Convergence Rates for Sparse Chaos Approximations of Elliptic Problems with Stochastic Coefficients[1]

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Abstract

A scalar, elliptic boundary value problem in divergence form with stochastic diffusion coefficient $a(x, \omega)$ in a bounded domain $D \subset \mathbb{R}^d$ is reformulated as a deterministic, infinite-dimensional, parametric problem by separation of deterministic ($x \in D$) and stochastic ($\omega \in \Omega$) variables in $a(x, \omega)$ via e.g. Karhunen-Loève or Legendre polynomial chaos expansion in the sense of N. Wiener [Wie38].

Deterministic, approximate solvers are based on projection of this problem into a product probability space of finite dimension M and sparse discretizations of the resulting M dimensional parametric problem.

Under regularity assumptions on the fluctuation of $a(x, \omega)$ in the deterministic variable x, the convergence rate of the deterministic solution algorithm is analyzed in terms of the chaos dimension M and of the number N of deterministic problems to be solved as both, dimension M and the multiresolution level of the sparse discretization resp. the degree of the polynomial chaos expansion increase simultaneously.

Based on analytic regularity estimates of the solution to the truncated parametric deterministic problem, new sparse FE spaces for the discretization in the parametric variable are proposed. Optimal convergence rates of the semi-discrete solution to the stochastic problem, *in terms of the number N of deterministic problems to be solved*, are proved for these spaces when the dimension M of the parameter space increases simultaneously with the multiresolution level in the sparse approximation resp. the spectral order in the polynomial chaos approximation.

Keywords: PDE's with Stochastic Data, Karhunen-Loève Expansion, Polynomial Chaos, Sparse Tensor Product Approximation

Subject Classification: 65N30

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1 Introduction

The numerical solution of elliptic partial differential equations with stochastic input data by deterministic methods has been employed in engineering for several decades now (see e.g. [GS97] and the references therein). We distinguish two broad classes of approaches to the deterministic numerical solution of elliptic stochastic partial differential equations – the perturbation approach and the spectral approach.

The perturbation approach is widely used in engineering applications (see [KH92] and references therein). There are several variants, of which the *first order second moment (FOSM)* (see e.g. [DW81]) technique became very popular, all based on *Neumann expansion* of the stochastic solution around its mean field (see [Kel64]), and succesive computation of (in general only) low order terms in this expansion.

The spectral approach is based on the *Wiener/generalized polynomial chaos (W/gPC)* expansion (see [Wie38], [Sch00], [XK02]) of the input random fields and of the random solution combined with either Galerkin projection or collocation in the stochastic variables of the input data. The numerical analysis of this approach has been started rather recently, see e.g. [BTZ04] and the references there. There, *exponential convergence rates* have been proved with respect to the spectral order of the stochastic discretization, *at fixed dimension M of the stochastic parametrization.* Since, however, the number M of stochastic variables in Wiener's gPC parametrization of random fields is unbounded, gPC type formulations approximate stochasticity of the random data and of the random solution by a deterministic problem in a finite number M of these 'stochastic' variables. It is essential here that the dimension M of the stochastic variables is a discretization parameter for the input and output random fields and can therefore be arbitrarily large.

Hence, *exponential convergence with respect to the polynomial degree* of tensor product type polynomial approximations in these M variables as shown, e.g., in [BTZ04] does *not*, for large M, imply low computational complexity of the spectral approach.

More precisely, if tensor product polynomial discretizations are used in the stochastic variables, to ensure consistency the number of stochastic degrees of freedom (and, hence, the number of deterministic BVPs to be solved) must increase at least exponentially with simultaneously increasing polynomial degree and stochastic dimension, even with adaptive and anisotropic polynomial degree selection. As a consequence, exponential convergence *in terms of the number of stochastic degrees of freedom and, hence, in the number of deterministic problems to be solved* is lost in space dimension larger than 1, and only subalgebraic convergence rate can be shown (see [FST05]).

In the present paper, we present new regularity estimates and sparse approximation error bounds for PDEs with stochastic data. As in other recent works (e.g. [BTZ04]), we consider as a model problem a diffusion process in a random medium occupying a bounded domain $D \subset \mathbb{R}^d$ with Lipschitz boundary $\Gamma = \partial D$. For brevity of exposition, we focus here only on the error analysis of semidiscretization in the stochastic variable.

The uncertainty in the diffusion coefficient a is modelled through the dependence on a stochastic parameter $\omega \in \Omega$, where (Ω, Σ, P) is a probability space.

Assumption 1.1 *We consider $a \in L^\infty(D \times \Omega)$ to be strictly positive, with lower and upper bounds a_- and a_+ respectively,*

$$a_- \leq a(x,\omega) \leq a_+ \quad \lambda \times P\text{-a.e. } (x,\omega) \in D \times \Omega. \tag{1.1}$$

The stochastic diffusion problem then reads,

$$\left\{ \begin{array}{rcll} -\text{div}(a(\cdot,\omega)\nabla u(\cdot,\omega)) & = & f(\cdot,\omega) & \text{in } D \\ u(\cdot,\omega) & = & 0 & \text{on } \partial D \end{array} \right. \quad P\text{-a.e. } \omega \in \Omega. \tag{1.2}$$

The coefficient $a(x,\omega)$ as well as the solution $u(x,\omega)$ are *random fields* in $D \subset \mathbb{R}^d$, i.e. jointly measurable functions from $D \times \Omega$ to $\mathbb{R}$. Whereas the random field $u(x,\omega)$ is a mathematically well-defined object (see Theorem 1.3 below), the task 'compute $u(x,\omega)$' is less obvious to realize numerically and of limited interest in practice. In applications only certain statistics and moments of $u(x,\omega)$ are of interest, and this is also our goal of computation, which we formulate as follows.

Problem 1.2 *Given statistics (w.r.t. $\omega \in \Omega$) of the stochastic data a, compute statistics of the random solution u to* (1.2), *like mean field,*

$$E_u : D \to \mathbb{R}, \quad E_u(x) := \int_\Omega u(x,\omega)\, dP(\omega) \quad x \in D,$$

2-point correlation (or higher order moments),

$$C_u : D \times D \to \mathbb{R}, \quad C_u(x,x') := \int_\Omega u(x,\omega)u(x',\omega)\, dP(\omega) \quad (x,x') \in D \times D,$$

or probabilistic level sets,

$$D_\varepsilon^\alpha := \{x \in D : P(|u(x,\cdot)| > \alpha) < \varepsilon\}.$$

As mentioned above, good performance of the perturbation approach has been demonstrated in practice (at least for small fluctuations, when the perturbation series could be truncated after the first order terms). The computation of higher order terms in the perturbation series (needed in the case of relatively large fluctuations) involves numerical approximation of higher order moments of the random solution. Using standard discretizations, this results in a loss of linear complexity[1].

Using sparse approximation of the higher order moments of the data, perturbation algorithms of linear complexity have been developed recently (see [Tod05b]). The results in the present work can be viewed as spectral counterparts of those in [Tod05b] on the convergence of the perturbation approach.

[1]Here and throughout the paper, linear complexity is understood as log-linear with respect to N, the number of degrees of freedom for a FE discretization of one deterministic version of the stochastic boundary value problem.

The parametrization of uncertainty is one of the key points in the numerical treatment of problems with stochastic data. A Karhunen-Loève expansion separating the deterministic and stochastic variables optimally in the mean-square sense (see e.g. [Loè77], [Loè78]) is a standard procedure to transform the original stochastic problem into a parametric deterministic one. The resulting parametrization belongs to a hypercube of dimension M *which is itself a discretization parameter.*

The parametric problem is then solved using e.g. a stochastic Galerkin (sG) method (variationally in both the parameter and the physical variable - note the need for numerical integration schemes in high dimensional domains) or by collocation and interpolation in the parametric variable. Backward substitution finally gives an approximation to the original stochastic problem and postprocessing is required to obtain statistical information on the random solution. Just as in the case of a MC simulation, detailed information on the joint probability densities of the input data is in general needed. Tensor product discretization/collocation grids in the parametric variable result however in *superalgebraic complexity* rates (see e.g. [BTZ04], [FST05], [MK05]). This is due to the unfavourable scaling of the required computational effort with the parameter dimension M.

The main result of this paper (Theorem 4.17) is an *explicit construction* of FE spaces in the parametric (i.e. stochastic) variable, which are *not of tensor product type*, and for which optimal convergence rates for the corresponding approximations of (1.2) hold (algebraic order $p+1$ for the h-FEM based construction, where p denotes the fixed polynomial degree, and superalgebraic for the p-FEM based polynomial chaos). Note that the rates are expressed *in terms of the number N of deterministic problems to be solved and are independent of the dimension M of stochastic variable* (see e.g. Theorem 4.17). We emphasize that our result gives in particular a concrete, explicit selection of basis functions in the chaos expansion.

We conclude this introductory part by noting that the problem (1.2) is well-posed. This follows trivially from (1.1) and the well-posedness of the deterministic diffusion problem (see also e.g. [Tod05b]).

Theorem 1.3 *If Assumption 1.1 holds and $p \geq 0$, then for any $f \in L^p(\Omega, H^{-1}(D))$, there exists a unique $u \in L^p(\Omega, H_0^1(D))$ solution to* (1.2) *(here $p = 0$ corresponds to measurability). Moreover, for $p \geq 1$ it holds*

$$\|u\|_{L^p(\Omega, H_0^1(D))} \leq c_a \|f\|_{L^p(\Omega, H^{-1}(D))}.$$

2 Separation of Deterministic and Stochastic Variables

To reduce (1.2) to a high-dimensional deterministic problem, we separate the deterministic and stochastic variables in the coeffcient $a(x, \omega)$ using an expansion in a deterministic basis, with random coefficients. Several choices are possible here, of which we mention and discuss the Legendre and the Karhunen-Loève (KL) expansion. We consider a splitting of the diffusion coefficient into a *deterministic expectation* e and a *random fluctuation* r and extend the positivity Assumption 1.1.

Assumption 2.1 *The random field* $a \in L^\infty(D \times \Omega)$ *satisfying* (1.1) *can be represented as*

$$a(x, \omega) = e(x) + r(x, \omega) \quad \forall (x, \omega) \in D \times \Omega \tag{2.1}$$

with a positive $e \in L^\infty(D)$ *(not necessarily equal to the mean field* E_a*),*

$$0 < e_- \leq e(x) \leq e_+ < \infty \quad \forall x \in D. \tag{2.2}$$

It follows from (2.2) that $r \in L^\infty(D \times \Omega)$ too, and we require that the fluctuation be pointwise smaller than the expectation.

