On Kolmogorov Equations for Anisotropic Multivariate Lévy Processes

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Abstract

For d-dimensional exponential Lévy models, variational formulations of the Kolmogorov equations arising in asset pricing are derived. Well-posedness of these equations is verified. Particular attention is paid to pure jump, d-variate Lévy processes built from parametric, copula dependence models in their jump structure. The domains of the associated Dirichlet forms are shown to be certain anisotropic Sobolev spaces. Representations of the Dirichlet forms are given which remain bounded for piecewise polynomial, continuous functions of finite element type. We prove that the variational problem can be localized to a bounded domain with explicit localization error bounds. Furthermore, we collect several analytical tools for further numerical analysis.

Keywords: Lévy-copulas, Lévy processes, integrodifferential equations, pseudo differential operators, Dirichlet forms, option pricing

1 Introduction

Consider a basket of $d \geq 1$ risky assets whose log returns X_t at time $t > 0$ are modeled by a Lévy process $X = \{X_t\}_{t\geq 0}$ with state space $\mathbb{R}^d$. By the fundamental theorem of asset pricing, arbitrage free prices u of European contingent claims on such baskets with "reasonable" payoffs $g(\cdot)$ and maturity T are given by the conditional expectation

$$u(t,x) = \mathbb{E}\left(e^{-r(T-t)}g(X_T) \mid X_t = x\right). \tag{1.0.1}$$

Here, the expectation is taken with respect to an a-priori chosen martingale measure equivalent to the historical measure (see e.g. [13, 14] for measure selection criteria).

It is well known that the family $\{T_t\}_{t\geq 0}$ of maps $T_t : g(\cdot) \to u(t,\cdot)$ is a one-parameter semigroup. We denote by $\mathcal{A}$ its associated infinitesimal generator, i.e.

$$\mathcal{A}u := \lim_{t\to 0+} \frac{1}{t}(T_t u - u), \tag{1.0.2}$$

for all functions $u \in \mathcal{D}(\mathcal{A})$ in the domain

$$\mathcal{D}(\mathcal{A}) := \left\{ u \in C_\infty(\mathbb{R}^d) : \lim_{t\to 0+} \frac{1}{t}(T_t u - u) \text{ exists as strong limit} \right\}.$$

Sufficiently smooth value functions u in (1.0.1) can be obtained as solutions of a partial integrodifferential equation (PIDE), the Kolmogorov equation

$$\frac{\partial u}{\partial t} + \mathcal{A}u - ru = 0\,, \tag{1.0.3}$$

where $\mathcal{A}$ is the infinitesimal generator of the process X defined by (1.0.2). Among several possible notions of solution (classical, variational and viscosity solutions, to name the most frequently employed), we opt for *variational solutions* which are the basis for variational discretization methods such as finite element discretizations. To convert (1.0.3) into variational form, we formally integrate against a test function v and obtain (assuming $r = 0$ for convenience)

$$\frac{d}{dt}(u,v) + \underbrace{\mathcal{E}(u,v)}_{(\mathcal{A}u,v)} = 0\,. \tag{1.0.4}$$

Here, the bilinear expression $\mathcal{E}(u,v)$ denotes the extension of the $L^2(\mathbb{R}^d)$ innerproduct $(\mathcal{A}u,v)$ corresponding to X from $u, v \in C_0^\infty(\mathbb{R}^d)$ by continuity to the domain $\mathcal{D}(\mathcal{E})$. For the class of Lévy processes considered in this paper we show that $\mathcal{E}(\cdot,\cdot)$ is in fact a Dirichlet form.

In the univariate case, i.e. for a Lévy process X with state space $\mathbb{R}$, equations (1.0.3), (1.0.4) and methods for their numerical solution have been studied by several authors, e.g. [6, 9, 20, 21] and the references therein. The numerical methods investigated were either finite difference methods [6, 9] approximating viscosity solutions or variational methods [20, 21] approximating weak (or variational) solutions. Both solution concepts coincide for sufficiently smooth solutions, but the resulting numerical schemes have essentially different properties. In [15], the univariate variational setting was extended to $d > 1$ dimensions for pure jump processes built from 1-homogeneous Lévy copulas and univariate marginal Lévy processes with symmetric tempered stable margins. The domain of the infinitesimal generator $\mathcal{A}$ was characterized and it was shown that the corresponding variational problem is well-posed.

The goal of this work is twofold: First, we extend [15] to the multivariate, nonsymmetric case, i.e. when the univariate marginal Lévy processes are tempered stable, but with possibly nonsymmetric margins. Second, we provide further analytical results that are required for an efficient numerical implementation of (1.0.4). We show that in the pure jump case the domain $\mathcal{D}(\mathcal{E})$ of the Dirichlet form $\mathcal{E}(\cdot,\cdot)$ of X belongs to a certain class of anisotropic Sobolev spaces and $\mathcal{E}(\cdot,\cdot)$ satisfies a Gårding inequality on these spaces. In addition, $\mathcal{E}(\cdot,\cdot)$ is cast into several forms which are equivalent on $C_0^\infty(\mathbb{R}^d)$ and which are well-defined for piecewise polynomial, globally Lipschitz continuous arguments. We show that these forms naturally compensate the singularity of the jump measure near zero arising from the square summable small jumps. There is no need to approximate the small jumps by a Brownian motion. These reformulations apply for any Lévy process with state space $\mathbb{R}^d$ and are the basis for a variational discretization of (1.0.3) by e.g. finite element methods. Furthermore, we derive the pricing PIDEs for d-dimensional Lévy models and obtain the corresponding variational formulation with explicit Sobolev characterization of the ansatz and test spaces. Extending [15], we establish sufficient conditions on X to render the bilinear form $\mathcal{E}(\cdot,\cdot)$ a nonsymmetric Dirichlet form in the sense of Berg and Forst [2]. We deduce the existence of a unique solution of the variational formulation of problem for a class of copulas and non-symmetric marginal processes. To allow the implementation of the variational problem, we furthermore localize it to the bounded domain $G_R = [-R,R]^d$ and show that the solution of the localized problem converges pointwise exponentially in R to the exact solution of the original problem.

Throughout this work, we write $x \lesssim y$ to express that x is bounded by a constant multiple of y. For $\mathcal{B} \subset \mathbb{R}^d$, by $1_\mathcal{B} : \mathbb{R}^d \to \{0,1\}$ we denote the indicator function of the set $\mathcal{B}$.

2 Preliminaries

We recapitulate several tools needed subsequently. First, we present some classical facts on Lévy processes and their generators and describe a class of parametric copula constructions for dependence in jumps of multivariate Lévy processes. Finally, we collect some abstract results on variational parabolic evolution and inequality problems.

2.1 Levy processes

We start by recalling essential definitions and properties of Lévy processes.

A càdlàg stochastic process $X = \{X_t : t > 0\}$ with state space $\mathbb{R}^d$ such that $X_0 = 0$ a.s. is called a Lévy process if it has independent and stationary increments and is stochastically continuous.

The characteristic exponent $\psi(\xi)$ of X is defined by

$$\mathbb{E}\left(e^{i\langle \xi, X_t\rangle}\right) = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d, \quad t > 0. \tag{2.1.1}$$

It is a continuous, negative definite function for which we have the Lévy-Khinchin representation (cf. e.g. [26, Theorem 8.1] or [17]),

$$\psi(\xi) = -i\langle \gamma, \xi\rangle + \frac{1}{2}\langle \xi, \mathcal{Q}\xi\rangle + \int_{\mathbb{R}^d} \left(1 - e^{i\langle \xi, z\rangle} + i\langle \xi, z\rangle 1_{\{|z|\leq 1\}}\right)\nu(\mathrm{d}z), \tag{2.1.2}$$

where $\mathcal{Q} \in \mathbb{R}^{d\times d}_{sym}$ denotes the covariance matrix of the continuous part of X, $\gamma \in \mathbb{R}^d$ the drift of X and ν is the Lévy measure which satisfies

$$\int_{\mathbb{R}^d} 1 \wedge |z|^2\,\nu(\mathrm{d}z) < \infty. \tag{2.1.3}$$

The triplet $(\mathcal{Q}, \nu, \gamma)$ is called characteristic triplet of the process X.

No arbitrage considerations require Lévy processes employed in mathematical finance to be martingales. The following result gives conditions on the characteristic triplet which ensure this.

Lemma 2.1. *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q},\nu,\gamma)$. Assume, $\int_{|z|>1} e^{z_j}\nu(\mathrm{d}z) < \infty$, $j = 1,\ldots,d$. Then e^{X^j} is a martingale with respect to the canonical filtration $\mathcal{F}$ of X if and only if*

$$\frac{\mathcal{Q}_{jj}}{2} + \gamma_j + \int_{\mathbb{R}^d} \left(e^{z_j} - 1 - z_j 1_{\{|z|\le 1\}}\right)\nu(\mathrm{d}z) = 0\,.$$

Proof. The result is obtained using the independent and stationary increments property and the Lévy-Khinchin formula (2.1.2),

$$\begin{aligned}\mathbb{E}\left(e^{X_s^j} \mid \mathcal{F}_t\right) &= \mathbb{E}\left(e^{X_t^j + X_s^j - X_t^j} \mid \mathcal{F}_t\right) = e^{X_t^j}\mathbb{E}\left(e^{X_s^j - X_t^j}\right)\\ &= e^{X_t^j}\mathbb{E}\left(e^{X_{s-t}^j}\right) = e^{X_t^j} e^{(t-s)\psi(-ie_j)}\,,\end{aligned}$$

with $s \ge t$. Here e_j denotes the j-th unit vector in $\mathbb{R}^d$. □

Based on $\psi(\xi)$ in (2.1.1), it is well-known that the infinitesimal generator $\mathcal{A}$ in (1.0.2) corresponding to the Lévy process X is a pseudodifferential operator acting on $u \in C_0^\infty(\mathbb{R}^d)$ by the (oscillatory) integral

$$(\mathcal{A}u)(x) = (\psi(D)u)(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}\psi(\xi)\hat{u}(\xi)\,\mathrm{d}\xi\,, \tag{2.1.4}$$

where $\hat{u}(\xi) := (2\pi)^{-d}\int e^{-i\langle\xi,z\rangle}u(z)\,\mathrm{d}z$ denotes the Fourier transform of u.

2.2 Lévy copulas

Since the law of a Lévy process X is time-homogeneous, it is completely characterized by its characteristic triplet $(\mathcal{Q},\nu,\gamma)$. The drift γ has no effect on dependence structure between components of X. The dependence structure of the Brownian motion part of X is given by its covariance matrix $\mathcal{Q}$. Since the continuous part and the jump part of a Lévy process are independent, it remains to determine the dependence structure of the purely discontinuous part of X. To this end, we next describe a class of parametric copula models of dependence in the jump components of a Lévy process X proposed first by Tankov [30] and further developed by Kallsen and Tankov [19].

At first, we introduce some notation and definitions. Let $\overline{\mathbb{R}} := (-\infty,\infty]$ and

$$\operatorname{sgn} x = \begin{cases} 1 & \text{for } x \ge 0\\ -1 & \text{for } x < 0\,.\end{cases}$$

For $a, b \in \overline{\mathbb{R}}^d$ we write $a \le b$ if $a_k \le b_k$, $1,\ldots,d$. In this case $(a,b]$ denotes a half-open interval

$$(a,b] := (a_1,b_1] \times \cdots \times (a_d,b_d]\,.$$

The F-volume of $(a,b]$ for a function $F : S \to \overline{\mathbb{R}}$, $S \subset \overline{\mathbb{R}}$ is defined by

$$V_F\left((a,b]\right) := \sum_{u\in\{a_1,b_1\}\times\cdots\times\{a_d,b_d\}} (-1)^{N(u)}F(u)\,,$$

where $N(u) = |\{k : u_k = a_k\}|$.

Definition 2.2. *A function $F : S \to \overline{\mathbb{R}}$, $S \subset \overline{\mathbb{R}}^d$ is called d-increasing if*

$$V_F\left((a,b]\right) \ge 0$$

for all $a, b \in S$ with $a \le b$ and $\overline{(a,b]} \subset S$.

For modeling dependence structure, margins play an important role.

Definition 2.3. *Let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be a d-increasing function which satisfies*

$$F(u_1, \ldots, u_d) = 0 \text{ if } u_i = 0 \text{ for at least one } i \in \{1, \ldots, d\}.$$

Furthermore, let $I \subset \{1, \ldots, d\}$ be a nonempty index set and denote with $I^c := \{1, \ldots, d\} \setminus I$ its complement. Then, the I-margin *of F is the function $F^I : \overline{\mathbb{R}}^{|I|} \to \overline{\mathbb{R}}$*

$$F^I\left((u_i)_{i \in I}\right) := \lim_{b \to \infty} \sum_{(u_j)_{j \in I^c} \in \{-b, \infty\}^{|I^c|}} \left(\prod_{j \in I^c} \operatorname{sgn} u_j \right) F(u_1, \ldots, u_d).$$

Since the Lévy measure is a measure on $\mathbb{R}^d$, it is possible to define a suitable notion of a copula. However, one has to take into account that the Lévy measure is possibly infinite at the origin.

Definition 2.4. *A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called Lévy copula if*

1. *$F(u_1, \ldots, u_d) \neq \infty$ for $(u_1, \ldots, u_d) \neq (\infty, \ldots, \infty)$,*
2. *$F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$,*
3. *F is d-increasing,*
4. *$F^{\{i\}}(u) = u$ for any $i \in \{1, \ldots, d\}$, $u \in \mathbb{R}$.*

We also need to introduce the tail integrals of a Lévy processes.

Definition 2.5. *Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν. The tail integral of X is the function $U : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ given by*

$$U(x_1, \ldots, x_d) = \prod_{j=1}^{d} \operatorname{sgn}(x_j) \nu \left(\prod_{j=1}^{d} \mathcal{I}(x_j) \right),$$

where

$$\mathcal{I}(x) = \begin{cases} (x, \infty) & \text{for } x \geq 0 \\ (-\infty, x] & \text{for } x < 0 \end{cases}.$$

Furthermore, for $I \subset \{1, \ldots, d\}$ nonempty the I-marginal tail integral U^I of X is the tail integral of the process $X^I := (X_i)_{i \in I}$. If $I = \{i\}$, we write $U_i = U^{\{i\}}$. We also use the notation $x^I := (x_i)_{i \in I}$ and for $x \in \mathbb{R}^d$, $y \in \mathbb{R}^{|I|}$

$$x + y^I = z \in \mathbb{R}^d \quad \text{with } z_i = \begin{cases} x_i, & \text{if } i \notin I, \\ x_i + y_i, & \text{else.} \end{cases}$$

The next result, [19, Theorem 3.6], shows that essentially any Lévy process $X \in \mathbb{R}^d$ can be built from univariate marginal processes X_i and Lévy copulas.

