Eidgenössische Technische Hochschule Zürich
Ecole polytechnique fédérale de Zurich
Politecnico federale di Zurigo
Swiss Federal Institute of Technology Zurich

Approximation by Plane Waves

A. Moiola, R. Hiptmair and I. Perugia*

Research Report No. 2009-27
August 2009

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

*Dipartimento di Matematica, Università di Pavia, ilaria.perugia@unipv.it

Approximation by Plane Waves

A. Moiola[1] R. Hiptmair[2] I. Perugia[3]

August 28, 2009

[1]SAM, ETH Zurich, CH-8092 Zürich, andrea.moiola@sam.math.ethz.ch
[2]SAM, ETH Zurich, CH-8092 Zürich, hiptmair@sam.math.ethz.ch
[3]Dipartimento di Matematica, Università di Pavia, ilaria.perugia@unipv.it

Contents

Abstract

This paper studies the approximation of the solutions of the homogeneous Helmholtz equation by finite dimensional spaces of plane, circular and spherical wave functions. The main results are the proofs of:

- algebraic convergence in the domain size h in two and three dimensions;
- algebraic convergence in the number p of approximating functions in two dimensions.

The approximation error is measured in weighted Sobolev norms; the dependence of all bounds on the wavenumber is made explicit.

The proofs rely on an explicit formulation of Vekua's theory for N-dimensional Helmholtz equation ($N \geq 2$) and on approximation properties for harmonic functions.

The obtained estimates can be used in the analysis of the convergence of several Trefftz-type finite elements methods.

Introduction

Standard polynomial finite element methods for the Helmholtz equation can be computationally very expensive, in particular when the size of the domain is much larger than the wavelength. This is due to the fact that a high number of degrees of freedom are necessary in order to resolve the oscillations of the analytical solution to be approximated.

A possible remedy to this problem is the use of Trefftz methods, where the finite element spaces are made by functions that are solution of the (adjoint) PDE in each element of the mesh.

For the homogeneous Helmholtz equation, different methods which exploit this idea are available in the literature. A first example is the Ultra Weak Variational Formulation (UWVF) introduced by Cessenat and Després in [13]. Then we mention the Discontinuous Enrichment Method (DEM, see for instance [21]) and the Plane Wave Discontinuous Galerkin method (PWDG, see [26, 28]), which is a generalization of the UWVF. In all these methods, the approximating spaces are made by plane wave functions, while in the Method of Fundamental Solutions (MFS, see [8]), singular solutions of the Helmholtz equation are used instead. Other methods employ plane wave functions modulated either by polynomials or by partition of unit functions; the Partition of Unity Method (PUM, see [5]) is based on this last approach.

The convergence analysis of each of these techniques requires a *best approximation estimate*: the finite element space must contain a function which approximates the analytic solution of the problem with an error that goes to zero when the mesh size h is reduced (h–convergence), or when the dimension p of the local approximating space is raised (p–convergence). This error is usually measured in Sobolev norms and an accurate estimate of the convergence rate with respect to the parameters h and p is also important.

Only few results of this kind for plane wave spaces are available in literature. The first one is contained in Theorem 3.7 of [13]: the proof was based on Taylor expansion and only h-convergence for two-dimensional domains was proved; moreover, the obtained order of convergence is not sharp. This result has been exploited in [12], combined with a duality technique, to bound the approximation error measured in L^2-norm of the UWVF. A more sophisticated result is Proposition 8.4.14 of [32]: in this case, p-estimates were obtained in the two-dimensional case by using complex analysis techniques and Vekua's theory. A similar approach was used in [35] to prove sharp estimates in h for the PWDG method; there, the dependence on the wavenumber was made explicit. In [26, Prop. 3.12, 3.13], a best approximation estimate for general H^2–functions in two space dimensions was proved. This was used to prove h-convergence of the PWDG method in both the L^2 and H^1–norms in the two-dimensional case.

In this paper, we adopt a similar approach to the one in [32, 35] and prove some more general and sharper best approximation estimates in weighted Sobolev norms. The major novelties are:

- the proof of sharp algebraic orders of convergence with respect to h in two and three dimensions;
- the proof of explicit algebraic orders of convergence with respect to p in

two dimensions;

- an explicit dependence of all the bounding constants on the wavenumber;
- the proof of best approximation estimates for both plane wave spaces and circular and spherical wave spaces.

These improvements with respect to [32] have been made possible by a more explicit definition of the Vekua operators (see Definition 1.1.4), and by the use of harmonic analysis techniques, instead of complex analysis techniques.

The final results (Theorems 2.2.1, 3.2.2 and 3.2.3) can be used in the analysis of the convergence of all the above cited Trefftz methods.

The approximation theory presented here is not completely satisfactory: in the three-dimensional case, two main gaps still have to be filled. Firstly, in Theorem 2.2.1 we show algebraic p-convergence for the approximation of solutions to the homogeneous Helmholtz equation by three-dimensional generalized harmonic polynomials, but we can not prove a reasonable explicit order of convergence. Secondly, we are not able to approximate a generalized harmonic polynomial using plane waves with an explicit dependence on the number of the approximating functions; this fact prevents us from proving p–convergence in three dimensions (see Lemma 3.1.6).

The outline of this paper is the following. In Chapter 1, we introduce Vekua's theory for the particular case of the N-dimensional Helmholtz equation and we prove some basic results. We also introduce a class of functions called *generalized harmonic polynomials*, that correspond to circular and spherical waves in two and three dimensions, respectively. In Chapter 2, we prove best approximation estimates of homogeneous Helmholtz solutions by generalized harmonic polynomials using harmonic analysis techniques. Finally, in Chapter 3, we use Jacobi-Anger expansions to approximate generalized harmonic polynomials by plane waves and obtain the final best approximation estimates of homogeneous Helmholtz solutions by plane wave functions.

Theorem 1.1.5 was already stated in [41], but the proof given in this paper is new; apart from Theorems 2.1.2, 2.1.4 and 2.1.8, all the other results presented in this paper are new, although many ideas in the first two chapters come from the work of M. Melenk (see [32, 33]).

Notation

In order to prove inequalities with constants that are explicit and sharp with respect to the indices, we need to fix the definition of Sobolev norms and semi-norms. Equivalent norms give different bounds. We denote by $\mathbb{N}$ the set of natural numbers, including 0. The definitions we need are the following:

$$
\begin{aligned}
B_r(x_0) &= \{x \in \mathbb{R}^N,\ |x - x_0| < r\}, && B_r = B_r(0),\\
S^{N-1} &= \partial B_1 \subset \mathbb{R}^N,\\
D^\alpha \phi &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}, \qquad |\alpha| = \sum_{j=1}^{N} \alpha_j && \forall\, \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N,\\
|u|_{W^{k,p}(\Omega)} &= \left(\sum_{\alpha \in \mathbb{N}^N, |\alpha| = k} \int_\Omega |D^\alpha u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}},\\
\|u\|_{W^{k,p}(\Omega)} &= \left(\sum_{j=1}^{k} |u|^p_{W^{j,p}(\Omega)} \right)^{\frac{1}{p}} = \left(\sum_{\alpha \in \mathbb{N}^N, |\alpha| \le k} \int_\Omega |D^\alpha u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}},\\
|u|_{k,\Omega} &= |u|_{W^{k,2}(\Omega)},\\
\|u\|_{k,\Omega} &= \|u\|_{W^{k,2}(\Omega)},\\
|u|_{W^{k,\infty}(\Omega)} &= \sup_{\alpha \in \mathbb{N}^N, |\alpha| = k} \operatorname*{ess\,sup}_{x \in \Omega} |D^\alpha u(x)|,\\
\|u\|_{W^{k,\infty}(\Omega)} &= \sup_{j = 0, \dots, k} |u|_{W^{j,\infty}(\Omega)},\\
\|u\|_{k,\omega,\Omega} &= \left(\sum_{j=0}^{k} \omega^{2(k-j)} \, |u|^2_{j,\Omega} \right)^{\frac{1}{2}} && \forall\, \omega > 0, \qquad (1)\\
\mathcal{H}^j(D) &:= \left\{ \phi \in H^j(D) :\ \Delta \phi = 0 \right\} && \forall\, j \in \mathbb{N},\\
\mathcal{H}^j_\omega(D) &:= \left\{ u \in H^j(D) :\ \Delta u + \omega^2 u = 0 \right\} && \forall\, j \in \mathbb{N},\ \omega \in \mathbb{C},\\
n(N,l) &:= \begin{cases} 1 & \text{if } l = 0,\\ \dfrac{(2l + N - 2)(l + N - 3)!}{l!\,(N-2)!} & \text{if } l \ge 1, \end{cases} \qquad (2)\\
&= \begin{cases} 1 & \text{if } l = 0,\\ N & \text{if } l = 1,\\ \dbinom{N + l - 1}{N - 1} - \dbinom{N + l - 3}{N - 1} & \text{if } l \ge 2. \end{cases}
\end{aligned}
$$

The last expression $n(N,l)$ is the number of the independent spherical harmonics of degree l in $\mathbb{R}^N$, see [36, eq. (11)] and [4, Prop. 5.8].

Chapter 1

Vekua's Theory for the Helmholtz Operator

Vekua's theory (see [27, 41]) is a tool for transferring properties from harmonic functions (solutions to the Laplace equation) to solutions to general second order elliptic PDE's. The so-called Vekua operators (inverse one of another) map harmonic functions to solutions of the second order elliptic PDE of interest and vice versa. Their continuity properties are essential in order to make explicit the dependence on the space dimension, on the considered domain and on the parameters appearing in the PDE of constants in estimates.

The original formulation takes into account elliptic PDEs with analytic coefficients in two space dimensions. Some generalizations to higher space dimensions have been made (see [14–16, 24, 29, 30] and the references therein) but the Vekua operators in these general case are not completely explicit.

Here, the PDE we are interested in is the homogeneous Helmholtz equation. In this particular case, simple explicit integral operators have been defined in the original work of Vekua in any space dimension $N \geq 2$ (see [39, 40], and [41, p. 59]), but there is no proof of their properties and, in our knowledge, these results have never been used later on.

Thus, we will start by defining the Vekua operators for Helmholtz equation and $N \geq 2$ and prove their basic properties, namely, that they are inverse to each other and map harmonic functions to solutions of the homogeneous Helmholtz equation and vice versa (see Theorem 1.1.5). Next, we establish their continuity properties in (weighted) Sobolev norms, like in [32], but with continuity constant explicit in the domain shape parameter, in the Sobolev regularity exponent and in the product of the wavenumber times the diameter of the domain (see Theorem 1.2.1). The main difficulty in proving these continuity estimates consists in establishing precise interior estimates. Finally, we introduce the generalized harmonic polynomials, which are the mapping through the direct Vekua operator of the harmonic polynomials, and derive their explicit expression. All the proofs are self-contained and do not need the use of other results concerning Vekua's theory.

1.1 N–Dimensional Vekua's Theory for the Helmholtz Operator

We will always consider a domain that satisfies the following assumption.

Assumption 1.1.1. *Let $D \subset \mathbb{R}^N$, $N \geq 2$, be a bounded open set such that*

- $\operatorname{diam} D = h$;
- *∂D is Lipschitz;*
- *there exists $\rho \in (0, 1/2]$ such that $B_{\rho h} \subseteq D$;*
- *there exists $0 < \rho_0 \leq \rho$ such that D is star-shaped with respect to $B_{\rho_0 h}$.*

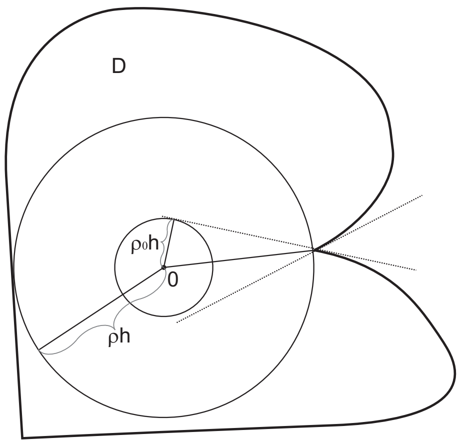
Not all these assumptions are necessary in order to establish the first results of this chapter (see Remark 1.1.7 below).

Remark 1.1.2. *If D is a domain as in Assumption 1.1.1, then*

$$B_{\rho h} \subseteq D \subseteq B_{(1-\rho)h}.$$

The maximum $1/2$ for the parameter ρ is achieved when the domain is a sphere: $D = B_{\frac{h}{2}}$.

Figure 1.1: A domain D that satisfies Assumption 1.1.1



Definition 1.1.3. *Given a positive number ω, we define two continuous functions*

$$M_1, M_2 : D \times [0, 1) \to \mathbb{R},$$

$$\begin{aligned} M_1(x,t) &= -\frac{\omega|x|}{2} \, \frac{\sqrt{t}^{N-2}}{\sqrt{1-t}} \, J_1(\omega|x|\sqrt{1-t}), \\ M_2(x,t) &= -\frac{i\omega|x|}{2} \, \frac{\sqrt{t}^{N-3}}{\sqrt{1-t}} \, J_1(i\omega|x|\sqrt{t(1-t)}), \end{aligned} \tag{1.1}$$

where J_1 is the 1-st order Bessel function of the first kind.

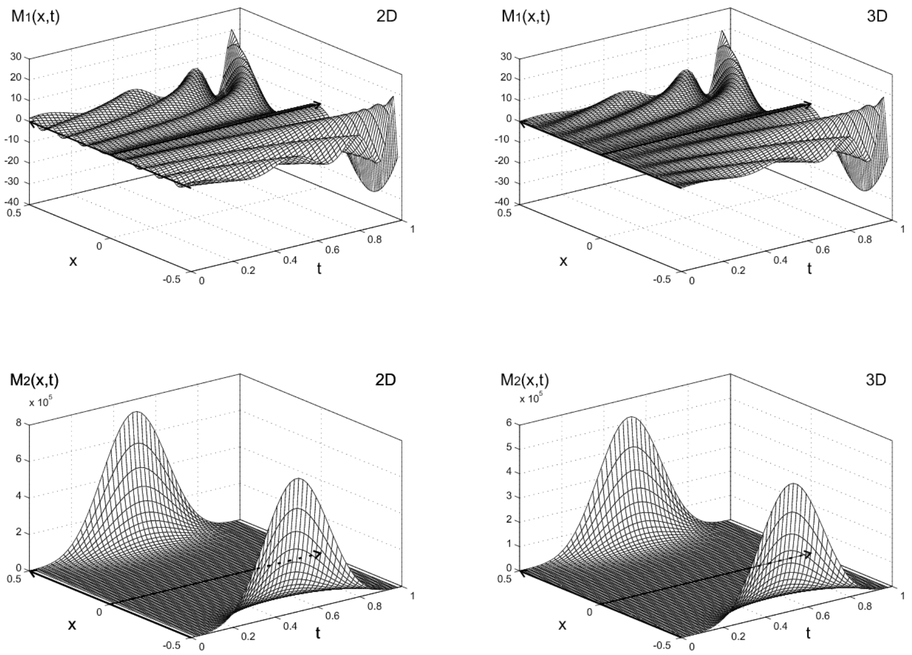
Using the expression (A.1) in the Appendix, we can write

$$M_1(x,t) = -t^{\frac{N}{2}-1} \sum_{k \geq 0} \frac{(-1)^k \left(\frac{\omega|x|}{2}\right)^{2k+2} (1-t)^k}{k!\,(k+1)!},$$

$$M_2(x,t) = \sum_{k \geq 0} \frac{\left(\frac{\omega|x|}{2}\right)^{2k+2} (1-t)^k \, t^{k+\frac{N}{2}-1}}{k!\,(k+1)!}.$$

Note that M_1 and M_2 are radially symmetric and belong to $C^\infty(D \times (0,1])$; if N is even, they are C^∞ in the whole domain.

Figure 1.2: The functions $M_1(x,t)$ and $M_2(x,t)$ in two and three dimensions with $\omega = 50$. Only a segment of D is represented.



Definition 1.1.4. *We define the Vekua operator V_1 and the inverse Vekua operator V_2 for the Helmholtz equation:*

$$V_1, V_2 : L^\infty(D) \to L^\infty(D),$$

$$V_j[\phi](x) = \phi(x) + \int_0^1 M_j(x,t)\phi(tx)\,\mathrm{d}t \qquad \forall \phi \in L^\infty(D),\ a.e.\ x \in D,\ j = 1, 2. \tag{1.2}$$

Notice that $M_j(x,\cdot)\phi(\cdot x)$, $j = 1, 2$, belong to $L^1([0,1])$ for almost every $x \in D$; consequently, V_1 and V_2 are well defined. The operators V_1 and V_2 can also be defined from the space of continuous function in the domain $C(D)$

to itself, or from $L^p(D)$ to $L^2(D)$, with a sufficiently high p depending on the dimension. We will call $V_1[\phi]$ the Vekua transform of ϕ.

In the following theorem, we summarize general results about the Vekua operators, while their continuity will be proved in Theorem 1.2.1 below.

Theorem 1.1.5. *Let D be a domain as in the Assumption 1.1.1; the Vekua operators satisfy:*

(i) *V_2 is the inverse of V_1:*

$$V_1\big[V_2[\phi]\big] = V_2\big[V_1[\phi]\big] = \phi \qquad\qquad \forall\, \phi \in L^\infty(D)\ .$$

(ii) *If ϕ is harmonic in D, i.e.,*

$$\Delta\phi = 0 \qquad in\ D\ , \tag{1.3}$$

then

$$\Delta V_1[\phi] + \omega^2 V_1[\phi] = 0 \qquad in\ D\ ;$$

if u is a solution of the homogeneous Helmholtz equation with wavenumber $\omega > 0$ in D, i.e.,

$$\Delta u + \omega^2 u = 0 \qquad in\ D\ , \tag{1.4}$$

then

$$\Delta V_2[u] = 0 \qquad in\ D\ .$$

Theorem 1.1.5 states that the operators V_1 and V_2 are inverse to each other and map harmonic functions to solutions of the homogeneous Helmholtz equation and vice versa.

The results of this theorem were stated in [41, Chapter 1, § 13.2-3]. In two space dimensions, the operator V_1 was introduced as a special case of the general Vekua's theory for elliptic PDEs; this implies that V_1 is a bijection between the space of complex harmonic function and the space of solutions of the homogeneous Helmholtz equation. The proof in higher space dimensions is probably contained only in the part of the paper [39] written in Georgian, that is not easy to obtain. The fact that the inverse of V_1 can be written as the operator V_2 (part *(i)* of Theorem 1.1.5) was stated in [40], and the proof was sketched as an "easy calculation", after reducing the problem to a one-dimensional Volterra integral equation. Here, we give a completely self-contained proof of Theorem 1.1.5 in its generality.

As in Theorem 1.1.5, in the following we will usually denote the solutions of the homogeneous Helmholtz equation with the letter u, and harmonic functions, as well as generic functions defined on D, with the letter ϕ.

Remark 1.1.6. *Theorem 1.1.5 holds with the same proof also for every $\omega \in \mathbb{C}$, i.e., for the Helmholtz equation in lossy materials.*

Remark 1.1.7. *Theorem 1.1.5 holds also for an unbounded or irregular domain: the only necessary hypotheses are that D has to be open and star-shaped with respect to the origin. In fact the proof relies only on the local properties of the functions on the segment $[0, x]$.*

The hypothesis of being star-shaped with respect to a ball ($\rho_0 > 0$) will be used only in the next chapters to prove approximation properties; here it is enough that D is star-shaped with respect to the origin.

Theorem 1.1.5 can be proved by using elementary mathematical analysis results. We proceed by proving the parts *(i)* and *(ii)* separately.

Proof of Theorem 1.1.5, part (i). We define a function

$$g : [0,\infty) \times [0,\infty) \to \mathbb{R},$$
$$g(r,t) = \frac{\omega\sqrt{r}\, t}{2\sqrt{r-t}}\, J_1(\omega\sqrt{r}\sqrt{r-t}).$$

Note that if $r < t$ the argument of the Bessel function J_1 is imaginary on the standard branch cut but the function g is always real-valued.

Using the change of variable $s = t|x|$, for every $\phi \in L^\infty(D)$ and for almost every $x \in D$, we can compute

$$\begin{aligned}
V_1[\phi](x) &= \phi(x) + \int_0^{|x|} M_1\Big(x, \frac{s}{|x|}\Big)\phi\Big(s\frac{x}{|x|}\Big)\frac{1}{|x|}\,\mathrm{d}s \\
&= \phi(x) - \int_0^{|x|} \frac{\omega|x|}{2}\sqrt{\frac{s}{|x|}}^{N-2} \frac{\sqrt{|x|}}{\sqrt{|x|-s}}\,\frac{1}{|x|}\, J_1\Big(\omega\sqrt{|x|}\sqrt{|x|-s}\Big)\,\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s \\
&= \phi(x) - \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(|x|,s)\;\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s,
\end{aligned}$$

$$\begin{aligned}
V_2[\phi](x) &= \phi(x) + \int_0^{|x|} M_2\Big(x, \frac{s}{|x|}\Big)\phi\Big(s\frac{x}{|x|}\Big)\frac{1}{|x|}\,\mathrm{d}s \\
&= \phi(x) - \int_0^{|x|} \frac{i\omega|x|}{2}\sqrt{\frac{s}{|x|}}^{N-3} \frac{\sqrt{|x|}}{\sqrt{|x|-s}}\,\frac{1}{|x|}\, J_1\Big(i\omega\sqrt{s}\sqrt{|x|-s}\Big)\,\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s \\
&= \phi(x) + \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(s,|x|)\;\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s
\end{aligned}$$

because $s \le |x|$ and we have fixed the sign $\sqrt{s-|x|} = i\sqrt{|x|-s}$. Note that in the expression of the two operators the arguments of the functions g are exchanged. Now we apply the first operator after the second one, switch the order of the integration in the obtained double integral and get

$$\begin{aligned}
V_1\big[V_2[\phi]\big](x) &= \left[\phi(x) + \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(s,|x|)\;\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s\right] \\
&\quad - \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\; g(|x|,s)\left[\phi\Big(s\frac{x}{|x|}\Big) + \int_0^s \frac{z^{\frac{N-4}{2}}}{s^{\frac{N-2}{2}}} g(z,s)\phi\Big(z\frac{x}{|x|}\Big)\,\mathrm{d}z\right]\mathrm{d}s \\
&= \phi(x) + \int_0^{|x|} \frac{s^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\;\big(g(s,|x|) - g(|x|,s)\big)\phi\Big(s\frac{x}{|x|}\Big)\,\mathrm{d}s \\
&\quad - \int_0^{|x|} \frac{z^{\frac{N-4}{2}}}{|x|^{\frac{N-2}{2}}}\;\phi\Big(z\frac{x}{|x|}\Big)\int_z^{|x|} \frac{1}{s}\, g(z,s)\, g(|x|,s)\;\mathrm{d}s\,\mathrm{d}z.
\end{aligned}$$

Notice that $V_1\big[V_2[\phi]\big] = V_2\big[V_1[\phi]\big]$, so we only have to show that V_2 is right inverse of V_1.

In order to prove that $V_1\big[V_2[\phi]\big] = \phi$ it is enough to show that

$$g(t,r) - g(r,t) = \int_t^r \frac{g(t,s)\; g(r,s)}{s}\; \mathrm{d}s \qquad\qquad \forall\, r \geq t \geq 0, \tag{1.5}$$

so that all the integrals in the previous expression vanish, and we are done. Using (A.1), we expand g in power series (recall that, for $k \geq 0$ integer, $\Gamma(k+1) = k!$):

$$g(r,t) = \frac{\omega^2\; r\; t}{4} \sum_{l \geq 0} \frac{(-1)^l\; \omega^{2l}\; r^l\; (r-t)^l}{2^{2l}\; l!\; (l+1)!},$$

from which we get

$$g(t,r) - g(r,t) = \frac{\omega^2\; r\; t}{4} \sum_{l \geq 0} \frac{(-1)^l\; \omega^{2l}\; (r-t)^l\; \big((-t)^l - r^l\big)}{2^{2l}\; l!\; (l+1)!}. \tag{1.6}$$

We compute the following integral using the change of variables $z = \frac{s-t}{r-t}$ and the expression of the beta integral $\int_0^1 (1-z)^p z^q\, \mathrm{d}z = B(p+1, q+1) = \dfrac{p!\; q!}{(p+q+1)!}$:

$$\begin{aligned}\int_t^r s(r-s)^j (t-s)^k\, \mathrm{d}s &= (-1)^k\; (r-t)^{j+k+1} \int_0^1 (1-z)^j\; z^k\; \big(zr + (1-z)t\big)\, \mathrm{d}z \\ &= (-1)^k\; (r-t)^{j+k+1}\; \frac{j!\; k!}{(j+k+2)!} \big(r(k+1) + t(j+1)\big).\end{aligned}$$

Thus, expanding the product of $g(t,s)\; g(r,s)$ in a double power series, integrating term by term and using the previous identity give

$$\begin{aligned}&\int_t^r \frac{g(t,s)\; g(r,s)}{s}\; \mathrm{d}s \\ &= \frac{\omega^2\; r\; t}{4} \sum_{j,k \geq 0} \frac{(-1)^{j+k}\; \omega^{2(j+k+1)}\; r^j\; t^k}{2^{2(j+k+1)}\; j!\; (j+1)!\; k!\; (k+1)!} \int_t^r \frac{s^2 (r-s)^j (t-s)^k}{s}\, \mathrm{d}s \\ &= \frac{\omega^2\; r\; t}{4} \sum_{j,k \geq 0} \frac{(-1)^j\; \omega^{2(j+k+1)}\; r^j\; t^k\; (r-t)^{j+k+1}}{2^{2(j+k+1)}\; (j+1)!\; (k+1)!\; (j+k+2)!} \big(r(k+1) + t(j+1)\big) \\ &\overset{(l=j+k+1)}{=} \frac{\omega^2\; r\; t}{4} \sum_{l \geq 1} \frac{\omega^{2l}\; (r-t)^l}{2^{2l}\; (l+1)!}\; \frac{1}{l!} \sum_{j=0}^{l-1} l!\; \frac{(-1)^j\; r^j\; t^{l-j-1}}{(j+1)!\; (l-j)!} \big(r(l-j) + t(j+1)\big) \\ &= \frac{\omega^2\; r\; t}{4} \sum_{l \geq 1} \frac{\omega^{2l}\; (r-t)^l}{2^{2l}\; (l+1)!\; l!} \sum_{j=0}^{l-1} \left[-\binom{l}{j+1} (-r)^{j+1}\; t^{l-j-1} + \binom{l}{j} (-r)^j\; t^{l-j} \right] \\ &= \frac{\omega^2\; r\; t}{4} \sum_{l \geq 1} \frac{\omega^{2l}\; (r-t)^l}{2^{2l}\; (l+1)!\; l!} \left[-(t-r)^l + t^l + (t-r)^l - (-r)^l \right] = g(t,r) - g(r,t),\end{aligned}$$

thanks to the binomial theorem and (1.6), where the term corresponding to $l = 0$ is zero. This proves (1.5), and the proof is complete. □

Proof of Theorem 1.1.5, part (ii). Let $\phi \in L^\infty(D)$ be a harmonic function, then $\phi \in C^\infty(D)$, thanks to the regularity theorem for harmonic functions (see, e.g., [20, Theorem 3, Section 6.3.1]). We prove that $(\Delta+\omega^2)V_1[\phi](x)=0$. In order to do that, we establish some useful identities.

We set $r := |x|$ and compute

$$\begin{aligned}
\frac{\partial}{\partial r}M_1(x,t) &= \omega\sqrt{1-t}\frac{\partial}{\partial(\omega r\sqrt{1-t})}\left[-\frac{\sqrt{t}^{N-2}}{2(1-t)}\;\omega r\sqrt{1-t}\;\;J_1(\omega r\sqrt{1-t})\right] \\
&\overset{(A.6)}{=} -\frac{\omega^2 r\sqrt{t}^{N-2}}{2}\;J_0(\omega r\sqrt{1-t}), \\
\Delta M_1(x,t) &= \frac{N-1}{r}\frac{\partial}{\partial r}M_1(x,t)+\frac{\partial^2}{\partial r^2}M_1(x,t) \\
&= -\frac{\omega^2\sqrt{t}^{N-2}}{2}\left(N\,J_0(\omega r\sqrt{1-t})-\omega r\sqrt{1-t}\,J_1(\omega r\sqrt{1-t})\right),
\end{aligned} \tag{1.7}$$

where the Laplacian acts on the x variable.

Since M_1 depends on x only through r, we can compute

$$\begin{aligned}
\Delta\Big(M_1(x,t)\phi(tx)\Big) &= \Delta M_1(x,t)\;\phi(tx)+2\nabla M_1(x,t)\cdot\nabla\phi(tx)+M_1(x,t)\Delta\phi(tx) \\
&= \Delta M_1(x,t)\;\phi(tx)+2\frac{\partial}{\partial r}M_1(x,t)\frac{x}{r}\cdot t\nabla\phi\Big|_{tx}+0 \\
&= \Delta M_1(x,t)\;\phi(tx)+2\frac{t}{r}\frac{\partial}{\partial r}M_1(x,t)\frac{\partial}{\partial t}\phi(tx),
\end{aligned}$$

because $\frac{\partial}{\partial t}\phi(tx)=x\cdot\nabla\phi\Big|_{tx}$.

Finally, we define an auxiliary function $f_1:[0,h]\times[0,1]\to\mathbb{R}$ by

$$f_1(r,t)=\sqrt{t}^N\;J_0(\omega r\sqrt{1-t}).$$

This function verifies

$$\begin{aligned}
\frac{\partial}{\partial t}f_1(r,t) &= \frac{N\sqrt{t}^{N-2}}{2}\;J_0(\omega r\sqrt{1-t})+\frac{\sqrt{t}^N\omega r}{2\sqrt{1-t}}J_1(\omega r\sqrt{1-t}), \\
f_1(r,0) &= 0, \qquad\qquad f_1(r,1)=1.
\end{aligned}$$

At this point, we can use all these identities to prove that $V_1[\phi]$ is a solution of the homogeneous Helmholtz equation:

$$\begin{aligned}
&(\Delta+\omega^2)V_1[\phi](x) \\
&= \Delta\phi(x)+\omega^2\phi(x)+\int_0^1\Delta\Big(M_1(x,t)\phi(tx)\Big)\,\mathrm{d}t+\int_0^1\omega^2 M_1(x,t)\phi(tx)\,\mathrm{d}t \\
&= \omega^2\phi(x)-\omega^2\int_0^1\sqrt{t}^N\;J_0(\omega r\sqrt{1-t})\frac{\partial}{\partial t}\phi(tx)\,\mathrm{d}t \\
&\quad-\omega^2\int_0^1\left(\frac{N\sqrt{t}^{N-2}}{2}\;J_0(\omega r\sqrt{1-t})-\frac{\omega r\sqrt{t}^{N-2}}{2}\frac{1-t}{\sqrt{1-t}}\;J_1(\omega r\sqrt{1-t})\right. \\
&\quad\left.+\frac{\omega r\sqrt{t}^{N-2}}{2\sqrt{1-t}}J_1(\omega r\sqrt{1-t})\right)\phi(tx)\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \omega^2\phi(x) - \omega^2 \int_0^1 \left(f_1(r,t)\frac{\partial}{\partial t}\phi(tx) + \frac{\partial}{\partial t} f_1(r,t)\phi(tx) \right) \mathrm{d}t \\
&= \omega^2 \left(\phi(x) - \Big[f_1(r,t)\phi(tx) \Big]_{t=0}^{t=1} \right) = 0.
\end{aligned}
$$

We have used the values assumed by ϕ only in the segment $[0, x]$ that lies inside D, because D is star-shaped with respect to 0. Thus, the values of the function ϕ and of its derivative are well defined and the fundamental theorem of calculus applies, thanks to the regularity theorem for harmonic functions.