Assumption 2.2 *For the representation* (2.1) *it holds*

$$0 \leq \sigma := \text{esssup}_{x \in D} \frac{\|r(x, \cdot)\|_{L^\infty(\Omega)}}{e(x)} < 1. \tag{2.3}$$

Remark 2.3 *The constant expectation choice*

$$e(x) := (a_- + a_+)/2 \quad \forall x \in D$$

satisfies Assumption 2.2 with $\sigma \leq (a_+ - a_-)/(a_+ + a_-) < 1$.
The more natural (from a statistical point of view) choice $e = E_a$ *satisfies* (2.3) *if the density function of* $r(x, \cdot)$ *is even for any* $x \in D$*, that is, if positive and negative fluctuations occur with equal probabilities.*

Concerning the fluctuation term r we formulate also a modelling assumption as well as a condition of regularity in the physical variable.

Assumption 2.4 *The random fluctuation* r *can be represented in* $L^\infty(D \times \Omega)$ *as a convergent series*

$$r = \sum_{m=1}^{\infty} \psi_m \otimes X_m \tag{2.4}$$

with known deterministic $\psi_m \in L^\infty(D)$ *and stochastic* $X_m \in L^\infty(\Omega)$*. W.l.o.g. we also assume that* $\psi_m, X_m \neq 0$ *for all* $m \in \mathbb{N}_+$*.*

The representation formula (2.4) describes the tensor product nature of the random field r, and achieves the separation of the deterministic and stochastic variables, $x \in D$ and $\omega \in \Omega$ respectively. Note also that we require uniform convergence of (2.4) in order to allow control of the error in the solution to (1.2) via Strang Lemma, after truncation of (2.4). The regularity of the random field r is quantified by the convergence rate of the series (2.4).

Assumption 2.5 *The random fluctuation* r *admits a representation* (2.4) *for which there exist constants* $c_r, c_{1,r}, \kappa > 0$ *such that*

$$\|\psi_m \otimes X_m\|_{L^\infty(\overline{D} \times \Omega)} \leq c_r \exp(-c_{1,r} m^\kappa) \quad \forall m \in \mathbb{N}_+. \tag{2.5}$$

In the following two sections Assumption 2.5 will be shown to hold with $\kappa = 1/d$ if the fluctuation r is piecewise analytic in the physical variable $x \in D \subset \mathbb{R}^d$. Two examples of separating expansions (2.4) will be presented and discussed in detail, the Legendre and the Karhunen-Loève expansion respectively.

We further assume that complete probabilistic information on the stochastic part of the separating expansion (2.4) is available, as follows.

Assumption 2.6 *The joint probability density functions of the family $\mathcal{X} := (X_m)_{m\in\mathbb{N}_+}$ are known.*

In fact, this will be only needed later for the postprocessing of the chaos solution to our stochastic problem (1.2).

2.1 Legendre Expansion

The validity of Assumption 2.5 with $\kappa = 1/d$ and the Legendre expansion as (2.4) follows from standard approximation theory of analytic functions (see e.g. [Dav63]), if the random fluctuation r is piecewise analytic in the physical variable, with values in $L^\infty(\Omega)$ ($r \in \mathcal{A}_{\mathrm{pw}}(\overline{D}, L^\infty(\Omega))$).

Example 2.7 *If $D \subset [-1,1]^d$ and $r \in \mathcal{A}([-1,1]^d, L^\infty(\Omega))$, then a representation (2.4) exists with $(\psi_m)_{m\in\mathbb{N}_+}$ the Legendre polynomials in $[-1,1]^d$ (tensor products of standard Legendre polynomials in $[-1,1]$, scaled to have L^2-norm equal to 1) and*

$$X_m(\omega) := \int_{[-1,1]^d} r(x,\omega)\psi_m(x)\,dx \quad P\text{-a.e. } \omega \in \Omega, \forall m \in \mathbb{N}_+.$$

Moreover, Assumption 2.5 holds with $\kappa := 1/d$ and $c_{1,r}$ depending on the size of the analyticity domain of r in a complex neighbourhood of $[-1,1]^d$.

2.2 Karhunen-Loève Expansion

An alternative to the Legendre expansion is the Karhunen-Loève series, which is known to be the $L^2(D \times \Omega)$ optimal representation satisfying the separation ansatz (2.4) (see also [ST05]). For analytic fluctuations r, the convergence rate of the Karhunen-Loève series is also exponential, that is, qualitatively similar to that of the Legendre expansion. However, determining it requires an additional eigenpair computation for the compact integral operator $\mathcal{C}_r$ with kernel C_r given by the two-point correlation of r,

$$C_r : D \times D \to \mathbb{R}, \quad C_r(x,x') := \int_\Omega r(x,\omega)r(x',\omega)\,dP(\omega) \quad (x,x') \in D \times D$$

We start by noting that $\mathcal{C}_r : L^2(D) \to L^2(D)$ given by

$$(\mathcal{C}_r u)(x) := \int_D C_r(x,x')u(x')dx' \quad \lambda\text{-a.e. } x \in D,\ \forall u \in L^2(D) \tag{2.6}$$

is a symmetric, nonnegative definite and compact integral operator. It possesses therefore a countable sequence $(\lambda_m, \phi_m)_{m\in\mathbb{N}_+}$ of eigenpairs with

$$\mathbb{R} \ni \lambda_m \searrow 0, \qquad \text{as } m \nearrow \infty.,$$

where the KL eigenvalues are enumerated in decreasing order of magnitude, with multiplicity counted. It then holds (see [Loè77]),

Theorem 2.8 *Under Assumption 2.1, there exists a sequence $\mathcal{X} := (X_m)_{m\in\mathbb{N}_+}$ of uncorrelated (and centered at 0 if $e = E_a$) random variables,*

$$\int_\Omega X_n(\omega)X_m(\omega)\,dP(\omega) = \delta_{nm} \quad \forall n, m \in \mathbb{N}_+, \tag{2.7}$$

such that the random field r can be expanded in $L^2(D \times \Omega)$ as

$$r(x,\omega) = a(x,\omega) - e(x) = \sum_{m\in\mathbb{N}_+} \sqrt{\lambda_m}\phi_m(x)X_m(\omega). \tag{2.8}$$

Note that the $L^2(D \times \Omega)$ convergence of the KL expansion is due to the trace-class condition

$$\sum_{m=1}^{\infty} \lambda_m = \mathrm{Tr}(\mathcal{C}_r) = \int_D \int_\Omega r(x,\omega)^2 < \infty. \tag{2.9}$$

Remark 2.9 *The convergence rate of the KL series in $L^2(D \times \Omega)$ is equal to the one of the eigenvalue sum in* (2.9).

Note that the $L^2(D\times\Omega)$ convergence of the Karhunen-Loève expansion (2.8) is not strong enough to allow control of the error in the solution of (1.2) via Strang Lemma, after truncation of (2.8). However, analytic regularity of r in the physical variable plus uniform boundedness of the family $\mathcal{X} = (X_m)_{m\in\mathbb{N}_+} \subset L^\infty(\Omega)$ will be next shown to ensure the uniform convergence of the Karhunen-Lòeve expansion (2.8).

Assumption 2.10 *The family $X = (X_m)_{m\in\mathbb{N}_+}$ of random variables is uniformly bounded in $L^\infty(\Omega)$, i.e. there exists $c_\mathcal{X} > 0$ such that*

$$\|X_m\|_{L^\infty(\Omega,dP)} \le c_\mathcal{X} \in \mathbb{R} \quad \forall m \in \mathbb{N}_+. \tag{2.10}$$

Eigenvalue and eigenfunction decay estimates derived in Propositions 2.13 and 2.16 of the following two sections immediately imply the desired strong convergence result.

Proposition 2.11 *If $D \subset [-1,1]^d$, $r \in \mathcal{A}([-1,1]^d, L^\infty(\Omega))$ with the associated Karhunen-Loève expansion given by* (2.8)*, and Assumption 2.10 holds, then Assumption 2.5 holds too, with $\kappa := 1/d$ and $\psi_m := \sqrt{\lambda_m}\phi_m$ for all $m \in \mathbb{N}_+$.*

2.2.1 Eigenvalue Decay

We state next decay rates for the KL eigenvalues in terms of regularity of the correlation kernel C_r. The results we present in this section are standard (see e.g. [Kön86], [Pie87]), following from the abstract theory of Weyl/approximation/entropy numbers via approximation of K by discrete, finite rank (separable w.r.t. (x, x')) kernels. Roughly speaking, the smoother the kernel the faster the eigenvalue decay, with finite Sobolev regularity implying algebraic decay and analyticity giving rise to quasi-exponential decay.

All these results hold for piecewise regular kernels on product subdomains of D, in the sense of Definition 2.12 below. Note that general piecewise regularity allowing singularities on the diagonal set of $D \times D$ ensure in general only a slower eigenvalue decay (see e.g. [Kön86] and [GS97] for examples with known exact eigenelements). We focus on the case of an analytic correlation kernel C_r, and refer the reader to [Tod05a] for a discussion of less regular kernels.

Definition 2.12 *If D is a bounded domain in $\mathbb{R}^d$ a measurable function $C_r : D \times D \to \mathbb{R}$ is said to be* **piecewise analytic on** $D \times D$ *if there exists a finite family $\mathcal{D} = (D_j)_{j\in\mathcal{J}}$ of subdomains of D such that*

i. $D_j \cap D_{j'} = \emptyset \quad \forall j, j' \in \mathcal{J}$ *with* $j \neq j'$

ii. $D \setminus \bigcup_{j\in\mathcal{J}} D_j$ *is a null set in* $\mathbb{R}^d$

iii. $\overline{D} \subset \bigcup_{j\in\mathcal{J}} \overline{D_j}$

iv. $C_r|_{D_j \times D_{j'}}$ *is analytic on* $\overline{D_j} \times \overline{D_{j'}} \quad \forall j, j' \in \mathcal{J}$.

We denote by $\mathcal{A}_{\mathcal{D}}(D^2)$ the space of piecewise analytic functions on $D \times D$ in the sense given above.
Moreover, if there exists also a finite family $\mathcal{G} = (G_j)_{j\in\mathcal{J}}$ of open sets in $\mathbb{R}^d$ such that

v. $\overline{D_j} \subset G_j \quad \forall j \in \mathcal{J}$

vi. $C_r|_{D_j \times D_{j'}}$ *has an analytic continuation to* $G_j \times G_{j'} \quad \forall j, j' \in \mathcal{J}$,

then we say that K is **piecewise analytic on a covering of** $D \times D$ *and we denote by $\mathcal{A}_{\mathcal{D},\mathcal{G}}(D^2)$ the corresponding space.*
Similarly we introduce spaces of piecewise analytic functions defined on D, which we denote by $\mathcal{A}_{\mathcal{D}}(D)$, $\mathcal{A}_{\mathcal{D},\mathcal{G}}(D)$ etc.