Theorem 2.6 (Sklar's theorem for Lévy copulas). *For any Lévy process X with state space $\mathbb{R}^d$ there exists a Lévy copula F such that the tail integrals of X satisfy*

$$U^I\left(x^I\right) = F^I\left((U_i(x_i))_{i \in I}\right), \tag{2.2.1}$$

for any nonempty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in \mathbb{R}^{|I|} \setminus \{0\}$. The Lévy copula F is unique on $\prod_{i=1}^d \overline{\text{Range } U_i}$.

Conversely, let F be a d-dimensional Lévy copula and U_i, $i = 1, \ldots, d$, tail integrals of univariate Lévy processes. Then, there exits a d-dimensional Lévy process X such that its components have tail integrals U_i and its marginal tail integrals satisfy (2.2.1). The Lévy measure ν of X is uniquely determined by F and U_i, $i = 1, \ldots, d$.

Lévy copulas F allow parametric constructions of multivariate jump densities from univariate ones.

Remark 2.7. *Let $U_1, \dots, U_d$ be one dimensional tail integrals with Lévy density $k_1, \dots, k_d$ and let F be a Lévy copula such that $\partial_1 \dots \partial_d F$ exists in the sense of distributions. Then,*

$$k(x_1, \dots, x_d) = \partial_1 \dots \partial_d F|_{\xi_1 = U_1(x_1), \dots, \xi_d = U_d(x_d)} k_1(x_1) \dots k_d(x_d)\,,$$

is the jump density of a d-variate Lévy measure with marginal Lévy densities $k_1, \dots, k_d$.

Using partial integration we can write the multidimensional Lévy density in terms of the Lévy copula.

Lemma 2.8. *Let $f \in C^\infty(\mathbb{R}^d)$ be bounded and vanishing on a neighborhood of the origin. Furthermore, let X be a d-dimensional Lévy process with Lévy measure ν, Lévy copula F and marginal Lévy measures ν_i, $i = 1, \dots, d$. Then,*

$$\begin{aligned}\int_{\mathbb{R}^d} f(z)\nu(\mathrm{d}z) = &\sum_{j=1}^{d} \int_{\mathbb{R}} f(0 + z_j)\nu_j(\mathrm{d}z_j) \\ &+ \sum_{j=2}^{d} \sum_{\substack{|I|=j \\ I_1 < \dots < I_j}} \int_{\mathbb{R}^j} \frac{\partial^j f}{\partial z^I}(0 + z^I) F^I\left((U_k(z_k))_{k \in I}\right) \mathrm{d}z^I\,.\end{aligned} \tag{2.2.2}$$

Proof. We proceed by induction with respect to the dimension d. For $d = 1$, integration by parts yields

$$\begin{aligned}\int_0^\infty f(z)\nu(\mathrm{d}z) &= -\lim_{b \to \infty} f(b)\nu(\mathcal{I}(b)) + \lim_{a \to 0+} f(a)\nu(\mathcal{I}(a)) + \int_0^\infty \frac{\partial f}{\partial z}(z)\nu(\mathcal{I}(z))\,\mathrm{d}z\,, \\ \int_{-\infty}^0 f(z)\nu(\mathrm{d}z) &= \lim_{a \to 0-} f(a)\nu(\mathcal{I}(a)) - \lim_{b \to -\infty} f(b)\nu(\mathcal{I}(b)) - \int_{-\infty}^0 \frac{\partial f}{\partial z}(z)\nu(\mathcal{I}(z))\,\mathrm{d}z\,,\end{aligned}$$

and since f is bounded

$$\int_{\mathbb{R}} f(z)k(z)\,\mathrm{d}z = f(0) \lim_{a \to 0+} \left(\nu(\mathcal{I}(a)) + \nu(\mathcal{I}(-a))\right) + \int_{\mathbb{R}} \frac{\partial f}{\partial z}(z)\mathrm{sgn}(z)\nu(\mathcal{I}(z))\,\mathrm{d}z\,.$$

Abusing notation, we write

$$\nu(\mathbb{R}) := \lim_{a \to 0+} \left(\nu(\mathcal{I}(a)) + \nu(\mathcal{I}(-a))\right)$$

With f vanishing on a neighborhood of 0 we therefore find $f(0)\nu(\mathbb{R}) = 0$.

For the multidimensional case we use that by [26, Proposition 11.10] the Lévy measure of X^I is given by

$$\nu^I(B) = \nu\left(\{x \in \mathbb{R}^d : (x_i)_{i \in I} \in B \setminus \{0\}\}\right), \quad B \in \mathcal{B}\left(\mathbb{R}^{|I|}\right)\,.$$

We show by induction with respect to the dimension d that

$$\begin{aligned}\int_{\mathbb{R}^d} f(z)\nu(\mathrm{d}z) = &f(0, \dots, 0)\nu(\mathbb{R}, \dots, \mathbb{R}) \\ &+ \sum_{i=1}^{d} \int_{\mathbb{R}} \frac{\partial f}{\partial z_i}(0, \dots, z_i, \dots, 0)\mathrm{sgn}(z_i)\nu_i(\mathcal{I}(z_i))\,\mathrm{d}z_i \\ &+ \sum_{i=2}^{d} \sum_{\substack{|I|=i \\ I_1 < \dots < I_i}} \int_{\mathbb{R}^i} \frac{\partial^i f}{\partial z^I}(0 + z^I) \prod_{j \in I} \mathrm{sgn}(z_j) \nu^I\left(\prod_{j \in I} \mathcal{I}(z_j)\right) \mathrm{d}z^I\,.\end{aligned}$$

With $f(0, \dots, 0)\nu(\mathbb{R}, \dots, \mathbb{R}) = 0$, the definition of the tail integrals and Theorem 2.6 we then have the required result.

For the induction step $d-1 \to d$, using integration by parts and the induction hypothesis we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} f(z)\nu(\mathrm{d}z) = \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} f(z', z_d)\nu(\mathrm{d}z', \mathrm{d}z_d) \\
&\quad = \int_{\mathbb{R}^{d-1}} f(z', 0)\nu(\mathrm{d}z', \mathbb{R}) \\
&\qquad + \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{\partial f}{\partial z_d}(z', z_d)\mathrm{sgn}(z_d)\nu\left(\mathrm{d}z', \mathcal{I}(z_d)\right) \mathrm{d}z_d \\
&\quad = f(0,\dots,0)\nu(\mathbb{R},\dots,\mathbb{R}) \\
&\qquad + \sum_{i=1}^{d-1} \int_{\mathbb{R}} \frac{\partial f}{\partial z_i}(0,\dots,z_i,\dots,0)\mathrm{sgn}(z_i)\nu_i\left(\mathcal{I}(z_i)\right) \mathrm{d}z_i \\
&\qquad + \sum_{i=2}^{d-1} \sum_{\substack{|I|=i \\ I_1<\dots<I_i}} \int_{\mathbb{R}^i} \frac{\partial^i f}{\partial z^I}(0+z^I) \prod_{j\in I} \mathrm{sgn}(z_j)\nu^I\left(\prod_{j\in I}\mathcal{I}(z_j)\right) \mathrm{d}z^I \\
&\qquad + \int_{\mathbb{R}} \frac{\partial f}{\partial z_d}(0,\dots,0,z_d)\mathrm{sgn}(z_d)\nu\left(\mathbb{R},\dots,\mathbb{R},\mathcal{I}(z_d)\right) \\
&\qquad + \sum_{i=1}^{d-1} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\partial^2 f}{\partial z_i z_d}(0,\dots,z_i,\dots,0,z_d)\mathrm{sgn}(z_i)\mathrm{sgn}(z_d)\nu_{i,d}\left(\mathcal{I}(z_i),\mathcal{I}(z_d)\right) \mathrm{d}z_i\,\mathrm{d}z_d \\
&\qquad + \sum_{i=2}^{d-1} \sum_{\substack{|I|=i \\ I_1<\dots<I_i}} \int_{\mathbb{R}^i} \int_{\mathbb{R}} \frac{\partial^{i+1} f}{\partial z^I z_d}(z^{\{I,d\}}) \prod_{j\in\{I,d\}} \mathrm{sgn}(z_j)\nu^{\{I,d\}}\left(\prod_{j\in\{I,d\}}\mathcal{I}(z_j)\right) \mathrm{d}z^I\,\mathrm{d}z_d\,,
\end{aligned}
$$

which is the claimed result. □

Remark 2.9. *The boundedness assumption on f in Lemma 2.8 can be weakened to certain unbounded $f \in C^d(\mathbb{R})$ if the Lévy measure ν decays sufficiently fast.*

We conclude this introductory section with examples of Lévy copulas.

Example 2.10. *Examples of Lévy copulas are:*

1. *Independence Lévy copula*

$$F(u_1,\dots,u_d) = \sum_{i=1}^{d} u_i \prod_{j\neq i} 1_{\{\infty\}}(u_j)\,. \tag{2.2.3}$$

2. *Complete dependence Lévy copula*

$$F(u_1,\dots,u_d) = \min(|u_1|,\dots,|u_d|)1_K(u_1,\dots,u_d)\prod_{j=1}^{d}\mathrm{sgn}\,u_j\,, \tag{2.2.4}$$

where $K := \{x\in\mathbb{R}^d : \mathrm{sgn}(x_1) = \ldots = \mathrm{sgn}(x_d)\}$.

3. *Clayton Lévy copulas*

$$F(u_1,\dots,u_d) = 2^{2-d}\left(\sum_{i=1}^{d}|u_i|^{-\theta}\right)^{-\frac{1}{\theta}}\left(\eta 1_{\{u_1\cdots u_d\geq 0\}} - (1-\eta)1_{\{u_1\cdots u_d\leq 0\}}\right), \tag{2.2.5}$$

where $\theta > 0$ and $\eta\in[0,1]$. For $\eta = 1$ and $\theta\to 0$, F converges to the independence Lévy copula, for $\eta = 1$ and $\theta\to\infty$ to the complete dependence Lévy copula.

For further details and examples of Lévy copulas, we refer to [15, 19].

2.3 Variational Parabolic Problems

The bilinear form $\mathcal{E}(\cdot,\cdot)$ associated to X is the basis for the variational formulation of the Kolmogorov equation (1.0.3) which we will now describe. The variational formulation is, in turn, the basis for Galerkin discretizations of the Kolmogorov equations.

To cover equations arising from optimal stopping (as e.g. for American style contracts) as well as from optimal control problems (as e.g. in portfolio optimization and for options of game-type), and in order to accommodate rough payoff functions, the rather general variational framework from [4, 5, 16] is adopted.

The variational setting will be based on the real Gelfand triple with Hilbert space $\mathcal{H}$:

$$\mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}^* \subset \mathcal{V}^* . \tag{2.3.1}$$

For $\mathcal{V}$, we have in mind the domain of $\mathcal{E}(\cdot,\cdot)$. For the infinitesimal generator $\mathcal{A}$ of X, and the corresponding bilinear form

$$\mathcal{E}(u,v) := (\mathcal{A}u, v), \qquad u,v \in \mathcal{V}, \tag{2.3.2}$$

we assume that there exist constants C_1, $C_2 > 0$ and $\lambda \geq 0$ such that for all $u, v \in \mathcal{V}$ there holds

$$\forall u,v \in \mathcal{V}: \quad \mathcal{E}(u,v) \quad \leq \quad C_1 \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}} , \tag{2.3.3}$$

$$\forall u \in \mathcal{V}: \quad \mathcal{E}(u,u) \quad \geq \quad C_2 \|u\|_{\mathcal{V}}^2 - \lambda \|u\|_{\mathcal{H}}^2 . \tag{2.3.4}$$

Moreover, we denote by $(\cdot,\cdot)$ the $\mathcal{H}$ innerproduct, which admits a unique extension by continuity to $\mathcal{V}^* \times \mathcal{V}$ in (2.3.1). For clarity, we denote this extension by $\langle \cdot,\cdot \rangle_{\mathcal{V}^*\times\mathcal{V}}$.