Let now $u \in L^2(D)$ be a solution of the homogeneous Helmholtz equation. Since the mentioned regularity theorem holds also for the solutions of the homogeneous Helmholtz equation, then $u \in C^\infty(D)$. In order to prove that $\Delta V_2[u] = 0$, we proceed as before. We compute

$$
\frac{\partial}{\partial r} M_2(x,t) = \frac{\omega^2 r \sqrt{t}^{N-2}}{2} \, J_0(i\omega r\sqrt{t(1-t)}), \tag{1.8}
$$

$$
\begin{aligned}
\Delta M_2(x,t) &= \frac{\omega^2 \sqrt{t}^{N-2}}{2} \Big(N\, J_0(i\omega r\sqrt{t(1-t)}) & (1.9) \\
&\quad - i\omega r\sqrt{t(1-t)}\, J_1(i\omega r\sqrt{t(1-t)}) \Big), & (1.10)
\end{aligned}
$$

$$
\begin{aligned}
\Delta \left(M_2(x,t)u(tx) \right) &= \Delta M_2(x,t)u(tx) + 2\frac{t}{r}\frac{\partial}{\partial r} M_2(x,t) \frac{\partial}{\partial t} u(tx) & (1.11) \\
&\quad - \omega^2 t^2 M_2(x,t)u(tx),
\end{aligned}
$$

and we define the function

$$
f_2(r,t) = \sqrt{t}^N \, J_0(i\omega r\sqrt{t(1-t)}),
$$

which verifies

$$
\begin{aligned}
&\frac{\partial}{\partial t} f_2(r,t) = \frac{N\sqrt{t}^{N-2}}{2} \, J_0(i\omega r\sqrt{t(1-t)}) - \frac{\sqrt{t}^N \, i\omega r(1-2t)}{2\sqrt{t(1-t)}} J_1(i\omega r\sqrt{t(1-t)}), \\
&f_2(r,0) = 0, \qquad\qquad f_2(r,1) = 1.
\end{aligned}
$$

We conclude by computing the Laplacian of $V_2[u]$:

$$
\begin{aligned}
&\Delta V_2[u](x) = \Delta u(x) + \int_0^1 \Delta\Big(M_2(x,t)u(tx) \Big) \,\mathrm{d}t \\
&= -\omega^2 u(x) + \omega^2 \int_0^1 \sqrt{t}^N \, J_0(i\omega r\sqrt{t(1-t)}) \frac{\partial}{\partial t} u(tx) \,\mathrm{d}t \\
&\quad + \omega^2 \int_0^1 \frac{\sqrt{t}^{N-2}}{2} \Big(N\, J_0(i\omega r\sqrt{t(1-t)}) \\
&\quad - i\omega r\sqrt{t}\frac{1-t}{\sqrt{1-t}} J_1(i\omega r\sqrt{t(1-t)}) + \frac{i\omega r t\sqrt{t}}{\sqrt{1-t}} \, J_1(i\omega r\sqrt{t(1-t)}) \Big) \, u(tx) \,\mathrm{d}t \\
&= -\omega^2 u(x) + \omega^2 \int_0^1 \left(f_2(r,t)\frac{\partial}{\partial t} u(tx) + \frac{\partial}{\partial t} f_2(r,t)u(tx) \right) \,\mathrm{d}t = 0.
\end{aligned}
$$

□

Remark 1.1.8. *With a slight modification in the proof, it is possible to show that V_1 transforms the solutions of the homogeneous Helmholtz equation*

$$\Delta\phi + \omega_0^2\phi = 0$$

into solutions of

$$\Delta\phi + (\omega_0^2 + \omega^2)\phi = 0$$

for every ω and $\omega_0 \in \mathbb{C}$, and V_2 does the converse.

1.2 Continuity of the Vekua Operators

In the following theorem, we establish the continuity of V_1 and V_2 in Sobolev norms with continuity constants of the most explicit possible nature.

Theorem 1.2.1. *Let D be a domain as in the Assumption 1.1.1; the Vekua operators*

$$V_1 : \mathcal{H}^j(D) \to \mathcal{H}^j_\omega(D),$$
$$V_2 : \mathcal{H}^j_\omega(D) \to \mathcal{H}^j(D),$$

with $\mathcal{H}^j(D)$ and $\mathcal{H}^j_\omega(D)$ both endowed with the norm $\|\cdot\|_{j,\omega,D}$ defined in (1), are continuous. More precisely, for all space dimensions $N \geq 2$, for all ϕ and u in $H^j(D)$, $j \geq 0$, solutions to (1.3) and (1.4), respectively, the following continuity estimates hold:

$$\|V_1[\phi]\|_{j,\omega,D} \leq C_1(N)\ \rho^{\frac{1-N}{2}}\ (1+j)^{\frac{3}{2}N+\frac{1}{2}}\ e^j\ \left(1+(\omega h)^2\right)\ \|\phi\|_{j,\omega,D}\,, \tag{1.12}$$

$$\|V_2[u]\|_{j,\omega,D} \leq C_2(N,\omega h,\rho)\ (1+j)^{\frac{3}{2}N-\frac{1}{2}}\ e^j\ \|u\|_{j,\omega,D}\,, \tag{1.13}$$

where the constant $C_1 > 0$ depends only on the space dimension N, and $C_2 > 0$ depends also on the product ωh and the shape parameter ρ. Moreover, we can establish the following continuity estimates for V_2 with constants dependent only on N:

$$\|V_2[u]\|_{0,D} \leq C_N\ \rho^{\frac{1-N}{2}}\left(1+(\omega h)^4\right)e^{\frac{1}{2}(1-\rho)\omega h}\left(\|u\|_{0,D} + h\,|u|_{1,D}\right) \tag{1.14}$$
$$\text{if } N = 2,\ldots,5,\ u \in H^1(D),$$

$$\|V_2[u]\|_{j,\omega,D} \leq C_N\ \rho^{\frac{1-N}{2}}\ (1+j)^{2N-1}\ e^j\ \left(1+(\omega h)^4\right)\ e^{\frac{3}{4}(1-\rho)\omega h}\ \|u\|_{j,\omega,D} \tag{1.15}$$
$$\text{if } N = 2,3,\ j \geq 1,\ u \in H^j(D),$$

$$\|V_2[u]\|_{L^\infty(D)} \leq \left(1 + \frac{\left((1-\rho)\omega h\right)^2}{4}\ e^{\frac{1}{2}(1-\rho)\omega h}\right)\|u\|_{L^\infty(D)} \tag{1.16}$$
$$\text{if } N \geq 2,\ u \in L^\infty(D).$$

Theorem 1.2.1 states that the operators V_1 and V_2 preserve the Sobolev regularity when applied to harmonic functions and solutions of the homogeneous

Helmholtz equation (see Theorem 1.1.5). For these functions, these operators are continuous from $H^j(D)$ to itself with continuity constants that depend on the wavenumber ω only through the product ωh. In two and three space dimensions, we can make explicit the dependence of the bounds on ωh. The only exception is the L^2–continuity of V_2 (see (1.14)), where a weighted H^1–norm appears on the right-hand side; this is due to the poor explicit interior estimates available for the solutions of the homogeneous Helmholtz equation.

All the continuity constants are explicit with respect to the order of the Sobolev norm and depend on D only through its shape parameter ρ and its diameter h, the latter only appearing within the product ωh.

In literature, there exist many proofs of the continuity of V_1 and V_2 in L^∞–norm (in two space dimensions); see, for example, [10,19]. To our knowledge, the only continuity result in Sobolev norms is the one given in [32, Section 4.2]: this holds for general PDEs and for norms with non integer indices, but is restricted to the two-dimensional case, and the constants in the bounds are not explicit in the various parameters.

In order to prove Theorem 1.2.1, we need some preliminary results. For here on, if β is multi-index in $\mathbb{N}^N$, we will denote by D^β the corresponding differential operator with respect to the space variable $x \in \mathbb{R}^N$.

Lemma 1.2.2. *For $\xi = 1, 2$, $j \geq 0$ and $\phi \in H^j(D)$, we have*

$$
\begin{aligned}
&|V_\xi[\phi]|^2_{j,D} \leq 2\,|\phi|^2_{j,D} \\
&\quad + 2(j+1)^{3N-2} e^{2j} \sum_{k=0}^{j} \sup_{t\in[0,1]} |M_\xi(\cdot,t)|^2_{W^{j-k,\infty}(D)} \sum_{|\beta|=k} \int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{1.17}
$$

Proof. From Definition 1.1.4, we have

$$
\begin{aligned}
\left|V_\xi[\phi]\right|^2_{j,D} &\leq 2\,|\phi|^2_{j,D} + 2 \sum_{|\alpha|=j} \int_D \left| \int_0^1 D^\alpha \left(M_\xi(x,t)\phi(tx)\right) \mathrm{d}t \right|^2 \mathrm{d}x \\
&\leq 2\,|\phi|^2_{j,D} + 2 \sum_{|\alpha|=j} \int_D \int_0^1 \left| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} M_\xi(x,t) D^\beta \phi(tx) \right|^2 \mathrm{d}t\,\mathrm{d}x \\
&\leq 2\,|\phi|^2_{j,D} + 2 \int_D \int_0^1 \left| \sum_{k=0}^{j} \sum_{|\beta|=k} \left|D^\beta \phi(tx)\right| \sum_{\substack{|\alpha|=j \\ \alpha\geq\beta}} \binom{\alpha}{\beta} \left|D^{\alpha-\beta} M_\xi(x,t)\right| \right|^2 \mathrm{d}t\,\mathrm{d}x\ ,
\end{aligned}
$$

where in the second inequality we have applied the Jensen inequality and the product (Leibniz) rule for multi-indices (see [1, Sec. 1.1]); here, the binomial coefficient for multi-indices is $\binom{\alpha}{\beta} = \prod_{i=1}^N \binom{\alpha_i}{\beta_i}$. We multiply by the number $\binom{N+k-1}{N-1}$ of the multi-indices β of length k in $\mathbb{N}^N$, in order to move the square inside the sum, and we obtain

$$
\begin{aligned}
&\left|V_\xi[\phi]\right|^2_{j,D} \leq 2\,|\phi|^2_{j,D} + \\
&2 \int_D \int_0^1 (j+1) \sum_{k=0}^{j} \binom{N+k-1}{N-1} \sum_{|\beta|=k} \left|D^\beta \phi(tx)\right|^2 \left| \sum_{\substack{|\alpha|=j \\ \alpha\geq\beta}} \binom{\alpha}{\beta} \left|D^{\alpha-\beta} M_\xi(x,t)\right| \right|^2 \mathrm{d}t\,\mathrm{d}x
\end{aligned}
$$

$$\leq 2\,|\phi|^2_{j,D} + 2(j+1)\tbinom{N+j-1}{N-1} \sum_{k=0}^{j} \sum_{|\beta|=k} \int_D \int_0^1 \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}t\,\mathrm{d}x$$

$$\cdot \sup_{t\in[0,1]} |M_\xi(\cdot,t)|^2_{W^{j-k,\infty}(D)} \sup_{|\beta|=k} \left[\sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \binom{\alpha}{\beta} \right]^2 ;$$

the last factor can be bounded as

$$\sup_{|\beta|=k} \sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \prod_{i=1}^{N} \binom{\alpha_i}{\beta_i} \leq \sup_{|\beta|=k} \sum_{\substack{|\alpha|=j\\ \alpha\geq\beta}} \prod_{i=1}^{N} \frac{\alpha_i^{\beta_i}}{\beta_i!} \leq \sum_{|\alpha|=j} e^{\sum_{i=1}^N \alpha_i}$$

$$\leq e^j \cdot \#\{\alpha \in \mathbb{N}^N,\ |\alpha| = j\} = e^j \tbinom{N+k-1}{N-1} \leq e^j \tbinom{N+j-1}{N-1}.$$

Finally, we note that, for every $j \in \mathbb{N}$, $N \geq 2$, we have

$$\binom{N+j-1}{N-1} = \frac{N+j-1}{N-1}\,\frac{N+j-2}{N-2}\cdots\frac{1+j}{1} \leq (1+j)^{N-1}\,, \tag{1.18}$$

from which the assertion follows. ☐

Now we need to bound the terms present in (1.17). The next lemma provides $W^{j,\infty}(D)$ estimates for M_1 and M_2 uniformly in t. The proof relies on some properties of Bessel functions.

Lemma 1.2.3. *The functions M_1 and M_2 satisfy the following bounds:*

$$\|M_1\|_{L^\infty(D\times[0,1])} \leq \frac{\big((1-\rho)\,\omega\,h\big)^2}{4}, \tag{1.19}$$

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{1,\infty}(D)} \leq \frac{(1-\rho)\,\omega^2\,h}{2}, \tag{1.20}$$

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{\omega^j}{2}\,(j + (1-\rho)\,\omega\,h) \qquad \forall j\geq 2, \tag{1.21}$$

$$\|M_2\|_{L^\infty(D\times[0,1])} \leq \frac{\big((1-\rho)\,\omega\,h\big)^2}{4}\, e^{\frac{1}{2}(1-\rho)\omega h}\,, \tag{1.22}$$

$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{1,\infty}(D)} \leq \frac{(1-\rho)\,\omega^2\,h}{2}\, e^{\frac{1}{2}(1-\rho)\omega h}, \tag{1.23}$$

$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{\omega^j}{2^{j-1}} \left(j + \frac{(1-\rho)\,\omega\,h}{2}\right) e^{\frac{3}{4}(1-\rho)\omega h} \qquad \forall j \geq 2. \tag{1.24}$$

$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{\omega^j}{2^{j-1}} \left(j + \frac{(1-\rho)\,\omega\,h}{2}\right) \cdot \left(1 + \left(\frac{(1-\rho)\,\omega\,h}{4}\right)^{j+1}\right) e^{\frac{1}{2}(1-\rho)\omega h}$$

Proof. Thanks to the Remark 1.1.2, we have that $\sup_{x\in D}|x| \leq (1-\rho)\,h$.

The L^∞ inequalities (1.19) and (1.22) follow directly from (A.4).

Since M_1 and M_2 depend on x only through $|x|$, we obtain the $W^{1,\infty}$ bounds (1.20) and (1.23):

$$\begin{aligned}
\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{1,\infty}(D)} &= \sup_{t\in[0,1],\, x\in D} \left|\frac{\partial}{\partial|x|} M_1(x,t)\right| \\
&\overset{(A.6)}{\leq} \sup_{\substack{t\in[0,1],\\ |x|\in[0,(1-\rho)h]}} \left|\frac{\omega^2|x|\sqrt{t}^{N-2}}{2}\ J_0(\omega|x|\sqrt{1-t})\right| \\
&\overset{(A.3)}{\leq} \frac{(1-\rho)\,\omega^2\,h}{2}, \\
\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{1,\infty}(D)} &\overset{(A.6)}{\leq} \sup_{\substack{t\in[0,1],\\ |x|\in[0,(1-\rho)h]}} \left|\frac{\omega^2|x|\sqrt{t}^{N-2}}{2}\ J_0(i\omega|x|\sqrt{t(1-t)})\right| \\
&\overset{(A.4)}{\leq} \frac{(1-\rho)\,\omega^2\,h}{2}\ e^{\frac{1}{2}(1-\rho)\omega h}.
\end{aligned}$$

In order to prove (1.21) and (1.24), we define an auxiliary complex-valued function

$$f(s) = s\ J_1(s).$$

It is easy to verify by induction that its derivative of order k is

$$\frac{\partial^k}{\partial s^k} f(s) = k\ \frac{\partial^{k-1}}{\partial s^{k-1}} J_1(s) + s\ \frac{\partial^k}{\partial s^k} J_1(s).$$

We can bound this derivative using (A.7) and the binomial theorem:

$$\begin{aligned}
&\left|\frac{\partial^k}{\partial s^k} f(s)\right| \\
&= \left| k\ \frac{1}{2^{k-1}} \sum_{m=0}^{k-1} (-1)^m \binom{k-1}{m} J_{2m-k+2}(s) + s\ \frac{1}{2^k} \sum_{m=0}^{k} (-1)^m \binom{k}{m} J_{2m-k+1}(s) \right| \\
&\leq (k+|s|) \max_{l=1-k,\dots,1+k} |J_l(s)|.
\end{aligned} \tag{1.25}$$

The functions M_1 and M_2 are related to f by

$$\begin{aligned}
M_1(x,t) &= -\frac{\sqrt{t}^{N-2}}{2(1-t)}\ f(\omega|x|\sqrt{1-t}), \\
M_2(x,t) &= -\frac{\sqrt{t}^{N-4}}{2(1-t)}\ f(i\omega|x|\sqrt{t(1-t)}),
\end{aligned}$$

so we can bound their derivatives of order $j \geq 2$:

$$\begin{aligned}
&\sup_{t\in[0,1]} |M_1|_{W^{j,\infty}(D)} \leq \sup_{t\in[0,1],\, x\in D} \left|\frac{\partial^j}{\partial|x|^j} M_1(x,t)\right| \\
&\quad\leq \sup_{t\in[0,1],\, x\in D} \left|\frac{\sqrt{t}^{N-2}}{2(1-t)} \left(\omega\sqrt{1-t}\right)^j \frac{\partial^j}{\partial(\omega|x|\sqrt{1-t})^j} f(\omega|x|\sqrt{1-t})\right|
\end{aligned}$$

$$\overset{(1.25),\,(A.3)}{\leq} \frac{\omega^j}{2}\,(j + (1-\rho)\omega h),$$

$$\sup_{t\in[0,1]} |M_2|_{W^{j,\infty}(D)}$$
$$\leq \sup_{t\in[0,1],\, x\in D} \left| \frac{\sqrt{t}^{N-4}}{2(1-t)} \big(i\omega\sqrt{t(1-t)}\big)^j \frac{\partial^j}{\partial(i\omega|x|\sqrt{t(1-t)})^j} f(i\omega|x|\sqrt{t(1-t)}) \right|$$
$$\overset{(1.25),\,(A.4)}{\leq} \frac{\omega^j}{2^{j-1}} \left(j + \frac{(1-\rho)\omega h}{2} \right) e^{\frac{3}{4}(1-\rho)\omega h}.$$

The last bound in the thesis of the lemma is obtained by modifying the last step of this chain of inequalities. □

Remark 1.2.4. *We can summarize the bounds of Lemma 1.2.3 for every $j \geq 0$ in less detailed estimates:*

$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{j,\infty}(D)} \leq \omega^j \,\big(j + (\omega h)^2\big), \tag{1.26}$$
$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{j,\infty}(D)} \leq \omega^j \,(1 + \omega h)\, e^{\frac{3}{4}(1-\rho)\omega h}. \tag{1.27}$$

We forget the algebraic dependence on ρ because it will be absorbed in a generic bounding constant. On the contrary, in a nice domain, $\rho \leq 2$ can be used to reduce the exponential dependence on ωh.

Remark 1.2.5. *If the wavenumber $\omega = \omega_R + i\omega_I$ is complex, the following more general estimates hold:*

$$\|M_1\|_{L^\infty(D\times[0,1])} \leq \frac{\big((1-\rho)|\omega|h\big)^2}{4}\, e^{(1-\rho)|\omega_I|h},$$
$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{1,\infty}(D)} \leq \frac{(1-\rho)|\omega|^2\, h}{2}\, e^{(1-\rho)|\omega_I|h},$$
$$\sup_{t\in[0,1]} |M_1(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{|\omega|^j}{2}\,\big(j + (1-\rho)|\omega|h\big)\, e^{\frac{3}{2}(1-\rho)|\omega|h} \qquad \forall j \geq 2,$$

$$\|M_2\|_{L^\infty(D\times[0,1])} \leq \frac{\big((1-\rho)|\omega|h\big)^2}{4}\, e^{\frac{1}{2}(1-\rho)|\omega_R|h},$$
$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{1,\infty}(D)} \leq \frac{(1-\rho)|\omega|^2 h}{2}\, e^{\frac{1}{2}(1-\rho)|\omega_R|h},$$
$$\sup_{t\in[0,1]} |M_2(\cdot,t)|_{W^{j,\infty}(D)} \leq \frac{|\omega|^j}{2^{j-1}} \left(j + \frac{(1-\rho)|\omega|h}{2} \right) e^{\frac{3}{4}(1-\rho)|\omega|h} \qquad \forall j \geq 2.$$

Remark 1.2.6. *By using the bounds in Remark 1.2.5, we can extend Theorem 1.2.1 to every $\omega \in \mathbb{C}$, similarly to Theorem 1.1.5 (see Remark 1.1.6). In fact, the case $\omega = 0$ is trivial, since V_1 and V_2 reduce to the identity, while in general, thanks to the Remark 1.2.5, Theorem 1.2.1 holds by substituting ω with $|\omega|$ in the estimates and in the definition of the weighted norm* (1), *and multiplying the right-hand side of* (1.12) *by $e^{\frac{3}{2}|\omega|h}$.*

Lemma 1.2.7. *Let $\phi \in H^k(D)$, $\beta \in \mathbb{N}^N$ be a multi-index of length $|\beta| = k$ and D^β be the corresponding differential operator in the variable x. Then*

$$\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t \tag{1.28}$$
$$\leq \begin{cases} \dfrac{1}{2k-N+1} \left\|D^\beta\phi\right\|^2_{0,D} & \text{if } 2k-N \geq 0, \\ K \left\|D^\beta\phi\right\|^2_{0,D} + \left(\dfrac{\rho}{2}\right)^{2k+1} \dfrac{|D|}{2k+1} \left\|D^\beta\phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} & \text{if } 2k-N < 0, \end{cases}$$

where $K = \log\frac{2}{\rho}$ if $2k - N = -1$, $K = \left(\frac{2}{\rho}\right)^{N-1}$ if $2k-N < -1$, $|D|$ denotes the measure of D and ρ is given in Assumption 1.1.1.

Proof. In the first case, we can simply compute the integral with respect to t with the change of variables $y = tx$:

$$\begin{aligned}\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^1 \int_{tD} t^{2|\beta|} \left|D^\beta \phi(y)\right|^2 \frac{\mathrm{d}y}{t^N}\,\mathrm{d}t \\ &= \frac{1}{2k-N+1}\left\|D^\beta\phi\right\|^2_{0,tD} \leq \frac{1}{2k-N+1}\left\|D^\beta\phi\right\|^2_{0,D};\end{aligned}$$

the set tD is included in D because this is star-shaped with respect to 0.

In the case $2k-N<0$, the integral in t is not bounded so we need to split it in two parts, treating the second part as before:

$$\begin{aligned}\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t &= \int_0^{\frac{\rho}{2}} \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_{\frac{\rho}{2}}^1 \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^{\frac{\rho}{2}} t^{2|\beta|}\,\mathrm{d}t |D| \left\|D^\beta\phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} + \int_{\frac{\rho}{2}}^1 t^{2k-N}\left\|D^\beta\phi\right\|^2_{0,tD}\,\mathrm{d}t \\ &= \frac{1}{2k+1}\left(\frac{\rho}{2}\right)^{2k+1} |D| \left\|D^\beta\phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} + \int_{\frac{\rho}{2}}^1 t^{2k-N}\left\|D^\beta\phi\right\|^2_{0,tD}\,\mathrm{d}t,\end{aligned}$$

and the assertion comes from the expression

$$\int_{\frac{\rho}{2}}^1 t^{2k-N}\,\mathrm{d}t = \begin{cases} \log\dfrac{2}{\rho} & \text{if } 2k-N=-1, \\ \dfrac{1}{2k-N+1}\left(1 - \left(\dfrac{\rho}{2}\right)^{2k-N+1}\right) \leq \left(\dfrac{2}{\rho}\right)^{N-1} & \text{if } 2k-N<-1. \end{cases}$$

□

Remark 1.2.8. *We can reduce the bounds of Lemma 1.2.7 for every value of the multi-index length k with the estimate*

$$\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \left(\frac{2}{\rho}\right)^{N-1} \left\|D^\beta\phi\right\|^2_{0,D} + \left(\frac{\rho}{2}\right)^{2k+1} \frac{|D|}{2k+1} \left\|D^\beta\phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})}. \tag{1.29}$$

From Lemma 1.2.7, it is clear that, in order to prove the continuity of V_1 and V_2 in the L^2–norm and in high-order Sobolev norms, we need interior estimates that bound the L^∞–norm of ϕ and its derivatives in a small ball contained in D with its L^2–norm and H^j–norms on D. It is easy to find such estimates for harmonic functions, thanks to the mean value theorem (see, e.g., Theorem 2.1 of [23]).

Notice that it is not possible to avoid the use of interior estimates for the continuity in $H^j(D)$ when $j \geq \frac{N}{2}$ as the assertion of Lemma 1.2.7 may suggest: in fact, Lemma 1.2.2 requires to estimate $\int_0^1 \int_D \left|D^\beta \phi(tx)\right|^2 \, \mathrm{d}x \, \mathrm{d}t$ for all the multi-index lengths $|\beta| = k \leq j$, so we necessarily end up in the cases $2k - N = -1$ and $2k - N < -1$.

Lemma 1.2.9 (Interior estimates for harmonic functions)**.** *Let ϕ be a harmonic function in $B_R(x)$, $R > 0$, then*

$$|\phi(x)|^2 \leq \frac{1}{R^N \, |B_1|} \, \|\phi\|^2_{0,B_R(x)} \,, \tag{1.30}$$

where $|B_1| = \frac{\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^N$. If $\phi \in H^k(D)$ and $\beta \in \mathbb{N}^N$, $|\beta| \leq k$, then

$$\left\|D^\beta \phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} \leq \frac{1}{|B_1|} \left(\frac{2}{\rho h}\right)^N \left\|D^\beta \phi\right\|^2_{0,D} \,, \tag{1.31}$$

Proof. By the mean value property of harmonic functions (see Theorem 2.1 of [23]) and the Jensen inequality, we get the first estimate:

$$\begin{aligned} |\phi(x)|^2 &= \left| \frac{1}{|B_R(x)|} \int_{B_R(x)} \phi(y) \, \mathrm{d}y \right|^2 \leq \frac{1}{|B_R|} \int_{B_R(x)} |\phi(y)|^2 \, \mathrm{d}y \\ &= \frac{1}{R^N \, |B_1|} \, \|\phi\|^2_{0,B_R(x)} \,. \end{aligned}$$

The second bound follows applying the first one to the derivatives of ϕ, which are harmonic in the ball $B_{\frac{\rho h}{2}}(x) \subset B_{\rho h} \subset D$. ☐

Remark 1.2.10. *The interior estimates for harmonic functions are related to Cauchy's estimates for their derivatives. Theorem 2.10 in [23] states that, given two domains $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ such that $d(\Omega_1, \partial\Omega_2) = d$, ϕ harmonic in Ω_2, for every multi-index α*

$$\|D^\alpha \phi\|_{L^\infty(\Omega_1)} \leq \left(\frac{N|\alpha|}{d}\right)^{|\alpha|} \|\phi\|_{L^\infty(\Omega_2)} \,. \tag{1.32}$$

In order to find analogous estimates for the Sobolev norms, we can combine (1.32) *and* (1.30) *using the intermediate domain $\{x \in \mathbb{R}^N \,:\, d(x, \Omega_1) < \frac{d}{2}\}$ and obtain*

$$\|D^\alpha \phi\|_{0,\Omega_1} \leq C_{N,\alpha} |\Omega_1|^{N/2} \, d^{-|\alpha| - N/2} \, \|\phi\|^2_{0,\Omega_2} \,,$$

but the order of the power of d is not satisfactory. In order to improve it, we represent the derivatives of a harmonic function ψ in $\overline{B}_1 \subset \mathbb{R}^N$ using the Poisson kernel P:

$$D^\alpha \psi(y) = \int_{S^{N-1}} \psi(z) \, D_1^\alpha P(y,z) \, \mathrm{d}\sigma(z) \qquad y \in B_1, \; \forall \, \alpha \in \mathbb{N}^N,$$

where the derivatives of P are taken with respect to the first variable (see (1.22) in [4]). Rewriting this formula in $y = 0$ and then translating in a point x, if ψ is harmonic in $\overline{B}_1(x)$, we have

$$D^\alpha \psi(x) = \int_{S^{N-1}} \psi(x+z)\, D_1^\alpha P(0,z)\, \mathrm{d}\sigma(z) \qquad \forall\, \alpha \in \mathbb{N}^N.$$

Given two domains $\hat{\Omega}_1 \subset \hat{\Omega}_2$ such that $d(\hat{\Omega}_1, \partial\hat{\Omega}_2) = 1$ and $\hat{\phi}$ harmonic in $\hat{\Omega}_2$

$$\begin{aligned}\left\|D^\alpha \hat{\phi}\right\|_{0,\hat{\Omega}_1} &= \int_{\hat{\Omega}_1} |D^\alpha \hat{\phi}(x)|^2\, \mathrm{d}x = \int_{\hat{\Omega}_1} \left| \int_{S^{N-1}} \hat{\phi}(x+z)\, D_1^\alpha P(0,z)\, \mathrm{d}\sigma(z)\right|^2 \mathrm{d}x \\ &\overset{y=x+z}{\leq} |S^{N-1}| \int_{S^{N-1}} \left(\int_{\hat{\Omega}_2} |\hat{\phi}(y)|^2\ \mathrm{d}y \right) |D_1^\alpha P(0,z)|^2\ \mathrm{d}\sigma(z) \leq\ C_{N,\alpha}\ \left\|\hat{\phi}\right\|_{0,\hat{\Omega}_2},\end{aligned}$$

where we have used the Jensen inequality and the Fubini theorem. By summing over all the multi-indices of the same length and scaling the domains such that $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ and $d(\Omega_1, \partial\Omega_2) = d$, we finally obtain

$$|\phi|_{j+k,\Omega_1} \leq C_{N,j,k}\ d^{-k}\ |\phi|_{j,\Omega_2}, \qquad j,\, k \in \mathbb{N}. \tag{1.33}$$

We will use the bicontinuity of the Vekua operator to prove an analogous result for the solutions of the Helmholtz equations, see Lemma 3.2.1.

The main tool used to prove the interior estimates for harmonic functions is the mean value theorem. For the solutions of the homogeneous Helmholtz equation, we have an analogous mean value formula [18, page 289] but it does not provide good estimates.