Proposition 2.13 *If $C_r \in \mathcal{A}_{\mathcal{D},\mathcal{G}}(D^2)$ and $(\lambda_m)_{m\in\mathbb{N}_+}$ is the eigenvalue sequence of its associated integral operator (2.6), then there exist constants $c_1, c_2 > 0$ such that*

$$0 \leq \lambda_m \leq c_1 \exp(-c_2 m^{1/d}) \quad \forall m \in \mathbb{N}_+. \tag{2.11}$$

Example 2.14 *One is often interested in Gaussian kernels of the form*

$$C_r(x,x') := \sigma^2 \exp(-|x-x'|^2/(\gamma^2\Lambda^2)) \quad \forall (x,x') \in D \times D, \tag{2.12}$$

where $\sigma, \gamma > 0$ *are real parameters (standard deviation and correlation length, respectively) and* Λ *is the diameter of the domain* D*.* C_r *given by* (2.12) *has an entire continuation to* $\mathbb{C}^d$ *and defines a nonnegative compact operator via* (2.6)*.*

Since C_r given by (2.12) admits an analytic continuation to the whole complex space $\mathbb{C}^d$, the eigenvalue decay is in this case even faster than in (2.11).

Proposition 2.15 *If* C_r *is given by* (2.12)*, then for the eigenvalue sequence* $(\lambda_m)_{m\in\mathbb{N}_+}$ *of the corresponding integral operator* $\mathcal{C}_r$ *defined by* (2.6) *it holds*

$$0 \le \lambda_m \le c_{\sigma,\gamma} \frac{(1/\gamma)^{m^{1/d}}}{\Gamma(m^{1/d}/2)} \quad \forall m \in \mathbb{N}_+. \tag{2.13}$$

Note that the decay estimate (2.13) is subexponential in dimension $d > 1$, and this is essentially due to the higher multiplicity of the eigenvalues in dimension larger than 1 (this can be explicity seen e.g. for the separable kernel (2.12) on a product domain D).

2.2.2 Eigenfunction Estimates

The smoothness assumption on the correlation kernel C_r allows also a good control of the eigenfunctions in terms of corresponding eigenvalues via the Gagliardo-Nirenberg inequalities. For a proof of the following result we refer the reader to [Tod05a], [ST05].

Proposition 2.16 *Let* $C_r \in L^2(D \times D)$ *be piecewise analytic on* $D \times D$*, such that all subdomains* D_j *in Definition 2.12 satisfy the uniform cone property. Denote by* $(\lambda_m, \phi_m)_{m\in\mathbb{N}_+}$ *the sequence of eigenpairs of the associated integral operator via* (2.6)*, such that* $\|\phi_m\|_{L^2(D)} = 1$ *for all* $m \in \mathbb{N}_+$*. Then for any* $s > 0$ *and any multiindex* $\alpha \in \mathbb{N}^d$ *there exists* $c_{r,\alpha,s} > 0$ *such that*

$$\|\partial^\alpha \phi_m\|_{L^\infty(D)} \le c_{r,\alpha,s} \, |\lambda_m|^{-s} \quad \forall m \in \mathbb{N}_+. \tag{2.14}$$

3 Uncertainty Parametrization

Throughout this section we suppose that the separating expansion (2.4) of the random fluctuation r satisfies the decay Assumption 2.5. As shown before, this is the case if r is piecewise analytic in the physical variable $x \in D$ ($\kappa = 1/d$ then).

3.1 Truncation of Fluctuation Expansion

Since computations can handle only finite data sets, we truncate the fluctuation expansion (2.4) and introduce for any $M \in \mathbb{N}$ the truncated stochastic diffusion coefficient

$$a_M(x,\omega) = e(x) + \sum_{m=1}^{M} \psi_m(x) X_m(\omega), \tag{3.1}$$

for which the following pointwise error estimate holds due to Assumption 2.5.

Proposition 3.1 *If Assumption 2.5 holds, then*

$$\|a - a_M\|_{L^\infty(D\times\Omega)} \leq c_r \exp(-c_{1,r} M^\kappa) \quad \forall M \in \mathbb{N}. \tag{3.2}$$

The diffusion problem with truncated coefficient a_M is therefore well-posed for M large enough (depending on a). This follows immediately from the Strang Lemma, which allows also explicit control of the error in the solution u to (1.2).

Corollary 3.2 *If the stochastic diffusion coefficient a satisfies Assumptions 2.1 and 2.5, then there exists a truncation order $M_{a,r} \in \mathbb{N}$ of the expansion (2.4) such that* (3.4) *below is well-posed in $L^\infty(\Omega, H_0^1(D))$ for any $M \geq M_{a,r}$. Moreover, if u and u_M are the unique solutions in $L^\infty(\Omega, H_0^1(D))$ of*

$$-\mathrm{div}(a(\cdot,\omega)\nabla u(\cdot,\omega)) = f(\cdot) \quad \text{in } H^{-1}(D), \quad P\text{-a.e. in } \Omega \tag{3.3}$$

and

$$-\mathrm{div}(a_M(\cdot,\omega)\nabla u_M(\cdot,\omega)) = f(\cdot) \quad \text{in } H^{-1}(D), \quad P\text{-a.e. in } \Omega \tag{3.4}$$

respectively, then

$$\|u - u_M\|_{L^\infty(\Omega,H_0^1(D))} \leq c_{a,r} \exp(-c_{1,r} M^\kappa) \cdot \|u\|_{L^\infty(\Omega,H_0^1(D))} \tag{3.5}$$

for all $M \geq M_{a,r}$.

Remark 3.3 *If the expectation e is chosen to be equal to the meanfield E_a and the family $\mathcal{X} = (X_m)_{m\in\mathbb{N}_+}$ is assumed to be independent, then* (3.4) *is well-posed for any $M \geq 0$, that is, $M_{a,r}$ can be chosen equal to 0 in Corollary 3.2. The possible loss of ellipticity in (3.4) - due to Gibbs' effect - is therefore not possible in the presence of an independent family $\mathcal{X} = (X_m)_{m\in\mathbb{N}_+}$, even in the case of slow, non-uniform convergence of the separating expansion* (2.4). *The typical example here is the Karhunen-Loève expansion of a fluctuation r with low regularity of its two-point correlation C_r, which exhibits only slow convergence in $L^2(D\times\Omega)$.*

Under the assumptions in Remark 3.3, the well-posedness of (3.4) can be seen for instance for the Karhunen-Loève expansion as follows. For any $N \in \mathbb{N}_+$ denote by $\Sigma_N \subset \Sigma$ the σ-algebra generated by the random variables $X_1, X_2, \ldots, X_N$. For any $M > N$, from (3.1) it follows (conditional expectations)

$$\mathbb{E}[a_M \mid \Sigma_N] = a_N, \tag{3.6}$$

since $(X_m)_{m\in\mathbb{N}_+}$ are assumed to be independent and, by construction of the KL expansion, centered at 0.

For any $\Omega_N \in \Sigma_N$, we use (3.6) and the defining property of the conditional expectation to write

$$\begin{aligned}\int_D \left(\int_{\Omega_N} (a_N - a)\, dP(\omega)\right)^2 dx &= \int_D \left(\int_{\Omega_N} (a_M - a)\, dP(\omega)\right)^2 dx \\ &\leq \int_D \int_{\Omega_N} (a_M - a)^2\, dP(\omega) dx \overset{M\to\infty}{\to} 0, \end{aligned} \tag{3.7}$$

due to $a_M \to a$ in $L^2(D \times \Omega)$ as $M \nearrow \infty$. Since $\Omega_N \in \Sigma_N$ was arbitrary, we conclude from (3.7) that

$$a_N = \mathbb{E}[a \mid \Sigma_N].$$

The positivity of the conditional expectation ensures then that the lower and upper bounds of a hold for a_N too.

3.2 Parametric Deterministic Problem

In this section we connect the equation (3.4) obtained by truncation at level $M \in \mathbb{N}$ of the separating expansion (2.4) of the random fluctuation r to an auxiliary, purely deterministic parametric problem. Without loss of generality, we suppose in the following that for $(X_m)_{m\in\mathbb{N}_+}$ in (2.4) it holds (this can be achieved by a rescaling of ψ_m and X_m),

$$\|X_m\|_{L^\infty(\Omega)} = 1/2 \quad \forall m \in \mathbb{N}_+, \tag{3.8}$$

so that

$$\text{Ran } X_m \subseteq I := [-1/2, 1/2] \quad \forall m \in \mathbb{N}_+.$$

To a_M we associate the function $\tilde{a}_M : D \times I^M \to \mathbb{R}$, defined by

$$\tilde{a}_M(x, y_1, y_2, \dots, y_M) := e(x) + \sum_{m=1}^{M} \psi_m(x) y_m \tag{3.9}$$

for all $y = (y_1, y_2, \dots, y_M) \in I^M$ and $x \in D$.

We now consider the purely deterministic, parametric elliptic problem of finding $\tilde{u}_M : I^M \to H_0^1(D)$ such that

$$-\text{div}(\tilde{a}_M(\cdot, y)\nabla \tilde{u}_M(\cdot, y)) = f(\cdot) \quad \text{in } H^{-1}(D) \quad \forall y \in I^M. \tag{3.10}$$

The uniform ellipticity of all truncates a_M for $M \geq M_{a,r}$, following from Corollary 3.2, ensures the well-posedness of (3.10). The solution of (3.4) can be obtained from the solution of (3.10) by backward substitution, as follows.

Proposition 3.4 *If $\tilde{u}_M$ is the solution of* (3.10) *and u_M solves* (3.4)*, then*

$$u_M(x, \omega) = \tilde{u}_M(x, X_1(\omega), X_2(\omega), \dots, X_M(\omega)), \tag{3.11}$$

$(\lambda \times P)$-a.e. $(x, \omega) \in D \times \Omega$.

The proof is immediate, observing that both the l.h.s. and the r.h.s. of (3.11) solve the well-posed problem (3.4).

Assuming that enough statistical information on the family $\mathcal{X} = (X_m)_{m\in\mathbb{N}_+}$ is available to allow the postprocessing (i.e. the computation of various statistics of u_M, see Assumption 2.6) via (3.11), Proposition 3.4 reduces the elliptic *problem with stochastic data* (1.2) to a question in *approximation theory* for the parametric (in $y \in I^M$) solution to (3.10), which we formulate as follows.

Problem 3.5 *For any M, compute the solution $\tilde{u}_M$ to* (3.10) *in $L^\infty(I^M, H_0^1(D))$ up to an error of* $\exp(-c_{2,r}M^\kappa)$.