As already illustrated in the introduction, prices of European style contracts contracts are solutions of Kolmogorov *equations*. Their abstract variational formulation reads: given an initial value $u_0 \in \mathcal{H}$ and $f \in L^2(0,T;\mathcal{V}^*)$,

$$\text{find } u \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}^*) \text{ such that}$$

$$\langle \frac{\partial u}{\partial t}, v \rangle_{\mathcal{V}^*\times\mathcal{V}} + \mathcal{E}(u,v) = \langle f, v\rangle_{\mathcal{V}^*\times\mathcal{V}} \quad \forall v \in \mathcal{V}, \quad \text{a.e. in } (0,T), \tag{2.3.5}$$

$$u(0) = u_0 \quad \text{in} \quad \mathcal{H}. \tag{2.3.6}$$

There holds

Theorem 2.11. *Assume the bilinear form $\mathcal{E}(\cdot,\cdot)$ satisfies* (2.3.3), (2.3.4). *Then, the abstract parabolic problem* (2.3.5)–(2.3.6) *admits a unique solution.*

Remark 2.12. *If instead of a Lévy process X, one considers a general strong Markov process with time dependent infinitesimal generator $\mathcal{A}(t)$ and corresponding bilinear form $\mathcal{E}(t;u,v) = (\mathcal{A}(t)u,v)$, then Theorem 2.11 remains valid provided that for all $u,v \in \mathcal{V}$ the mapping $t \mapsto \mathcal{E}(t,u,v)$ is measurable.*

Remark 2.13. *The initial condition $u(0) = u_0$ is required to hold in $\mathcal{H}$, not in $\mathcal{V}$. Due to the embedding $L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}^*) \subset C^0([0,T];\mathcal{H})$, the initial condition* (2.3.6) *makes sense and the parabolic evolution problem is well-posed even for initial data u_0 belonging to $\mathcal{H}$, but not to $\mathcal{V}$. For example, this is the case in European derivative contracts with discontinuous payoffs, such as binary options.*

For the study of optimal stopping problems which arise e.g. from American contracts we require variational formulations of parabolic variational *inequalities*. To this end, let $\emptyset \neq \mathcal{K} \subset \mathcal{V}$ be a closed, non-empty and convex subset of $\mathcal{V}$ with *indicator function*

$$\phi(v) := I_{\mathcal{K}}(v) = \begin{cases} 0, & \text{if } v \in \mathcal{K}, \\ +\infty, & \text{else.} \end{cases} \tag{2.3.7}$$

This is a proper, convex lower semicontinuous (l.s.c.) function $\phi : \mathcal{V} \to \overline{\mathbb{R}}$ with domain $\mathcal{D}(\phi) = \{v \in \mathcal{V} : \phi(v) < \infty\}$. We denote by $\overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}}$ the closure of $\mathcal{D}(\phi)$ in $\mathcal{H}$ and consider the following variational problem: given $f \in L^2(0,T;\mathcal{V}^*)$, $u_0 \in \overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}} \subset \mathcal{H}$,

find $u \in L^2(0,T;\mathcal{V}) \cap H^1(0,T;\mathcal{V}^*)$ such that $u \in \mathcal{D}(\phi)$ a.e. in $(0,T)$ and

$$\langle \frac{\partial u}{\partial t} + \mathcal{A}u - f, u - v\rangle_{\mathcal{V}^* \times \mathcal{V}} + \phi(u) - \phi(v) \geq 0 \quad \forall v \in \mathcal{D}(\phi), \quad \text{a.e. in } (0,T), \tag{2.3.8}$$

$$u(0) = u_0 \quad \text{in} \quad \mathcal{H}. \tag{2.3.9}$$

Existence and uniqueness results for solutions $u \in L^2(0,T;\mathcal{V})$ of (2.3.8)–(2.3.9) can be obtained from e.g. [16, Theorem 6.2.1] under rather strict conditions on the data $f(t)$. To derive the well-posedness of (2.3.8)–(2.3.9) under minimal regularity conditions on $f(t)$, u_0 and ϕ, the problem needs to be replaced by a *weak variational formulation*. To state it, introduce the integral functional Φ on $L^2(0,T;\mathcal{V})$

$$\Phi(v) = \begin{cases} \displaystyle\int_0^T \phi(v(t))e^{-2\lambda t}dt, & \text{if } \phi(v) \in L^1(0,T), \\ +\infty, & \text{else,} \end{cases} \tag{2.3.10}$$

with $\lambda \geq 0$ as in (2.3.4).

Note that $\Phi(\cdot)$ is proper convex and l.s.c. with domain

$$\mathcal{D}(\Phi) = \{v \in L^2(0,T;\mathcal{V}) : \phi(v) \in L^1(0,T)\}. \tag{2.3.11}$$

Herewith, the weak variational formulation of (2.3.8)–(2.3.9) reads (cf. [1, 27]): given $u_0 \in \overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}} \subset \mathcal{H}$ and $f \in L^2(0,T;\mathcal{V}^*)$,

find $u \in L^\infty(0,T;\mathcal{H}) \cap \mathcal{D}(\Phi)$ such that $u(0) = u_0$ in $\mathcal{H}$ and

$$\int_0^T \langle \frac{\partial v}{\partial t}(t) + (\mathcal{A}+\lambda)u(t) - (f+\lambda v), u(t) - v(t)\rangle \cdot e^{-2\lambda t}dt + \Phi(u) - \Phi(v) \leq \frac{1}{2}\|u_0 - v(0)\|^2_{\mathcal{H}}, \tag{2.3.12}$$

for all $v \in \mathcal{D}(\Phi)$ with $\frac{\partial v}{\partial t} \in L^2(0,T;\mathcal{V}^*)$.

The well-posedness of (2.3.12) is ensured by [27]:

Theorem 2.14. *Assume that the bilinear form $\mathcal{E}(\cdot,\cdot)$ satisfies* (2.3.3)–(2.3.4)*. Then problem* (2.3.12) *admits a unique solution*

$$u \in L^2(0,T;\mathcal{V}) \cap L^\infty(0,T;\mathcal{H}) \quad \text{such that } t \mapsto \phi(u(t,\cdot)) \in L^1(0,T).$$

Remark 2.15. *As for the parabolic equality problem* (2.3.5)–(2.3.6)*, also for* (2.3.12) *the initial condition is only required to hold in $\mathcal{H}$. In addition, however, in* (2.3.12) *the data u_0 must belong to the closure $\overline{\mathcal{K}}^{\|\circ\|_{\mathcal{H}}}$ of $\mathcal{K}$ in $\mathcal{H}$.*

Remark 2.16. *Convergence rates of backward Euler time discretizations of* (2.3.12) *for American style contracts under minimal regularity are given in [1, 23, 27].*

3 Properties of Lévy measures built from Lévy copulas

In the present section, we verify properties of Lévy measures corresponding to multivariate Lévy processes X with state space $\mathbb{R}^d$ built from so-called tempered stable, univariate Lévy processes X^i by 1-homogeneous Lévy copulas as constructed in Section 2.2 above. For the (in general nonsymmetric) bilinear form $\mathcal{E}(\cdot,\cdot)$ corresponding to the generator $\mathcal{A}$ of X, we verify the so-called *sector condition*. Due to a classical result of Berg and Forst [2] (see also [17, Chapter 4.7]) this, in conjunction with the translation invariance of X, implies that $\mathcal{E}(\cdot,\cdot)$ is a nonsymmetric Dirichlet form. It also allows to give an explicit characterization of the domains $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{E})$ of $\mathcal{A}$ and $\mathcal{E}(\cdot,\cdot)$ in terms of anisotropic Sobolev spaces.

3.1 Semiheavy tails

At first, we show that the tails of the multivariate Lévy processes stemming from the copula construction decay exponentially fast provided the one-dimensional marginal processes are of tempered stable type in the sense of [3], i.e. the corresponding densities decay exponentially at infinity.

We use the following assumptions on the marginal Lévy measures ν_i, $i = 1, \dots, d$. These are satisfied by a wide range of Lévy models [21].

Assumption 3.1. *Let X be a Lévy process with state space $\mathbb{R}^d$, characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ and marginal Lévy measures ν_i, $i = 1, \dots, d$, with densities k_i. There are constants $G_i > 0$, $M_i > 0$, $i = 1, \dots, d$ such that*

$$k_i(z) \lesssim \begin{cases} e^{-G_i|z|}, & z < -1, \\ e^{-M_i|z|}, & z > 1. \end{cases} \tag{3.1.1}$$

The tail behavior (3.1.1) carries over to the d-variate case.

Proposition 3.2. *Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν such that the marginal measures ν_i satisfy* (3.1.1)*. Then, the Lévy measure ν also decays exponentially*

$$\int_{|z|>1} e^{\eta(z)} \nu(\mathrm{d}z) < \infty \,, \qquad \textit{with} \quad \eta(z) = \sum_{i=1}^{d} \left(\mu_i^+ 1_{\{z_i>0\}} + \mu_i^- 1_{\{z_i<0\}} \right) |z_i| \,,$$

where $0 < \mu_i^- < \frac{G_i}{d}$, $0 < \mu_i^+ < \frac{M_i}{d}$, $i = 1, \dots, d$. Furthermore, for each $i = 1, \dots, d$ there holds

$$\int_{|z|>1} e^{\eta_i(z)} \nu(\mathrm{d}z) < \infty \,, \qquad \textit{with} \quad \eta_i(z) = \left(\mu_i^+ 1_{\{z_i>0\}} + \mu_i^- 1_{\{z_i<0\}} \right) |z_i| \,,$$

where now $0 < \mu_i^- < G_i$ and $0 < \mu_i^+ < M_i$, $i = 1, \dots, d$.

Proof. Using [26, Proposition 11.10] as in Lemma 2.8 we obtain

$$\int_{|z|>1} e^{\sum_{i=1}^d \mu_i |z_i|} \nu(\mathrm{d}z) \lesssim \sum_{i=1}^{d} \int_{|z|>1} e^{d\mu_i|z_i|} \nu(\mathrm{d}z) \lesssim \sum_{i=1}^{d} \int_{|z_i|>1} e^{d\mu_i|z_i|} \nu_i(\mathrm{d}z_i) < \infty \,.$$

□

3.2 Sector condition

We prove the so-called sector condition

$$|\mathrm{Im}\, \psi(\xi)| \lesssim \mathrm{Re}\, \psi(\xi), \qquad \text{for all } \xi \in \mathbb{R}^d. \tag{3.2.1}$$

Since the Lévy process' bilinear form is in general a nonsymmetric bilinear form due to the asymmetric jump structure in financial models, this condition is necessary for the bilinear form to be a Dirichlet form. Additionally, it allows to give an explicit characterization of the domains $\mathcal{D}(\mathcal{A})$ and $\mathcal{D}(\mathcal{E})$ of the infinitesimal generator and bilinear form of X, cf. [2] and [17, Example 4.7.32].

Assumption 3.3. *Let X be a Lévy process with state space $\mathbb{R}^d$, characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ and marginal Lévy measures ν_i, $i = 1, \dots, d$ with densities k_i. There are constants $0 < Y_i < 2$ and $c_i^+, c_i^- \geq 0$, $c_i^+ + c_i^- > 0$, $i = 1, \dots, d$ such that*

$$k_i(z) \gtrsim c_i^- \frac{1}{|z|^{1+Y_i}} 1_{\{z<0\}}(z) + c_i^+ \frac{1}{|z|^{1+Y_i}} 1_{\{0 \leq z\}}(z) \quad 0 < |z| \leq 1 \,, \tag{3.2.2}$$

$$k_i(z) \lesssim c_i^- \frac{1}{|z|^{1+Y_i}} 1_{\{z<0\}}(z) + c_i^+ \frac{1}{|z|^{1+Y_i}} 1_{\{0 \leq z\}}(z) \quad 0 < |z| \leq 1 \,. \tag{3.2.3}$$

Example 3.4. *For instance, Assumption 3.3 is satisfied by non-symmetric tempered stable (CGMY) margins as in [7] and spectrally negative margins, i.e.*

$$k_i(z) = \begin{cases} C_i \dfrac{e^{-G_i|z|}}{|z|^{1+Y_i}}, & z<0, \\ C_i \dfrac{e^{-M_i|z|}}{|z|^{1+Y_i}}, & z>0, \end{cases} \quad \text{and} \quad k_i(z) = \begin{cases} C_i \dfrac{e^{-G_i|z|}}{|z|^{1+Y_i}}, & z<0, \\ 0, & z>0, \end{cases}$$

with $G_i,\ M_i \geq 0$*,* $i = 1, \dots, d$*.*

The following proposition provides an upper bound for $|\psi(\xi)|$ and hence for $|\operatorname{Im}\psi(\xi)|$.

Proposition 3.5. *Let X be a Lévy process with state space $\mathbb{R}^d$, characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ and characteristic exponent ψ. Assume $\mathcal{Q} = 0$ and $\gamma_i = 0$, $i = 1, \dots, d$, and the marginal Lévy measures ν_i, $i = 1, \dots, d$, satisfy* (3.2.3)*. Then for $\|\xi\|_\infty > 1$ there holds*

$$|\psi(\xi)| \lesssim \sum_{j=1}^{d} |\xi_j|^{Y_j} \,. \tag{3.2.4}$$

Proof. For notational convenience we assume without loss of generality that there are only positive jumps. We distinguish the cases Y_i smaller or larger than 1. After possibly re-numbering coordinates, let $0 \leq j \leq d$ be such that

$$Y_1, \dots, Y_j < 1, \quad 1 \leq Y_{j+1}, \dots, Y_d < 2\,.$$

Then the characteristic exponent ψ can be written as

$$\psi(\xi) = \int_{\mathbb{R}^d_{\geq 0}} \left(1 - e^{i\langle \xi, z\rangle} + \sum_{k=j+1}^{d} i\xi_k z_k 1_{|z|\leq 1} \right) \nu(\mathrm{d}z) + i \sum_{k=1}^{j} \widetilde{\gamma}_k \xi_k \,.$$

Without loss of generality, we set $\widetilde{\gamma}_k$, $k = 1, \dots, j$, to zero.

With $B = [0, \frac{1}{d|\xi_1|}] \times \cdots \times [0, \frac{1}{d|\xi_d|}]$ we obtain

$$\begin{aligned} |\psi(\xi)| &\lesssim \int_{[0,1]^d} \left| 1 - e^{i\langle \xi, z\rangle} + \sum_{k=j+1}^{d} i\xi_k z_k \right| \nu(\mathrm{d}z) + 1 \\ &\lesssim \int_B \left| 1 - e^{i\langle \xi, z\rangle} + \sum_{k=j+1}^{d} i\xi_k z_k \right| \nu(\mathrm{d}z) + \int_{[0,1]^d \setminus B} \left(1 + \sum_{k=j+1}^{d} |\xi_k z_k| \right) \nu(\mathrm{d}z) + 1. \end{aligned} \tag{3.2.5}$$

We estimate the first term in (3.2.5) as follows:

$$\begin{aligned} \int_B \left| 1 - e^{i\langle \xi, z\rangle} + \sum_{k=j+1}^{d} i\xi_k z_k \right| \nu(\mathrm{d}z) &\lesssim \int_B \left(\sum_{k=1}^{j} |\xi_k z_k| + \sum_{k=j+1}^{d} \xi_k^2 z_k^2 \right) \nu(\mathrm{d}z) \\ &\lesssim \sum_{k=1}^{j} \int_0^{\frac{1}{|\xi_k|}} |\xi_k z_k| \, \nu_k(\mathrm{d}z_k) + \sum_{k=j+1}^{d} \int_0^{\frac{1}{|\xi_k|}} \xi_k^2 z_k^2 \nu_k(\mathrm{d}z_k) \\ &\lesssim \sum_{k=1}^{j} \int_0^{\frac{1}{|\xi_k|}} |\xi_k z_k| \, \frac{1}{z_k^{Y_k+1}} \mathrm{d}z_k + \sum_{k=j+1}^{d} \int_0^{\frac{1}{|\xi_k|}} \xi_k^2 z_k^2 \frac{1}{z_k^{Y_k+1}} \mathrm{d}z_k \\ &\lesssim \sum_{k=1}^{d} |\xi_k|^{Y_k} \,. \end{aligned}$$

To estimate the second term in (3.2.5), note that if $z \in [0,1]^d \backslash B$ with $z_k \leq \frac{1}{d|\xi_k|}$, there exists l_k such that $z_{l_k} \geq \frac{1}{d|\xi_{l_k}|}$.

