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R. Hiptmair, J. Li* and J. Zou†

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Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

*Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong; jzli@math.cuhk.edu.hk.

†Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong, zou@math.cuhk.edu.hk. The work of this author was substantially supported by Hong Kong RGC grants (Projects 404606 and and 404407).

Universal Extension for Sobolev Spaces of Differential Forms and Applications

Ralf Hiptmair* Jingzhi Li† Jun Zou‡

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Abstract

This article is devoted to the construction of a family of universal extension operators for the Sobolev spaces $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ of differential forms of degree l $(0 \leq l \leq d)$ in a Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d \in \mathbb{N},\ d \geq 2)$ for any $k \in \mathbb{N}_0$. It generalizes the construction of the first universal extension operator for standard Sobolev spaces $H^k(\Omega)$, $k \in \mathbb{N}_0$, on Lipschitz domains, introduced by Stein [Theorem 5, pp. 181, E. M. STEIN, *Singular integrals and differentiability properties of functions*, Princeton University Press, N. J., 1970]. This corresponds to the case $l = 0$ of our theory. We adapt Stein's idea in the form of integral averaging over the pullback of a parametrized reflection mapping.

The new theory covers extension operators for $\boldsymbol{H}^k(\mathbf{curl}; \Omega)$ and $\boldsymbol{H}^k(\mathrm{div}; \Omega)$ in $\mathbb{R}^3$ as special cases for $l = 1, 2$, respectively. Of considerable mathematical interest in its own right, the new theoretical results have many important applications: we elaborate existence proofs for generalized regular decompositions

Key words. Universal (Stein) extension, Sobolev spaces of differential forms, Lipschitz domains, integral averaging, parametrized reflection mapping, generalized regular decomposition.

AMS subject classification 2000. 46B70, 47A57, 54D35

1 Introduction

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ $(d \in \mathbb{N}_0, d \geq 2)$, Stein [34, Theorem 5, pp.181] constructed a celebrated *extension mapping*

$$\mathscr{E} : C^\infty(\overline{\Omega}) \mapsto C^\infty(\mathbb{R}^d) \ , \quad \mathscr{E}u(\mathbf{x}) = u(\mathbf{x}) \quad \forall \mathbf{x} \in \overline{\Omega} \ ,$$

which fulfills that for any $m \in \mathbb{N}_0$, $1 \leq p < \infty$,

$$\exists\, C = C(m, p, \Omega) > 0 : \quad \|\mathscr{E}u\|_{W^{m,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{m,p}(\Omega)} \quad \forall\, u \in C^\infty(\overline{\Omega}) \ . \tag{1.1}$$

Thus, it can be naturally extended to a continuous extension operator for *any* classical Sobolev space $W^{m,p}(\Omega)$, $m \in \mathbb{N}_0$, $1 \leq p < \infty$. Thanks to its "one formula fits all (Sobolev spaces)" property, the operator $\mathscr{E}$ is called a *universal* (or degree-independent [31]) extension operator. This makes it exceptional, because other designs of extension operators for Sobolev spaces by, for instance, the successive reflection method [19, 33, 37], or the singular integral method [9], rely on different

*SAM, ETH, Zürich, CH-8092 Zürich, Switzerland (hiptmair@sam.math.ethz.ch).

†Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong (jzli@math.cuhk.edu.hk).

‡Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong. The work of this author was substantially supported by Hong Kong RGC grants (Projects 404606 and and 404407). (zou@math.cuhk.edu.hk)

formulas for different orders m and may hinge on smoothness of the boundary. It goes without saying that universality renders $\mathscr{E}$ a valuable tool in the theory of Sobolev spaces and their applications.

Beyond the classical articles, a few modern publications are devoted to extension operators for various function spaces on Lipschitz domains and beyond, e.g, [18,22,31,36]. Extension results can find wide applications to such as interpolation spaces and regularity estimates in PDEs, see, e.g., [23,27].

The main purpose of this paper is to construct a new family of universal extension operators for Sobolev spaces $\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)$ of differential forms, for $l,k \in \mathbb{N}_0$, and $0 \le l \le d$ in Lipschitz domains $\Omega \subset \mathbb{R}^d$, see Section 2 for the precise definition. To keep our presentation succinct, we study only Hilbert spaces, that is, the case $p=2$. We would like to point out that Sobolev spaces of differential forms are fundamental to the theoretical analysis of, e.g., electromagnetic phenomena governed by Maxwell's equation [3,20,26,28], the Navier-Stokes equation [17], and interpolation theory [23].

In this paper, we will prepare some necessary notations and materials in Section 2. In Section 3, we briefly recall Stein's approach, i.e., an integral averaging method based on local parametrized reflection mappings and then present our construction. Guided by the commuting relationship of the pullback and the exterior derivative of differential forms, the gist of our construction is to apply Stein's integral averaging to the pullback operators induced by the reflection mappings. This offers a natural generalization of Stein's formula to differential forms, see Formula (3.9). With some technical effort, Stein's original analysis can be adapted, which is also done in Section 3 of this article, see Lemma 3.4 and Theorems 3.5, 3.6. From the perspective of vector fields, we demonstrate the explicit construction of those extension operators in terms of Euclidean vector proxies in $\mathbb{R}^3$ in Section 4. We point out that universal extension operators for the Sobolev spaces $\boldsymbol{H}^k(\mathbf{curl};\Omega)$ and $\boldsymbol{H}^k(\mathrm{div};\Omega)$ of vector fields in $\mathbb{R}^3$ are covered by our universal extension theorem as special cases for $l=1,2$, respectively. These new theoretical results are not only of mathematical interest in their own right, but also have important applications. We elaborate existence proofs for generalized regular decompositions in Section 5.

2 Notation and preliminaries

Throughout the paper, $\mathbb{R}^d$ stands for the classical Euclidean space ($d \in \mathbb{N}$, $d \ge 2$), equipped with the canonical orthonormal bases e_j's, $1 \le j \le d$, and norm $|\mathbf{x}| := \sqrt{x_1^2+\cdots+x_d^2}$, if $\mathbf{x}=(x_1,\ldots,x_d)^T \in \mathbb{R}^d$. The canonical orthonormal basis of $\mathbb{R}^d$ corresponds to a dual basis of $(\mathbb{R}^d)^*$, i.e., $\boldsymbol{d}x_1, \boldsymbol{d}x_2, \ldots, \boldsymbol{d}x_d$ with $\boldsymbol{d}x_i(e_j)=1$ if $i=j$ and zero otherwise.

Recall that a function $f : D \mapsto \mathbb{R}$, $D \subset \mathbb{R}^{d-1}$ is called Lipschitz if there exists a finite constant $C>0$ such that

$$|f(\mathbf{x})-f(\mathbf{y})| \le C|\mathbf{x}-\mathbf{y}| \quad \forall \mathbf{x},\mathbf{y} \in D.$$

A Lipschitz epigraph $\Omega \subset \mathbb{R}^d$ is defined as a domain lying above the graph of a Lipschitz function $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, i.e., $\Omega = \{\, (\widehat{\mathbf{x}}, x_d) \mid \phi(\widehat{\mathbf{x}}) < x_d \,\}$ with $\widehat{\mathbf{x}}=(x_1,\ldots,x_{d-1})$. See [34] and Figure 1 for illustration.

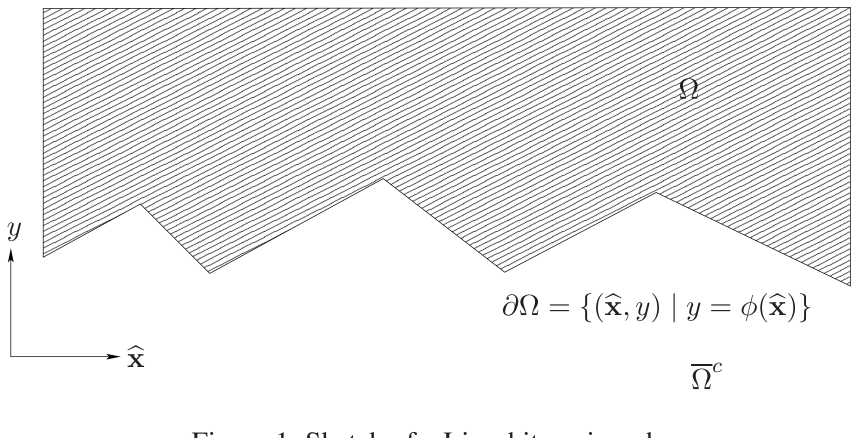

Figure 1: Sketch of a Lipschitz epigraph.

A bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ is a bounded domain whose boundary $\partial\Omega$ can be covered by a finite number of open balls B_i, $1 \le i \le m$, so that, possibly after a proper rigid motion, $\partial\Omega \cap B_i$ is part of the graph of a Lipschitz function, above which $\Omega \cap B_i$ lies, for all i's,

Next, we introduce differential forms and associated Sobolev spaces. We will adopt some standard notations, and refer to [3, 6, 10, 11, 14, 15, 20, 21, 29, 32] for more details. For $l \in \mathbb{N}_0$ and $0 \le l \le d$, we denote by Λ^l the vector space of real-valued (or complex-valued), alternating, l-multilinear maps on $\mathbb{R}^d$. In particular, Λ^0 and Λ^1 can be identified with $\mathbb{R}$ and $\mathbb{R}^d$, respectively. Given $\omega \in \Lambda^l$ and $\eta \in \Lambda^k$, the *exterior product* $\omega \wedge \eta \in \Lambda^{l+k}$ is defined by [1]

$$(\omega \wedge \eta)(v_1, \ldots, v_{l+k}) = \sum_{\sigma} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(l)})\eta(v_{\sigma(l+1)}, \ldots, v_{\sigma(l+k)}),$$

for any $v_1, \ldots, v_{l+k} \in \mathbb{R}^d$ where $\operatorname{sgn}(\sigma)$ indicates the signature of σ and the sum is taken over all the permutations σ of $\{1, \ldots, l+k\}$ such that $\sigma(1) < \ldots < \sigma(l)$ and $\sigma(l+1) < \ldots < \sigma(l+k)$.

Given a vector $\mathbf{a} = (a_1, a_2, \ldots, a_d)^T$ and a basis l-form ($l \ge 1$) $\boldsymbol{\omega} = \boldsymbol{d}x_{j_1} \wedge \boldsymbol{d}x_{j_2} \wedge \cdots \wedge \boldsymbol{d}x_{j_l}$ with $j_1 < j_2 < \ldots < j_l$, the *interior product* $\mathbf{a}\lrcorner\, \boldsymbol{\omega} \in \Lambda^{l-1}$ and is defined by

$$\mathbf{a}\lrcorner\, \boldsymbol{\omega} = \sum_{k=1}^{l} (-1)^{k-1} a_{j_k} \boldsymbol{d}x_{j_1} \wedge \cdots \wedge \check{\boldsymbol{d}x}_{j_k} \wedge \cdots \wedge \boldsymbol{d}x_{j_l} \in \Lambda^{l-1} \ ,$$

where $\check{\cdot}$ indicates that $\cdot$ is dropped.

