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Research Report No. 2009-20
May 2009

Seminar für Angewandte Mathematik
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PLANE WAVE DISCONTINUOUS GALERKIN METHODS FOR THE 2D HELMHOLTZ EQUATION: ANALYSIS OF THE P–VERSION

R. HIPTMAIR*, A. MOIOLA†, AND I. PERUGIA‡

Abstract. Plane wave discontinuous Galerkin methods (PWDG) are a class of Trefftz-type methods for the spatial discretization of boundary value problems for the Helmholtz operator $-\Delta - \omega^2$, $\omega > 0$. They include the so-called ultra weak variational formulation from [O. Cessenat and B. Després, *Application of an ultra weak variational formulation of elliptic PDEs to the two-dimensional Helmholtz equation*, SIAM J. Numer. Anal., 35 (1998), pp. 255–299].

This paper is concerned with the a priori convergence analysis of PWDG in the case of p-refinement, that is, the study of the asymptotic behavior of relevant error norms as the number of plane wave directions in the local trial spaces is increased. For convex domains in two space dimensions, we derive convergence rates, employing mesh skeleton based norms, duality techniques from [P. Monk and D. Wang, *A least squares method for the Helmholtz equation*, Computer Methods in Applied Mechanics and Engineering, 175 (1999), pp. 121–136], and plane wave approximation theory.

Key words. Helmholtz equation, wave propagation, discontinuous Galerkin (DG) methods, plane waves, p–version error analysis, duality estimates

AMS subject classifications. 65N15, 65N30, 35J05

1. Introduction.

Standard low order Lagrangian finite element discretizations of boundary value problems for the Helmholtz equation $-\Delta u - \omega^2 u = f$ are afflicted with the so-called pollution phenomenon [6]: though for sufficiently small ωh, h being the mesh size, an accurate approximation of u is possible, the Galerkin procedure fails to provide it. Attempts to remedy this have focused on incorporating extra information in the form of plane wave functions $\boldsymbol{x} \mapsto \exp(i\omega \boldsymbol{d} \cdot \boldsymbol{x})$, $|\boldsymbol{d}| = 1$, into the trial spaces. Prominent examples of such methods are the plane wave partition of unity finite element method of Babuška and Melenk [5], see also [28], the discontinuous enrichment method [4, 30, 13], the VCTR (variational theory of complex rays) [29], and the ultra weak Galerkin discretization due to Cessenat and Despres [11], see also [24, 23, 22, 20, 21] for further discussion and applications.

It turns out that the latter method can be recast as a special discontinuous Galerkin (DG) method employing local trial spaces spanned by a few plane waves, as pointed out in [10, 15, 14]. In a sense, this is a special case of a Trefftz-type approximation, as the local trial functions are solutions of the homogeneous Helmholtz equation $-\Delta u - \omega^2 u = 0$. This perspective paves the way for marrying plane wave approximation with many of the various DG methods developed for 2nd-order elliptic boundary value problems. We have pursued this in [15,19,26] for a class of primal and mixed DG methods, which generalize the ultra-weak scheme, and which differ from each other in the choice of the numerical fluxes; we refer to these methods as "plane wave discontinuous Galerkin methods", PWDG, in short.

In particular, in [15], an h–version error analysis for the PWDG method applied to the 2D inhomogeneous Helmholtz problem was carried out. In that case, independently of how many plane waves are used in the local approximation spaces, only first order convergence can be achieved in general. The analysis was restricted to a class of PWDG methods with flux parameters depending on the product ωh (not including

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the classical ultra weak variational formulation of [11]). Key elements of this analysis are local approximation estimates and inverse estimates for plane waves, and a duality technique. This involves estimating the approximation error of the solution of an inhomogeneous dual problem by plane waves. High-order convergence as $h \to 0$ can be expected in the homogeneous case $f \equiv 0$ only [26].

The application of a duality argument in the error analysis entails a threshold condition on the mesh size: quasi-optimality of the PWDG solution is guaranteed only if $\omega^2 h$ is "sufficiently small", see [15, Theorem 4.10]. In numerical experiments this is observed as a widening gap between discretization error and plane wave best approximation error. Thus, the notorious pollution effect that haunts local discretizations of wave propagation problems manifests itself in the theoretical estimates.

For polynomial schemes, their p–versions, also called the spectral approach, are immune to the pollution effect [1, 2, 3]. Thus, we believe that the spectral version/p–version of PWDG, which strives for better accuracy by enlarging the local trial spaces, will also possess this desirable property. Besides, practical experience suggests that (well balanced) p–refinement is highly advisable [24]. Since aspects of implementation are not covered in this paper, we will gloss over the notorious issue of ill-conditioned linear systems arising from spectral PWDG approaches.

The aim of this paper is the derivation of p–version error estimates for the PWDG method applied to the two-dimensional homogeneous Helmholtz equation. The used approach has little in common with the duality techniques pursued in [15,26], because p–refinement does not yield any useful approximation of the solution of the inhomogeneous dual problem, since plane waves fail to approximate general functions.

Moreover, we cannot rely on coercivity in seminorm of the bilinear form defining the PWDG method for general functions. Instead, we consider a weaker skeleton-based energy norm (i.e., containing interelement jump terms and boundary terms only) which is a norm on the space of local Trefftz' functions only. We prove a coercivity result in this norm. This grants more freedom in the choice of the flux parameters; in particular, constant flux parameters are allowed so that also the classical ultra weak variational formulation of [11] is covered by our analysis.

Our argument is based on an estimate of the L^2–norm of Trefftz' functions by their skeleton-based norm, which was discovered in the context of least squares Trefftz methods in [27]. We re-derive this estimate in order to establish the dependence of the constants in front of the estimate explicitly not only on the mesh width h, but also on the wave number ω. In parts, the analysis is carried out along the lines of [10]. On the other hand, we do not rewrite the PWDG bilinear form in terms of impedance traces, but stay closer to the DG setting and our arguments are substantially simpler than those of [10].

We point out that the constant in front of the final p–version error estimates depends on the product ωh. This is inevitable, because no accuracy can be expected unless the underlying wavelength is resolved by the trial space. Yet, in contrast to the h–version estimates of [10, Sect. 4], the error bounds do not hinge on the assumption that ωh is "sufficiently small".

The abstract analysis contained in this paper could be extended directly to the 3D case. Also the elliptic regularity and the stability estimates for the inhomogeneous Helmholtz problem, obtained in [25] in the 2D case, have been proved also in 3D in [12] and [17]. What is still missing are p–version projection error estimates for functions satisfying the homogeneous Helmholtz equation onto plane wave spaces in 3D. While for the h–version these estimates obtained in 2D in [26] have already been extended to

3D in [18] by using a different argument which considers both the 2D and 3D cases at once, the 3D case for the p–version is still under investigation. As soon as this result will be available, our p–version analysis will cover the 3D case.

The outline of this paper is as follows: In Section 2, we report the derivation of the PWDG method for the homogeneous Helmholtz equation with impedance boundary conditions. Next, we derive p–version error estimates in Section 3. They hinge on certain assumptions on the domain, mesh, and trial spaces, in particular the uniformity of element sizes. In Section 3.1 we state a coercivity property and continuity of the PWDG bilinear form. Then, we prove quasi-optimality of the approximation error in a mesh skeleton-based norm and derive a bound for the L^2–norm of the error in Section 3.2. There we follow [27] and [10]. Subsequently, from an approximation result proved in [18], we derive best approximation estimates in the skeleton-based norm in Section 3.3. We continue by stating energy-norm error estimates in Section 3.4. In Section 3.5, we derive error estimates in a stronger norm, also containing the difference between the gradient of the analytical solution and the gradient of a (computable) projection of the PWDG solution. The final section studies the PWDG discretization error numerically for some model problems.

2. The PWDG Method. In this section, we introduce the plane wave discontinuous Galerkin (PWDG) method for the homogeneous Helmholtz equation, following [15].

Assume Ω to be a bounded Lipschitz domain in $\mathbb{R}^d$, $d = 2, 3$ (since the abstract error analysis developed in Section 3 below holds true in both 2D and 3D, we do not restrict to 2D at this point). Consider the Helmholtz boundary value problem

$$\begin{aligned} -\Delta u - \omega^2 u &= 0 && \text{in } \Omega\,, \\ \nabla u \cdot \boldsymbol{n} + i\omega\, u &= g && \text{on } \partial\Omega\,. \end{aligned} \tag{2.1}$$

Here, $\omega > 0$ is a fixed wave number (the corresponding wavelength is $\lambda = 2\pi/\omega$), $\boldsymbol{n}$ is the outer normal unit vector to $\partial\Omega$, and i is the imaginary unit. Inhomogeneous first order absorbing boundary conditions in the form of impedance boundary conditions are used in (2.1), with boundary data $g \in L^2(\partial\Omega)$.

Let $\mathcal{T}_h$ be a finite element partition of Ω, with possible hanging nodes, of mesh width h (i.e, $h = \max_{K\in\mathcal{T}_h} h_K$, with $h_K := \operatorname{diam}(K)$) on which we define our PWDG method, and let $V_p(\mathcal{T}_h)$ be the plane wave space on $\mathcal{T}_h$, locally spanned by p plane waves per element, $p \in \mathbb{N}$:

$$V_p(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K(\boldsymbol{x}) = \sum_{j=1}^{p} \alpha_j \exp(i\omega \boldsymbol{d}_j \cdot \boldsymbol{x}),\ \alpha_j \in \mathbb{C},\ \forall K \in \mathcal{T}_h\}\,,$$

where $\boldsymbol{d}_j$, $|\boldsymbol{d}_j| = 1$, are p different directions.

