Sparse p-version BEM for first kind boundary integral equations with random loading

A. Chernov and C. Schwab

Research Report No. 2008-02
March 2008

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Sparse p-version BEM for first kind boundary integral equations with random loading

Dedicated to Professor Ernst P. Stephan on the occasion of his 60th birthday

A. Chernov and C. Schwab

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 2008-02 March 2008

Abstract

We consider the weakly singular boundary integral equation $\mathcal{V}u = g(\omega)$ on a deterministic smooth closed curve $\Gamma \subset \mathbb{R}^2$ with random loading $g(\omega)$. The statistical moments of g up to order k are assumed to be known. The aim is the efficient deterministic computation of statistical moments $\mathcal{M}^k u := \mathbb{E}[\bigotimes_{i=1}^k u]$, $k \geq 1$.

We derive a deterministic formulation for the kth statistical moment. It is posed in the tensor product Sobolev space and involves the k-fold tensor product operator $\mathcal{V}^{(k)} := \bigotimes_{i=1}^k \mathcal{V}$. The standard full tensor product Galerkin BEM requires $\mathcal{O}(N^k)$ unknowns for the kth moment problem, where N is the number of unknowns needed to discretize Γ. Extending ideas of [21], we develop the p-sparse grid Galerkin BEM to reduce the number of unknowns from $\mathcal{O}(N^k)$ to $\mathcal{O}(N(\log N)^{k-1})$.

1 Introduction

Due to growth of computer power in the recent decades large numerical simulations became possible, which found many applications in e.g. structural mechanics, computational chemistry and finance. Normally, the value of interest u is computed based on the input data g, known from experimental measurements or a priori knowledge. It turns out that in many practical applications the input data are not known precisely, e.g. due to error in experiments and/or inexact measurements. Then the input data can be considered as a random field $g(\omega)$. Writing the problem in mathematical terms leads to the operator equation

$$Au = g(\omega). \tag{1}$$

The efficient solution of the operator equation (1) with stochastic data was investigated over the last years [17, 18, 24]. Possible applications are e.g. the electrostatic potential problem with random charge distribution or problem in linearized elasticity with random boundary traction. We mention that the problems with randomness in the operator A, i.e. $A(\kappa(\omega))u = g$ can be approximated by a series of problems (1), if it is known that $\kappa(\omega)$ fluctuates around the "nominal" value κ with small (in suitable norm) amplitude [22]. Examples are diffusion problems with small random perturbations of the diffusion coefficient [7] and potential problems in random domain perturbed around the nominal domain with small amplitude perturbations [9].

The operator A may be an elliptic differential operator or an elliptic nonlocal pseudodifferential operator. In this paper we consider the case $A = \mathcal{V}$, where $\mathcal{V}$ is the single layer potential operator of the Laplacian on a smooth closed curve $\Gamma := \partial D$ and D is an open, simply connected, bounded domain in $\mathbb{R}^2$. This corresponds e.g. to the potential problem in stationary heat conduction in D with the random temperature $g(\omega)$ prescribed on the boundary Γ.

Normally, a particular solution $u(\omega)$ of (1) is of less interest than the statistical moments of u: mean field $\mathbb{E}_u$, two-point correlation C_u and higher order moments. The simplest and most widely used approach for solving the problem (1) is the Monte Carlo (MC) method. The method consists of sampling particular realizations $\mathcal{G} = \{g_1, \ldots, g_M\}$ of $g(\omega)$, finding (possibly in parallel) corresponding solutions $\mathcal{U} = \{u_1, \ldots, u_M\}$ and computation of the statistical moments of u based on $\mathcal{U}$. The major drawback of the method is the slow convergence with respect to the number of samples M, e.g. in [24, Theorem 4.6] it was shown that the convergence rate is $\mathcal{O}(M^{-1/2})$ up to logarithmic terms.

An alternative possibility is given by deterministic methods, which allow direct computation of the statistical moments of u if corresponding statistical moments of the input data are known. The key ingredient here is the deterministic moment formulation derived by tensorization [17, 18]. One possible approach is based on Karhunen-Loève expansion for the random input data (KL method) and allows to determine approximately the second moment C_u for given C_g. The method consists of 1) approximate computation of the first Q eigenvalues $\Lambda_Q := \{\lambda_1, \ldots, \lambda_Q\}$ and eigenvectors $\Phi_Q := \{\varphi_1, \ldots, \varphi_Q\}$ of the correlation kernel C_g, e.g. with the fast multipole method [19], then 2) possibly in parallel, numerically solving the moment equations $\Psi_Q = A^{-1}\Phi_Q$ and 3) evaluation of C_u based on Ψ_Q. In this approach there is an additional parameter Q – the number of terms in the KL approximation. The optimal value of Q depends on the decay rate of the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$. It was proven in [19] that the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ decay exponentially to zero provided C_g is piecewise analytic, i.e. in this case only a few terms in the KL expansion must be computed. Then the total complexity of the KL method is $\mathcal{O}(N(\log N)^c)$ for some constant $c > 0$, where N is the number of unknowns in Ψ_Q. Thus, in the case of piecewise analytic correlation function C_g the KL method has the same asymptotic complexity as deterministic sparse grid (SG) methods (cf. [17, 18, 24] and discussion below). On the other hand, if C_g is piecewise smooth or has finite Sobolev regularity, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ decay only algebraically to zero [19], consequently more terms in the KL expansion must be involved. This impacts the total complexity of the KL method, which becomes superlinear in N, whereas the complexity of the deterministic SG methods remains logarithmic-linear in N. In other words, if C_g is piecewise smooth or has finite Sobolev regularity, the deterministic SG methods are more advantageous than the KL method.

The deterministic SG approach is based on direct solution of the tensorized equation in the product domain. Here the dimension of the computational (product) domain grows linearly with the moment index and a naive full tensor product discretization would lead to a prohibitive number of unknowns already for the second moment problem (curse of dimensionality). On the other hand we can exploit the tensor product structure of the equation and leave out most of the unknowns and preserve the same up to the logarithmic factor convergence rate. As a result we obtain that the problem of finding the kth statistical moment has up to logarithms the same complexity as the problem of finding the mean field. One possibility is to apply the h-sparse wavelet FEM [24] with wavelets of some fixed polynomial degree. Here in the case $A = \mathcal{V}$ the wavelet basis admits matrix compression and the whole complexity of the problem is $\mathcal{O}(N(\log N)^c)$ with a constant $c > 0$, where N is the number of unknowns needed for discretization of original (not tensorized) physical domain. In this paper we develop an alternative p-sparse technique. Here the convergence is achieved by increasing the polynomial degree with basis functions of fixed support. Applying ideas of [21] from the theory of periodic approximation we define p-sparse discretization spaces for kth moment problem, which allow to solve the kth moment problem with the same (up to logarithmic terms) complexity as the mean field problem. We prove a priori error estimates in the energy norm and complexity bounds. The question of optimal complexity of the mean field problem is not addressed in this paper.

The SG technique became a well established approach for overcoming the curse of dimensionality for many problems of practical interest. In [8, 5] (see also references therein) the h-version of SG technique was applied to *nontensorized* differential and integral equations in product domains arising mainly from Laplace and Navier-Stokes problems. In this paper we suggest and investigate a p-version SG BEM for an elliptic equation governed by a *tensor product* integral operator.

The paper is organized as follows. In Section 2 we formulate the problem under consideration and derive the deterministic tensor product integral equation for the kth statistical moment. This formulation is posed in the tensor product Sobolev spaces. Section 3 is devoted to the definition of tensor product Sobolev spaces and their properties. The question of well-posedness of the kth moment problem is addressed in Section 4. Section 5 contains the key theorems of this paper. Here, based on the notion of the hyperbolic cross, we define p-sparse discrete spaces and study approximation properties of the corresponding L^2-projection operator in the L^2-norm and in the norms of negative order tensor product Sobolev spaces. We give the p-sparse Galerkin formulation for the kth moment problem in Section 6 and prove the corresponding version of Céa's lemma. Then we apply approximation results from Section 5 and obtain a priori error estimates in the energy norm. Section 7 contains remarks on efficient implementation and possible generalizations. Convergence of the suggested p-version of sparse grid method is illustrated on several numerical examples in Section 8. Technical results concerning interpolation between tensor product

Sobolev spaces are proven in the Appendix.

2 Symm's integral equation with random loading

Let $D \subset \mathbb{R}^2$ be an open, simply connected, bounded domain with smooth boundary $\Gamma := \partial D$. Suppose we are given a function $g \in H^{s+1/2}(\Gamma)$ for $s \in \mathbb{R}_{\geq 0}$. We are interested in the unknown function $u \in H^{-1/2}(\Gamma)$ satisfying the Symm's integral equation

$$\mathcal{V}u = g \qquad \text{in } H^{1/2}(\Gamma) \tag{2}$$

for the single layer operator $\mathcal{V}$

$$\mathcal{V}u(\mathbf{x}) := \int_\Gamma G(\mathbf{x}, \mathbf{y})u(\mathbf{y})\, ds_y,$$

where the kernel $G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{y}\|$ is the fundamental solution of the Laplace operator in $\mathbb{R}^2$ and ds_y denotes the curve integration on Γ with respect to the variable $\mathbf{y}$.

The operator $\mathcal{V}$ was extensively studied in the theory of boundary element methods. If $\Gamma \in C^\infty$ and $\text{cap}(\Gamma) < 1$ the operator $\mathcal{V} : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is a bounded and bijective operator for arbitrary real s. In particular, cf. [10, Theorem 3], there exists a constant $C_S > 0$, such that for arbitrary real s there holds

$$C_S^{-1}\|u\|_{H^s(\Gamma)} \leq \|\mathcal{V}u\|_{H^{s+1}(\Gamma)} \leq C_S\|u\|_{H^s(\Gamma)}, \qquad \forall u \in H^s(\Gamma). \tag{3}$$

Furthermore, the operator $\mathcal{V}$ is elliptic on $H^{-1/2}(\Gamma)$, cf. [10, Corollary 1], i.e. there exists a constant $C_E > 0$ such that

$$C_E\|u\|^2_{H^{-1/2}(\Gamma)} \leq \langle \mathcal{V}u, u\rangle, \qquad \forall u \in H^{-1/2}(\Gamma). \tag{4}$$

Here $\langle \cdot, \cdot \rangle$ stands for the duality pairing on $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

In what follows we consider (2) for random loading functions g leading to random solutions u of (2). Let (Ω, Σ, P) be a probability space consisting of the space of "events" Ω, σ-algebra of its subsets Σ and the probability measure P on Σ. Then u and g are defined as functions on $\Gamma \times \Omega$, and for fixed $\omega \in \Omega$ there holds $u(\cdot, \omega) \in H^{-1/2}(\Gamma)$ if $g(\cdot, \omega) \in H^{1/2}(\Gamma)$.