For simplicity, we will frequently use the increasing l-permutation $I = (i_1, \ldots, i_l)$, with $1 \le i_1 < \cdots < i_l \le d$, and denote $\boldsymbol{d}\mathbf{x}_I = \boldsymbol{d}x_{i_1} \wedge \cdots \wedge \boldsymbol{d}x_{i_l}$. Σ_I always means the summation over all the increasing l-permutations I. Therefore Λ^l can be viewed as a vector space of dimension $\binom{d}{l}$ with bases $\{ \boldsymbol{d}x_I \}$ for all increasing l-permutations I.

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, spaces of differential forms are equivalent to those in the componentwise sense. We use standard function spaces $C^m(\overline{\Omega})$, $C^\infty(\overline{\Omega})$, $C_0^\infty(\Omega)$, $L^2(\Omega)$ and $H^s(\Omega)$, $s \in \mathbb{R}_0^+$ (see [1] for more details).

A differential form ω of degree l, $l \in \mathbb{N}_0$, and class C^m, $m \in \mathbb{N}_0$, in Ω is a l-form valued mapping

$$\boldsymbol{\omega} = \sum\nolimits_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I : \mathbf{x} \in \Omega \subset \mathbb{R}^d \mapsto \boldsymbol{\omega}(\mathbf{x}) \in \Lambda^l,$$

where all the components $\boldsymbol{\omega}_I(\mathbf{x}) \in C^m(\overline{\Omega})$. Hence we write $\boldsymbol{\omega} \in \mathcal{DF}^{l,m}(\overline{\Omega})$. In an analogous way, we can define $\mathcal{DF}^{l,\infty}(\overline{\Omega})$ if all $\boldsymbol{\omega}_I(\mathbf{x}) \in C^\infty(\overline{\Omega})$, and $\mathcal{DF}_0^{l,\infty}(\Omega)$ if all $\boldsymbol{\omega}_I(\mathbf{x}) \in C_0^\infty(\Omega)$. Note that the exterior and interior products can be extended as pointwise operations to differential forms on domains in $\mathbb{R}^d$.

Likewise, $\boldsymbol{H}^s(\Omega; \Lambda^l)$ ($s \in \mathbb{R}_0^+$) denotes the space consisting of all differential forms with each component in $H^s(\Omega)$, which can be viewed as the Hilbert space obtained by means of the completion of $\mathcal{DF}^{l,\infty}(\overline{\Omega})$ with respect to the norm

$$\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\Omega;\Lambda^l)} := \sum_I \|\boldsymbol{\omega}_I\|^2_{H^s(\Omega)}$$

In particular we use $\boldsymbol{L}^2(\Omega; \Lambda^l)$ instead of $\boldsymbol{H}^0(\Omega; \Lambda^l)$.

If $\mathscr{T} : \widehat{\Omega} \mapsto \Omega$, is a diffeomorphism between two manifolds in $\mathbb{R}^d$, then the pullback $\mathscr{T}^* : \mathcal{DF}^{l,\infty}(\overline{\Omega}) \mapsto \mathcal{DF}^{l,\infty}(\overline{\Omega})$ is given by

$$((\mathscr{T}^*\boldsymbol{\omega})(\widehat{\mathbf{x}}))(\mathbf{v}_1, \ldots, \mathbf{v}_l) = (\boldsymbol{\omega}(\mathscr{T}(\widehat{\mathbf{x}})))(D\mathscr{T}(\widehat{\mathbf{x}})\mathbf{v}_1, \ldots, D\mathscr{T}(\widehat{\mathbf{x}})\mathbf{v}_l),$$

where $\mathbf{v}_1, \ldots, \mathbf{v}_l \in \mathbb{R}^d$ and the linear map $D\mathscr{T}(\widehat{\mathbf{x}}) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is the derivative (Jacobian) of $\mathscr{T}$ at $\widehat{\mathbf{x}}$.

For a differential l-form $\boldsymbol{\omega} = \sum_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I \in \mathcal{DF}^{l,\infty}(\overline{\Omega})$, its *exterior derivative* $\boldsymbol{d\omega}$ is defined by

$$\boldsymbol{d\omega} := \sum_{i=1}^{d} \sum_I \frac{\partial \omega_I}{\partial x_i} \boldsymbol{d}x_i \wedge \boldsymbol{d}\mathbf{x}_I \in \mathcal{DF}^{l+1,\infty}(\overline{\Omega}) \,, \tag{2.1}$$

[1] We adopt the convention that roman letters denote scalar functions, and their associated spaces etc., while bold letters represent vector-valued functions, and their associated spaces etc. In particular, bold greek letters, $\boldsymbol{\omega}$, $\boldsymbol{\eta}$, $\boldsymbol{\nu}$ and $\boldsymbol{\rho}$, are reserved for differential forms, except that $\boldsymbol{\xi}$ stands for the independent variable in the frequency domain $\mathbb{R}^d$.

and if $l \geq d$, we let $\boldsymbol{d\omega} = 0$.

We recall the fact that the pullback commutes with the exterior derivative, i.e.,

$$\mathscr{T}^*(\boldsymbol{d\omega}) = \boldsymbol{d}(\mathscr{T}^*\boldsymbol{\omega}), \quad \forall\, \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega}) \tag{2.2}$$

and with the wedge product

$$\mathscr{T}^*(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = \mathscr{T}^*\boldsymbol{\omega} \wedge \mathscr{T}^*\boldsymbol{\eta}, \quad \forall\, \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega}),\ \boldsymbol{\eta} \in \mathcal{DF}^{k,\infty}(\overline{\Omega})\ . \tag{2.3}$$

The crucial Hilbert spaces of differential forms are

$$\boldsymbol{H}^s(\boldsymbol{d}, \Omega, \Lambda^l) := \left\{ \boldsymbol{\omega} \in \boldsymbol{H}^s(\Omega; \Lambda^l) \mid \boldsymbol{d\omega} \in \boldsymbol{H}^s(\Omega; \Lambda^{l+1}) \right\}, \quad s \in \mathbb{R}_0^+\ ,$$

with the natural graph norms

$$\|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\boldsymbol{d},\Omega,\Lambda^l)} := \|\boldsymbol{\omega}\|^2_{\boldsymbol{H}^s(\Omega,\Lambda^l)} + \|\boldsymbol{d\omega}\|^2_{\boldsymbol{H}^s(\Omega,\Lambda^{l+1})}\ .$$

Specifically, we simply put $\boldsymbol{H}(\boldsymbol{d}, \Omega, \Lambda^l)$ when $s = 0$.

Moreover, we define some important subspaces of $\boldsymbol{H}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ and $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$, $k \in \mathbb{N}$, respectively:

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l) &:= \left\{ \boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l) \mid \boldsymbol{d\omega} = 0 \right\}, \\
\boldsymbol{H}_0^k(\boldsymbol{d}, \Omega, \Lambda^l) &:= \text{the closure of } \mathcal{DF}_0^{l,\infty}(\Omega) \text{ in the space } \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)\ .
\end{aligned}$$

In the sequel, we denote by c and C generic positive constants which may depend on the domain Ω, space dimension d, the degree of differential forms l and the order of differentiability k, but independent of the differential forms involved.

3 Universal extension of differential forms

In this section, we present in detail our construction of the universal extension operators for Sobolev spaces of differential forms. After briefly recalling essential ingredients of Stein's approach for constructing the universal extension operator for standard Sobolev spaces $H^k(\Omega)$ ($k \in \mathbb{N}_0$) (cf. [34, Chap. VI]), we first show the extension for the case of a Lipschitz epigraph with most key ingredients, and then generalize to bounded Lipschitz domains by the partition of unity.

3.1 Some technical lemmas

For a closed domain $\overline{\Omega}$, let $\delta(\mathbf{x}) := \operatorname{dist}(\mathbf{x}; \overline{\Omega})$ denote the distance of $\mathbf{x} \in \mathbb{R}^d$ from $\overline{\Omega}$. The function $\delta(\mathbf{x})$ vanishes in $\overline{\Omega}$, and, in general, will only be Lipschitz continuous, as $|\delta(\mathbf{x}) - \delta(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \overline{\Omega}^c$, the complement of $\overline{\Omega}$. The next lemma introduces a regularized distance with enhanced smoothness as a replacement for $\delta(\mathbf{x})$.

Lemma 3.1. *[Regularized distance [34, Thm. 2, pp. 171]] For a closed domain $\overline{\Omega} \in \mathbb{R}^d$, there exists a regularized distance function $\Delta(\mathbf{x}) = \Delta(\mathbf{x}, \overline{\Omega})$ such that for $\mathbf{x} \in \overline{\Omega}^c$*

i). $c\delta(\mathbf{x}) \leq \Delta(\mathbf{x}) \leq C\delta(\mathbf{x})$;

ii). $\Delta(\mathbf{x})$ is C^∞-smooth in $\overline{\Omega}^c$ and $\left| \frac{\partial^\alpha}{\partial \mathbf{x}^\alpha} \Delta(\mathbf{x}) \right| \leq C_\alpha (\delta(\mathbf{x}))^{1-|\alpha|}$,

where $c > 0$ and $C > 0$ are constants independent of $\overline{\Omega}$ and $C_\alpha > 0$ depends on the multi-index α [2].

The following two technical lemmas are key tools to construct universal extension operators. The first lemma introduces a suitable weighting function, in terms of which the weighted averaging integral for the construction of extension operators will be defined.

[2] $\frac{\partial^\alpha}{\partial \mathbf{x}^\alpha}$ stands for $\frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$ with $\alpha = (\, \alpha_1, \ldots, \alpha_d \,)$ being a multi-index, $\alpha_i \geq 0$ for $1 \leq i \leq d$, and $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$

Lemma 3.2. *[Weighting function [34, Lemma 1, pp. 182]] The weighting function* [3]

$$\psi(\lambda) := \frac{e}{\pi\lambda}\Im\Big(\exp(\frac{1}{2}\sqrt{2}(-1+i)(\lambda-1)^{1/4}) \Big) \tag{3.1}$$

is defined in $[1,\infty)$*, and satisfies the decay property*

$$\psi(\lambda) = O(\lambda^{-n}) \quad \text{as } \lambda \to \infty,\ \forall\, n \in \mathbb{N}, \tag{3.2}$$

and all its higher moments vanish

$$\int_1^\infty \lambda^k \psi(\lambda)\, \mathrm{d}\lambda = \begin{cases} 1, & \text{for } k=0\,, \\ 0, & \text{for } k \in \mathbb{N}\,. \end{cases} \tag{3.3}$$

Now we consider the special case that Ω is a Lipschitz epigraph with its boundary defined by a Lipschitz function $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$, see Figure 1. We split position vectors according to $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \mathbb{R}^d$, where $\widehat{\mathbf{x}} \in \mathbb{R}^{d-1}$ and $y \in \mathbb{R}$.