$$\begin{aligned}
\int_{[0,1]^d \backslash B} \left(1 + \sum_{k=j+1}^{d} |\xi_k z_k|\right) \nu(\mathrm{d}z) &\leq \sum_{k=j+1}^{d} \int_{-\infty}^{\infty} \cdots \int_{\frac{1}{d|\xi_k|}}^{1} \cdots \int_{-\infty}^{\infty} (1 + |\xi_k z_k|)\, \nu(\mathrm{d}z) \\
&\quad + \sum_{k=j+1}^{d} \int_{-\infty}^{\infty} \cdots \int_{0}^{\frac{1}{d|\xi_k|}} \cdots \int_{\frac{1}{d|\xi_{l_k}|}}^{1} \cdots \int_{-\infty}^{\infty} (1 + |\xi_k z_k|)\, \nu(\mathrm{d}z) \\
&\leq \sum_{k=j+1}^{d} \int_{\frac{1}{d|\xi_k|}}^{1} (1 + |\xi_k z_k|)\, \nu_k(\mathrm{d}z_k) \\
&\quad + \sum_{k=j+1}^{d} \int_{-\infty}^{\infty} \cdots \int_{0}^{\frac{1}{d|\xi_k|}} \cdots \int_{\frac{1}{d|\xi_{l_k}|}}^{1} \cdots \int_{-\infty}^{\infty} \left(1 + \frac{1}{d}\right) \nu(\mathrm{d}z) \\
&\lesssim 1 + \sum_{k=j+1}^{d} |\xi_k|^{Y_k} + \sum_{k=j+1}^{d} |\xi_k| + \sum_{k=j+1}^{d} \int_{\frac{1}{d|\xi_{l_k}|}}^{1} \nu_{l_k}(\mathrm{d}z_{l_k}) \\
&\lesssim 1 + \sum_{k=1}^{d} |\xi_k|^{Y_k} + \sum_{k=j+1}^{d} |\xi_k| \, .
\end{aligned}$$

Therefore, we obtain for $\|\xi\|_\infty > 1$

$$|\psi(\xi)| \lesssim \sum_{k=1}^{d} |\xi_k|^{Y_k} \, .$$

□

In order to prove (3.2.1), we also require a lower bound on $\operatorname{Re}\psi(\xi)$. For this, we need to make a few technical assumptions on the underlying copula F. To state these assumptions we introduce some notation.

Definition 3.6. *For $m \in \mathbb{R}$, a function $F : \mathbb{R}^d \to \mathbb{R}$ is called* m-homogeneous *if for any $r > 0$ there holds*

$$F(rx_1, \ldots, rx_d) = r^m F(x_1, \ldots, x_d), \qquad \text{for all } (x_1, \ldots, x_d) \in \mathbb{R}^d \backslash \{0\}.$$

Definition 3.7. *Let $\mathcal{I} \subset \mathbb{R}$. Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are called equivalent on $\mathcal{I}$ if there exists a constant $c > 0$ such that*

$$c \cdot |f(x)| \leq |g(x)| \leq c^{-1} \cdot |f(x)|, \qquad \text{for all } x \in \mathcal{I}.$$

We denote the equivalence of f and g by $f \sim g$.

Definition 3.8. *A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called equivalence preserving if for any two families of equivalent functions $f_i \sim g_i$, $i = 1, \ldots, d$, on some $\mathcal{I} \subset \mathbb{R}$, there exists a constants $C > 0$ such that*

$$C \cdot F(f_1(x_1), \ldots, f_d(x_d)) \leq F(g_1(x_1), \ldots, g_d(x_d)) \leq C^{-1} \cdot F(f_1(x_1), \ldots, f_d(x_d)),$$

for all $x \in \mathcal{I}^d$.

We can now state the sufficient assumptions on the Lévy copula.

Assumption 3.9. *Let X be a Lévy process with state space $\mathbb{R}^d$ and 1-homogeneous Lévy copula F. The derivative $\partial_1 \ldots \partial_d F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ exists in the sense of distributions (i.e. the multivariate process admits a Lévy kernel) and is equivalence preserving.*

One readily infers that for instance the independence copula (2.2.3) satisfies Assumption 3.9. Nonetheless, the equivalence preserving property of $\partial_1 \dots \partial_d F$ is non-trivial in general. We prove it for a wide class of Lévy copulas in Appendix A. But first, under Assumption 3.9, one obtains the required lower bound of $\operatorname{Re}\psi(\xi)$:

Proposition 3.10. *Let X be a Lévy process with state space $\mathbb{R}^d$, characteristic triplet $(\mathcal{Q}, \nu, \gamma)$ and characteristic exponent ψ. Assume $\mathcal{Q} = 0$ and that the marginal Lévy measures ν_i, $i = 1, \dots, d$, satisfy (3.2.2), and the Lévy copula F satisfies Assumption 3.9. Then, for $\|\xi\|_\infty$ sufficiently large*

$$\operatorname{Re}\psi(\xi) \gtrsim \sum_{j=1}^{d} |\xi_j|^{Y_j} \, . \tag{3.2.6}$$

Proof. At first, let $d = 1$. Using $1 - \cos(z) = 2(\sin \frac{z}{2})^2 \gtrsim z^2$ for $|z| \le 1$, we obtain for $|\xi| > 1$

$$\operatorname{Re}\psi(\xi) = \int_{\mathbb{R}} (1 - \cos(\xi z))\, k(z)(\mathrm{d}z) \gtrsim \int_{-\frac{1}{|\xi|}}^{\frac{1}{|\xi|}} \xi^2 z^2 k(z)\, \mathrm{d}z \gtrsim \int_0^{\frac{1}{|\xi|}} \xi^2 z^2 \frac{1}{z^{1+Y}}\, \mathrm{d}z \gtrsim |\xi|^Y \, .$$

Now let $d > 1$ and suppose Assumption 3.9 is satisfied. Consider the kernels

$$k_i^0(z) := c_i^- \frac{1}{z^{1+Y_i}} 1_{\{z<0\}}(z) + c_i^+ \frac{1}{z^{1+Y_i}} 1_{\{0 \le z\}}(z), \quad i = 1, \dots, d,$$

where Y_i, c_i^+, c_i^- are the constants of (3.2.2). Denote by $U_i : \mathbb{R} \to \mathbb{R}$ the marginal tail integrals of X and let U_i^0 be the tail integral corresponding to k_i^0. From (3.2.2)–(3.2.3) one infers $k_i \sim k_i^0$ and $U_i \sim U_i^0$ on $[-1, 1]$, $i = 1, \dots, d$. By Assumption 3.9, Remark 2.7 yields that the Lévy measure ν of X admits a kernel representation $\nu(dx) = k(x)dx$ with

$$k(x_1, \dots, x_d) = (\partial_1 \dots \partial_d F)(\underline{U}(x)) k_1(x_1) \dots k_d(x_d),$$

where we have set $\underline{U}(x) = (U_1(x_1), \dots, U_d(x_d))$. Thus, using the equivalence preserving property of $\partial_1 \dots \partial_d F$ one obtains

$$\begin{aligned} \operatorname{Re}\psi(\xi) &= \int_{\mathbb{R}^d} (1 - \cos\langle \xi, x\rangle) k(x) dx \\ &\ge \int_{B_1(0)} (1 - \cos\langle \xi, x\rangle)(\partial_1 \dots \partial_d F)(\underline{U}(x)) k_1(x_1) \dots k_d(x_d) dx \\ &\ge C \cdot \int_{B_1(0)} (1 - \cos\langle \xi, x\rangle)(\partial_1 \dots \partial_d F)(\underline{U}^0(x)) k_1^0(x_1) \dots k_d^0(x_d) dx. \end{aligned} \tag{3.2.7}$$

Now define $k^0(x_1, \dots, x_d) := (\partial_1 \dots \partial_d F)(\underline{U}^0(x)) k_1^0(x_1) \dots k_d^0(x_d)$. Since F is 1-homogeneous and the marginal kernels k_i^0 satisfy the homogeneity condition

$$k_i^0(rz) = r^{-1-Y_i} k_i^0(z), \quad \text{for all } r > 0,\ z \in \mathbb{R}\backslash\{0\},$$

by [15, Theorem 3.2] there holds

$$k^0(r^{-\frac{1}{Y_1}} x_1, \dots, r^{-\frac{1}{Y_d}} x_d) = r^{1 + \frac{1}{Y_1} + \dots + \frac{1}{Y_d}} k^0(x_1, \dots, x_d),$$

for all $r > 0$ and $x \in \mathbb{R}^d$ such that $x_i \neq 0$. Using [15, Theorem 3.3] one obtains that $\psi^0(\xi) := \int_{\mathbb{R}^d} (1 - \cos\langle \xi, z\rangle) k^0(x) dx$ is an anisotropic distance function of order $(1/Y_1, \dots, 1/Y_d)$. Since all anisotropic distance functions of the same order are equivalent (cf. e.g. [12, Lemma 2.2]), there exists some constant $C_1 > 0$ such that

$$\psi^0(\xi) \ge C_1(|\xi_1|^{Y_1} + \dots + |\xi_d|^{Y_d}), \quad \text{for all } \xi \in \mathbb{R}^d.$$

Hence, by (3.2.7),

$$\begin{aligned}\operatorname{Re}\psi(\xi) &\geq C\psi^0(\xi) - C\cdot\int_{\mathbb{R}^d\setminus B_1(0)}(1-\cos\langle\xi,x\rangle)k^0(x)dx\\ &\geq C\psi^0(\xi) - 2C\cdot\int_{\mathbb{R}^d\setminus B_1(0)}k^0(x)dx\\ &\geq C\psi^0(\xi) - C'\\ &\geq C\cdot C_1\sum_{i=1}^{d}|\xi_i|^{Y_i} - C'.\end{aligned}$$

□

Since ψ is continuous we immediately obtain the sector condition.

Theorem 3.11. *Let X be a Lévy process with state space $\mathbb{R}^d$, characteristic triplet $(\mathcal{Q},\nu,\gamma)$ and characteristic exponent ψ. Assume either $\mathcal{Q}>0$ or $\mathcal{Q}=0$ and $\gamma_i=0$, $i=1,\dots,d$, and that the marginal Lévy measures ν_i, $i=1,\dots,d$, satisfy* (3.2.2)–(3.2.3) *and the Lévy copula F satisfies Assumption 3.9. Then*

$$|\operatorname{Im}\psi(\xi)| \lesssim \operatorname{Re}\psi(\xi)\,,\quad \forall\xi\in\mathbb{R}^d\,.$$

Proof. For $\mathcal{Q}=0$ the result follows with Propositions 3.5 and 3.10. For $\mathcal{Q}>0$ we have

$$\operatorname{Re}\psi(\xi) = \frac{1}{2}\langle\xi,\mathcal{Q}\xi\rangle + \int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\,\nu(\mathrm{d}z) \gtrsim \sum_{j=1}^{d}\xi_j^2\,, \tag{3.2.8}$$

and for $\|\xi\|_\infty > 1$

$$|\psi(\xi)| \lesssim |\langle\gamma,\xi\rangle| + \langle\xi,\mathcal{Q}\xi\rangle + \int_{\mathbb{R}^d}\left|e^{i\langle\xi,z\rangle} - 1 - i\langle\xi,z\rangle 1_{|z|\leq 1}\right|\nu(\mathrm{d}z) \lesssim \sum_{j=1}^{d}\xi_j^2\,. \tag{3.2.9}$$

Thus, the result follows from the continuity of ψ. □

4 Option pricing

Assume the risk-neutral dynamics of $d\geq 1$ assets are given by

$$S_t^i = S_0^i e^{rt+X_t^i}\,,\quad i=1,\dots,d\,,$$

where X is a d-variate Lévy process and characteristic triplet $(\mathcal{Q},\nu_{\mathbb{Q}},\gamma)$ under a risk-neutral measure $\mathbb{Q}$ such that e^{X^i} is a martingale with respect to the canonical filtration $\mathcal{F}_t^0 := \sigma(X_s,\, s\leq t)$, $t\geq 0$, of the multivariate process X. As shown in Lemma 2.1 this martingale condition implies

$$\int_{|z|>1} e^{z_i}\nu_{\mathbb{Q}}(\mathrm{d}z) < \infty \quad i=1,\dots,d\,.$$

This equation holds for semiheavy tails satisfying (3.1.1) with $M_i>1$, $i=1,\dots,d$, as shown in Proposition 3.2. We drop the subscript $\mathbb{Q}$ in the following.

Remark 4.1. *With respect to the restriction $\mathcal{F}^{0,i}$ of $\mathcal{F}^0$ to the i-th margin, e^{X^i} is again a martingale. There holds $\sigma(X_s^i,\, s\leq t)\subset\mathcal{F}_t^{0,i}$ for all $t\geq 0$. Unless the marginal processes are independent this inclusion is proper.*

4.1 Partial Integrodifferential Equations (PIDEs) for European Contracts

We consider a European option with maturity $T < \infty$ and payoff $g(S)$ which is assumed to be Lipschitz. The value $V(t,S)$ of this option is given by

$$V(t,S) = \mathbb{E}\left(e^{-r(T-t)} g(S_T) | S_t = S\right) . \tag{4.1.1}$$

It can be characterized as solution of a PIDE.