In order to introduce a notion of kth statistical moment of a random field we define following [18, 24] for an integer $k \geq 1$ and a separable Hilbert space X the Bochner spaces $L^k(\Omega, X)$ of functions $u : \Omega \to X$ endowed with the norm

$$\|u\|_{L^k(\Omega,X)} := \left(\int_\Omega \|u(\omega)\|^k_X \, dP(\omega)\right)^{1/k}.$$

Further, we define the k-fold tensor products $X^{(k)} := \bigotimes_{i=1}^k X$ with the induced norms $\|\cdot\|_{X^{(k)}}$ (see Section 3 for tensor products of Sobolev spaces). Note that $u \in L^k(\Omega, X)$ yields $(\bigotimes_{i=1}^k u) \in L^1(\Omega, X^{(k)})$, cf. [24].

Definition 2.1. *Let $k \geq 1$ be an integer. Then for $u \in L^k(\Omega, X)$ its kth moment is defined by*

$$\mathcal{M}^k u := \int_\Omega (\bigotimes_{i=1}^k u(\omega))\, dP(\omega) \in X^{(k)}. \tag{5}$$

We write in particular $\mathbb{E}_u := \mathcal{M}^1 u \in X$, $\mathrm{C}_u := \mathcal{M}^2 u \in X \otimes X$.

Note that $\mathcal{M}^k u$ is well defined, since

$$\|\mathcal{M}^k u\|_{X^{(k)}} \leq \|\bigotimes_{i=1}^{k} u\|_{L^1(\Omega,X^{(k)})} = \|u\|^k_{L^k(\Omega,X)}.$$

Let $u(\omega)$ be a random solution of (2) with a random right-hand side $g(\omega)$. Now we derive a deterministic tensor product formulation for finding $\mathcal{M}^k u$ from known $\mathcal{M}^k g$. For this reason we introduce a tensor product operator $\mathcal{V}^{(k)} := \bigotimes_{i=1}^k \mathcal{V}$, which is a linear mapping (see [24, Proposition 2.4] for more details)

$$\mathcal{V}^{(k)} : (H^{-1/2}(\Gamma))^{(k)} \to (H^{1/2}(\Gamma))^{(k)}.$$

Then, tensorization of (2) yields for every fixed $\omega \in \Omega$

$$\mathcal{V}^{(k)}(\bigotimes_{i=1}^{k} u(\omega)) = \bigotimes_{i=1}^{k} g(\omega) \qquad \text{in } (H^{1/2}(\Gamma))^{(k)}. \tag{6}$$

Taking the expectation of (6) we obtain

$$\mathcal{V}^{(k)}\mathcal{M}^k u = \mathcal{M}^k g \qquad \text{in } (H^{1/2}(\Gamma))^{(k)}. \tag{7}$$

The above equation allows to compute *deterministically* the kth moments of u if corresponding moments of g are known. Equation (7) has a very special tensor product structure. It inherits many properties from (2), e.g. it admits a unique solution if the right-hand side is sufficiently smooth and the shift theorem also holds true. In order to work with (7) we need several properties of tensor product Sobolev spaces from Section 3. We address questions of well-posedness of (7) and investigate properties of the operator $\mathcal{V}^{(k)}$ in Section 4.

3 Tensor products of Sobolev spaces

We recall that $D \subset \mathbb{R}^2$ is an open, simply connected, bounded domain with smooth boundary $\Gamma := \partial\Omega$. In this section we introduce tensor products of Sobolev spaces on Γ and investigate its interpolation and duality properties.

We use the standard Sobolev spaces $H^m(\mathbb{R}), m \in \mathbb{N}, H^0(\mathbb{R}) := L^2(\mathbb{R})$, and define $H^s(\mathbb{R}), s \in \mathbb{R}$ by duality and interpolation. Classical result [12, Theorem 7.1] gives that

$$\|u\|_{H^s(\mathbb{R})} := \left(\int_{\mathbb{R}} (1+|x|^2)^s |\hat{u}|^2 \, dx\right)^{1/2}, \qquad s \in \mathbb{R}. \tag{8}$$

is an equivalent norm on $H^s(\mathbb{R})$ and $(u,v)_{H^s(\mathbb{R})} := \int_{\mathbb{R}} (1+|x|^2)^s \hat{u}\bar{\hat{v}} \, dx$ is the induced scalar product, cf. [25, Corollary I.5.1]. Here $\hat{u}$ is the Fourier transform of u. By classical procedure (cf. [12, I.7.3]) involving finite coverings of Γ we define $H^s(\Gamma)$ based on (8).

We define by $X \otimes Y$ the tensor product of two separable Hilbert spaces X and Y, cf. [1, Definition 12.3.2]. For the corresponding scalar products there holds [1, p. 296]

$$(u_1 \otimes v_1, u_2 \otimes v_2)_{X \otimes Y} = (u_1, u_2)_X (v_1, v_2)_Y \qquad \forall u_1, u_2 \in X, \quad \forall v_1, v_2 \in Y. \tag{9}$$

Let $\|u\|_{H^{s_1,s_2}(\mathbb{R}^2)} := \left(\int_{\mathbb{R}^2} (1+|x|^2)^{s_1}(1+|y|^2)^{s_2} |\hat{u}|^2 \, dxdy\right)^{1/2}$ for $(s_1, s_2) \in \mathbb{R}^2$. Then relation (9) provides the isometry $H^{s_1}(\Gamma) \otimes H^{s_2}(\Gamma) = H^{s_1,s_2}(\Gamma^2)$. We define for arbitrary $\mathbf{s} \in \mathbb{R}^k$ the tensor product Sobolev spaces on $\Gamma^k := \times_{i=1}^k \Gamma$ by

$$\mathbf{H}^{\mathbf{s}}(\Gamma^k) := H^{s_1,\dots,s_k}(\Gamma^k) = \bigotimes_{i=1}^{k} H^{s_i}(\Gamma). \tag{10}$$

Note that for arbitrary $s \in \mathbb{R}$ we have $(H^s(\Gamma))^{(k)} = H^{s,\dots,s}(\Gamma^k) = \mathbf{H}^{s\mathbf{1}}(\Gamma^k)$, with $\mathbf{1} := (1,\dots,1) \in \mathbb{N}^k$. The family of tensor product Sobolev spaces (10) will be extensively used in the forthcoming analysis. However, several important questions arise regarding these spaces. Does this family constitute an interpolation scale? What are their dual spaces? And what is the duality pairing for these spaces?

We consider the duals of the tensor product spaces. We denote by $\langle \cdot,\cdot \rangle_{X'\times X}$ and $\langle \cdot,\cdot \rangle_{Y'\times Y}$ the duality pairings in the corresponding spaces. According to [1, p. 296], the bilinear form $\langle\langle u_1 \otimes v_1, u_2 \otimes v_2 \rangle\rangle := \langle u_1, u_2 \rangle_{X\times X'} \langle v_1, v_2 \rangle_{Y\times Y'}$ extends by continuity to $(X \otimes Y) \times (X' \otimes Y')$, and [1, Theorem 12.3.2] yields in particular

$$(\mathbf{H}^{\mathbf{s}}(\Gamma^k))' = \bigotimes_{i=1}^{k} (H^{s_i}(\Gamma))' = \mathbf{H}^{-\mathbf{s}}(\Gamma^k), \qquad \forall \mathbf{s} \in \mathbb{R}^k. \tag{11}$$

From now on we denote by $\langle\langle \cdot,\cdot \rangle\rangle$ the duality pairing on $\mathbf{H}^{\mathbf{s}}(\Gamma^k) \times \mathbf{H}^{-\mathbf{s}}(\Gamma^k)$.

The family of spaces (10) forms an interpolation scale, i.e. with standard notations of real interpolation theory (cf. [23]) there holds

$$(\mathbf{H}^{\mathbf{s}_0}(\Gamma^k), \mathbf{H}^{\mathbf{s}_1}(\Gamma^k))_{\theta,2} = \mathbf{H}^{\mathbf{s}}(\Gamma^k), \quad \text{for } \mathbf{s} = \mathbf{s}_0(1-\theta) - \mathbf{s}_1\theta, \quad \forall \mathbf{s}_0, \mathbf{s}_1 \in \mathbb{R}^k. \tag{12}$$

We give the proof of (12) in the Appendix.

4 Well-posedness and regularity of the kth moment problem

Properties (3), (4) of the single layer operator can be easily carried over to the tensor product operator $\mathcal{V}^{(k)}$.

Proposition 4.1. *Assume that (3) and (4) hold true for $\mathcal{V}$. Then the operator $\mathcal{V}^{(k)} := \bigotimes_{i=1}^k \mathcal{V}$ is bounded and elliptic with constants C_S^k and C_E^k respectively and there holds for arbitrary $s \in \mathbb{R}$*

$$C_S^{-k} \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)} \le \|\mathcal{V}^{(k)} u\|_{\mathbf{H}^{(s+1)\mathbf{1}}(\Gamma^k)} \le C_S^k \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)}, \qquad \forall u \in \mathbf{H}^{s\mathbf{1}}(\Gamma^k), \tag{13}$$

$$C_E^k \|u\|^2_{\mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)} \le \langle\langle \mathcal{V}^{(k)} u, u \rangle\rangle, \qquad \forall u \in \mathbf{H}^{-\mathbf{1}/2}(\Gamma^k). \tag{14}$$

Proof. Note that the operator $\mathcal{V}^{(k)}$ can be rewritten as a chain

$$\mathcal{V}^{(k)} = (\mathcal{V} \otimes I^{(k-1)}) \circ (I \otimes \mathcal{V} \otimes I^{(k-2)}) \circ \cdots \circ (I^{(k-1)} \otimes \mathcal{V}). \tag{15}$$

Then norm equivalence (3) provides e.g. for the last term in (15)

$$C_S^{-1} \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)} \le \|I^{(k-1)} \otimes \mathcal{V} u\|_{H^{s,\dots,s,s+1}(\Gamma^k)} \le C_S \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)}, \quad \forall u \in \mathbf{H}^{s\mathbf{1}}(\Gamma^k) \tag{16}$$

and similar inequalities for the other terms in (15). Then (13) follows by iteration.