Another way to prove interior estimates for the solutions of the homogeneous Helmholtz equation is to use the Green formula for the Laplacian in a ball, but this gives estimates that either involve the H^1–norm of u on the right-hand side of the bound or give bad order in the domain diameter R.

A third way is to use the technique presented in Lemma 4.2.7 of [32] for the two-dimensional case. This method can be generalized only to three space dimensions, and does not provide estimates with only the L^2–norm of u on the right-hand side. On the other hand, it is possible to make explicit the dependence of the bounding constants on ωR. We will prove these interior estimates in Lemma 1.2.12 and we will use them in the estimates of the approximation error of a generic homogeneous Helmholtz solution by plane waves.

A more general way is to use Theorem 8.17 of [23]. This holds in every space dimension with the desired norms and the desired order in R. The only shortcoming of this result is that the bounding constant still depends on the product ωR but this dependence is not explicit. We report this result in Theorem 1.2.11.

Summarizing: we are able to prove interior estimates for homogeneous Helmholtz solutions with sharp order in R in two fashions. Theorem 1.2.11 works in any space dimension and with good norms (L^2). Lemma 1.2.12 works only in low space dimensions and with different norms but the constant in front of the estimates is explicit in ωR. Both techniques, however, allow to prove the final best approximation results we are looking for with the same order and in the same norms.

Theorem 1.2.11 (Interior estimates for Helmholtz solutions, version 1)**.** *For every $N \geq 2$, let $u \in H^1(B_R(x_0))$ be a solution of the homogeneous Helmholtz* *equation. Then there exists a constant $C > 0$ depending only on the product ωR and the dimension N, such that*

$$\|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C(\omega R, N)\, R^{-\frac{N}{2}}\ \|u\|_{0,B_R(x_0)}\,. \tag{1.34}$$

This is exactly Theorem 8.17 of [23]; with that notation, for the homogeneous Helmholtz equation we have $k(R) = 0$, $\lambda = 1$, $\Lambda = \sqrt{N}$ $\nu = \omega$ and $p = 2$ (q is not relevant for the homogeneous problem); see also [23], p. 178.

Lemma 1.2.12 (Interior estimates for Helmholtz solutions, version 2)**.** *Let $u \in H^1(B_R(x_0))$ be a solution of the inhomogeneous Helmholtz equation*

$$-\Delta u - \omega^2 u = f,$$

with $f \in H^1(B_R(x_0))$. Then there exists a constant $C > 0$ depending only on the space dimension N such that

if $N = 2$:

$$\|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-1} \Big(\big(1+\omega^2R^2\big)\, \|u\|_{0,B_R(x_0)} + R\ \|\nabla u\|_{0,B_R(x_0)} + R^2\, \|f\|_{0,B_R(x_0)} \Big)\,, \tag{1.35}$$

if $N = 3, 4, 5$:

$$\|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-\frac{N}{2}} \Big((1+\omega^2R^2)\ (\|u\|_{0,B_R(x_0)} + R\ \|\nabla u\|_{0,B_R(x_0)}) + R^2\, \|f\|_{0,B_R(x_0)} + R^3\, \|\nabla f\|_{0,B_R(x_0)} \Big)\,, \tag{1.36}$$

if $N = 2, 3$:

$$\|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-\frac{N}{2}} \Big(\omega^2 R\, \|u\|_{0,B_R(x_0)} + (1+\omega^2R^2)\ \|\nabla u\|_{0,B_R(x_0)} + R\, \|f\|_{0,B_R(x_0)} + R^2\, \|\nabla f\|_{0,B_R(x_0)} \Big)\,. \tag{1.37}$$

Remark 1.2.13. *In the homogeneous case, Lemma 1.2.12 reads as follows. Let $u \in H^1(B_R(x_0))$ be a solution of the homogeneous Helmholtz equation. Then there exists a constant $C > 0$ depending only on the space dimension N such that*

if $N = 2, 3, 4, 5$:

$$\|u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-\frac{N}{2}} (1+\omega^2R^2)\ (\|u\|_{0,B_R(x_0)} + R\ \|\nabla u\|_{0,B_R(x_0)}), \tag{1.38}$$

if $N = 2, 3$:

$$\|\nabla u\|_{L^\infty(B_{\frac{R}{2}}(x_0))} \leq C\, R^{-\frac{N}{2}} \Big(\omega^2 R\, \|u\|_{0,B_R(x_0)} + (1+\omega^2R^2)\ \|\nabla u\|_{0,B_R(x_0)} \Big)\,. \tag{1.39}$$

Proof of Lemma 1.2.12. It is enough to bound $|u(x_0)|$ and $|\nabla u(x_0)|$, because for all $x \in B_{\frac{R}{2}}(x_0)$ we can repeat the proof using $B_{\frac{R}{2}}(x)$ instead of $B_R(x_0)$ with the same constants. We can also fix $x_0 = 0$.

Let $\varphi : \mathbb{R}^+ \to [0,1]$ be a smooth cut-off function such that

$$\varphi(r) = \begin{cases} 1 & |r| \le \frac{1}{4}, \\ 0 & |r| \ge \frac{3}{4}, \end{cases}$$

and $\varphi_R : \mathbb{R}^N \to [0,1]$, $\varphi_R(x) := \varphi\big(\frac{|x|}{R}\big)$. Then

$$\nabla\varphi_R(x) = \varphi'\left(\frac{|x|}{R}\right)\frac{x}{R|x|}, \qquad \Delta\varphi_R(x) = \frac{1}{R^2}\varphi''\left(\frac{|x|}{R}\right) + \frac{N-1}{R|x|}\varphi'\left(\frac{|x|}{R}\right).$$

We define the average of u and two auxiliary functions on B_R:

$$\overline{u} := \frac{1}{|B_R|}\int_{B_R} u(y)\,\mathrm{d}y, \qquad g(x) := u(x)\,\varphi_R(x), \qquad \overline{g}(x) := (u(x) - \overline{u})\,\varphi_R(x);$$

their Laplacians are:

$$\begin{aligned}
\tilde{f}(x) :&= \tilde{f}_1(x) + \tilde{f}_2(x) + \tilde{f}_3(x) := -\Delta g(x) \\
&= -\left[\frac{1}{R^2}\varphi''\big(\tfrac{|x|}{R}\big) + \frac{N-1}{R|x|}\varphi'\big(\tfrac{|x|}{R}\big)\right]u(x) - 2\varphi'\big(\tfrac{|x|}{R}\big)\frac{x}{R|x|}\cdot\nabla u(x) \\
&\quad + \varphi\big(\tfrac{|x|}{R}\big)\,(\omega^2 u(x) + f(x)), \\
\overline{f}(x) :&= \overline{f}_1(x) + \overline{f}_2(x) + \overline{f}_3(x) := -\Delta\overline{g}(x) \\
&= -\left[\frac{1}{R^2}\varphi''\big(\tfrac{|x|}{R}\big) + \frac{N-1}{R|x|}\varphi'\big(\tfrac{|x|}{R}\big)\right](u(x) - \overline{u}) - 2\varphi'\big(\tfrac{|x|}{R}\big)\frac{x}{R|x|}\cdot\nabla u(x) \\
&\quad + \varphi\big(\tfrac{|x|}{R}\big)\,(\omega^2 u(x) + f(x)).
\end{aligned}$$

The fundamental solution formula for Poisson equation states that, if $-\Delta a = b$ in $\mathbb{R}^N$, then

$$a(x) = \int_{\mathbb{R}^N}\Phi(x-y)\,b(y)\,\mathrm{d}y, \quad \text{with} \quad \Phi(x) = \begin{cases} -\dfrac{1}{2\pi}\,\log|x| & N = 2, \\ \dfrac{1}{N(N-2)|B_1|}\,|x|^{2-N} & N \ge 3. \end{cases} \tag{1.40}$$

The identity (1.40) holds for all $b \in L^2(B_R)$, thanks to Theorem 9.9 of [23]. We notice that

$$|\nabla\Phi(x)| = \left|-\frac{1}{N\,|B_1|}\,\frac{x}{|x|^N}\right| = \frac{1}{N\,|B_1|}\,|x|^{1-N} \qquad \forall\, N \ge 2.$$

We start by bounding $|u(0)|$ for $N = 2$. In this case, it is easy to see that, for all $R > 0$, we have

$$\int_{B_R}\big(\log|x| - \log R\big)^2\,\mathrm{d}x = \frac{\pi}{2}R^2. \tag{1.41}$$

We note that from the divergence theorem

$$\int_{B_R}\tilde{f}(y)\,\mathrm{d}y = -\int_{B_R}\Delta g(y)\,\mathrm{d}y = -\int_{\partial B_R}\nabla g(s)\cdot\boldsymbol{n}\,\mathrm{d}s = 0,$$

because $g \equiv 0$ in $\mathbb{R}^2 \setminus B_{\frac{3}{4}R}$ and, since $\tilde{f} = 0$ outside $B_{\frac{3}{4}R}$ then $\tilde{f}$ has zero mean value in the whole $\mathbb{R}^2$.

We apply (1.40) with $a = g$ and $b = \tilde{f}$; using the Cauchy-Schwartz inequality, the identity (1.41) and the fact that $\tilde{f}$ has zero mean value in $\mathbb{R}^2$, we obtain:

$$
\begin{aligned}
|u(0)| = |g(0)| &= \left| -\frac{1}{2\pi} \int_{\mathbb{R}^2} \big(\log|y| - \log R \big) \tilde{f}(y)\, \mathrm{d}y \right| \le \frac{1}{2\pi} \sqrt{\frac{\pi}{2}}\, R\, \|\tilde{f}\|_{0, B_{\frac{3}{4}R}} \\
&\le C_{N,\varphi} R \left(\frac{1}{R^2} \|u\|_{0,B_R} + \frac{1}{R} \|\nabla u\|_{0,B_R} + \omega^2 \|u\|_{0,B_R} + \|f\|_{0,B_R} \right),
\end{aligned}
$$

where the constant $C_{N,\varphi}$ depends only on N and φ; in the last step we have used the definition of $\tilde{f}$ and the fact that $\varphi'(\frac{|x|}{R}) = 0$ in $B_{\frac{R}{4}}$. The estimate (1.35) easily follows.

Proving all the other bounds (on $|u(0)|$ for $N \ge 2$ and on $|\nabla u(0)|$ for $N \ge 2$) is more involved. We fix p, $p' > 1$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $\alpha > 0$, we calculate

$$
\begin{aligned}
\||y|^\alpha\|_{L^{p'}(B_R)} &= \left(\int_{S^{N-1}} \int_0^R r^{\alpha p'} r^{N-1}\, \mathrm{d}r\, \mathrm{d}S \right)^{\frac{1}{p'}} \\
&= \left(\frac{|S^{N-1}|}{\alpha p' + N} \right)^{\frac{1}{p'}} R^{\alpha + \frac{N}{p'}} = C_{N,p',\alpha} R^{\alpha + N - \frac{N}{p}},
\end{aligned}
\tag{1.42}
$$

that holds if $\alpha p' + N \neq 0$, that is equivalent to $(\alpha + N)p \neq N$, for every $N \ge 2$. We compute also

$$
\begin{aligned}
\|\Phi\|_{L^p(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})} &= C_{N,p} \left(|S^{N-1}| \int_{\frac{1}{4}R}^{\frac{3}{4}R} r^{(2-N)p}\, r^{N-1}\, \mathrm{d}r \right)^{\frac{1}{p}} \\
&= C_{N,p}\, |S^{N-1}|^{\frac{1}{p}} \left(\left(\frac{3}{4} R \right)^{(2-N)p+N} - \left(\frac{1}{4} R \right)^{(2-N)p+N} \right)^{\frac{1}{p}} \\
&= C_{N,p}\, R^{2-N+\frac{N}{p}},
\end{aligned}
\tag{1.43}
$$

for every $p \neq \frac{N}{N-2}$, $N \ge 3$, and the analogue

$$
\begin{aligned}
\|\nabla \Phi\|_{L^p(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})} &= C_{N,p} \left(|S^{N-1}| \int_{\frac{1}{4}R}^{\frac{3}{4}R} r^{(1-N)p}\, r^{N-1}\, \mathrm{d}r \right)^{\frac{1}{p}} \\
&= C_{N,p}\, R^{1-N+\frac{N}{p}},
\end{aligned}
\tag{1.44}
$$

that holds for every $p \neq \frac{N}{N-1}$, $N \ge 2$.

For all $\psi \in H_0^1(B_R)$, using scaling arguments, the continuity of the Sobolev embeddings $H_0^1(B_1) \hookrightarrow L^p(B_1)$ which hold provided that $2 \le p \le \frac{2N}{N-2}$, if $N \ge 3$, and $2 \le p < \infty$, if $N = 2$ (see [1, Th. 5.4,I,A-B]), and the Poincaré inequality, we obtain

$$
\begin{aligned}
\|\psi\|_{L^p(B_R)} &= R^{\frac{N}{p}}\, \|\hat{\psi}\|_{L^p(B_1)} \le C_{N,p}\, R^{\frac{N}{p}}\, \|\hat{\psi}\|_{1,B_1} \\
&\le C_{N,p}\, R^{\frac{N}{p}}\, \|\nabla \hat{\psi}\|_{0,B_1} \le C_{N,p}\, R^{\frac{N}{p}+1-\frac{N}{2}}\, \|\nabla \psi\|_{0,B_R}.
\end{aligned}
\tag{1.45}
$$

Now we can estimate u in the case $N \geq 3$. From the Hölder inequality for the pair of spaces $L^{p'}$, L^p, $p > 2$ (thus, $p < 2$), and the fact that $\tilde{f}_1 \equiv \tilde{f}_2 \equiv 0$ in $B_{\frac{1}{4}R}$ (see the definition of $\tilde{f}$), we can write

$$\begin{aligned}|u(0)| = |g(0)| &= \left| \int_{\mathbb{R}^N} \Phi(x) \tilde{f}(x)\, \mathrm{d}x \right| \\ &\leq \|\Phi\|_{L^p(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})} \|\tilde{f}_1 + \tilde{f}_2\|_{L^{p'}(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})} \\ &\quad + \|\Phi\|_{L^{p'}(B_R)} \|\tilde{f}_3\|_{L^p(B_R)}.\end{aligned}$$

Using (1.43) to bound the L^p–norm of Φ, the continuity of the embedding of $L^{p'}(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})$ into $L^2(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})$ (recall that $1 < p' < 2$) with constant $|B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R}|^{\frac{1}{p'} - \frac{1}{2}}$ for the norm of $\tilde{f}_1 + \tilde{f}_2$, the definition (1.40) of Φ and (1.42) with $\alpha = 2 - N$, which requires $p > \frac{N}{2}$, to bound the $L^{p'}$–norm of Φ, and finally (1.45)which requires $2 \leq p \leq \frac{2N}{N-2}$, to bound the norm of $\tilde{f}_3$ (recall that $\tilde{f}_3 \in H_0^1(B_R)$), we have

$$\begin{aligned}|u(0)| &\leq C_{N,p} R^{2-N+\frac{N}{p}}\ |B_{\frac{3}{4}R}|^{\frac{1}{p'} - \frac{1}{2}} \left\| \tilde{f}_1 + \tilde{f}_2 \right\|_{0, B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R}} \\ &\quad + C_{N,p} R^{2-\frac{N}{p}}\ R^{\frac{N}{p}+1-\frac{N}{2}} \left\| \nabla \tilde{f}_3 \right\|_{0, B_R}\end{aligned}$$

Finally, using the definitions of the $\tilde{f}_i$'s, $|\nabla \varphi_R| \leq \frac{1}{R} C_\varphi$ and $\frac{1}{p} + \frac{1}{p'} = 1$ we obtain

$$\begin{aligned}|u(0)| &\leq C_{N,p,\varphi} R^{2-N+\frac{N}{p}}\ R^{\frac{N}{p'} - \frac{N}{2}} \left(\frac{1}{R^2} \|u\|_{0,B_R} + \frac{1}{R} \|\nabla u\|_{0,B_R} \right) \\ &\quad + C_{N,p,\varphi} R^{3-\frac{N}{2}} \left(\omega^2 \|\nabla u\|_{0,B_R} + \|\nabla f\|_{0,B_R} + \frac{1}{R} \omega^2 \|u\|_{0,B_R} + \frac{1}{R} \|f\|_{0,B_R} \right) \\ &\leq C_{N,p,\varphi}\ R^{-\frac{N}{2}} \Big((1 + \omega^2 R^2) \|u\|_{0,B_R} + R\,(1 + \omega^2 R^2) \|\nabla u\|_{0,B_R} \\ &\quad + R^2 \|f\|_{0,B_R} + R^3 \|\nabla f\|_{0,B_R} \Big).\end{aligned}$$

The previous argument for bounding $|u(0)|$ requires that there exists p such that $\frac{N}{2} < p \leq \frac{2N}{N-2}$, which is possible only if $N < 6$; this is the reason of the upper bound on the space dimension in the statement.

In order to conclude this proof, we have to estimate $|\nabla u(0)|$. We use the same technique as before, after differentiating the relation (1.40) with $a = \overline{g}$ and $b = \overline{f}$. For every $N \geq 2$, thanks to (1.44), the embedding of $L^{p'}(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})$ into $L^2(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})$, (1.42) with $\alpha = 1 - N$ and (1.45), that require $N < p \leq \frac{2N}{N-2}$, we have

$$\begin{aligned}|\nabla u(0)| = |\nabla \overline{g}(0)| &= \left| \int_{\mathbb{R}^N} \nabla \Phi(x) \overline{f}(x)\, \mathrm{d}x \right| \\ &\leq \|\nabla \Phi\|_{L^p(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})} \|\overline{f}_1 + \overline{f}_2\|_{L^{p'}(B_{\frac{3}{4}R} \setminus B_{\frac{1}{4}R})} \\ &\quad + \|\nabla \Phi\|_{L^{p'}(B_R)} \|\overline{f}_3\|_{L^p(B_R)}\end{aligned}$$

$$\leq C_{N,p} R^{1-N+\frac{N}{p}} \, |B_{\frac{3}{4}R}|^{\frac{1}{p'}-\frac{1}{2}} \|\overline{f}_1 + \overline{f}_2\|_{0,B_{\frac{3}{4}R}\setminus B_{\frac{1}{4}R}}$$
$$+ C_{N,p} R^{1-\frac{N}{p}} \, R^{\frac{N}{p}+1-\frac{N}{2}} \|\nabla \tilde{f}_3\|_{0,B_R}.$$

By using the Poincaré–Wirtinger inequality, whose constant scales with R, to bound $\|u-\overline{u}\|_{0,B_R}$, we obtain

$$\begin{aligned}
|\nabla u(0)| &\leq C_{N,p,\varphi}\, R^{-1-\frac{N}{2}} \left(R^{-2} \|u-\overline{u}\|_{0,B_R} + R^{-1} \|\nabla u\|_{0,B_R} \right) \\
&\quad + C_{N,p,\varphi}\, R^{2-\frac{N}{2}} \left(R^{-1} \left\|\omega^2 u + f\right\|_{0,B_R} + \left\|\nabla(\omega^2 u + f)\right\|_{0,B_R} \right) \\
&\leq C_{N,p,\varphi}\, R^{-\frac{N}{2}} \Big(\omega^2 R \|u\|_{0,B_R} + (1+\omega^2 R^2) \|\nabla u\|_{0,B_R} \\
&\quad + R \|f\|_{0,B_R} + R^2 \|\nabla f\|_{0,B_R} \Big),
\end{aligned}$$

The requirement that there exists p such that $N < p \leq \frac{2N}{N-2}$ can be satisfied only if $N < 4$. □

Lemma 1.2.12 is the only result in this section which we are not able to generalize to all the space dimensions $N \geq 2$. This is because in its proof we make use of a pair of conjugate exponents p and p' such that the fundamental solution Φ of the Laplace equation (together with its gradient) belongs to $L^{p'}(B_R)$ and, at the same time, $H^1(B_R)$ is continuously embedded in $L^p(B_R)$. This requirement yields the upper bounds on the space dimension we have required in the statement of Lemma 1.2.12.

Combining the results of the previous lemmas, we can now prove Theorem 1.2.1.

Proof of Theorem 1.2.1. We start by proving the continuity bound (1.12) for V_1. For every $j \in \mathbb{N}$, $N \geq 2$, $\phi \in \mathcal{H}^j(D)$, inserting (1.26) and (1.29) into (1.17) with $\xi = 1$, we have

$$\begin{aligned}
|V_1[\phi]|_{j,D} \leq \Bigg[& 2\,|\phi|^2_{j,D} + 2(1+j)^{3N-2} e^{2j} \sum_{k=0}^{j} \omega^{2(j-k)} \big(j-k+(\omega h)^2\big)^2 \\
& \cdot \left(\left(\frac{2}{\rho}\right)^{N-1} |\phi|^2_{k,D} + \left(\frac{\rho}{2}\right)^{2k+1} \frac{|D|}{2k+1} \sum_{|\beta|=k} \left\|D^\beta \phi\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} \right) \Bigg]^{\frac{1}{2}}.
\end{aligned}$$

Then, using the interior estimates (1.31), we get

$$\begin{aligned}
|V_1[\phi]|_{j,D} &\leq C_N \,(1+j)^{\frac{3}{2}N-1+1}\, e^j \left(1+(\omega h)^2\right) \\
&\quad \cdot \left[\sum_{k=0}^{j} \omega^{2(j-k)} \left(\rho^{1-N} + \rho^{2k+1} \frac{|D|}{(\rho h)^N} \right) |\phi|^2_{k,D} \right]^{\frac{1}{2}} \\
&\leq C_N\, \rho^{\frac{1-N}{2}}\, (1+j)^{\frac{3}{2}N}\, e^j \left(1+(\omega h)^2\right) \|\phi\|_{j,\omega,D},
\end{aligned}$$

by the definition of weighted Sobolev norms (1), and because $|D| \leq h^N$ and $\rho < 1$. The constant C_N depends only on the dimension N of the space. Passing from the seminorms to the complete Sobolev norms gives an extra coefficient $(1+j)^{1/2}$ and the bound (1.12) follows.

In order to prove the continuity bound (1.13) for V_2, we proceed similarly. For every $j \in \mathbb{N}$, $N \geq 2$, $u \in \mathcal{H}^j_\omega(D)$, inserting (1.27) and (1.29) into (1.17) with $\xi = 2$, we have

$$\begin{aligned}
|V_2[u]|_{j,D} \leq & \Bigg[2\,|u|^2_{j,D} + 2(1+j)^{3N-2} e^{2j} \sum_{k=0}^{j} \omega^{2(j-k)} (1+\omega h)^2 e^{\frac{3}{2}(1-\rho)\omega h} \\
& \cdot \Bigg(\left(\frac{2}{\rho}\right)^{N-1} |u|^2_{k,D} + \left(\frac{\rho}{2}\right)^{2k+1} \frac{|D|}{2k+1} \sum_{|\beta|=k} \left\| D^\beta u \right\|^2_{L^\infty(B_{\frac{\rho h}{2}})} \Bigg) \Bigg]^{\frac{1}{2}} \\
& \overset{(1.34)}{\leq} C(N,\omega h,\omega\rho h)\; (1+j)^{\frac{3}{2}N-1}\; e^j \\
& \cdot \left[\sum_{k=0}^{j} \omega^{2(j-k)} \left(\rho^{1-N} + \rho^{2k+1} \frac{|D|}{(\rho h)^N} \right) |u|^2_{k,D} \right]^{\frac{1}{2}} \\
& \leq C(N,\omega h,\rho)\; (1+j)^{\frac{3}{2}N-1}\; e^j \;\; \|u\|_{j,\omega,D}\, .
\end{aligned}$$

Again, passing from the seminorms to the complete Sobolev norms gives an extra coefficient $(1+j)^{1/2}$ and the bound (1.13) follows.

Now we proceed by proving the bounds (1.14), (1.15) and (1.16) for V_2 with constants only depending on N.

For the continuity bound (1.14) for the V_2 operator from $H^1(D)$ to $L^2(D)$, we repeat the same reasoning as above. If $u \in \mathcal{H}^1_\omega(D)$, $N = 2, \ldots, 5$, using the definition of V_2, (1.22), (1.29) and (1.38), we have

$$\begin{aligned}
\|V_2[u]\|_{0,D} \leq & \left[2\,\|u\|^2_{0,D} + 2\,\|M_2\|^2_{L^\infty(D\times[0,1])} \int_0^1 \int_D |u(tx)|^2 \,\mathrm{d}x\,\mathrm{d}t \right]^{\frac{1}{2}} \\
\leq & \Bigg[2\,\|u\|^2_{0,D} + 2\left(\frac{(\omega h)^2}{4} e^{\frac{1}{2}(1-\rho)\omega h} \right)^2 \Bigg[\left(\frac{2}{\rho}\right)^{N-1} \|u\|^2_{0,D} \\
& + \frac{\rho}{2}|D| \Big(C_N (\rho h)^{-\frac{N}{2}} \big(1+(\omega\rho h)^2\big) \big(\|u\|_{0,D} + \rho h \,\|\nabla u\|_{0,D} \big) \Big)^2 \Bigg] \Bigg]^{\frac{1}{2}} \\
\leq & \; C_N \; \rho^{\frac{1-N}{2}} \; \big(1+(\omega h)^4\big)\; e^{\frac{1}{2}(1-\rho)\omega h} \big(\|u\|_{0,D} + \rho h \,\|\nabla u\|_{0,D} \big),
\end{aligned}$$

which immediately gives (1.14).

Let us prove now (1.15). To this aim, given a multi-index $\beta \in \mathbb{N}^N$, we need to bound $\left\| D^\beta u \right\|_{L^\infty(B_{\frac{\rho h}{2}})}$. If $|\beta| = 0$, for $N = 2,3,4,5$, we simply use (1.38) and get

$$\begin{aligned}
\left\| D^\beta u \right\|_{L^\infty(B_{\frac{\rho h}{2}})} &= \|u\|_{L^\infty(B_{\frac{\rho h}{2}})} \\
&\leq C_N (\rho h)^{-\frac{N}{2}} (1+\omega^2\rho^2 h^2) \left(\|u\|_{0,D} + \rho h\,\|\nabla u\|_{0,D} \right) .
\end{aligned} \tag{1.46}$$

If $|\beta| = j \geq 1$, we note that there exists another multi-index $\alpha \in \mathbb{N}^N$ of length $|\alpha| = j-1$, such that for $N = 2, 3$ and $u \in \mathcal{H}^j_\omega(D)$ it holds

$$\begin{aligned}\left\|D^\beta u\right\|_{L^\infty(B_{\frac{\rho h}{2}})} &\leq \|\nabla D^\alpha u\|_{L^\infty(B_{\frac{\rho h}{2}})} \qquad (1.47)\\ &\leq C_N(\rho h)^{-\frac{N}{2}}\Big(\omega^2\rho h \|D^\alpha u\|_{0,D} + \left(1+(\omega\rho h)^2\right)\|\nabla D^\alpha u\|_{0,D}\Big),\end{aligned}$$

thanks to (1.39). Notice that the restriction to $N = 2, 3$ in this proof is due to the use of (1.39). Again, inserting (1.27) and (1.29) into (1.17) with $\xi = 2$ gives

$$\begin{aligned}|V_2[u]|_{j,D} &\leq C_N \Bigg[|u|^2_{j,D} + (1+j)^{3N-2}\, e^{2j} \sum_{k=0}^{j}\omega^{2(j-k)}(1+\omega h)^2 e^{\frac{3}{2}(1-\rho)\omega h}\\ &\qquad\cdot\Bigg(\rho^{1-N}|u|^2_{k,D} + \rho^{2k+1}|D|\sum_{|\beta|=k}\left\|D^\beta u\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})}\Bigg)\Bigg]^{\frac{1}{2}}\\ &\leq C_N\,(1+j)^{\frac{3}{2}N-1}\,e^j\,(1+\omega h)\,e^{\frac{3}{4}(1-\rho)\omega h}\\ &\qquad\cdot\Bigg[\sum_{k=0}^{j}\omega^{2(j-k)}\Bigg(\rho^{1-N}|u|^2_{k,D} + \rho^{2k+1}|D|\sum_{|\beta|=k}\left\|D^\beta u\right\|^2_{L^\infty(B_{\frac{\rho h}{2}})}\Bigg)\Bigg]^{\frac{1}{2}},\end{aligned}$$

and thus, as a consequence of (1.46) and (1.47), we obtain

$$\begin{aligned}|V_2[u]|_{j,D} &\leq C_N\;(1+j)^{\frac{3}{2}N-1}\,e^j\,(1+\omega h)\,e^{\frac{3}{4}(1-\rho)\omega h}\\ &\quad\cdot\Bigg[\omega^{2j}\rho^{1-N}\left(\|u\|^2_{0,D} + \frac{|D|}{h^N}(1+\omega^2\rho^2h^2)^2\Big(\|u\|_{0,D} + \rho h\|\nabla u\|_{0,D}\Big)^2\right)\\ &\qquad+\sum_{k=1}^{j}\omega^{2(j-k)}\rho^{1-N}\;\Bigg(|u|^2_{k,D} + \rho^{2k}\binom{N+k-1}{N-1}\frac{|D|}{h^N}\\ &\qquad\qquad\cdot\Big(\omega^2\rho h\,|u|_{k-1,D} + (1+\omega^2\rho^2h^2)\,|u|_{k,D}\Big)^2\Bigg)\Bigg]^{\frac{1}{2}}\\ &\leq C_N\;(1+j)^{\frac{3}{2}N-1}\,\rho^{\frac{1-N}{2}}\,e^j\,(1+\omega h)\,e^{\frac{3}{4}(1-\rho)\omega h}\\ &\quad\cdot\Bigg[\omega^{2j}(1+\omega^2h^2)^2\Big(\|u\|_{0,D} + h\|\nabla u\|_{0,D}\Big)^2\\ &\qquad+\sum_{k=1}^{j}\omega^{2(j-k)}(1+k)^{N-1}\Big(\omega^2h\,|u|_{k-1,D} + (1+\omega^2h^2)\,|u|_{k,D}\Big)^2\Bigg]^{\frac{1}{2}}\\ &\leq C_N\;(1+j)^{2N-\frac{3}{2}}\,\rho^{\frac{1-N}{2}}\,e^j\,(1+\omega h)\,e^{\frac{3}{4}(1-\rho)\omega h}\\ &\quad\cdot\Bigg[\left(1+(\omega h)^2\right)^2\omega^{2j}\|u\|^2_{0,D} + \left((\omega h)^2 + (\omega h)^6\right)\omega^{2(j-1)}|u|^2_{1,D}\\ &\qquad+(\omega h)^2\sum_{k=1}^{j}\omega^{2(j-k+1)}|u|^2_{k-1,D} + \left(1+(\omega h)^2\right)^2\sum_{k=1}^{j}\omega^{2(j-k)}|u|^2_{k,D}\Bigg]^{\frac{1}{2}}\\ &\leq C_N\;(1+j)^{2N-\frac{3}{2}}\,\rho^{\frac{1-N}{2}}\,e^j\left(1+(\omega h)^4\right)e^{\frac{3}{4}(1-\rho)\omega h}\,\|u\|_{j,\omega,D}\,,\end{aligned}$$

where the binomial coefficient comes from the number of the multi-indices $|\beta| = k$ and is bounded by (1.18). As before, passing from the seminorms to the complete Sobolev norms gives an extra coefficient $(1+j)^{1/2}$ and the bound (1.15) follows.