In the p–version setting, we assume the mesh $\mathcal{T}_h$ to be fixed, and we only vary p.

Further assumptions on the problem domain, on the mesh $\mathcal{T}_h$ and on the approximation spaces $V_p(\mathcal{T}_h)$ will be made precise at the beginning of Section 3 and in Section 3.3.

In order to derive the PWDG method, we start by writing problem (2.1) as a first order system:

$$\begin{aligned} i\omega\, \boldsymbol{\sigma} &= \nabla u && \text{in } \Omega\,, \\ i\omega\, u - \nabla \cdot \boldsymbol{\sigma} &= 0 && \text{in } \Omega\,, \\ i\omega \boldsymbol{\sigma} \cdot \boldsymbol{n} + i\omega\, u &= g && \text{on } \partial\Omega\,. \end{aligned} \tag{2.2}$$

By multiplying the first and second equation of (2.2) by smooth test functions $\boldsymbol{\tau}$ and v, respectively, and integrating by parts on each $K \in \mathcal{T}_h$, we obtain

$$\begin{aligned} &\int_K i\omega\, \boldsymbol{\sigma}\cdot\overline{\boldsymbol{\tau}}\,\mathrm{d}V + \int_K u\,\overline{\nabla\cdot\boldsymbol{\tau}}\,\mathrm{d}V - \int_{\partial K} u\,\overline{\boldsymbol{\tau}\cdot\boldsymbol{n}}\,\mathrm{d}S = 0 \quad \forall \boldsymbol{\tau}\in\mathbf{H}(\mathrm{div};K)\,,\\ &\int_K i\omega\, u\,\overline{v}\,\mathrm{d}V + \int_K \boldsymbol{\sigma}\cdot\overline{\nabla v}\,\mathrm{d}V - \int_{\partial K} \boldsymbol{\sigma}\cdot\boldsymbol{n}\,\overline{v}\,\mathrm{d}S = 0 \quad \forall v\in H^1(K)\,. \end{aligned} \tag{2.3}$$

Replace u, v by $u_p, v_p \in V_p(\mathcal{T}_h)$ and $\boldsymbol{\sigma}, \boldsymbol{\tau}$ by $\boldsymbol{\sigma}_p, \boldsymbol{\tau}_p \in V_p(\mathcal{T}_h)^d$. Then, approximate the traces of u and $\boldsymbol{\sigma}$ across interelement boundaries by the so-called *numerical fluxes* denoted by $\widehat{u}_p$ and $\widehat{\boldsymbol{\sigma}}_p$, respectively, and obtain

$$\begin{aligned} &\int_K i\omega\, \boldsymbol{\sigma}_p\cdot\overline{\boldsymbol{\tau}}_p\,\mathrm{d}V + \int_K u_p\,\overline{\nabla\cdot\boldsymbol{\tau}}_p\,\mathrm{d}V - \int_{\partial K} \widehat{u}_p\,\overline{\boldsymbol{\tau}_p\cdot\boldsymbol{n}}\,\mathrm{d}S = 0 \quad \forall \boldsymbol{\tau}_p\in V_p(\mathcal{T}_h)(K)^d\,,\\ &\int_K i\omega\, u_p\,\overline{v}_p\,\mathrm{d}V + \int_K \boldsymbol{\sigma}_p\cdot\overline{\nabla v}_p\,\mathrm{d}V - \int_{\partial K} \widehat{\boldsymbol{\sigma}}_p\cdot\boldsymbol{n}\,\overline{v}_p\,\mathrm{d}S = 0 \quad \forall v_p\in V_p(\mathcal{T}_h)(K)\,. \end{aligned} \tag{2.4}$$

The numerical fluxes will be defined below; they also take into account the inhomogeneous boundary conditions.

Integrating again by parts the first equation of (2.4), we obtain

$$\int_K \boldsymbol{\sigma}_p\cdot\overline{\boldsymbol{\tau}}_p\,\mathrm{d}V = \frac{1}{i\omega}\int_K \nabla u_p\cdot\overline{\boldsymbol{\tau}}_p\,\mathrm{d}V - \frac{1}{i\omega}\int_{\partial K}(u_p-\widehat{u}_p)\,\overline{\boldsymbol{\tau}_p\cdot\boldsymbol{n}}\,\mathrm{d}S\,. \tag{2.5}$$

Since $\nabla_h V_p(\mathcal{T}_h) \subseteq V_p(\mathcal{T}_h)^d$, we can take $\boldsymbol{\tau}_p = \nabla v_p$ in each element. Inserting the resulting expression for $\int_K \boldsymbol{\sigma}_p\cdot\overline{\nabla v}_p\,\mathrm{d}V$ into the second equation of (2.4), we arrive at

$$\int_K (\nabla u_p\cdot\overline{\nabla v}_p - \omega^2 u_p\overline{v}_p)\,\mathrm{d}V - \int_{\partial K}(u_p-\widehat{u}_p)\,\overline{\nabla v_p\cdot\boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega\widehat{\boldsymbol{\sigma}}_p\cdot\boldsymbol{n}\,\overline{v}_p\,\mathrm{d}S = 0\,. \tag{2.6}$$

Notice that the formulation (2.6) is equivalent to (2.4) in the sense that their u_p solution components coincide and the $\boldsymbol{\sigma}_p$ solution component of (2.4) can be recovered from u_p by using (2.5).

Another equivalent formulation can be obtained by integrating by parts once more the first term in (2.6) (notice that the boundary term appearing in this integration by parts cancels out with a boundary term already present in (2.6)):

$$\int_K \overline{(-\Delta v_p - \omega^2 v_p)}\,u_p\,\mathrm{d}V + \int_{\partial K}\widehat{u}_p\,\overline{\nabla v_p\cdot\boldsymbol{n}}\,\mathrm{d}S - \int_{\partial K} i\omega\widehat{\boldsymbol{\sigma}}_p\cdot\boldsymbol{n}\,\overline{v}_p\,\mathrm{d}S = 0\,. \tag{2.7}$$

Since $v_p \in V_p(\mathcal{T}_h)$, the volume term in (2.7) vanishes and, adding over all elements $K \in \mathcal{T}_h$, we obtain the PWDG formulation: find $u_p \in V_p(\mathcal{T}_h)$ such that, for all $v_p \in V_p(\mathcal{T}_h)$,

$$\begin{aligned} &\int_{\mathcal{F}_h^I} \left(\widehat{u}_p\,[\![\overline{\nabla_h v_P}]\!]_N - i\omega\,\widehat{\boldsymbol{\sigma}}_p\cdot[\![\overline{v}_p]\!]_N\right)\mathrm{d}S\\ &\qquad + \int_{\mathcal{F}_h^B}\left(\widehat{u}_p\,\overline{\nabla_h v_P\cdot\boldsymbol{n}}\,\mathrm{d}S - i\omega\,\widehat{\boldsymbol{\sigma}}_p\cdot\boldsymbol{n}\,\overline{v}_p\right)\mathrm{d}S = 0\,. \end{aligned}$$

In order to define the numerical fluxes, we recall some standard DG notation. Write $\boldsymbol{n}^+$, $\boldsymbol{n}^-$ for the exterior unit normals on ∂K^+ and ∂K^-, respectively. Let u_p and $\boldsymbol{\sigma}_p$ be a piecewise smooth function and vector field on $\mathcal{T}_h$, respectively. On $\partial K^- \cap \partial K^+$, we define

$$\begin{aligned} \text{the averages:} \quad & \{\!\{u_p\}\!\} := \tfrac{1}{2}(u_p^+ + u_p^-) \quad , \quad \{\!\{\boldsymbol{\sigma}_p\}\!\} := \tfrac{1}{2}(\boldsymbol{\sigma}_p^+ + \boldsymbol{\sigma}_p^-) \, , \\ \text{the jumps:} \quad & [\![u_p]\!]_N := u_p^+ \boldsymbol{n}^+ + u_p^- \boldsymbol{n}^- \quad , \quad [\![\boldsymbol{\sigma}_p]\!]_N := \boldsymbol{\sigma}_p^+ \cdot \boldsymbol{n}^+ + \boldsymbol{\sigma}_p^- \cdot \boldsymbol{n}^- \, . \end{aligned}$$

We define the PWDG fluxes by setting

$$\begin{cases} \widehat{\boldsymbol{\sigma}}_p = \dfrac{1}{i\omega} \{\!\{\nabla_h u_p\}\!\} - \alpha \, [\![u_p]\!]_N \, , \\ \widehat{u}_p = \{\!\{u_p\}\!\} - \beta \, \dfrac{1}{i\omega} [\![\nabla_h u_p]\!]_N \, , \end{cases}$$

on interior faces, and

$$\begin{cases} \widehat{\boldsymbol{\sigma}}_p = \dfrac{1}{i\omega} \nabla_h u_p - (1-\delta) \left(\dfrac{1}{i\omega} \nabla_h u_p + u_p \boldsymbol{n} - \dfrac{1}{i\omega} \, g \boldsymbol{n} \right) , \\ \widehat{u}_p = u_p - \delta \left(\dfrac{1}{i\omega} \nabla_h u_p \cdot \boldsymbol{n} + u_p - \dfrac{1}{i\omega} \, g \right) , \end{cases}$$

on boundary faces, where the parameters α, β and δ are the so-called flux parameters; assumption on them will be made in Section 3 below.