Inequality (14) follows with similar arguments.

□

Theorem 4.1. *Let us assume that $\mathcal{M}^k g \in \mathbf{H}^{\mathbf{1}/2}(\Gamma^k)$. Then problem (7) has a unique solution $\mathcal{M}^k u \in \mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)$. Furthermore, if $\mathcal{M}^k g \in \mathbf{H}^{(s+1)\mathbf{1}}(\Gamma^k)$, $s \ge -1/2$, then $\mathcal{M}^k u \in \mathbf{H}^{s\mathbf{1}}(\Gamma^k)$ and there holds the following shift theorem*

$$\|\mathcal{M}^k u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)} \le C_S^k \|\mathcal{M}^k g\|_{\mathbf{H}^{(s+1)\mathbf{1}}(\Gamma^k)}. \tag{17}$$

Proof. The proof follows directly from Proposition 4.1, Lax-Milgram lemma and the left inequality in (13).

□

5 Sparse polynomial tensor approximation

5.1 Preliminaries and Notations

Let $\{L_n(x)\}_{n=0}^{\infty}$ be the set of Legendre polynomials on $I := [-1,1]$ normalized such that $L_n(1) = 1$. The well-known $L^2(I)$ orthogonality property of Legendre polynomials reads, cf. [16, 3.3.9]

$$\int_I (1-x^2)^n L_i^{(n)}(x) L_j^{(n)}(x)\, dx = (n+1/2)^{-1} \frac{(i+n)!}{(i-n)!} \delta_{ij}, \qquad i,j,n \in \mathbb{N}_0. \tag{18}$$

As before, we denote by k the moment index in (7). Note that (7) is formulated in the domain Γ^k of dimension k. We use small bold face letters for multiindices of dimension k, e.g. $\mathbf{n} := (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ and two special multiindices $\mathbf{0}, \mathbf{1}$ consisting only of zero and unit components respectively. We denote by

$$|\mathbf{n}|_1 := \sum_{i=1}^{d} n_i$$

the l_1-norm of $\mathbf{n}$. Throughout the paper we use standard multiindex notations. With $\mathbf{x} := (x_1, \ldots, x_k) \in \mathbb{R}^k$ we designate a point in I^k. If $u \in L^2(I^k)$, it can be expanded into a Legendre series

$$u(\mathbf{x}) = \sum_{\mathbf{n}=0}^{\infty} u_{\mathbf{n}} L_{\mathbf{n}}(\mathbf{x}), \qquad \text{where} \quad L_{\mathbf{n}}(\mathbf{x}) := \prod_{i=1}^{k} L_{n_i}(x_i).$$

Property (18) yields the following coefficient representation

$$u_{\mathbf{n}} = (\mathbf{n} + \frac{1}{2}\mathbf{1})^{\mathbf{1}} \int_{I^k} u(\mathbf{x}) L_{\mathbf{n}}(\mathbf{x})\, d\mathbf{x} \tag{19}$$

and the Parseval equation

$$\|u\|_{L^2(I^k)}^2 = \sum_{\mathbf{n}=0}^{\infty} |u_{\mathbf{n}}|^2 (\mathbf{n} + \frac{1}{2}\mathbf{1})^{-\mathbf{1}}. \tag{20}$$

Orthogonality property (18) admits a generalization of (20) on tensor products of certain weighted Sobolev spaces. Following [16, 3.3.10] for $v \in L^2(I)$ we define a family of seminorms and the corresponding weighted Sobolev spaces on I for $j, m \in \mathbb{N}_0, j \le m$

$$|v|_{V_j^m(I)}^2 := \sum_{i=j}^{m} \int_I (1-x^2)^i |v^{(i)}(x)|^2\, dx, \qquad V_j^m(I) := \{v \in L^2(I) : |v|_{V_j^m(I)} < \infty\}.$$

For arbitrary $\mathbf{j}, \mathbf{m} \in \mathbb{N}_0^k$ we define the tensor product weighted Sobolev spaces on I^k

$$\mathbf{V}_{\mathbf{j}}^{\mathbf{m}}(I^k) := \bigotimes_{i=1}^{k} V_{j_i}^{m_i}(I).$$

The seminorm induced on $\mathbf{V}_{\mathbf{j}}^{\mathbf{m}}(I^k)$ is given by

$$|u|_{\mathbf{V}_{\mathbf{j}}^{\mathbf{m}}(I^k)}^2 := \sum_{\mathbf{i}=\mathbf{j}}^{\mathbf{m}} \int_{I^k} (\mathbf{1}-\mathbf{x})^{\mathbf{i}} (\mathbf{1}+\mathbf{x})^{\mathbf{i}} |D^{\mathbf{i}} u(\mathbf{x})|^2\, d\mathbf{x}.$$

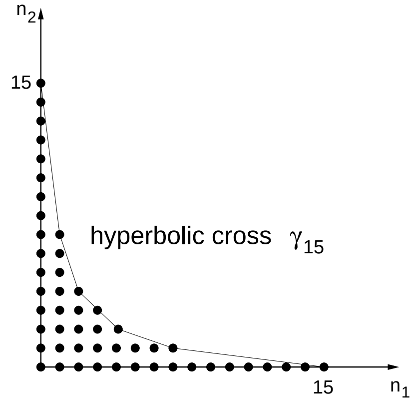

Figure 1: Hyperbolic cross for $k = 2$, $p = 15$

The orthogonality property (18) yields, cf. [16, Lemma 3.10]

$$|u|^2_{\mathbf{V}^{\mathbf{m}}_{\mathbf{j}}(I^k)} := \sum_{\mathbf{i}=\mathbf{j}}^{\mathbf{m}} \sum_{\mathbf{n}\geq\mathbf{i}} |u_{\mathbf{n}}|^2 (\mathbf{n} + \frac{1}{2}\mathbf{1})^{-\mathbf{1}} \frac{(\mathbf{n}+\mathbf{i})!}{(\mathbf{n}-\mathbf{i})!}.$$

In particular, for $\mathbf{m} = \mathbf{j}$ there holds

$$|u|^2_{\mathbf{V}^{\mathbf{m}}_{\mathbf{m}}(I^k)} := \sum_{\mathbf{n}\geq\mathbf{m}} |u_{\mathbf{n}}|^2 (\mathbf{n} + \frac{1}{2}\mathbf{1})^{-\mathbf{1}} \frac{(\mathbf{n}+\mathbf{m})!}{(\mathbf{n}-\mathbf{m})!}. \tag{21}$$

Similarly as in [16] we denote $\mathbf{V}^{\mathbf{m}}(I^k) := \mathbf{V}^{\mathbf{m}}_{\mathbf{0}}(I^k)$. Note that $\mathbf{V}^{\mathbf{0}}(I^k) = L^2(I^k)$.

5.2 Hyperbolic cross and p-sparse approximation in $L^2(I^k)$

The choice of the discrete (finite element) space is one of the main ingredients of an efficient computation. On the one hand the discrete space must be large enough to provide good approximation, on the other hand taking unnecessarily many degrees of freedom requires more computational effort without significant improvement of the approximation. In other words, the finite element space must be adapted to the solution of the problem, cf. [14].

In this section we construct an efficient approximation of a function $u \in \mathbf{H}^{s\mathbf{1}}(I^k)$ by polynomials in the $L^2(I^k)$-norm. The standard full tensor product approximation of $u \in \mathbf{H}^{s\mathbf{1}}(I^k)$ yields the convergence rate $(p+1)^{-s}$ and demands $(p+1)^k$ unknowns. This is the best possible construction for functions $u \in H^s(I^k) \setminus H^{s+\epsilon}(I^k)$ for any $\epsilon > 0$. In our case $\mathbf{H}^{s\mathbf{1}}(I^k)$ is a proper subspace of $H^s(I^k)$. Extending ideas of [21] we develop the hyperbolic cross approximation, which provides the same approximation order with $\mathcal{O}(p(\log p)^{k-1})$ unknowns.

Definition 5.1. *For $p \in \mathbb{N}_0$ we define the index set*

$$\gamma_p := \left\{ \mathbf{n} \in \mathbb{N}_0^k : (\mathbf{n}+\mathbf{1})^{\mathbf{1}} \leq p+1 \right\}, \tag{22}$$

which relates to the hyperbolic cross, cf. [21, p. 5], [11, Definition 3.4].

Definition 5.2. *We define*

$$S_p^\gamma := \mathrm{span}\{L_{\mathbf{n}}(\mathbf{x}) : \mathbf{n} \in \gamma_p, \mathbf{x} \in I^k\} \subset L^2(I^k)$$

and let $P_p^\gamma : L^2(I^k) \to S_p^\gamma$ be the $L^2(I^k)$-orthogonal projection onto S_p^γ.

Remark 5.1. *Note that*

$$|\gamma_p| = \dim(S_p^\gamma) = \mathcal{O}(p(\log p)^{k-1}).$$

The following auxiliary lemma is a key tool in the proof of Theorem 5.1.

Lemma 5.1. *(cf. [16, Corollary 3.12]) There exists a constant $\theta > 0$ such that for all $n, s \in \mathbb{N}_0$, $n \geq s \geq 0$ there holds*

$$\frac{(n-s)!}{(n+s)!} \leq \left(\frac{\theta}{n+1}\right)^{2s}. \tag{23}$$

Proof. We recall Stirling's formula, cf. [6, (9.15)]

$$\sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m+(12m+1)^{-1}} < m! < \sqrt{2\pi} m^{m+\frac{1}{2}} e^{-m+(12m)^{-1}} \qquad m \in \mathbb{N}. \tag{24}$$

I. Assume that $n-1 \geq s \geq 1$. Then (24) yields

$$\frac{(n-s)!}{(n+s)!} < \left(\frac{n-s}{n+s}\right)^{n-s+\frac{1}{2}} (n+s)^{-2s} e^{2s+\frac{1}{12(n-s)}} \leq \left(\frac{e^{1+\frac{1}{24s(n-s)}}}{k+1}\right)^{2s},$$

since $\frac{1}{s(n-s)} \leq 1$ by assumption. Thus (23) holds with $\theta = e^{\frac{25}{24}}$.