Finally, we prove the continuity of V_2 in the L^∞–norm stated in (1.16). Thanks to the definition of V_2 and (1.22), we have

$$\begin{aligned}\|V_2[u]\|_{L^\infty(D)} &\leq \Big(1 + \|M_2\|_{L^\infty(D\times[0,1])}\Big)\,\|u\|_{L^\infty(D)} \\ &\leq \left(1 + \frac{\big((1-\rho)\omega h\big)^2}{4} e^{\frac{1}{2}(1-\rho)\omega h}\right)\|u\|_{L^\infty(D)}\,,\end{aligned}$$

that holds for every $\phi \in L^\infty(D)$ and for every $N \geq 2$. This proves (1.16) and the proof of Theorem 1.2.1 is complete. □

1.3 Generalized Harmonic Polynomials

We want to use Vekua's theory to derive approximation estimates for the solutions of the homogeneous Helmholtz equation using finite dimensional spaces of particular functions. The approximating functions we want to use are the *generalized harmonic polynomials*.

Definition 1.3.1. *Given $D \subset \mathbb{R}^N$, we denote with $\mathbb{P}^L(D)$ the space of the ordinary homogeneous polynomials of degree $L \in \mathbb{N}$ defined in the domain D. The subspace of harmonic polynomials is denoted as*

$$\mathbb{H}^L(D) = \{P \in \mathbb{P}^L(D) :\ \Delta P = 0\}.$$

Its image under the operator V_1 is

$$\mathbb{H}^L_\omega(D) = V_1[\mathbb{H}^L(D)] = \{Q \in L^2(D) :\ \exists\, P \in \mathbb{H}^L(D) \ s.t.\ \ Q = V_1[P]\}.$$

The elements of $\mathbb{H}^L_\omega(D)$ are called generalized harmonic polynomials of degree L.

Notice that $\mathbb{P}^L(D)$, $\mathbb{H}^L(D)$, $\mathbb{H}^L_\omega(D)$ are vector spaces of dimensions

$$\begin{aligned}&\dim \mathbb{P}^0(D) = \dim \mathbb{H}^0(D) = \dim \mathbb{H}^0_\omega(D) = 1, \\ &\dim \mathbb{P}^L(D) = \binom{N+L-1}{N-1}, \\ &\dim \mathbb{H}^L(D) = \dim \mathbb{H}^L_\omega(D) = n(N,L) = \frac{(2L+N-2)(L+N-3)!}{L!\,(N-2)!} \qquad L \geq 1,\end{aligned}$$

(see (2) and [36, eq. (11)]). In particular, if $N = 2$ then $\dim \mathbb{H}^0_\omega(D) = 1$ and $\dim \mathbb{H}^L_\omega(D) = 2$ for $L \geq 1$, while if $N = 3$ then $\dim \mathbb{H}^L_\omega(D) = 2L + 1$.

Thanks to the results of the previous sections, the generalized harmonic polynomials are solution of the homogeneous Helmholtz equation with wavenumber ω and belong to $H^k(D)$ for every $k \in \mathbb{N}$, so they are also in $C^\infty(D)$.

In order to write explicitly the generalized harmonic polynomials we prove the following lemma for homogeneous functions.

Lemma 1.3.2. *If $\phi \in L^2(D)$ is an l-homogeneous function with $l \in \mathbb{R}$, $l > -\frac{N}{2}$, i.e., there exists $g \in L^2(S^{N-1})$ such that*

$$\phi(x) = g\Big(\frac{x}{|x|}\Big)\;|x|^l, \qquad a.e.\ x \in D,$$

then its Vekua transform is

$$V_1[\phi](x) = \Gamma\Big(l+\frac{N}{2}\Big)\;\Big(\frac{2}{\omega}\Big)^{l+\frac{N}{2}-1}\;g\Big(\frac{x}{|x|}\Big)\;|x|^{1-\frac{N}{2}}\;J_{l+\frac{N}{2}-1}(\omega|x|) \quad a.e.\ x \in D. \tag{1.48}$$

Proof. Using the Beta integral $\int_0^1 t^a(1-t)^b\,\mathrm{d}t = \frac{\Gamma(a+1)\,\Gamma(b+1)}{\Gamma(a+b+2)}$, $a, b > -1$, we can compute directly the Vekua transform from the definition of V_1:

$$\begin{aligned}
V_1[\phi](x) &= g\Big(\frac{x}{|x|}\Big)\;|x|^l + \int_0^1 g\Big(\frac{x}{|x|}\Big)\;(|x|t)^l\;M_1(x,t)\,\mathrm{d}t\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\,\Big(1+\int_0^1 t^l M_1(x,t)\,\mathrm{d}t\Big)\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\,\left(1-\int_0^1 t^{l+\frac{N}{2}-1}\;\sum_{j\ge 0}\frac{(-1)^j\left(\frac{\omega|x|}{2}\right)^{2j+2}(1-t)^j}{j!\,(j+1)!}\,\mathrm{d}t\right)\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\,\left(1-\sum_{j\ge 0}\frac{(-1)^j\left(\frac{\omega|x|}{2}\right)^{2j+2}}{j!\,(j+1)!}\;\frac{\Gamma\Big(l+\frac{N}{2}\Big)\,\Gamma(j+1)}{\Gamma\Big(l+\frac{N}{2}+j+1\Big)}\right)\\
&\overset{k=j+1}{=}\; g\Big(\frac{x}{|x|}\Big)\,|x|^l\,\left(1+\sum_{k\ge 1}\frac{(-1)^k\left(\frac{\omega|x|}{2}\right)^{2k}}{k!\,\Gamma\Big(l+\frac{N}{2}+k\Big)}\;\Gamma\Big(l+\frac{N}{2}\Big)\right)\\
&= g\Big(\frac{x}{|x|}\Big)\,|x|^l\sum_{k\ge 0}\frac{(-1)^k\left(\frac{\omega|x|}{2}\right)^{2k}}{k!\,\Gamma\Big(l+\frac{N}{2}+k\Big)}\;\Gamma\Big(l+\frac{N}{2}\Big)\\
&= \Gamma\Big(l+\frac{N}{2}\Big)\;g\Big(\frac{x}{|x|}\Big)|x|^{1-\frac{N}{2}}\;\Big(\frac{2}{\omega}\Big)^{l+\frac{N}{2}-1}\;\sum_{k\ge 0}\frac{(-1)^k\left(\frac{\omega|x|}{2}\right)^{2k+l+\frac{N}{2}-1}}{k!\,\Gamma\Big(l+\frac{N}{2}+k\Big)}\\
&= \Gamma\Big(l+\frac{N}{2}\Big)\;\Big(\frac{2}{\omega}\Big)^{l+\frac{N}{2}-1}\;g\Big(\frac{x}{|x|}\Big)\;|x|^{1-\frac{N}{2}}\;J_{l+\frac{N}{2}-1}(\omega|x|).
\end{aligned}$$

The condition $l > -\frac{N}{2}$ is necessary to compute the integral $\int_0^1 t^{l+\frac{N}{2}-1}(1-t)^j\,\mathrm{d}t$. ∎

As a consequence, the general (non homogeneous) harmonic polynomial of degree L and its Vekua transform can be written, in terms of spherical harmonics and hyperspherical Bessel functions (see the Appendix), by

$$P(x) = \sum_{l=0}^{L}\sum_{m=1}^{n(N,l)} a_{l,m}\;|x|^l\;Y_{l,m}\Big(\frac{x}{|x|}\Big), \tag{1.49}$$

$$
\begin{aligned}
V_1[P](x) &= |x|^{1-\frac{N}{2}} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m}\, \Gamma\Big(l+\frac{N}{2}\Big) \left(\frac{2}{\omega}\right)^{l+\frac{N}{2}-1} Y_{l,m}\Big(\frac{x}{|x|}\Big) J_{l+\frac{N}{2}-1}(\omega|x|) \\
&= \frac{2\,(N-4)!!}{\Gamma\left(\frac{N}{2}-1\right)} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m}\, \Gamma\left(l+\frac{N}{2}\right) \left(\frac{2}{\omega}\right)^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big)\, j_l^N(\omega|x|).
\end{aligned}
\tag{1.50}
$$

If $N = 2$, identifying $\mathbb{R}^2 = \mathbb{C}$ and using the complex variable $z = re^{i\psi}$, using directly (1.48), we have

$$
P(z) = \sum_{l=-L}^{L} a_l\, r^{|l|}\, e^{il\psi}, \tag{1.51}
$$

$$
V_1[P](z) = \sum_{l=-L}^{L} a_l\, |l|! \left(\frac{2}{\omega}\right)^{|l|} e^{il\psi}\, J_{|l|}(\omega r). \tag{1.52}
$$

If $N = 3$, we use the definition of spherical Bessel function (A.8) to get

$$
P(x) = \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m}\, |x|^l\, Y_{l,m}\Big(\frac{x}{|x|}\Big), \tag{1.53}
$$

$$
\begin{aligned}
V_1[P](x) &= \frac{2}{\sqrt{\pi}} \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m}\; \Gamma\Big(l+\frac{3}{2}\Big) \left(\frac{2}{\omega}\right)^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big)\, j_l(\omega|x|) \qquad (1.54)\\
&= \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m}\; \frac{(2l+1)!}{l!} \left(\frac{1}{2\omega}\right)^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big)\, j_l(\omega|x|),
\end{aligned}
$$

where $\{Y_{l,m}\}_{m=-l,\dots,l}$ are a basis of spherical harmonics of order l, and we have used $\Gamma(l+\frac{3}{2}) = \frac{\sqrt{\pi}\,(2l+1)!}{2^{2l+1}\,l!}$, which follows from $\Gamma(s+1) = s\,\Gamma(s)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. This means that the generalized harmonic polynomials in 2D and 3D are the well-known circular and spherical waves, respectively.

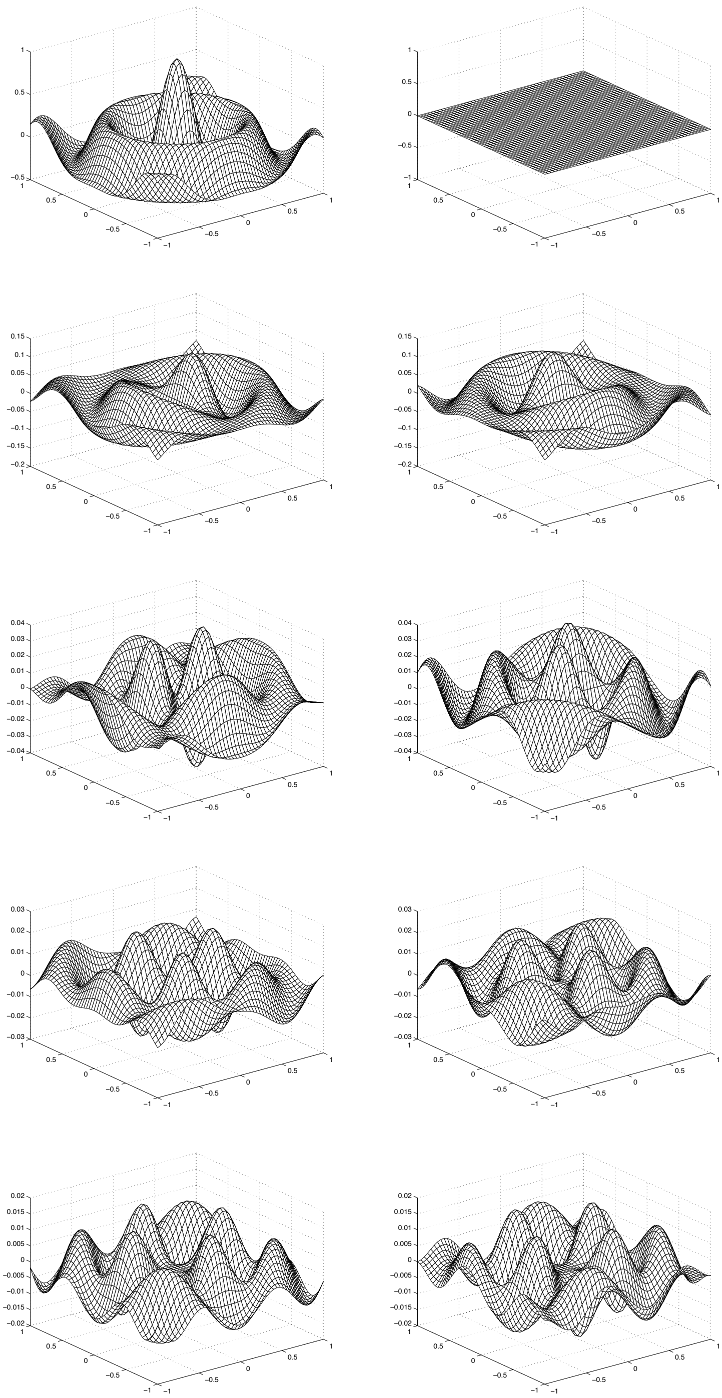
1.3.1 Generalized Harmonic Polynomials as Herglotz Functions

In this section, we define an important family of solutions of the homogeneous Helmholtz equation: the Herglotz functions (see [17, Def. 3.14]), and we see that the generalized harmonic polynomials belong to this class. This result could be used to prove approximation properties for plane waves, as in [32, Prop. 8.4.14]. In the following, we will adopt a different approach: by using the Jacobi-Anger expansions, we can directly correlate the generalized harmonic polynomials to the plane waves and find in this way sharper bound in an easier way. On the other hand, we report the result on generalized harmonic polynomials as Herglotz functions for the sake of completeness.

Definition 1.3.3. *Given a function $g \in L^2(S^{N-1})$ we define the Herglotz function with Herglotz kernel g and wavenumber ω as the the function in $C^\infty(\mathbb{R}^N)$*

$$
w_g(x) = \int_{S^{N-1}} g(d)\, e^{i\omega x\cdot d}\, \mathrm{d}\sigma(d) \qquad\qquad x \in \mathbb{R}^N. \tag{1.55}
$$

Figure 1.3: The real and imaginary parts of the 2-dimensional generalized harmonic polynomials $V_1[z^l]$, $l = 0, \ldots, 4$, $\omega = 10$, in $[-1, 1]^2$.



The Herglotz functions are entire solutions of the homogeneous Helmholtz equation; it is known that they are dense in $\mathcal{H}^k_\omega(\mathcal{D})$ with respect to the $H^k(\mathcal{D})$–norm or the $C^\infty(\mathcal{D})$ topology, where $\mathcal{D}$ is a $C^{k-1,1}$ domain; the proof is given in Theorem 2 of [45]. In part *(iv)* of Theorem 2.2.1 we will prove that the generalized harmonic polynomials, which are Herglotz functions, are dense in $\mathcal{H}^k_\omega(D)$ in two and three dimensions. This means that, for $k \geq 2$, we slightly generalize the result of [45] to domains that satisfy only Assumption 1.1.1: we weaken the regularity assumption $C^{k-1,1}$ to Lipschitz continuity, but we require the star-shapedness.

Lemma 1.3.4. *Let P be a harmonic polynomial of degree $L \in \mathbb{N}$ in $\mathbb{R}^2$ or $\mathbb{R}^N$, $N \geq 3$, defined as in* (1.51) *or in* (1.49)*, respectively. Then the corresponding generalized harmonic polynomial $V_1[P]$ is a Herglotz function w_g with Herglotz kernel*

$$g(\theta) = \sum_{l=-L}^{L} a_l \, \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} e^{il\theta} \qquad N = 2,$$

$$g(d) = \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m} \, \frac{\Gamma\Big(l + \frac{N}{2}\Big)\left(\frac{N}{2} - 1\right)}{\pi^{\frac{N}{2}}(N-2)} \left(\frac{2}{i\omega}\right)^{l} Y_{l,m}(d) \qquad N \geq 3.$$

Proof. We only have to use the Jacobi-Anger expansion and the summation theorem for spherical harmonics to verify that the Herglotz functions with these kernels correspond to (1.52) and (1.50), respectively.

In two space dimensions with the polar coordinates $z = r\,e^{i\psi}$ we have

$$\begin{aligned}
w_g(z) &= \int_0^{2\pi} \sum_{l=-L}^{L} a_l \, \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} e^{il\theta}\, e^{i\omega r(\cos\psi,\sin\psi)\cdot(\cos\theta,\sin\theta)} \,\mathrm{d}\theta \\
&= \sum_{l=-L}^{L} a_l \, \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} \int_0^{2\pi} e^{il\theta}\, e^{i\omega r\cos(\psi-\theta)} \,\mathrm{d}\theta \\
&\overset{(A.15)}{=} \sum_{l=-L}^{L} a_l \, \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} \int_0^{2\pi} e^{il\theta} \sum_{l'\in\mathbb{Z}} i^{l'}\, J_{l'}(\omega r)\, e^{il'(\psi-\theta)} \,\mathrm{d}\theta \\
&= \sum_{l=-L}^{L} \sum_{l'\in\mathbb{Z}} a_l \, \frac{|l|!}{2\pi} \left(\frac{2}{i\omega}\right)^{|l|} i^{l'}\, J_{l'}(\omega r)\, e^{il\psi} \int_0^{2\pi} e^{i(l-l')\theta} \,\mathrm{d}\theta \\
&= \sum_{l=-L}^{L} a_l \, |l|! \left(\frac{2}{\omega}\right)^{|l|} J_l(\omega r)\, e^{il\psi} = V_1[P](z),
\end{aligned}$$

where in the second last step we have used the identity $\int_0^{2\pi} e^{i(l-l')\theta}\,\mathrm{d}\theta = 2\pi\,\delta_{l,l'}$.

In higher space dimensions, we use the orthonormality of the spherical harmonics $\int_{S^{N-1}} Y_{l,m}\overline{Y_{l',m'}} = \delta_{l,l'}\delta_{m,m'}$:

$$\begin{aligned}
w_g(x) &= \int_{S^{N-1}} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m} \, \frac{\Gamma\Big(l + \frac{N}{2}\Big)\left(\frac{N}{2} - 1\right)}{\pi^{\frac{N}{2}}(N-2)} \left(\frac{2}{i\omega}\right)^{l} Y_{l,m}(d)\, e^{i\omega x\cdot d} \,\mathrm{d}\sigma(d) \\
&\overset{(A.17)}{=} \int_{S^{N-1}} \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m} \, \frac{\Gamma\Big(l + \frac{N}{2}\Big)\left(\frac{N}{2} - 1\right)}{\pi^{\frac{N}{2}}(N-2)} \left(\frac{2}{i\omega}\right)^{l} Y_{l,m}(d)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{l' \geq 0} \sum_{m'=1}^{n(N,l')} (N-2)!!\, |S^{N-1}|\, i^{l'}\, j_{l'}^N(\omega|x|)\, Y_{l',m'}\Big(\frac{x}{|x|}\Big)\, \overline{Y_{l',m'}(d)}\, \mathrm{d}\sigma(d) \\
&= \frac{(N-2)!!\left(\frac{N}{2}-1\right) 2\pi^{\frac{N}{2}}}{\pi^{\frac{N}{2}}(N-2)\,\Gamma\left(\frac{N}{2}\right)} \\
&\cdot \sum_{l=0}^{L} \sum_{m=1}^{n(N,l)} a_{l,m}\, \Gamma\Big(l+\frac{N}{2}\Big)\, \left(\frac{2}{\omega}\right)^{l}\, Y_{l,m}\Big(\frac{x}{|x|}\Big)\, j_l^N(\omega|x|) \\
&\overset{(1.50)}{=} V_1[P](x),
\end{aligned}
$$

where in the second last step we have used the formula $|S^{N-1}| = 2\pi^{\frac{N}{2}}/\Gamma\left(\frac{N}{2}\right)$. □

Chapter 2

Approximation by Harmonic Polynomials

We want to provide hp-estimates for the approximation of solutions of the homogeneous Helmholtz equation using finite dimensional spaces of generalized harmonic polynomials. Thanks to the continuity of the Vekua operator and its inverse, in order to find approximation estimates for $\mathbb{H}^L_\omega(D)$ in $\mathcal{H}^j_\omega(D)$ it is enough to derive approximation estimates in Sobolev norms for $\mathbb{H}^L(D)$ in $\mathcal{H}^j(D)$. This means that we have to approximate harmonic functions using harmonic polynomials, with respect to both decreasing diameters h and increasing orders L.

In Section 2.1, we state three theorems in this direction: *i)* Theorem 2.1.2 (Bramble-Hilbert argument), which gives sharp error estimates with respect to h in every space dimension (these estimates do not converge in the polynomial degree L); *ii)* Theorem 2.1.4 (which is Theorem 2.9 of [33]), where sharp estimates both in h and L are established in the case $N = 2$ (its proof is based on complex analysis techniques and thus can not be extended to $N \geq 3$); *iii)* Theorem 2.1.10, which gives hp-estimates for every $N \geq 2$, but the dependence of the algebraic order of convergence in L on the shape of the domain is not explicit.

In Section 2.2, we use the continuity of the Vekua operators in order to state analogous results for the approximation of solutions of the homogeneous Helmholtz equation by generalized harmonic polynomials. Theorem 2.2.1 contains the main results of this chapter.

2.1 Approximation of Harmonic Functions

2.1.1 h-Estimates

The first result is Bramble-Hilbert theorem, as presented in [11, Lemma 4.3.8]. We rewrite the proof for harmonic functions, in the Hilbert case ($p = 2$) and making explicit the dependence of the bounding constants on the degree of the approximating polynomial and the order of the norms.

We introduce the averaged Taylor polynomials, following [11, Section 4.1].

Definition 2.1.1. *Let $\phi \in H^{m-1}(D)$, with D as in Assumption 1.1.1, and ψ be a smooth cut-off function such that*

$$\operatorname{supp}\psi = B_{\rho_0 h}, \qquad \int_{B_{\rho_0 h}} \psi = 1, \qquad \|\psi\|_{L^\infty(B_{\rho_0 h})} \le C(\rho_0 h)^{-N},$$

with C independent of ρ_0 and h.

Let $y \in B_{\rho_0 h}$. If $T^m_y[\phi](x) = \sum_{|\alpha|<m} \frac{1}{\alpha!} D^\alpha \phi(y)(x-y)^\alpha$ is the value in x of the Taylor polynomial of order m of ϕ centered at y, we define the averaged Taylor polynomial of order m of ϕ as

$$\begin{aligned} Q^m\phi(x) &= \int_{B_{\rho_0 h}} T^m_y[\phi](x)\ \psi(y)\,\mathrm{d}y \\ &= \int_{B_{\rho_0 h}} \sum_{|\alpha|<m} \frac{1}{\alpha!}\ D^\alpha\phi(y)\ (x-y)^\alpha\ \psi(y)\,\mathrm{d}y. \end{aligned} \tag{2.1}$$

$Q^m\phi$ is a polynomial of degree at most $m-1$ and it is possible to define it for every $\phi \in L^1(B_{\rho_0 h})$ (see [11, Proposition 4.1.9]).

For every multi-index β such that $|\beta| \le m-1$,

$$D^\beta Q^m\phi(x) = \int_{B_{\rho_0 h}} \sum_{\substack{|\alpha|<m \\ \alpha\ge\beta}} \frac{1}{\alpha!}\ D^\alpha\phi(y)\ \frac{\alpha!}{(\alpha-\beta)!}\ (x-y)^{\alpha-\beta}\ \psi(y)\,\mathrm{d}y \tag{2.2}$$

$$\overset{\gamma=\alpha-\beta}{=} \int_{B_{\rho_0 h}} \sum_{|\gamma|<m-|\beta|} \frac{1}{\gamma!}\ D^{\beta+\gamma}\phi(y)\ (x-y)^\gamma\ \psi(y)\,\mathrm{d}y \tag{2.3}$$

$$= Q^{m-|\beta|} D^\beta\phi(x).$$

This fact, with the linearity of Q^m, implies that if ϕ is harmonic then the polynomials $Q^m\phi$ are harmonic for every $m \in \mathbb{N}$:

$$\Delta Q^m\phi = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} Q^m\phi = \sum_{i=1}^N Q^{m-2} \frac{\partial^2}{\partial x_i^2}\phi = Q^{m-2}\Delta\phi = 0. \tag{2.4}$$

The version of the Bramble-Hilbert theorem (see [11, Lemma 4.3.8]) we need is the following.

Theorem 2.1.2 (Bramble-Hilbert for harmonic functions)**.** *Let D be a domain as in Assumption 1.1.1 and $\phi \in H^m(D)$ be a harmonic function. Then the harmonic polynomial $Q^m\phi$ approximates ϕ with the estimates*

$$|\phi - Q^m\phi|_{j,D} \le C_{\rho_0,N}\ (1+j)^{\frac{N-1}{2}}\ h^{m-j}\ |\phi|_{m,D} \qquad j = 0,\dots,m, \tag{2.5}$$

where the constant C depends only on ρ_0 and N, but is independent of h, m, j and ϕ.

Proof. If $j = m$ the thesis is trivial: $|\phi - Q^m\phi|_{m,D} = |\phi|_{m,D}$. Then, assume $0 \le j < m$.

For $z \in D$, we compute

$$\int_D |x-z|^{m-N}\ \mathrm{d}x \le |S^{N-1}| \int_0^h r^{m-N}\ r^{N-1}\,\mathrm{d}r = |S^{N-1}|\ \frac{h^m}{m}.$$

If $f \in L^2(D)$ we define

$$g(x) = \int_D |x-z|^{m-N} |f(z)| \, \mathrm{d}z,$$

then for every $m > 0$:

$$\begin{aligned}
\|g\|_{0,D}^2 &= \int_D \left(\int_D |x-z|^{m-N} |f(z)| \, \mathrm{d}z \right)^2 \mathrm{d}x \\
&\le \int_D \left(\int_D |x-z|^{m-N} |f(z)|^2 \, \mathrm{d}z \right) \left(\int_D |x-z|^{m-N} \, \mathrm{d}z \right) \mathrm{d}x \\
&\le \int_D \left(\int_D |x-z|^{m-N} \, \mathrm{d}x \right) |f(z)|^2 \, \mathrm{d}z \ |S^{N-1}| \frac{h^m}{m} \\
&\le |S^{N-1}|^2 \, \frac{h^{2m}}{m^2} \, \|f\|_{0,D}^2 \, .
\end{aligned} \tag{2.6}$$

Proposition 4.2.8 of [11] states that

$$\phi(x) - Q^m\phi(x) = m \sum_{|\alpha|=m} \int_{C_x} \frac{1}{\alpha!} (x-z)^\alpha \ k(x,z) \ D^\alpha \phi(z) \, \mathrm{d}z \qquad x \in D,$$

where C_x is the convex hull of $\{x\} \cup B_{\rho_0 h}$ (that is a subset of D because this is star-shaped with respect to $B_{\rho_0 h}$) and

$$|k(x,z)| \le C_{N,\psi} \left(1 + \frac{1}{\rho_0 h} |x| \right)^N \ |x-z|^{-N} \le C_{N,\psi} \left(1 + \frac{1}{\rho_0} \right)^N \ |x-z|^{-N},$$

where the constant $C_{N,\psi}$ is independent of m, ϕ, ρ_0 and h.

We restrict ourselves to a domain D with $\operatorname{diam} D = h = 1$.

For $j = 0$, we have

$$\begin{aligned}
\|\phi - Q^m\phi\|_{0,D} &\le m \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| \int_{C_x} (x-z)^\alpha \ k(x,z) \ D^\alpha \phi(z) \, \mathrm{d}z \right\|_{0,D} \\
&\le m \ C_{N,\psi} \left(1 + \frac{1}{\rho_0} \right)^N \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| \int_D |x-z|^{m-N} \ |D^\alpha \phi(z)| \, \mathrm{d}z \right\|_{0,D} \\
&\overset{(2.6)}{\le} m \ C_{N,\rho_0,\psi} \sum_{|\alpha|=m} \frac{1}{\alpha!} \ \frac{|S^{N-1}|}{m} \ \|D^\alpha \phi\|_{0,D} \\
&\le C_{N,\rho_0,\psi} \frac{(1+m)^{N-1}}{\left(\lfloor \frac{m}{N} \rfloor ! \right)^N} \ |\phi|_{m,D} \le C_{N,\rho_0,\psi} \ |\phi|_{m,D} \, ,
\end{aligned} \tag{2.7}$$

because $\alpha! \ge \left(\lfloor \frac{m}{N} \rfloor ! \right)^N$ where $\lfloor \cdot \rfloor$ is the integer part, and thanks to the (1.18) that controls the number of the multi-indices of length m.

For $0 < j < m$, we obtain

$$|\phi - Q^m\phi|_{j,D} \overset{(2.2)}{=} \left[\sum_{|\beta|=j} \left\| D^\beta \phi - Q^{m-j} D^\beta \phi \right\|_{0,D}^2 \right]^{\frac{1}{2}}$$

$$\overset{(2.7)}{\leq} C_{N,\rho_0,\psi} \left[\sum_{|\beta|=j} \left| D^\beta \phi \right|^2_{m-j,D} \right]^{\frac{1}{2}} \leq C_{N,\rho_0,\psi} \, (1+j)^{\frac{N-1}{2}} \, |\phi|_{m,D} \, .$$

Finally the assertion holds for a generic domain D by a standard scaling argument. ☐

2.1.2 p-Estimates in Two Space Dimensions

The second theorem is Theorem 2.9 of [33]. Its proof uses complex analysis techniques and thus that cannot be directly generalized to dimensions higher than two: $\mathbb{R}^2$ is identified with $\mathbb{C}$ and the harmonic function to be approximated is considered as the sum of a holomorphic function and the complex conjugate of another holomorphic function. These two functions can be approximated by complex polynomials and their conjugates.

Definition 2.1.3. *We say that the domain D satisfies the exterior cone condition with angle $\lambda\pi$, $\lambda \in (0,1]$ if for every $z \in \mathbb{C} \setminus D$ there is a cone $C \subset \mathbb{C} \setminus D$ with vertex in z and congruent to*

$$C_0(\lambda\pi, r) = \{x \in \mathbb{C} \mid 0 < \arg x < \lambda\pi, \ |x| < r\}.$$

A convex domain satisfies the exterior cone condition with angle π ($\lambda = 1$).

Theorem 2.1.4 ([33, Theorem 2.9]). *Let $D \in \mathbb{R}^2$ be a domain as in Assumption 1.1.1 that satisfies the exterior cone condition with angle $\lambda\pi$ and $\phi \in \mathcal{H}^{k+1}(D)$, k integer ≥ -1. Then for every $L \geq k$ there exists a harmonic polynomial P_L of degree L such that*

$$|\phi - P_L|_{j,D} \leq C \, h^{k+1-j} \left(\frac{\log(L+2)}{L+2} \right)^{\lambda(k+1-j)} |\phi|_{k+1,D} \qquad j = 0, \ldots, k+1, \tag{2.8}$$

where the constant C depends only on k and the shape of D.