Lemma 3.3. *[Existence of smoothed distance function [34, Lemma. 2, pp. 182]] For a Lipschitz epigraph* Ω*, let* $\Delta(\mathbf{x})$ *be the regularized distance given in Lemma 3.1. Then there exists a constant* $C_\delta = C_\delta(\phi) > 0$ *such that for* $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c$,

$$C_\delta \Delta(\mathbf{x}) \geq \phi(\widehat{\mathbf{x}}) - y\,. \tag{3.4}$$

We define a *scaled smoothed distance* $\delta^*(\mathbf{x}) := 2C_\delta\Delta(\mathbf{x})$ with smoothness inherited from $\Delta(\mathbf{x})$. From (3.4) it is immediate to see that

$$\delta^*(\mathbf{x}) \geq 2(\phi(\widehat{\mathbf{x}}) - y)\,. \tag{3.5}$$

3.2 Extension formula for epigraphs

The classical Stein extension formula [34] for compactly supported[4] smooth functions f on a Lipschitz epigraph $\overline{\Omega}$ reads

$$\mathscr{E}(f)(\mathbf{x}) = \int_1^\infty f(\widehat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x}))\psi(\lambda)\, \mathrm{d}\lambda\,. \tag{3.6}$$

To generalize this formula, let us first define a parametrized reflection mapping (see Figure 2) for $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c \in \mathbb{R}^d$,

$$\mathscr{R}_\lambda(\mathbf{x}) = (\widehat{\mathbf{x}}, y + \lambda\delta^*(\mathbf{x})) = \mathbf{x} + \lambda\delta^*(\mathbf{x})e_d\,. \tag{3.7}$$

Note that for points $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}$ we have, using the fact that $\delta^*(\mathbf{x}) = 0$,

$$\mathscr{R}_\lambda(\mathbf{x}) = (\widehat{\mathbf{x}}, y + 0) = \mathbf{x}\,.$$

In other words, $\mathscr{R}_\lambda$ reduces to the identity operator in $\overline{\Omega}$. However, for $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c$ with $y < \phi(\widehat{\mathbf{x}})$, due to (3.5) and the fact that $\lambda \geq 1$, we see that

$$y + \lambda\delta^*(\mathbf{x}) \geq y + 2(\phi(\widehat{\mathbf{x}}) - y) \geq \phi(\widehat{\mathbf{x}}) + (\phi(\widehat{\mathbf{x}}) - y) > \phi(\widehat{\mathbf{x}})\,.$$

Thus, the parametrized reflection mapping $\mathscr{R}_\lambda$ always maps $\mathbf{x} \in \overline{\Omega}^c$ into Ω for any $\lambda \in [1,\infty)$.

It is straightforward to calculate the Jacobian of the parametrized reflection mapping

$$D\mathscr{R}_\lambda(\mathbf{x}) = \begin{pmatrix} Id_{d-1} & 0 \\ \lambda\, \mathbf{grad}_{\widehat{\mathbf{x}}}\, \delta^*(\mathbf{x})^T & 1 + \lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_d} \end{pmatrix}, \tag{3.8}$$

where $\mathbf{grad}_{\widehat{\mathbf{x}}}\, \delta^*(\mathbf{x}) = (\frac{\partial\delta^*(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial\delta^*(\mathbf{x})}{\partial x_{d-1}})^T$ and 0 represents a column vector with $(d-1)$ zeros.

[3] $\Im$ in (3.1) means taking the imaginary part.

[4] It is understood in the sequel as functions or differential forms compactly supported in $\mathbb{R}^d$ with restriction on Ω.

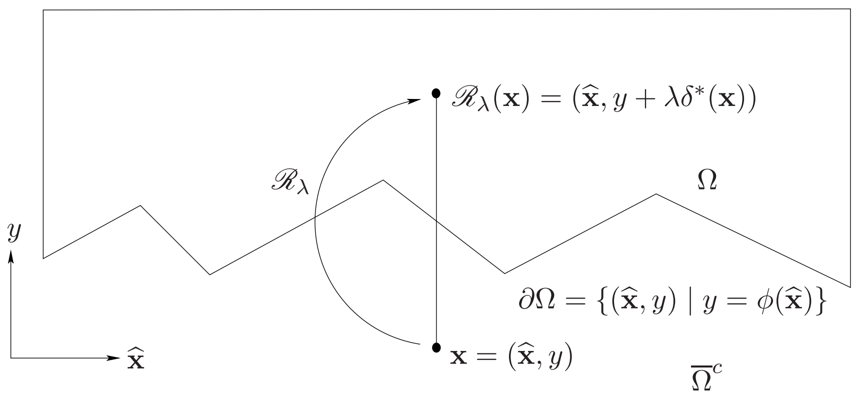

Figure 2: Parametrized reflection mapping.

The function f in (3.6) can be regarded as a vector proxy of a compactly supported 0-form ω on $\overline{\Omega}$. From this perspective, $\mathbf{x} \mapsto f(\widehat{\mathbf{x}}, y+\lambda\delta^*(\mathbf{x}))$ turns out to be the vector proxy of the pullback $\mathscr{R}_\lambda^* f$. This immediately suggests the following generalization of (3.6) to a universal extension operator for smooth compactly supported l-forms on $\overline{\Omega}$:

$$\boxed{(\mathscr{E}_l\boldsymbol{\omega})(\mathbf{x}) := \begin{cases} \boldsymbol{\omega}(\mathbf{x}), & \mathbf{x} \in \overline{\Omega}; \\ \displaystyle\int_1^\infty (\mathscr{R}_\lambda^*\boldsymbol{\omega})(\mathbf{x})\psi(\lambda)\,\mathrm{d}\lambda, & \mathbf{x} \in \overline{\Omega}^c. \end{cases}} \tag{3.9}$$

For the remainder of this section we fix an increasing l-permutation $I = (i_1, \ldots, i_l)$ with $1 \leq i_1 < \cdots < i_l \leq d$. For a compactly supported differential l-form $\boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega})$ we have

$$\begin{aligned} (\mathscr{E}_l\boldsymbol{\omega})_I(\mathbf{x}) :=\ & (\mathscr{E}_l\boldsymbol{\omega})(\mathbf{x})(e_{i_1}, \ldots, e_{i_l}) = \int_1^\infty (\mathscr{R}_\lambda^*\boldsymbol{\omega})(\mathbf{x})(e_{i_1}, \ldots, e_{i_l})\psi(\lambda)\,\mathrm{d}\lambda \\ &= \int_1^\infty (\,\boldsymbol{\omega}(\mathscr{R}_\lambda(\mathbf{x}))\,)(D\mathscr{R}_\lambda(\mathbf{x})e_{i_1}, \ldots, D\mathscr{R}_\lambda(\mathbf{x})e_{i_l})\psi(\lambda)\,\mathrm{d}\lambda\,. \end{aligned}$$

From (3.8) we infer

$$(D\mathscr{R}_\lambda(\mathbf{x}))e_{i_k} = e_{i_k} + \lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_{i_k}}e_d \quad \text{for } 1 \leq k \leq l, \tag{3.10}$$

which yields

$$(\mathscr{E}_l\boldsymbol{\omega})_I(\mathbf{x}) = \mathfrak{K} + \sum_{k=1}^{l}(-1)^{l-k}\mathfrak{J}_{i_k}\ , \tag{3.11}$$

where we have used the abbreviations

$$\begin{aligned} \mathfrak{K} &:= \int_1^\infty (\,\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))\,)\psi(\lambda)\,\mathrm{d}\lambda, \\ \mathfrak{J}_i &:= \frac{\partial\delta^*(\mathbf{x})}{\partial x_i}\int_1^\infty \Big(\boldsymbol{\omega}_{\check{I}_i\cup\{d\}}(\mathscr{R}_\lambda(\mathbf{x}))\Big)\lambda\psi(\lambda)\,\mathrm{d}\lambda, \quad i = 1, 2, \ldots, d\,, \end{aligned}$$

and by $\check{I}_{i_k} \cup \{d\}$ we designate the increasing l-permutation $1 \leq i_1 < \ldots < \check{i}_k < \cdots < i_l < d$ with i_k dropped and d included. For $i_l = d$, we have a simpler representation, viz,

$$(\mathscr{E}_l\boldsymbol{\omega})_I(\mathbf{x}) = \mathfrak{K} + \mathfrak{J}_d\ . \tag{3.12}$$

For $\boldsymbol{d}\boldsymbol{\omega}$, using the commuting diagram property of exterior derivative and the parametrized reflection mapping $\mathscr{R}_\lambda$ used in $\mathscr{E}_l$, we derive

$$\boldsymbol{d}(\mathscr{E}_l\boldsymbol{\omega})(\mathbf{x}) = \int_1^\infty \boldsymbol{d}(\mathscr{R}_\lambda^*\boldsymbol{\omega})(\mathbf{x})\psi(\lambda)\,\mathrm{d}\lambda = \int_1^\infty \mathscr{R}_\lambda^*(\boldsymbol{d}\boldsymbol{\omega})(\mathbf{x})\psi(\lambda)\,\mathrm{d}\lambda\,, \tag{3.13}$$

which implies

$$\boldsymbol{d} \circ \mathscr{E}_l = \mathscr{E}_{l+1} \circ \boldsymbol{d}\,. \tag{3.14}$$

Before we proceed, we have to verify that $\mathscr{E}_l\boldsymbol{\omega}$ provides well-defined differential l-forms.

Lemma 3.4. *For a Lipschitz epigraph Ω, the extension formula in* (3.9) *is well-defined in the sense that for compactly supported $\boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega})$,*

$$\mathscr{E}_l\boldsymbol{\omega} = \boldsymbol{\omega} \quad \textit{in } \overline{\Omega} \quad \textit{and} \quad \mathscr{E}_l\boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\mathbb{R}^d)\;.$$

Proof. The bounded support and smoothness of $\boldsymbol{\omega}$ guarantee that $\mathscr{E}_l\boldsymbol{\omega}$ is well-defined everywhere in $\mathbb{R}^d$. In particular, $\mathscr{E}_l\boldsymbol{\omega} = \boldsymbol{\omega}$ in $\overline{\Omega}$ due to the fact that the reflection mapping $\mathscr{R}_\lambda$ reduces to the identity operator.

The smoothness of δ^* and $\boldsymbol{\omega}$ along with the compact support of $\boldsymbol{\omega}$ ensures that $\mathscr{E}_l\boldsymbol{\omega}$ belongs to $\mathcal{DF}^{l,\infty}(\Omega \cup \Omega^c)$. It remains to prove that all partial derivatives $\frac{\partial^\alpha(\mathscr{E}_l\boldsymbol{\omega})_I}{\partial \mathbf{x}^\alpha}$ for any multi-index α and any component index I are continuous across $\partial\Omega$.

The argument is similar for all partial derivatives of every component. Thus, we demonstrate the technique of the proof for the typical case of $\frac{\partial^2((\mathscr{E}_l\boldsymbol{\omega})_I)}{\partial x_j^2}$ and appeal to analogy as far as the treatment of other partial derivatives is concerned.