II. Assume that $n = s \geq 1$. Then

$$\frac{(n-s)!}{(n+s)!} = \frac{1}{(2n)!} < \frac{1}{\sqrt{2\pi}} (2n)^{-2n-\frac{1}{2}} e^{2n-(24n)^{-1}} \leq \left(\frac{e}{2n}\right)^{2n} \leq \left(\frac{e}{n+1}\right)^{2s},$$

since $2n \geq n+1$ for $n \geq 1$.

III. For $n \geq s = 0$ the assertion is trivial.

□

For simplicity of notation we introduce the following partition of $\mathbb{N}_0$.

Definition 5.3. *For arbitrary $s \in \mathbb{N}$ and $\boldsymbol{\sigma} \in \{0,1\}^k$ we define*

$$A_s^0 := \{0, \dots, s-1\}, \qquad A_s^1 := \mathbb{N}_0 \setminus A_s^0, \qquad A_s^{\boldsymbol{\sigma}} := A_s^{\sigma_1} \times \dots \times A_s^{\sigma_k}. \tag{25}$$

Corollary 5.1. *With above definition we have the following nonoverlapping decomposition*

$$\mathbb{N}_0^k = \bigsqcup_{\mathbf{0} \leq \boldsymbol{\sigma} \leq \mathbf{1}} A_s^{\boldsymbol{\sigma}}.$$

Now, we prove the main result of this section.

Theorem 5.1. *Let $k, s, p \in \mathbb{N}$, $(p+1)^{1/k} \geq s$. Assume that $u \in \mathbf{V}^{s\mathbf{1}}(I^k)$. Then there exists a constant $C(s,k) > 0$, independent of p, such that*

$$\|u - P_p^\gamma u\|_{L^2(I^k)} \leq C(s,k)\,(p+1)^{-s} \left(\sum_{\mathbf{0} < \boldsymbol{\sigma} \leq \mathbf{1}} |u|^2_{\mathbf{V}^{s\boldsymbol{\sigma}}_{s\boldsymbol{\sigma}}(I^k)}\right)^{1/2}. \tag{26}$$

Proof. The assumption $(p+1)^{1/k} \geq s$ yields the inclusion $A_s^{\mathbf{0}} \subset \gamma_p$. Using the Parseval equation (20) and Corollary 5.1 we obtain

$$\|u - P_p^\gamma u\|^2_{L^2(I^k)} = \sum_{\mathbf{0} < \boldsymbol{\sigma} \leq \mathbf{1}} \sum_{\mathbf{n} \in A_s^{\boldsymbol{\sigma}} \setminus \gamma_p} u_{\mathbf{n}}^2 (\mathbf{n} + \frac{1}{2}\mathbf{1})^{-\mathbf{1}}. \tag{27}$$

Lemma 5.1 yields for arbitrary $\mathbf{n} \in A_s^{\boldsymbol{\sigma}} \setminus \gamma_p$, $\mathbf{0} < \boldsymbol{\sigma} \leq \mathbf{1}$

$$\begin{aligned}\frac{(\mathbf{n}-s\boldsymbol{\sigma})!}{(\mathbf{n}+s\boldsymbol{\sigma})!} &\leq \prod_{i:\,\sigma_i=1}\left(\frac{\theta}{n_i+1}\right)^{2s}\\ &= \theta^{2s|\boldsymbol{\sigma}|_1}\prod_{i:\,\sigma_i=1}(n_i+1)^{-2s}\\ &\leq \theta^{2s|\boldsymbol{\sigma}|_1}(p+1)^{-2s}\prod_{i:\,\sigma_i=0}(n_i+1)^{2s} \leq C_1(s,k,\boldsymbol{\sigma})(p+1)^{-2s}\end{aligned}$$

with $C_1(s,k,\boldsymbol{\sigma}) := \theta^{2s|\boldsymbol{\sigma}|_1}s^{2s(k-|\boldsymbol{\sigma}|_1)}$. Thus, recalling (21) we obtain the following estimate for the interior sum in (27)

$$\begin{aligned}\sum_{\mathbf{n}\in A_s^{\boldsymbol{\sigma}}\setminus\gamma_p} u_{\mathbf{n}}^2(\mathbf{n}+\frac{1}{2}\mathbf{1})^{-\mathbf{1}} &\leq C_1(s,k,\boldsymbol{\sigma})(p+1)^{-2s}\sum_{\mathbf{n}\in A_s^{\boldsymbol{\sigma}}\setminus\gamma_p} u_{\mathbf{n}}^2(\mathbf{n}+\frac{1}{2}\mathbf{1})^{-\mathbf{1}}\frac{(\mathbf{n}+s\boldsymbol{\sigma})!}{(\mathbf{n}-s\boldsymbol{\sigma})!}\\ &\leq C_1(s,k,\boldsymbol{\sigma})(p+1)^{-2s}\sum_{\mathbf{n}\geq s\boldsymbol{\sigma}} u_{\mathbf{n}}^2(\mathbf{n}+\frac{1}{2}\mathbf{1})^{-\mathbf{1}}\frac{(\mathbf{n}+s\boldsymbol{\sigma})!}{(\mathbf{n}-s\boldsymbol{\sigma})!}\\ &\overset{(21)}{=} C_1(s,k,\boldsymbol{\sigma})(p+1)^{-2s}|u|^2_{\mathbf{V}^{s\boldsymbol{\sigma}}_{s\boldsymbol{\sigma}}(I^k)}.\end{aligned} \tag{28}$$

Inserting (28) in (27) yields (26) with

$$C(s,k) := \max_{\mathbf{0}<\boldsymbol{\sigma}\leq\mathbf{1}} C_1(s,k,\boldsymbol{\sigma})^{1/2} \leq \theta^{sk}s^{s(k-1)}.$$

□

Corollary 5.2. *Let $k, s, p \in \mathbb{N}$, $(p+1)^{1/k} \geq s$. Assume that $u \in \mathbf{H}^{s\mathbf{1}}(I^k)$. Then there exists a constant $C(s,k) > 0$, independent of p, such that*

$$\|u - P_p^{\gamma}u\|_{L^2(I^k)} \leq C(s,k)(p+1)^{-s}\left(\sum_{\mathbf{0}<\boldsymbol{\sigma}\leq\mathbf{1}}|u|^2_{\mathbf{H}^{s\boldsymbol{\sigma}}(I^k)}\right)^{1/2}. \tag{29}$$

5.3 p-Sparse approximation on quasi-uniform meshes

Let $\mathcal{T}_h$ be a quasi-uniform partition of Γ into a union of open curves $e \in \mathcal{T}_h$ of characteristic size h. Thus $|\mathcal{T}_h| = \mathcal{O}(h^{-1})$. The above partition induces a partition of Γ^k into elements $\mathbf{e} \in \boldsymbol{\mathcal{T}}_h := (\mathcal{T}_h)^k$. The volume of an element $\mathbf{e} \in \boldsymbol{\mathcal{T}}_h$ is $|\mathbf{e}| = \mathcal{O}(h^k)$.

We pick an arbitrary $s \in \mathbb{N}_0$, $\boldsymbol{\sigma} \in \{0,1\}^k$ and an arbitrary element $\mathbf{e} \in \boldsymbol{\mathcal{T}}_h$. Let $\mathbf{f_e} : I^k \to \mathbf{e}$ be the bijective mapping, defined by $x_i = f_i(\xi_i)$, $i = 1, \ldots, k$. The Jacobi determinant of $\mathbf{f_e}$ is

$$J := \mathrm{Det}\frac{\partial\mathbf{f_e}}{\partial\boldsymbol{\xi}} \sim \left(\frac{h}{2}\right)^k$$

with an equivalence constant depending only on k. Assume that $u \in \mathbf{H}^{s\boldsymbol{\sigma}}(\mathbf{e})$. Then

$$\begin{aligned}|u|^2_{\mathbf{H}^{s\boldsymbol{\sigma}}(\mathbf{e})} = \int_{\mathbf{e}}|D_{\mathbf{x}}^{s\boldsymbol{\sigma}}u(\mathbf{x})|^2d\mathbf{x} &= \int_{I^k}|D_{\boldsymbol{\xi}}^{s\boldsymbol{\sigma}}u(\mathbf{f_e}(\boldsymbol{\xi}))|^2\prod_{i:\sigma_1=1}\left|\frac{d^s\xi_i}{dx_i^s}\right|^2 Jd\boldsymbol{\xi}\\ &\sim \left(\frac{h}{2}\right)^{k-2s|\boldsymbol{\sigma}|_1}|u\circ\mathbf{f_e}|^2_{\mathbf{H}^{s\boldsymbol{\sigma}}(I^k)}.\end{aligned} \tag{30}$$

Furthermore, since $s \in \mathbb{N}_0, \boldsymbol{\sigma} \in \{0,1\}^k$ and the integral is additive with respect to the domain of integration, there holds

$$|u|^2_{\mathbf{H}^{s\boldsymbol{\sigma}}(\Gamma^k)} = \sum_{\mathbf{e}\in\boldsymbol{\mathcal{T}}_h}|u|^2_{\mathbf{H}^{s\boldsymbol{\sigma}}(\mathbf{e})}. \tag{31}$$

Definition 5.4. *Define*

$$S_p^\gamma(\mathbf{e}) := \mathrm{span}\{\hat{L}_\mathbf{k}(\mathbf{x}) := L_\mathbf{k} \circ \mathbf{f}_\mathbf{e}^{-1}(\mathbf{x}) : \mathbf{k} \in \gamma_p, \mathbf{x} \in \mathbf{e}\},$$
$$S_p(\mathbf{e}) := \mathrm{span}\{\hat{L}_\mathbf{k}(\mathbf{x}) := L_\mathbf{k} \circ \mathbf{f}_\mathbf{e}^{-1}(\mathbf{x}) : \mathbf{k} \le p\mathbf{1}, \mathbf{x} \in \mathbf{e}\}$$

and

$$S_{hp}^\gamma(\Gamma^k) := \{u \in L^2(\Gamma^k) : u|_\mathbf{e} \in S_p^\gamma(\mathbf{e}),\ \forall \mathbf{e} \in \boldsymbol{\mathcal{T}}_h\},$$
$$S_{hp}(\Gamma^k) := \{u \in L^2(\Gamma^k) : u|_\mathbf{e} \in S_p(\mathbf{e}),\ \forall \mathbf{e} \in \boldsymbol{\mathcal{T}}_h\}.$$

Let $P_{hp}^\gamma : L^2(\Gamma^k) \to S_{hp}^\gamma$ *be the* $L^2(\Gamma^k)$*-orthogonal projection onto* S_{hp}^γ.