With these definitions, we can write the PWDG method as follows: find $u_p \in V_p(\mathcal{T}_h)$ such that, for all $v_p \in V_p(\mathcal{T}_h)$,

$$\mathcal{A}_h(u_p, v_p) = \ell_h(v_p) \, ,$$

where

$$\begin{aligned} \mathcal{A}_h(u,v) = & \int_{\mathcal{F}_h^I} \{\!\{u\}\!\} [\![\overline{\nabla_h v}]\!]_N \, \mathrm{d}S + i\omega^{-1} \int_{\mathcal{F}_h^I} \beta \, [\![\nabla_h u]\!]_N [\![\overline{\nabla_h v}]\!]_N \, \mathrm{d}S \\ & - \int_{\mathcal{F}_h^I} \{\!\{\nabla_h u\}\!\} \cdot [\![\overline{v}]\!]_N \, \mathrm{d}S + i\omega \int_{\mathcal{F}_h^I} \alpha \, [\![u]\!]_N \cdot [\![\overline{v}]\!]_N \, \mathrm{d}S \\ & + \int_{\mathcal{F}_h^B} (1-\delta) \, u \, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S + i\omega^{-1} \int_{\mathcal{F}_h^B} \delta \, \nabla_h u \cdot \boldsymbol{n} \, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S \\ & - \int_{\mathcal{F}_h^B} \delta \, \nabla_h u \cdot \boldsymbol{n} \, \overline{v} \, \mathrm{d}S + i\omega \int_{\mathcal{F}_h^B} (1-\delta) u \, \overline{v} \, \mathrm{d}S \, , \end{aligned}$$

and

$$\ell_h(v) = i\omega^{-1} \int_{\mathcal{F}_h^B} \delta \, g \, \overline{\nabla_h v \cdot \boldsymbol{n}} \, \mathrm{d}S + \int_{\mathcal{F}_h^B} (1-\delta) g \, \overline{v} \, \mathrm{d}S \, .$$

The PWDG formulation is consistent by construction; thus, if $u \in H^2(\Omega)$ solves (2.1), then it holds

$$\mathcal{A}_h(u, v_p) = \ell_h(v_p) \quad \forall v_p \in V_p(\mathcal{T}_h) \, . \tag{2.8}$$

3. Error Analysis. We develop our a priori error analysis under the following additional assumptions:

Assumptions on the problem domain:

- Ω is convex.

Assumptions on the mesh:

- each element K of $\mathcal{T}_h$ is a convex Lipschitz domain;
- there exists a constant $\rho \in (0,1)$ such that each element $K \in \mathcal{T}_h$ contains a ball of radius $\rho\, h_K$ (shape regularity);
- there exists a constant $\mu \in (0,1)$ such that, for each $K \in \mathcal{T}_h$, $h_K \geq \mu\, h$ (quasi-uniformity).

Assumptions on the flux parameters:

- α, β and δ are real, strictly positive, independent of p, h and ω, with $\delta \in (0,1)$.

REMARK 3.1. *A choice of flux parameters that depends on p and on the product $\omega\, h$, in the spirit of standard DG methods and of the PWDG method of [15] will be discussed in Remark 3.15 below. The choice $\alpha = \beta = \delta = 1/2$ gives rise to the original ultra weak variational formulation by Cessenat and Despres (see [11] and [10]).*

Define the broken Sobolev spaces

$$H^s(\mathcal{T}_h) = \{w \in L^2(\Omega) \,:\, w_{|_K} \in H^s(K)\ \forall K \in \mathcal{T}_h\}\,.$$

Let $T(\mathcal{T}_h)$ be the piecewise Trefftz' space defined on $\mathcal{T}_h$ by

$$T(\mathcal{T}_h) = \{w \in H^2(\mathcal{T}_h) \,:\, \Delta w + \omega^2 w = 0 \text{ in each } K \in \mathcal{T}_h\}\,,$$

and endow it with the norm (see Proposition 3.2 below)

$$\begin{aligned} |||w|||^2_{\mathcal{F}_h} =& \omega^{-1} \|\beta^{1/2} [\![\nabla_h w]\!]_N\|^2_{0,\mathcal{F}_h^I} + \omega\, \|\alpha^{1/2} [\![w]\!]_N\|^2_{0,\mathcal{F}_h^I} \\ &+ \omega^{-1} \|\delta^{1/2} \nabla_h w \cdot \boldsymbol{n}\|^2_{0,\mathcal{F}_h^B} + \omega\, \|(1-\delta)^{1/2} w\|^2_{0,\mathcal{F}_h^B}\,. \end{aligned} \tag{3.1}$$

In the following, we will also make use of the augmented norm

$$\begin{aligned} |||w|||^2_{\mathcal{F}_h^+} =& |||w|||^2_{\mathcal{F}_h} + \omega\, \|\beta^{-1/2} \{\!\{w\}\!\}\|^2_{0,\mathcal{F}_h^I} \\ &+ \omega^{-1} \|\alpha^{-1/2} \{\!\{\nabla_h w\}\!\}\|^2_{0,\mathcal{F}_h^I} + \omega\, \|\delta^{-1/2} w\|^2_{0,\mathcal{F}_h^B}\,. \end{aligned} \tag{3.2}$$

3.1. Auxiliary Results. Here we collect technical prerequisites for the convergence analysis.

PROPOSITION 3.2. *The seminorm* (3.1) *is actually a norm on $T(\mathcal{T}_h)$.*

Proof. Let $w \in T(\mathcal{T}_h)$ be such that $|||w|||^2_{\mathcal{F}_h} = 0$. Then $w \in H^2(\Omega)$ and satisfies $\Delta w + \omega^2 w = 0$ in Ω and $\nabla w \cdot \boldsymbol{n} + i\omega u = 0$ on $\partial\Omega$, which implies $w = 0$. ∎

PROPOSITION 3.3. *If $w \in T(\mathcal{T}_h)$, then*

$$\text{Im}\ [\mathcal{A}_h(w,w)] = |||w|||^2_{\mathcal{F}_h}\,.$$

Proof. Provided that $u, v \in T(\mathcal{T}_h)$, local integration by parts permits us to rewrite the bilinear form $\mathcal{A}_h(u,v)$ as

$$\begin{aligned}\mathcal{A}_h(u,v) =&(\nabla_h u, \nabla_h v)_{0,\Omega} - \int_{\mathcal{F}_h^{\mathcal{I}}} [\![u]\!]_N \cdot \{\!\{\overline{\nabla_h v}\}\!\}\,\mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{I}}} \{\!\{\nabla_h u\}\!\} \cdot [\![\overline{v}]\!]_N\,\mathrm{d}S \\ &- \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, u\, \overline{\nabla_h v \cdot \boldsymbol{n}}\,\mathrm{d}S - \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, \nabla_h u \cdot \boldsymbol{n}\, \overline{v}\,\mathrm{d}S \\ &+ i\omega^{-1} \int_{\mathcal{F}_h^{\mathcal{I}}} \beta [\![\nabla_h u]\!]_N [\![\overline{\nabla_h v}]\!]_N\,\mathrm{d}S + i\omega^{-1} \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, \nabla_h u \cdot \boldsymbol{n}\, \overline{\nabla_h v \cdot \boldsymbol{n}}\,\mathrm{d}S \\ &+ i\omega \int_{\mathcal{F}_h^{\mathcal{I}}} \alpha\, [\![u]\!]_N \cdot [\![\overline{v}]\!]_N\,\mathrm{d}S + i\omega \int_{\mathcal{F}_h^{\mathcal{B}}} (1-\delta)\, u\, \overline{v}\,\mathrm{d}S - \omega^2 (u,v)_{0,\Omega}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{A}_h(w,w) =&\|\nabla_h w\|_{0,\Omega}^2 - 2\,\mathrm{Re}\left[\int_{\mathcal{F}_h^{\mathcal{I}}} [\![w]\!]_N \cdot \{\!\{\overline{\nabla_h w}\}\!\}\,\mathrm{d}S + \int_{\mathcal{F}_h^{\mathcal{B}}} \delta\, w\, \overline{\nabla_h w \cdot \boldsymbol{n}}\,\mathrm{d}S\right] \\ &+ i\omega^{-1} \|\beta^{1/2} [\![\nabla_h w]\!]_N\|_{0,\mathcal{F}_h^I}^2 + i\omega^{-1} \|\delta^{1/2} \nabla_h w \cdot \boldsymbol{n}\|_{0,\mathcal{F}_h^B}^2 \\ &+ i\omega \|\alpha^{1/2} [\![w]\!]_N\|_{0,\mathcal{F}_h^I}^2 + i\omega \|(1-\delta)^{1/2} w\|_{0,\mathcal{F}_h^B}^2 - \omega^2 \|w\|_{0,\Omega}^2\,,\end{aligned}$$

from which, by taking the imaginary part, we get the result. □

REMARK 3.4. *Well-posedness of the PWDG method follows from Proposition 3.3 and Proposition 3.2. In fact, if $\mathcal{A}_h(u_p, v_p) = 0$ for all $v_p \in V_p(\mathcal{T}_h)$, then $\mathcal{A}_h(u_p, u_p) = 0$ and thus $|||u_p|||_{\mathcal{F}_h} = 0$ which implies $u_p = 0$.*

PROPOSITION 3.5. *There exists a constant $C > 0$ independent of h, p and ω such that, for all $w_1, w_2 \in H^2(\mathcal{T}_h)$,*

$$|\mathcal{A}_h(w_1, w_2)| \le C\, |||w_1|||_{\mathcal{F}_h^+} |||w_2|||_{\mathcal{F}_h}\,.$$

Proof. The result follows from the definition of $\mathcal{A}_h(\cdot,\cdot)$ and repeated applications of the (weighted) Cauchy-Schwarz inequality. □

3.2. Abstract Error Estimates. In the next proposition, we prove quasi-optimality of the PWDG method in the $|||\cdot|||_{\mathcal{F}_h}$–norm.