Theorem 4.2. *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q}, \nu, \gamma)$. Assume that the function $V(t,S)$ in (4.1.1) satisfies*

$$V(t,S) \in C^{1,2}\left((0,T) \times \mathbb{R}^d_{>0}\right) \cap C^0\left([0,T] \times \mathbb{R}^d_{\geq 0}\right) .$$

Then, $V(t,S)$ is a classical solution of the backward Kolmogorov equation*:*

$$\begin{aligned} &\frac{\partial V}{\partial t}(t,S) + \frac{1}{2}\sum_{i,j=1}^{d} S_i S_j \mathcal{Q}_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + r\sum_{i=1}^{d} S_i \frac{\partial V}{\partial S_i}(t,S) - rV(t,S) \\ &+ \int_{\mathbb{R}^d} \left(V(t, Se^z) - V(t,S) - \sum_{i=1}^{d} S_i \left(e^{z_i} - 1\right) \frac{\partial V}{\partial S_i}(t,S) \right) \nu(\mathrm{d}z) = 0 , \end{aligned} \tag{4.1.2}$$

on $(0,T) \times \mathbb{R}^d_{\geq 0}$ where $V(t, Se^z) := V(t, S_1 e^{z_1}, \ldots, S_d e^{z_d})$, and the terminal condition is given by

$$V(T,S) = g(S), \quad \forall S \in \mathbb{R}^d_{\geq 0} . \tag{4.1.3}$$

Proof. We first need the risk-neutral dynamics of S^i_t. With the Itô formula for multidimensional Lévy processes and the Lévy-Itô decomposition we obtain

$$\begin{aligned} \mathrm{d}S^i_t &= rS^i_t \,\mathrm{d}t + S^i_{t-} \,\mathrm{d}X^i_t + \frac{1}{2}\mathcal{Q}_{ii} S^i_t \,\mathrm{d}t + S^i_{t-} e^{\Delta X^i_t} - S^i_{t-} - \Delta X^i_t S^i_{t-} \\ &= rS^i_t \,\mathrm{d}t + S^i_{t-}\gamma_i \,\mathrm{d}t + S^i_{t-} \sum_{k=1}^{d} \Sigma_{ik} \,\mathrm{d}W^k_t + \int_{|z|<1} S^i_{t-}\, z_i \tilde{J}(\mathrm{d}t, \mathrm{d}z) + \frac{1}{2}\mathcal{Q}_{ii} S^i_t \,\mathrm{d}t \\ &\quad + S^i_{t-}(e^{\Delta X^i_t} - 1 \underbrace{-\Delta X^i_t + \Delta X^i_t 1_{\{|\Delta X_t| \geq 1\}}}_{-\Delta X^i_t 1_{\{|\Delta X_t|<1\}}}) \\ &= rS^i_t \,\mathrm{d}t + S^i_{t-}\gamma_i \,\mathrm{d}t + S^i_{t-} \sum_{k=1}^{d} \Sigma_{ik} \,\mathrm{d}W^k_t + \frac{1}{2}\mathcal{Q}_{ii} S^i_t \,\mathrm{d}t \\ &\quad + \int_{\mathbb{R}^d} S^i_{t-} \left(e^{z_i} - 1\right) \tilde{J}(\mathrm{d}t, \mathrm{d}z) + \int_{\mathbb{R}^d} S^i_{t-} \left(e^{z_i} - 1 - z_i 1_{\{|z|<1\}}\right) \nu(\mathrm{d}z) \,\mathrm{d}t . \end{aligned}$$

Since $e^{X^i_t}$ is a martingale, we have

$$\mathrm{d}S^i_t = rS^i_t \,\mathrm{d}t + S^i_{t-} \sum_{k=1}^{d} \Sigma_{ik} \,\mathrm{d}W^k_t + \int_{\mathbb{R}^d} S^i_{t-} \left(e^{z_i} - 1\right) \tilde{J}(\mathrm{d}t, \mathrm{d}z) .$$

We now apply the Itô formula for semimartingales [18, Theorem 4.57] to the discounted values $e^{-rt} V_t$.

$$d(e^{-rt} V_t) = -re^{-rt} V \,\mathrm{d}t + e^{-rt} \left(\frac{\partial V}{\partial t}(t, S_t) \,\mathrm{d}t + \sum_{i=1}^{d} \frac{\partial V}{\partial S_i}(t, S_{t-}) \,\mathrm{d}S^i_t \right.$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2 V}{\partial S_i \partial S_j}(t,S_{t-})\,\mathrm{d}[S^i,S^j]^c_t + V(t,S_{t-}e^{\Delta X_t})\\
&-V(t,S_{t-})-\sum_{i=1}^{d} S^i_{t-}\left(e^{\Delta X^i_t}-1\right)\frac{\partial V}{\partial S_i}(t,S_{t-})\Bigg)\\
&= a(t)\,\mathrm{d}t + \mathrm{d}M_t\,,
\end{aligned}
$$

where

$$
\begin{aligned}
a(t) &= -re^{-rt}V + e^{-rt}\left(\frac{\partial V}{\partial t}+\sum_{i=1}^{d}\frac{\partial V}{\partial S_i} r S^i_{t-} + \frac{1}{2}\sum_{i,j=1}^{d}\mathcal{Q}_{ij}S^i_{t-}S^j_{t-}\frac{\partial^2 V}{\partial S_i\partial S_j}\right.\\
&\quad \left. + \int_{\mathbb{R}^d}\left(V(t,S_{t-}e^z)-V(t,S_{t-})-\sum_{i=1}^{d}S^i_{t-}\left(e^{z_i}-1\right)\frac{\partial V}{\partial S_i}(t,S_{t-})\right)\nu(\mathrm{d}z)\right)\\
\mathrm{d}M_t &= e^{-rt}\left(\sum_{i=1}^{d}\frac{\partial V}{\partial S_i}(t,S_{t-})S^i_{t-}\sum_{k=1}^{d}\Sigma_{ik}\,\mathrm{d}W^k_t\right.\\
&\quad \left. + \int_{\mathbb{R}^d}\left(V(t,S_{t-}e^z)-V(t,S_{t-})\right)\tilde{J}(\mathrm{d}t,\mathrm{d}z)\right)\,.
\end{aligned}
$$

Since g is Lipschitz, V is also Lipschitz with respect to S and $\frac{\partial V}{\partial S_i}$ is bounded $i=1,\ldots,d$. With

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T\int_{\mathbb{R}^d}\left(V(t,S_{t-}e^z)-V(t,S_{t-})\right)^2\nu(\mathrm{d}z)\,\mathrm{d}t\right)\\
&\lesssim \quad \mathbb{E}\left(\int_0^T\int_{\mathbb{R}^d}\sum_{i=1}^{d}(S^i_{t-})^2(e^{2z_i}+1)\nu(\mathrm{d}z)\,\mathrm{d}t\right)\\
&\lesssim \quad \sum_{i=1}^{d}\int_{\mathbb{R}}(e^{2z_i}+1)\nu_i(\mathrm{d}z_i)\mathbb{E}\left(\int_0^T(S^i_{t-})^2\,\mathrm{d}t\right)<\infty\,,
\end{aligned}
$$

and

$$
\mathbb{E}\left(\int_0^T (S^i_{t-})^2\left|\frac{\partial V}{\partial S_i}(t,S_{t-})\right|\mathrm{d}t\right)\lesssim \mathbb{E}\left(\int_0^T(S^i_{t-})^2\,\mathrm{d}t\right)<\infty\,,
$$

for $i=1,\ldots,d$, M_t is a square-integrable martingale, by [8, Proposition 8.6]. Therefore $e^{-rt}V_t - M_t$ is a martingale and since $e^{-rt}V_t - M_t = \int_0^t a(s)\,\mathrm{d}s$ is also a continuous process with bounded variation, we have $a(t)=0$ almost surely, by [8, Proposition 8.9]. This yields the desired PIDE. ☐

The PIDE (4.1.2) can further be transformed into a simpler form:

Corollary 4.3. *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q},\nu,\gamma)$ and marginal Lévy measures ν_i, $i=1,\ldots,d$ satisfying (3.1.1) with $M_i>1$, $G_i>0$, $i=1,\ldots,d$. Furthermore, let*

$$
u(\tau,x)=e^{r\tau}V\left(T-\tau,e^{x_1+(\gamma_1-r)\tau},\ldots,e^{x_d+(\gamma_d-r)\tau}\right)\,, \tag{4.1.4}
$$

where

$$
\gamma_i=\frac{\mathcal{Q}_{ii}}{2}+\int_{\mathbb{R}}(e^{z_i}-1-z_i)\,\nu_i(\mathrm{d}z_i)\,.
$$

Then, u satisfies the PIDE

$$
\frac{\partial u}{\partial \tau}+\mathcal{A}_{\mathrm{BS}}[u]+\mathcal{A}_{\mathrm{J}}[u]=0\,, \tag{4.1.5}
$$

in $(0,T)\times\mathbb{R}^d$ with initial condition $u(0,x):=u_0$. The differential operator $\mathcal{A}_{BS}$ is defined for $\varphi\in C^2_0(\mathbb{R}^d)$ by

$$
\mathcal{A}_{\mathrm{BS}}[\varphi]=-\frac{1}{2}\sum_{i,j=1}^{d}\mathcal{Q}_{ij}\frac{\partial^2\varphi}{\partial x_i\partial x_j}\,, \tag{4.1.6}
$$

and the integrodifferential operator $\mathcal{A}_{\mathrm{J}}$ *is given by*

$$\mathcal{A}_{\mathrm{J}}[\varphi] = -\int_{\mathbb{R}^d} \left(\varphi(x+z) - \varphi(x) - z \cdot \nabla_x \varphi(x)\right) \nu(\mathrm{d}z)\,. \tag{4.1.7}$$

The initial condition is given by

$$u_0 = g(e^x) := g(e^{x_1}, \ldots, e^{x_d})\,. \tag{4.1.8}$$

Proof. We proceed in several steps. To obtain constant coefficients we set $x_i = \log S_i$. Furthermore, we change to time to maturity $\tau = T - t$ and set $u(\tau, x) = V(T - \tau, e^{x_1}, \ldots, e^{x_d})$. The resulting differential operator is given by

$$\mathcal{A}_{\mathrm{BS}}[\varphi] = -\frac{1}{2}\sum_{i,j=1}^{d} \mathcal{Q}_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \left(\frac{1}{2}\mathcal{Q}_{ii} - r\right) \frac{\partial \varphi}{\partial x_i} + r\varphi\,,$$

and the integrodifferential operator by

$$\mathcal{A}_{\mathrm{J}}[\varphi] = -\int_{\mathbb{R}^d} \left(\varphi(x+z) - \varphi(x) - \sum_{i=1}^{d} (e^{z_i} - 1) \frac{\partial \varphi}{\partial x_i}(x)\right) \nu(\mathrm{d}z)\,.$$

The interest rate r can be set to zero by by transforming u to $\tilde{u}$ using

$$u(\tau, x) = e^{-r\tau}\, \tilde{u}(\tau, x + r\tau)\,.$$

Furthermore, the integrodifferential operator can be rewritten as

$$\mathcal{A}_{\mathrm{J}}[\varphi] = -\int_{\mathbb{R}^d} \left(\varphi(x+z) - \varphi(x) - z \cdot \nabla_x \varphi(x)\right) \nu(\mathrm{d}z) + \tilde{\gamma} \cdot \nabla_x \varphi(x)\,,$$

where the coefficients of the drift vector $\tilde{\gamma}$ are given by

$$\tilde{\gamma}_i = \int_{\mathbb{R}} (e^{z_i} - 1 - z_i)\, \nu_i(\mathrm{d}z_i)\,, \quad i = 1, \ldots, d\,.$$

We remove the drift in the integrodifferential and in the diffusion operator by setting

$$u(\tau, x) = \check{u}(\tau, x_1 - \gamma_1 \tau, \ldots, x_d - \gamma_d \tau)\,.$$

□

4.2 Barrier Contracts

In this section we derive the PIDE for knock-out barrier options (see e.g. [8, Section 12.1.2] for the one-dimensional case). The prices of corresponding knock-in and other barrier contracts with the same barrier can herewith be obtained using superposition and linearity arguments (see e.g. [3, Section 6]). Let $G \subset \mathbb{R}^d_{\geq 0}$ be an open subset and let $\tau_G = \inf\{s \geq 0 | X_s \in G^c\}$ be the first hitting time of the complement set $G^c = \mathbb{R}^d \backslash G$ by X. Then, the price of a knock-out barrier option with payoff g is given by

$$V_G(t, S) = \mathbb{E}\left(e^{-r(T-t)} g(S_T) 1_{\{T < \tau_G\}} | S_t = S\right)\,. \tag{4.2.1}$$

If V_G is sufficiently smooth it can be computed as the solution of a PIDE.

Theorem 4.4. *Assume that the function* $V_G(t, S)$ *satisfies*

$$V_G(t, S) \in C^{1,2}\left((0, T) \times \mathbb{R}^d_{>0}\right) \cap C^0\left([0, T] \times \mathbb{R}^d_{\geq 0}\right)\,. \tag{4.2.2}$$

Then, $V_G(t,S)$ satisfies the following PIDE.

$$\begin{aligned}\frac{\partial V_G}{\partial t}(t,S) + \frac{1}{2}\sum_{i,j=1}^{d} S_i S_j \mathcal{Q}_{ij}\frac{\partial^2 V_G}{\partial S_i \partial S_j} + r\sum_{i=1}^{d} S_i \frac{\partial V_G}{\partial S_i}(t,S) - rV_G(t,S) \\ + \int_{\mathbb{R}^d}\left(V_G(t,Se^z) - V_G(t,S) - \sum_{i=1}^{d} S_i\,(e^{z_i}-1)\,\frac{\partial V_G}{\partial S_i}(t,S)\right)\nu(\mathrm{d}z) = 0\,,\end{aligned} \tag{4.2.3}$$

on $(0,T)\times G$ where the terminal condition is given by

$$V_G(T,S) = g(S), \quad \forall S \in G\,, \tag{4.2.4}$$

and the "boundary" condition reads

$$V_G(t,S) = 0, \qquad \text{for all } (t,S) \in (0,T)\times G^c. \tag{4.2.5}$$

Proof. Define the deterministic function $\widetilde{g}(S) := g(S)1_{\{S\in G\}}$, and consider the European vanilla-type price function

$$\widetilde{V}(t,S) = \mathbb{E}\left(e^{-r(T-t)}\widetilde{g}(S_{T\wedge\tau_G})\,|\,S_t = S\right).$$

Since S is a strong Markov process, there holds $V_G(t,S) = \widetilde{V}(t,S)$ for all $t \le T\wedge\tau_G$. Thus, applying the Itô formula as in the proof of Theorem 4.2 one obtains that V_G satisfies (4.2.3) on $(0,T)\times G$. By definition there holds $V_G(t,S) = 0$ for all $(t,S)\in(0,T)\times G^c$. □

Remark 4.5. *Note that in contrast to plain European vanilla contracts, the price V_G of a barrier contract does not satisfy the smoothness condition* (4.2.2) *for general Lévy models. The validity of* (4.2.2) *can however be shown in case the process X admits a non-vanishing diffusion component, i.e. $\mathcal{Q} > 0$. Also for market models satisfying the ACP condition of [26, Definition 41.11] Theorem 4.4 can be shown to hold, see [3].*

4.3 American Contracts

Using the notation of the previous sections, we now consider an American option with maturity $T < \infty$ and Lipschitz continuous payoff $g(S)$. Its price $V_{\mathrm{A}}(t,S)$ is given by the optimal stopping problem

$$V_{\mathrm{A}}(t,S) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(e^{-r(T-\tau)}g(S_\tau)|S_t = S\right), \tag{4.3.1}$$

where $\mathcal{T}_{t,T}$ denotes the set of all stopping times between t and T.