As regards $\frac{\partial^2((\mathscr{E}_l\boldsymbol{\omega})_I)}{\partial x_j^2}$ for $j < d$ (note that $j = d$ is an easier case and can be treated in the same way), in light of (3.11) and (3.12), it suffices to check whether $\frac{\partial^2\mathfrak{K}}{\partial x_j^2}$ and $\frac{\partial^2\mathfrak{J}_i}{\partial x_j^2}$ are continuous across $\partial\Omega$. A straightforward differentiation of $\mathfrak{K}$ and $\mathfrak{J}_i$ by the chain rule yields

$$\begin{aligned}\frac{\partial^2\mathfrak{K}}{\partial x_j^2} =& \int_1^\infty \left(\frac{\partial^2\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j^2}\right)\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \int_1^\infty \left(\frac{\partial^2\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j \partial x_d}\right)\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_j}\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \int_1^\infty \left(\frac{\partial^2\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d^2}\right)(\lambda\frac{\partial\delta^*(\mathbf{x})}{\partial x_j})^2\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \int_1^\infty \left(\frac{\partial\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d}\right)\lambda\frac{\partial^2\delta^*(\mathbf{x})}{\partial x_j^2}\psi(\lambda)\,\mathrm{d}\lambda\,,\end{aligned} \tag{3.15}$$

$$\begin{aligned}\frac{\partial^2\mathfrak{J}_i}{\partial x_j^2} =& \frac{\partial^3\delta^*(\mathbf{x})}{\partial x_i x_j^2}\int_1^\infty \left(\boldsymbol{\omega}_{\check{I}_i\cup\{d\}}(\mathscr{R}_\lambda(\mathbf{x}))\right)\lambda\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \frac{\partial^2\delta^*(\mathbf{x})}{\partial x_i\partial x_j}\int_1^\infty \left(\frac{\partial\boldsymbol{\omega}_{\check{I}_i\cup\{d\}}(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j}\right)\lambda\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \frac{\partial^2\delta^*(\mathbf{x})}{\partial x_i\partial x_j}\int_1^\infty \left(\frac{\partial\boldsymbol{\omega}_{\check{I}_i\cup\{d\}}(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d}\right)\frac{\partial\delta^*(\mathbf{x})}{\partial x_j}\lambda^2\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \frac{\partial\delta^*(\mathbf{x})}{\partial x_i}\int_1^\infty \left(\frac{\partial^2\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j^2}\right)\lambda\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \frac{\partial\delta^*(\mathbf{x})}{\partial x_i}\int_1^\infty \left(\frac{\partial^2\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_j\partial x_d}\right)\frac{\partial\delta^*(\mathbf{x})}{\partial x_j}\lambda^2\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \frac{\partial\delta^*(\mathbf{x})}{\partial x_i}\int_1^\infty \left(\frac{\partial^2\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d^2}\right)(\frac{\partial\delta^*(\mathbf{x})}{\partial x_j})^2\lambda^3\psi(\lambda)\,\mathrm{d}\lambda \\ &+ \frac{\partial\delta^*(\mathbf{x})}{\partial x_i}\int_1^\infty \left(\frac{\partial\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d}\right)\frac{\partial^2\delta^*(\mathbf{x})}{\partial x_j^2}\lambda^2\psi(\lambda)\,\mathrm{d}\lambda\,.\end{aligned} \tag{3.16}$$

Now, we establish continuity of both $\frac{\partial^2\mathfrak{K}}{\partial x_j^2}$ and $\frac{\partial^2\mathfrak{J}_i}{\partial x_j^2}$ across $\partial\Omega$: Let $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c$ tend to some point $\mathbf{x}^0 = (\widehat{\mathbf{x}}^0, y^0)$ on the boundary $\partial\Omega$, that is, $y^0 = \phi(\widehat{\mathbf{x}}^0)$. Then by Lemma 3.3, $\delta^*(\mathbf{x}) \to 0$ and the derivatives $\frac{\partial \delta^*(\mathbf{x})}{\partial x_j}$, $1 \leq j \leq d$, are bounded uniformly as $\mathbf{x} \to \mathbf{x}^0$. By Lemma 3.2 the first three terms on the right hand side of (3.15) converge to $\frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j^2}(\mathbf{x}^0)$, 0 and 0, respectively.

As for the last term in (3.15), the difficulty involving the unboundedness of the higher order derivatives of δ^* can be circumvented by using the Taylor expansion of $\boldsymbol{\omega}_I$ about $(\widehat{\mathbf{x}}, y + \delta^*)$:

$$\frac{\partial \boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))}{\partial x_d} = \frac{\partial \boldsymbol{\omega}_I(\widehat{\mathbf{x}}, y + \delta^*(\mathbf{x}))}{\partial x_d} + (\lambda - 1)\delta^*(\mathbf{x}) \frac{\partial^2 \boldsymbol{\omega}_I(\widehat{\mathbf{x}}, y + \delta^*(\mathbf{x}))}{\partial x_d^2} + r(\lambda, \mathbf{x}) \,, \tag{3.17}$$

with a remainder term $r(\lambda, \mathbf{x})$ that satisfies

$$|r(\lambda, \mathbf{x})| \leq C[(\lambda - 1)\delta^*(\mathbf{x})]^2 \quad \forall \mathbf{x} \in \Omega^c, \ \lambda > 1 \,. \tag{3.18}$$

Thanks to Lemma 3.1, we conclude

$$|r(\lambda, \mathbf{x}) \frac{\partial^2 \delta^*(\mathbf{x})}{\partial x_j^2}| \leq C(\lambda - 1)^2 \delta^*(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega^c, \ \lambda > 1 \,.$$

Hence, substituting the identity (3.17) into (3.15) gives two more vanishing integrals plus a remainder

$$\left| \int_1^\infty r(\lambda, \mathbf{x}) \lambda \frac{\partial^2 \delta^*(\mathbf{x})}{\partial x_j^2} \psi(\lambda) \,\mathrm{d}\lambda \right| \leq C\delta^*(\mathbf{x}) \int_1^\infty (\lambda - 1)^2 \lambda |\psi(\lambda)| \,\mathrm{d}\lambda \to 0 \,,$$

since the last integral is uniformly bounded by (3.2) and $\delta^*(\mathbf{x}) \to 0$ as $\mathbf{x} \to \mathbf{x}_0$.

Similar arguments can be applied to show that all the terms in (3.16) vanish as $\mathbf{x} \to \mathbf{x}^0$. Either we end up with an integral involving the factor term $\lambda^k \psi(\lambda)$ for some k or we resort to a Taylor expansion like (3.17), thus eliminating possible blow-ups in higher partial derivatives of δ^*. Summing up, thus we have shown the continuity of $\frac{\partial^2 \mathscr{E}_l \boldsymbol{\omega}}{\partial x_j^2}$. $\square$

3.3 Continuity of extension operators

Theorem 3.5. *Let Ω be a Lipschitz epigraph in $\mathbb{R}^d$, $k \in \mathbb{N}_0$ and $0 \leq l \leq d$. Then the extension operator* (3.9) *satisfies*

$$\|\mathscr{E}_l \boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)} \leq C \, \|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)} \quad \forall \text{ compactly supported } \boldsymbol{\omega} \in \mathcal{DF}^{l,\infty}(\overline{\Omega}) \,,$$

with a constant $C = C(\Omega, d, k, l) > 0$. Thus, $\mathscr{E}_l$ can be extended to a continuous extension operator

$$\mathscr{E}_l : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l) \,.$$

Proof. The second assertion relies on density argument, because compactly supported differential forms in $\mathcal{DF}^{l,\infty}(\overline{\Omega}) \cap \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ form a dense subset of $\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$.

It remains to show that the continuity of the extension operator. Let us first consider the case when $k = 0$. Consider a boundary point $(\widehat{\mathbf{x}}^0, y^0) \in \partial\Omega$, and assume without loss of generality that $0 = y^0 = \phi(\widehat{\mathbf{x}}^0)$. Then from (3.11) (the argument for (3.12) is the same)

$$\begin{aligned} |(\mathscr{E}_l \boldsymbol{\omega})_I(\widehat{\mathbf{x}}^0, y)| \leq C \Bigg(&\int_1^\infty |\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x}))| \, \frac{1}{\lambda^2} \,\mathrm{d}\lambda \\ &+ \sum_{k=1}^{l} \int_1^\infty \left| \boldsymbol{\omega}_{\check{I}_{i_k} \cup \{d\}}(\mathscr{R}_\lambda(\mathbf{x})) \right| \frac{1}{\lambda^2} \,\mathrm{d}\lambda \Bigg) \quad \text{for } y < 0, \end{aligned} \tag{3.19}$$

where we have used the facts that $|\psi| \leq C_1/\lambda^2$ for the first term and $|\psi| \leq C_2/\lambda^3$ for the summation term in the right hand side, and that $|\frac{\partial \delta^*(\mathbf{x})}{x_{i_k}}| \leq C$ from Lemma 3.1.

For fixed $\mathbf{x} = (\widehat{\mathbf{x}}, y)$ with $y < 0$, we have $\delta^*(\mathbf{x}) \geq 2|y|$ for $\phi(\widehat{\mathbf{x}}^0) = 0$ by (3.5). On the other hand, by Lemma 3.1, we see $\delta^*(\mathbf{x}) = C\Delta(\mathbf{x}) \leq C \cdot \tilde{C}\delta(\mathbf{x}) \leq C \cdot \tilde{C}|y|$ since $\phi(\widehat{\mathbf{x}}^0) - y$ is not less than the distance $\delta(\widehat{\mathbf{x}}, y)$ of $(\widehat{\mathbf{x}}, y)$ from $\overline{\Omega}$. By performing the change of variables $s = y + \lambda\delta^*(\mathbf{x})$, then we have $\mathrm{d}s = \delta^*(\mathbf{x})\mathrm{d}\lambda$ and

$$\int_1^\infty |(\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\mathbf{x})))| \frac{1}{\lambda^2}\,\mathrm{d}\lambda \le C|y| \int_{|y|}^\infty |(\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, s))| \frac{1}{s^2}\,\mathrm{d}s\,.$$

Note that $(s - y) \ge s$ for $y < 0$ and $s > 0$. Likewise, it still holds when we replace I by $\check{I}_{i_k} \cup \{d\}$ for $k = 1, \dots, l$.

Recall the Hardy inequality [34, pp. 272]:, for any $f \ge 0, p \ge 1$ and $r > 0$,

$$\left(\int_0^\infty \left(\int_x^\infty f(y)\,\mathrm{d}y\right)^p x^{r-1}\,\mathrm{d}x\right)^{1/p} \le \frac{p}{r}\left(\int_0^\infty (\,yf(y)\,)^p y^{r-1}\,\mathrm{d}y\right)^{1/p}. \tag{3.20}$$

We can apply (3.20) for the case that $r = 3$ and $p = 2$ to obtain

$$\begin{aligned}
&\left(\int_{-\infty}^0 |(\mathscr{E}\boldsymbol{\omega})_I(\widehat{\mathbf{x}}^0, y)|^2\,\mathrm{d}y\right)^{1/2} \\
&\le C\left(\int_{-\infty}^0 \left(\int_{|y|}^\infty \left(\left|\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, s)\right| + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\widehat{\mathbf{x}}^0, s)\right|\right)\frac{1}{s^2}\,\mathrm{d}s\right)^2 |y|^2\,\mathrm{d}y\right)^{1/2} \\
&\le C\left(\int_{-\infty}^0 \left(\left(\left|\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, |y|)\right| + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\widehat{\mathbf{x}}^0, |y|)\right|\right)\frac{|y|}{|y|^2}\right)^2 |y|^2\,\mathrm{d}y\right)^{1/2} \\
&\le C\left(\int_0^\infty \left(\left|\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, y)\right|^2 + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\widehat{\mathbf{x}}^0, y)\right|^2\right)\mathrm{d}y\right)^{1/2},
\end{aligned}$$

where we use the Hardy inequality for the second inequality, the Cauchy-Schwarz inequality and the change of variable $y \to -y$ in the last.