Note that the truncation order M of the separating expansion (2.4) is the dimension of the parameter space I^M and, in fact, a discretization parameter. In the following section the aim will be therefore to solve Problem 3.5, by developing efficient approximations for $\tilde{u}_M$ as a function of $y \in I^M$. The keypoint of our analysis will be the regularity of $\tilde{u}_M$ with respect to the "stochastic parameter" y, to which we shall refer as "stochastic regularity". While it is easy to see that the solution $\tilde{u}_M$'s dependence on y is analytic, we shall prove that the domain of analyticity of $\tilde{u}_M$ as a function of coordinate y_m increases in size as $m \nearrow \infty$. Our estimates indicate in particular that $\tilde{u}_M$ as function of $y_m \in I$ becomes 'flat' as m increases at a rate which is governed by the convergence rate of the separating expansion (2.4).

To see this, we note that the decay rate of the expansion (2.4) of the random fluctuation r shows the decreasing sensitivity of $\tilde{a}_M$ w.r.t. y_m as $m \nearrow M$. Intuitively, $\tilde{u}_M$ is then expected to exhibit a similar behaviour.

Note that we are not interested in approximating $\tilde{u}_M$ with arbitrarily high accuracy, but only up to an error which matches the truncation error $O(\exp(-c_{1,r}M^\kappa))$ in the separating expansion (2.4). The needed accuracy depends thus in fact on the dimension M of the domain I^M on which the function $\tilde{u}_M$ to be approximated is defined.

4 Sparse Approximation Results

For the solution of the approximation Problem 3.5 we propose non-linear, adapted approach. To describe it, let $(\phi_{M,\alpha})_{\alpha\in\Lambda} \subset L^\infty(I^M)$ be a family of real-valued functions defined on the hypercube I^M such that $\tilde{u}_M$ admits the expansion

$$\tilde{u}_M = \sum_{\alpha\in\Lambda} \phi_{M,\alpha} \otimes c_{M,\alpha} \quad \text{in } L^\infty(I^M, H_0^1(D)), \tag{4.1}$$

with $c_{M,\alpha} \in H_0^1(D)$ for all $\alpha \in \Lambda$.

Definition 4.1 *If* (4.1) *holds, we call the series on the r.h.s. of* (4.1) chaos expansion of dimension M of u *solution to* (1.2).

For a finite index set $\Sigma \subset \Lambda$ we define the corresponding truncation of (4.1),

$$\tilde{u}_{M,\Sigma} := \sum_{\alpha\in\Sigma} \phi_{M,\alpha} \otimes c_{M,\alpha} \in L^\infty(I^M, H_0^1(D)). \tag{4.2}$$

In the spirit of the theory of adaptive/best N-term approximation, we consider the most economical chaos truncate (4.2) which achieves an accuracy comparable with that obtained after truncation of the separating expansion of r (see Corollary 3.2).

Definition 4.2 *If* (4.1) *holds, we define*

$$\Sigma_M := \operatorname{argmin}\{|\Sigma| : \|\tilde{u}_M - \tilde{u}_{M,\Sigma}\|_{L^\infty(I^M,H_0^1(D))} \le \|u - u_M\|_{L^\infty(\Omega,H_0^1(D))}\} \tag{4.3}$$

and call the truncate $\tilde{u}_{M,\Sigma_M}$ *the* adapted chaos expansion of dimension M of u *solution to* (1.2)*.*

Due to explicit control of the truncation error in the case of a diffusion coefficient with known decay rate of the fluctuation expansion (2.4), it is more convenient to work with the following more explicit version of Definition 4.2.

Definition 4.3 *If Assumption 2.5 and* (4.1) *hold, we set*

$$\Sigma_M := \operatorname{argmin}\{|\Sigma| : \|\tilde{u}_M - \tilde{u}_{M,\Sigma}\|_{L^\infty(I^M,H_0^1(D))} \le \exp(-c_{1,r}M^\kappa)\} \tag{4.4}$$

and call the truncate $\tilde{u}_{M,\Sigma_M}$ *the* adapted chaos expansion of dimension M of u *solution to* (1.2)*.*

The aim of the following sections is the (approximate) identification of the index set Σ_M, based on the regularity properties of $\tilde{u}_M$ w.r.t. y (analyticity and explicit bounds for all derivatives), if the family $(\phi_{M,\alpha})_{\alpha\in\Lambda} \subset L^\infty(I^M)$ is chosen to generate the standard finite element spaces (piecewise polynomials of fixed degree on regular meshes), corresponding to h and p FEM in I^M respectively.

4.1 Stochastic Regularity

We start by observing that Assumption 2.5 and (3.8) trivially ensure the following norm estimates

$$0 \le \rho_m := \|\psi_m\|_{L^\infty(D)} \le c_r \exp(-c_{1,r}m^\kappa) \quad \forall m \in \mathbb{N}_+. \tag{4.5}$$

Explicit bounds of all derivatives of $\tilde{u}_M$ are then obtained either using equations (3.10), (3.9) (as shown below) or by Cauchy formula (see e.g. [Tod05b]).

Proposition 4.4 *If* $\tilde{u}_M$ *solves* (3.10)*, then*

$$\|\partial_y^\alpha \tilde{u}_M(y,\cdot)\|_{H_0^1(D)} \le c_{a,r}^{|\alpha|} \cdot |\alpha|! \cdot \prod_{m=1}^M \rho_m^{\alpha_m} \cdot \|\tilde{u}_M(y,\cdot)\|_{H_0^1(D)}, \tag{4.6}$$

$\forall y \in I^M, \forall \alpha \in \mathbb{N}^M, \forall M \in \mathbb{N}, M \ge M_{a,r}$.

Proof. We prove the estimate (4.6) by induction on $|\alpha|$. Since (4.6) is clear for $|\alpha| = 0$, we assume it to hold also for all $\alpha \in \mathbb{N}^M$ such that $|\alpha| \le k$, for some $k \in \mathbb{N}$. We consider a multiindex α such that $|\alpha| = k+1$ and we apply ∂_y^α to (3.10). We obtain

$$-\operatorname{div}(\tilde{a}_M(\cdot,y)\nabla\partial_y^\alpha \tilde{u}_M(\cdot,y)) = \sum_{m=1}^M \alpha_m \operatorname{div}(\psi_m(\cdot)\nabla\partial_y^{\alpha-e_m}\tilde{u}_M(\cdot,y))$$

from which it follows

$$c_{a,r}\|\partial_y^\alpha \tilde{u}_M(\cdot,y)\|_{H_0^1(D)} \le \sum_{m=1}^M \alpha_m\rho_m\|\partial_y^{\alpha-e_m}\tilde{u}_M(\cdot,y)\|_{H_0^1(D)} \tag{4.7}$$

The desired estimate follows then by using (4.6) in (4.7) for all multiindices $\alpha - e_m$, $1 \le m \le M$, whose length equals k. □

4.2 Sparse Chaos Approximations

Using Proposition 4.4 we investigate next convergence rates of adapted chaos approximations for $\tilde{u}_M : I^M \to H_0^1(D)$, if tensor product families $(\phi_{M,\alpha})_{\alpha\in\Lambda} \subset L^\infty(I^M, H_0^1(D))$ corresponding to standard h or p FEM in I^M are chosen in (4.1) to expand $\tilde{u}_M$.

4.2.1 h-FEM Based Adapted Approximation

For $p \in \mathbb{N}_+$ and $l \in \mathbb{N}$, let $V^{l,p}$ be the space of piecewise polynomials of degree at most $p-1$ on a regular mesh of width 2^{-l} in I. We set $V^{-1,p} := \{0\}$, and by

$$W^{l,p} := V^{l,p} \cap (V^{l-1,p})^\perp$$

we define the hierarchical excess of the scale $(V^{l,p})_{l\in\mathbb{N}}$, where the orthogonal complement is taken in the sense of $L^2(I)$. In this way we obtain an $L^2(I)$ orthogonal decomposition

$$L^2(I) = \bigoplus_{l=0}^{\infty} W^{l,p}. \tag{4.8}$$

If H is an arbitrary Hilbert space and P_V denotes the $L^2(I,H) \simeq L^2(I) \otimes H$ projection onto the closed subspace $V \otimes H$ of $L^2(I)$, the standard (vector-valued) approximation property of the scale $(V^{l,p})_{l\in\mathbb{N}}$ reads

$$\|v - P_{V^{l,p}} v\|_{L^2(I,H)} \le c_p 2^{-lp} \|\partial^p v\|_{L^2(I,H)} \quad \forall v \in H^p(I,H), \tag{4.9}$$

with some constant $c_p > 0$.

Remark 4.5 *Note that an estimate similar to* (4.9) *holds also in the* $L^\infty(I,H)$ *norm, for* $v \in W^{p,\infty}(I,H)$.

Using the FE spaces $V^{l,p}$ in I we build FE spaces in I^M as tensor products. More precisely, for any multiindex $\mathbf{l} = (l_1, l_2, \ldots, l_M) \in \mathbb{N}^M$ we define

$$W^{\mathbf{l},p} := \bigotimes_{m=1}^{M} W^{l_m,p},$$

which enables us via (4.8) to decompose $L^2(I^M)$ as

$$L^2(I^M) = \bigoplus_{\mathbf{l}\in\mathbb{N}^M} W^{\mathbf{l},p}. \tag{4.10}$$

In $L^2(I^M, H)$ we then have,

$$v = \sum_{\mathbf{l}\in\mathbb{N}^M} v^{\mathbf{l}}, \quad v^{\mathbf{l}} := P_{W^{\mathbf{l},p}\otimes H} v \quad \forall v \in L^2(I^M, H). \tag{4.11}$$

For $\tilde{u}_M \in L^\infty(I^M, H_0^1(D)) \subset L^2(I^M, H_0^1(D))$ solution to (3.10) we estimate next the size of the general term (detail of $\tilde{u}_M$ at level $\mathbf{l}$) in the corresponding orthogonal decomposition (4.11) with $H := H_0^1(D)$. To this end we introduce first several notations. We define the length $|\mathbf{l}|$ of a multiindex $\mathbf{l} = (l_1, l_2, \ldots, l_M) \in \mathbb{N}^M$ by

$$|\mathbf{l}| := l_1 + l_2 + \cdots + l_M. \tag{4.12}$$

Further, the support of $\mathbf{l}$ will be denoted by

$$\mathcal{J}_\mathbf{l} := \text{ supp}(\mathbf{l}) = \{m : 1 \leq m \leq M,\, l_m > 0\}, \tag{4.13}$$

and its length by $j_\mathbf{l} := |\mathcal{J}_\mathbf{l}|$, so that $\mathcal{J}_\mathbf{l} = \{m_1, m_2, \ldots, m_{j_\mathbf{l}}\}$.

Proposition 4.6 *If $\tilde{u}_M$ solves* (3.10) *and Assumption 2.5 holds, then*

$$\|\tilde{u}_M^\mathbf{l}\|_{L^2(I^M, H_0^1(D))} \leq c_{a,p}^{j_\mathbf{l}} \cdot 2^{-|\mathbf{l}|p} \cdot (pj_\mathbf{l})! \cdot \prod_{j=1}^{j_\mathbf{l}} \rho_{m_j}^p \cdot \|\tilde{u}_M\|_{L^2(I^M, H_0^1(D))}, \tag{4.14}$$

where $\tilde{u}_M^\mathbf{l} := P_{W^{\mathbf{l},p} \otimes H_0^1(D)} \tilde{u}_M$ *for all* $\mathbf{l} \in \mathbb{N}^M$.