In [24, 25] it is shown how the price $V_{\mathrm{A}}(t,S)$ can be characterized as the *viscosity solution* of a corresponding Bellman equation (for details on viscosity solutions we refer to e.g. [10] and the original sources [11, 28, 29]):

Theorem 4.6. *The price $V_{\mathrm{A}}(t,S)$ of an American option defined in* (4.3.1) *is a viscosity solution of*

$$\min\left\{\begin{aligned} & rV_{\mathrm{A}}(t,S) - \frac{\partial V_{\mathrm{A}}}{\partial t}(t,S) - \frac{1}{2}\sum_{i,j=1}^{d} S_i S_j \mathcal{Q}_{ij}\frac{\partial^2 V_{\mathrm{A}}}{\partial S_i\partial S_j} - r\sum_{i=1}^{d} S_i\frac{\partial V_{\mathrm{A}}}{\partial S_i}(t,S) \\ & \quad - \int_{\mathbb{R}^d}\left(V_{\mathrm{A}}(t,Se^z) - V_{\mathrm{A}}(t,S) - \sum_{i=1}^{d} S_i\,(e^{z_i}-1)\,\frac{\partial V_{\mathrm{A}}}{\partial S_i}(t,S)\right)\nu(\mathrm{d}z), \\ & V_{\mathrm{A}}(t,S) - g(S) \end{aligned}\right\} = 0. \tag{4.3.2}$$

If $V_{\mathrm{A}}(t,S)$ is uniformly continuous and there holds

$$\sup_{[0,T]\times\mathbb{R}^d_{>0}} \frac{V_{\mathrm{A}}(t,S)}{1+S} < \infty, \tag{4.3.3}$$

this solution is unique.

Proof. Existence of the viscosity solution follows from [25, Theorems 3.1] and its uniqueness is ensured by [25, Theorems 4.1] and [28]. □

Analogously to Corollary 4.3, by setting

$$\begin{aligned} u_{\mathrm{A}}(\tau,x) &= e^{r\tau} V_{\mathrm{A}}\left(T-\tau, e^{x_1+(\gamma_1-r)\tau},\dots,e^{x_d+(\gamma_d-r)\tau}\right), && \tau\in[0,T],\ x\in\mathbb{R}^d,\\ \widetilde{g}_\tau(x) &= g\left(e^{x_1+(\gamma_1-r)\tau},\dots,e^{x_d+(\gamma_d-r)\tau}\right), && \tau\in[0,T],\ x\in\mathbb{R}^d, \end{aligned} \tag{4.3.4}$$

with γ_i, $i=1,\dots,d$, as in (4.1.4), the Bellman equation (4.3.2) can equivalently be restated as the following linear complementarity problem:

$$\begin{aligned} &\frac{\partial u_{\mathrm{A}}}{\partial\tau}(\tau,x)+\mathcal{A}_{\mathrm{BS}}[u_{\mathrm{A}}](\tau,x)+\mathcal{A}_{\mathrm{J}}[u_{\mathrm{A}}](\tau,x)\le 0,\\ &u_{\mathrm{A}}(\tau,x)-e^{r\tau}\widetilde{g}_\tau(x)\ge 0,\\ &\left(\frac{\partial u_{\mathrm{A}}}{\partial\tau}(\tau,x)+\mathcal{A}_{\mathrm{BS}}[u_{\mathrm{A}}](\tau,x)+\mathcal{A}_{\mathrm{J}}[u_{\mathrm{A}}](\tau,x)\right)\left(u_{\mathrm{A}}(\tau,x)-e^{r\tau}\widetilde{g}_\tau(\tau,x)\right)=0, \end{aligned} \tag{4.3.5}$$

on $[0,T]\times\mathbb{R}^d$ with $\mathcal{A}_{\mathrm{BS}}$ and $\mathcal{A}_{\mathrm{J}}$ defined in (4.1.6) and (4.1.7). As in (4.1.8), the initial condition is given by $u_{\mathrm{A},0}=g(e^x)$, i.e. $u_{\mathrm{A},0}=u_0$.

4.4 Variational formulation

For $u,v\in C_0^\infty(\mathbb{R}^d)$ we associate with $\mathcal{A}_{\mathrm{BS}}$ the bilinear form

$$\mathcal{E}_{\mathrm{BS}}(u,v)=\frac{1}{2}\sum_{i,j=1}^{d}\mathcal{Q}_{ij}\int_{\mathbb{R}^d}\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)\,\mathrm{d}x\,. \tag{4.4.1}$$

To the jump part $\mathcal{A}_{\mathrm{J}}$ we associate the bilinear *canonical* jump form

$$\mathcal{E}_{\mathrm{J}}^{\mathrm{C}}(u,v)=-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(u(x+z)-u(x)-\sum_{i=1}^{d}z_i\frac{\partial u}{\partial x_i}(x)\right)v(x)\,\mathrm{d}x\,\nu(\mathrm{d}z)\,, \tag{4.4.2}$$

and set

$$\mathcal{E}(u,v)=\mathcal{E}_{\mathrm{BS}}(u,v)+\mathcal{E}_{\mathrm{J}}^{\mathrm{C}}(u,v)\,,$$

We can now formulate the realization of the abstract problem (2.3.5) for European contracts with $\mathcal{V}=\mathcal{D}(\mathcal{E})$ and $\mathcal{H}=L^2(\mathbb{R}^d)$:

$$\begin{aligned} &\text{Find } u\in L^2((0,T);\mathcal{D}(\mathcal{E}))\cap H^1((0,T);\mathcal{D}(\mathcal{E})^*) \text{ such that}\\ &\langle\frac{\partial u}{\partial\tau},v\rangle_{\mathcal{D}(\mathcal{E})^*,\mathcal{D}(\mathcal{E})}+\mathcal{E}(u,v)=0\,,\ \tau\in(0,T),\ \forall v\in\mathcal{D}(\mathcal{E})\,,\\ &u(0)=u_0\,. \end{aligned} \tag{4.4.3}$$

where u_0 is defined as in (4.1.8).

Furthermore, if the solution u_{A} of (4.3.5) satisfies $u_{\mathrm{A}}\in L^2((0,T);\mathcal{D}(\mathcal{E}))\cap H^1((0,T);\mathcal{D}(\mathcal{E})^*)$ it can be identified with the solution of the following realization of the abstract variational inequality (2.3.8)–(2.3.9):

$$\begin{aligned} &\text{Find } u_{\mathrm{A}}\in L^2((0,T);\mathcal{D}(\mathcal{E}))\cap H^1((0,T);\mathcal{D}(\mathcal{E})^*) \text{ such that } u_{\mathrm{A}}\in\mathcal{D}(\phi_\tau) \text{ a.e. in } (0,T) \text{ and}\\ &\langle\frac{\partial u_{\mathrm{A}}}{\partial\tau},v-u_{\mathrm{A}}\rangle_{\mathcal{D}(\mathcal{E})^*,\mathcal{D}(\mathcal{E})}+\mathcal{E}(u_{\mathrm{A}},v-u_{\mathrm{A}})-\phi_\tau(u)+\phi_\tau(v)\ge 0\,, \end{aligned} \tag{4.4.4}$$

for all $v \in \mathcal{D}(\phi_\tau)$, a.e. in $(0,T)$, and $u_{\mathrm{A}}(0) = u_0$,

with $\phi_\tau := I_{\mathcal{K}_\tau}$ as in (2.3.7) and convex sets

$$\mathcal{K}_\tau := \{v \in \mathcal{D}(\mathcal{E}) \,:\, v \geq e^{r\tau}\widetilde{g}_\tau\} \subset \mathcal{D}(\mathcal{E}), \qquad \tau \in (0,T),$$

where $\widetilde{g}_\tau : \mathbb{R}^d \to \mathbb{R}$ is given by (4.3.4).

As illustrated in Section 2.3, in weak form the variational problem (4.4.4) reads:

Find $u_{\mathrm{A}} \in L^\infty((0,T);\mathcal{D}(\mathcal{E})) \cap H^1((0,T);\mathcal{D}(\mathcal{E})^*)$ such that $u_{\mathrm{A}} \in \mathcal{D}(\Phi)$ a.e. in $(0,T)$ and

$$\int_0^T \langle \frac{\partial v}{\partial \tau}(\tau) + (\mathcal{A}+\lambda)u_{\mathrm{A}}(\tau) - \lambda v(\tau), u_{\mathrm{A}}(\tau) - v(\tau)\rangle \cdot e^{-2\lambda\tau} d\tau + \Phi(u_{\mathrm{A}}) - \Phi(v) \leq \frac{1}{2}\|u_0 - v(0)\|^2_{\mathcal{H}},$$

for all $v \in \mathcal{D}(\Phi)$ with $\dfrac{\partial v}{\partial \tau} \in L^2(0,T;\mathcal{V}^*)$. (4.4.5)

Here Φ and $\mathcal{D}(\Phi)$ are depending on ϕ_τ as defined in Section 2.3.

Remark 4.7. *In* (4.4.3)–(4.4.5)*, it is required that $u_0 \in \mathcal{H} = L^2(\mathbb{R}^d)$ which implies a growth condition on the payoff g. In Section 4.5 we reformulate the problem on a bounded domain where this condition can be weakend. The weaker growth condition is given explicitly in* (4.5.1)*.*

The well-posedness of (4.4.3) and (4.4.5) is ensured by

Theorem 4.8. *Let X be a Lévy process with state space $\mathbb{R}^d$ and characteristic triplet $(\mathcal{Q},\nu,\gamma)$ and Dirichlet form $\mathcal{E}(\cdot,\cdot)$. Assume that either $\mathcal{Q} > 0$ or Assumptions 3.3 and 3.9 hold in conjunction with $\gamma = 0$. Then, the variational equation* (4.4.3) *and the weak variational inequality* (4.4.5) *with $u_0 \in L^2(\mathbb{R}^d)$ admit a unique solution in $\mathcal{D}(\mathcal{E})$.*

For $\mathcal{Q} > 0$ there holds $\mathcal{D}(\mathcal{E}) = H^1(\mathbb{R}^d)$ and for $\mathcal{Q} = 0$ one obtains $\mathcal{D}(\mathcal{E}) = H^{(Y_1/2,\ldots,Y_d/2)}(\mathbb{R}^d)$ where

$$H^{(s_1,\ldots,s_d)}(\mathbb{R}^d) = \left\{u \in L^2(\mathbb{R}^d) \,:\, \int_{\mathbb{R}^d} \sum_{j=1}^d \left(1+\xi_j^2\right)^{s_j} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty\right\},$$

is an anisotropic Sobolev space.

Proof. Since a Lévy process X is stationary, its infinitesimal generator is translation invariant. We also have with Theorem 3.11 that the characteristic exponent ψ of X satisfies the sector condition (3.2.1). Therefore, the bilinear form $\mathcal{E}(u,v)$ is a Dirichlet form and, by [17, Example 4.7.32], it can be written as

$$|\mathcal{E}(u,v)| = (2\pi)^d \left| \int_{\mathbb{R}^d} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi \right|.$$

By Theorem 2.11, for existence and uniqueness of a solution of (4.4.3) we need to show that $\mathcal{E}(\cdot,\cdot)$ satisfies the continuity condition (2.3.3) and the Gårding inequality (2.3.4).

At first, consider the case $\mathcal{Q} = 0$. By Propositions 3.5 and 3.10, there exist some constants $C_1, C_2, C_3 > 0$ such that

$$\operatorname{Re}\psi(\xi) \geq C_1 \sum_{j=1}^d |\xi_j|^{Y_j} - C_2, \quad |\psi(\xi)| \leq C_3 \left(\sum_{j=1}^d |\xi_j|^{Y_j} + 1\right) \quad \text{for all } \xi \in \mathbb{R}^d. \tag{4.4.6}$$

Herewith, the continuity of $\mathcal{E}(\cdot,\cdot)$ is ensured by

$$|\mathcal{E}(u,v)| = \left| \int_{\mathbb{R}^d} \psi(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi \right| \leq C_3 \int_{\mathbb{R}^d} \left(1 + \sum_{i=1}^d |\xi_i|^{Y_i}\right) \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi$$

$$
\begin{aligned}
&\leq \widetilde{C}_3 \int_{\mathbb{R}^d} \sum_{i=1}^d \left(1+|\xi_i|^2\right)^{Y_i/2} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi \\
&\leq \widetilde{C}_3 \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \left(1+|\xi_i|^2\right)^{Y_i/2} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} \left(\int_{\mathbb{R}^d} \sum_{i=1}^d \left(1+|\xi_i|^2\right)^{Y_i/2} |\widehat{v}(\xi)|^2 \,\mathrm{d}\xi\right)^{1/2} \\
&\lesssim \|u\|_{H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d)} \cdot \|v\|_{H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d)},
\end{aligned}
$$

where we used $\sum_{i=1}^d (1+|\xi_i|^2)^{Y_i/2} \sim (1+\sum_{i=1}^d |\xi_i|^{Y_i})$. Furthermore, for the Gårding inequality one finds

$$
\mathcal{E}(u,u) = \int_{\mathbb{R}^d} \operatorname{Re}\psi(\xi)|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi = \int_{\mathbb{R}^d} (C_1+C_2+\operatorname{Re}\psi(\xi))\,|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi - (C_1+C_2)\int_{\mathbb{R}^d} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi\,,
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^d} (C_1+C_2+\operatorname{Re}\psi(\xi))\,|\widehat{u}(\xi)|^2 \,\mathrm{d}\xi &\geq C_1 \int_{\mathbb{R}^d} \left(1+\sum_{i=1}^d |\xi_i|^{Y_i}\right) |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi \\
&\geq \widetilde{C}_1 \int_{\mathbb{R}^d} \sum_{i=1}^d \left(1+|\xi_i|^2\right)^{Y_i/2} |\widehat{u}(\xi)|^2 \,\mathrm{d}\xi\,.
\end{aligned}
$$

Theorem 2.11 thus yields the existence and uniqueness of a solution $u \in H^{(Y_1/2,\dots,Y_d/2)}(\mathbb{R}^d) = \mathcal{D}(\mathcal{E})$ of (4.4.3). One obtains the existence and uniqueness of the solution u_{A} of (4.4.5) analogously from Theorem 2.14 in conjunction with e.g. [5, Remarque 3] (to account for the smooth time dependence of the convex set $\mathcal{K}_\tau$).