PROPOSITION 3.6. *Let u be the analytical solution to* (2.1) *and let u_p be the PWDG solution. Then, there exists a constant $C > 0$ independent of h, p and ω such that*

$$|||u - u_p|||_{\mathcal{F}_h} \le C \inf_{v_p \in V_p(\mathcal{T}_h)} |||u - v_p|||_{\mathcal{F}_h^+}\,,$$

where $|||\cdot|||_{\mathcal{F}_h^+}$ is defined by (3.2).

Proof. We apply the triangle inequality and write

$$|||u - u_p|||_{\mathcal{F}_h} \le |||u - v_p|||_{\mathcal{F}_h} + |||u_p - v_p|||_{\mathcal{F}_h} \tag{3.3}$$

for all $v_p \in V_p(\mathcal{T}_h)$. Since $u_p - v_p \in T(\mathcal{T}_h)$, Proposition 3.3 gives

$$|||u_p - v_p|||_{\mathcal{F}_h}^2 = \mathrm{Im}\ [\mathcal{A}_h(u_p - v_p, u_p - v_p)]\,.$$

From Galerkin orthogonality and continuity of $\mathcal{A}_h(\cdot,\cdot)$ (see Proposition 3.5), we have

$$|||u_p - v_p|||^2_{\mathcal{F}_h} \le C\,|||u - v_p|||_{\mathcal{F}_h^+}|||u_p - v_p|||_{\mathcal{F}_h}\,,$$

which, inserted into (3.3) gives the result. ∎

Following [27, 10], we bound the L^2–norm of any Trefftz' function by using a duality argument.

LEMMA 3.7. *There exists a constant $C > 0$ independent of h, p and ω such that, for any $w \in T(\mathcal{T}_h)$,*

$$\|w\|_{0,\Omega} \le C\ \mathrm{diam}(\Omega)\ \left(\omega^{-1/2}h^{-1/2} + \omega^{1/2}h^{1/2}\right)\ |||w|||_{\mathcal{F}_h}\,.$$

Proof. Let φ be in $L^2(\Omega)$. Consider the adjoint problem:

$$\begin{aligned} -\Delta v - \omega^2 v &= \varphi && \text{in } \Omega\,,\\ \nabla v\cdot\boldsymbol{n} - i\omega\, v &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{3.4}$$

The solution v belongs to $H^2(\Omega)$ and the stability estimates

$$\begin{aligned} &|v|_{1,\Omega} + \omega\|v\|_{0,\Omega} \le C_1\,\mathrm{diam}(\Omega)\|\varphi\|_{0,\Omega}\,,\\ &|v|_{2,\Omega} \le C_2(1+\omega\ \mathrm{diam}(\Omega))\|\varphi\|_{0,\Omega}\,, \end{aligned} \tag{3.5}$$

hold, with $C_1, C_2 > 0$ depending only on the shape of Ω (see [25, Proposition 8.1.4], [12] and [17, Propositions 3.3, 3.5, and 3.6]).

Multiplying by $w \in T(\mathcal{T}_h)$, integrating by parts twice the first equation of (3.4) element by element (using $\Delta w + \omega^2 w = 0$ in each $K \in \mathcal{T}_h$), and taking into account that $\nabla v\cdot\boldsymbol{n} = i\omega v$ on $\partial\Omega$, we obtain

$$\begin{aligned} (w,\varphi)_{0,\Omega} &= \sum_{K\in\mathcal{T}_h}\int_{\partial K}\left(\nabla w\cdot\boldsymbol{n}\,\overline{v} - w\,\overline{\nabla v\cdot\boldsymbol{n}}\right)\,\mathrm{d}S\\ &= \int_{\mathcal{F}_h^I}\left([\![\nabla_h w]\!]_N\overline{v} - [\![w]\!]_N\cdot\overline{\nabla v}\right)\,\mathrm{d}S + \int_{\mathcal{F}_h^B}\left(\nabla_h w\cdot\boldsymbol{n} + i\omega w\right)\,\overline{v}\,\mathrm{d}S\\ &\le \sum_{f\in\mathcal{F}_h^I}\left(\|\beta^{1/2}[\![\nabla_h w]\!]_N\|_{0,f}\|\beta^{-1/2}v\|_{0,f} + \|\alpha^{1/2}[\![w]\!]_N\|_{0,f}\|\alpha^{-1/2}\nabla_h v\|_{0,f}\right)\\ &\quad + \sum_{f\in\mathcal{F}_h^B}\left(\|\delta^{1/2}\nabla w\cdot\boldsymbol{n}\|_{0,f}\|\delta^{-1/2}v\|_{0,f} + \omega^{1/2}\|\delta^{1/2}w\|_{0,f}\omega^{1/2}\|\delta^{-1/2}v\|_{0,f}\right)\\ &\le |||w|||_{\mathcal{F}_h}\Big[\sum_{f\in\mathcal{F}_h^I}\left(\omega\|\beta^{-1/2}v\|^2_{0,f} + \omega^{-1}\|\alpha^{-1/2}\nabla_h v\|^2_{0,f}\right)\\ &\qquad\qquad + \sum_{f\in\mathcal{F}_h^B}\omega\|\delta^{-1/2}v\|^2_{0,f}\Big]^{1/2}\cdot\\ &=: |||w|||_{\mathcal{F}_h}\mathcal{G}(v)^{1/2}\,. \end{aligned}$$

Introducing, for convenience, a parameter γ defined by $\gamma = \beta$ on interior faces and $\gamma = \delta$ on boundary faces, we have

$$\mathcal{G}(v) \le \sum_{K\in\mathcal{T}_h}\left(\omega\|\gamma^{-1/2}v\|^2_{0,\partial K} + \omega^{-1}\|\alpha^{-1/2}\nabla v\|^2_{0,\partial K}\right)\,.$$

We recall that, for any $K \in \mathcal{T}_h$, the trace inequality [9, Theorem 1.6.6]

$$\|u\|^2_{0,\partial K} \leq C \, \|u\|_{0,\Omega} \left(h_K^{-1} \|u\|_{0,K} + |u|_{1,K} \right) \qquad \forall u \in H^1(K) \, , \tag{3.6}$$

holds with a constant $C > 0$ depending only on the "shape regularity measure" ρ. Since $v \in H^2(\Omega)$, using the definition of the flux parameters, the trace estimate (3.6) and the stability estimates (3.5), we can bound $\mathcal{G}(v)$ as follows:

$$\begin{aligned}
\mathcal{G}(v) &\leq C \sum_{K \in \mathcal{T}_h} \left[\omega h^{-1} \|v\|^2_{0,K} + \omega \, \|v\|_{0,K} |v|_{1,K} + \omega^{-1} h^{-1} |v|^2_{1,K} + \omega^{-1} |v|_{1,K} |v|_{2,K} \right] \\
&\leq C \sum_{K \in \mathcal{T}_h} \left[\omega h^{-1} \|v\|^2_{0,K} + \omega h \, |v|^2_{1,K} + \omega^{-1} h^{-1} |v|^2_{1,K} + \omega^{-1} h \, |v|^2_{2,K} \right] \\
&\leq C \left[\mathrm{diam}(\Omega)^2 \omega^{-1} h^{-1} + \mathrm{diam}(\Omega)^2 \omega h + (1 + \omega \, \mathrm{diam}(\Omega))^2 \omega^{-1} h \right] \|\varphi\|^2_{0,\Omega} \\
&\leq C \, \mathrm{diam}(\Omega)^2 \left(\omega^{-1} h^{-1} + \omega h \right) \|\varphi\|^2_{0,\Omega}
\end{aligned}$$

(we have used also the obvious inequality $h \leq \mathrm{diam}(\Omega)$), with a constant $C > 0$ independent of h, p and ω. Consequently, for all $\varphi \in L^2(\Omega)$, we obtain

$$\frac{(w, \varphi)_{0,\Omega}}{\|\varphi\|_{0,\Omega}} \leq C \, \mathrm{diam}(\Omega) \, \left(\omega^{-1/2} h^{-1/2} + \omega^{1/2} h^{1/2} \right) \, |||w|||_{\mathcal{F}_h} \, ,$$

and the result readily follows. □

By applying Lemma 3.7 to $u - u_p \in T(\mathcal{T}_h)$ we can bound the L^2–norm of the error by its $|||\cdot|||_{\mathcal{F}_h}$–norm, like in [10].

COROLLARY 3.8. *Let u be the analytical solution to (2.1) and let u_p be the PWDG solution. Then, there exists a constant $C > 0$ independent of h, p and ω such that*

$$\|u - u_p\|_{0,\Omega} \leq C \, \mathrm{diam}(\Omega) \, \left(\omega^{-1/2} h^{-1/2} + \omega^{1/2} h^{1/2} \right) \, |||u - u_p|||_{\mathcal{F}_h} \, .$$

3.3. Best Approximation Estimates. From now on, we restrict ourselves to the two-dimensional case $d = 2$. We also make the following *assumptions on the approximation spaces*:

- $p = 2m + 1$, with $m \in \mathbb{N} \setminus \{0\}$, is constant;
- the p directions $\{\boldsymbol{d}_\ell = (\cos\theta_\ell, \sin\theta_\ell)\}_{\ell=-m}^{m}$ satisfy the following condition: there exists $\eta \in (0, 1]$ such that the minimum angle between two different directions is greater than or equal to $2\pi \, \eta / p$.