Proof. For a fixed multiindex $\mathbf{l} \in \mathbb{N}^M$ we define its support multiindex $\mathbf{e} := (e_1, e_2, \ldots, e_M) \in \mathbb{N}^M$ (depending on $\mathbf{l}$) by

$$e_m := \begin{cases} 1 & \text{if} \quad l_m > 0 \\ 0 & \text{if} \quad l_m = 0 \end{cases} \qquad \forall 1 \leq m \leq M,$$

and write

$$\tilde{u}_M^\mathbf{l} = P_{W^{\mathbf{l},p}} \tilde{u}_M = \bigotimes_{m=1}^M (P_{V^{l_m,p}} - P_{V^{l_m - 1,p}}) \tilde{u}_M.$$

Replacing $P_{V^{l_m,p}} - P_{V^{l_m-1,p}}$ by $P_{V^{l_m,p}} - I + I - P_{V^{l_m-1,p}}$ for all m in the support of $\mathbf{l}$ and expanding the resulting product we obtain

$$\tilde{u}_M^\mathbf{l} = \sum_{\mathbf{f} \in \mathbb{N}^M,\, \mathbf{f} \leq \mathbf{e}} (-1)^{M - |\mathbf{f}|} \left(\bigotimes_{m=1}^M Q_{l_m, f_m} \right) \tilde{u}_M, \tag{4.15}$$

where

$$Q_{l_m, f_m} := \begin{cases} P_{V^{0,p}} & \text{if} \quad l_m = 0 \\ I - P_{V^{l_m - f_m, p}} & \text{if} \quad l_m > 0 \end{cases}.$$

Using the approximation property (4.9) and noting that the sum in (4.15) consists of $2^{j_\mathbf{l}}$ terms, we deduce

$$\begin{aligned} \|\tilde{u}_M^\mathbf{l}\|_{L^2(I^M, H_0^1(D))} &\leq \sum_{\mathbf{f} \in \mathbb{N}^M,\, \mathbf{f} \leq \mathbf{e}} c_p^{j_\mathbf{l}} 2^{-(|\mathbf{l}| - |\mathbf{f}|)p} \cdot \|\partial_y^{p \cdot \mathbf{e}} \tilde{u}_M\|_{L^2(I^M, H_0^1(D))} \\ &\leq \sum_{\mathbf{f} \in \mathbb{N}^M,\, \mathbf{f} \leq \mathbf{e}} (2^p c_p)^{j_\mathbf{l}} 2^{-|\mathbf{l}|p} \cdot \|\partial_y^{p \cdot \mathbf{e}} \tilde{u}_M\|_{L^2(I^M, H_0^1(D))} \\ &\leq (2^{p+1} c_p)^{j_\mathbf{l}} 2^{-|\mathbf{l}|p} \cdot \|\partial_y^{p \cdot \mathbf{e}} \tilde{u}_M\|_{L^2(I^M, H_0^1(D))}. \end{aligned} \tag{4.16}$$

Proposition 4.4 coupled with (4.16) leads now to the desired estimate (4.14). □

Remark 4.7 *Based on Remark 4.5, an estimate similar to* (4.14) *can be shown also in the* $L^\infty(I^M, H_0^1(D))$ *norm.*

We next define a scale of sparse FE spaces in I^M which will be shown to achieve an almost optimal convergence rate of the corresponding approximations of $\tilde u_M$. We in fact prescribe an index set in $\mathbb{N}^M$ corresponding intuitively to the largest details in the orthogonal decomposition (4.11) of $\tilde u_M$. To this end we introduce for $\mu, \nu \in \mathbb{N}$ the set of all multiindices $\mathbf{l} \in \mathbb{N}^M$ not exceeding μ in length and having at most ν nontrivial entries,

$$\Sigma_{\mu,\nu} \subset \mathbb{N}^M, \quad \Sigma_{\mu,\nu} := \{\mathbf{l} \in \mathbb{N}^M : |\mathbf{l}| \le \mu, j_{\mathbf{l}} \le \nu\}. \tag{4.17}$$

Correspondingly we define, in view of (4.10), the following finite dimensional subspace of $L^2(I^M)$,

$$\hat V^{\mu,\nu} := \bigoplus_{\mathbf{l} \in \Sigma_{\mu,\nu}} W^{\mathbf{l},p}.$$

Using $\hat V^{\mu,\nu} \otimes H_0^1(D) \subset L^2(I, H_0^1(D))$ as semidiscretization space to approximate $\tilde u_M$, we now prove the main approximation result of this section. Here and in the following $P_{\hat V^{\mu,\nu}}$ denotes the $L^2(I, H_0^1(D))$ projection onto $\hat V^{\mu,\nu} \otimes H_0^1(D)$.

Proposition 4.8 *If* $\mu, \nu \in \mathbb{N}$ *and Assumption 2.5 holds, then for* $\tilde u_M$ *solution to* (3.10) *we have*

$$\begin{aligned}\|\tilde u_M - P_{\hat V^{\mu,\nu}} \tilde u_M\|_{L^2(I^M, H_0^1(D))} &\le c_{a,r,p,\theta}(e^{-\nu^{1+\kappa} c_{1,r} p/2(1+\kappa)} + \\ &+ 2^{-p(\mu+1)} \cdot e^{\nu \log(M+1) + \nu \log(\mu+2)}) \cdot \|\tilde u_M\|_{L^2(I^M, H_0^1(D))}.\end{aligned} \tag{4.18}$$

Besides,

$$\dim \hat V^{\mu,\nu} \le p(M+1)^\nu (\mu+1)^{\nu+1} 2^\mu. \tag{4.19}$$

Proof. For notational ease and since in the following arguments all functions are evaluated in the standard norm of $L^2(I^M, H_0^1(D))$, we drop the corresponding subscript from all estimates. For arbitrary $\mu, \nu \in \mathbb{N}$ we write

$$\|\tilde u_M - P_{\hat V^{\mu,\nu}} \tilde u_M\| \le \sum_{\mathbf{l} \in \mathbb{N}^M \setminus \Lambda^{\mu,\nu}} \|\tilde u_M^{\mathbf{l}}\| = \sum_{\substack{\mathbf{l} \in \mathbb{N}^M \\ j_{\mathbf{l}} > \nu}} \|\tilde u_M^{\mathbf{l}}\| + \sum_{\substack{\mathbf{l} \in \mathbb{N}^M \\ j_{\mathbf{l}} \le \nu \\ |\mathbf{l}| > \mu}} \|\tilde u_M^{\mathbf{l}}\| \tag{4.20}$$

and estimate next the two sums S_1, S_2 on the r.h.s. of (4.20) separately. In both cases we use Proposition 4.6 and the notations (4.12), (4.13). We start with S_1 and write

$$S_1 = \sum_{j=\nu+1}^{M} \sum_{\substack{\mathbf{l} \in \mathbb{N}^M \\ |j_{\mathbf{l}}| = j}} \|\tilde u_M^{\mathbf{l}}\| \overset{(4.14)}{\le} \sum_{j=\nu+1}^{M} c_{a,p}^j \cdot (pj)! \cdot \sum_{\substack{\mathbf{l} \in \mathbb{N}^M \\ |j_{\mathbf{l}}| = j}} 2^{-|\mathbf{l}|p} \cdot \prod_{k=1}^{j} \rho_{m_k}^p \cdot \|\tilde u_M\|. \tag{4.21}$$

Indexing the multiindices in the second sum on the r.h.s. of (4.21) over their support, we have that

$$\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ |j_{\mathbf{l}}|=j}} 2^{-|\mathbf{l}|p}\cdot\prod_{k=1}^{j}\rho_{m_k}^p \overset{(4.5)}{\leq} c_r^j\cdot\sum_{1\leq m_1<\cdots<m_j\leq M}\prod_{k=1}^{j}e^{-c_{1,r}m_k^\kappa p}\cdot\sum_{l_{m_1},\ldots,l_{m_j}=1}^{\infty}2^{-p(l_{m_1}+\cdots+l_{m_j})}$$

$$\leq \sum_{j=\nu+1}^{M}c_r^j\cdot\sum_{1\leq m_1<\cdots<m_j\leq M}\prod_{k=1}^{j}e^{-c_{1,r}m_k^\kappa p} \tag{4.22}$$

We then use Lemma A.2 (with $y=c_{1,r}p$ and $z=(1+\kappa)\theta p$) in (4.22) to obtain from (4.21),

$$S_1\leq c_{r,p,\theta}\sum_{j=\nu+1}^{M}c_{a,r,p}^j\cdot(pj)!\cdot e^{-j^{1+\kappa}\theta p}\cdot\|\tilde{u}_M\|, \tag{4.23}$$

for any $\theta\in]0,c_{1,r}/(1+\kappa)[$. The fast, supergeometrical decay of the third factor on the r.h.s. of (4.23) as $j\nearrow\infty$ (due to $\kappa>0$), allows absorbtion of the first two (exponential and factorial). We conclude

$$S_1\leq c_{a,r,p,\theta}e^{-\nu^{1+\kappa}\theta p}\cdot\|\tilde{u}_M\|\quad\forall\theta\in]0,c_{1,r}/(1+\kappa)[. \tag{4.24}$$

We turn now to the second sum S_2 in (4.20). Using again Proposition 4.6 and Lemma A.2 we similarly deduce

$$S_2 \overset{(4.14)}{\leq} \sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_{\mathbf{l}}\leq\nu \\ |\mathbf{l}|>\mu}} c_{a,p}^{j_{\mathbf{l}}}\cdot 2^{-|\mathbf{l}|p}\cdot(pj_{\mathbf{l}})!\cdot\prod_{j=1}^{j_{\mathbf{l}}}\rho_{m_j}^p\cdot\|\tilde{u}_M\|$$

$$\leq c_{a,r,p,\theta}\sum_{\substack{\mathbf{l}\in\mathbb{N}^M \\ j_{\mathbf{l}}\leq\nu \\ |\mathbf{l}|>\mu}} e^{-j_{\mathbf{l}}^{1+\kappa}\theta p}\cdot 2^{-|\mathbf{l}|p}\cdot\|\tilde{u}_M\|, \tag{4.25}$$

for any $\theta\in]0,c_{1,r}/(1+\kappa)[$. Using now a counting argument on the r.h.s. of (4.25) and then Lemma A.1 with $t=2^{-p}$, we obtain

$$\begin{aligned} S_2 &\leq c_{a,r,p,\theta}\sum_{j=1}^{\nu}\binom{M}{j}e^{-j^{1+\kappa}\theta p}\cdot\sum_{l=\mu+1}^{\infty}\binom{l}{j}2^{-pl}\cdot\|\tilde{u}_M\| \\ &\leq c_{a,r,p,\theta}2^{-p(\mu+1)}\cdot\sum_{j=1}^{\nu}\binom{M}{j}e^{-j^{1+\kappa}\theta p}\cdot(1-2^{-p})^{-j-1}\cdot(\mu+2)^j\cdot\|\tilde{u}_M\| \\ &\leq c_{a,r,p,\theta}2^{-p(\mu+1)}\cdot(M+1)^\nu\cdot(\mu+2)^\nu\cdot\|\tilde{u}_M\|, \end{aligned} \tag{4.26}$$

since $\binom{M}{j}\leq(M+1)^j$. (4.18) follows now from (4.24) and (4.26) by choosing $\theta=c_{1,r}/2(1+\kappa)$.