We refer to $S_{hp}(\Gamma^k)$ as to the full tensor product discrete space and to $S_{hp}^\gamma(\Gamma^k)$ as to the p-sparse tensor product discrete space. Note that for fixed h and $k \ge 2$

$$|S_{hp}^\gamma(\Gamma^k)| = \mathcal{O}(p(\log p)^{k-1}) \ll |S_{hp}(\Gamma^k)| = \mathcal{O}(p^k).$$

Remark 5.2. *Note that for* $\forall u \in L^2(\Gamma^k)$, $\forall \mathbf{e} \in \boldsymbol{\mathcal{T}}_h$ *there holds*

$$(P_{hp}^\gamma u) \circ \mathbf{f}_\mathbf{e} = P_p^\gamma(u \circ \mathbf{f}_\mathbf{e}). \tag{32}$$

Scaling and summing properties (30), (31) together with the reference element estimate (29) provide the following $L^2(\Gamma^k)$-approximation result. Define $\lceil s \rceil := \min\{n \in \mathbb{Z} : n \ge s\}$ for real s.

Theorem 5.2. *Let* $k, p \in \mathbb{N}$, $s \in \mathbb{R}_{\ge 0}$ $(p+1)^{1/k} \ge \lceil s \rceil$. *Assume that* $u \in \mathbf{H}^{s\mathbf{1}}(\Gamma^k)$. *Further, let* $\boldsymbol{\mathcal{T}}_h$ *be a quasi-uniform partition of* Γ^k. *Then there exists a constant* $C(s,k) > 0$, *independent of* h, p, *such that*

$$\|u - P_{hp}^\gamma u\|_{L^2(\Gamma^k)} \le C(s,k) \left(\frac{h}{p+1}\right)^s \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)}. \tag{33}$$

Proof. Define $\hat{u}_\mathbf{e} := u \circ \mathbf{f}_\mathbf{e} \in L^2(I^k)$. Then

$$\begin{aligned}
\|u - P_{hp}^\gamma u\|_{L^2(\mathbf{e})}^2 &\overset{(30)}{\sim} \left(\frac{h}{2}\right)^k \|(u - P_{hp}^\gamma u) \circ \mathbf{f}_e\|_{L^2(I^k)}^2 \\
&\overset{(32)}{=} \left(\frac{h}{2}\right)^k \|\hat{u}_\mathbf{e} - P_p^\gamma \hat{u}_\mathbf{e}\|_{L^2(I^k)}^2 \\
&\overset{(29)}{\le} C(s,k)^2 (p+1)^{-2s} \left(\frac{h}{2}\right)^k \sum_{\mathbf{0} < \boldsymbol{\sigma} \le \mathbf{1}} |\hat{u}_\mathbf{e}|_{\mathbf{H}^{s\boldsymbol{\sigma}}(I^k)}^2 \\
&\overset{(30)}{\sim} C(s,k)^2 (p+1)^{-2s} \sum_{\mathbf{0} < \boldsymbol{\sigma} \le \mathbf{1}} \left(\frac{h}{2}\right)^{2s|\boldsymbol{\sigma}|_1} |u|_{\mathbf{H}^{s\boldsymbol{\sigma}}(\mathbf{e})}^2 \\
&\le C(s,k)^2 \left(\frac{h/2}{p+1}\right)^{2s} \sum_{\mathbf{0} < \boldsymbol{\sigma} \le \mathbf{1}} |u|_{\mathbf{H}^{s\boldsymbol{\sigma}}(\mathbf{e})}^2.
\end{aligned} \tag{34}$$

Therefore

$$\begin{aligned}
\|u - P_{hp}^\gamma u\|_{L^2(\Gamma^k)}^2 &\overset{(31)}{=} \sum_{\mathbf{e} \in \boldsymbol{\mathcal{T}}_h} \|u - P_{hp}^\gamma u\|_{L^2(\mathbf{e})}^2 \\
&\overset{(34)}{\le} \sum_{\mathbf{e} \in \boldsymbol{\mathcal{T}}_h} C(s,k)^2 \left(\frac{h/2}{p+1}\right)^{2s} \sum_{\mathbf{0} < \boldsymbol{\sigma} \le \mathbf{1}} |u|_{\mathbf{H}^{s\boldsymbol{\sigma}}(\mathbf{e})}^2 \\
&\overset{(31)}{\le} C(s,k)^2 \left(\frac{h/2}{p+1}\right)^{2s} \sum_{\mathbf{0} < \boldsymbol{\sigma} \le \mathbf{1}} |u|_{\mathbf{H}^{s\boldsymbol{\sigma}}(\Gamma^k)}^2.
\end{aligned}$$

This gives (33) for $s \in \mathbb{N}$. In case $s = 0$ the estimate (33) holds with $C(s,k) = 1$, since $(I - P_{hp}^{\gamma}) \in \mathcal{L}(L^2(\Gamma^k), L^2(\Gamma^k))$. Finally, we obtain (33) for noninteger $s \in \mathbb{R}_{\geq 0}$ by the real method of interpolation, see Appendix for more details.

□

The main corollary of the above theorem is the estimate for negative tensor product Sobolev norms.

Corollary 5.3. *Under assumptions of Theorem 5.2 for arbitrary $r, s \in \mathbb{R}_{\geq 0}$ there exists a constant $C(s,k) > 0$, independent of h, p, such that*

$$\|u - P_{hp}^{\gamma} u\|_{\mathbf{H}^{-r\mathbf{1}}(\Gamma^k)} \leq C(s,k) \left(\frac{h}{p+1}\right)^{r+s} \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)}. \tag{35}$$

Proof. The proof follows from Definition 5.4, an Aubin-Nitsche duality argument and Theorem 5.2. For arbitrary $r \in \mathbb{R}_{\geq 0}$ there holds

$$\begin{aligned}
\|u - P_{hp}^{\gamma} u\|_{\mathbf{H}^{-r\mathbf{1}}(\Gamma^k)} &= \sup_{v \in \mathbf{H}^{r\mathbf{1}}(\Gamma^k)} \frac{\langle\langle u - P_{hp}^{\gamma} u, v \rangle\rangle}{\|v\|_{\mathbf{H}^{r\mathbf{1}}(\Gamma^k)}} \\
&\leq \|u - P_{hp}^{\gamma} u\|_{L^2(\Gamma^k)} \sup_{v \in \mathbf{H}^{r\mathbf{1}}(\Gamma^k)} \frac{\|v - P_{hp}^{\gamma} v\|_{L^2(\Gamma^k)}}{\|v\|_{\mathbf{H}^{r\mathbf{1}}(\Gamma^k)}} \\
&\leq C(s,k) \left(\frac{h}{p+1}\right)^{r+s} \|u\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)}.
\end{aligned}$$

□

In other words, from Corollary 5.3 it follows that the approximation order of $S_{hp}^{\gamma}(\Gamma^k)$ in $\mathbf{H}^{-r\mathbf{1}}(\Gamma^k)$, $r \in \mathbb{R}_{\geq 0}$ is the same as the approximation order of $S_{hp}(\Gamma)$ in $H^{-r}(\Gamma)$, $r \in \mathbb{R}_{\geq 0}$, cf. [20, Theorem 3.2]; the cardinality of $S_{hp}^{\gamma}(\Gamma^k)$ is up to logarithmic terms the same as the cardinality of $S_{hp}(\Gamma)$.

6 p-Sparse Galerkin BEM

Summarizing results of the previous section, we have proven that the p-sparse tensor product discrete space $S_{hp}^{\gamma}(\Gamma^k)$ has *the same* approximation properties as the full tensor product space $S_{hp}(\Gamma^k)$ in the anisotropic Sobolev spaces $\mathbf{H}^{s\mathbf{1}}(\Gamma^k)$. Therefore we suggest the p-sparse Galerkin BEM based on $S_{hp}^{\gamma}(\Gamma^k)$. The Galerkin formulation of (7) reads as follows.

Given $\mathcal{M}^k g \in \mathbf{H}^{\mathbf{1}/2}(\Gamma^k)$, find $\mathcal{U} \in S_{hp}^{\gamma}(\Gamma^k)$ such that

$$\langle\langle \mathcal{V}^{(k)} \mathcal{U}, v \rangle\rangle = \langle\langle \mathcal{M}^k g, v \rangle\rangle, \qquad \forall v \in S_{hp}^{\gamma}(\Gamma^k). \tag{36}$$

There holds $S_{hp}^{\gamma} \subset L^2(\Gamma^k) \subset \mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)$. Therefore, recalling (7), we have the following Galerkin orthogonality property and the Céa's Lemma.

Corollary 6.1. *Let $\mathcal{M}^k u \in \mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)$ and $\mathcal{U} \in S_{hp}^{\gamma}(\Gamma^k)$ be the unique solutions of (7) and (36) respectively, $k \in \mathbb{N}$. Then*

$$\langle\langle \mathcal{V}^{(k)}(\mathcal{M}^k u - \mathcal{U}), v \rangle\rangle = 0, \qquad \forall v \in S_{hp}^{\gamma}(\Gamma^k). \tag{37}$$

Lemma 6.1. *(Céa) Let $\mathcal{M}^k u \in \mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)$ and $\mathcal{U} \in S_{hp}^{\gamma}(\Gamma^k)$ be the unique solutions of (7) and (36) respectively, $k \in \mathbb{N}$. Then there holds the best approximation property*

$$\|\mathcal{M}^k u - \mathcal{U}\|_{\mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)} \leq \left(\frac{C_S}{C_E}\right)^k \inf_{v \in S_{hp}^{\gamma}(\Gamma^k)} \|\mathcal{M}^k u - v\|_{\mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)}, \tag{38}$$

where C_S and C_E are continuity and ellipticity constants of $\mathcal{V}$, cf. (3), (4).

Now we are in the position to prove our main result.

