imath\xi_p\rho+\imath\pi/4}}{\sqrt{\rho}}$$

$$H_2^{2\pm} + \mathsf{z}_{2,\rho}^{-1} E_3^{2\pm} t_1(\varphi) \sim \left[-\frac{\omega\epsilon_2}{\xi_p} t_1(\varphi) + \mathsf{z}_{2,\rho}^{-1} t_1(\varphi)\right] \Lambda_{\epsilon,j,r}^{E,2\pm} X_\epsilon^{2\pm} \frac{e^{\imath\xi_p\rho+\imath\pi/4}}{\sqrt{\rho}}$$

$$\mathsf{z}_{2,\rho}^{-1}\left[E_1^{2\pm} t_2(\varphi) - E_2^{2\pm} t_1(\varphi)\right] \sim -\imath \frac{-\imath\sqrt{\xi_p^2 - k_2^2}}{z_{2,\rho}\xi_p} \left[-t_1(\varphi)t_2(\varphi) + t_2(\varphi)t_1(\varphi)\right] \Lambda_{\epsilon,j,r}^{E,2\pm} X_\epsilon^{2\pm} \frac{e^{\imath\xi_p\rho+\imath\pi/4}}{\sqrt{\rho}}$$

Since $\mathsf{z}_{2,\rho} = \xi_p/(\omega\epsilon_2)$, only terms decreasing as $\rho^{-3/2}$ remain. □

3 Conclusion and Extensions

We have extended radiation conditions for compactly supported excitations in layered isotropic media. This allows the construction of suitable bases for both theoretical and numerical use. Furthermore, one can extend this results via the same methodology to more layers or excitations outside the guide. However, the requirement of modal decompositions is crucial for the conditions to hold. Numerically, this can be implemented to enhance PML performance and constitutes a future line of work.

A Appendix

A.1 The method of steepest descents

We use the method of the steepest descents, to calculate the asymptotics for the residual terms of the form:

$$I(\lambda) \sim \int_C g(z) e^{\lambda\Phi(z)}\, dz \tag{73}$$

Theorem A.1. *Let all derivatives up to order $n-1$ vanish at a point z_0, i.e.,*

$$\left.\frac{d^q\Phi}{dz^q}\right|_{z=z_0} = 0 \qquad q = 1, \dots, n-1, \qquad \frac{1}{n!}\left.\frac{d^n\Phi}{dz^n}\right|_{z=z_0} = ae^{\imath\alpha} \quad a > 0$$

If $z - z_0 = \rho e^{\imath\theta}$, then the directions of steepest descent are given by

$$\Theta_p = -\frac{\alpha}{n} + (2p+1)\frac{\pi}{n} \qquad p = 0, \dots n-1 \tag{74}$$

Proof. See the proofs in [1], [4]. □

Remark A.1. A generalization of the above for non-integer n is obtained by setting:

$$\Phi(z) \sim \Phi(z_0) + ae^{\imath\alpha}(z - z_0)^n$$

in some sector of the z-plane with apex in z_0. Then the directions of steepest descent at z_0 are also given by (74).

A.2 Procedure and formulae

The method can be divided into the following steps:

1. We identify the potentially critical points in the integrand such as: integration endpoints; poles; branch points; and saddle points.
2. We find paths of steepest descent for each point – except for poles. These must satisfy $\mathrm{Im}\,(\Phi(z)) = \mathrm{Im}\,(\Phi(z_0))$.
3. The original contours are deformed using Cauchy's integral theorem onto paths of steepest descents.
4. Far-field expressions are found for each path required, and then added so as to obtain the total integral asymptotic.

According to the following cases, we present the associated asymptotics:

- *Saddle point at regular point of* $g(z)$:

$$I(\lambda) \sim \frac{g(z_0)}{n} \left[\frac{n!}{\lambda \left| \Phi^{(n)}(z_0) \right|} \right]^{1/n} \Gamma\left(\frac{1}{n}\right) e^{\lambda \Phi(z_0) + \imath \Theta_p} \tag{75}$$

- *Saddle point in* $\Phi(z)$ *and branch point in* $g(z)$: we write

$$g(z) \sim g_0 (z - z_0)^{\beta - 1} \qquad z \to z_0$$

  yielding

$$I(\lambda) \sim \frac{g_0}{n} \left[\frac{n!}{\lambda \left| \Phi^{(n)}(z_0) \right|} \right]^{\beta/n} \Gamma\left(\frac{\beta}{n}\right) e^{\lambda \Phi(z_0) + \imath \beta \Theta_p} \tag{76}$$

- *Branch point in both* $\Phi(z)$ *and* $g(z)$: we write

$$\Phi(z) \sim \Phi(z_0) + a e^{\imath \alpha} (z - z_0)^n \tag{77}$$

  and the approximation becomes, where now $n \in \mathbb{R}$

$$I(\lambda) \sim \frac{g_0}{n} \frac{1}{(\lambda a)^{\beta/n}} \Gamma\left(\frac{\beta}{n}\right) e^{\lambda \Phi(z_0) + \imath \beta \Theta_p} \tag{78}$$

- *Only a branch point in* $g(z)$ *and* $n = 1$: we write

$$I(\lambda) \sim \frac{g_0}{(\lambda |\Phi'(z_0)|)^{\beta}} \Gamma(\beta)\, e^{\lambda \Phi(z_0) + \imath \beta \Theta_1} \tag{79}$$

  where $\Theta_1 = \pi - \alpha$.

- *Branch point only in* $\Phi(z)$:

$$I(r) \sim \frac{g(z_0)}{n} \frac{1}{(\lambda a)^{1/n}} \Gamma\left(\frac{1}{n}\right) e^{\lambda \Phi(z_0) + \imath \Theta_p} \tag{80}$$

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