It remains to estimate the dimension of $\hat{V}^{\mu,\nu}$. Taking into account that the dimension of the detail space $W^{l,p}$ equals $p2^l$, we have

$$\begin{aligned}\dim \hat{V}^{\mu,\nu} &= p\sum_{q=0}^{\nu}\sum_{l=0}^{\mu}\binom{M}{q}\binom{l}{q}2^l \le p(M+1)^{\nu}\sum_{q=0}^{\nu}\sum_{l=0}^{\mu}\binom{l}{q}2^l \\ &\le p(M+1)^{\nu}\sum_{l=0}^{\mu}(l+1)^{\nu}2^l \le p(M+1)^{\nu}(\mu+1)^{\nu+1}2^{\mu},\end{aligned}$$

which concludes the proof. □

Corollary 4.9 *Under Assumption 2.5, there exist positive constants* c_1, c_2 *such that by choosing*

$$\mu := \lceil c_1 M^{\kappa}\rceil, \quad \nu := \lceil c_2 M^{\kappa/(\kappa+1)}\rceil, \tag{4.27}$$

for $\tilde{u}_M$ *solution to* (3.10) *we have*

$$\|\tilde{u}_M - P_{\hat{V}^{\mu,\nu}}\tilde{u}_M\|_{L^2(I^M,H_0^1(D))} \le c_{a,r,p}\exp(-c_{1,r}M^{\kappa}+o(M^{\kappa})) \tag{4.28}$$

with

$$N_{ace} := \dim \hat{V}^{\mu,\nu} \le c_{\kappa,p}\exp(\frac{c_{1,r}}{p}M^{\kappa}+o(M^{\kappa})), \tag{4.29}$$

as $M \nearrow \infty$, *and with the same constant* $c_{1,r}$ *as in* (3.5). *Here the subscript 'ace' abbreviates 'adapted chaos expansion'.*

Proof. We simply choose in Proposition 4.8

$$\mu = \lceil (2(1+\kappa)/p)^{1/(1+\kappa)}M^{\kappa/(\kappa+1)}\rceil, \quad \nu = \lceil c_{1,r}M^{\kappa}/p\log 2\rceil, \tag{4.30}$$

so that (4.28), (4.29) follow directly from (4.18) and (4.19) respectively. □

Remark 4.10 *The proof of Corollary 4.9 offers (see* (4.30)*) also explicit values for the constants* c_1, c_2 *in* (4.27). *Note that* c_1 *depends only on* κ, p *and never exceeds 3, whereas* c_2 *scales linearly with* $c_{1,r}$.

Combining (4.28) and (4.29), we reformulate the main approximation result of this section (optimality of the adapted chaos expansion) as follows.

Theorem 4.11 *If Assumption 2.5 holds, then*

$$\inf_{v\in\hat{V}^{\mu,\nu}\otimes H_0^1(D)}\|\tilde{u}_M - v\|_{L^2(I^M,H_0^1(D))} \le c_{a,r,p}N_{ace}^{-p+o(1)} \quad \text{as } M\nearrow\infty, \tag{4.31}$$

and for the parameter choice (4.27), *where* $N_{ace} = \dim \hat{V}^{\mu,\nu}$ *is the number of deterministic diffusion problems in* D *to be solved.*

Remark 4.12 *The convergence rate* (4.31) *of the h-FEM based adapted chaos expansion is, already for* $p = 1$ *(corresponding to piecewise constant elements), faster than the Monte Carlo* ($O(N^{-1/2})$) *or quasi-Monte Carlo* ($O(N^{-1}(\log N)^{c_M})$) *rates, where* N *denotes here the number of samples.*

4.2.2 p-FEM Based Adapted Approximation

The analyticity of $\tilde{u}_M$ in $y \in I^M$ following e.g. from Proposition 4.4 suggests the use of polynomial approximation (sometimes called *polynomial chaos*, corresponding to a polynomial basis $(\phi_{M,\alpha})_{\alpha\in\Lambda}$ in (4.1), see e.g. [Wie38]) in the stochastic variable y. In this section we give, for any $M \in \mathbb{N}_+$, the construction of a polynomial space of low dimension in y, and in which $\tilde{u}^M : I^M \to H_0^1(D)$ can be approximated with the desired accuracy, that is, up to an error of $O(e^{-c_{1,r}M^\kappa})$. The construction is based, just as in the case of h FEM discussed in the previous section, on a-priori estimation of the coefficients $c_{M,\alpha}$ in (4.1) using a tensor product basis in I^M. Selection of the largest estimated coefficients leads then to an upper estimate of the optimal index set Σ_M in Definition 4.3.
The tensor product basis we use to represent $\tilde{u}_M$ is given by the monomials in $y_1, \dots, y_M$,

$$\Lambda := \mathbb{N}^M, \quad \phi_{M,\alpha}(y_1, y_2, \dots, y_M) := y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_M^{\alpha_M} \quad \forall \alpha \in \Lambda.$$

The chaos expansion (4.1) holds then as the Taylor expansion of $\tilde{u}_M$ around $y = 0$, due to Proposition 4.4. Moreover, it can be shown (see also [Tod05b]) that $\tilde{u}_M$ as function of y admits a complex analytic extension to a cylinder complex neighbourhood $U^M \times i\mathbb{R}$ of I^M, where $I \subset U \subset \mathbb{R}$.

In analogy with the construction of the FE space $\hat{V}^{\mu,\nu}$ in Section 4.2.1 we consider, for $M, M', \eta, \mu, \nu \in \mathbb{N}$ with $M' \le M$ and in the context of the p FEM, the polynomial space $\mathcal{P}_{M',\eta,\mu,\nu}$ in the M variables $y_1, y_2, \dots, y_M$ spanned by all monomials satisfying three additional properties, as follows. First, we require that the monomials have degree at most η in each of the first M' variables $y_1, y_2, \dots, y_{M'}$. Second, their total degree in $y_{M'+1}, y_{M'+2}, \dots, y_M$ is at most μ. Finally, each monomial is nonconstant in at most ν variables taken from $y_{M'+1}, y_{M'+2}, \dots, y_M$. Formally we have,

Definition 4.13 *For $M, M', \eta, \mu, \nu \in \mathbb{N}$ with $M' \le M$ and $\nu \le M'' := M - M'$ we set*

$$\mathcal{P}_{M',\eta,\mu,\nu} := \operatorname{span}\{\phi_{M,\alpha} : \alpha \in \Sigma_{M',\eta,\mu,\nu}\}, \tag{4.32}$$

where the index set $\Sigma_{M',\eta,\mu,\nu} \subset \mathbb{N}^M$ is given by

$$\Sigma_{M',\eta,\mu,\nu} := \{\alpha = (\alpha', \alpha'') \in \mathbb{N}^{M'} \times \mathbb{N}^{M''} : |\alpha'|_\infty \le \eta, |\alpha''| \le \mu, \, |\operatorname{supp}(\alpha'')| \le \nu\}. \tag{4.33}$$

In order to prove an approximation property for the polynomial space $\mathcal{P}_{M',\eta,\mu,\nu} \otimes H_0^1(D)$ similar to the one derived in Proposition 4.8 in the context of the h FEM, we first recall that the solution $\tilde{u}_M$ of (3.10) satisfies the estimate (4.6), which we reformulate as follows.

Proposition 4.14 *If $\tilde{u}_M$ solves* (3.10)*, then*

$$\|\partial_y^\alpha \tilde{u}_M\|_{L^\infty(I^M, H_0^1(D))} \le c_{a,r,f}^{|\alpha|} |\alpha|! \, \rho^\alpha \quad \forall \alpha \in \mathbb{N}^M, \tag{4.34}$$

where $\rho^\alpha := \prod_{m=1}^M \rho_m^{\alpha_m}$.

Based on (4.34) we prove next the main approximation result for $\tilde{u}_M$ in the space $\mathcal{P}_{M',\eta,\mu,\nu}$.

Proposition 4.15 *If $\tilde{u}_M$ solves (3.10), then there exist $M' \in \mathbb{N}$ and constants c_1, c_2, c_3 depending only on the data a, r, f, such that (recall notation (4.2))*

$$\|\tilde{u}_M - \tilde{u}_{M,\Sigma_{M',\eta,\mu,\nu}}\|_{L^\infty(I^M, H_0^1(D))} \le c_1(e^{-c_3\eta} + e^{c_2\eta - c_3\mu} + e^{c_2\eta - c_3\nu^{1+\kappa}}) \qquad (4.35)$$

for any $M, \eta, \mu, \nu \in \mathbb{N}$ with $\nu \le M'' = M - M'$. Besides,

$$\dim \mathcal{P}_{M',\eta,\mu,\nu} \le (\eta+1)^{M'}(M''+1)^{\nu}(\mu+2)^{\nu+1}(\nu+1). \qquad (4.36)$$

Proof. Let us introduce the notation $y = (y', y'')$ corresponding to the following splitting of the stochastic variable y,

$$y' = (y_1, y_2, \dots, y_{M'}) \in I^{M'} \quad y'' = (y_{M'+1}, y_{M'+2}, \dots, y_M) \in I^{M''},$$

where M' will be chosen later. We consider the Taylor expansion of $\tilde{u}_M$ w.r.t. y around $y = 0$,

$$\tilde{u}_M(y', y'') = \sum_{\substack{\alpha' \in \mathbb{N}^{M'} \\ \alpha'' \in \mathbb{N}^{M''}}} \frac{\partial_{y'}^{\alpha'} \partial_{y''}^{\alpha''} \tilde{u}_M(0)}{\alpha!} y'^{\alpha'} y''^{\alpha''}, \qquad (4.37)$$

which converges absolutely for y in a neighbourhood of I^M.