Given a domain $D \subset \mathbb{R}^2$, let $\|\cdot\|_{s,\omega,D}$ be the ω–weighted Sobolev norm defined by

$$\|v\|^2_{s,\omega,D} = \sum_{j=0}^{s} \omega^{2(s-j)} |v|^2_{j,D} \, .$$

We quote the following result proved in [18].

THEOREM 3.9. *Let $D \in \mathbb{R}^2$ be a bounded, convex Lipschitz domain of diameter h_D which contains a ball of radius ρh_D, with $\rho \in (0, 1)$. Let $u \in H^{k+1}(D)$, $k \geq 0$, be such that $\Delta u + \omega^2 u = 0$ in D.*

Fix $m \geq 1$, set $p = 2m + 1$, and let the directions $\{\boldsymbol{d}_\ell = (\cos\theta_\ell, \sin\theta_\ell)\}_{\ell=-m}^{m}$ satisfy the following condition: there exists $\eta \in (0, 1]$ such that the minimum angle between two different directions is greater than or equal to $2\pi \, \eta / p$.

Finally, assume that

$$k \le \left\lceil \frac{m+1}{2} \right\rceil .$$

Then, there exists $\sigma_1, \dots, \sigma_p \in \mathbb{C}$ *such that, if we set* $P_\omega u := \sum_{\ell=1}^{p} \sigma_\ell e^{i\omega \boldsymbol{x}\cdot\boldsymbol{d}_k}$*, for every* $0 \le j \le k$*, we have*

$$\begin{aligned}\|u - P_\omega u\|_{j,\omega,D} \le{}& C\, e^{11\omega h_D/4} \left(1 + (\omega h_D)^{j+8}\right) h_D^{k+1-j} \\ &\cdot \left[\left(\frac{\log\left(k+1+\lfloor \frac{m+1}{2} \rfloor\right)}{k+1+\lfloor \frac{m+1}{2} \rfloor} \right)^{k+1-j} + \left(\frac{e^{5/2}}{2\eta^2}\, \frac{1+\omega h_D}{\sqrt{m+1}} \right)^m \right] \|u\|_{k+1,\omega,D} ,\end{aligned}$$

with a constant $C > 0$ *only depending on the shape of* D*,* j *and* k*, but not on* h*,* p*,* ω*,* η *and* u*.*

We point out that, thanks to the assumptions made before, Theorem 3.9 can be applied to the elements $K \in \mathcal{T}_h$. Also note that $p = 2m+1$ is the spectral discretization parameter.

Set, for simplicity,

$$\mathcal{L}(k,m) = \frac{\log\left(k+1+\lfloor \frac{m+1}{2} \rfloor\right)}{k+1+\lfloor \frac{m+1}{2} \rfloor} ;$$

notice that $0 < \mathcal{L}(k,m) < 1$.

Lemma 3.10. *We have the following estimates:*

$$\begin{aligned}&\omega \|u - P_\omega u\|^2_{0,\mathcal{F}_h} \\ &\quad \le C\, e^{11\omega h_K/2} \left(1 + (\omega h)^{17}\right) \omega h^{2k+1} \left[\mathcal{L}(k,m)^{k+1} + \left(\frac{1+\omega h}{\sqrt{m+1}} \right)^m \right] \\ &\qquad \cdot \left[\mathcal{L}(k,m)^{k} + \left(\frac{1+\omega h}{\sqrt{m+1}} \right)^m \right] \|u\|^2_{k+1,\omega,\Omega} , \qquad (3.7)\\ &\omega^{-1} \|\nabla_h (u - P_\omega u)\|^2_{0,\mathcal{F}_h} \\ &\quad \le C\, e^{11\omega h/2} \left(1 + (\omega h)^{19}\right) \omega^{-1} h^{2k-1} \left[\mathcal{L}(k,m)^{k} + \left(\frac{1+\omega h}{\sqrt{m+1}} \right)^m \right] \\ &\qquad \cdot \left[\mathcal{L}(k,m)^{k-1} + \left(\frac{1+\omega h}{\sqrt{m+1}} \right)^m \right] \|u\|^2_{k+1,\omega,\Omega} , \qquad (3.8)\end{aligned}$$

with a constant $C > 0$ *independent of* h*,* p*,* ω *and* u*.*

Proof. Using (3.6), Theorem 3.9, the fact that $(1 + \omega h) > 1$ and $\mathcal{L}(k,m) < 1$, we have

$$\begin{aligned}\|u - P_\omega u\|^2_{0,\partial K} \le{}& C \left(h_K^{-1} \|u - P_\omega u\|^2_{0,K} + \|u - P_\omega u\|_{0,K} |u - P_\omega u|_{1,K} \right) \\ \le{}& C\, e^{11\omega h_K/2} \left(1 + (\omega h_K)^{17}\right) h_K^{2k+1} \left[\mathcal{L}(k,m)^{k+1} + \left(\frac{1+\omega h_K}{\sqrt{m+1}} \right)^m \right] \\ &\cdot \left[\mathcal{L}(k,m)^{k} + \left(\frac{1+\omega h_K}{\sqrt{m+1}} \right)^m \right] \|u\|^2_{k+1,\omega,K} ,\end{aligned}$$

and

$$\begin{aligned}\|\nabla(u-P_\omega u)\|^2_{0,\partial K} \leq & C\,(h_K^{-1}|u-P_\omega u|^2_{1,K}+|u-P_\omega u|_{1,K}|u-P_\omega u|_{2,K})\\ \leq & C\,e^{11\omega h_K/2}\big(1+(\omega h_K)^{19}\big)\,h_K^{2k-1}\left[\mathcal{L}(k,m)^k+\left(\frac{1+\omega h_K}{\sqrt{m+1}}\right)^m\right]\\ & \cdot\left[\mathcal{L}(k,m)^{k-1}+\left(\frac{1+\omega h_K}{\sqrt{m+1}}\right)^m\right]\|u\|^2_{k+1,\omega,K}\,,\end{aligned}$$

Adding over all elements and multiplying by $\omega^{\pm 1}$ give the two bounds. ∎

The following theorem is a straightforward consequence of Lemma 3.10.

THEOREM 3.11. *Assuming $\omega h \leq c_0$, with a constant $c_0>0$, we have the following estimate:*

$$\begin{aligned}&|||u-P_\omega u|||^2_{\mathcal{F}_h^+}\\ &\quad\leq C\,h^{2k}\left\{\omega^{-1}h^{-1}\left[\mathcal{L}(k,m)^{k+1}+\left(\frac{1}{\sqrt{m+1}}\right)^m\right]\left[\mathcal{L}(k,m)^{k}+\left(\frac{1}{\sqrt{m+1}}\right)^m\right]\right.\\ &\quad\left.+\left[\mathcal{L}(k,m)^{k}+\left(\frac{1}{\sqrt{m+1}}\right)^m\right]\left[\mathcal{L}(k,m)^{k-1}+\left(\frac{1}{\sqrt{m+1}}\right)^m\right]\right\}\|u\|^2_{k+1,\omega,\Omega}\,,\end{aligned}$$

with a constant $C>0$ independent of p and u, but depending on c_0 (i.e., it depends on ω and h only through their product ωh).

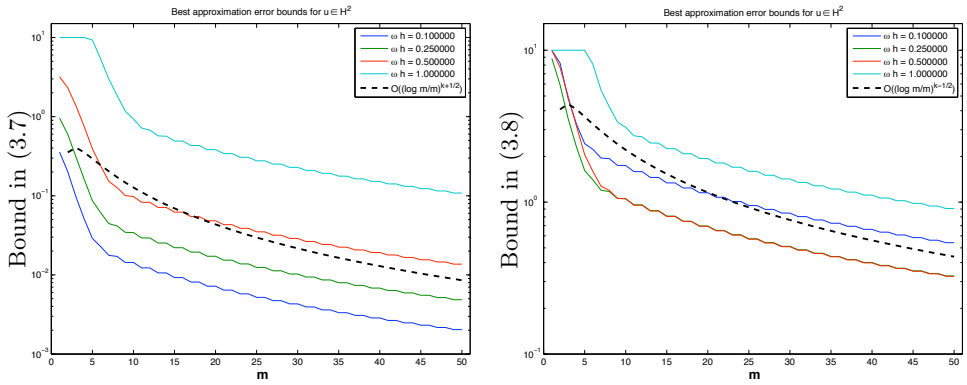

FIG. 3.1. *Qualitative plots of the bounds given in Lemma 3.10 (with cutoff at* 10*).*

REMARK 3.12. *The graphs of the bounds of Lemma 3.10 given in Figure 3.1 highlight the pronounced increase of the constants for large ωh and small p. This is evidence of a threshold condition, that is, a minimal resolution requirement on the plane wave space before any reasonable approximation can be expected.*

REMARK 3.13. *For $k \leq \left\lceil\frac{m+1}{2}\right\rceil$ (see Theorem 3.9), $\mathcal{L}(k,m)$ asymptotically behaves, for increasing m, like $\log(m)/m$ (and thus like $\log(p)/p$), see Figure 3.1, whereas the term $\left(\frac{1}{\sqrt{m+1}}\right)^m$ decays faster than exponentially. Therefore, the estimate of Theorem 3.11, for large p, can be written as*

$$|||u-P_\omega u|||_{\mathcal{F}_h^+} \leq C\,\omega^{-1/2}\,h^{k-1/2}\left(\frac{\log(p)}{p}\right)^{k-1/2}\|u\|_{k+1,\omega,\Omega}\,. \tag{3.9}$$

No better estimate can be obtained for the weaker $|||\cdot|||_{\mathcal{F}_h}$–norm.