The term $(L+2)^{-\lambda(k+1-j)}$ gives the algebraic convergence of the approximation when the degree of the polynomials is raised. These orders are sharp as shown in the numerical examples provided in [33, Section 2.4]. The speed of convergence can be improved when the singularity is located on a convex corner of the domain (see [33, Corollary 2.13]) and becomes exponential when the error is measured on a compact subset (see [34, Section 2.5]).

For complete polynomial spaces, the term $(\log(L+2))^{\lambda(k+1-j)}$ can be avoided in the best approximation spectral estimates, but it is not guaranteed that, given a harmonic function, this sharper estimate is attained with a harmonic polynomial.

2.1.3 p-Estimates in N Space Dimensions

In two space dimensions, there are several results concerning the approximation of harmonic functions by harmonic polynomials; see for example [43]. Since all the proofs are based on complex analysis techniques, only very few of them have been generalized to higher space dimensions. The proof of the density of three dimensional harmonic polynomials dates back to the work of Bergmann and Walsh (see [9, 38, 42]) but the first estimates of the order of convergence are much more recent (see [2, 7]).

The technique used by Melenk in the proof of Theorem 2.1.4 is based on a deformation of the harmonic (holomorphic in two dimensions) function to a function defined in a larger domain. Then, a classical result gives exponential convergence in the original domain, since it is compactly contained in the enlarged one; the dilation reduce the speed of convergence to an algebraic order.

In order to exploit the same idea in higher space dimensions, we need a result that gives exponential convergence in compact subdomains with a suitable dependence on the size of the extended domain. This result is provided by [6] and reported here in Theorem 2.1.8. This fact allows to prove Theorem 2.1.10 below, which generalizes Theorem 2.1.4 to higher space dimensions. For L large enough, both statements give an algebraic order of convergence in L equal to $\lambda(k+1-j)$. The main difference between the two results is that the geometric constant λ for $N \geq 3$ is not explicit, even for convex domains. This fact prevents the *hp*-estimates from being fully explicit. We are currently investigating how to find an explicit bound for λ, at least for (three dimensional) convex domain, by revisiting the proof of Theorem 2.1.8 given in [6] in this particular case.

In order to apply the compact subset convergence theorem, we need to require that our domain D is the interior of the complement of a John domain. We report the definition of John domain, according to [6].

Definition 2.1.5. *A domain $\Omega \subset \mathbb{R}^N$ is called a John domain if $\mathbb{R}^N \setminus \Omega$ is nonempty and compact and there is a constant $0 < J \leq 1$ such that for every $y \in \Omega$ there exists a locally rectifiable curve $\gamma(s) \subset \Omega$, parametrized by the arclength, with $\gamma(0) = y$ and $\gamma(\infty) = \infty$, such that $d(\gamma(s), \mathbb{R}^N \setminus \Omega) \geq sJ$, for every positive s.*

In two dimensions, if Ω is a John domain with constant J, then the interior of its complement $D = \mathbb{R}^2 \setminus \overline{\Omega}$, satisfies the exterior cone condition with constant $\lambda = {}^2\!/\!_\pi \arcsin J$. The converse is not true, in general, but it depends on the star-shapedness of D.

Remark 2.1.6. *Let $D \subset \mathbb{R}^N$ be a domain as in Assumption 1.1.1; the exterior $\mathbb{R}^N \setminus \overline{D}$ is a John domain with constant $J \geq \rho_0/\rho$: for every $y \notin \overline{D}$ it is possible to choose the curve γ of Definition 2.1.5 as the half line $\gamma(s) = (1 + s/|y|)y$. In two dimensions, the cone $\{B_{\rho_0 s/\rho}(\gamma(s))\}_{s \geq 0}$ lies outside D, as shown in Figure 2.1.*

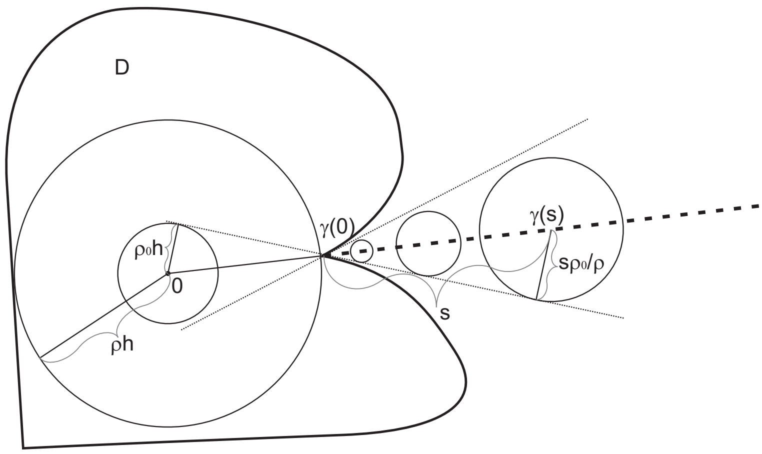
Lemma 2.1.7. *In any dimension $N \geq 2$ an open bounded set $D \subset \mathbb{R}^N$ is convex if and only if the interior of its complement $\mathbb{R}^N \setminus \overline{D}$ is a John domain with constant $J = 1$.*

Proof. If D is convex, we suppose without loss of generality that $0 \in D$. For every $y \notin D$ the curve $\gamma(s) = (1 + s/|y|)y$ satisfies Definition 2.1.5 with $J = 1$.

We prove the converse by contradiction: we assume D to be non-convex and $\mathbb{R}^N \setminus \overline{D}$ to be a John domain with $J = 1$. Since D is non-convex there exist w_1 and $w_2 \in D$ such that $(w_1 + w_2)/2 \notin D$ and since D is also open there exists $r \in (0, |w_1 - w_2|/2)$ such that $B_r(w_1) \cup B_r(w_2) \subset D$. We assume without loss of generality that $w_1 = (0, \ldots, 0, z)$ and $w_2 = -w_1$; $z > r$ follows.

By definition of John domain, there exists a curve $\gamma(s)$ in the arclength s such that $\gamma(0) = (w_1 + w_2)/2 = 0$ and $d\big(\gamma(s), B_r(w_1) \cup B_r(w_2)\big) \geq s$ for every

Figure 2.1: The exterior of D is a John domain with $J = \rho_0/\rho$. Given a point $y = \gamma(0)$ inside the re-entrant corner, the curve $\gamma(s)$ is the dashed half line.



real $s > 0$. We fix $s_* = z^2/r - r$ and we have that $\gamma(s_*) \in \overline{B}_{s_*}$ because γ is parametrized by the arclength. We have:

$$\begin{aligned} s_* &\le d\big(\gamma(s_*), B_r(w_1) \cup B_r(w_2)\big) & &\le \sup_{y \in \overline{B}_{s_*}} d\big(y, B_r(w_1) \cup B_r(w_2)\big) \\ &= d\big((s_*, 0, \dots, 0), B_r(w_1) \cup B_r(w_2)\big) & &= |(s_*, 0, \dots, 0) - w_1| - r \\ &= \sqrt{s_*^2 + z^2} - r & &= \sqrt{\frac{z^4}{r^2} + r^2 - 2z^2 + z^2} - r \\ &= \sqrt{\frac{z^4 + r^2(r^2 - z^2)}{r^2}} - r & &\overset{r<z}{<} \frac{z^2}{r} - r \quad = \quad s_*, \end{aligned}$$

that is a contradiction because the last inequality is strict. This implies that if J is equal to 1, then D must be convex. □

We report Theorem 1 of [6], where the best approximation error for harmonic functions with harmonic polynomials in the L^∞–norm on the domain D is bounded in terms of the L^∞–norm of the harmonic functions on an enlarged domain. We will use the following notation:

$$D^\delta = \{x \in \mathbb{R}^N : d(x, D) < \delta h\}.$$

Theorem 2.1.8 ([6, Theorem 1])**.** *Let $D \subset \mathbb{R}^N$ be an open set such that its exterior $\mathbb{R}^N \setminus \overline{D}$ is a John domain. Then there exist constants $p > 0$, $b > 1$, $q > 0$ and $C > 0$ depending only on D, such that, for every $\delta \in (0, 1)$, for every ϕ harmonic in D^δ, and for every polynomial degree $L > 0$, it holds*

$$\inf_{P \in \mathbb{H}^L} \|\phi - P\|_{L^\infty(D)} \le C\, (\delta h)^{-p}\, b^{-L(\delta h)^q}\, \|\phi\|_{L^\infty(D^\delta)}\,. \tag{2.9}$$

We cannot expect that the function ϕ we want to approximate can be extended outside the domain D because a singularity can be present on the boundary of D. In order to use this Theorem 2.1.8, we need to introduce a function

$T\phi$ defined on a neighborhood of D such that: *i)* $T\phi$ has the same regularity of ϕ; *ii)* is harmonic; *iii)* $T\phi$ approximates ϕ in the different Sobolev norms. In the next lemma we build a function that satisfies this requirements using a technique analogue to the one used in [33, Lemma 2.11].

Lemma 2.1.9. *Let $D \subset \mathbb{R}^N$ be a domain as in Assumption 1.1.1, $\phi \in H^{k+1}(D)$, $k \in \mathbb{N}$, $\epsilon \in (0, 1/2)$. Denote by $D_\epsilon \supset D$ the enlarged domain*

$$D_\epsilon := \frac{1}{1-\epsilon} D = \left(1 + \frac{\epsilon}{1-\epsilon}\right) D,$$

and by $T_l[\phi](x)$ the functions defined on D_ϵ by

$$T_l[\phi](x) := T^{l+1}_{(1-\epsilon)x}[\phi](x) = \sum_{|\alpha|\le l} \frac{1}{\alpha!}\, D^\alpha \phi\left((1-\epsilon)x\right) (\epsilon x)^\alpha \qquad l = 0, \dots, k. \tag{2.10}$$

Then:

(i)

$$\rho_0\, h\, \epsilon \le d(D, \partial D_\epsilon) \le 2\, h\, \epsilon; \tag{2.11}$$

(ii) there exist a constant $C_{N,k}$ independent of ϵ, D and ϕ such that

$$\|T_k[\phi]\|_{0,D_\epsilon} \le C_{N,k} \sum_{l=0}^{k} (\epsilon h)^l\, |\phi|_{l,D}\,; \tag{2.12}$$

(iii) for every multi-index β, $|\beta| \le k$

$$D^\beta\, T_k[\phi] = \sum_{l=0}^{|\beta|} \binom{|\beta|}{l}\, \epsilon^l\, (1-\epsilon)^{|\beta|-l}\, T_{k-l}[D^\beta \phi], \tag{2.13}$$

which also implies that if ϕ is harmonic in D then $T_k[\phi]$ is harmonic in D_ϵ;

(iv) if ϕ is harmonic in D, there exist a constant $C_{N,k}$ independent of ϵ, D and ϕ such that

$$|\phi - T_k[\phi]|_{j,D} \le C_{N,k}\, \rho_0^{-j}\, (\epsilon h)^{k+1-j}\, |\phi|_{k+1,D} \qquad \forall\, j = 0, \dots, k. \tag{2.14}$$

Proof. Item *(i)* follows from the bound

$$\rho_0 h\epsilon \le \frac{\rho_0 h\epsilon}{1-\epsilon} \le d(D, \partial D_\epsilon) \le \sup_{x\in D} d\Big(x, \frac{1}{1-\epsilon}x\Big) \le h\Big(\frac{1}{1-\epsilon} - 1\Big) = \frac{h\epsilon}{1-\epsilon} \le 2h\epsilon,$$

where the second inequality is proved in [32, Appendix A.3] (due to the slightly different definitions of D_ϵ, the ϵ of [32, Appendix A.3] corresponds to our $\frac{\epsilon}{1-\epsilon}$).

The bound (2.12) in item *(ii)* is straightforward:

$$\|T_k[\phi]\|^2_{0,D_\epsilon} \le \int_{D_\epsilon} \sum_{|\alpha|\le k} \frac{1}{(\alpha!)^2} \left|D^\alpha \phi\left((1-\epsilon)x\right)\right|^2 |\epsilon x|^{2|\alpha|}\, \mathrm{d}x\ (\#\{\alpha : |\alpha| \le k\})$$

$$\overset{y=(1-\epsilon)x}{\leq} \int_D \sum_{|\alpha|\leq k} \frac{1}{(\alpha!)^2} \left|D^\alpha \phi\,(y)\right|^2 \left|\frac{\epsilon h}{1-\epsilon}\right|^{2|\alpha|} \frac{\mathrm{d}y}{(1-\epsilon)^N}\ (\#\{\alpha:\ |\alpha|\leq k\})$$

$$\overset{l=|\alpha|}{\leq} C_{N,k} \sum_{l=0}^{k} (\epsilon h)^{2l}\ |\phi|^2_{l,D}.$$

For item *(iii)*, we proceed by induction on $|\beta|$. For the case $|\beta| = 1$, given $m \in \{1,\dots,N\}$, we denote $e_m = (\underbrace{0,\dots,0}_{m-1},1,0,\dots,0) \in \mathbb{N}^N$ and by α_m the m–th component of α; then

$$\begin{aligned}
D_{x_m} T_k[\phi](x) &= \sum_{|\alpha|\leq k} \frac{(1-\epsilon)}{\alpha!}\ (D_{x_m} D^\alpha)\phi\ \big((1-\epsilon)x\big)\ (\epsilon x)^\alpha \\
&\quad + \sum_{\substack{|\alpha|\leq k \\ \alpha_m \geq 1}} \frac{1}{\alpha!}\ D^\alpha \phi\ \big((1-\epsilon)x\big)\ \epsilon \alpha_m\ (\epsilon x)^{\alpha - e_m} \\
&\overset{\gamma=\alpha-e_m}{=} (1-\epsilon)\ T_k[D_{x_m}\phi](x) + \sum_{|\gamma|\leq k-1} \frac{\epsilon(\gamma_m+1)}{(\gamma_m+1)\gamma!} D^{\gamma+e_m}\phi\big((1-\epsilon)x\big)\ (\epsilon x)^\gamma \\
&= (1-\epsilon)\ T_k[D_{x_m}\phi](x) + \epsilon\ T_{k-1}[D_{x_m}\phi](x),
\end{aligned} \tag{2.15}$$

this gives (2.13) in the case $|\beta| = 1$. Now we proceed by induction for $2 \leq |\beta| \leq k$. Let assume that (2.13) holds for every multi-index γ such that $1 \leq |\gamma| < |\beta| \leq k$. Given β, there exists $m \in 1,\dots,N$ and $\gamma \in \mathbb{N}^N$ such that $\beta = \gamma + e_m$; then

$$\begin{aligned}
D^\beta T_k[\phi] =& D_{x_m} D^\gamma T_k[\phi] \overset{\substack{\text{induction} \\ (2.13)}}{=} \sum_{l=0}^{|\beta|-1} \binom{|\beta|-1}{l}\ \epsilon^l\ (1-\epsilon)^{|\beta|-1-l}\ D_{x_m} T_{k-l}[D^\gamma \phi] \\
&\overset{(2.15)}{=} \sum_{l=0}^{|\beta|-1} \binom{|\beta|-1}{l}\ \epsilon^l (1-\epsilon)^{|\beta|-1-l}\ \big[(1-\epsilon)\ T_{k-l}[D^\beta\phi] + \epsilon\ T_{k-l-1}[D^\beta \phi]\big] \\
&= \sum_{l=0}^{|\beta|} \binom{|\beta|}{l}\ \epsilon^l\ (1-\epsilon)^{|\beta|-l}\ T_{k-l}[D^\beta \phi]
\end{aligned}$$

where the last identity follows from Pascal's rule $\binom{j-1}{l} + \binom{j-1}{l-1} = \binom{j}{l}$.

In order to prove (2.14) of item *(iv)*, we fix a multi-index β and a integer l, $0 \leq l \leq |\beta| = j \leq k$. From the formula for the remainder of the multivariate Taylor polynomial, we have

$$\begin{aligned}
&\left\|D^\beta \phi - T_{k-l}[D^\beta\phi]\right\|^2_{0,D} \\
&\quad = \int_D \left| \sum_{|\alpha|=k-l+1} \frac{k-l+1}{\alpha!}\ (x\epsilon)^\alpha \int_0^1 (1-t)^{k-l}\ D^\alpha D^\beta \phi\big((1-\epsilon+t\epsilon)x\big)\,\mathrm{d}t \right|^2\ \mathrm{d}x \\
&\quad \leq C_{k,N}\ (h\epsilon)^{2(k-l+1)} \\
&\qquad\qquad \int_0^1 (1-t)^{2(k-l)} \sum_{|\alpha|=k-l+1} \int_D \left|D^\alpha D^\beta \phi\big((1-\epsilon+t\epsilon)x\big)\right|^2\ \mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$\leq C_{k,N}\ (h\epsilon)^{2(k-l+1)} \int_0^1 (1-t)^{2(k-l)}\ |\phi|^2_{k-l+1+j,(1-\epsilon+t\epsilon)D}\ \mathrm{d}t,$$

where the seminorm on the right-hand side is well defined, though ϕ belongs only to $H^{k+1}(D)$, because since it is harmonic, it is C^∞ in the interior of D. Thus,

$$\begin{aligned}
&\left\|D^\beta\phi - T_{k-l}[D^\beta\phi]\right\|^2_{0,D}\\
&\overset{(1.33)}{\leq} C_{k,N}\ (h\epsilon)^{2(k-l+1)} \int_0^1 (1-t)^{2(k-l)} d\big((1-\epsilon+t\epsilon)D, \partial D\big)^{-2(j-l)}\ |\phi|^2_{k+1,D}\ \mathrm{d}t\\
&\leq C_{k,N}\ \rho_0^{-2j}\ (h\epsilon)^{2(k-j+1)}\ |\phi|^2_{k+1,D}\ ,
\end{aligned}$$

because $(1-\epsilon+t\epsilon)D$ is star-shaped with respect to $B_{\rho_0 h(1-\epsilon+t\epsilon)}$, $d\big((1-\epsilon+t\epsilon)D, \partial D\big) \geq \rho_0 h(1-t)\epsilon$ thanks to [32, Appendix A.3] and the remaining integral is $\int_0^1 (1-t)^{2(k-j)}\ \mathrm{d}t \leq 1$.

Finally we use the fact that the sum of the coefficients in (2.13) is equal to 1 and obtain

$$\begin{aligned}
|\phi - T_k[\phi]|_{j,D} &\leq \sum_{|\beta|=j} \left\|D^\beta\phi - D^\beta T_k[\phi]\right\|_{0,D}\\
&\overset{(2.13)}{=} \sum_{|\beta|=j} \left\| \sum_{l=0}^{j} \binom{j}{l}\ \epsilon^l\ (1-\epsilon)^{j-l}\ (D^\beta\phi - T_{k-l}[D^\beta\phi]) \right\|_{0,D}\\
&\leq \sum_{|\beta|=j} \sum_{l=0}^{j} \binom{j}{l}\ \epsilon^l\ (1-\epsilon)^{j-l}\ \left\|D^\beta\phi - T_{k-l}[D^\beta\phi]\right\|_{0,D}\\
&\leq C_{k,N}\ \rho_0^{-j}\ (h\epsilon)^{k+1-j}\ |\phi|_{k+1,D}\ .
\end{aligned}$$

□

This lemma allows to apply Theorem 2.1.8 to harmonic functions with given Sobolev regularity in D, regardless of their possibilities of extension outside this set. For L large enough, the obtained order of convergence is algebraic and depends on the difference between the norms on the right- and left-hand side (namely, $k+1-j$), and on a parameter λ that depends on the geometry of the domain. The following theorem is the three-dimensional analogue of the two-dimensional result given in Theorem 2.1.4. Without any further assumption on D, we cannot expect to find a fully explicit speed of convergence. As mentioned before, estimates of λ at least for convex domains are under study.

Theorem 2.1.10. *Fix $k \in \mathbb{N}$ and let $D \subset \mathbb{R}^N$, $N \geq 2$, be a domain as in Assumption 1.1.1. Then there exist three constants:*

$$\begin{gathered}
C > 0 \textit{ depending only on } k,\ N \textit{ and the shape of } D,\\
q > 0,\ b > 1 \textit{ depending only on the shape of } D
\end{gathered}$$

such that

$$\textit{for every } L \geq \max\{k,\ 2^q\} \textit{ and for every } \phi \in H^{k+1}(D) \textit{ harmonic in } D,$$

there exists a harmonic polynomial P of degree L that satisfies

$$|\phi - P|_{j,D} \le C\, h^{k+1-j} \left(L^{-\lambda(k+1-j)} + b^{-L^{1-\lambda q}}\, L^{\lambda(1+j+\frac{N}{2})} \right) |\phi|_{k+1,D} \qquad \forall\, 0 \le j \le k, \quad \forall\, \lambda \in (\log 2/\log L, 1/q). \tag{2.16}$$

If the degree L is large enough, $1-\lambda q$ is positive, the second term on the right-hand side is smaller than the first one and the convergence in L is algebraic with order $\lambda(k+1-j)$. The coefficient λ depends only on the shape of D (through the constant q of Theorem 2.1.8).

Proof of Theorem 2.1.10. Firstly, we fix three small positive constants ϵ_1, ϵ_2, ϵ_3 in the interval $(0, 1/2)$ and define $\epsilon_* := 1-(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3) < \epsilon_1+\epsilon_2+\epsilon_3$. For every domain Ω we can define

$$\Omega'_\epsilon := \frac{1}{1-\epsilon_1}\Omega, \qquad \Omega''_\epsilon := \frac{1}{1-\epsilon_2}\Omega'_\epsilon = \frac{1}{(1-\epsilon_1)(1-\epsilon_2)}\Omega,$$
$$\Omega'''_\epsilon := \frac{1}{1-\epsilon_3}\Omega''_\epsilon = \frac{1}{(1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)}\Omega = \frac{1}{1-\epsilon_*}\Omega,$$
$$\hat\Omega := \frac{1}{h}\Omega.$$

For every function f defined on Ω, we also define $\hat f(\hat x) = f(h\hat x)$ on $\hat\Omega$.

Thanks to Remark 2.1.6, we can apply Theorem 2.1.8: for every $T \in H^j(D'''_\epsilon)$ and harmonic, there exists a harmonic polynomial $\tilde P^L$ of degree at most L such that

$$\begin{aligned}
\left|T - \tilde P^L\right|_{j,D} &\le C_{N,j}\, h^{\frac{N}{2}-j} \left|\hat T - \hat{\tilde P}^L\right|_{j,\hat D} \\
&\overset{(1.33)}{\underset{(2.11)}{\le}} C_{N,j}\, h^{\frac{N}{2}-j} (\rho_0\epsilon_1)^{-j} \left\|\hat T - \hat{\tilde P}^L\right\|_{0,\hat D'_\epsilon} \\
&\le C_{N,j}\, h^{\frac{N}{2}-j} |\hat D'_\epsilon|^{\frac12} (\rho_0\epsilon_1)^{-j} \left\|\hat T - \hat{\tilde P}^L\right\|_{L^\infty(\hat D'_\epsilon)} \\
&\overset{(2.9)}{\le} C_{N,j,\hat D}\, h^{\frac{N}{2}-j} \left(\frac{1}{1-\epsilon_1}\right)^{\frac{N}{2}} (\rho_0\epsilon_1)^{-j} \epsilon_2^{-p}\, b^{-L\epsilon_2^q} \left\|\hat T\right\|_{L^\infty(\hat D''_\epsilon)} \\
&\overset{(1.30)}{\le} C_{N,j,\hat D}\, h^{\frac{N}{2}-j} (\rho_0\epsilon_1)^{-j} \epsilon_2^{-p}\, b^{-L\epsilon_2^q}\, \epsilon_3^{-\frac{N}{2}} \left\|\hat T\right\|_{0,\hat D'''_\epsilon} \\
&\le C_{N,j,\hat D}\, h^{-j}\, \epsilon_1^{-j} \epsilon_2^{-p}\, b^{-L\epsilon_2^q}\, \epsilon_3^{-\frac{N}{2}} \|T\|_{0,D'''_\epsilon}\,.
\end{aligned} \tag{2.17}$$

Now we define

$$\tilde\phi := \phi - Q^{k+1}\phi,$$

where $Q^{k+1}\phi$ is the averaged Taylor polynomial of ϕ from Definition 2.1.1. We choose

$$T := T_k[\tilde\phi]$$

from Lemma 2.1.9, using $\epsilon = \epsilon_*$. Let $\tilde P^L$ be the polynomial that approximate T on D from Theorem 2.1.8 as above, so that (2.17) is satisfied. Finally we define

$$P^L := \tilde P^L + Q^{k+1}\phi$$

that is a harmonic polynomial of degree at most L, because $k \le L$ and thanks to (2.4).

These definitions allow to gather all the approximation results proved so far in the following estimate:

$$
\begin{aligned}
\left|\phi - P^L\right|_{j,D} &= \left|\tilde{\phi} + Q^{k+1}\phi - \tilde{P}^L - Q^{k+1}\phi\right|_{j,D} \\
&\le \left|\tilde{\phi} - T_k[\tilde{\phi}]\right|_{j,D} + \left|T_k[\tilde{\phi}] - \tilde{P}^L\right|_{j,D} \\
&\overset{\substack{(2.14)\\(2.17)}}{\le} C_{N,k}\, \rho_0^{-j} (\epsilon_* h)^{k+1-j} \left|\tilde{\phi}\right|_{k+1,D} + C_{N,j,\hat{D}}\, h^{-j} \frac{\epsilon_1^{-j}\, \epsilon_2^{-p}\, \epsilon_3^{-\frac{N}{2}}}{b^{L\epsilon_2^q}} \left\|T_k[\tilde{\phi}]\right\|_{0,D_\epsilon'''} \\
&\overset{(2.12)}{\le} C_{N,j,k,\hat{D}} \left((\epsilon_* h)^{k+1-j} \left|\tilde{\phi}\right|_{k+1,D} + \frac{\epsilon_1^{-j} \epsilon_2^{-p}\, \epsilon_3^{-\frac{N}{2}}}{b^{L\epsilon_2^q}} \sum_{l=0}^{k} \epsilon_*^l h^{l-j} \left|\tilde{\phi}\right|_{l,D} \right) \\
&\overset{(2.5)}{\le} C_{N,j,k,\hat{D}} \left(\epsilon_*^{k+1-j} + \frac{\epsilon_1^{-j} \epsilon_2^{-p}\, \epsilon_3^{-\frac{N}{2}}}{b^{L\epsilon_2^q}} \sum_{l=0}^{k} \epsilon_*^l \right) h^{k+1-j} \left|\tilde{\phi}\right|_{k+1,D} \\
&\le C_{N,j,k,\hat{D}} \left(\epsilon_*^{k+1-j} + \frac{\epsilon_1^{-j} \epsilon_2^{-p}\, \epsilon_3^{-\frac{N}{2}}}{b^{L\epsilon_2^q}} \right) h^{k+1-j}\ \left|\phi\right|_{k+1,D}.
\end{aligned}
$$

Now, for every $\lambda \in (\log 2/\log L, 1/q)$ we can fix $\epsilon_1 = \epsilon_2 = \epsilon_3 = L^{-\lambda} < \frac{1}{2}$. This gives

$$
\left|\phi - P^L\right|_{j,D} \le C_{N,j,k,\hat{D}} \left(L^{-\lambda(k+1-j)} + \frac{L^{\lambda(j+p+\frac{N}{2})}}{b^{L^{1-\lambda q}}} \right) h^{k+1-j}\ \left|\phi\right|_{k+1,D},
$$

and we obtain the thesis. □

2.2 Approximation of Homogeneous Helmholtz Solutions

Now we are able to bound the error in the approximation of solutions of the homogeneous Helmholtz equation by generalized harmonics polynomials. These estimates guarantee the convergence in the diameter h and in the degree L. We only have to combine the results of Theorems 1.2.1, 2.1.2, 2.1.4 and 2.1.10.

Theorem 2.2.1. *Let $D \subset \mathbb{R}^N$ be a domain as in Assumption 1.1.1, $k \in \mathbb{N}$ and $u \in H^{k+1}(D)$ be a solution of the homogeneous Helmholtz equation* (1.4) *in D. Then the following results hold.*

(i) h-estimates:
For every $N \ge 2$ and for every $L \le k$ there exists a generalized harmonic polynomial Q_L of degree at most L such that, for every $j \le L+1$, it holds

$$
\|u - Q_L\|_{j,\omega,D} \le C\, (1+L)^{\frac{7}{2}N}\, e^{j+L}\, h^{L+1-j}\ \|u\|_{L+1,\omega,D}, \qquad (2.18)
$$

where the constant C depends only on ωh, ρ, ρ_0 and N, but is independent of L, j and u. In particular, this holds when $Q_L = V_1\left[Q^{L+1}V_2[u]\right]$, where $Q^{L+1}V_2[u]$ denote the averaged Taylor polynomial of degree $L+1$ of $V_2[u]$ (see Definition 2.1.1 and Theorem 2.1.2).

(ii) h-estimates, explicit in ωh:
If $N = 2, 3$, for every $L \leq k$ there exists a generalized harmonic polynomial Q_L of degree at most L such that, for every $j \leq L+1$, it holds

$$\|u - Q_L\|_{j,\omega,D} \leq C\,(1+L)^{4N-\frac{1}{2}}\, e^{j+L} \left(1 + (\omega h)^{j+6}\right) e^{\frac{3}{4}(1-\rho)\omega h}\, h^{L+1-j}\, \|u\|_{L+1,\omega,D}\,, \tag{2.19}$$

where the constant C depends only on ρ, ρ_0 and N, but is independent of h, ω, L, j and u. Again, this holds when $Q_L = V_1\left[Q^{L+1}V_2[u]\right]$.

(iii) Explicit hp-estimates in two space dimensions:
If $N = 2$ and D satisfies the exterior cone condition with angle $\lambda\pi$ (see Definition 2.1.3), then for every $L \geq k$ there exist a generalized harmonic polynomial Q'_L of degree at most L such that, for every $j \leq k+1$, it holds

$$\|u - Q'_L\|_{j,\omega,D} \leq C \left(1 + (\omega h)^{j+6}\right) e^{\frac{3}{4}(1-\rho)\omega h} \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(k+1-j)} h^{k+1-j}\, \|u\|_{k+1,\omega,D}\,, \tag{2.20}$$

where the constant C depends only on the shape of D, j and k, but is independent of h, ω, L and u. This holds when $Q'_L = V_1[P'^L]$, where P'^L is the harmonic polynomial approximating $V_2[u]$ provided by Theorem 2.1.4; notice that (2.20) *holds also for $k = -1$.*

(iv) hp-estimates in two and three space dimensions:
If $N = 2, 3$, for every $L \geq \max\{k, t_D\}$, where t_D is a threshold depending only on the shape of D (see Theorem 2.1.10), there exist a generalized harmonic polynomial Q''_L of degree at most L such that, for every $j \leq k+1$, it holds

$$\|u - Q''_L\|_{j,\omega,D} \leq C \left(1 + (\omega h)^{j+6}\right) e^{\frac{3}{4}(1-\rho)\omega h}\, L^{-\lambda(k+1-j)}\, h^{k+1-j}\, \|u\|_{k+1,\omega,D}\,, \tag{2.21}$$

where $\lambda > 0$ depends only on the shape of D and the constant C depends only on the shape of D, j, k and N, but is independent of h, ω, L and u. In particular, this holds when $Q''_L = V_1[P''^L]$, where P''^L is the harmonic polynomial approximating $V_2[u]$ provided by Theorem 2.1.10.