We next estimate using Proposition 4.14 the size of that part of the expansion (4.37) which corresponds to the complement of the index set $\Sigma_{\eta,\mu,\nu} \subset \mathbb{N}^M$ given by

$$\Sigma_{\eta,\mu,\nu} := \Sigma'_\eta \times (\Sigma''_\mu \cap \Sigma''_\nu) \qquad (4.38)$$

with

$$\begin{aligned} \Sigma'_\eta &:= \{\alpha' \in \mathbb{N}^{M'} : |\alpha'| \le \eta\} \subset \mathbb{N}^{M'} \\ \Sigma''_\mu &:= \{\alpha'' \in \mathbb{N}^{M''} : |\alpha''| \le \mu\} \subset \mathbb{N}^{M''} \\ \Sigma''_\nu &:= \{\alpha'' \in \mathbb{N}^{M''} : |\mathrm{supp}(\alpha'')| \le \nu\} \subset \mathbb{N}^{M''}. \end{aligned}$$

Note that, due to (4.38),

$$\mathbb{N}^M \setminus \Sigma_{\eta,\mu,\nu} = ((\mathbb{N}^{M'} \setminus \Sigma'_\eta) \times \mathbb{N}^{M''}) \cup (\Sigma'_\eta \times (\mathbb{N}^{M''} \setminus \Sigma''_\mu)) \cup (\Sigma'_\eta \times (\mathbb{N}^{M''} \setminus \Sigma''_\nu)). \qquad (4.39)$$

Let us denote by T_1, T_2, T_3 those parts of the Taylor expansion (4.37) corresponding to the three disjoint index sets in (4.39) respectively.
An upper bound for T_1 (corresponding to the index set $(\mathbb{N}^{M'} \setminus \Sigma'_\eta) \times \mathbb{N}^{M''}$) follows by standard p FEM estimate (or using Cauchy formula), due to the analyticity of $\tilde{u}_M(\cdot, y'')$ in a neighbourhood $U_{M'} \times i\mathbb{R}$ of $I^{M'}$ in $\mathbb{C}^{M'}$, uniformly in $y'' \in I^{M''}$,

$$\|\sum_{\substack{\alpha' \in \mathbb{N}^{M'} \setminus \Sigma'_\eta \\ \alpha'' \in \mathbb{N}^{M''}}} \frac{\partial_{y'}^{\alpha'} \partial_{y''}^{\alpha''} \tilde{u}_M(0)}{\alpha!} y'^{\alpha'} y''^{\alpha''}\|_{L^\infty(I^M, H_0^1(D))} \le c_{a,f,M'} e^{-c_{1,a,f,M'}\eta}. \qquad (4.40)$$

Concerning T_2 (corresponding to the index set $\Sigma'_\eta \times (\mathbb{N}^{M''} \setminus \Sigma''_\mu)$) we have, by (4.34) and the multinomial formula, with $\rho' := (\rho_1, \rho_2, \ldots, \rho_{M'})$, $\rho'' := (\rho_{M'+1}, \rho_{M'+2}, \ldots, \rho_M)$,

$$\| \sum_{\substack{\alpha' \in \Sigma'_\eta \\ \alpha'' \in \mathbb{N}^{M''} \setminus \Sigma''_\mu}} \frac{\partial^{\alpha'}_{y'} \partial^{\alpha''}_{y''} \tilde{u}_M(0)}{\alpha!} y'^{\alpha'} y''^{\alpha''} \|_{L^\infty(I^M, H^1_0(D))} \leq \sum_{\substack{\alpha' \in \mathbb{N}^{M'}, \alpha'' \in \mathbb{N}^{M''} \\ |\alpha'| \leq \eta, |\alpha''| \geq \mu+1}} c^{|\alpha|}_{a,f} \frac{|\alpha|!}{\alpha!} \rho^\alpha. \quad (4.41)$$

Using the inequality $|\alpha|! \leq |\alpha'|! \cdot |\alpha''|! \cdot 2^{|\alpha|}$ we separate the variables α', α'' in the summation on the r.h.s. of (4.41) and obtain

$$\begin{aligned} \|T_2\|_{L^\infty(I^M, H^1_0(D))} &\leq \sum_{\substack{\alpha' \in \mathbb{N}^{M'} \\ |\alpha'| \leq \eta}} c^{|\alpha'|}_{a,f} \frac{|\alpha'|!}{\alpha'!} \rho'^{\alpha'} \cdot \sum_{l=\mu+1}^{\infty} \sum_{\substack{\alpha'' \in \mathbb{N}^{M''} \\ |\alpha''| = l}} c^l_{a,f} \frac{l!}{\alpha''!} \rho''^{\alpha''} \\ &= \sum_{n=0}^{\eta} (c_{a,f} |\rho'|)^n \cdot \sum_{l=\mu+1}^{\infty} (c_{a,f} |\rho''|)^l \\ &\leq c^{\eta+1}_{a,r,f} (c_{a,f} |\rho''|)^{\mu+1} \leq c_{a,r,f} e^{c_{2,a,r,f}\eta - c_{3,a,f}\mu}, \end{aligned} \quad (4.42)$$

where the last two estimates hold if M' is chosen in such a way that

$$c_{a,f}(\rho_{M'+1} + \rho_{M'+2} + \cdots) < 1/2.$$

Note that such a choice is always possible due to the decay condition (4.5), and that this is how we determine M', depending therefore only on the data a, r, f. In turn, the dependence on M' of the two constants in the upper bound (4.40) can be replaced by dependence on r.

Next we estimate T_3, corresponding to the index set $\Sigma'_\eta \times (\mathbb{N}^{M''} \setminus \Sigma''_\nu)$. From (4.34) we deduce,

$$\begin{aligned} \|T_3\|_{L^\infty(I^M, H^1_0(D))} &\leq \sum_{\substack{\alpha' \in \mathbb{N}^{M'}, \alpha'' \in \mathbb{N}^{M''} \\ |\alpha'| \leq \eta, |\mathrm{supp}(\alpha'')| \geq \nu+1}} c^{|\alpha|}_{a,f} \frac{|\alpha|!}{\alpha!} \rho^\alpha \\ &\leq \sum_{\substack{\alpha' \in \mathbb{N}^{M'} \\ |\alpha'| \leq \eta}} c^{|\alpha'|}_{a,f} \frac{|\alpha'|!}{\alpha'!} \rho'^{\alpha'} \cdot \sum_{l=0}^{\infty} \sum_{\substack{\alpha'' \in \mathbb{N}^{M''} \\ |\mathrm{supp}(\alpha'')| \geq \nu+1, |\alpha''| = l}} c^l_{a,f} \frac{l!}{\alpha!} \rho''^{\alpha''} \end{aligned} \quad (4.43)$$

The first sum (over α') on the r.h.s. of (4.43) can be evaluated just as in (4.42), so we only analyze the second one (over l and α''), which we denote in the following by S. To this end, we parametrize the indices α'' through their support (consisting of at least $\nu+1$ integers between 1 and M''), and obtain

$$\begin{aligned} S &= \sum_{l=0}^{\infty}\sum_{j=\nu+1}^{M''}\sum_{1\le m_1<\cdots<m_j\le M''}\sum_{\substack{\alpha''\in\mathbb{N}^{M''},|\alpha''|=l\\ \mathrm{supp}(\alpha'')=\{m_1,\dots,m_j\}}} c_{a,f}^l\, l!\prod_{k=1}^{j}\frac{\rho_{M'+m_k}^{\alpha''_{m_k}}}{\alpha''_{m_k}!} \\ &= \sum_{l=0}^{\infty}\sum_{j=\nu+1}^{M''}\sum_{1\le m_1<\cdots<m_j\le M''}\sum_{\substack{\alpha''\in\mathbb{N}^{M''},|\alpha''|=l\\ \mathrm{supp}\alpha''=\{m_1,\dots,m_j\}}} c_{\alpha''} c_{a,f}^l\, l!\prod_{k=1}^{j}\frac{\rho_{M'+m_k}^{\alpha''_{m_k}-1}}{(\alpha''_{m_k}-1)!} \end{aligned}$$

with

$$c_{\alpha''} := \prod_{k=1}^{j}\frac{\rho_{M'+m_k}}{\alpha''_{m_k}} \le \prod_{k=1}^{j}\rho_{M'+m_k}. \tag{4.44}$$

Using (4.44) and the multinomial formula we obtain

$$\begin{aligned} S &\le \sum_{l=0}^{\infty}\sum_{j=\nu+1}^{M''}\frac{l!}{(l-j)!}\sum_{1\le m_1<\cdots<m_j\le M''} c_{a,f}^l|\rho''|^{l-j}\prod_{k=1}^{j}\rho_{M'+m_k} \\ &\le \sum_{l=0}^{\infty}\sum_{j=\nu+1}^{M''} j!\, c_{a,f}^j\binom{l}{j}\varepsilon^{l-j}\sum_{1\le m_1<\cdots<m_j\le M''}\rho_{M'+m_1}\cdots\rho_{M'+m_j} \end{aligned} \tag{4.45}$$

where

$$\varepsilon := c_{a,f}(\rho_{M'+1}+\rho_{M'+2}+\cdots+\rho_M).$$

Note that, by increasing M' if necessary (still depending only on the data a,r,f), we can assume w.l.o.g. $\varepsilon<1/2$.
From the decay estimate (4.5) and Lemma A.2 we use here to bound the last sum in (4.45) we obtain

$$S \le \sum_{l=0}^{\infty}\sum_{j=\nu+1}^{M''} j!\, c_{a,r,f}^j\binom{l}{j}\varepsilon^{l-j}e^{-c_r j^{1+\kappa}}.$$

Performing first the sum over l via (A.3) in Lemma A.1 and absorbing then the factorial and the exponential functions of j in the last factor we arrive at ($\varepsilon<1/2$)

$$S \le \sum_{j=\nu+1}^{M''} j!\, c_{a,r,f}^j\frac{1}{(1-\varepsilon)^{j+1}}e^{-c_r j^{1+\kappa}} \le c_{a,r,f}e^{-c_{4,a,r,f}\nu^{1+\kappa}}, \tag{4.46}$$

which ensures via (4.43),

$$\|T_3\|_{L^\infty(I^M,H_0^1(D))} \le c_{a,r,f}e^{c_{2,a,r,f}\eta-c_{4,a,r,f}\nu^{1+\kappa}}. \tag{4.47}$$

(4.35) follows now from (4.40), (4.42) and (4.47).