Furthermore, the assumption $\phi(\widehat{\mathbf{x}}^0) = 0$ can be dropped by using an appropriate translation in y, which yields

$$\begin{aligned}
&\left(\int_{-\infty}^\infty |(\mathscr{E}_l\boldsymbol{\omega})_I(\widehat{\mathbf{x}}^0, y)|^2\,\mathrm{d}y\right)^{1/2} \le \left(\left(\int_{-\infty}^{\phi(\widehat{\mathbf{x}}^0)} + \int_{\phi(\widehat{\mathbf{x}}^0)}^\infty\right)|(\mathscr{E}\boldsymbol{\omega})_I(\widehat{\mathbf{x}}^0, y)|^2\,\mathrm{d}y\right)^{1/2} \\
&\le C\left(\int_{\phi(\widehat{\mathbf{x}}^0)}^\infty \left(\left|\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, y)\right|^2 + \sum_{k=1}^l \left|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}(\widehat{\mathbf{x}}^0, y)\right|^2\right)\mathrm{d}y\right)^{1/2}.
\end{aligned}$$

Taking the square on both sides and integrating over all $\widehat{\mathbf{x}} \in \mathbb{R}^{d-1}$ yields

$$\|(\mathscr{E}_l\boldsymbol{\omega})_I\|^2_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^l)} \le C\left(\|\boldsymbol{\omega}_I\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)} + \sum_{k=1}^l \left\|\boldsymbol{\omega}_{\check{I}_{i_k}\cup\{d\}}\right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)}\right).$$

Summing over all indices I we derive

$$\|\mathscr{E}_l\boldsymbol{\omega}\|^2_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^l)} \le C\,\|\boldsymbol{\omega}\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)}\,.$$

In exactly the same way, we can show that, in view of the commuting diagram property (3.14),

$$\|\boldsymbol{d}(\mathscr{E}_l\boldsymbol{\omega})\|_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^{l+1})} = \|\mathscr{E}_{l+1}(\boldsymbol{d}\boldsymbol{\omega})\|_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^{l+1})} \le C\,\|\boldsymbol{d}\boldsymbol{\omega}\|_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})}\,,$$

which completes the proof in the case $k = 0$.

The proof for $k > 0$ is again done for one representative special case. Let us take $k = 2$ with $\frac{\partial^2((\mathscr{E}_l\boldsymbol{\omega})_I)}{\partial x_j^2}$ as our specimen. Using $\psi(\lambda) \le C_1/\lambda^2, C_2/\lambda^3, C_3/\lambda^4$, respectively, the terms in (3.15) can be bounded as follows:

$$\begin{aligned}
\left|\frac{\partial^2(\mathscr{E}_l\boldsymbol{\omega})_I(\widehat{\mathbf{x}}^0, y)}{\partial x_j^2}\right| &\le C\int_1^\infty \left(\left|\left(\frac{\partial^2\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, y)}{\partial x_j^2}\right)\right| + \left|\left(\frac{\partial^2\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, y)}{\partial x_j x_d}\right)\right| + \left|\left(\frac{\partial^2\boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, y)}{\partial x_d^2}\right)\right|\right)\frac{1}{\lambda^2}\,\mathrm{d}\lambda \\
&\quad + \left|\int_1^\infty \left(\frac{\partial\boldsymbol{\omega}_I(\mathscr{R}_\lambda(\widehat{\mathbf{x}}^0, y))}{\partial x_d}\right)\lambda\frac{\partial^2\delta^*(\widehat{\mathbf{x}}^0, y)}{\partial x_j^2}\psi(\lambda)\,\mathrm{d}\lambda\right|.
\end{aligned} \tag{3.21}$$

Only the last term has to be dealt with separately. Using the Taylor expansion with integral residual, we have

$$\frac{\partial \boldsymbol{\omega}_I}{\partial x_d}(\mathscr{R}_\lambda(\widehat{\mathbf{x}}^0, y)) = \frac{\partial \boldsymbol{\omega}_I}{\partial x_d}(\widehat{\mathbf{x}}^0, y + \delta^*(\widehat{\mathbf{x}}^0, y)) + \int_{y+\delta^*(\widehat{\mathbf{x}}^0,y)}^{y+\lambda\delta^*(\widehat{\mathbf{x}}^0,y)} \frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_d^2}(\widehat{\mathbf{x}}^0, s)\,\mathrm{d}s.$$

Substituting this in (3.21), we know that the integral term involving $\frac{\partial \boldsymbol{\omega}_I}{\partial x_d}(\widehat{\mathbf{x}}^0, y + \delta^*(\widehat{\mathbf{x}}^0, y))$ vanishes due to Lemma 3.2. Hence, it suffices to show the following bound (Note that $|\frac{\partial^2 \delta^*}{x_j^2}(\widehat{\mathbf{x}}^0, y)| \leq C|\delta(\widehat{\mathbf{x}}^0, y)|^{-1} \leq C|y|^{-1}$. We assume $\phi(\widehat{\mathbf{x}}^0) = 0$ without loss of generality):

$$\begin{aligned}
&|y|^{-1}\int_1^\infty \left\{ \int_{y+\delta^*(\widehat{\mathbf{x}}^0,y)}^{y+\lambda\delta^*(\widehat{\mathbf{x}}^0,y)} \left| \frac{\partial^2 \boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, s)}{\partial x_d^2} \right| \mathrm{d}s \right\} \frac{1}{\lambda^3}\,\mathrm{d}\lambda \\
&= |y|^{-1}\int_{y+\delta^*(\widehat{\mathbf{x}}^0,y)}^\infty \left\{ \int_{(s-y)/\delta^*(\widehat{\mathbf{x}}^0,y)}^\infty \left| \frac{\partial^2 \boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, s)}{\partial x_d^2} \right| \frac{1}{\lambda^3}\,\mathrm{d}\lambda \right\} \mathrm{d}s \\
&\leq |y|^{-1}\big(\delta^*(\widehat{\mathbf{x}}^0, y)\big)^2 \int_{y+\delta^*(\widehat{\mathbf{x}}^0,y)}^\infty \left\{ \left| \frac{\partial^2 \boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, s)}{\partial x_d^2} \right| \right\} \frac{1}{(s-y)^2}\,\mathrm{d}s \\
&\leq C|y| \int_{|y|}^\infty \left\{ \left| \frac{\partial^2 \boldsymbol{\omega}_I(\widehat{\mathbf{x}}^0, s)}{\partial x_d^2} \right| \right\} \frac{1}{s^2}\,\mathrm{d}s\,,
\end{aligned}$$

where we have interchanged the order of integration for the first equality, and used that $\delta^*(\widehat{\mathbf{x}}^0, y) \leq C|y|$, $\delta^*(\widehat{\mathbf{x}}^0, y) \geq 2|y|$ and $s - y \geq s$ when $y < 0$ for the second inequality. Thus we can appeal to the Hardy inequality once again for (3.21) and integrate over all $\widehat{\mathbf{x}} \in \mathbb{R}^{d-1}$ to obtain

$$\left\| \frac{\partial^2 (\mathscr{E}_l \boldsymbol{\omega})_I}{\partial x_j^2} \right\|^2_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^l)} \leq C \left(\left\| \frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j^2} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)} + \left\| \frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_j x_d} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)} + \left\| \frac{\partial^2 \boldsymbol{\omega}_I}{\partial x_d^2} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^l)} \right).$$

Analogously by the commuting diagram property, we have

$$\left\| \frac{\partial^2 \boldsymbol{d}(\mathscr{E} \boldsymbol{\omega})_I}{\partial x_j^2} \right\|^2_{\boldsymbol{L}^2(\mathbb{R}^d;\Lambda^{l+1})} \leq C \left(\left\| \frac{\partial^2 \boldsymbol{d}\boldsymbol{\omega}_I}{\partial x_j^2} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})} + \left\| \frac{\partial^2 \boldsymbol{d}\boldsymbol{\omega}_I}{\partial x_j \partial x_d} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})} + \left\| \frac{\partial^2 \boldsymbol{d}\boldsymbol{\omega}_I}{\partial x_d^2} \right\|^2_{\boldsymbol{L}^2(\Omega;\Lambda^{l+1})} \right).$$

Thus we have proved the assertion for the case $k = 2$.

Now, for general k, differentiating (3.9) gives various order partial derivatives of the components of $\boldsymbol{\omega}$. Whenever the total differential order of $\boldsymbol{\omega}$ is less than k, we always use the Taylor expansion around the point $(\widehat{\mathbf{x}}^0, y + \delta^*(\widehat{\mathbf{x}}^0, y))$ and carry it up to order k with integral remainders and proceed the arguments as above. We note that the constant C involved only depends on the domain Ω, the dimension d, the order of differentiability k and the degree of differential forms l. This finishes the proof. ☐

The general situation of a compact Lipschitz boundary can be tackled by a partition of unity subordinate to a finite cover of $\partial\Omega$ in the usual way. This yields our main result:

Theorem 3.6. *Let Ω be a domain with a bounded Lipschitz domain, $k \in \mathbb{N}_0$ and $0 \leq l \leq d$. Then there exists a universal extension operator*

$$\mathscr{E}_l : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$$

satisfying

1. *$\mathscr{E}_l \boldsymbol{\omega} = \boldsymbol{\omega}$ a.e. in Ω, and*
2. *the extension operator is continuous*

$$\|\mathscr{E}_l \boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\mathbb{R}^d,\Lambda^l)} \leq C\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\boldsymbol{d},\Omega,\Lambda^l)} \quad \forall\, \boldsymbol{\omega} \in \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l),$$

with the constant $C = C(\Omega, d, k, l)$, but independent of the differential forms involved.

Remark 3.1. Theorem 3.6 also holds for a domain Ω whose complement $\overline{\Omega}^c$ is a bounded Lipschitz domain. It is further pointed out that the commuting diagram property (3.14) no longer holds for $\mathscr{E}_l$ for general bounded Lipschitz domains due to the use of a partition of unity.

Remark 3.2. Costabel and McIntosh have recently introduced some so-called smoothed Poincaré liftings in [13]. Those offer an alternative way to define universal extension operators based on standard extension for the Sobolev spaces $H^s(\Omega)$.

4 Vector field perspective

In three-dimensional Euclidean space, we may represent the differential forms in terms of their so-called vector proxies, as shown in Table 1.