Theorem 6.1. *(A priori error estimate) Assume that $\mathcal{M}^k g \in \mathbf{H}^{s\mathbf{1}}(\Gamma^k)$, where $s \geq 1/2$ and $k \in \mathbb{N}$ are given. Further, let $\mathcal{M}^k u \in \mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)$ and $\mathcal{U} \in S^{\gamma}_{hp}(\Gamma^k)$ be the unique solutions of (7) and (36) respectively. Then there exists a constant $C(s,k) > 0$ independent of the discretization parameters h and p, such that*

$$\|\mathcal{M}^k u - \mathcal{U}\|_{\mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)} \leq C(s,k) \left(\frac{h}{p+1}\right)^{s-1/2} \|\mathcal{M}^k g\|_{\mathbf{H}^{s\mathbf{1}}(\Gamma^k)}. \tag{39}$$

Proof. Application of Céa's lemma and of the approximation result of Corollary 5.3 yield for $s \geq 1$

$$\|\mathcal{M}^k u - \mathcal{U}\|_{\mathbf{H}^{-\mathbf{1}/2}(\Gamma^k)} \leq C(s,k) \left(\frac{h}{p+1}\right)^{s-1/2} \|\mathcal{M}^k u\|_{\mathbf{H}^{(s-1)\mathbf{1}}(\Gamma^k)}. \tag{40}$$

Continuity and ellipticity of $\mathcal{V}^{(k)}$ (Proposition 4.1) and Galerkin orthogonality (Corollary 6.1) give (40) for $s = 1/2$. An interpolation argument yields (40) also for $s \geq 1/2$. Finally, (39) follows from (40) and the shift theorem (Theorem 4.1).

□

7 Implementation and generalizations

7.1 Matrix-vector multiplication

The discrete formulation (36) is equivalent to solving a linear system of equations

$$\hat{V}^{(k)}_p \hat{\mathcal{U}}_p = \hat{\mathcal{G}}_p, \tag{41}$$

where $''\,\hat{}\,''$ designates the matrix and the vectors obtained by the p-sparse tensor product discretization. So, $\hat{V}^{(k)}_p$ is the system matrix, $\hat{\mathcal{U}}_p$ is the vector of unknowns and $\hat{\mathcal{G}}_p$ is the right-hand side obtained by the p-sparse tensor product discretization (36). The sequence of hyperbolic crosses is nested by construction (cf. Definition 5.1), i.e. $\gamma_{p_1} \subset \gamma_{p_2} \Leftrightarrow p_1 < p_2$. Therefore for fixed h and variable p the discrete spaces $S^{\gamma}_{hp}(\Gamma^k)$ are nested and the set of scaled Legendre polynomials (cf. Definition 5.4) forms a hierarchical basis in $S^{\gamma}_{hp}(\Gamma^k)$. Thus, the subscript p in (41), representing the characteristic polynomial degree in $S^{\gamma}_{hp}(\Gamma^k)$, can be viewed as a level of the hierarchical discretization.

We use the conjugate gradient (CG) method to solve (41) iteratively. The matrix $\hat{V}^{(k)}_p$ is dense and requires $\mathcal{O}(p^2(\log p)^{2k-2})$ memory units. This might be rather costly, particularly for high k. CG does not require explicit computation and storage of the whole matrix $\hat{V}^{(k)}_p$; it is enough to realize the matrix-vector multiplication on a procedural level.

Similarly as in [17, 18] we exploit the anisotropic tensor product structure of $\hat{V}^{(k)}_p$. We suggest an algorithm for the matrix-vector multiplication requiring $\mathcal{O}(p \max\{p, (\log p)^{k-1}\})$ storage and of complexity $\mathcal{O}(p^2(\log p)^{2k-2})$. We store only the matrix $V := V^{(1)}_p$, which is the system matrix of the mean field problem, $V = [V_{ij}]^p_{i,j=0}$. We are interested in the product

$$\hat{\mathcal{Y}}_p = \hat{V}^{(k)}_p \hat{\mathcal{X}}_p, \tag{42}$$

where $\hat{\mathcal{X}}_p = [\mathcal{X}_{\mathbf{j}}]_{\mathbf{j} \in \gamma_p}$ and $\hat{\mathcal{Y}}_p = [\mathcal{Y}_{\mathbf{i}}]_{\mathbf{i} \in \gamma_p}$. Then the matrix-vector multiplication (42) can be realized as follows.

Algorithm 7.1.

for $\mathbf{i}$ satisfying $\mathbf{i} \in \gamma_p$

compute $\mathcal{Y}_{\mathbf{i}} := \sum_{\mathbf{j} \in \gamma_p} \mathcal{X}_{\mathbf{j}} \prod_{m=1}^{k} V_{i_m j_m}$

end

Computation of $\mathcal{Y}_{\mathbf{i}}$ has complexity $\mathcal{O}(p(\log p)^{k-1})$, which leads to the overall complexity $\mathcal{O}(p^2(\log p)^{2k-2})$. The storage expenses for the matrix V are $\mathcal{O}(p^2)$ and $\mathcal{O}(p(\log p)^{k-1})$ for the vector $\hat{\mathcal{X}}_p$. Thus the total storage is $\mathcal{O}(p \max\{p, (\log p)^{k-1}\})$.

Note that the complexity of the matrix-vector multiplication with the tensor product matrix $\hat{V}_p^{(k)}$ is up to the logarithmic factor the same as the complexity of the matrix-vector multiplication with the matrix V, corresponding to the mean field problem.

7.2 Strongly elliptic systems

The suggested sparse p-version of BEM admits straightforward generalizations to other strongly elliptic systems as soon as a direct conforming discretization of the mean value problem is possible. Let $D \subset \mathbb{R}^2$ be an open, simply connected, bounded domain with smooth boundary $\Gamma := \partial D$ and consider the following integral operator on Γ

$$\mathcal{V}_\kappa u(\mathbf{x}) := \int_\Gamma G_\kappa(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) \, ds_y$$

with kernel $G_\kappa(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(\kappa \|\mathbf{x} - \mathbf{y}\|)$, where $H_0^{(1)}$ is the Hankel function of the first kind.

Lemma 7.1. *[15, Proposition 3.5.5] The operator $\mathcal{V}_\kappa$ satisfies a generalized Gårding inequality, i.e. there exists a constant $C > 0$ and a compact operator $T : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ such that*

$$\operatorname{Re} \langle (\mathcal{V}_\kappa + T) u, u \rangle \geq C \|u\|^2_{H^{-1/2}(\Gamma)}, \qquad \forall u \in H^{-1/2}(\Gamma). \tag{43}$$

Further, consider the following problem: for given $g \in H^{1/2}(\Gamma)$ find $u \in H^{-1/2}(\Gamma)$ such that

$$\mathcal{V}_\kappa u = g \qquad \text{in } H^{1/2}(\Gamma). \tag{44}$$

It is well known that the Helmholtz problem for a sound-soft obstacle is equivalent to the integral equation (44) when $-k^2$ is not an eigenvalue of the corresponding Dirichlet problem for the Laplacian in the interior domain D (i.e. $-k^2 \notin \sigma(\Delta)$) [13]. In this case the operator $\mathcal{V}_\kappa : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is injective, which together with the generalized Gårding inequality (43) yields unique solvability of (44) [15]. Similarly as in Section 2 the corresponding kth moment problem can be derived (cf. (7)) and its well-posedness can be shown. Further, for $-k^2 \notin \sigma(\Delta)$ the integral equation (44) admits stable conforming discretization with p-version boundary elements provided $p \geq p_0(\kappa)$ is sufficiently large. Alternatively, the BIE (44) must be stabilized (cf. [3, 4] for convenient choices of T). Thus, the sparse p-version of BEM can be applied for efficiently solving the kth moment problem with a priori error estimates similar to that in Theorem 6.1.

8 Numerical examples

In this section we give several numerical examples illustrating convergence of the suggested p-version of sparse grids for the second moment problem ($k = 2$, recall $\mathrm{C}_g := \mathcal{M}^2 g$). We consider equation (7) for with different right-hand sides C_g and examine convergence of the discrete solutions of (36) depending on the smoothness of C_g. Let (r, φ) be the polar coordinates in $\mathbb{R}^2$. We choose the computational domain Γ to be the circle $\{(r, \varphi) \in \mathbb{R}^2 : r = 1/4, \varphi \in [0, 2\pi)\}$.

8.1 Finite Sobolev regularity, tensor product solution

We consider

$$f(\mathbf{x}) := \sin \frac{\varphi(\mathbf{x})}{2}, \qquad C_g := f \otimes f.$$

The function g has a finite Sobolev regularity, namely $C_g \in H^{3/2-\epsilon}(\Gamma) \otimes H^{3/2-\epsilon}(\Gamma)$ for any positive ϵ. The tensor product structure of (7) gives that $C_u = w \otimes w$, where $w \in H^{1/2-\epsilon}(\Gamma)$ solves

$$\mathcal{V} w = f, \qquad \text{in } H^{1/2}(\Gamma).$$

The function C_u clearly allows a sparse tensor product approximation. Convergence of the p-version of the sparse grid method and convergence of the full tensor product method in the energy norm $\|\cdot\|_{\mathcal{V}\otimes\mathcal{V}} := \langle\langle(\mathcal{V} \otimes \mathcal{V})\cdot, \cdot\rangle\rangle^{1/2}$ are compared in Fig. 2. The approximate value of the energy norm of the exact solution $\|C_u\|_{\mathcal{V}\otimes\mathcal{V}} \approx 9.67295$ is obtained by extrapolation. We observe that the convergence rate with respect to the polynomial degree is nearly the same for both methods (Fig. 2, left), which corresponds to the statement of the Theorem 6.1. On the other hand, the p-sparse discretization requires only $\mathcal{O}(N \log N)$ unknowns, whereas the full tensor product discretization requires $\mathcal{O}(N^2)$ unknowns, which leads to a better convergence rate of the suggested p-sparse discretization with respect to the total number of unknowns (Fig. 2, right).