3.4. Error Estimates. In the following theorem, we state error estimates for the PWDG method in the following energy-type norm:

$$\|w\|_{DG}^2 := |||u-u_p|||_{\mathcal{F}_h}^2 + \omega^2 \|w\|_{0,\Omega}^2 \,.$$

THEOREM 3.14. *Let $u \in H^{k+1}(\Omega)$ be the analytical solution to (2.1) and let u_p be the PWDG solution. Assuming $\omega h \le c_0$, with a constant $c_0 > 0$, there exists a constant $C > 0$ independent of p and u, but depending on c_0 (i.e., it depends on ω and h only through their product ωh), such that*

$$\begin{aligned} |||u-u_p|||_{\mathcal{F}_h} &\le C\, \omega^{-1/2}\, h^{k-1/2} \left(\frac{\log(p)}{p}\right)^{k-1/2} \|u\|_{k+1,\omega,\Omega} \,, \\ \omega\, \|u-u_p\|_{0,\Omega} &\le C\, \operatorname{diam}(\Omega)\, h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-1/2} \|u\|_{k+1,\omega,\Omega} \,, \end{aligned}$$

and thus

$$\begin{aligned} &\|u-u_p\|_{DG} \\ &\qquad \le C\, \operatorname{diam}(\Omega)^{1/2} \left[\omega^{-1/2} + \operatorname{diam}(\Omega)^{1/2}\right] h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-1/2} \|u\|_{k+1,\omega,\Omega} \,. \end{aligned}$$

Proof. The first two bounds follow from Proposition 3.6, Remark 3.13 and Corollary 3.8. The third bound is a direct consequence of the first two. ∎

REMARK 3.15. *If we choose the flux parameters depending on p and ωh in the following way:*

$$\alpha = \frac{\mathtt{a}}{\omega h}\, \frac{p}{\log(p)} \,, \qquad \beta = \mathtt{b}\, \omega h\, \frac{\log(p)}{p} \,, \qquad \delta = \mathtt{d}\, \omega h\, \frac{\log(p)}{p} \,,$$

with $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{d}$ strictly positive and independent of h, ω and p, again with $\delta \in (0,1)$, the result of Lemma 3.7 becomes

$$\|w\|_{0,\Omega} \le C\, \operatorname{diam}(\Omega) \left[\left(\frac{p}{\log(p)}\right)^{1/2} \omega^{-1} h^{-1} + \left(\frac{p}{\log(p)}\right)^{1/2} + \omega h\right] |||w|||_{\mathcal{F}_h} \,,$$

and the best approximation estimate of Theorem 3.11 is

$$\begin{aligned} &|||u - P_\omega u|||_{\mathcal{F}_h^+}^2 \\ &\le C\, h^{2k} \left\{ \frac{p}{\log(p)} \left[\mathcal{L}(k,m)^{k+1} + \left(\frac{1}{\sqrt{m+1}}\right)^m\right] \left[\mathcal{L}(k,m)^{k} + \left(\frac{1}{\sqrt{m+1}}\right)^m\right] \right. \\ &\quad \left. + \frac{\log(p)}{p} \left[\mathcal{L}(k,m)^{k} + \left(\frac{1}{\sqrt{m+1}}\right)^m\right] \left[\mathcal{L}(k,m)^{k-1} + \left(\frac{1}{\sqrt{m+1}}\right)^m\right] \right\} \|u\|_{k+1,\omega,\Omega}^2 \,; \end{aligned}$$

consequently, reasoning as in Remark 3.13 yields the optimal estimate

$$|||u - P_\omega u|||_{\mathcal{F}_h^+} \le C\, h^k \left(\frac{\log(p)}{p}\right)^k \|u\|_{k+1,\omega,\Omega} \,.$$

Thus, the final error bounds (see Theorem 3.14) for this choice of flux parameters are

$$
\begin{aligned}
|||u-u_p|||_{\mathcal{F}_h} \leq & C\, h^k \left(\frac{\log(p)}{p}\right)^k \|u\|_{k+1,\omega,\Omega}\ , \\
\omega\, \|u-u_p\|_{0,\Omega} \leq & C\ \mathrm{diam}(\Omega)\, h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-1/2} \|u\|_{k+1,\omega,\Omega}\ , \\
\|u-u_p\|_{DG} \leq & C\ \mathrm{diam}(\Omega)\, h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-1/2} \|u\|_{k+1,\omega,\Omega}\ .
\end{aligned}
$$

The gain of half a power of $log(p)/p$ in the best approximation estimate, with respect to the case of constant flux parameters, is compensated by a loss of half a power of $log(p)/p$ in the result of Lemma 3.7, and thus the order of convergence in the energy-norm is the same as in the case of constant flux parameters.

REMARK 3.16. *Matching the final estimate of Theorem 3.14 with the best approximation estimate* (3.9) *we find that the bounds feature optimal asymptotic behavior with respect to p, but half a power of h is lost.*

REMARK 3.17. *The proof of the "coercivity" result of Proposition 3.3 does not involve inverse trace inequalities. This allows to choose either constant flux parameters or the variable flux parameters discussed in Remark 3.15, which, in both cases, give convergence in the energy-norm of order $(\log(p)/p)^{k-1/2}$.*

On the other hand, the bound of the L^2–norm of the trace of a discrete function on the boundary of an element K by the L^2–norm of the discrete function within K involves a constant proportional to $p\, h_K^{-1/2}$ (see numerics in [15]). Therefore, the use of inverse trace inequalities would have required a choice of the flux parameters similar to the one in Remark 3.15, but with p^2 instead of $p/log(p)$, resulting in a deterioration of the order of convergence of the energy norm by a factor $p\log(p)$.

3.5. Error Estimates in Stronger Norms. It would be desirable to derive an asymptotically quasi-optimal estimate of $\|\nabla_h(u-u_p)\|_{0,\Omega}$ as it could be achieved for the h–version of PWDG in [15]. The duality technique employed in our approach does not provide such estimates. We have to settle for weaker results.

Define the following $H^1(\mathcal{T}_h)$–orthogonal projection onto the space $\mathbb{P}^p(\mathcal{T}_h) \subset H^1(\Omega)$ of globally continuous, $\mathcal{T}_h$–piecewise polynomial functions of degree $\leq p$: $\mathcal{P}: H^1(\mathcal{T}_h) \to \mathbb{P}^p(\mathcal{T}_h)$ is such that, if $w \in H^1(\mathcal{T}_h)$,

$$
\mathcal{L}_h(\mathcal{P}(w), v) = \mathcal{L}_h(w, v) \quad \forall v \in \mathbb{P}^p(\mathcal{T}_h)\ , \tag{3.10}
$$

where

$$
\mathcal{L}_h(w,v) := \int_\Omega (\nabla_h w \cdot \overline{\nabla_h v} + \omega^2 w\, \overline{v})\, dV \qquad \forall w, v \in H^1(\mathcal{T}_h)\ . \tag{3.11}
$$

Note that, given w, the computation of $\mathcal{P}$ amounts to solving a Neumann boundary value problem for $-\Delta + \omega^2$ by means of p–degree Lagrangian finite elements. Thus, in principle, $\mathcal{P}u_p$ can be obtained from the PWDG solution $u_p \in V_p(\mathcal{T}_h)$ by means of solving a discrete positive definite 2nd-order elliptic boundary value problem in a post-processing step.

PROPOSITION 3.18. *With the same assumptions as in Theorem 3.14, we have*

$$
\|\nabla\,(u - \mathcal{P}(u_p))\,\|_{0,\Omega} \leq C\,(\mathrm{diam}(\Omega) + \omega^{-1})\, h^{k-1} \left(\frac{\log(p)}{p}\right)^{k-1/2} \|u\|_{k+1,\omega,\Omega}\ ,
$$

with a constant $C > 0$ independent of p and u, but depending on an upper bound of the product ωh (i.e., depending on ω and h only through their product).

Proof. By the triangle inequality, we can write

$$\|\nabla\left(u-\mathcal{P}(u_p)\right)\|_{0,\Omega} \leq \|\nabla\left(u-\mathcal{P}(u)\right)\|_{0,\Omega} + \|\nabla\left(\mathcal{P}(u-u_p)\right)\|_{0,\Omega}\,. \tag{3.12}$$

We bound the second term on the right-hand side. By the definition of $\mathcal{P}$, for all $v \in H^1(\Omega)$, local integration by parts gives

$$\begin{aligned}
\mathcal{L}_h(\mathcal{P}(u-u_p),v) &= \sum_{K\in\mathcal{T}_h}\int_K \nabla(u-u_p)\cdot\overline{\nabla v}\,\mathrm{d}V + \omega^2(u-u_p,v)_{0,\Omega}\\
&= -\sum_{K\in\mathcal{T}_h}\int_K \Delta(u-u_p)\,\overline{v}\,\mathrm{d}V + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\nabla(u-u_p)\cdot\boldsymbol{n}_K\,\overline{v}\,\mathrm{d}S + \omega^2(u-u_p,v)_{0,\Omega}\\
&= 2\omega^2(u-u_p,v)_{0,\Omega} + \int_{\mathcal{F}_h^I}[\![\nabla_h(u-u_p)]\!]_N\,\overline{v}\,\mathrm{d}S + \int_{\mathcal{F}_h^B}\nabla_h(u-u_p)\cdot\boldsymbol{n}\,\overline{v}\,\mathrm{d}S\,.
\end{aligned}$$