Differential form	Related function u/vectorfield $\mathbf{u}$
$\mathbf{x} \mapsto \boldsymbol{\omega}(\mathbf{x})$	$u(\mathbf{x}) := \boldsymbol{\omega}(\mathbf{x})$
$\mathbf{x} \mapsto \{\mathbf{v} \mapsto \boldsymbol{\omega}(\mathbf{x})(\mathbf{v})\}$	$\mathbf{u}(\mathbf{x}) \cdot \mathbf{v} := \boldsymbol{\omega}(\mathbf{x})(\mathbf{v})$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(\mathbf{x}) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2)$
$\mathbf{x} \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(\mathbf{x}) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \boldsymbol{\omega}(\mathbf{x})(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Table 1: Relationship between differential forms and vectorfields ("vector proxies") in three-dimensional Euclidean space ($\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$). The operation $\cdot$ is the canonical inner product in Euclidean space.

The concept of Euclidean vector proxies establishes a one-to-one correspondence between Sobolev spaces of scalar/vector functions and Sobolev spaces of differential forms, see Table 2.

l	Sobolev spaces of functions	Sobolev spaces of differential forms
0	$H^{k+1}(\Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^0)$
1	$\boldsymbol{H}^k(\mathbf{curl}; \Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^1)$
2	$\boldsymbol{H}^k(\mathrm{div}; \Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^2)$
3	$H^k(\Omega)$	$\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^3)$

Table 2: Correspondence between Sobolev spaces of functions/fields and Sobolev spaces of differential forms in $\mathbb{R}^3$.

Now we give special incarnations of the extension operators $\mathscr{E}_l$, $0 \le l \le 3$, for Lipschitz epigraphs $\Omega \subset \mathbb{R}^3$ from (3.9) in terms of vector proxies in $\mathbb{R}^3$. Of course, for $l = 0$ we recover Stein's formula (3.6).

In the case $l = 1$, that is, for a covector field $\mathbf{u} \in \boldsymbol{H}^k(\mathbf{curl}; \Omega)$, $k \in \mathbb{N}_0$, we have for $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c$,

$$\begin{aligned} \mathscr{E}_1 \mathbf{u}(\mathbf{x}) &= \int_1^\infty \left(D\mathscr{R}_\lambda(\mathbf{x}) \right)^T \mathbf{u}(\widehat{\mathbf{x}}, \bullet) \psi(\lambda) \, \mathrm{d}\lambda \\ &= \int_1^\infty \left(\mathbf{u}(\widehat{\mathbf{x}}, \bullet) + \lambda u_3(\widehat{\mathbf{x}}, \bullet) \, \mathbf{grad} \, \delta^*(\mathbf{x}) \right) \psi(\lambda) \, \mathrm{d}\lambda \, , \end{aligned} \tag{4.1}$$

where $\bullet$ stands for $y + \lambda \delta^*(\mathbf{x})$ and u_3 is the third component of $\mathbf{u}$.

For $l = 2$, that is, a bivector field $\mathbf{u} \in \boldsymbol{H}^k(\mathrm{div}; \Omega)$, $k \in \mathbb{N}_0$, we have for $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c$,

$$\begin{aligned}\mathscr{E}_2\mathbf{u}(\mathbf{x}) &= \int_1^\infty (\, D\mathscr{R}_\lambda(\mathbf{x})\,)^{-1} \det(\, D\mathscr{R}_\lambda(\mathbf{x})\,)\mathbf{u}(\widehat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda \\ &= \int_1^\infty \left((1 + \lambda \frac{\partial \delta^*}{\partial x_3}(\mathbf{x}))\mathbf{u}(\widehat{\mathbf{x}}, \bullet) - \begin{pmatrix} 0 \\ 0 \\ \lambda\, \mathbf{grad}\, \delta^*(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}, \bullet) \end{pmatrix} \right) \psi(\lambda)\,\mathrm{d}\lambda\,. \end{aligned} \tag{4.2}$$

Here, the occurrence of $D\mathscr{R}_\lambda(\mathbf{x})^{-1}$ is merely formal, because, in fact, we need the adjunct Jacobian matrix of the paremetrized reflection mapping

$$\det(\, D\mathscr{R}_\lambda(\mathbf{x})\,)(\, D\mathscr{R}_\lambda(\mathbf{x})\,)^{-1} = \begin{pmatrix} 1 + \lambda \dfrac{\partial \delta^*(\mathbf{x})}{\partial x_3} & 0 & 0 \\ 0 & 1 + \lambda \frac{\partial \delta^*(\mathbf{x})}{\partial x_3} & 0 \\ -\lambda \dfrac{\partial \delta^*(\mathbf{x})}{\partial x_1} & -\lambda \dfrac{\partial \delta^*(\mathbf{x})}{\partial x_2} & 1 \end{pmatrix}. \tag{4.3}$$

Lastly, for any density function $u \in H^k(\Omega)$, $k \in \mathbb{N}_0$, we have for $\mathbf{x} = (\widehat{\mathbf{x}}, y) \in \overline{\Omega}^c$

$$\begin{aligned}\mathscr{E}_3 u(\mathbf{x}) &= \int_1^\infty \det(\, D\mathscr{R}_\lambda(\mathbf{x})\,) u(\widehat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda \\ &= \int_1^\infty \left(1 + \lambda \frac{\partial \delta^*}{\partial x_3}(\mathbf{x})\right) u(\widehat{\mathbf{x}}, \bullet)\psi(\lambda)\,\mathrm{d}\lambda\,. \end{aligned} \tag{4.4}$$

To the best knowledge of the authors, The latter three formulae (4.1), (4.2) and (4.4) seem new to the mathematical community. Applying Theorem 3.6 for the Euclidean space $\mathbb{R}^3$, it is immediate to obtain the following "vector analytic" specialization.

Corollary 4.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and $k \in \mathbb{N}_0$. Then there exist universal extension operators*

$$\begin{aligned}
&\mathscr{E}_0 : H^{k+1}(\Omega) \mapsto H^{k+1}(\mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_0 u = u, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_0 u\|_{H^k(\mathbb{R}^3)} \le C \|u\|_{H^k(\Omega)}\,; \end{cases} \\
&\mathscr{E}_1 : \boldsymbol{H}^k(\mathbf{curl}; \Omega) \mapsto \boldsymbol{H}^k(\mathbf{curl}; \mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_1 \mathbf{u} = \mathbf{u}, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_1 \mathbf{u}\|_{\boldsymbol{H}^k(\mathbf{curl}; \mathbb{R}^3)} \le C \|\mathbf{u}\|_{\boldsymbol{H}^k(\mathbf{curl}; \Omega)}\,; \end{cases} \\
&\mathscr{E}_2 : \boldsymbol{H}^k(\mathrm{div}; \Omega) \mapsto \boldsymbol{H}^k(\mathrm{div}; \mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_2 \mathbf{u} = \mathbf{u}, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_2 \mathbf{u}\|_{\boldsymbol{H}^k(\mathrm{div}; \mathbb{R}^3)} \le C \|\mathbf{u}\|_{\boldsymbol{H}^k(\mathrm{div}; \Omega)}\,; \end{cases} \\
&\mathscr{E}_3 : H^k(\Omega) \mapsto H^k(\mathbb{R}^3) && \textit{satisfying} \begin{cases} \mathscr{E}_3 u = u, \ \textit{a.e. in } \Omega, \textit{ and} \\ \|\mathscr{E}_3 u\|_{H^k(\mathbb{R}^3)} \le C \|u\|_{H^k(\Omega)}\,; \end{cases}
\end{aligned}$$

with all the constants $C = C(k, \Omega)$, but independent of the functions/fields involved.

5 Application: Regular decompositions

Regular decomposition results for $\boldsymbol{H}(\mathrm{div}; \Omega)$ and $\boldsymbol{H}(\mathbf{curl}; \Omega)$ and related spaces assert that those can be split into the kernel of the underlying differential operator and a complement space of H^1-regular functions. Regular decompositions, pioneered in [5], have become a powerful tool in mathematical analysis [8,12] and numerical analysis, see [20, Sect. 2.4] and the references given there.

In this section, we apply the universal extension result to establish regular decompositions of Sobolev spaces of differential forms. As a consequence, a well-known lifting lemma can be generalized to Sobolev spaces of differential forms. Throughout this section we only consider $d \ge 3$ and $\Omega \subset \mathbb{R}^d$ is always assumed to be a bounded Lipschitz domain.

Regular decomposition rely on the existence of regular potentials in $\mathbb{R}^d$.

Lemma 5.1 (Existence of regular potentials in $\mathbb{R}^d$)**.** *For $1 \le l \le d$, $l \in \mathbb{N}$ and every $k \in \mathbb{N}_0$ there is a continuous lifting mapping*

$$\mathscr{L} : \boldsymbol{H}(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l) \cap \boldsymbol{H}^k(\mathbb{R}^d, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}_{loc}(\mathbb{R}^d, \Lambda^{l-1})$$

such that for all $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l) \cap \boldsymbol{H}^k(\mathbb{R}^d, \Lambda^l)$,

$$\boldsymbol{d}\mathscr{L}\boldsymbol{\omega} = \boldsymbol{\omega}. \tag{5.1}$$

As a tool for the proof, we introduce the Fourier transform of functions, denoted by $\mathscr{F}$, mapping from $L^2(\mathbb{R}^d)$ into itself, and let $\mathscr{F}^{-1}$ stand for its inverse.

The Fourier transform (cf. [35]) of a differential l-form $\boldsymbol{\omega} = \sum_I \boldsymbol{\omega}_I \boldsymbol{d}\mathbf{x}_I \in \boldsymbol{L}^2(\mathbb{R}^d; \Lambda^l)$, still denoted by $\mathscr{F}$, is defined componentwise by

$$\widehat{\boldsymbol{\omega}}(\boldsymbol{\xi}) := \mathscr{F}(\boldsymbol{\omega})(\boldsymbol{\xi}) = \sum_I \widehat{\boldsymbol{\omega}}_I(\boldsymbol{\xi})\boldsymbol{d}\boldsymbol{\xi}_I\,,$$

where

$$\widehat{\boldsymbol{\omega}}_I(\boldsymbol{\xi}) := \mathscr{F}(\boldsymbol{\omega}_I)(\boldsymbol{\xi}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(-\imath\boldsymbol{\xi}\cdot\mathbf{x})\boldsymbol{\omega}_I(\mathbf{x})\,\mathrm{d}\mathbf{x}\,,$$

and $\imath$ is the imaginary unit, $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_d)^T$ is the vectorial angular frequency in $\mathbb{R}^d$ and $\boldsymbol{d}\boldsymbol{\xi}_I = \boldsymbol{d}\xi_{i_1} \wedge \cdots \wedge \boldsymbol{d}\xi_{i_l}$. with I being an increasing l-permutation.