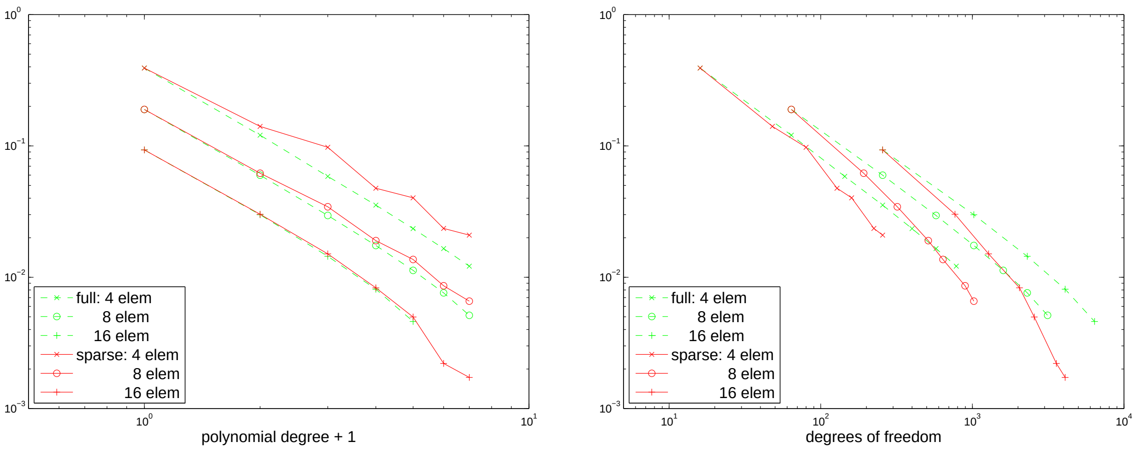

Figure 2: Relative error in the energy norm for $C_g = \sin \frac{\varphi(\mathbf{x})}{2} \sin \frac{\varphi(\mathbf{y})}{2}$

8.2 Finite Sobolev regularity, not tensorized solution

As a second example we take

$$f(\mathbf{x}) := \sin \frac{\varphi(\mathbf{x})}{2}, \qquad C_g := \exp(f \otimes f).$$

The function C_g possesses again a finite Sobolev regularity, but the solution C_u can not be written in a form of a tensor product. Nonetheless, convergence behavior of the suggested p-version of the sparse grid method repeats the convergence behavior from the previous example (cf. Fig. 3), which was predicted by Theorem 6.1. The extrapolated value of the energy norm is $\|C_u\|_{\mathcal{V}\otimes\mathcal{V}} \approx 67.3368$.

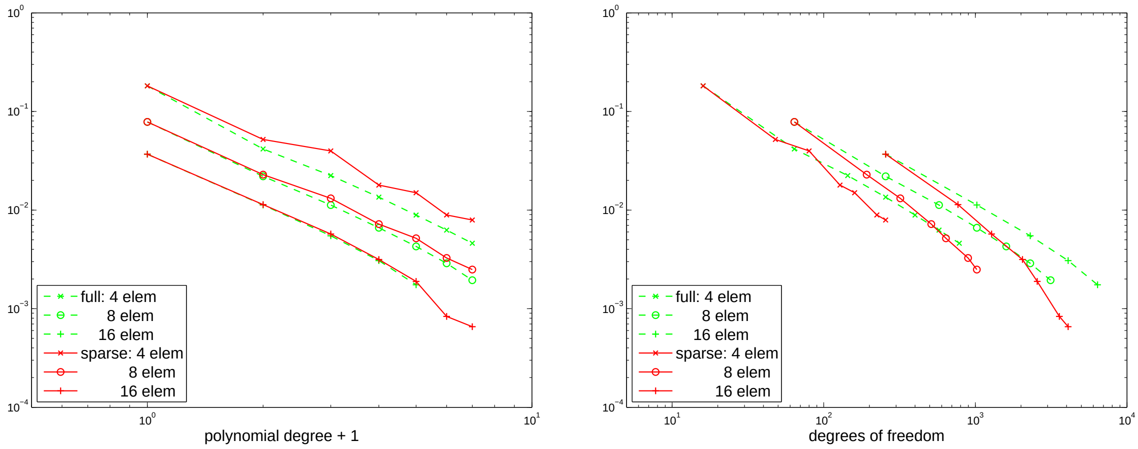

Figure 3: Relative error in the energy norm for $\mathrm{C}_g = \exp(\sin\frac{\varphi(\mathbf{x})}{2}\sin\frac{\varphi(\mathbf{y})}{2})$

8.3 Smooth solution

In the last example we examine the applicability limits of the p-sparse discretization and of the a priori error estimates (39). We take a Gaussian distribution as a right-hand side in (7)

$$\mathrm{C}_g = \exp\frac{-\|\mathbf{x}-\mathbf{y}\|^2}{2}.$$

For such a function Theorem 6.1 does not guarantee the advantage of the p-sparse discretization, and indeed, numerical experiments show (Fig. 4) that the convergence of the p-version of the sparse grid method is still exponential, but does not dominate the convergence rate of the full tensor product discretization. The extrapolated value of the energy norm is $\|\mathrm{C}_u\|_{\mathcal{V}\otimes\mathcal{V}} \approx 18.43665$.

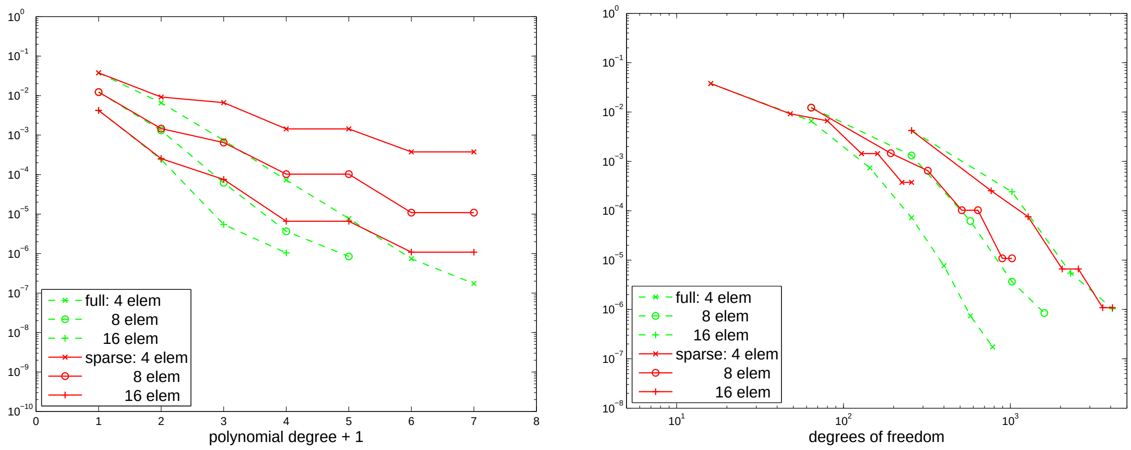

Figure 4: Relative error in the energy norm for $\mathrm{C}_g = \exp\frac{-\|\mathbf{x}-\mathbf{y}\|^2}{2}$

9 Appendix: Interpolation of tensor product Sobolev spaces

In this section we investigate interpolation properties of the tensor product Sobolev spaces $\mathbf{H}^{\mathbf{s}}(\Gamma^k)$, $\mathbf{s} \in \mathbb{R}^k$. The following standard lemma allows us to work with the weighted sequence spaces instead of the function spaces.

Lemma 9.1. *Let $k \in \mathbb{N}$, $\mathbf{s} \in \mathbb{R}^k$. Then the norm*

$$|||u|||_{\mathbf{H}^{\mathbf{s}}(\Gamma^k)} := \left(\sum_{\mathbf{n} \in \mathbb{Z}^k} \mathbf{g}_{\mathbf{n}}^{\mathbf{s}} |\hat{u}_{\mathbf{n}}|^2 \right)^{1/2}, \quad (\mathbf{g}_{\mathbf{n}})_i := 1 + n_i^2, \qquad \hat{u}_{\mathbf{n}} := |\Gamma|^{-k} \int_{\Gamma^k} u(\mathbf{x}) e^{-i\mathbf{n}\cdot\mathbf{x}}\, d\mathbf{x} \tag{45}$$

is an equivalent norm on $\mathbf{H}^{\mathbf{s}}(\Gamma^k)$.

In what follows we omit the domain Γ^k and write $\mathbf{H}^{\mathbf{s}} = \mathbf{H}^{\mathbf{s}}(\Gamma^k)$ for short. We use the standard notations and results of the classical interpolation theory. Let A_0, A_1 be two compatible Banach spaces, $u \in A_0 + A_1$ (cf. [23, Section 1.3]) and $t \in \mathbb{R}, t > 0$. The K-functional is defined by

$$K(t, u; (A_0, A_1)) := \inf_{u = u_0 + u_1} (\|u_0\|_{A_0} + t\|u_1\|_{A_1}). \tag{46}$$

Then the norm in the intermediate space $(A_0, A_1)_{\theta,2}$ for $\theta \in (0, 1)$ is given by

$$\|u\|_{(A_0,A_1)_{\theta,2}} := \left(\int_0^\infty \left(t^{-\theta} K(t, u; (A_0, A_1)) \right)^2 \frac{dt}{t} \right)^{1/2}. \tag{47}$$

Definition 9.1. *For arbitrary fixed $\mathbf{s}_0, \mathbf{s}_1 \in \mathbb{R}^k$ we define the interpolation couple $\overline{\mathbf{H}} := (\mathbf{H}^{\mathbf{s}_0}, \mathbf{H}^{\mathbf{s}_1})$ and for some $\theta \in (0, 1)$ the interpolation space $\overline{\mathbf{H}}_\theta := (\mathbf{H}^{\mathbf{s}_0}, \mathbf{H}^{\mathbf{s}_1})_{\theta,2}$.*

The aim of this section is to show that the spaces $\overline{\mathbf{H}}_\theta$ and $\mathbf{H}^{\mathbf{s}}$ have equivalent norms, shortly $\overline{\mathbf{H}}_\theta \sim \mathbf{H}^{\mathbf{s}}$, if $\mathbf{s} = (1 - \theta)\mathbf{s}_0 + \theta\mathbf{s}_1$.

It turns out that the discretized version of (47) is more convenient to work with.

Lemma 9.2. *[2, Lemma 3.1.3] Let $\bar{A} := (A_0, A_1)$, $u \in \bar{A}_{\theta,2}$. Then the norm*

$$\|u\|_\theta := \left(\sum_{\nu \in \mathbb{Z}} \left(2^{-\nu\theta} K(2^\nu, u; \bar{A}) \right)^2 \right)^{1/2}$$

is an equivalent norm on $\bar{A}_{\theta,2}$. Moreover, there holds

$$2^{-\theta} (\log 2)^{1/2} \|u\|_\theta \le \|u\|_{\bar{A}_{\theta,2}} \le 2(\log 2)^{1/2} \|u\|_\theta. \tag{48}$$

The following lemma gives an equivalent representation of the K-functional for the weighted sequence spaces $\mathbf{H}^{\mathbf{s}}$ (cf. [2, p. 122]).

Lemma 9.3. *Let $u \in \mathbf{H}^{\mathbf{s}_0} + \mathbf{H}^{\mathbf{s}_1}$, $\overline{\mathbf{H}} := (\mathbf{H}^{\mathbf{s}_0}, \mathbf{H}^{\mathbf{s}_1})$. Then*

$$K(t, u; \overline{\mathbf{H}}) \sim \left(\sum_{\mathbf{n} \in \mathbb{Z}^k} \min(\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0}, t^2 \mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1}) |u_{\mathbf{n}}|^2 \right)^{1/2} \tag{49}$$

with an absolute constant of equivalence.