If $\mathcal{Q} > 0$ one obtains the required results using the same arguments: By (3.2.8) and (3.2.9), instead of (4.4.6) in this case there holds

$$
\operatorname{Re}\psi(\xi) \gtrsim \sum_{j=1}^d |\xi_j|^2\,, \quad |\psi(\xi)| \lesssim \sum_{j=1}^d |\xi_j|^2\,, \quad \text{for all } \|\xi\|_\infty > 1,
$$

and the result follows as above. □

Remark 4.9. *We omitted the partially degenerate case $\mathcal{Q} \neq 0$ but $\mathcal{Q} \not> 0$ in Theorem 4.8. Here, the domain $\mathcal{D}(\mathcal{E})$ can be obtained by writing*

$$
\mathcal{Q} = \mathbf{\Sigma}\mathbf{\Sigma}^\top = (\sigma_i\sigma_j\rho_{ij})_{1\leq i,j\leq d}
$$

where ρ_{ij} is the correlation of the Brownian motion W_i and W_j. Suppose $\sigma_i = 0$ for all $i \in \mathcal{I} \subset \{1,\dots,d\}$ and $\sigma_j > 0$ for all $j \notin \mathcal{I}$. By [22, Section 9.2] the anisotropic Sobolev spaces in Theorem 4.8 admit an intersection structure

$$
H^{(s_1,\dots,s_d)}(\mathbb{R}^d) = \bigcap_{j=1}^d H_j^{s_j}(\mathbb{R}^d), \quad (s_1,\dots,s_d) \in \mathbb{R}^d,
$$

with

$$
H_j^{s_j}(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{H_j^{s_j}(\mathbb{R}^d)} = \left\|(1+\xi_j^2)^{s_j/2}\widehat{f}\right\|_{L^2(\mathbb{R}^d)} < \infty \right\}.
$$

Using the above arguments, one obtains

$$
\mathcal{D}(\mathcal{E}) = \bigcap_{i\in\mathcal{I}} H_i^{\frac{Y_i}{2}}(\mathbb{R}^d) \cap \bigcap_{j\notin\mathcal{I}} H_j^1(\mathbb{R}^d).
$$

Remark 4.10. *For European contracts, Theorem 4.8 was already obtained in dimension $d = 1$ in Matache et al. [21]. For $d > 1$, Farkas et al. [15] proved Theorem 4.8 for symmetric tempered stable margins.*

For the numerical implementation of (4.4.3) it is important to note that all integrals in (4.4.2) exist in Lebesgue sense even for functions $u, v \in H^1(\mathbb{R}^d)$ with compact supports.

Proposition 4.11. *If $u, v \in H^1(\mathbb{R}^d)$ with compact supports then $|\mathcal{E}_{\mathrm{J}}^{\mathrm{C}}(u,v)| < \infty$, where the bilinear form $\mathcal{E}_{\mathrm{J}}^{\mathrm{C}}(u,v)$ is given by* (4.4.2).

Proof. Since $\int_{\mathbb{R}^d} |z|^2 \, \nu(\mathrm{d}z) < \infty$, we need to show that

$$\left| \int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \sum_{i=1}^{d} z_i \frac{\partial u}{\partial x_i}(x) \right) v(x) \, \mathrm{d}x \right| \lesssim |z|^2 \, \|u\|_{H^1(\mathbb{R}^d)} \, \|v\|_{H^1(\mathbb{R}^d)} \, .$$

Using integration by parts and the Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\left| \int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \sum_{i=1}^{d} z_i \frac{\partial u}{\partial x_i}(x) \right) v(x) \, \mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^d} \sum_{i=1}^{d} z_i \int_0^1 \frac{\partial u}{\partial x_i}(x_1, \ldots, x_i + \theta_i z_i, x_{i+1} + z_{i+1}, \ldots, x_d + z_d) \, \mathrm{d}\theta_i v(x) \, \mathrm{d}x \right. \\
&\quad \left. - \int_{\mathbb{R}^d} \sum_{i=1}^{d} z_i \frac{\partial u}{\partial x_i}(x) v(x) \, \mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^d} \sum_{i=1}^{d} z_i \int_0^1 u(x_1, \ldots, x_i + \theta_i z_i, x_{i+1} + z_{i+1}, \ldots, x_d + z_d) \, \mathrm{d}\theta_i \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x \right. \\
&\quad \left. - \int_{\mathbb{R}^d} \sum_{i=1}^{d} z_i u(x) \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x \right| \\
&= \left| \int_{\mathbb{R}^d} \sum_{i=1}^{d} \sum_{j=i+1}^{d} z_i z_j \int_0^1 \int_0^1 u(x_1, \ldots, x_i + \theta_i z_i, x_{i+1}, \right. \\
&\qquad \ldots, x_{j-1}, x_j + \theta z_j, x_{j+1} + z_{i+1}, \ldots, x_d + z_d) \, \mathrm{d}\theta_j \, \mathrm{d}\theta_i \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x \\
&\quad \left. + \int_{\mathbb{R}^d} \sum_{i=1}^{d} z_i \left(\int_0^1 u(x_1, \ldots, x_i + \theta_i z_i, \ldots, x_d) \, \mathrm{d}\theta_i - u(x) \right) \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x \right| \\
&\lesssim \sum_{i=1}^{d} \sum_{j=i+1}^{d} |z_i z_j| \, \|u\|_{L^2(\mathbb{R}^d)} \, \|v\|_{H^1(\mathbb{R}^d)} \\
&\quad + \left| \int_{\mathbb{R}^d} \sum_{i=1}^{d} z_i^2 \int_0^1 (1 - \theta_i) \frac{\partial u}{\partial x_i}(x_1, \ldots, x_i + \theta_i z_i, \ldots, x_d) \, \mathrm{d}\theta_i \frac{\partial v}{\partial x_i}(x) \, \mathrm{d}x \right| \\
&\lesssim \sum_{i=1}^{d} \sum_{j=i+1}^{d} |z_i z_j| \, \|u\|_{L^2(\mathbb{R}^d)} \, \|v\|_{H^1(\mathbb{R}^d)} + \sum_{i=1}^{d} z_i^2 \, \|u\|_{H^1(\mathbb{R}^d)} \, \|v\|_{H^1(\mathbb{R}^d)} \\
&\lesssim |z|^2 \, \|u\|_{H^1(\mathbb{R}^d)} \, \|v\|_{H^1(\mathbb{R}^d)} \, .
\end{aligned}$$

□

We can also convert the canonical form $\mathcal{E}_{\mathrm{J}}^{\mathrm{C}}(\cdot,\cdot)$ of (4.4.2) into the *integrated* jump form $\mathcal{E}_{\mathrm{J}}^{\mathrm{I}}(\cdot,\cdot)$ by using Lemma 2.8,

$$\mathcal{E}_{\mathrm{J}}^{\mathrm{I}}(u,v) = -\sum_{i=1}^{d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(u(x + z_i) - u(x) - z_i \frac{\partial u}{\partial x_i}(x) \right) v(x) k_i(z_i) \, \mathrm{d}x \, \mathrm{d}z_i$$

$$-\sum_{i=2}^{d}\sum_{\substack{|I|=i\\ I_1<\cdots<I_i}}\int_{\mathbb{R}^i}\int_{\mathbb{R}^d}\frac{\partial^i u}{\partial x^I}(x+z^I)v(x)F^I\left((U_j(z_j))_{j\in I}\right)\mathrm{d}x\,\mathrm{d}z^I\,. \tag{4.4.7}$$

For the integrals in (4.4.7) to exist we need that $u \in \mathcal{H}^1_{mix}(\mathbb{R}^d) = H^1(\mathbb{R})\otimes\cdots\otimes H^1(\mathbb{R})$ and that u has compact support. Note that tensor products of one-dimensional continuous, piecewise linear finite element basis functions are contained in $\mathcal{H}^1_{mix}(\mathbb{R}^d)$ and satisfy the support condition.

Proposition 4.12. *For $u, v \in \mathcal{H}^1_{mix}(\mathbb{R}^d)$ with compact support, there holds $|\mathcal{E}^{\mathrm{I}}_{\mathrm{J}}(u,v)| < \infty$.*

Proof. Analogously to Proposition 4.11 for $u, v \in H^1_{comp}(\mathbb{R}^d)$ holds

$$\int_{\mathbb{R}^d}\left(u(x+z_i)-u(x)-z_i\frac{\partial u}{\partial x_i}(x)\right)v(x)\,\mathrm{d}x \lesssim z_i^2\,\|u\|_{H^1(\mathbb{R}^d)}\,\|v\|_{H^1(\mathbb{R}^d)}\qquad i=1,\ldots,d\,.$$

With

$$\int_{\mathbb{R}^d}\frac{\partial^{|I|}u}{\partial x^I}(x+z^I)v(x)\,\mathrm{d}x \le \|u\|_{\mathcal{H}^1_{mix}(\mathbb{R}^d)}\,\|v\|_{H^1(\mathbb{R}^d)}\qquad \forall z\in\mathbb{R}^d,\ I\subset\{1,\ldots,d\}\,,$$

and

$$\int_{\mathbb{R}^{|I|}}F^I\left((U_i(z_i))_{i\in I}\right)\mathrm{d}z^I < \infty \quad \forall I\subset\{1,\ldots,d\}\,,$$

we obtain the asserted result. □

Finally, one may also split the canonical jump form $\mathcal{E}^{\mathrm{C}}_{\mathrm{J}}(\cdot,\cdot)$ defined in (4.4.2) into its symmetric part $\mathcal{E}^{\mathrm{sym}}_{\mathrm{J}}(\cdot,\cdot)$ and its antisymmetric part $\mathcal{E}^{\mathrm{asym}}_{\mathrm{J}}(\cdot,\cdot)$ which are defined by

$$\begin{aligned}
\mathcal{E}^{\mathrm{sym}}_{\mathrm{J}}(u,v) &= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x+z)-u(x))\cdot(v(x+z)-v(x))\,\mathrm{d}x\,k^{\mathrm{sym}}(z)\,\mathrm{d}z,\\
\mathcal{E}^{\mathrm{asym}}_{\mathrm{J}}(u,v) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{u(x+z)-u(x-z)}{2}-z\cdot\nabla_x u(x)\right)v(x)\,\mathrm{d}x\,k^{\mathrm{asym}}(z)\,\mathrm{d}z,
\end{aligned}$$

with $k^{\mathrm{sym}}(z) := \frac{1}{2}(k(z)+k(-z))$ and $k^{\mathrm{asym}}(z) := \frac{1}{2}(k(z)-k(-z))$.

Lemma 4.13. *Under the assumptions of Theorem 4.8, for $u,\, v \in C_0^\infty(\mathbb{R}^d)$ there holds*

$$\mathcal{E}^{\mathrm{C}}_{\mathrm{J}}(u,v) = \mathcal{E}^{\mathrm{sym}}_{\mathrm{J}}(u,v) + \mathcal{E}^{\mathrm{asym}}_{\mathrm{J}}(u,v).$$

Proof. The bilinear form $\mathcal{E}^{\mathrm{C}}_{\mathrm{J}}$ is a translation invariant Dirichlet form. Hence, by [17, Example 4.7.32], it can be written as

$$\begin{aligned}
\mathcal{E}^{\mathrm{C}}_{\mathrm{J}}(u,v) &= (2\pi)^d\int_{\mathbb{R}^d}\psi_{\mathrm{J}}(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi\\
&= (2\pi)^d\int_{\mathbb{R}^d}\mathrm{Re}\,\psi_{\mathrm{J}}(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi + i(2\pi)^d\int_{\mathbb{R}^d}\mathrm{Im}\,\psi_{\mathrm{J}}(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi,
\end{aligned}\tag{4.4.8}$$

where $\psi_{\mathrm{J}}(\xi) = \int_{\mathbb{R}^d}\left(1-e^{i\langle\xi,z\rangle}+i\langle\xi,z\rangle\right)\nu(\mathrm{d}z)$ denotes the jump part of the Lévy symbol ψ in (2.1.2). Recall the convolution theorem

$$\widehat{u}(\xi)\overline{\widehat{v}(\xi)} = (2\pi)^{-d}\widehat{u*\widetilde{v}}(\xi),\quad \xi\in\mathbb{R}^d,$$

where $\widetilde{v}(\cdot) := v(-\cdot)$. Denoting by $B_\varepsilon(0)$ the ball of radius $\varepsilon > 0$ around the origin and using Plancherel's theorem one obtains

$$\int_{\mathbb{R}^d}\mathrm{Re}\,\psi_{\mathrm{J}}(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0+}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)}\int_{\mathbb{R}^d}(1-\cos\langle\xi,z\rangle)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&= \lim_{\varepsilon\to 0+}(2\pi)^{-d}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)}\int_{\mathbb{R}^d}u(x)v(x)-u(x+z)v(x)dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&= (2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(x+z))\,v(x)dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&= \frac{1}{2}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(x+z))\,v(x)dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&\qquad +\frac{1}{2}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x-z)-u(x))\,v(x-z)dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&= \frac{1}{2}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(x+z))\,v(x)dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&\qquad +\frac{1}{2}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x+z)-u(x))\,v(x+z)dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z\\
&= \frac{1}{2}(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(u(x)-u(x+z))\,(v(x)-v(x+z)\,dx\, k^{\mathrm{sym}}(z)\,\mathrm{d}z,
\end{aligned}
$$

where we have used that k^{sym} is symmetric with respect to each coordinate axis. With analogous arguments, one also obtains

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\mathrm{Im}\,\psi_{\mathrm{J}}(\xi)\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi\\
=\;&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\langle\xi,z\rangle-\sin\langle\xi,z\rangle)\,\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi\, k^{\mathrm{asym}}(z)\,\mathrm{d}z\\
=\;&\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)}\Bigg[\int_{\mathbb{R}^d}\langle\xi,z\rangle\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi\\
&\qquad -i(2\pi)^{-d}\int_{\mathbb{R}^d}\frac{e^{i\langle\xi,z\rangle}-e^{-i\langle\xi,z\rangle}}{2}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi\Bigg]k^{\mathrm{asym}}(z)\,\mathrm{d}z\\
=\;&\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^d\setminus B_\varepsilon(0)}\Bigg[i(2\pi)^{-d}\int_{\mathbb{R}^d}\sum_{i=1}^d z_i\frac{\partial u}{\partial x_i}(x)\,v(x)\,\mathrm{d}x\\
&\qquad -i(2\pi)^{-d}\int_{\mathbb{R}^d}\frac{u(x+z)-u(x-z)}{2}v(x)\,\mathrm{d}x\Bigg]k^{\mathrm{asym}}(z)\,\mathrm{d}z\\
=\;& i(2\pi)^{-d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{u(x-z)-u(x+z)}{2}+\sum_{i=1}^d z_i\frac{\partial u}{\partial x_i}(x)\right)v(x)\,\mathrm{d}x\, k^{\mathrm{asym}}(z)\,\mathrm{d}z.
\end{aligned}
$$

Substituting these results back into (4.4.8) one obtains $\mathcal{E}_{\mathrm{J}}^{\mathrm{C}}(u,v)=\mathcal{E}_{\mathrm{J}}^{\mathrm{sym}}(u,v)+\mathcal{E}_{\mathrm{J}}^{\mathrm{asym}}(u,v)$. □

4.5 Formulation on bounded domain

In this section we show how one may localize the unbounded log-price space domain $\mathbb{R}^d$ to a bounded domain. To analyze the effect of this localization procedure on the option price, we require the following growth condition on the payoff function: There exists some $q \geq 1$ such that

$$
g(S) \lesssim \left(\sum_{i=1}^d S_i + 1\right)^q, \quad \text{for all } S \in \mathbb{R}^d_{\geq 0}. \tag{4.5.1}
$$

This condition is satisfied by all standard multi-asset options like basket, maximum or best-of options.