Accordingly, the inverse Fourier transform of $\boldsymbol{\omega}$, also denoted by $\mathscr{F}^{-1}$, is defined by

$$\boldsymbol{\omega}(\mathbf{x}) := \mathscr{F}^{-1}(\widehat{\boldsymbol{\omega}})(\mathbf{x}) = \sum_I \boldsymbol{\omega}_I(\mathbf{x})\boldsymbol{d}\mathbf{x}_I\,,$$

where

$$\boldsymbol{\omega}_I(\mathbf{x}) := \mathscr{F}^{-1}(\widehat{\boldsymbol{\omega}}_I)(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(\imath\boldsymbol{\xi}\cdot\mathbf{x})\widehat{\boldsymbol{\omega}}_I(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}\,.$$

It is easy to see that the Fourier transform converts the exterior derivative into an exterior product:

Lemma 5.2. *For any $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}, \Omega, \Lambda^l)$, we have*

$$\mathscr{F}(\boldsymbol{d}\boldsymbol{\omega}) = \imath\widehat{\boldsymbol{\xi}} \wedge \mathscr{F}(\boldsymbol{\omega}), \tag{5.2}$$

where $\widehat{\boldsymbol{\xi}}$ is the differential 1-form in the frequency domain, namely $\widehat{\boldsymbol{\xi}} = \xi_1\boldsymbol{d}\xi_1 + \xi_2\boldsymbol{d}\xi_2 + \cdots + \xi_d\boldsymbol{d}\xi_d$.

Proof of Lemma 5.1. We follows the idea in the proof of [2, Lemma 3.5, pp. 837]. It boils down to straightforward calculations with Fourier transforms of differential forms.

Let $\boldsymbol{\omega} \in \boldsymbol{H}(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^l) \cap \boldsymbol{H}^k(\mathbb{R}^d, \Lambda^l)$, i.e., $\boldsymbol{d}\boldsymbol{\omega} = 0$. We try to seek a $\boldsymbol{\eta} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$ such that for any compact $D \subset \mathbb{R}^d$,

$$\boldsymbol{d}\boldsymbol{\eta} = \boldsymbol{\omega} \quad \text{and} \quad \|\boldsymbol{\eta}\|_{\boldsymbol{H}^{k+1}(D,\Lambda^{l-1})} \le C\,\|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(D,\Lambda^l)} \tag{5.3}$$

Taking the Fourier transform on both sides of the equations $\boldsymbol{d}\boldsymbol{\eta} = \boldsymbol{\omega}$ and $\boldsymbol{d}\boldsymbol{\omega} = 0$, we get from (5.2)

$$\imath\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{\omega}}, \qquad \imath\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\omega}} = 0\,. \tag{5.4}$$

This linear system has a solution given by

$$\widehat{\boldsymbol{\eta}}(\boldsymbol{\xi}) := \frac{-\imath\boldsymbol{\xi} \lrcorner\, \widehat{\boldsymbol{\omega}}(\boldsymbol{\xi})}{|\boldsymbol{\xi}|^2}\,. \tag{5.5}$$

To see this note that $\imath\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}} = 0$ is a direct consequence of $\imath\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\eta}} = \widehat{\boldsymbol{\omega}}$ for whatever $\widehat{\boldsymbol{\eta}}$ is. For the l-form $\widehat{\boldsymbol{\omega}}$, using (5.5) we can write

$$\begin{aligned}
\imath\widehat{\boldsymbol{\xi}} \wedge \widehat{\boldsymbol{\eta}} &= \frac{\widehat{\boldsymbol{\xi}}\wedge(\boldsymbol{\xi} \lrcorner\, \widehat{\boldsymbol{\omega}})}{|\boldsymbol{\xi}|^2} = \sum_J \frac{\widehat{\boldsymbol{\xi}}\wedge(\boldsymbol{\xi} \lrcorner\, \widehat{\boldsymbol{\omega}}_J\boldsymbol{d}\boldsymbol{\xi}_J)}{|\boldsymbol{\xi}|^2} = \sum_J \frac{\widehat{\boldsymbol{\xi}}\wedge(\sum_{k=1}^l (-1)^{k-1}\xi_{j_k}\boldsymbol{d}\boldsymbol{\xi}_{\check{J}_{j_k}})}{|\boldsymbol{\xi}|^2}\boldsymbol{\omega}_J \\
&= \sum_J \frac{\sum_{k=1}^l (-1)^{k-1}\widehat{\boldsymbol{\xi}}\wedge(\xi_{j_k}\boldsymbol{d}\boldsymbol{\xi}_{\check{J}_{j_k}})}{|\boldsymbol{\xi}|^2}\boldsymbol{\omega}_J\,,
\end{aligned}$$

where by $\check{J}_{j_k}$ we mean the the index $j_k, 1 \le k \le l$, is dropped in the increasing l-permutation J with $1 \le j_1 < \ldots, j_l \le d$. Observe that

$$\widehat{\boldsymbol{\xi}}\wedge(\xi_{j_k}\boldsymbol{d\xi}_{\check{J}_{j_k}}) = (-1)^{k-1}\xi_{j_k}^2\boldsymbol{d\xi}_J + \sum_{m\notin J,m=1}^{d}\xi_m\xi_{j_k}\boldsymbol{d\xi}_m\wedge\boldsymbol{d\xi}_{\check{J}_{j_k}}\,.$$

Thus we have

$$\imath\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\eta}} = \sum_J \frac{\sum_{m\in J}\xi_m^2 + \mathfrak{A}}{|\boldsymbol{\xi}|^2}\boldsymbol{\omega}_J\,, \tag{5.6}$$

where

$$\mathfrak{A} = \sum_{k=1}^{l}(-1)^{k-1}\sum_{m\notin J,m=1}^{d}\left(\xi_m\xi_{j_k}\boldsymbol{d\xi}_m\wedge\boldsymbol{d\xi}_{\check{J}_{j_k}}\right).$$

We also have

$$\boldsymbol{\xi}\lrcorner\left(\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}\right) = \sum_J\boldsymbol{\omega}_J\Big(\boldsymbol{\xi}\lrcorner\left(\widehat{\boldsymbol{\xi}}\wedge\boldsymbol{d\xi}_J\right)\Big) = \sum_J\boldsymbol{\omega}_J\left(\sum_{m\notin J}\boldsymbol{\xi}\lrcorner\left(\xi_m\boldsymbol{d}\xi_m\wedge\boldsymbol{d\xi}_J\right)\right).$$

Without loss of generality, we assume that $j_1 < \ldots < j_{i_m} < m < j_{i_m+1} < \ldots < j_l$ for $m \notin J$ and denote $J\cup\{m\}$ by the increasing $l+1$-permutation $\{j_1,\ldots,j_{i_m},m,j_{i_m+1},\ldots,j_l\}$, then

$$\begin{aligned}
\boldsymbol{\xi}\lrcorner\left(\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}\right) &= \sum_J\boldsymbol{\omega}_J\left(\sum_{m\notin J}\boldsymbol{\xi}\lrcorner\left(\xi_m\boldsymbol{d}\xi_m\wedge\boldsymbol{d\xi}_J\right)\right) = \sum_J\boldsymbol{\omega}_J\left(\sum_{m\notin J}(-1)^{i_m}\boldsymbol{\xi}\lrcorner\left(\xi_m\boldsymbol{d\xi}_{J\cup\{m\}}\right)\right)\\
&= \sum_J\boldsymbol{\omega}_J\left\{\sum_{m\notin J}(-1)^{i_m}\left(\sum_{k=1}^{i_m}(-1)^{k-1}\xi_{j_k}\xi_m\boldsymbol{d\xi}_{\check{J}_{j_k}\cup\{m\}} + (-1)^{i_m}\xi_m^2\boldsymbol{d\xi}_J\right.\right.\\
&\qquad\left.\left. + \sum_{k=i_m+1}^{l}(-1)^k\xi_{j_k}\xi_m\boldsymbol{d\xi}_{\check{J}_{j_k}\cup\{m\}}\right)\right\}\\
&= \sum_J\boldsymbol{\omega}_J\left\{\sum_{m\notin J}\xi_m^2\boldsymbol{d\xi}_J + \sum_{m\notin J}(-1)^{i_m}\left((-1)^{i_m-1}\sum_{k=1}^{i_m}(-1)^{k-1}\xi_{j_k}\xi_m\boldsymbol{d\xi}_m\wedge\boldsymbol{d\xi}_{\check{J}_{j_k}}\right)\right.\\
&\qquad\left. + \sum_{m\notin J}(-1)^{i_m}\left((-1)^{i_m}\sum_{k=i_m+1}^{l}(-1)^k\xi_{j_k}\xi_m\boldsymbol{d\xi}_m\wedge\boldsymbol{d\xi}_{\check{J}_{j_k}}\right)\right\}\\
&= \sum_J\boldsymbol{\omega}_J\left\{\sum_{m\notin J}\xi_m^2\boldsymbol{d\xi}_J - \sum_{m\notin J}\left(\sum_{k=1}^{l}(-1)^{k-1}\xi_{j_k}\xi_m\boldsymbol{d\xi}_m\wedge\boldsymbol{d\xi}_{\check{J}_{j_k}}\right)\right\}\\
&= \sum_J\boldsymbol{\omega}_J\left\{\sum_{m\notin J}\xi_m^2\boldsymbol{d\xi}_J - \mathfrak{A}\right\}.
\end{aligned}$$

Solving $\mathfrak{A}$ from the above equation and plugging it into (5.6), we have

$$\imath\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\eta}} = \sum_J\frac{\sum_{m\in J}\xi_m^2 + \sum_{m\notin J}\xi_m^2 - \boldsymbol{\xi}\lrcorner\left(\widehat{\boldsymbol{\xi}}\wedge\widehat{\boldsymbol{\omega}}\right)}{|\boldsymbol{\xi}|^2}\boldsymbol{\omega}_J = \sum_J\frac{|\boldsymbol{\xi}|^2}{|\boldsymbol{\xi}|^2}\boldsymbol{\omega}_J = \widehat{\boldsymbol{\omega}}\,,$$

where we have used the second relation in (5.4). Hence we have shown that $\widehat{\boldsymbol{\eta}}$ is a solution of (5.4).

It remains to show that $\boldsymbol{\eta} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$. We will use the cut-off technique as in the proof of [2, Lemma 3.5]. First we observe that for any increasing l-permutation I and any j with $1 \le j \le d$, we get from (5.5)

$$|\xi_j\widehat{\boldsymbol{\eta}}_I(\boldsymbol{\xi})| \le \sum_J|\widehat{\boldsymbol{\omega}}_J(\boldsymbol{\xi})|\,. \tag{5.7}$$

Appealing to the Fourier representation of Sobolev norms on $\mathbb{R}^d$, we can conclude $\frac{\partial \boldsymbol{\eta}_I}{\partial x_j} \in H^k(\mathbb{R}^d)$ for all combinations of I and j.