Proof. Definition of the K-functional (46) yields the equivalence

$$\begin{aligned} K(t, u; \overline{\mathbf{H}}) &= \inf_{v \in \mathbf{H}^{\mathbf{s}_0}} \left\{ \left(\sum_{\mathbf{n} \in \mathbb{Z}^k} \mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0} |v_{\mathbf{n}}|^2 \right)^{1/2} + t \left(\sum_{\mathbf{k} \in \mathbb{Z}^k} \mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1} |u_{\mathbf{n}} - v_{\mathbf{n}}|^2 \right)^{1/2} \right\} \\ &\sim \inf_{v \in \mathbf{H}^{\mathbf{s}_0}} \left(\sum_{\mathbf{n} \in \mathbb{Z}^k} \left(\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0} |v_{\mathbf{n}}|^2 + t^2 \mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1} |u_{\mathbf{n}} - v_{\mathbf{n}}|^2 \right) \right)^{1/2}. \end{aligned} \tag{50}$$

with absolute constant of equivalence $\sqrt{2}$. This yields the assertion of the lemma, since

$$\inf_{w\in\mathbb{R}}\left(\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0}|w|^2 + t^2\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1}|u_{\mathbf{n}} - w|^2\right) = \min(\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0}, t^2\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1})|u_{\mathbf{n}}|^2$$

for arbitrary fixed $\mathbf{n} \in \mathbb{Z}^k$. □

The next lemma is based on the simple properties of geometrical series. It plays the key role in the proof of Theorem 9.1.

Lemma 9.4. *Let* $a, b > 1$, $\theta \in (0,1)$. *Then*

$$\frac{1}{a^\theta - 1} + \frac{1}{a^{1-\theta} - 1} \le \sum_{\nu\in\mathbb{Z}} (ba^\nu)^{-\theta} \min(1, ba^\nu) \le \frac{a^\theta}{a^\theta - 1} + \frac{a^{1-\theta}}{a^{1-\theta} - 1} \tag{51}$$

independently of b.

Proof. We prove the lemma by decomposing the sum in (51) into two parts.

$$\sum_{\nu\in\mathbb{Z}} (ba^\nu)^{-\theta}\min(1, ba^\nu) = \sum_{\nu\in\mathbb{Z},\,\nu\ge -\log_a b} (ba^\nu)^{-\theta} + \sum_{\nu\in\mathbb{Z},\,\nu< -\log_a b} (ba^\nu)^{1-\theta} \tag{52}$$

For $q \in (0,1), \mu \ge 0$ the simple properties of geometrical series provide

$$\sum_{\nu\in\mathbb{Z},\,\nu\ge -\mu} q^\nu = \frac{q^{-\lfloor\mu\rfloor}}{1-q}, \qquad \sum_{\nu\in\mathbb{Z},\,\nu>\mu} q^\nu = \frac{q^{\lfloor\mu\rfloor+1}}{1-q}, \tag{53}$$

where $\lfloor\mu\rfloor := \max\{n \in \mathbb{Z} : n \le \mu\}$. Since $\log_a b \ge 0$ for $a, b > 1$ we apply (53) and obtain

$$\sum_{\nu\in\mathbb{Z}} (ba^\nu)^{-\theta}\min(1, ba^\nu) = \frac{b^{-\theta}a^{\theta\lfloor\log_a b\rfloor}}{1-a^{-\theta}} + \frac{b^{1-\theta}a^{(\theta-1)(\lfloor\log_a b\rfloor+1)}}{1-a^{\theta-1}}.$$

Both summands admit the following upper and lower bounds

$$\frac{1}{a^\theta - 1} \le \frac{b^{-\theta}a^{\theta\lfloor\log_a b\rfloor}}{1-a^{-\theta}} \le \frac{a^\theta}{a^\theta - 1},$$

$$\frac{1}{a^{1-\theta} - 1} \le \frac{b^{1-\theta}a^{(\theta-1)(\lfloor\log_a b\rfloor+1)}}{1-a^{\theta-1}} \le \frac{a^{1-\theta}}{a^{1-\theta} - 1}$$

independently of b, which finishes the proof. □

Theorem 9.1. *The spaces* $\overline{\mathbf{H}}_\theta$ *and* $\mathbf{H}^{\mathbf{s}}$ *are isomorphic and have equivalent norms.*

Proof. Applying consequently Lemma 9.2, Lemma 9.3 and changing the order of summation we obtain

$$\begin{aligned}
\|u\|^2_{\overline{\mathbf{H}}_\theta} &\overset{(48)}{\sim} \sum_{\nu\in\mathbb{Z}} \left(2^{-\nu\theta}K(2^\nu, u; \overline{\mathbf{H}})\right)^2 \\
&\overset{(49)}{\sim} \sum_{\nu\in\mathbb{Z}} 2^{-2\nu\theta} \sum_{\mathbf{n}\in\mathbb{Z}^k} \min(\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0}, 2^{2\nu}\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1})|u_{\mathbf{n}}|^2 \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^k} \mathbf{g}_{\mathbf{n}}^{\mathbf{s}}|u_{\mathbf{n}}|^2 \sum_{\nu\in\mathbb{Z}} 2^{-2\nu\theta}\mathbf{g}_{\mathbf{n}}^{-\mathbf{s}} \min(\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_0}, 2^{2\nu}\mathbf{g}_{\mathbf{n}}^{\mathbf{s}_1}).
\end{aligned}$$

Recalling $\mathbf{s} = (1-\theta)\mathbf{s}_0 + \theta\mathbf{s}_1$ and Lemma 9.4 we obtain for the inner sum

$$\sum_{\nu\in\mathbb{Z}} 2^{-2\nu\theta}\mathbf{g}_{\mathbf{k}}^{-\mathbf{s}}\min(\mathbf{g}_{\mathbf{k}}^{\mathbf{s}_0}, 2^{2\nu}\mathbf{g}_{\mathbf{k}}^{\mathbf{s}_1}) = \sum_{\nu\in\mathbb{Z}} (2^{2\nu}\mathbf{g}_{\mathbf{k}}^{\mathbf{s}_1-\mathbf{s}_0})^{-\theta}\min(1, 2^{2\nu}\mathbf{g}_{\mathbf{k}}^{\mathbf{s}_1-\mathbf{s}_0}) \sim 1$$

independently of $\mathbf{k}$. Finally, Lemma 9.1 yields the asserted norm equivalence.

□

References

[1] J.-P. Aubin, *Applied functional analysis*, Pure and Applied Mathematics (New York), Wiley-Interscience, New York, second ed., 2000.

[2] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976. Grundlehren der Mathematischen Wissenschaften, No. 223.

[3] A. Buffa and R. Hiptmair, *Coercive combined field integral equations*, Tech. Rep. 2003-08, SAM, ETH Zürich, Switzerland, 2003.

[4] A. Buffa and S. Sauter, *On the acoustic single layer potential: stabilization and Fourier analysis*, SIAM J. Sci. Comput., 28 (2006), pp. 1974–1999 (electronic).

[5] H.-J. Bungartz and M. Griebel, *Sparse grids*, Acta Numer., 13 (2004), pp. 147–269.

[6] W. Feller, *An introduction to probability theory and its applications. Vol. I*, Third edition, John Wiley & Sons Inc., New York, 1968.

[7] P. Frauenfelder, C. Schwab, and R. A. Todor, *Finite elements for elliptic problems with stochastic coefficients*, Comput. Methods Appl. Mech. Engrg., 194 (2005), pp. 205–228.

[8] M. Griebel, P. Oswald, and T. Schiekofer, *Sparse grids for boundary integral equations*, Numer. Math., 83 (1999), pp. 279–312.

[9] H. Harbrecht, R. Schneider, and C. Schwab, *Sparse second moment analysis for elliptic problems in stochastic domains*, Numer. Math., (2008), to appear.

[10] G. C. Hsiao and W. L. Wendland, *A finite element method for some integral equations of the first kind*, J. Math. Anal. Appl., 58 (1977), pp. 449–481.

[11] F. Kupka, *Sparse grid spectral methods for the numerical solution of partial differential equations with periodic boundary conditions*, PhD thesis, Universität Wien, 1997.

[12] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972. Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[13] J.-C. Nédélec, *Acoustic and electromagnetic equations*, vol. 144 of Applied Mathematical Sciences, Springer-Verlag, New York, 2001.

[14] P.-A. Nitsche, *Best N term approximation spaces for tensor product wavelet bases*, Constr. Approx., 24 (2006), pp. 49–70.

[15] S. Sauter and C. Schwab, *Randelementmethoden*, B. G. Teubner, 2004.

[16] C. Schwab, *p- and hp-finite element methods*, Numerical Mathematics and Scientific Computation, The Clarendon Press Oxford University Press, New York, 1998.

[17] C. Schwab and R.-A. Todor, *Sparse finite elements for elliptic problems with stochastic loading*, Numer. Math., 95 (2003), pp. 707–734.

[18] C. Schwab and R. A. Todor, *Sparse finite elements for stochastic elliptic problems—higher order moments*, Computing, 71 (2003), pp. 43–63.

[19] C. Schwab and R. A. Todor, *Karhunen-Loève approximation of random fields by generalized fast multipole methods*, J. Comput. Phys., 217 (2006), pp. 100–122.

[20] E. P. Stephan and M. Suri, *On the convergence of the p-version of the boundary element Galerkin method*, Math. Comp., 52 (1989), pp. 31–48.

[21] V. N. Temlyakov, *Approximation of functions with bounded mixed derivative*, Proc. Steklov Inst. Math., (1989), pp. vi+121. A translation of Trudy Mat. Inst. Steklov 178 (1986)

[22] R. A. Todor, *Sparse perturbation algorithms for elliptic PDE's with stochastic data*, PhD thesis, ETH Zürich, 2005.

[23] H. Triebel, *Interpolation theory, function spaces, differential operators*, Johann Ambrosius Barth, Heidelberg, second ed., 1995.

[24] T. von Petersdorff and C. Schwab, *Sparse finite element methods for operator equations with stochastic data*, Appl. Math., 51 (2006), pp. 145–180.

[25] K. Yosida, *Functional analysis*, Die Grundlehren der Mathematischen Wissenschaften, Band 123, Academic Press Inc., New York, 1965.