4.5.1 Localization

The unbounded log-price domain $\mathbb{R}^d$ of the variable x is truncated to a bounded domain $G_R \supset [-R,R]^d$. In finance, this corresponds to approximating the solution V of the problem (4.1.2) by a barrier option V_R which is the solution of the problem (4.2.3), similarly for American options. In log price the European and American barrier option is given by

$$u_R(t,x) = \mathbb{E}\left(g(e^{X_T})1_{\{T<\tau_{G_R}\}}|X_t = x\right),\ u_{\mathrm{A},R}(t,x) = \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(g(e^{X_\tau})1_{\{\tau<\tau_{G_R}\}}|X_t = x\right),$$

where, for notational convenience, we have set $r = 0$. We show that for semiheavy tails the solution of the localized problem converges pointwise exponentially to the solution of the original problem.

Lemma 4.14. *Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν such that the marginal measures ν_i satisfy* (3.1.1)*. Then, the density $p_t(x)$ of the process X_t, $t > 0$ decays exponentially independent of time t*

$$\int_{\mathbb{R}^d} e^{\eta_i(x)} p_t(x)\,\mathrm{d}x < \infty\,, \quad \text{with} \quad \eta_i(x) = \left(\mu_i^+ 1_{\{x_i>0\}} + \mu_i^- 1_{\{x_i<0\}}\right)|x_i|\,, \tag{4.5.2}$$

and $0 < \mu_i^- < G_i$ and $0 < \mu_i^+ < M_i$, $i = 1,\dots,d$.

Proof. Using Sato [26, Theorem 25.3], we know (4.5.2) is true if and only if

$$\int_{|z|>1} e^{\eta_i(z)}\nu(\mathrm{d}z) < \infty\,.$$

The result follows from Proposition 3.2. □

Theorem 4.15. *Suppose the payoff function $g : \mathbb{R}^d \to \mathbb{R}$ satisfies* (4.5.1)*. Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν such that the marginal measures ν_i satisfy* (3.1.1) *with $M_i > q$, $G_i > q$, $i = 1,\dots,d$, with q as in* (4.5.1)*. Then,*

$$|u(t,x) - u_R(t,x)|\,,\left|u_{\mathrm{A}}(t,x) - u_{\mathrm{A},R}(t,x)\right| \lesssim e^{-\alpha R+\beta\|x\|_\infty}\,,$$

with $0 < \alpha < \min_i \min(G_i, M_i) - q$ and $\beta = \alpha + q$.

Proof. We only consider the American case in detail, since this also implies the case of European contracts. Let $\eta_i(x)$ be as in (4.5.2) and $M_\tau = \sup_{s\in[t,\tau]}\|X_s\|_\infty$. Then with (4.5.1)

$$\begin{aligned}\left|u_{\mathrm{A}}(t,x) - u_{\mathrm{A},R}(t,x)\right| &\le \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(g(e^{X_\tau})1_{\{\tau\ge\tau_{G_R}\}}|X_t = x\right)\\ &\lesssim \sup_{\tau\in\mathcal{T}_{t,T}} \mathbb{E}\left(e^{qM_\tau}1_{\{M_\tau>R\}}|X_t = x\right).\end{aligned}$$

Using Sato [26, Theorem 25.18] it suffices to show for $t \le \tau \le T$

$$\begin{aligned}&\mathbb{E}\left(e^{q\|X_\tau\|_\infty}1_{\{\|X_\tau\|_\infty>R\}}|X_t = x\right) = \int_{\mathbb{R}^d} e^{q\|z+x\|_\infty}1_{\{\|z+x\|_\infty>R\}}p_{\tau-t}(z)\,\mathrm{d}z\\ &\lesssim e^{q\|x\|_\infty}\sum_{i=1}^d \int_{\mathbb{R}^d} e^{q|z|_i}e^{-\eta_i(z)}1_{\{\|z+x\|_\infty>R\}}e^{\eta_i(z)}p_{\tau-t}(z)\,\mathrm{d}z\\ &\lesssim e^{q\|x\|_\infty}\sum_{i=1}^d \int_{\mathbb{R}^d} e^{-\left(\min_j\min(\mu_j^+,\mu_j^-)-q\right)(R-\|x\|_\infty)}e^{\eta_i(z)}p_{\tau-t}(z)\,\mathrm{d}z\\ &\lesssim e^{-\alpha R+\beta\|x\|_\infty}\sum_{i=1}^d \int_{\mathbb{R}^d} e^{\eta_i(z)}p_{\tau-t}(z)\,\mathrm{d}z\,.\end{aligned}$$

The result follows with (4.5.2). □

The domain of integration $\mathbb{R}^d$ of the variable z in e.g. (4.4.2) can also be truncated to a bounded domain $\Lambda_B = [-B,B]^d$. For this, consider the truncated Lévy measure $\nu_B = \nu 1_{\{\|z\|_\infty \le B\}}$ and the corresponding Lévy process X_B with characteristic triplet $(\mathcal{Q}, \nu_B, \gamma_B)$. Here γ_B is defined such that $e^{X_B^i}$ is a martingale, $i = 1, \dots, d$. Denote by $\widetilde{X} = X - X_B$ the Lévy process with characteristic triplet $(0, \widetilde{\nu}, \widetilde{\gamma})$ where $\widetilde{\nu} = \nu 1_{\{\|z\|_\infty > B\}}$. Let u_B, $u_{\mathrm{A},B}$ be the solution of

$$u_B(t,x) = \mathbb{E}\left(g(e^{X_{B,T}})|X_{B,t} = x\right),\ u_{\mathrm{A},B}(t,x) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(g(e^{X_{B,\tau}})|X_{B,t} = x\right),$$

where, again for notational convenience, we have set $r = 0$.

Theorem 4.16. *Let X be a Lévy process with state space $\mathbb{R}^d$ and Lévy measure ν such that the marginal measures ν_i satisfy* (3.1.1) *with $M_i > 1$, $G_i > 0$, $i = 1, \dots, d$. Then,*

$$|u(t,x) - u_B(t,x)|, \left|u_{\mathrm{A}}(t,x) - u_{\mathrm{A},B}(t,x)\right| \lesssim e^{-\alpha B + \|x\|_\infty}.$$

with $0 < \alpha < \min_i \min(G_i, M_i - 1)$.

Proof. Since g is Lipschitz and X_B, $\widetilde{X}$ are independent, we have

$$\begin{aligned}
\left|u_{\mathrm{A}}(t,x) - u_{\mathrm{A},B}(t,x)\right| &\le \sup_{\tau \in \mathcal{T}_{t,T}} \left|\mathbb{E}\left(g(e^{x+X_{\tau-t}})\right) - \mathbb{E}\left(g(e^{x+X_{B,\tau-t}})\right)\right| \\
&\lesssim \sup_{\tau \in \mathcal{T}_{t,T}} \sum_{i=1}^d \mathbb{E}\left(\left|e^{x_i + X^i_{\tau-t}} - e^{x_i + X^i_{B,\tau-t}}\right|\right) \\
&\lesssim \sup_{\tau \in \mathcal{T}_{t,T}} \sum_{i=1}^d e^{\|x\|_\infty} \mathbb{E}\left(e^{X^i_{B,\tau-t}}\left|e^{\widetilde{X}^i_{\tau-t}} - 1\right|\right) \\
&\lesssim \sup_{\tau \in \mathcal{T}_{t,T}} e^{\|x\|_\infty} \sum_{i=1}^d \mathbb{E}\left(\left|e^{\widetilde{X}^i_{\tau-t}} - 1\right|\right).
\end{aligned}$$

Using the same argumentation as in the proof of [9, Proposition 4.2] for $d = 1$, we obtain the required result. □

4.5.2 Variational formulation on the bounded domain

For any function u with support in G_R we denote by $\widetilde{u}$ its extension by zero to all of $\mathbb{R}^d$ and define

$$\mathcal{E}_R(u,v) = \mathcal{E}(\widetilde{u}, \widetilde{v}).$$

Thus, we obtain continuity and a Gårding inequality of $\mathcal{E}_R(u,v)$ on

$$\mathcal{D}(\mathcal{E}_R) := \overline{\{\widetilde{u} \mid u \in C_0^\infty(G_R)\}},$$

where the closure is taken with respect to the norm of $\mathcal{D}(\mathcal{E})$ as given explicitly in Theorem 4.8. Now we can restate the problem (4.4.3) on bounded domain:

$$\begin{aligned}
&\text{Find } u_R \in L^2((0,T); \mathcal{D}(\mathcal{E}_R)) \cap H^1((0,T); \mathcal{D}(\mathcal{E}_R)^*) \text{ such that} \\
&\left(\frac{\partial u_R}{\partial \tau}, v\right) + \mathcal{E}_R(u_R, v) = 0,\ \forall \tau \in (0,T),\ \forall v \in \mathcal{D}(\mathcal{E}_R), \\
&u_R(0) = u_0|_{G_R}.
\end{aligned} \tag{4.5.3}$$

By Theorem 4.8, the problem (4.5.3) is well-posed, i.e. there exists a unique solution $u_R \in L^2(0,T; \mathcal{D}(\mathcal{E}_R)) \cap C^0([0,T]; L^2(G_R))$.

The solution of (4.5.3) can now be approximated by a finite element Galerkin scheme (as introduced in [15, 21, 20]).

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A Equivalence preserving copulas

In view of Assumption 3.9, there remains to show the equivalence preserving property of $H := \partial_1 \dots \partial_d F$ for a large class of 1-homogeneous copulas F. The following lemmas provide such a class.

Lemma A.1. *Suppose $G_1, G_2 : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}_{\geq 0}$ are two equivalence preserving functions. Then*

(i) For any $\gamma \geq 0$, the power $G_1(\cdot)^\gamma : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}_{\geq 0}$ is equivalence preserving on $\overline{\mathbb{R}}^d$.

(ii) The product $G_1 \cdot G_2 : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}_{\geq 0}$ is equivalence preserving on $\overline{\mathbb{R}}^d$.

(ii) The quotient $G_1/G_2 : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}_{\geq 0}$ is equivalence preserving on any subset $\mathcal{J} \subset \mathbb{R}^d$ such that $\overline{\mathcal{J}}$ does not contain any poles of G_1/G_2.

Proof. The claims follow directly from Definition 3.8. □

Lemma A.2. *Consider any quasi-polynomial $P : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ of the form*

$$P(x_1, \dots, x_d) = \sum_{i_1,\dots,i_d=0}^{N} \alpha_{i_1,\dots,i_d} \cdot |x_1|^{\beta_{i_1}} \dots |x_d|^{\beta_{i_d}}, \tag{A.0.4}$$

with coefficients $\alpha_{i_1,\dots,i_d} \geq 0$ and $\beta_{i_k} \geq 0$. Then P is an equivalence preserving function.

Proof. Let $\mathcal{I} \subset \mathbb{R}$ and consider two families of equivalent functions $f_i \sim g_i$, $i = 1, \dots, d$, on $\mathcal{I}$. There exist constants c_i, $d_i > 0$ such that

$$c_i|f_i(x)| \leq |g_i(x)| \leq d_i|f_i(x)|, \quad \text{for all } x \in \mathcal{I},\ i = 1, \dots, d.$$

Thus, for any $x = (x_1, \dots, x_d) \in \mathcal{I}^d$ there holds

$$\begin{aligned} P(g_1(x_1), \dots, g_d(x_d)) &= \sum_{i_1,\dots,i_d=0}^{N} d_{i_1}^{\beta^1_{i_1}} \dots d_{i_d}^{\beta^1_{i_d}} \cdot \alpha_{i_1,\dots,i_d} \cdot |f_1(x_1)^{\beta_{i_1}} \dots f_d(x_d)^{\beta_{i_d}}| \\ &\leq \max_{0 \leq i_1,\dots,i_d \leq N_1} \{d_{i_1}^{\beta^1_{i_1}} \dots d_{i_d}^{\beta^1_{i_d}}\} \cdot P(f_1(x_1), \dots, f_d(x_d)) \\ &=: D \cdot P(f_1(x_1), \dots, f_d(x_d)). \end{aligned}$$

Analogously one obtains that there exists some $C > 0$ such that

$$C \cdot P(f_1(x_1), \dots, f_d(x_d)) \leq P(g_1(x_1), \dots, g_d(x_d)).$$

□

Corollary A.3. *For any $\theta > 0$, the Clayton Lévy copula F of Example 2.10 satisfies Assumption 3.9.*

Proof. Clearly, F is 1-homogeneous and $H := \partial_1 \dots \partial_d F$ exists. There holds

$$F(x_1, \dots, x_d) = \frac{P_1(x_1, \dots, x_d)^{\gamma_1}}{P_2(x_1, \dots, x_d)^{\gamma_2}}$$

where $\gamma_1, \gamma_2 \geq 0$ and P_1, $P_2 : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ are two quasi-polynomials of the form (A.0.4). Due to the polynomial structure, an analogous representation naturally holds for H. Thus, by Lemma A.1, the derivative H is equivalence preserving. □

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