Proof. In order to prove both items *(i)* and *(ii)*, we choose the same $Q_L = V_1\left[Q^{L+1}V_2[u]\right]$, and we use the continuity of the Vekua operators (1.12), (1.13), (1.15) and the Bramble-Hilbert Theorem 2.1.2. For every $N \geq 2$ we have

$$\begin{aligned} &\|u - Q_L\|^2_{j,\omega,D} \\ &\overset{(1.12)}{\leq} C(1+j)^{3N+1} e^{2j} (1 + (\omega h)^2)^2 \sum_{l=0}^{j} \omega^{2(j-l)} \left|V_2[u] - Q^{L+1}V_2[u]\right|^2_{l,D} \\ &\overset{(2.5)}{\leq} C(1+j)^{3N+1} e^{2j} (1 + (\omega h)^2)^2 \end{aligned}$$

$$
\begin{aligned}
&\cdot \sum_{l=0}^{j} \omega^{2(j-l)} (1+L)^{N-1}\ h^{2(L+1-l)}\ |V_2[u]|^2_{L+1,D} \\
&\leq C(1+j)^{4N+1} e^{2j} (1+(\omega h)^{j+2})^2\ h^{2(L+1-j)}\ |V_2[u]|^2_{L+1,D} \\
&\overset{(1.13)}{\leq} C_{\omega h,\rho}\ (1+j)^{4N+1} e^{2j}\ h^{2(L+1-j)}\ (L+1)^{3N-1} e^{2(L+1)}\ \|u\|^2_{L+1,\omega,D} \\
&\leq C(1+L)^{7N} e^{2(j+L)}\ h^{2(L+1-j)}\ \|u\|^2_{L+1,\omega,D}\,,
\end{aligned}
$$

and for $N = 2, 3$ we obtain

$$
\begin{aligned}
\|u - Q_L\|^2_{j,\omega,D} &\leq C(1+j)^{4N+1} e^{2j} (1+(\omega h)^{j+2})^2\ h^{2(L+1-j)}\ |V_2[u]|^2_{L+1,D} \\
&\overset{(1.15)}{\leq} C(1+j)^{4N+1} e^{2j} (1+(\omega h)^{j+2+4})^2\ h^{2(L+1-j)} \\
&\qquad \cdot (L+1)^{4N-2} e^{2(L+1)} e^{\frac{3}{2}(1-\rho)\omega h}\ \|u\|^2_{L+1,\omega,D} \\
&\leq C(1+L)^{8N-1} e^{2(j+L)} (1+(\omega h)^{j+6})^2\ h^{2(L+1-j)}\ e^{\frac{3}{2}(1-\rho)\omega h}\ \|u\|^2_{L+1,\omega,D}\,.
\end{aligned}
$$

Items *(iii)* and *(iv)* can be proved in a similar way by choosing $Q'_L = V_1[P'^L]$ and $Q''_L = V_1[P''^L]$, with P'^L and P''^L approximations to $V_2[u]$ provided by Theorems 2.1.4 and 2.1.10, respectively. For $N = 2$ we have

$$
\begin{aligned}
\|u - Q'_L\|^2_{j,\omega,D} &\overset{(1.12)}{\leq} C\ (1+j)^7 e^{2j} (1+(\omega h)^2)^2 \sum_{l=0}^{j} \omega^{2(j-l)} \left|V_2[u] - P'^L\right|^2_{l,D} \\
&\overset{(2.8)}{\leq} C_{j,k,\hat{D}}\ (1+(\omega h)^2)^2 \sum_{l=0}^{j} \omega^{2(j-l)} h^{2(k+1-l)} \\
&\qquad \cdot \left(\frac{\log(L+2)}{L+2}\right)^{2\lambda(k+1-l)} |V_2[u]|^2_{k+1,D} \\
&\leq C_{j,k,\hat{D}}\ (1+(\omega h)^{j+2})^2 \left(\frac{\log(L+2)}{L+2}\right)^{2\lambda(k+1-j)} h^{2(k+1-j)}\ |V_2[u]|^2_{k+1,D} \\
&\overset{(1.15)}{\leq} C_{j,k,\hat{D}}\ (1+(\omega h)^{j+6})^2 e^{\frac{3}{2}(1-\rho)\omega h} \\
&\qquad \cdot \left(\frac{\log(L+2)}{L+2}\right)^{2\lambda(k+1-j)} h^{2(k+1-j)}\ \|u\|^2_{k+1,\omega,D}\,,
\end{aligned}
$$

while for $N = 2, 3$ we obtain

$$
\begin{aligned}
\|u - Q''_L\|^2_{j,\omega,D} &\overset{(1.12)}{\leq} C\ (1+j)^{3N+1} e^{2j} (1+(\omega h)^2)^2 \sum_{l=0}^{j} \omega^{2(j-l)} \left|V_2[u] - P''^L\right|^2_{l,D} \\
&\overset{(2.16)}{\leq} C_{j,k,\hat{D},N}\ (1+(\omega h)^2)^2 \sum_{l=0}^{j} \omega^{2(j-l)} h^{2(k+1-l)}\ L^{-2\lambda(k+1-l)}\, |V_2[u]|^2_{k+1,D} \\
&\leq C_{j,k,\hat{D},N}\ (1+(\omega h)^{j+2})^2 L^{-2\lambda(k+1-j)}\ h^{2(k+1-j)}\ |V_2[u]|^2_{k+1,D} \\
&\overset{(1.15)}{\leq} C_{j,k,\hat{D},N}\ (1+(\omega h)^{j+6})^2 e^{\frac{3}{2}(1-\rho)\omega h} L^{-2\lambda(k+1-j)}\ h^{2(k+1-j)}\ \|u\|^2_{k+1,\omega,D}\,.
\end{aligned}
$$

□

Chapter 3

Plane Wave Approximation Estimates

In order to approximate a generic solution of the homogeneous Helmholtz equation by using a finite dimensional space of plane wave functions, we proceed in two steps: first, we approximate the homogeneous Helmholtz solution by a generalized harmonic polynomial, then we approximate the generalized harmonic polynomial by a linear combination of plane waves.

The first step has been dealt with in Chapter 2, and we study the second one in Section 3.1. Since we have to link plane waves to circular and spherical waves, the proof of the approximation estimates relies on the use of the Jacobi-Anger expansion and on a careful bound of all the resulting terms. This proof is closely related to the existence of a basis of the plane waves space that does not degenerate for small wavenumbers.

For the two dimensional case, a stable basis was already introduced in [26]. Here, with the same technique based on the Jacobi-Anger expansion and the definition of the generalized harmonic polynomials, we introduce new stable bases both in two and three dimensions. By using these new stable bases, we derive best approximation error estimates for generalized harmonic polynomials in plane wave spaces. More precisely, in two space dimensions, we can prove error estimate with sharp algebraic order of convergence in h, the diameter of the domain, and a faster than exponential speed of convergence in p, the number of plane waves used (see Lemma 3.1.3). In the three-dimensional case, we are able to prove only h–estimates (see Lemma 3.1.6).

In Section 3.2, we combine these results with the ones obtained in Chapter 2 and get approximation estimates for solutions of the homogeneous Helmholtz equation by using plane waves. We prove sharp h–estimates in two and three dimensions and hp–estimates in two dimensions. All the speeds of convergence are algebraic and the rates are given explicitly in terms of the Sobolev regularity of the approximating function, the number of plane waves in the approximation space and the order of the Sobolev norm used to measure the error (see Theorems 3.2.2 and 3.2.3).

3.1 Approximation of Generalized Harmonic Polynomials by Plane Waves

We fix $p \in \mathbb{N}$, $p \geq 1$, different directions $\{d_k\}_{k=1,\dots,p}$ in S^{N-1} and define the approximating plane wave space

$$PW_{\omega,p}(\mathbb{R}^N) := \left\{ u \in C^\infty(\mathbb{R}^N) : \; u(x) = \sum_{k=1}^{p} \alpha_k \, e^{i\omega x \cdot d_k}, \; \alpha_k \in \mathbb{C} \right\}.$$

We will always choose

$$\begin{aligned} p = \dim PW_{\omega,p}(\mathbb{R}^N) &= \sum_{l=0}^{q} n(N,l) \\ &= \dim\{N\text{–dimensional harmonic polynomials of degree } \leq q\}. \end{aligned}$$

for some $q \in \mathbb{N}$. This means

$$p = \begin{cases} 2q+1 & \text{in two dimensions ,} \\ (q+1)^2 & \text{in three dimensions .} \end{cases}$$

We will see that we need at least p plane waves in order to approximate all the generalized harmonic polynomials of degree at most q.

We study only the interesting cases $N = 2, 3$, but it is clear that everything can be extended to higher space dimensions.

3.1.1 Stable Bases

It is well-known that the plane waves Galerkin matrix associated with the $L^2(D)$ inner product is very ill-conditioned when the wavenumber is low or when the size of the domain is small. In fact the plane waves tend to be linearly dependent in the limit $\omega \to 0$. In order to cope with this problem it is possible to introduce a basis for the space $PW_{\omega,p}(\mathbb{R}^N)$ that is stable with respect to this limit. For the two dimensional case, a stable basis was already introduced in [26, Sec. 3.1].

In three dimensions, thanks to the Jacobi-Anger expansion and the definition of the generalized harmonic polynomials, we can easily find a stable basis for $PW_{\omega,p}(\mathbb{R}^3)$.

We fix $q \in \mathbb{N}$, $p = (q+1)^2$ and the p directions $\{d_{l,m}\}_{l=0,\dots,q;\ |m|\leq l}$ that define $PW_{\omega,p}(\mathbb{R}^3)$ such that the $p \times p$ matrix

$$M = \left\{ M_{l,m;l',m'} \right\}_{\substack{l=0,\dots,q,\ |m|\leq l, \\ l'=0,\dots,q,\ |m'|\leq l'}} = \left\{ Y_{l,m}(d_{l',m'}) \right\}_{\substack{l=0,\dots,q,\ |m|\leq l, \\ l'=0,\dots,q,\ |m'|\leq l'}} \tag{3.1}$$

is invertible.

Since vector indices are often denoted by a pair of integers separated by a comma (e.g., $d_{l,m}$), here and in the following we use the semicolon to separate the row and column indices of second order matrices (e.g., $M_{l,m;l',m'}$). The components of vectors and matrices will be denoted by round brackets with subscripts, whenever their names are composite (e.g., $(Md)_{l,m}$ or $(\overline{M^{-1}})_{l,m;l',m'}$). The superscript $^{-t}$ will be used to denote the transpose of the inverse (i.e., $M^{-t} = (M^{-1})^t$).

We define the p elements of $PW_{\omega,p}(\mathbb{R}^3)$

$$b_{l,m}(x) = \frac{\Gamma\left(l+\frac{3}{2}\right)}{2\pi^{\frac{3}{2}}} \left(\frac{2}{\omega}\right)^l \sum_{\substack{l'=0,\dots,q,\\ |m'|\le l'}} (\overline{M^{-t}})_{l,m;l',m'} \;\; e^{i\omega x\cdot d_{l',m'}} \qquad l=0,\dots,q,\; |m|\le l. \tag{3.2}$$

We can compute with (A.16):

$$\begin{aligned}
b_{l,m}(x) &= 4\pi\, \frac{\Gamma\left(l+\frac{3}{2}\right)}{2\pi^{\frac{3}{2}}} \left(\frac{2}{\omega}\right)^l \sum_{\substack{\tilde l\in\mathbb{N},\\ |\tilde m|\le \tilde l}} i^{\tilde l}\, j_{\tilde l}(\omega|x|)\, Y_{\tilde l,\tilde m}\Big(\frac{x}{|x|}\Big) \\
&\qquad \cdot \sum_{\substack{l'=0,\dots,q,\\ |m'|\le l'}} (\overline{M^{-1}})_{l',m';l,m} \overline{Y_{\tilde l,\tilde m}(d_{l',m'})} \\
&= \frac{2\,\Gamma\left(l+\frac{3}{2}\right)}{\sqrt{\pi}} \left(\frac{2}{\omega}\right)^l \Bigg[i^l\, j_l(\omega|x|)\, Y_{l,m}\Big(\frac{x}{|x|}\Big) \\
&\qquad + \sum_{\substack{\tilde l> q,\\ |\tilde m|\le \tilde l}} i^{\tilde l}\, j_{\tilde l}(\omega|x|)\, Y_{\tilde l,\tilde m}\Big(\frac{x}{|x|}\Big) \sum_{\substack{l'=0,\dots,q,\\ |m'|\le l'}} (\overline{M^{-1}})_{l',m';l,m} \overline{Y_{\tilde l,\tilde m}(d_{l',m'})} \Bigg] \\
&\overset{(1.54)}{=} V_1\Big[|x|^l\, Y_{l,m}\Big(\frac{x}{|x|}\Big)\Big] + O(\omega^{q+1-l})_{\omega\to 0},
\end{aligned}$$

thanks to to the asymptotic properties of the spherical Bessel functions (A.10) and to

$$\begin{aligned}
\sum_{\substack{l'=0,\dots,q,\\ |m'|\le l'}} (M^{-1})_{l',m';l,m} Y_{\tilde l,\tilde m}(d_{l',m'}) &= \sum_{\substack{l'=0,\dots,q,\\ |m'|\le l'}} (M^{-1})_{l',m';l,m} (M)_{\tilde l,\tilde m;l',m'} \\
&= \delta_{l,\tilde l}\, \delta_{m,\tilde m}, \qquad \text{if } |\tilde m|\le \tilde l\le q.
\end{aligned}$$

The functions $b_{l,m}$ constitute a basis in $PW_{\omega,p}(\mathbb{R}^3)$; since

$$b_{l,m}(x) \overset{\omega\to 0}{\longrightarrow} |x|^l\, Y_{l,m}\Big(\frac{x}{|x|}\Big)$$

uniformly on compact sets, this basis does not degenerate for small positive ω and its mass matrix is well conditioned.

In two dimensions, using this same technique, it is possible to define a stable basis in a simpler way than in [26]:

$$b_l(x) := (-i)^l\, \gamma_l\, |l|! \left(\frac{2}{\omega}\right)^{|l|} \sum_{l'=-q}^{q} (A^{-t})_{l;l'}\; e^{i\omega x\cdot d_{l'}} \qquad l=-q,\dots,q, \tag{3.3}$$

where $\gamma_l = 1$ if $l\ge 0$ and $\gamma_l = (-1)^l$ if $l<0$, the plane waves directions are

$$d_l = (\cos\theta_l, \sin\theta_l) \qquad l=-q,\dots,q, \qquad\qquad d_l\ne d_k \qquad \forall\, l\ne k,$$

and the matrix A is

$$A = \big\{A_{l;l'}\big\}_{\substack{l=-q,\dots,q\\ l'=-q,\dots,q}} = \big\{e^{-il\theta_{l'}}\big\}_{\substack{l=-q,\dots,q\\ l'=-q,\dots,q}}.$$

With this definition, using the polar coordinates $x = r(\cos\psi, \sin\psi)$, we can easily prove that

$$\begin{aligned} b_l(x) &= (-i)^l\, \gamma_l\, |l|! \left(\frac{2}{\omega}\right)^{|l|} \sum_{l'=-q}^{q} (A^{-t})_{l;l'}\, e^{i\omega r \cos(\psi-\theta_{l'})} \\ &\overset{(A.15)}{=} (-i)^l\, \gamma_l\, |l|! \left(\frac{2}{\omega}\right)^{|l|} \sum_{\tilde{l}\in\mathbb{Z}} i^{\tilde{l}}\, J_{\tilde{l}}(\omega r)\, e^{i\tilde{l}\psi} \sum_{l'=-q}^{q} (A^{-t})_{l;l'}\, e^{-i\tilde{l}\theta_{l'}} \\ &= (-i)^l\, \gamma_l\, |l|! \left(\frac{2}{\omega}\right)^{|l|} \left(i^l\, J_l(\omega r)\, e^{il\psi} + \sum_{|\tilde{l}|>q} i^{\tilde{l}}\, J_{\tilde{l}}(\omega r)\, e^{i\tilde{l}\psi} \sum_{l'=-q}^{q} (A^{-t})_{l;l'}\, e^{-i\tilde{l}\theta_{l'}} \right) \\ &\overset{1.52}{\underset{}{\overset{(A.2)}{=}}} V_1 \left[r^{|l|} e^{il\psi} \right] + O(\omega^{q+1-|l|})_{\omega\to 0}. \end{aligned}$$

The existence of a stable basis and the proof of the convergence of the plane wave approximation require the matrices A and M to be invertible. This is the case if and only if the sets of directions $\{d_l\}$ or $\{d_{l,m}\}$ (in two or three dimensions, respectively) constitute a fundamental system for the harmonic polynomials of degree at most q. In two dimensions, if the directions d_l are all different from each other, this is always true, as we will see in the proof of Lemma 3.1.3. In three dimensions, we prove in the following two lemmas that there exist many configurations of directions that make M invertible and provide an example.

Lemma 3.1.1. *Let the matrix M be defined as in* (3.1). *The set of the configuration of directions $\{d_{l,m}\}_{l=0,\dots,q,\ |m|\le l}$ that makes M invertible is a dense open subset of $(S^2)^p$.*

Proof. The spherical harmonics $Y_{l,m} = Y_{l,m}(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$ are polynomial functions of $\sin\theta$, $\cos\theta$, $\sin\varphi$, $\cos\varphi$, and the same is the determinant $\det(M) : (S^2)^p \to \mathbb{C}$. This implies that it is continuous and then the pre-image $[\det(M)]^{-1}\{\mathbb{C}\setminus 0\}$ is an open set.

The existence of at least one configuration of directions $\{d_{l,m}\}_{l=0,\dots,q;\ |m|\le l}$ such that M is invertible is guaranteed by a simple generalization (to non constant degrees n) of Lemma 6 of [36], or by Lemma 3.1.2 below. Since a trigonometric polynomial is equal to zero in an open set of $\mathbb{R}^{2p}$ if and only if it is zero everywhere, then $\det(M)$ is zero only in a closed subset of $(S^2)^p$ with empty interior, which means that M is invertible on a dense set. □

Lemma 3.1.2. *Given $q \in \mathbb{N}$, let the $p = (q+1)^2$ directions on S^2 be chosen as*

$$d_{l,m} = \big(\sin\theta_l \cos\varphi_{l,m},\ \sin\theta_l \sin\varphi_{l,m},\ \cos\theta_l\big)$$

for all $l = 0,\dots,q$, $|m| \le l$, where the $q+1$ colatitude angles $\{\theta_l\}_{l=0,\dots,q} \subset (0,\pi)$ are all different from each other, and the azimuths $\{\varphi_{l,m}\}_{l=0,\dots,q;|m|\le l} \subset [0, 2\pi)$ satisfy $\varphi_{l,m} \ne \varphi_{l,m'}$ for every $m \ne m'$.

Then the matrix M defined in (3.1) *is invertible.*

Proof. We define

$$c_l = \cos\theta_l \qquad\qquad l = 0,\dots,q,$$

$$N_{l,m} = \sqrt{\frac{(2l+1)(l-m)!}{4\pi\ (l+m)!}} \qquad |m| \le l \le q.$$

We notice that the values c_l are all different in $(-1,1)$ and, thanks to (A.14), it is possible to write the elements of the matrix in the form

$$M_{l,m;l',m'} = N_{l,m}\ P_l^m(c_{l'})\ e^{im\varphi_{l',m'}},$$

where P_l^m denote the Legendre function of order m associated with the Legendre polynomial of degree l (see (A.12)).

For every $m \in \{0,\dots,q\}$, we define the square matrix of dimension $q-m+1$

$$S^m = \big\{S^m_{j;l}\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}} = \big\{D^m P_l(c_j)\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}},$$

where $D^m P_l$ are the m^{th} derivatives of the Legendre polynomials of degree l defined in (A.11) and constitute a basis of the space of the polynomials of degree $q-m$. If the vector $\eta \in \mathbb{R}^{q-m+1}$ belongs to the kernel of S^m, i.e., $S^m\eta = 0$, then we have

$$0 = (S^m\eta)_j = \sum_{l=m}^{q} D^m P_l(c_j)\ \eta_l \qquad \forall\ j = m,\dots,q,$$

that means the polynomial $\sum_{l=m}^{q} D^m P_l(x)\eta_l$ of degree $q-m$ has $q-m+1$ distinct zeroes. This implies that $\eta = 0$ and hence the matrix S^m is invertible.

This fact also implies the invertibility of the matrices

$$\begin{aligned}
\big\{R^m_{j;l}\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}} &= (-1)^m \operatorname{diag}\big(\{(1-c_j^2)^{\frac{m}{2}}\}_{j=m,\dots,q}\big)\cdot S^m \cdot \operatorname{diag}\big(\{N_{l,m}\}_{l=m,\dots,q}\big)\\
&= \big\{(-1)^m\ N_{l,m}\ (1-c_j^2)^{\frac{m}{2}}\ D^m P_l(c_j)\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}}\\
&\overset{(A.12)}{=} \big\{N_{l,m}\ P_l^m(c_j)\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}} \qquad m = 0,\dots,q,
\end{aligned}$$

where P_l^m are the associated Legendre functions. Similarly, also the matrices

$$\begin{aligned}
\big\{R^{-m}_{j;l}\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}} &= \operatorname{diag}\big(\{(1-c_j^2)^{\frac{m}{2}}\}_{j=m,\dots,q}\big)\cdot S^m\\
&\qquad \cdot \operatorname{diag}\left(\left\{\frac{(l-m)!}{(l+m)!}\ N_{l,-m}\right\}_{l=m,\dots,q}\right)\\
&= \left\{N_{l,-m}\ \frac{(l-m)!}{(l+m)!}\ (1-c_j^2)^{\frac{m}{2}}\ D^m P_l(c_j)\right\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}}\\
&\overset{(A.12)}{=} \big\{N_{l,-m}\ P_l^{-m}(c_j)\big\}_{\substack{j=m,\dots,q\\ l=m,\dots,q}} \qquad m = 1,\dots,q,
\end{aligned}$$

are invertible.

We fix a vector ξ in $\mathbb{C}^p$ such that

$$(M^t\xi)_{l',m'} = \sum_{\substack{l=0,\dots,q\\ |m|\le l}} Y_{l,m}(d_{l',m'})\ \xi_{l,m} = 0 \qquad \forall\ l' = 0\dots,q, \quad |m'| \le l'.$$

If we show that $\xi_{l,m} = 0$ for all $l = 0, \dots, q$ and $m = -l, \dots, l$, then M^t (and thus M) is invertible and the proof is complete.

We define the functions

$$a_m(\theta) = \sum_{l=|m|}^{q} \xi_{l,m}\, N_{l,m}\, P_l^m(\cos\theta) \qquad \forall\, m = -q, \dots, q, \;\; \theta \in (0, \pi), \tag{3.4}$$

so that, owing to (A.14), we have

$$(M^t \xi)_{l',m'} = \sum_{m=-q}^{q} a_m(\theta_{l'})\, e^{im\varphi_{l',m'}} = 0 \qquad \forall\, l' = 0, \dots, q, \quad |m'| \le l'. \tag{3.5}$$

The last expression in the case $l' = q$ reads

$$\sum_{m=-q}^{q} a_m(\theta_q)\, e^{im\varphi_{q,m'}} = 0 \qquad \forall\, m' = -q, \dots, q.$$

Thus, the function $\sum_{m=-q}^{q} a_m(\theta_q)\, e^{im\varphi}$ is a trigonometric polynomial of degree q in the variable φ with $2q + 1$ zeroes, so its coefficients vanish:

$$a_m(\theta_q) = 0 \qquad \forall\, m = -q, \dots, q. \tag{3.6}$$

Take $m = q$; thanks to (3.4) and (A.13), we have

$$0 = a_q(\theta_q) = \xi_{q,q}\, N_{q,q}\, P_q^q(\cos\theta_q) = \xi_{q,q}\, N_{q,q}\, (-1)^q\, \frac{(2q)!}{2^q\, q!}\, (1 - \cos^2\theta_q)^{\frac{q}{2}},$$

that implies $\xi_{q,q} = 0$ and also $a_q(\theta) = 0$ for every $\theta \in (0, \pi)$. Similarly we can prove that $\xi_{q,-q} = 0$ and $a_{-q}(\theta) = 0$ for every $\theta \in (0, \pi)$.

Now we proceed by induction on the index $\overline{m}$ decreasing from $q - 1$ to 0:

$$\text{induction hypotheses} \quad \begin{cases} \xi_{l,m} = 0 & \overline{m} < |m| \le l \le q, \qquad \text{(A)} \\ a_m(\theta_j) = 0 & |m| \le \overline{m} < j \le q. \qquad \text{(B)} \end{cases}$$

We have already verified the induction hypotheses at the initial step $\overline{m} = q - 1$: $\xi_{q,\pm q} = 0$ and $a_m(\theta_q) = 0$ for all $|m| \le q$ (see (3.6)), and in particular for all $|m| \le q - 1$.

Let us suppose that (A) and (B) hold for a fixed $\overline{m} \in \{0, \dots, q - 1\}$. We have to prove

$$\text{induction assertions} \quad \begin{cases} \xi_{l,m} = 0 & \overline{m} = |m| \le l \le q, \qquad \text{(A')} \\ a_m(\theta_j) = 0 & |m| \le \overline{m} = j. \qquad \text{(B')} \end{cases}$$

The equation (3.5) for $l' = \overline{m}$ reads

$$\sum_{m=-\overline{m}}^{\overline{m}} a_m(\theta_{\overline{m}})\, e^{im\varphi_{\overline{m},m'}} = 0, \qquad \forall\, |m'| \le \overline{m},$$

since, thanks to (A) and (3.4), $a_m(\theta_{\overline{m}}) = 0$ for $|m| > \overline{m}$. This is a trigonometric polynomial in φ of degree $\overline{m}$ having $2\overline{m} + 1$ zeroes $\{\varphi_{\overline{m},m'}\}_{m'=-\overline{m},\dots,\overline{m}}$, so it is identically zero and $a_m(\theta_{\overline{m}}) = 0$ for every $|m| \le \overline{m}$, that is (B').

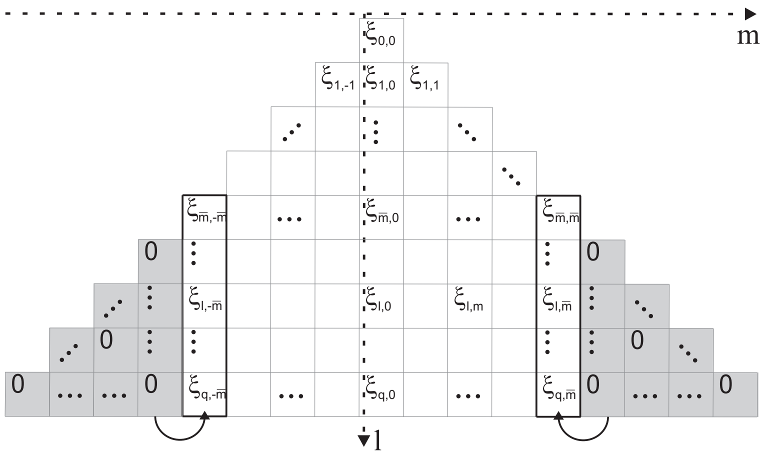
Thanks to (B) and (B'), for every $j \in \{\overline{m}, \ldots, q\}$ holds

$$0 = a_{\overline{m}}(\theta_j) \overset{(3.4)}{=} \sum_{l=\overline{m}}^{q} \xi_{l,\overline{m}}\, N_{l,\overline{m}}\, P_l^{\overline{m}}(\cos\theta_j) = \sum_{l=\overline{m}}^{q} R_{j,l}^{\overline{m}}\, \xi_{l,\overline{m}},$$

and the analogous is true with the index $-\overline{m}$. Since $R^{\pm\overline{m}}$ are invertible, we have (A') and the induction argument is complete.

We conclude that all the coefficient $\xi_{l,m}$ are equal to zero, thus M is invertible. □

Figure 3.1: A graphical representation of the backward induction on the index $\overline{m}$ in the proof of Lemma 3.1.2 with $q = 8$ and $p = 81$. At the step $\overline{m} = 4$ the coefficients in the grey squares are zero (hypothesis (A)). The induction step shows that also the coefficients in the two boxes are equal to zero (assertion (A')).



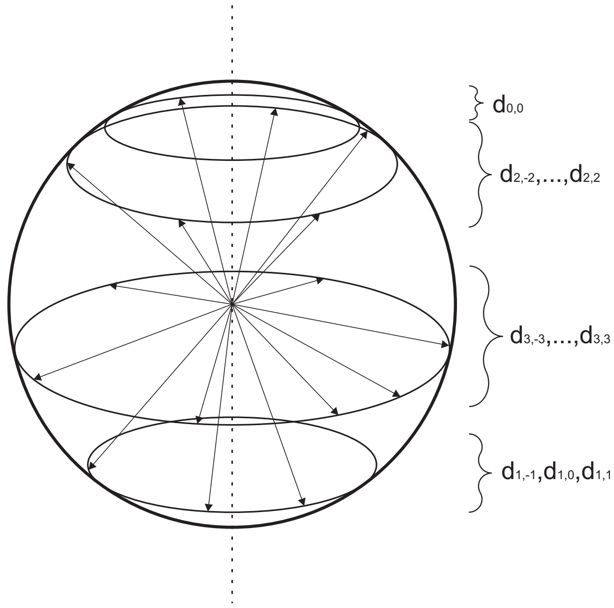
Lemma 3.1.2 provides a quite general class of configurations of plane wave directions $\{d_{l,m}\}_{l=0,\ldots,q;\ |m|\le l}$ that renders the matrix M invertible. This implies the existence of a stable basis in $PW_{\omega,p}(\mathbb{R}^3)$ and allows to prove the approximation estimates in Section 3.1.3. In order to fulfill the hypotheses of Lemma 3.1.2, the directions have to satisfy only the following geometric requirement: there exists $q + 1$ different heights $z_j \in (-1, 1)$ such that exactly $2j + 1$ different vectors $d_{l,m}$ belong to $S^2 \cap \{(x, y, z),\ z = z_j\}_{j=0,\ldots,q}$.

An example of directions satisfying this condition with $q = 3$ is shown in Figure 3.2.