Aiming for the $|||w|||_{\mathcal{F}_h}$–norm, we use the Cauchy-Schwarz inequality and get

$$\begin{aligned}
\mathcal{L}_h(\mathcal{P}(u-u_p),v) \leq\;& 2\omega\|u-u_p\|_{0,\Omega}\,\omega\|v\|_{0,\Omega}\\
&+\omega^{-1/2}\|\beta^{1/2}[\![\nabla_h(u-u_p)]\!]_N\|_{0,\mathcal{F}_h^I}\omega^{1/2}\|\beta^{-1/2}v\|_{0,\mathcal{F}_h^I}\\
&+\omega^{-1/2}\|\delta^{1/2}\nabla_h(u-u_p)\cdot\boldsymbol{n}\|_{0,\mathcal{F}_h^B}\omega^{1/2}\|\delta^{-1/2}v\|_{0,\mathcal{F}_h^B}\\
\leq\;& 2\omega\|u-u_p\|_{0,\Omega}\,\omega\|v\|_{0,\Omega}\\
&+|||u-u_p|||_{\mathcal{F}_h}\omega^{1/2}\max\{\delta^{-1/2},\beta^{-1/2}\}\|v\|_{0,\mathcal{F}_h}\,.
\end{aligned}$$

Now, the trace inequality (3.6) gives

$$\begin{aligned}
\mathcal{L}_h(\mathcal{P}(u-u_p),v) \leq\;& 2\omega\|u-u_p\|_{0,\Omega}\,\omega\|v\|_{0,\Omega} + C(\omega h)^{-1/2}|||u-u_p|||_{\mathcal{F}_h}\\
&\cdot\max\{\delta^{-1/2},\beta^{-1/2}\}\cdot(\omega\|v\|_{0,\Omega}+\omega h\|\nabla v\|_{0,\Omega})\;,\\
\leq\;& \left(\omega^2\|u-u_p\|_{0,\Omega}^2 + (\omega h)^{-1}|||u-u_p|||_{\mathcal{F}_h}^2\right)^{1/2}\\
&\cdot C\max\{\delta^{-1/2},\beta^{-1/2}\}\max\{\omega h,1\}\\
&\cdot\left(\omega^2\|v\|_{0.\Omega}^2+\|\nabla v\|_{0,\Omega}^2\right)^{1/2}
\end{aligned}$$

where $C > 0$ only depends on the shape-regularity of the mesh $\mathcal{T}_h$. Setting $v := \mathcal{P}(u-u_p)$ yields the estimate

$$\begin{aligned}
\|\nabla\mathcal{P}(u-u_p)\|_{0,\Omega}^2 + \omega^2\|\mathcal{P}(u-u_p)\|_{0.\Omega}^2 \leq\;& C\left(\min\{\delta,\beta\}\right)^{-1}\max\{\omega h,1\}^2\\
&\cdot\left(\omega^2\|u-u_p\|_{0,\Omega}^2 + (\omega h)^{-1}|||u-u_p|||_{\mathcal{F}_h}^2\right)\;.
\end{aligned}$$

We plug in the estimates of Theorem 3.14 and allow $C > 0$ to depend on an upper bound for ωh and the (constant) flux parameters, in addition. Thus, we arrive at

$$\|\nabla\mathcal{P}(u-u_p)\|_{0,\Omega} \leq C(\mathrm{diam}(\Omega)+\omega^{-1})\,h^{k-1}\left(\frac{\log(p)}{p}\right)^{k-1/2}\|u\|_{k+1,\omega,\Omega}\,. \tag{3.13}$$

Further, standard error estimates for H^1–conforming Lagrangian finite element spaces [7] provide

$$\|\nabla(u-\mathcal{P}u)\|_{0,\Omega} \leq C\frac{h^k}{p^k}\|u\|_{k+1,\omega,\Omega}\,, \tag{3.14}$$

where $C > 0$ depends on the shape-regularity of $\mathcal{T}_h$ and Ω.

Inserting (3.13) and (3.14) into (3.12) yields the assertion of the theorem. □

4. Numerical Experiments. In this section, we numerically investigate the p–convergence of the PWDG method for regular and singular solutions of the Helmholtz equation in 2D.

We consider a square domain $\Omega = [0,1] \times [-0.5, 0.5]$, partitioned by a mesh consisting of 8 triangles (see Figure 4.1, upper-left plot), so that $h = 1/\sqrt{2}$. For the time being, we fix $\omega = 10$, such that an entire wavelength $\lambda = 2\pi/\omega \simeq 0.628$ is completely contained in Ω. All the computations have been done in *MATLAB*, and the system matrix was computed by exact integration on the mesh skeleton.

We choose the inhomogeneous boundary conditions in such a way that the analytical solutions are the circular waves given, in polar coordinates $\boldsymbol{x} = (r\cos\theta, r\sin\theta)$, by

$$u(\boldsymbol{x}) = J_\xi(\omega r)\ \cos(\xi\theta)\ , \qquad \xi \geq 0\ ;$$

here, J_ξ denotes the Bessel function of the first kind and order ξ. For $t \ll 1$, these functions behave like

$$J_\xi(t) \approx \frac{1}{\Gamma(\xi+1)} \left(\frac{t}{2}\right)^\xi\ .$$

Thus, if $\xi \in \mathbb{N}$, u can be analytically extended to a Helmholtz solution in $\mathbb{R}^2$, while, if $\xi \notin \mathbb{N}$, its derivatives have a singularity at the origin. Then $u \in H^{\xi+1-\epsilon}(\Omega)$ for every $\epsilon > 0$, but $u \notin H^{\xi+1}(\Omega)$ (see [16, Theorem 1.4.5.3]).

We compute the numerical solutions in the regular case $\xi = 1$ and in the singular cases $\xi = 2/3$ and $\xi = 3/2$. The profiles of the analytical solutions corresponding to these three cases are displayed in Figure 4.1, upper-right and lower plots.

We consider two choices of numerical fluxes: with constant parameters, as in the original ultra weak variational formulation (UWVF) of Cessenat and Despres [11] ($\alpha = \beta = \delta = 1/2$; dashed line in the plots), or depending on p, h and ω as in Remark 3.15: $\alpha = \beta^{-1} = \delta^{-1} = \mathtt{a}_0\, p/(\omega h \log p)$, with $\mathtt{a}_0 = 10$ (PWDG from here on; dashed-dotted lines in the plots). We also plot the error of the L^2–projection of u onto $V_p(\mathcal{T}_h)$ (solid line). For every case, we compute the L^2–norm of the error, the broken H^1–seminorm and the L^2–norm of the jumps on the skeleton of the mesh. The errors are plotted in Figures 4.2–4.5.

These plots highlight three different regimes for increasing p: *i)* a pre-asymptotic region with slow convergence, *ii)* a region of faster convergence, and finally *iii)* a sudden stalling of convergence, due to the impact of round-off.

With a parameter $\mathtt{a}_0 \geq 5$ in the definition of the fluxes, such that the condition $\delta < 1$ (and thus $1 - \delta > 0$) is satisfied for all the considered p, the PWDG method is slightly superior to the one with constant fluxes (UWVF) in the L^2– and H^1–norms; the difference in the jumps norm is even more significant.

The most evident outcome is that, for both methods, the numerical errors are always close to L^2–approximation error of the analytical solution, that is, the *p-version is not affected by the pollution effect*.

The discretization error for $\xi = 1$ (analytic solution) converges in all the considered norms with exponential rate (see Figure 4.2). This behavior is not a surprise: the algebraic convergence in the theoretical estimates is only due to the best approximation error and becomes exponential when the analytical solution of the problem

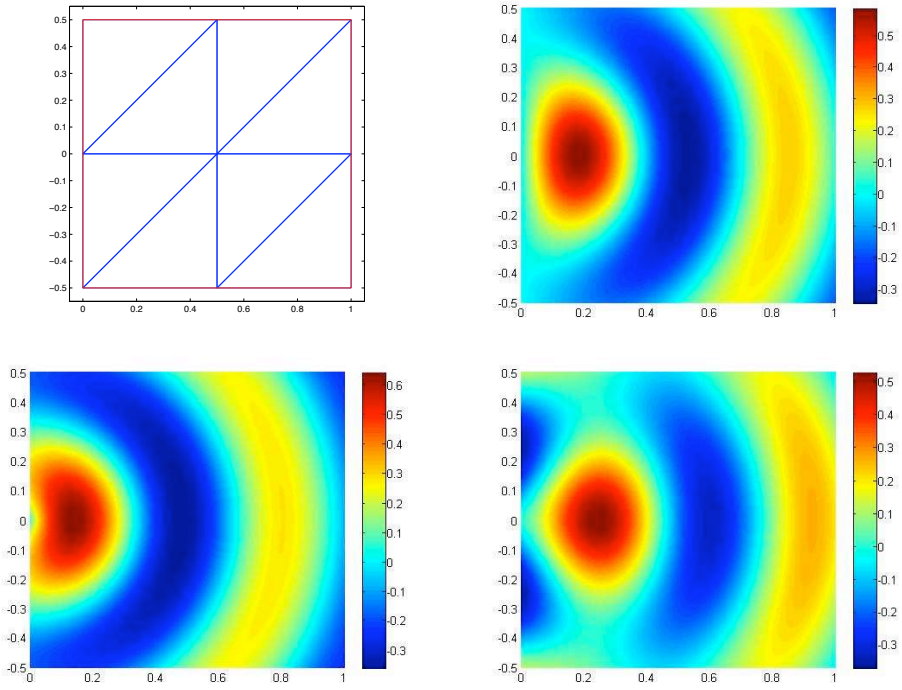

FIG. 4.1. *The mesh used for the numerical experiments and the analytical solutions for* $\xi = 1,\ 2/3,\ 3/2$.

can be extended analytically outside the domain (see [25, Section 2.4] and [8, Theorem 6.3.3], where the results refer to the approximation of holomorphic functions by complex polynomials, but they can be transferred to the approximation by plane waves).