Finally, the dimension estimate (4.36) follows by a counting argument, based on the combinatorial fact that the equation $x_1 + x_2 + \cdots + x_q = l$ has exactly $\binom{l}{q}$ solutions $(x_1, x_2, \ldots, x_q) \in \mathbb{N}_+^q$, which ensures,

$$
\begin{aligned}
|\Sigma''_\mu \cap \Sigma''_\nu| = \sum_{l=0}^{\mu}\sum_{q=0}^{\nu}\binom{M''}{q}\binom{l}{q} &\leq (M''+1)^\nu \sum_{l=0}^{\mu}\sum_{q=0}^{\nu}\binom{l}{q} \\
&= (M''+1)^\nu \sum_{q=0}^{\nu}\binom{\mu+1}{q+1} \\
&\leq (M''+1)^\nu (\mu+2)^{\nu+1}(\nu+1),
\end{aligned}
$$

and the proof is concluded. □

We recall that we do not ask for an arbitrarily high accuracy in the computation of $\tilde{u}_M$, since the truncation of the diffusion coefficient expansion (2.4) already resulted in an error between u and u_M of order $O(e^{-c_{1,r}M^\kappa})$ (see Problem 3.5). Making therefore an appropriate choice for the parameters η, μ, ν in order to match this accuracy, we arrive at superalgebraic (though subexponential) convergence rate of the semidiscrete solution of (3.10) w.r.t. y.

Corollary 4.16 *If Assumption 2.5 is satisfied and* $\tilde{u}_M$ *solves* (3.10)*, then there exist* $M' \in \mathbb{N}$ *and positive constants* c_4, c_5, c_6 *depending only on* a, r, f*, such that for*

$$\eta := \lceil c_4 M^\kappa \rceil, \ \mu := \lceil c_5 M^\kappa \rceil, \ \nu := \lceil c_6 M^{\kappa/(\kappa+1)} \rceil \tag{4.48}$$

we have

$$\|\tilde{u}_M - \tilde{u}_{M,\Sigma_{M,\eta,\mu,\nu}}\|_{L^\infty(I^M, H_0^1(D))} \leq c_{a,r,f} \exp(-c_{1,r}M^\kappa) \tag{4.49}$$

for all $M \in \mathbb{N}, M \geq M_{a,r}$*, with*

$$N_{ace} := \dim \mathcal{P}_{M',\eta,\mu,\nu} \leq \exp(c_{a,r,f} M^{\kappa/(\kappa+1)} \log(M+2)). \tag{4.50}$$

Here the subscript 'ace' abbreviates 'adapted chaos expansion'.

Proof. We first choose c_4 so that the first term in the upper bound (4.35) matches (4.49). Then we choose also c_5, c_6 (depending on c_2, c_3 in (4.35) and c_4) so that the other two error terms on the r.h.s. of (4.35) match (4.49). The dimension estimate (4.50) follows then from (4.36). □

Combining (4.49) and (4.50), we reformulate the main approximation result of this section as follows.

Theorem 4.17 *If Assumption 2.5 holds, then*

$$\inf_{v \in \mathcal{P}_{M',\eta,\mu,\nu} \otimes H_0^1(D)} \|\tilde{u}_M - v\|_{L^\infty(I^M, H_0^1(D))} \leq c_{1,a,r,f} \exp(-c_{2,a,r,f} (\log N_{ace})^{1+\kappa-o(1)}) \tag{4.51}$$

as $M \nearrow \infty$ *and for the parameter choice* (4.48)*, where* $N_{ace} = \dim \mathcal{P}_{M',\eta,\mu,\nu}$ *is the number of deterministic diffusion problems in* D *to be solved.*

Note that the estimated convergence rate (4.51) is asymptotically superalgebraic in the number of deterministic problems N_{ace} to be solved (due to $\kappa > 0$), but not asymptotically exponential, as $M \nearrow \infty$ (or, equivalently, $N_{ace} \nearrow \infty$).

Remark 4.18 *Our proof of Theorem 4.17 is based on the Taylor expansion of $\tilde{u}_M$ around $y = 0$ (expansion in the standard monomial basis). A similar result can be obtained for the Legendre expansion, by explicitly estimating its coefficients using Proposition 4.4.*

5 Postprocessing

For brevity of exposition, we only consider here the p-FEM based adapted approximation discussed in Section 4.2.2. Analogous results hold for the h-FEM based chaos approximation of Section 4.2.1.
We show that Theorem 4.17 allows control of the chaos expansion error in the solution to the initial problem (1.2) w.r.t. a strong (L^∞) topology in the stochastic variable ω.

Theorem 5.1 *Under Assumption 2.5, for*

$$u_{M,\Sigma_{M,\eta,\mu,\nu}}(\cdot,\omega) := \tilde{u}_{M,\Sigma_{M,\eta,\mu,\nu}}(\cdot, X_1(\omega), \dots, X_M(\omega)) \in H_0^1(D) \quad P\text{-a.e. } \omega \in \Omega$$

it holds

$$\|u - u_{M,\Sigma_{M,\eta,\mu,\nu}}\|_{L^\infty(\Omega, H_0^1(D))} \le c_{1,a,r,f} \exp(-c_{2,a,r,f}(\log N_{ace})^{1+\kappa - o(1)}) \tag{5.1}$$

as $M \nearrow \infty$ and for the parameter choice (4.48).

Proof. The claim follows immediately from (4.51) (adapted chaos error estimate) and (3.5) (fluctuation truncation error estimate), taking into account the relationship between M and N_{ace} given by (4.50). □

Remark 5.2 *The boundedness of the k-th moment operator between $L^\infty(\Omega, H_0^1(D))$ and $\underbrace{H_0^1(D) \otimes \cdots \otimes H_0^1(D)}_{k \text{ times}}$ ensures an upper bound similar to* (5.1) *also for the y-semidiscretization error in these moments.*

A Appendix

Lemma A.1 *For any $t \in [0,1)$ and $j, L \in \mathbb{N}$ with $j \le L$ it holds*

$$\sum_{n=0}^{\infty} \binom{L+n}{j} t^n \le (L+1)^j (1-t)^{-j-1}. \tag{A.1}$$

Proof. Using the factorial representation of the binomial coefficients it is easy to see that

$$\binom{L+n}{j} \leq (L+1)^j \binom{j+n}{j} \quad \forall n \in \mathbb{N},$$

which ensures

$$\sum_{n=0}^{\infty} \binom{L+n}{j} t^n \leq (L+1)^j \sum_{n=0}^{\infty} \binom{j+n}{j} t^n. \tag{A.2}$$

Denoting by S_j the sum on the r.h.s. of (A.2), the binomial identity

$$\binom{j+n}{j} = \binom{j+n-1}{j} + \binom{j+n-1}{j-1}$$

leads to the recursive formula $S_j = tS_j + S_{j-1}$, which shows that $(S_0 = (1-t)^{-1})$

$$S_j = (1-t)^{-j-1}. \tag{A.3}$$

(A.1) follows then from (A.2) and (A.3). □

We now prove that, if $y, \kappa > 0$ and $j \in \mathbb{N}_+$, the sum of the series with general term $\exp(-y\sum_{i=1}^{j} m_i^\kappa)$ indexed over $1 \leq m_1 < \cdots < m_j < \infty$ is, qualitatively and uniformly in $j \in \mathbb{N}_+$, just as large as the leading term, corresponding to $m_i = i$ for all $1 \leq i \leq j$. More precisely, it holds

Lemma A.2 *If $\kappa > 0$ and $x > y > z > 0$, then there exist $c_{\kappa,x,y}, c_{\kappa,y,z} > 0$ such that*

$$c_{\kappa,x,y} \exp(-x\frac{1}{1+\kappa} j^{1+\kappa}) \leq \sum_{1 \leq m_1 < \cdots < m_j < \infty} \prod_{i=1}^{j} \exp(-y m_i^\kappa) \leq c_{\kappa,y,z} \exp(-z\frac{1}{1+\kappa} j^{1+\kappa}) \tag{A.4}$$

for all $j \in \mathbb{N}_+$.

Proof. For $y > 0$ and $j \in \mathbb{N}_+, J \in \mathbb{N}_+ \cup \{\infty\}$ with $j \leq J$ we set

$$S_{j,J} := \sum_{1 \leq m_1 < \cdots < m_j \leq J} \prod_{i=1}^{j} \exp(-y m_i^\kappa). \tag{A.5}$$

The lower bound in (A.4) follows by observing that the sum in (A.5) contains the term corresponding to $m_i = i$ for all $1 \leq i \leq j$, so that

$$S_{j,J} \geq \exp(-y\sum_{i=1}^{j} i^\kappa),$$

where

$$\sum_{i=1}^{j} i^\kappa \leq (j+1)^{1+\kappa} \int_0^1 x^\kappa \, dx = \frac{1}{1+\kappa}(j+1)^{1+\kappa}.$$

It remains to prove the upper bound of the sum in (A.4). From (A.5) it follows that the sequence $(S_{j,j})_{j\in\mathbb{N}_+}$ is rapidly decaying, that is

$$S_{j,j} \le c_{\kappa,y,\beta}\beta^j \quad \forall j \in \mathbb{N}_+, \forall \beta > 0. \tag{A.6}$$

From (A.5) we also derive the recursive formula

$$\begin{aligned} S_{j,J+1} &= S_{j,J} + \exp(-y(J+1)^\kappa) \sum_{1\le m_1<\ldots<m_{j-1}\le J} \prod_{k=1}^{j-1} \exp(-y m_k^\kappa) \\ &= S_{j,J} + \exp(-y(J+1)^\kappa) S_{j-1,J} \end{aligned} \tag{A.7}$$

By induction on on j in (A.7) we immediately see that

$$S_{j,J} < S_{j,\infty} = \lim_{J\nearrow\infty} S_{j,J} < \infty \quad \forall j \in \mathbb{N}_+,$$

and that

$$S_{j,\infty} \le S_{j,j} + \sum_{i=j+1}^{\infty} \exp(-yi^\kappa) \cdot S_{j-1,\infty}. \tag{A.8}$$

Now, for an arbitrary $\gamma \in]0,1[$ we have, for j large enough ($j \ge j_0$, with j_0 depending on y, κ, γ),

$$\sum_{i=j+1}^{\infty} \exp(-yi^\kappa) \le \gamma,$$

which ensures via (A.8)

$$S_{j,\infty} \le S_{j,j} + \gamma S_{j-1,\infty} \quad \forall j \ge j_0. \tag{A.9}$$

From (A.6) and (A.9) we deduce that

$$S_{j,\infty} \le c_{\kappa,y,\beta}(\gamma+\beta)^j + \gamma^{j-j_0+1} S_{j_0-1,\infty},$$

which shows that $S_{j,\infty} \to 0$ as $j \nearrow \infty$, by choosing β such that $\gamma+\beta < 1$. The sequence $(S_{j,\infty})_{j\in\mathbb{N}_+}$ is in particular bounded, that is

$$S_{j,\infty} \le c_{\kappa,y} \quad \forall j \in \mathbb{N}_+. \tag{A.10}$$

Since this inequality holds for any $y > 0$, the conclusion follows then from (A.10) upon replacing y by $y - z$ and noting that

$$\sum_{i=1}^{j} i^\kappa \ge j^{1+\kappa} \int_0^1 x^\kappa \, dx = \frac{1}{1+\kappa} j^{1+\kappa}.$$

□

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