3.1.2 The Two-Dimensional Case

In two space dimensions, thanks to the Jacobi-Anger expansion and the special properties of the circular harmonics $Y_l(\theta) = e^{il\theta}$ (see Section A.1 in the Appendix), we are able to approximate a generalized harmonic polynomial in $PW_{\omega,p}(\mathbb{R}^2)$, with approximation error estimates that converge both in h and in p. The order of convergence with respect to h is sharp, as can be seen from

Figure 3.2: A choice of directions $\{d_{l,m}\}_{l=0,\ldots,q;\ |m|\leq l}$ that satisfies the hypothesis of Lemma 3.1.2 with $q = 3$, $p = 16$. Notice that 1 direction belongs to level 0, 3 directions to level 1, 5 to level 2 and 7 to level 3.



simple numerical experiments [12, 25, 26, 35]. The proof given below simplifies the one given in [35], and also gives the convergence in p.

We prove a completely explicit estimate of the L^∞–norm of the approximation error.

Lemma 3.1.3. *Let $D \subset \mathbb{R}^2$ be a domain as in Assumption 1.1.1. Let P be a harmonic polynomial of degree L and let*

$$\{d_k = (\cos\theta_k, \sin\theta_k)\}_{k=-q,\ldots,q}$$

be the different directions in the definition of $PW_{\omega,p}(\mathbb{R}^2)$, $p = 2q+1$. We assume that there exists $0 < \delta \leq 1$ such that

$$\min_{\substack{j,k=-q,\ldots,q\\ j\neq k}} \left|\theta_j - \theta_k\right| \geq \frac{2\pi}{p}\,\delta\,. \tag{3.7}$$

Let the conditions on the indices

$$0 \leq K \leq L \leq q, \qquad\qquad L - K + 1 \leq \left\lfloor \frac{q+1}{2} \right\rfloor, \tag{3.8}$$

be satisfied. Then there exists a vector $\alpha \in \mathbb{C}^p$ such that, for every $R > 0$,

$$\left\| V_1[P] - \sum_{k=-q}^{q} \alpha_k\, e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_R)} \leq C(\omega, \delta, \rho, h, R, q, K, L)\ \|P\|_{K,\omega,D}\ , \tag{3.9}$$

where we have set, for brevity,

$$C(\omega,\delta,\rho,h,R,q,K,L) = \frac{e^3}{\pi^{\frac{3}{2}}\rho^{\lfloor\frac{q+1}{2}\rfloor}} \left(\frac{e^{\frac{5}{2}}}{2\sqrt{2}\,\delta^2}\right)^q \left(2^L\sqrt{L+1}\right) e^{\frac{\omega R}{2}} \cdot (\omega R)^{q+1}\left(1+(\omega h)^{-L}\right)\, h^{K-1}\, \frac{1}{(q+1)^{\frac{q+1}{2}}}\,.$$

Proof. We write the polynomial

$$P(z) = \sum_{l=-L}^{L} a_l\, r^{|l|}\, e^{il\psi}$$

with the usual identification $\mathbb{R}^2 = \mathbb{C}$ and $z = re^{i\psi}$. We have

$$\begin{aligned} V_1[P](z) - \sum_{k=-q}^{q} \alpha_k\, e^{i\omega(r\cos\psi, r\sin\psi)\cdot d_k} &\overset{(1.52)}{=} \sum_{l=-L}^{L} a_l\, |l|!\, \left(\frac{2}{\omega}\right)^{|l|} e^{il\psi}\, J_{|l|}(\omega r) - \sum_{k=-q}^{q} \alpha_k\, e^{i\omega r\cos(\psi-\theta_k)} \\ &\overset{(A.15)}{=} \sum_{l=-L}^{L} a_l\, |l|!\, \left(\frac{2}{\omega}\right)^{|l|} e^{il\psi}\, \gamma_l\, J_l(\omega r) - \sum_{l\in\mathbb{Z}} i^l\, J_l(\omega r)\, e^{il\psi} \sum_{k=-q}^{q} \alpha_k\, e^{-il\theta_k}, \end{aligned}$$

where $\gamma_l = 1$ if $l \geq 0$ and $\gamma_l = (-1)^l$ if $l < 0$ thanks to (A.2). Define the $p\times p$ matrix A by

$$A = \{A_{l;k}\}_{l,k=-q,\dots,q} = \{e^{-il\theta_k}\}_{l,k=-q,\dots,q},$$

and the vector $\beta \in \mathbb{C}^p$ by

$$\beta_l = \begin{cases} a_l\, |l|!\, \left(\frac{2}{\omega}\right)^{|l|} i^{-l}\,\gamma_l & l = -L,\dots,L, \\ 0, & l = -q,\dots,-L-1,\ L+1,\dots,q. \end{cases}$$

The matrix A is non-singular because it is the product of a Vandermonde matrix and a diagonal matrix:

$$A = \{e^{-ij\theta_k}\}_{\substack{j=0,\dots,2q \\ k=-q,\dots,q}} \cdot \operatorname{diag}\left(\{e^{iq\theta_k}\}_{k=-q,\dots,q}\right) = V_A \cdot D_A.$$

By choosing the p–dimensional vector α as the solution of the linear system

$$A\,\alpha = \beta,$$

we have

$$V_1[P](z) - \sum_{k=-q}^{q} \alpha_k\, e^{i\omega(r\cos\psi, r\sin\psi)\cdot d_k} = -\sum_{|l|>q} i^l\, J_l(\omega r)\, e^{il\psi} \sum_{k=-q}^{q} \alpha_k\, e^{-il\theta_k},$$

and thus the L^∞–norm of the error is controlled by

$$\left\| V_1[P] - \sum_{k=-q}^{q} \alpha_k\, e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_R)} \leq \sup_{t\in[0,\omega R]} 2\sum_{l>q} |J_l(t)|\, \left\|A^{-1}\right\|_1\, \|\beta\|_1\,. \quad (3.10)$$

We have to bound each of the three factors on the right-hand side of (3.10).

The first factor is the easiest one:

$$\sup_{t\in[0,\omega R]} \sum_{l>q} |J_l(t)| \overset{(A.4)}{\leq} \sup_{t\in[0,\omega R]} \sum_{l>q} \left(\frac{t}{2}\right)^l \frac{1}{l!}$$
$$\leq \sup_{t\in[0,\omega R]} \left(\frac{t}{2}\right)^{q+1} \frac{1}{(q+1)!} \sum_{j\geq 0} \left(\frac{t}{2}\right)^j \frac{1}{(j)!} \tag{3.11}$$
$$= \left(\frac{\omega R}{2}\right)^{q+1} \frac{e^{\frac{\omega R}{2}}}{(q+1)!} .$$

For $\|A^{-1}\|_1$, we observe that the 1-norm of the inverse of the diagonal matrix D_A is one, while the norm of the inverse of the Vandermonde matrix V_A can be bounded using Theorem 1 of [22]:

$$\|A^{-1}\|_1 \leq \|V_A^{-1}\|_1 \|D_A^{-1}\|_1 \leq p \ \|V_A^{-1}\|_\infty \cdot 1$$
$$\leq p \max_{k=-q,\dots,q} \prod_{\substack{s=-q,\dots,q\\ s\neq k}} \frac{1+|e^{-i\theta_s}|}{|e^{-i\theta_s} - e^{-i\theta_k}|} .$$

With simple geometric considerations, it is easy to see that, under the constraint (3.7), the product on the right-hand side is bounded by its value when

$$\theta_s^* = \theta_0^* + \frac{2\pi}{p}\,\delta\, s \qquad\qquad s=-q,\dots,q,$$

and the maximum is obtained for $k=0$. A simple trigonometric calculation gives

$$|e^{-i\theta_s^*} - e^{-i\theta_0^*}| = \sqrt{2}\,\sqrt{1-\cos(\theta_s^*-\theta_0^*)} \geq \sqrt{2}\,\frac{\sqrt{2}}{\pi}\,|\theta_s^*-\theta_0^*| = \frac{4}{p}\,\delta\,|s|,$$

because $1-\cos t \geq \frac{2}{\pi^2}\,t^2$ for every $t\in[-\pi,\pi]$. This leads to the bound

$$\|A^{-1}\|_1 \leq p \prod_{\substack{s=-q,\dots,q\\ s\neq k}} \frac{2p}{4\,\delta\,|s|} \leq \frac{p^p}{(2\delta)^{2q}\,(q!)^2}. \tag{3.12}$$

In order to bound $\|\beta\|_1$, we need to bound from below the Sobolev seminorm of order m of P for every $m=0,\dots,L$. Recalling that $B_{\rho h}\subseteq D$ and taking into account the expression of P, we have

$$|P|^2_{m,D} \geq \left\|\frac{\partial^m}{\partial r^m} P\right\|^2_{0,B_{\rho h}} = \left\|\sum_{|j|=m}^{L} a_j \frac{|j|!}{(|j|-K)!} r^{|j|-K} e^{ij\psi}\right\|^2_{0,B_{\rho h}}$$
$$= \int_0^{\rho h} \sum_{|j|,|j'|=m}^{L} \frac{a_j \overline{a_{j'}}\ |j|!\ |j'|!}{(|j|-m)!\ (|j'|-m)!}\ r^{|j|+|j'|-2m} \int_0^{2\pi} e^{i(j-j')\psi}\,\mathrm{d}\psi\ \ r\,\mathrm{d}r$$
$$= 2\pi \sum_{|j|=m}^{L} |a_j|^2 \frac{(|j|!)^2}{\big((|j|-m)!\big)^2}\ \frac{(\rho h)^{2(|j|-m+1)}}{2(|j|-m+1)} , \tag{3.13}$$

where in the last step we have used the identity

$$\int_0^{2\pi} e^{i(j-j')\psi}\, \mathrm{d}\psi = 2\pi\, \delta_{jj'} \ .$$

All the terms in the sum on the right-hand side of the previous bound (3.13) are non-negative, so we can invert the estimate. Thus, considering (3.13) for $m = |l| < K$ and $m = K$, we obtain, respectively,

$$\begin{aligned}
|a_l| &\le \frac{1}{\sqrt{\pi}} \frac{1}{|l|!\,(\rho h)}\ \ |P|_{|l|,D} && \text{if } |l| < K\ ,\\
|a_l| &\le \frac{1}{\sqrt{\pi}} \frac{(|l|-K)!\,\sqrt{|l|-K+1}}{|l|!\,(\rho h)^{|l|-K+1}}\ \ |P|_{K,D} && \text{if } K \le |l| \le L\ .
\end{aligned}$$

We plug these bounds into the definition of the coefficients of β, with $K \le L$:

$$\begin{aligned}
\|\beta\|_1 &= \sum_{l=-L}^{L} |a_l| \left(\frac{2}{\omega}\right)^{|l|} |l|! \\
&\le \sum_{l=-K}^{K} \frac{1}{\sqrt{\pi}\,\rho h}\left(\frac{2}{\omega}\right)^{|l|} |P|_{|l|,D} \\
&\quad + \sum_{|l|=K+1}^{L} \frac{1}{\sqrt{\pi}} \left(\frac{2}{\omega}\right)^{|l|}\ \frac{(|l|-K)!\,\sqrt{|l|-K+1}}{(\rho h)^{|l|-K+1}}\, |P|_{K,D} \\
&\le \frac{\sqrt{2K+1}}{\sqrt{\pi}\rho} \left(\frac{1}{\omega h}\right)^{K} 2^{K+\frac{1}{2}}\ h^{K-1}\, \|P\|_{K,\omega,D} \\
&\quad + \frac{2}{\sqrt{\pi}}\ \frac{2^L h^{K-1}}{\rho^{L-K+1}} \left(\sum_{l=K+1}^{L} \frac{(l-K)!\,\sqrt{l-K+1}}{(\omega h)^l} \right) |P|_{K,D} \\
&\le \left\{ \frac{2^{L+1}}{\sqrt{\pi}\rho^{L-K+1}}\ h^{K-1} \left(1 + (\omega h)^{-L}\right) \right. \\
&\quad \left. \cdot \left(\sqrt{K+1} + (L-K)(L-K)!\sqrt{L-K+1}\right) \right\}\ \|P\|_{K,\omega,D}\ .
\end{aligned} \tag{3.14}$$

Now we can combine the bound of the sum of the Bessel functions (3.11) with the one of $\left\|A^{-1}\right\|_1$ given by (3.12) and the one of $\|\beta\|_1$ given by (3.14); inserting everything inside (3.10) gives

$$\begin{aligned}
&\left\| V_1[P] - \sum_{k=-q}^{q} \alpha_k\, e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_R)} \le 2 \left\{ \left(\frac{\omega R}{2}\right)^{q+1}\ \frac{e^{\frac{\omega R}{2}}}{(q+1)!} \right\}\ \cdot\ \left\{ \frac{p^p}{(2\delta)^{2q}\,(q!)^2} \right\} \\
&\qquad \cdot \left\{ \frac{2^{L+1}}{\sqrt{\pi}\rho^{L-K+1}}\ h^{K-1}\ \left(1 + (\omega h)^{-L}\right) \sqrt{L+1}\,(L-K+1)! \right\}\ \|P\|_{K,\omega,D} \\
&\le \left\{ \left(\frac{1}{8\delta^2}\right)^{q} (\omega R)^{q+1} e^{\frac{\omega R}{2}}\ \frac{p^p}{(q!)^2 (q+1)!} \right\} \\
&\qquad \cdot \left\{ \frac{2^{L+1}}{\sqrt{\pi}\rho^{L-K+1}}\ h^{K-1}\ \left(1 + (\omega h)^{-L}\right) \sqrt{L+1}\,(L-K+1)! \right\}\ \|P\|_{K,\omega,D}
\end{aligned}$$

$$\overset{(3.8)}{\leq} \frac{2}{\sqrt{\pi}\rho^{\lfloor\frac{q+1}{2}\rfloor}} \left(\frac{1}{8\delta^2}\right)^q \left(2^L\sqrt{L+1}\right) e^{\frac{\omega R}{2}} (\omega R)^{q+1}$$
$$\cdot \left(1+(\omega h)^{-L}\right)\ h^{K-1}\ \frac{p^p \lfloor\frac{q+1}{2}\rfloor!}{(q!)^2(q+1)!} \|P\|_{K,\omega,D}\ .$$

The Stirling formula says that

$$\sqrt{2\pi}\sqrt{n}\, n^n e^{-n} e^{\frac{1}{12n+1}} \ <\ n!\ <\ \sqrt{2\pi}\sqrt{n}\, n^n e^{-n} e^{\frac{1}{12n}}, \qquad n \geq 1.$$

We use it to bound

$$\frac{p^p\lfloor\frac{q+1}{2}\rfloor!}{(q!)^2(q+1)!} \leq \frac{(2q+2)^{2q+1}\lfloor\frac{q+1}{2}\rfloor!}{\left((q+1)!\right)^3}\ (q+1)^2$$
$$< \frac{2^{2q+1}}{2\pi}\ \frac{(q+1)^{2q+3}\left(\frac{q+1}{2}\right)^{\left(\frac{q+1}{2}\right)+\frac{1}{2}}}{(q+1)^{3(q+1)+\frac{3}{2}}} e^{3(q+1)-\frac{q}{2}} e^{-\frac{3}{12(q+1)+1}+\frac{1}{6q}}\ .$$

For $q \geq 3$, since the exponent in the last factor on the right-hand side of the last inequality is negative, we get

$$\frac{p^p\lfloor\frac{q+1}{2}\rfloor!}{(q!)^2(q+1)!} \leq \frac{e^3}{2\pi}\ \left(2\sqrt{2}\, e^{\frac{5}{2}}\right)^q\ (q+1)^{-\frac{q+1}{2}}\ .$$

For $q = 1, 2$, one can see directly that the same bound holds true, thus we can use it for any $q \geq 1$ and obtain

$$\left\| V_1[P] - \sum_{k=-q}^{q} \alpha_k\ e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_R)} \leq \frac{e^3}{\pi^{\frac{3}{2}}\rho^{\lfloor\frac{q+1}{2}\rfloor}} \left(\frac{e^{\frac{5}{2}}}{2\sqrt{2}\ \delta^2}\right)^q \left(2^L\sqrt{L+1}\right) e^{\frac{\omega R}{2}}$$
$$\cdot (\omega R)^{q+1} \left(1+(\omega h)^{-L}\right)\ h^{K-1}\ \frac{1}{(q+1)^{\frac{q+1}{2}}}\ \ \|P\|_{K,\omega,D}\ ;$$

this concludes the proof. □

Notice that, thanks to the properties of the polynomials, the assertion of Lemma 3.1.3 holds for every $R > 0$, which is not related to the size of D.

Lemma 3.1.3 provides the order of convergence of the approximation error both in h and p. We will see in Section 3.2 how to link this bound to the general Sobolev norm of the error and to the Theorem 2.2.1.

Remark 3.1.4. *When $\delta = 1$ in (3.7), we have uniformly spaced directions $\theta_j = \theta_0 + \frac{2\pi}{p}\ j$ in S^1. In this case, we see that $\left\|A^{-1}\right\|_1 = \left\|\frac{1}{p}\overline{A}^t\right\|_1 = 1$:*

$$(A\overline{A}^t)_{l;k} = \sum_{j=-q}^{q} e^{-il\theta_j} e^{ik\theta_j} = \sum_{j=-q}^{q} e^{-i(l-k)(\theta_0+\frac{2\pi}{p}j)}$$
$$= \begin{cases} e^{-i(l-k)\theta_0} e^{i(l-k)\frac{2\pi}{p}q} \dfrac{1-e^{-i(l-k)\frac{2\pi}{p}p}}{1-e^{-i(l-k)\frac{2\pi}{p}}} = 0 & l \neq k, \\ p & l = k. \end{cases}$$

In this case, the bounding constant in Lemma 3.1.3 becomes slightly smaller, but the orders of convergence remain unchanged:

$$\left\| V_1[P] - \sum_{k=-q}^{q} \alpha_k \, e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_R)} \leq \frac{e^{\frac{7}{6}}}{\sqrt{\pi}\rho^{\lfloor\frac{q+1}{2}\rfloor}} \left(\frac{e^{\frac{1}{2}}}{2\sqrt{2}}\right)^q$$

$$\cdot \left(2^L\sqrt{L+1}\right) e^{\frac{\omega R}{2}} (\omega R)^{q+1} \left(1 + (\omega h)^{-L}\right) \; h^{K-1} \; \frac{1}{(q+1)^{\frac{q+1}{2}}} \; \|P\|_{K,\omega,D} \,,$$

using $\frac{\lfloor\frac{q+1}{2}\rfloor!}{(q+1)!} \leq e^{\frac{1}{6}} \left(\frac{e}{2}\right)^{\frac{q}{2}+1} (q+1)^{-\frac{q+1}{2}}$. *The constant has been reduced by a factor* $e^{\frac{11}{6}+2q}/\pi \simeq 2e^{2q}$.

Remark 3.1.5. *We can modify the condition* (3.8) *with* $L - K + 1 \leq \alpha(q+1)$, $\alpha \in (0,1)$. *This allows to choose higher order generalized harmonic polynomials in the final p-estimate and modify the constants in Theorem 3.2.3 and in Remark 3.2.4. However, this does not affect the general order of convergence of the approximation by generalized harmonic polynomials.*

3.1.3 The Three-Dimensional Case

Now we would like to prove an approximation estimate similar to Lemma 3.1.3 in a three-dimensional setting. In two dimensions we were able to prove the order of convergence with respect to q using a sharp bound on the norm of the inverse of the matrix A, since this is of Vandermonde type. In three dimensions the corresponding matrix is M, defined in (3.1), we can guarantee its invertibility (Lemmas 3.1.1 and 3.1.2) but not bound the norm of M^{-1} with a reasonable dependence on q. This is the reason why here we have only h and not p estimate.

Lemma 3.1.6. *Let* $D \subset \mathbb{R}^3$ *be a domain that satisfies Assumption 1.1.1,* P *be a harmonic polynomial of degree* $L \leq q$, $p = (q+1)^2$ *and* $\{d_{l,m}\}_{l=0,\dots,q;\,|m|\leq l} \subset S^2$ *be different directions of the basis plane waves of* $PW_{\omega,p}(D)$ *such that* M *is invertible. Then there exists a vector* $\alpha \in \mathbb{C}^p$ *such that*

$$\left\| V_1[P] - \sum_{\substack{l=0,\dots,q;\\ |m|\leq l}} \alpha_{l,m} \, e^{i\omega x\cdot d_{l,m}} \right\|_{L^\infty(B_R)}$$
$$\leq \frac{\sqrt{3}}{2\pi\rho^{\frac{3}{2}}} \, \frac{(L+1)^{\frac{3}{2}}}{2^{q-L} \; q!} \; e^{\frac{\omega R}{2}} \; (\omega R)^{q+1-L} \; \frac{R^L}{h^{\frac{3}{2}}} \; \left\|M^{-1}\right\|_1 \; \|P\|_{L,\omega,D} \,, \tag{3.15}$$

for every $R > 0$.

Proof. As in two dimensions we write the polynomial

$$P(x) = \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} \; |x|^l \; Y_{l,m}\Big(\frac{x}{|x|}\Big),$$

and we use the Jacobi-Anger expansion:

$$
\begin{aligned}
&V_1[P](x) - \sum_{\substack{l'=0,\dots,q;\\ |m'|\le l'}} \alpha_{l',m'}\, e^{i\omega x\cdot d_{l',m'}} \\
&\overset{\substack{(1.54),\\(A.16)}}{=} \sum_{l=0}^{L}\sum_{m=-l}^{l} a_{l,m} \quad \left(\frac{1}{2\omega}\right)^{l} \frac{(2l+1)!}{l!}\, Y_{l,m}\Big(\frac{x}{|x|}\Big)\, j_l(\omega|x|) \\
&\qquad - 4\pi \sum_{l\ge 0} i^l\, j_l(\omega|x|) \sum_{m=-l}^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big) \sum_{\substack{l'=0,\dots,q;\\ |m'|\le l'}} \alpha_{l',m'}\, \overline{Y_{l,m}(d_{l',m'})} \\
&= -4\pi \sum_{l\ge q+1} i^l\, j_l(\omega|x|) \sum_{m=-l}^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big) \sum_{\substack{l'=0,\dots,q;\\ |m'|\le l'}} \alpha_{l',m'}\, \overline{Y_{l,m}(d_{l',m'})},
\end{aligned} \tag{3.16}
$$

where the vector $\alpha \in \mathbb{C}^p$ is the solution of the linear system $M \cdot \alpha = \beta$ with

$$
\beta_{l,m} = \begin{cases} \dfrac{1}{4\pi}\left(\dfrac{1}{2i\omega}\right)^{l} \dfrac{(2l+1)!}{l!} a_{l,m}, & l = 0,\dots,L;\ |m|\le l, \\ 0, & l = L+1,\dots,q;\ |m|\le l. \end{cases} \tag{3.17}
$$

Now we can bound the coefficients $a_{l,m}$ with the norms of the polynomial P, denoting $r = |x|$:

$$
\begin{aligned}
|P|^2_{K,D} \ge \left\| \frac{\partial^K}{\partial r^K} P \right\|^2_{0,B_{\rho h}} &= \left\| \sum_{l=K}^{L}\sum_{m=-l}^{l} a_{l,m} \frac{l!}{(l-K)!} r^{l-K} Y_{l,m}\Big(\frac{x}{|x|}\Big) \right\|^2_{0,B_{\rho h}} \\
&= \int_0^{\rho h} \sum_{l=K}^{L}\sum_{m=-l}^{l}\sum_{l'=K}^{L}\sum_{m'=-l'}^{l'} a_{l,m}\overline{a_{l',m'}} \frac{l!\, l'!}{(l-K)!\,(l'-K)!}\; r^{l+l'-2K} \\
&\qquad \cdot \int_{S^2} Y_{l,m}(d)\overline{Y_{l',m'}(d)}\, \mathrm{d}d\; r^2\, \mathrm{d}r \\
&= \sum_{l=K}^{L}\sum_{m=-l}^{l} |a_{l,m}|^2 \frac{(l!)^2}{\big((l-K)!\big)^2}\; \frac{(\rho h)^{2(l-K)+3}}{2(l-K)+3}
\end{aligned}
$$

thanks to the orthonormality of the spherical harmonics. This gives:

$$
|a_{l,m}| \le \frac{(l-K)!\,\sqrt{2(l-K)+3}}{l!\,(\rho h)^{l-K+\frac{3}{2}}}\; |P|_{K,D}\,, \qquad 0 \le K \le l \le L. \tag{3.18}
$$

Now, for every $d_{l',m'}$ and for every $x \in D$ we have

$$\begin{aligned}
&\left| 4\pi \sum_{l \geq q+1} i^l\, j_l(\omega|x|) \sum_{m=-l}^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big)\, \overline{Y_{l,m}(d_{l',m'})} \right| \\
&\overset{(A.8)}{\leq} 4\pi \sum_{l \geq q+1} \sqrt{\frac{\pi}{2\omega|x|}} \big|J_{l+\frac{1}{2}}(\omega|x|)\big| \sqrt{\sum_{m=-l}^{l} \left|Y_{l,m}\Big(\frac{x}{|x|}\Big)\right|^2} \sqrt{\sum_{m=-l}^{l} |Y_{l,m}(d_{l',m'})|^2} \\
&\overset{(A.4)}{\leq} 4\pi \sqrt{\frac{\pi}{2\omega|x|}} \sum_{l \geq q+1} \frac{(\omega|x|)^{l+\frac{1}{2}}}{\Gamma\left(l+\frac{3}{2}\right)\, 2^{l+\frac{1}{2}}}\, \frac{2l+1}{4\pi} \\
&\overset{j=l-q-1}{\leq} \frac{\sqrt{\pi}}{2} \Big(\frac{\omega x}{2}\Big)^{q+1} \sum_{j=0}^{\infty} \frac{\left(\frac{\omega|x|}{2}\right)^j\, 2\left(q+j+1+\frac{1}{2}\right)}{\Gamma(q+j+1+\frac{3}{2})} \\
&\leq \frac{\sqrt{\pi}}{\Gamma(q+\frac{3}{2})} \Big(\frac{\omega x}{2}\Big)^{q+1} \sum_{j=0}^{\infty} \frac{\left(\frac{\omega|x|}{2}\right)^j}{\Gamma(j+1)} \\
&\leq \frac{q!\, 2^q}{(2q+1)!} (\omega R)^{q+1}\, e^{\frac{\omega R}{2}},
\end{aligned} \tag{3.19}$$

where we have bounded the sum of the spherical harmonics with (2.4.105) of [37] and used

$$\frac{(q+j+\frac{3}{2})}{\Gamma(q+j+1+\frac{3}{2})} = \frac{1}{\Gamma(q+j+\frac{3}{2})} \leq \frac{1}{\Gamma(q+\frac{3}{2})\Gamma(j+1)} = \frac{q!\, 2^{2q+1}}{\sqrt{\pi}(2q+1)!\, \Gamma(j+1)}.$$

Now we plug (3.19) in (3.16) with the definition of β and the bound (3.18) on the coefficients $a_{l,m}$ with $K = l$:

$$\begin{aligned}
&\left\| V_1[P] - \sum_{\substack{l=0,\dots,q;\\ |m|\leq l}} \alpha_{l,m}\, e^{i\omega x\cdot d_{l,m}} \right\|_{L^\infty(B_R)} \\
&\leq \sup_{\substack{x\in B_R\\ l'=0,\dots,q,\\ m'=-l',\dots,l'}} \left| 4\pi \sum_{l\geq q+1} i^l\, j_l(\omega|x|) \sum_{m=-l}^{l} Y_{l,m}\Big(\frac{x}{|x|}\Big)\, \overline{Y_{l,m}(d_{l',m'})} \right| \cdot \|\alpha\|_1 \\
&\leq \frac{q!\, 2^q}{(2q+1)!} (\omega R)^{q+1}\, e^{\frac{\omega R}{2}} \left\|M^{-1}\right\|_1 \|\beta\|_1 \\
&\overset{(3.17)\,(3.18)}{\leq} \frac{q!\, 2^q}{(2q+1)!} (\omega R)^{q+1}\, e^{\frac{\omega R}{2}} \sum_{l=0}^{L} \sum_{m=-l}^{l} \frac{(2l+1)!}{4\pi l!} \Big(\frac{1}{2\omega}\Big)^l \frac{\sqrt{3}}{l!\, (\rho h)^{\frac{3}{2}}} |P|_{l,D} \left\|M^{-1}\right\|_1 \\
&\leq \frac{\sqrt{3}}{4\pi\rho^{\frac{3}{2}}} \frac{q!\, 2^q}{(2q+1)!} \frac{(2L+1)!}{2^L\, L!\, L!}\, e^{\frac{\omega R}{2}}\, (\omega R)^{q+1-L} \frac{R^L}{h^{\frac{3}{2}}} \sum_{l=0}^{L} (2l+1)\omega^{L-l} |P|_{l,D} \left\|M^{-1}\right\|_1 \\
&\leq \frac{\sqrt{3}}{4\pi\rho^{\frac{3}{2}}} \frac{q!\, 2^q}{(2q+1)!} \frac{(2L+1)!}{2^L\, L!\, L!} (2L+1)\sqrt{L+1}\, e^{\frac{\omega R}{2}} (\omega R)^{q+1-L} \frac{R^L}{h^{\frac{3}{2}}} \|P\|_{L,\omega,D} \left\|M^{-1}\right\|_1
\end{aligned}$$

$$\leq \frac{\sqrt{3}}{2\pi\rho^{\frac{3}{2}}} \frac{(L+1)^{\frac{3}{2}}\, 2^L}{q!\, 2^q} \frac{\frac{(2L+1)!}{4^L\, L!\, L!}}{\frac{(2q+1)!}{4^q\, q!\, q!}} e^{\frac{\omega R}{2}} (\omega R)^{q+1-L} \frac{R^L}{h^{\frac{3}{2}}} \left\|M^{-1}\right\|_1 \|P\|_{L,\omega,D}$$

$$\leq \frac{\sqrt{3}}{2\pi\rho^{\frac{3}{2}}} \frac{(L+1)^{\frac{3}{2}}}{2^{q-L}\, q!} e^{\frac{\omega R}{2}} (\omega R)^{q+1-L} \frac{R^L}{h^{\frac{3}{2}}} \left\|M^{-1}\right\|_1 \|P\|_{L,\omega,D}\,,$$

where we have used the monotonicity of the sequences $l \mapsto \frac{(2l+1)!}{2^l\, l!\, l!}$ and $l \mapsto \frac{(2l+1)!}{4^l\, l!\, l!}$. □

In the last chain of inequalities we have bounded each coefficient a_l of P using the seminorm of the same order $|P|_{l,D}$, that means we have used (3.18) with $K = l$. This makes the proof simpler but ties together the order of the norm on the right-hand side and the degree of the polynomial. In the case of the h-estimate this fact does not affect the final result; when a bound on $\left\|M^{-1}\right\|$ will be available, in order to obtain an useful p-estimate, this proof has to be modified following the line of Lemma 3.1.3.

3.2 Approximation of Homogeneous Helmholtz Solutions by Plane Waves

In order to use Lemma 3.1.3 and Lemma 3.1.6 to approximate a solution of the homogeneous Helmholtz equation in $PW_{\omega,p}(\mathbb{R}^N)$, we need to link the Sobolev norms to the L^∞–norm of the error. This is done in the following lemma, that generalizes the usual Cauchy estimates for harmonic functions to the Helmholtz case. The result is a simple consequence of the continuity of the Vekua transform.