For $\xi = 2/3$ and $\xi = 3/2$, the solution u has a singularity located in a boundary node of the mesh. It corresponds to the typical corner singularities arising from reentrant corners in scattering problems. In this case, as expected, the convergence is not exponential but algebraic, altough the orders of convergence are not clear. In the region of faster convergence, the orders are significantly better then the ones expected from the theory; for higher p numerical instability prevents us from obtaining a neat slope in the logarithmic plot. In all the considered norms, the orders of convergence are clearly better for the solution with higher Sobolev regularity (with $\xi = 3/2$, $u \in H^2(\Omega)$).

By decreasing the wavenumber ω, keeping the mesh fixed, we obtain a faster convergence in all the norms for both methods; see Figure 4.6. On the other hand, the instability appears for smaller p because the plane waves are closer to be linearly dependent. Of course in this case the domain accommodates fewer wavelengths.

Vice versa, if we increase ω, again with the same mesh, the preasymptotic region becomes larger and larger (more plane waves are needed before the onset of convergence) and the instability reduces the maximum possible accuracy we can reach.

5. Conclusions. We have presented the first p–version a priori error analysis of the plane wave discontinuous Galerkin (PWDG) method for the 2D homogeneous

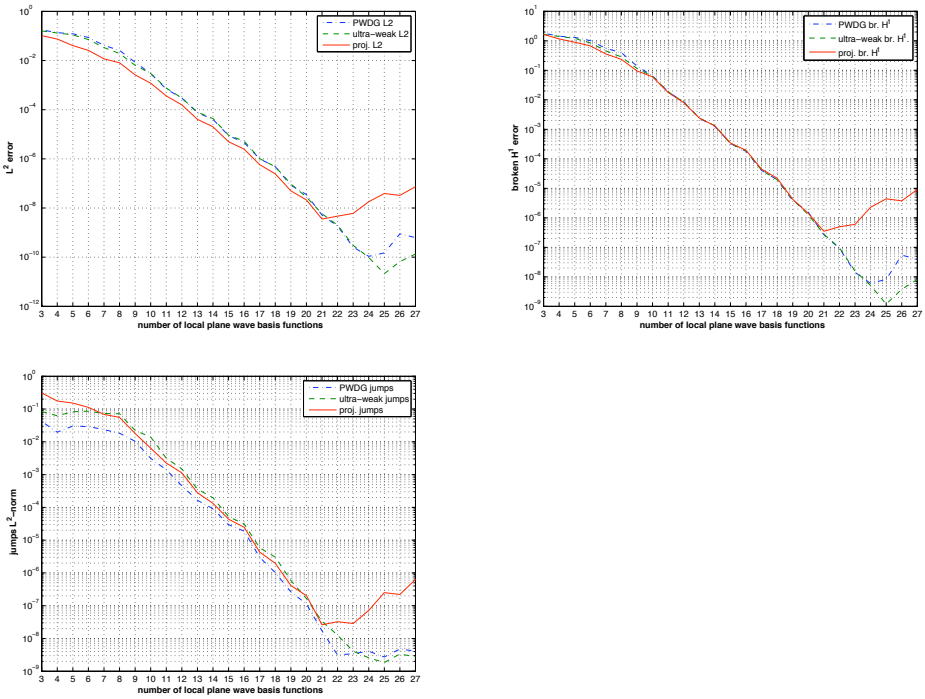

FIG. 4.2. *The errors in L^2–norm, H^1–seminorm, and L^2–norm for the jumps for the regular solution $u = J_1(\omega r)\cos(\theta)$ plotted against $p \in \{3, \dots, 27\}$. The convergence is exponential before the onset of numerical instability, and the discretization error is very close to the L^2–projection error.*

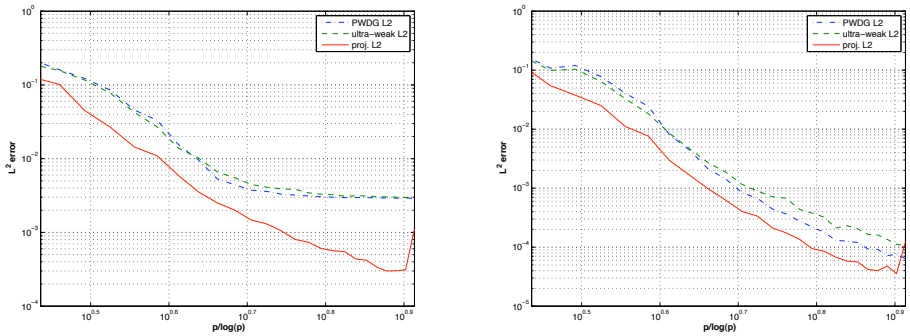

FIG. 4.3. *The errors in L^2–norm for the two singular solutions ($\xi = 2/3$ on the left and $\xi = 3/2$ on the right) in logarithmic scale with respect to $p/\log p$, $p \in \{3, \dots, 27\}$.*

Helmholtz equation. Quasi-optimal error estimates in an energy-type norm are established by using a modified duality argument due to [27, 10]. On the other hand, our analysis does not provide an estimate of the gradient of the error (we were only able to estimate the gradient of the difference between the analytical solution and a computable projection of the PWDG solution), nor a quasi-optimal estimate of the L^2–norm of the error. These problems remain open.

As far as the 3D case is concerned, the only missing point are projection error

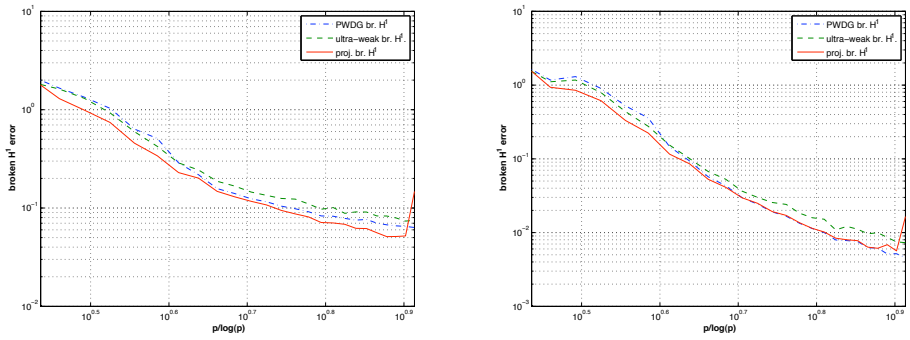

FIG. 4.4. *The errors in broken H^1–seminorm for the two singular solutions ($\xi = 2/3$ on the left and $\xi = 3/2$ on the right) in logarithmic scale with respect to $p/\log p$, $p \in \{3, \dots, 27\}$.*

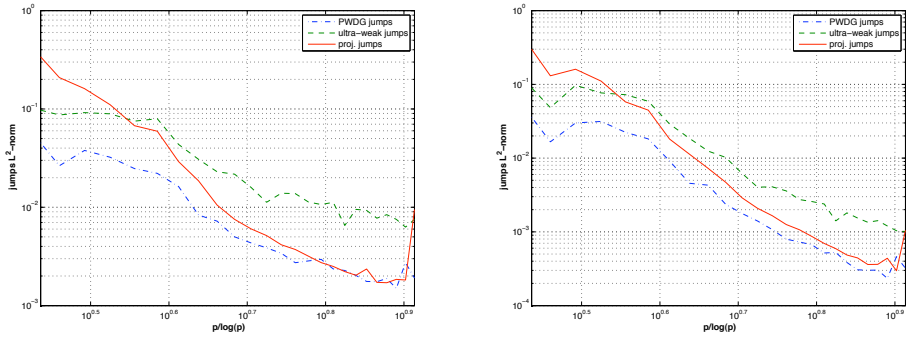

FIG. 4.5. *The errors in L^2–norm on the skeleton for the jumps of the two singular solutions ($\xi = 2/3$ on the left and $\xi = 3/2$ on the right) in logarithmic scale with respect to $p/\log p$, $p \in \{3, \dots, 27\}$.*

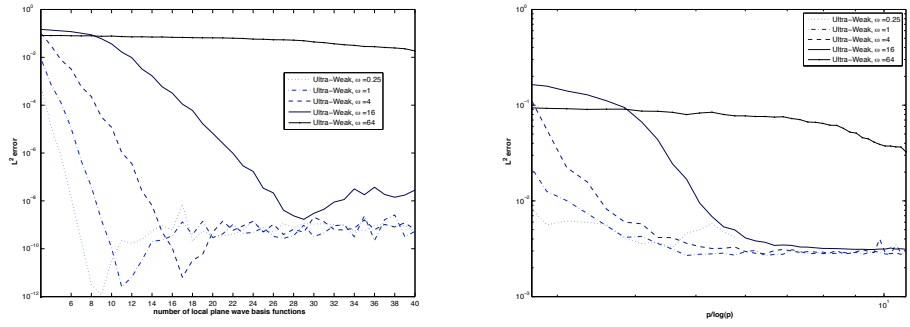

FIG. 4.6. *The errors in L^2–norm for the regular solution ($\xi = 1$, on the left) and the singular one ($\xi = 2/3$, on the right, in logarithmic scale with respect to $p/\log p$) for different values of ω (0.25, 1, 4, 16, 64), $p \in \{3, \dots, 40\}$.*

estimates for solutions to the homogeneous Helmholtz equation onto plane wave spaces in 3D; as soon as these estimates will be established, the error analysis for the 3D case will be complete, since our analysis framework covers 2D and 3D at once.

Acknowledgment. The authors wish to thank Claude Gittelson for providing his MATLAB implementation of PWDG as a basis for the numerical experiments.

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