Next we can choose a cut-off function $\psi \in C_0^\infty(\mathbb{R}^d)$ with $\psi(\boldsymbol{\xi}) = 1$ for $|\boldsymbol{\xi}| \leq 1$, and $\psi(\boldsymbol{\xi}) = 0$ for $|\boldsymbol{\xi}| \geq 2$. Then split $\widehat{\boldsymbol{\omega}}$ according to

$$\widehat{\boldsymbol{\eta}}(\boldsymbol{\xi}) = \psi(\boldsymbol{\xi})\widehat{\boldsymbol{\eta}}(\boldsymbol{\xi}) + (1 - \psi(\boldsymbol{\xi}))\widehat{\boldsymbol{\eta}}(\boldsymbol{\xi}) \,. \tag{5.8}$$

Note that each component of the differential form $\psi(\boldsymbol{\xi})\widehat{\boldsymbol{\eta}}(\boldsymbol{\xi})$ has a compact support and belongs to $L^1(\mathbb{R}^d)$ ($d \geq 3!$), so that its inverse Fourier transform is analytic. Hence, the restriction of $\mathscr{F}^{-1}(\psi(\cdot)\widehat{\boldsymbol{\eta}}(\cdot))$ to any compact $D \subset \mathbb{R}^d$ belongs to $H^m(D)$ for any $m \in \mathbb{N}_0$. It goes without saying that the inverse Fourier transform of the second term $(1-\psi(\boldsymbol{\xi}))\widehat{\boldsymbol{\eta}}(\boldsymbol{\xi})$ yields a form in $\boldsymbol{H}^k(\mathbb{R}^d, \Lambda^{l-1})$ Summing up, we have shown that $\mathscr{F}^{-1}(\widehat{\boldsymbol{\eta}}(\cdot)) \in \boldsymbol{H}^k_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$. This completes our proof. $\square$

The following theorem is a fairly straightforward generalization of the regular decomposition lemma [20, Lemma 2.4]:

Theorem 5.3. *(Lifted regular decompositions) For every $k \in \mathbb{N}_0$, $1 \leq l \leq d$, there exist continuous maps $\mathsf{R} : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}(\Omega, \Lambda^l)$ and $\mathsf{N} : \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}(\Omega, \Lambda^{l-1})$ such that*

$$\mathsf{R} + \boldsymbol{d} \circ \mathsf{N} = Id \qquad \text{on } \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l) \,. \tag{5.9}$$

In addition, there are continuous maps $\mathsf{R}_0 : \boldsymbol{H}^k_0(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}_0(\Omega, \Lambda^l)$ and $\mathsf{N}_0 : \boldsymbol{H}^k_0(\boldsymbol{d}, \Omega, \Lambda^l) \mapsto \boldsymbol{H}^{k+1}_0(\Omega, \Lambda^{l-1})$ such that

$$\mathsf{R}_0 + \boldsymbol{d} \circ \mathsf{N}_0 = Id \qquad \text{on } \boldsymbol{H}^k_0(\boldsymbol{d}, \Omega, \Lambda^l) \,. \tag{5.10}$$

Proof. (i) Proof of (5.9): Pick $\boldsymbol{\omega} \in \boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^l)$ and extend it to $\widetilde{\boldsymbol{\omega}} \in \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ using the universal extension of Theorem 3.6. Then set

$$\mathsf{R}\boldsymbol{\omega} := (\mathscr{L}\boldsymbol{d}\widetilde{\boldsymbol{\omega}})|_\Omega \,, \quad \mathsf{N}\boldsymbol{\omega} := \mathscr{L}(\widetilde{\boldsymbol{\omega}} - \mathscr{L}\boldsymbol{d}\widetilde{\boldsymbol{\omega}})|_\Omega \,. \tag{5.11}$$

It is easy to check that $\boldsymbol{d}(\widetilde{\boldsymbol{\omega}} - \mathsf{R}\widetilde{\boldsymbol{\omega}}) = 0$ in Ω in view of (5.1). The continuity properties of these operators and (5.9) are straightforward from Lemma 5.1.

(ii) Proof of (5.10), *cf.* proof of Lemma 2.4 of [20]: For any $\boldsymbol{\mu} \in \boldsymbol{H}^k_0(\boldsymbol{d}, \Omega, \Lambda^l)$, let us extend it by zero to $\widetilde{\boldsymbol{\mu}} \in \boldsymbol{H}^k(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ and define $\widetilde{\boldsymbol{\omega}} = \boldsymbol{d}\widetilde{\boldsymbol{\mu}} \in \boldsymbol{H}^k(\boldsymbol{d}0, \mathbb{R}^d, \Lambda^{l+1})$. There exists from Lemma 5.1 $\boldsymbol{\eta} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^l)$ and $\widetilde{\boldsymbol{\omega}} = \boldsymbol{d}\boldsymbol{\eta}$, which implies that $\boldsymbol{d}(\widetilde{\boldsymbol{\mu}} - \boldsymbol{\eta}) = 0$. Applying Lemma 5.1 again yields $\boldsymbol{\rho} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^l)$ satisfying $\widetilde{\boldsymbol{\mu}} - \boldsymbol{\eta} = \boldsymbol{d}\boldsymbol{\rho}$. Using the fact that $\widetilde{\boldsymbol{\mu}} = 0 \quad \text{in } \mathbb{R}^d \setminus \Omega$ leads to $\boldsymbol{\rho} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\boldsymbol{d}, \mathbb{R}^d \setminus \Omega, \Lambda^l)$. Use Theorem 3.6 to extend $\boldsymbol{\rho}|_{\mathbb{R}^d \setminus \Omega}$ into the *interior* of Ω and write $\widetilde{\boldsymbol{\rho}} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\boldsymbol{d}, \mathbb{R}^d, \Lambda^l)$ for the extension. Then define

$$\mathsf{R}_0\boldsymbol{\mu} := \boldsymbol{\eta} + \boldsymbol{d}\widetilde{\boldsymbol{\rho}}_{|\Omega} \in \boldsymbol{H}^{k+1}(\Omega, \Lambda^l) \quad , \quad \mathsf{N}_0\boldsymbol{\mu} := \boldsymbol{\rho} - \widetilde{\boldsymbol{\rho}} \in \boldsymbol{H}^{k+1}(\Omega, \Lambda^{l-1}) \,.$$

The identity (5.10) is a consequence of the construction. Continuity of the extension translates into the asserted continuity properties of the operators. Finally, note that $\boldsymbol{\rho} - \widetilde{\boldsymbol{\rho}} = 0$ on $\mathbb{R}^d \setminus \Omega$. In light of $\boldsymbol{\rho} - \widetilde{\boldsymbol{\rho}} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^{l-1})$ this implies the homogeneous boundary conditions for $\mathsf{N}_0\boldsymbol{\mu}$. Similarly, $\boldsymbol{\eta} + \boldsymbol{d}\widetilde{\boldsymbol{\rho}} = 0$ in $\mathbb{R}^d \setminus \Omega$, and $\boldsymbol{\eta} + \boldsymbol{d}\widetilde{\boldsymbol{\rho}} \in \boldsymbol{H}^{k+1}_{\mathrm{loc}}(\mathbb{R}^d, \Lambda^l)$ means that $\mathsf{R}_0\boldsymbol{\mu} \in \boldsymbol{H}^{k+1}_0(\Omega, \Lambda^l)$. $\square$

Let us recall the classical version of a lifting lemma important in the analysis of the Navier-Stokes equations, see Corollary 2.4 and Lemma 2.2 in [17].

Lemma 5.4 (Classical lifting Lemma)**.** *Assuming that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then*

1. *there exists a positive constant C such that for all $p \in L^2(\Omega)$ there is a $\mathbf{v} \in \boldsymbol{H}^1(\Omega)$ satisfying*

$$\operatorname{div} \mathbf{v} = p \quad \textit{and} \quad \|\mathbf{v}\|_{H^1(\Omega)} \leq C \, \|p\|_{L^2(\Omega)} \,; \tag{5.12}$$

2. *if $\int_\Omega p \; \mathrm{d}\mathbf{x} = 0$, then there exists a $\mathbf{v} \in \boldsymbol{H}^1_0(\Omega)$ such that* (5.12) *holds.*

It is remarked that the original proof due to Nečas establishes an equivalent assertion by showing that the range space of $\mathbf{grad}$, the adjoint operator of div, is closed in $\boldsymbol{H}_0^1(\Omega)$, which is the dual space of $\boldsymbol{H}^{-1}(\Omega)$ (cf. [30]). Nečas' proof is rather lengthy and quite complicated for the technical treatment of Lipschitz boundary. The lemma is of crucial importance for the treatment of the constraints of compressibility and incompressibility in mechanics and fluid mechanics. With the regular decomposition lemma we have established earlier, a generalized version of Lemma 5.4 lemma can be deduced easily.

Corollary 5.5 (General lifting lemma)**.** *Let $k \in \mathbb{N}_0$ and $1 \le l \le d$. For a bounded Lipschitz domain $\Omega \in \mathbb{R}^d$ of full topological generality and all $\boldsymbol{\omega} \in \boldsymbol{d}\boldsymbol{H}^k(\boldsymbol{d}, \Omega, \Lambda^{l-1})$, then there is a $\boldsymbol{\eta} \in \boldsymbol{H}^{k+1}(\Omega, \Lambda^{l-1})$ and a positive constant C independent of $\boldsymbol{\eta}$ such that*

$$\boldsymbol{d}\boldsymbol{\eta} = \boldsymbol{\omega}, \tag{5.13}$$

$$\|\boldsymbol{\eta}\|_{\boldsymbol{H}^{k+1}(\Omega,\Lambda^{l-1})} \le C \|\boldsymbol{\omega}\|_{\boldsymbol{H}^k(\Omega,\Lambda^l)} . \tag{5.14}$$

Moreover, for all $\boldsymbol{\omega} \in \boldsymbol{d}\boldsymbol{H}_0^s(\boldsymbol{d}, \Omega, \Lambda^{l-1})$ for $1 \le l < d$, and $\int_\Omega \boldsymbol{\omega} = 0$ if $l = d$, there is a $\boldsymbol{\eta} \in \boldsymbol{H}_0^{s+1}(\Omega, \Lambda^{l-1})$ and a positive constant C independent of $\boldsymbol{\eta}$ such that (5.13) *and* (5.14) *holds.*

Proof of Corollary 5.5. By Theorem 5.3, we prove the desired result by defining $\boldsymbol{\eta} = \mathsf{R}\boldsymbol{\omega}$ or $\boldsymbol{\eta} = \mathsf{R}_0\boldsymbol{\omega}$, which show the first and second parts, respectively. $\square$

It is natural to derive from Corollary 5.5 a similar result for the $\mathbf{curl}$ operator to Lemma 5.4.

Corollary 5.6. *Assuming that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, then*

1. *there exists a positive constant C such that for all $\mathbf{v} \in \mathbf{curl}\,\boldsymbol{H}(\mathbf{curl};\Omega)$, one can find $\mathbf{u} \in \boldsymbol{H}^1(\Omega)$ satisfying*
$$\mathbf{curl}\,\mathbf{u} = \mathbf{v} \quad \textit{and} \quad \|\mathbf{u}\|_{\boldsymbol{H}^1(\Omega)} \le C \|\mathbf{v}\|_{\boldsymbol{L}^2(\Omega)} ; \tag{5.15}$$
2. *if $\mathbf{v} \in \mathbf{curl}\,\boldsymbol{H}_0(\mathbf{curl};\Omega)$, we can find a $\mathbf{u} \in \boldsymbol{H}_0^1(\Omega)$ such that* (5.15) *holds.*

Remark 5.1. Theorem 5.3 also holds for $d = 2$, albeit with a different proof invoking analyticity of Fourier transforms, *cf.* [17, Sect. I.3.1].

Remark 5.2. We acknowledge that Theorem 5.3 can also be deduced from [13, Theorem 4.6] by appealing to its assertions on so-called Bogovskii operators.

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