Lemma 3.2.1. *Let ϕ be a harmonic function in $H^j(B_h)$, $j \in \mathbb{N}$, $\omega > 0$, $N \geq 2$ and D a domain as in Assumption 1.1.1. Then we have*

$$\|V_1[\phi]\|_{j,\omega,D} \leq C_{N,j}\, \rho^{\frac{1-N}{2}-j} \left(1+(\omega h)^{j+4}\right) e^{\frac{1}{2}\omega h}\, h^{\frac{N}{2}-j} \|V_1[\phi]\|_{L^\infty(B_h)}\,. \quad (3.20)$$

where the constant C depends only on N and j.

Proof. Remark 1.1.2 implies that $d(D, \partial B_h) \geq \rho h$. Using the Cauchy estimates for harmonic functions and the continuity of the Vekua operators we have

$$\begin{aligned}
\|V_1[\phi]\|_{j,\omega,D} &\overset{(1.12)}{\leq} C_N\, \rho^{\frac{1-N}{2}} (1+j)^{\frac{3}{2}N+\frac{1}{2}} e^j \left(1+(\omega h)^2\right) \|\phi\|_{j,\omega,D} \\
&\leq C_{N,j}\, \rho^{\frac{1-N}{2}} \left(1+(\omega h)^2\right) \sum_{l=0}^{j} \omega^{j-l} |\phi|_{l,D} \\
&\leq C_{N,j}\, \rho^{\frac{1-N}{2}} \left(1+(\omega h)^2\right) \sum_{l=0}^{j} \omega^{j-l}\, h^{\frac{N}{2}} |\phi|_{W^{l,\infty}(D)} \\
&\overset{(1.32)}{\leq} C_{N,j}\, \rho^{\frac{1-N}{2}} \left(1+(\omega h)^2\right) \sum_{l=0}^{j} \omega^{j-l}\, h^{\frac{N}{2}} (\rho h)^{-l} \|\phi\|_{L^\infty(B_h)} \\
&\leq C_{N,j}\, \rho^{\frac{1-N}{2}-j} \left(1+(\omega h)^{j+2}\right) h^{\frac{N}{2}-j} \|\phi\|_{L^\infty(B_h)}
\end{aligned}$$

$$\overset{(1.16)\text{ on } B_h}{\leq} C_{N,j}\,\rho^{\frac{1-N}{2}-j}\,(1+(\omega h)^{j+4})\,e^{\frac{1}{2}\omega h}\,h^{\frac{N}{2}-j}\,\|V_1[\phi]\|_{L^\infty(B_h)}\,,$$

where in the last step, the exponential has coefficient 1/2 because ball B_h has diameter $2h$ and shape parameter $\rho(B_h) = 1/2$. □

Now we can state the two final results: the approximation estimates for homogeneous Helmholtz solutions in $\mathcal{H}^j_\omega(D)$ with plane waves in $PW_{\omega,p}(D)$, with respect to h for $N = 2, 3$ and with respect to both h and p for $N = 2$ only.

Theorem 3.2.2 (h-estimates, $N = 2, 3$)**.** *Let $u \in H^{K+1}(D)$ be a solution of the homogeneous Helmholtz equation in a domain $D \subset \mathbb{R}^N$, $N = 2, 3$, satisfying Assumption 1.1.1. Fix $q \geq 1$, set*

$$p = \begin{cases} 2q+1 & \text{if } N = 2, \\ (q+1)^2 & \text{if } N = 3, \end{cases}$$

and let the directions $\{d_k\}_{k=1,\dots,p} \subset S^{N-1}$ be such that the matrix

$$\{Y_{l,m}(d_k)\}_{\substack{l\geq 0,\, m=1,\dots,n(N,l)\\ k=1,\dots,p}}$$

is invertible. Then for every $1 \leq L \leq \min(q, K)$, there exists $\alpha \in \mathbb{C}^p$ such that, for every $j \leq L$,

$$\left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{j,\omega,D} \leq C\left(1+(\omega h)^{j+q+8}\right) e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega h}\, h^{L+1-j}\, \|u\|_{L+1,\omega,D}\,, \tag{3.21}$$

where the constant $C > 0$ depends only on q, j, L, N, ρ, ρ_0 and the directions $\{d_k\}$, but is independent of ω, h and u.

Proof. Let Q be the generalized harmonic polynomial of degree at most L equal to Q_L from Theorem 2.2.1, item *(ii)*. $V_2(Q)$ is the averaged Taylor polynomial of $V_2[u]$, as in Theorem 2.1.2. From Lemmas 3.1.3 and 3.1.6 with $R = h$, we know that there exists vector $\alpha \in \mathbb{C}^p$ such that

$$\begin{aligned}
&\left\| Q - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_h)} \\
&\overset{(3.9),\,(3.15)}{\leq} C_{(N,\rho,L,q,\{d_k\})}\, e^{\frac{\omega h}{2}} \left((\omega h)^{q-L} + (\omega h)^q\right) h^{L+1-\frac{N}{2}}\omega \,\|V_2[Q]\|_{L,\omega,D} \\
&\leq C_{(N,\rho,L,q,\{d_k\})}\, e^{\frac{\omega h}{2}} \left((\omega h)^{q-L} + (\omega h)^q\right) h^{L+1-\frac{N}{2}} \,\|V_2[Q]\|_{L+1,\omega,D} \\
&\leq C_{(N,\rho,L,q,\{d_k\})}\, e^{\frac{\omega h}{2}} \left(1 + (\omega h)^q\right) h^{L+1-\frac{N}{2}} \\
&\qquad\qquad \cdot \left[\|V_2[u]\|_{L+1,\omega,D} + \|V_2[u]-V_2[Q]\|_{L+1,\omega,D}\right] \\
&\overset{(2.5)}{\leq} C_{(N,\rho,L,q,\{d_k\},\rho_0)}\, e^{\frac{\omega h}{2}} \left(1 + (\omega h)^q\right) h^{L+1-\frac{N}{2}} \,\|V_2[u]\|_{L+1,\omega,D} \\
&\overset{(1.15)}{\leq} C_{(N,\rho,L,q,\{d_k\},\rho_0)}\, e^{\left(\frac{1}{2}+\frac{3}{4}(1-\rho)\right)\omega h} \left(1 + (\omega h)^{q+4}\right) h^{L+1-\frac{N}{2}} \,\|u\|_{L+1,\omega,D}\,.
\end{aligned}$$

Now we use the triangle inequality and the approximation results from Theorem 2.2.1 and from Lemma 3.2.1 with $\phi = V_2[Q - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k}]$:

$$\begin{aligned}
\left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{j,\omega,D} &\leq \|u - Q\|_{j,\omega,D} + \left\| Q - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{j,\omega,D} \\
&\overset{(2.19),\,(3.20)}{\leq} C\,\left(1 + (\omega h)^{j+6}\right)\, e^{\frac{3}{4}(1-\rho)\omega h}\; h^{L+1-j}\;\; \|u\|_{L+1,\omega,D} \\
&\qquad + C\,\left(1 + (\omega h)^{j+4}\right)\, e^{\frac{1}{2}\omega h}\; h^{\frac{N}{2}-j} \left\| Q - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{L^\infty(B_h)} \\
&\leq C\,\left(1 + (\omega h)^{j+q+8}\right)\, e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega h}\; h^{L+1-j}\;\; \|u\|_{L+1,\omega,D}\,.
\end{aligned}$$

□

Theorem 3.2.3 (hp-estimates, $N = 2$)**.** *Let $u \in H^{K+1}(D)$ be a solution of the homogeneous Helmholtz equation in a domain $D \subset \mathbb{R}^2$ satisfying Assumption 1.1.1 and the exterior cone condition with angle $\lambda\pi$ (see Definition 2.1.3). Fix $q \geq 1$, set $p = 2q + 1$ and let the directions $\{d_k = (\cos\theta_k, \sin\theta_k)\}_{k=-q,\dots,q}$ satisfy the condition* (3.7)*.*

Then for every L satisfying

$$0 \leq K \leq L \leq q, \qquad\qquad L - K + 1 \leq \left\lfloor \frac{q+1}{2} \right\rfloor,$$

there exists $\alpha \in \mathbb{C}^p$ such that, for every $0 \leq j \leq K$,

$$\begin{aligned}
\left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{j,\omega,D} &\leq C\, e^{(\frac{7}{4}-\frac{3}{4}\rho)\omega h}\, \left(1 + (\omega h)^{j+8}\right)\; h^{K+1-j} \\
\cdot \left\{ \left(\frac{\log(L+2)}{L+2} \right)^{\lambda(K+1-j)} + 2^L \sqrt{\frac{L+1}{q+1}} \left(\frac{e^{\frac{5}{2}}}{2\sqrt{2}\delta^2\rho^{\frac{1}{2}}}\; \frac{(1+\omega h)}{\sqrt{q+1}} \right)^q \right\} & \|u\|_{K+1,\omega,D}\,,
\end{aligned} \tag{3.22}$$

where the constant $C > 0$ depends only on j, K and the shape of D, but is independent of q, L, δ, ω, h and u.

Proof. Let Q be the generalized harmonic polynomial of degree at most L equal to Q'_L from Theorem 2.2.1, item *(iii)*.

Since $V_2[Q]$ approximates $V_2[u]$ as in Theorem 2.1.4, we notice that, for $K \geq 1$,

$$\begin{aligned}
\|V_2[Q]\|_{K,\omega,D} &\leq \|V_2[u]\|_{K,\omega,D} + \|V_2[u] - V_2[Q]\|_{K,\omega,D} \\
&\overset{(2.8),\ \text{j=k+1=K}}{\leq} (1 + C)\; \|V_2[u]\|_{K,\omega,D} \\
&\overset{(1.15)}{\leq} C\,\left(1 + (\omega h)^4\right)\, e^{\frac{3}{4}(1-\rho)\omega h}\, \|u\|_{K,\omega,D}\,,
\end{aligned} \tag{3.23}$$

where C depends only on K and the shape of D.

Finally, we combine all the ingredients, in the case $K \geq 1$:

$$\left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{j,\omega,D} \leq \|u - Q\|_{j,\omega,D} + \left\| Q - \sum_{k=1}^{p} \alpha_k e^{i\omega x\cdot d_k} \right\|_{j,\omega,D}$$

$$\overset{\substack{(2.20),\\(3.20)}}{\leq} C_{(K,j,\hat{D})} \, (1+(\omega h)^{j+6}) \, e^{\frac{3}{4}(1-\rho)\omega h} \, \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(K+1-j)}$$
$$\cdot \, h^{K+1-j} \, \|u\|_{K+1,\omega,D}$$
$$+ C_{(j,\rho)} \, (1+(\omega h)^{j+4}) \, e^{\frac{1}{2}\omega h} \, h^{1-j} \left\| Q - \sum_{k=1}^{p} \alpha_k e^{i\omega x \cdot d_k} \right\|_{L^\infty(B_h)}$$
$$\overset{\substack{(3.9),\\R=h}}{\leq} C_{(K,j,\hat{D})} \, (1+(\omega h)^{j+6}) \, e^{\frac{3}{4}(1-\rho)\omega h} \, \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(K+1-j)}$$
$$\cdot \, h^{K+1-j} \, \|u\|_{K+1,\omega,D}$$
$$+ C_{(j,\rho)} \left(\frac{e^{\frac{5}{2}}}{2\sqrt{2}\delta^2 \, \rho^{\frac{1}{2}}}\right)^q (2^L \, \sqrt{L+1})(1+(\omega h)^{q+j+4}) \, e^{\omega h} \, h^{K+1-j}$$
$$\cdot \, \frac{1}{(q+1)^{\frac{q+1}{2}}} \, \omega \, \|V_2[Q]\|_{K,\omega,D}$$
$$\overset{(3.23)}{\leq} C_{(K,\hat{D},j)} \, e^{\left(1+\frac{3}{4}(1-\rho)\right)\omega h} \, (1+(\omega h)^{j+8}) \, h^{K+1-j}$$
$$\cdot \left\{ \left(\frac{\log(L+2)}{L+2}\right)^{\lambda(K+1-j)} + 2^L \sqrt{\frac{L+1}{q+1}} \left(\frac{e^{\frac{5}{2}}}{2\sqrt{2}\delta^2 \, \rho^{\frac{1}{2}}} \, \frac{(1+\omega h)}{\sqrt{q+1}}\right)^q \right\} \|u\|_{K+1,\omega,D} \, .$$

If $K = j = 0$, in (3.23) we have to use (1.14) instead of (1.15), so that (3.23) becomes

$$\|V_2[Q]\|_{0,D} \leq C(1+(\omega h)^4) e^{\frac{1}{2}(1-\rho)\omega h} (\|u\|_{0,D} + h \, |u|_{1,D}).$$

The rest of the proof continues as in the case $K \geq 1$ until the last but one step. For the last step, since

$$\begin{aligned}
\omega \, \|V_2[Q]\|_{0,D} &\leq C(1+(\omega h)^4) e^{\frac{1}{2}(1-\rho)\omega h} \omega (\|u\|_{0,D} + h \, |u|_{1,D}) \\
&\leq C(1+(\omega h)^4) e^{\frac{1}{2}(1-\rho)\omega h} (1+\omega h) \, \|u\|_{1,\omega,D} \\
&\leq C(1+(\omega h)^4) e^{\frac{3}{4}(1-\rho)\omega h} \, \|u\|_{1,\omega,D} \, ,
\end{aligned}$$

we get exactly the same conclusion as in the case $K \geq 1$. □

Remark 3.2.4. *In the two-dimensional case, if*

$$K \leq q + 1 - \left\lfloor \frac{q+1}{2} \right\rfloor = \left\lceil \frac{q+1}{2} \right\rceil$$

we can choose

$$L = K - 1 + \left\lfloor \frac{q+1}{2} \right\rfloor$$

and we get the simpler estimate

$$\left\| u - \sum_{k=1}^{p} \alpha_k e^{i\omega x \cdot d_k} \right\|_{j,\omega,D} \leq C_{(K,\hat{D},j)} \, e^{\left(\frac{7}{4} - \frac{3}{4}\rho\right)\omega h} \, \left(1+(\omega h)^{j+8}\right) \, h^{K+1-j}$$

$$\cdot \left\{ \left(\frac{\log(K+1+\lfloor \frac{q+1}{2} \rfloor)}{K+1+\lfloor \frac{q+1}{2} \rfloor} \right)^{\lambda(K+1-j)} + \left(\frac{e^{\frac{5}{2}}}{2\delta^2 \, \rho^{\frac{1}{2}}} \; \frac{(1+\omega h)}{\sqrt{q+1}} \right)^{q} \right\} \|u\|_{K+1,\omega,D} \, .$$

The second term within the curly brackets converges to zero in q faster than exponentially, the first one only algebraically.

Appendix A

Special Functions

A.1 Bessel Functions

We denote the usual Bessel functions of the first kind by $J_\nu(z)$ and the spherical Bessel functions of the first kind by $j_\nu(z)$. The first ones are defined, for every $\nu, z \in \mathbb{C}$, as

$$J_\nu(z) = \sum_{t=0}^{\infty} \frac{(-1)^t}{t!\,\Gamma(t+\nu+1)} \left(\frac{z}{2}\right)^{2t+\nu}, \tag{A.1}$$

where Γ is the gamma function. When $\nu \notin \mathbb{Z}$ and z belongs to the segment $[-\infty, 0]$, $J_\nu(z)$ is not single-valued. When $\nu \in \mathbb{Z}$, J_ν is an entire function.

We list some properties of these functions (references can be found in [31,44]):

$$J_{-k}(z) = (-1)^k J_k(z) \qquad \forall\, k \in \mathbb{Z}, \tag{A.2}$$

$$\operatorname{Im}\big(J_k(t)\big) = 0, \qquad \operatorname{Re}\big(J_k(it)\big) = 0 \qquad \forall\, k \in \mathbb{Z},\ t \in \mathbb{R},$$

$$|J_k(t)| \le 1 \qquad \forall\, k \in \mathbb{Z},\ t \in \mathbb{R}, \tag{A.3}$$

$$|J_\nu(z)| \le \frac{e^{|\operatorname{Im} z|}}{\Gamma(\nu+1)} \left(\frac{|z|}{2}\right)^\nu \qquad \forall\, \nu > -\frac{1}{2},\ z \in \mathbb{C}, \tag{A.4}$$

$$J_0(0) = 1, \qquad J_k(0) = 0 \qquad \forall\, k \in \mathbb{Z} \setminus \{0\},$$

$$\frac{\partial}{\partial z} J_\nu(z) = \frac{1}{2}\left(J_{\nu-1}(z) - J_{\nu+1}(z)\right), \tag{A.5}$$

$$\frac{\partial}{\partial z}\big(z^k J_k(z)\big) = z^k J_{k-1}(z),$$

$$\frac{\partial}{\partial z} J_0(z) = -J_1(z), \qquad \frac{\partial}{\partial z}\left(z J_1(z)\right) = z J_0(z), \tag{A.6}$$

$$\frac{\partial^l}{\partial z^l} J_k(z) = \frac{1}{2^l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} J_{2m-l+k}(z). \tag{A.7}$$

The last equality can be easily proved by induction from (A.5).

The spherical Bessel functions are defined as

$$j_\nu(z) = \sqrt{\frac{\pi}{2z}}\, J_{\nu+\frac{1}{2}}(z). \tag{A.8}$$

These functions are a particular case of the so-called hyperspherical Bessel functions (see [3] p. 52):

$$j_k^N(z) = \sum_{t=0}^{\infty} \frac{(-1)^t \; z^{2t+k}}{(2t)!! \; (N+2t+2k-2)!!} = \frac{\Gamma\left(\frac{N}{2}-1\right) \, 2^{\frac{N}{2}-2}}{(N-4)!!} \; \frac{J_{k+\frac{N}{2}-1}(z)}{z^{\frac{N}{2}-1}}, \quad \text{(A.9)}$$

$$J_k(z) = j_k^2(z), \qquad\qquad j_k(z) = j_k^3(z).$$

The asymptotic forms of the Bessel functions for small arguments are:

$$J_k(z) \approx \frac{1}{k!}\left(\frac{z}{2}\right)^k, \qquad j_k(z) \approx \frac{2^k \; k!}{(2k+1)!}\, z^k \qquad |z| << 1, \; k \in \mathbb{Z}. \qquad \text{(A.10)}$$

A.2 Legendre Functions and Spherical Harmonics

For every natural l the Legendre polynomial of degree l is defined as

$$P_l(t) = \frac{1}{2^l \; l!} \, \frac{\partial^l}{\partial t^l}\left[(t^2-1)^l\right]. \qquad \text{(A.11)}$$

The associate Legendre function are

$$\begin{aligned} P_l^m(t) &= (-1)^m \; (1-t^2)^{\frac{m}{2}} \; \frac{\partial^m}{\partial t^m} P_l(t) \quad = \frac{(-1)^m}{2^l \; l!} \; (1-t^2)^{\frac{m}{2}} \; \frac{\partial^{l+m}}{\partial t^{l+m}}\left[(t^2-1)^l\right], \\ P_l^{-m}(t) &= (-1)^m \; \frac{(l-m)!}{(l+m)!} \; P_l^m(t) \qquad\qquad 0 \le m \le l. \end{aligned} \qquad \text{(A.12)}$$

From these follows

$$P_l^l(t) = (-1)^l \, \frac{(2l)!}{2^l \; l!} \, (1-t^2)^{\frac{l}{2}}, \qquad P_l^{-l}(t) = \frac{1}{2^l \; l!} \; (1-t^2)^{\frac{l}{2}} \qquad \forall \, l \in \mathbb{N}. \quad \text{(A.13)}$$

For every $N \in \mathbb{N}$, $N \ge 2$, the N-dimensional spherical harmonics are a set of complex functions $\{Y_{l,m}\}_{l \ge 0, m=1,\dots,n(N,l)}$ defined on S^{N-1} that constitutes an orthonormal basis of $L^2(S^{N-1})$. The set $\{|x|^l Y_{l,m}(\frac{x}{|x|})\}_{m=1,\dots,n(N,l)}$ is a basis of the space of the homogeneous N–dimensional harmonic polynomials of degree l.

If $N = 2$, the number $n(2,l)$ of linearly independent spherical harmonics of degree l is 1, if $l = 0$, and 2, if $l \ge 1$; we will use only one index l running over $\mathbb{Z}$ and define

$$Y_l(e^{i\theta}) = e^{il\theta} \qquad\qquad \forall l \in \mathbb{Z}.$$

If $N = 3$, the number of linearly independent spherical harmonics of degree l is $n(3,l) = 2l+1$, so we use $m \in \{-l, \dots, l\}$:

$$Y_{l,m}(d) = \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \; P_l^m(\cos\theta) \; e^{im\varphi} \qquad \text{(A.14)}$$

$$l \in \mathbb{N}, \quad m = -l, \dots, l, \quad d = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in S^2,$$

where P_l^m is an associated Legendre function.

Other useful identities are the Jacobi-Anger expansions, combined with the addition theorem for spherical harmonics, see [3, 17, 36]:

$$e^{iz\cos\theta} = \sum_{l\in\mathbb{Z}} i^l J_l(z)\, e^{il\theta}, \tag{A.15}$$

$$\begin{aligned} e^{ir\xi\cdot\eta} &= \sum_{l\geq 0}(2l+1)\, i^l\, j_l(r)\, P_l(\xi\cdot\eta) \qquad\qquad (A.16)\\ &= 4\pi \sum_{l\geq 0} i^l\, j_l(r) \sum_{m=-l}^{l} Y_{l,m}(\xi)\overline{Y_{l,m}(\eta)} \qquad \forall\, \xi,\, \eta\in S^2,\ r\geq 0, \end{aligned}$$

$$e^{ir\xi\cdot\eta} = (N-2)!!\,|S^{N-1}| \sum_{l\geq 0} i^l\, j_l^N(r) \sum_{m=1}^{n(N,l)} Y_{l,m}(\xi)\overline{Y_{l,m}(\eta)} \tag{A.17}$$

$$\forall\, \xi,\, \eta\in S^{N-1},\ r\geq 0,\ N\geq 3.$$

Bibliography

[1] R. Adams, *Sobolev spaces*, Pure and Applied Mathematics, Academic Press, 1975.

[2] V. Andrievskii, *Uniform harmonic approximation on compact sets in* $\mathbf{R}^k$, $k \geq 3$, SIAM J. Math. Anal., 24 (1993), pp. 216–222.

[3] J. Avery, *Hyperspherical Harmonics: applications in quantum theory*, Reidel texts in the mathematical sciences, Kluwer Academic Publishers, 1989.

[4] S. Axler, P. Bourdon, and W. Ramey, *Harmonic function theory*, Graduate Texts in Mathematics, Springer-Verlag, New York, 2001.

[5] I. Babuška and J. Melenk, *The partition of unity method*, International Journal for Numerical Methods in Engineering, 40 (1997), pp. 727–758.

[6] T. Bagby, L. Bos, and N. Levenberg, *Quantitative approximation theorems for elliptic operators*, J. Approx. Theory, 85 (1996), pp. 69–87.

[7] T. Bagby and N. Levenberg, *Bernstein theorems for harmonic functions*, in Methods of approximation theory in complex analysis and mathematical physics (Leningrad, 1991), vol. 1550 of Lecture Notes in Math., Springer, Berlin, 1993, pp. 7–18.

[8] A. H. Barnett and T. Betcke, *Stability and convergence of the method of fundamental solutions for Helmholtz problems on analytic domains*, J. Comput. Phys., 227 (2008), pp. 7003–7026.

[9] S. Bergmann, *Über die Entwicklung der harmonischen Funktionen der Ebene und des Raumes nach Orthogonalfunktionen*, Math. Ann., 86 (1922), pp. 238–271.

[10] T. Betcke, *Numerical computation of eigenfunctions of planar regions*, PhD thesis, University of Oxford, 2005.

[11] S. Brenner and R. Scott, *Mathematical theory of finite element methods*, Texts in Applied Mathematics, Springer–Verlag, New York, 3rd ed., 2007.

[12] A. Buffa and P. Monk, *Error estimates for the ultra weak variational formulation of the Helmholtz equation*, Mathematical Modelling and Numerical Analysis, 42 (2008), pp. 925–940.

[13] O. Cessenat and B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM Journal of Numerical Analysis, 35 (1998), pp. 255–299.

[14] D. Colton, *Bergman operators for elliptic equations in three independent variables*, Bull. Amer. Math. Soc., 77 (1971), pp. 752–756.

[15] ——, *Integral operators for elliptic equations in three independent variables. I*, Applicable Anal., 4 (1974/75), pp. 77–95.

[16] ——, *Integral operators for elliptic equations in three independent variables. II*, Applicable Anal., 4 (1974/75), pp. 283–295.

[17] D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, vol. 93 of Applied Mathematical Sciences, Springer, Heidelberg, 2nd ed., 1998.

[18] R. Courant and D. Hilbert, *Methods of mathematical physics. Volume II: partial differential equations*, Interscience Publishers, 1962.

[19] S. C. Eisenstat, *On the rate of convergence of the Bergman-Vekua method for the numerical solution of elliptic boundary value problems*, SIAM J. Numer. Anal., 11 (1974), pp. 654–680.

[20] L. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics, American Mathematical Society, Providence, 3rd ed., 2002.

[21] C. Farhat, I.Harari, and L. Franca, *The discontinuous enrichment method*, Computer Methods in Applied Mechanics and Engineering, 190 (2001), pp. 6455–6479.

[22] W. Gautschi, *On inverses of Vandermonde and confluents Vandermonde determinants*, Numerische Mathematik, 4 (1962), pp. 117–123.

[23] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics, Springer-Verlag, 2nd ed., 1983.

[24] R. P. Gilbert and C. Y. Lo, *On the approximation of solutions of elliptic partial differential equations in two and three dimensions*, SIAM J. Math. Anal., 2 (1971), pp. 17–30.

[25] C. Gittelson, *Plane wave discontinuous Galerkin methods*, master's thesis, SAM, ETH Zürich, Zürich, Switzerland, 2008.

[26] C. Gittelson, R. Hiptmair, and I. Perugia, *Plane wave discontinuous Galerkin methods: analysis of the h-version*, Mathematical Modelling and Numerical Analysis, 43 (2009), pp. 297–332.

[27] P. Henrici, *A survey of I. N. Vekua's theory of elliptic partial differential equations with analytic coefficients*, Zeitschrift für angewandte Mathematik und Physik, 8 (1957), pp. 169–202.

[28] R. Hiptmair, A. Moiola, and I. Perugia, *Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p-version*, Preprint 2009-20, SAM Report, ETH Zürich, Switzerland, 2009.

[29] N. T. HOP, *Some extensions of I. N. Vekua's method in higher dimensions. I. Integral representation of solutions of an elliptic system. Properties of solutions*, Math. Nachr., 130 (1987), pp. 17–34.

[30] ——, *Some extensions of I. N. Vekua's method in higher dimensions. II. Boundary value problems*, Math. Nachr., 130 (1987), pp. 35–46.

[31] N. LEBEDEV, *Special functions and their applications*, Prentice–Hall, Englewood Cliffs, N.J., 1965.

[32] J. MELENK, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, USA, 1995.

[33] ——, *Operator adapted spectral element methods I: harmonic and generalized harmonic polynomials*, Numerische Mathematik, 84 (1999), pp. 35–69.

[34] ——, *Operator adapted spectral element methods I: harmonic and generalized harmonic polynomials*, Tech. Rep. 97-07, SAM Report, ETH Zürich, Switzerland, 1999.

[35] A. MOIOLA, *Approximation properties of plane wave spaces and application to the analysis of the plane wave discontinuous Galerkin method*, Tech. Rep. 2009-06, SAM Report, ETH Zürich, Switzerland, 2009.

[36] C. MÜLLER, *Spherical harmonics*, Lecture notes in Mathematics, Springer-Verlag, 1966.

[37] J.-C. NÉDÉLEC, *Acoustic and Electromagnetic Equations: Integral Representations for Harmonic Problems*, vol. 44 of Applied Mathematical Sciences, Springer-Verlag, Berlin, 2001.

[38] M. NICOLESCO, *Approssimazione delle funzioni armoniche in più variabili mediante polinomi armonici*, Univ. Roma. Ist. Nat. Alta Mat. Rend. Mat. e Appl. (5), 6 (1947), pp. 410–423.

[39] I. VEKUA, *Solutions of the equation* $\Delta u+\lambda^2 u = 0$, Soobshcheniya Akademii Nauk Gruz. SSR, 3 (1942), pp. 307–314.

[40] ——, *Inversion of an integral transformation and some applications*, Soobshcheniya Akademii Nauk Gruz. SSR, 6 (1945), pp. 177–183.

[41] ——, *New methods for solving elliptic equations*, North Holland, 1967.

[42] J. L. WALSH, *The approximation of harmonic functions by harmonic polynomials and by harmonic rational functions*, Bull. Amer. Math. Soc., 35 (1929), pp. 499–544.

[43] ——, *Interpolation and approximation by rational functions in the complex domain*, Fifth edition. American Mathematical Society Colloquium Publications, Vol. XX, American Mathematical Society, Providence, R.I., 1969.

[44] G. WATSON, *Theory of Bessel functions*, Cambridge University Press, 1966.

[45] N. WECK, *Approximation by Herglotz wave functions*, Math. Methods Appl. Sci., 27 (2004), pp. 155–162.

Research Reports

No.	Authors/Title
09-27	*A. Moiola, R. Hiptmair, I. Perugia* Approximation by plane waves
09-26	*M. Karow, E. Kokiopoulou, D. Kressner* On the computation of structured singular values and pseudospectra
09-25	*M. Durán, M. Guarini, C.F. Jerez-Hanckes* Hybrid FEM/BEM modeling of finite-sized photonic crystals for semiconductor laser beams
09-24	*A. Bespalov, N. Heuer, R. Hiptmair* Convergence of the natural hp-BEM for the electric field integral equation on polyhedral surfaces
09-23	*R. Hiptmair, J. Li, J. Zou* Real interpolation of spaces of differential forms
09-22	*R. Hiptmair, J. Li, J. Zou* Universal extension for Sobolev spaces of differential forms and applications
09-21	*T. Betcke, D. Kressner* Perturbation, computation and refinement of invariant pairs for matrix polynomials
09-20	*R. Hiptmair, A. Moiola and I. Perugia* Plane wave discontinuous Galerkin methods for the 2D Helmholtz equation: analysis of the p–version
09-19	*C. Winter* Wavelet Galerkin schemes for multidimensional anisotropic integrodifferential operators
09-18	*C.J. Gittelson* Stochastic Galerkin discretization of the lognormal isotropic diffusion problem
09-17	*A. Bendali, A. Tizaoui, S. Tordeux, J. P. Vila* Matching of Asymptotic Expansions for a 2-D eigenvalue problem with two cavities linked by a narrow hole
09-16	*D. Kressner, C. Tobler* Krylov subspace methods for linear systems with tensor product structure
09-15	*R. Granat, B. Kågström, D. Kressner* A novel parallel QR algorithm for hybrid distributed memory HPC systems