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Abstract

In this paper we consider the Laplace-Dirichlet equation in a polygonal domain which has been slightly perturbed to the scale ε near one vertex. We derive and validate an asymptotic expansion of the solution via two different techniques: matched asymptotic expansion and multiscale expansion. We present a simple approach of both methods and compare the obtained expansions.

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1 Introduction

1.1 Motivations

The solutions of elliptic problems in polygons are known to be singular near the vertices, see [7, 4, 3]. In applications, however, domains are always smooth, provided you look at them to a sufficiently large scale: for example a closer look to corners let them appear more or less rounded. The smoothness of the solution resulting from the regularity theory of elliptic problems is nevertheless misleading. Indeed, if the corner is rounded to a scale ε, the singularities are ready to come up as ε goes to 0: for the Laplace operator, the gradient of the solution – even bounded foreach $\varepsilon > 0$ – blows up when $\varepsilon \to 0$.

This question is also from high mathematical interest since the mechanism by which the singularities are hiding is worth understanding. Some authors have investigated this subject in particular cases – see [8, 9], giving a construction of the asymptotic expansion of the solution of the Laplace equation in the domain with a slightly perturbed corner with respect to the parameter ε. The most common technique is known as *matched asymptotic expansions*, and was first introduced by the english and russian schools, see [11] and [5], respectively. One can also refer to [10, 6] for a more recent work. It consists in building two expansions: one near the corner and the other away from the corner, satisfying matching conditions to fit each other in the intermediate region. Another possibility is to use a *multiscale technique* – see [2, 12, 1] – which is based on a superposition of terms via cut-off functions, which involve different scales.

We aim in the present paper at describing the two later methods in the very simple framework of the Laplace-Dirichlet equation in a polygonal domain, slightly perturbed near one of its vertices. We derive in a natural way the construction of the terms, as well as the matching conditions. Moreover, we validate the expansions by giving optimal estimates of the remainder in both cases. Finally we compare the two techniques and show how to switch from one expansion to the other. We hope this article to give a clear view of these techniques and the associated tools, which apply to much more complex problems.

1.2 Geometry and equations

The unperturbed domain. Let ω be a non-degenerate bounded polygon of $\mathbb{R}^2$, the origin O being one of its corners; the angle associated with O is denoted by α. Any point x in ω will be parametrized via its radial coordinates centered in O: $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi]$. The sector $(0, +\infty)_r \times (0, \alpha)_\theta$ coincides with ω in a neighborhood of O: let r^* be a real number such that $(0, r^*)_r \times (0, \alpha)_\theta \subset \omega$.

The perturbing pattern. Let Ω be a smooth connected unbounded domain of $\mathbb{R}^2$ parametrized by the polar coordinates centered in O: $(R, \theta) \in \mathbb{R}^+ \times [0, 2\pi]$. We assume that the sector $(0, +\infty)_R \times (0, \alpha)_\theta$ coincides with Ω in a neighborhood of infinity: R^* is a real number satisfying $(R^*, +\infty)_R \times (0, \alpha)_\theta \subset \Omega$. We emphasize that there is *no need for inclusion* of Ω into ω near the origin (though it is the case in Figure 1).

The perturbed domain. For ε small enough, ω_ε denotes the open bounded domain

$$\omega_\varepsilon \ = \ \{x \in \omega \ ; \ |x| > \varepsilon R^*\} \cup \{x \in \varepsilon\Omega \ ; \ |x| < r^*\} \,. \tag{1}$$

The domain ω_ε coincides with the polygon ω everywhere except near the origin, where its shape is given by the ε-dilation of the domain Ω, see Figure 1. For instance, depending on the pattern Ω, we can see ω_ε either as ω with a chopped-off corner, or as ω with a rounded corner (both to the scale ε). It is clear that this domain is piecewise regular and tends to ω when the parameter ε goes to 0.

The Laplace-Dirichlet problem. We are interested in the derivation and the mathematical validation of the asymptotic expansion with respect to ε of the solution u^ε of the problem:

$$\text{Find } u_\varepsilon \in \mathrm{H}^1_0(\omega_\varepsilon) \text{ such that } \ -\Delta u_\varepsilon \ = \ f \text{ in } \omega_\varepsilon, \tag{2}$$

where f is some function belonging to $\mathrm{L}^2(\omega)$ with compact support in ω.

Outline. When ε goes to 0, the solution u_ε of (2) clearly tends to $u_0 \in \mathrm{H}^1_0(\omega)$ solution of $-\Delta u_0 = f$ in the limit domain ω. In the following, we shall derive a full asymptotic expansion of u_ε in powers of ε, whose first terms read

$$u_\varepsilon \simeq \chi(\tfrac{x}{\varepsilon}) u_0 + \psi(x) \varepsilon^{\frac{\pi}{\alpha}} U^0(\tfrac{x}{\varepsilon}), \tag{3}$$

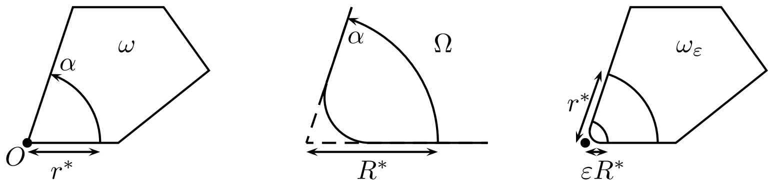

Figure 1: Definition of the domain ω_ε.

where χ is a smooth cut-off function vanishing at point O and ψ a smooth cut-off function localized near O; the *profile* U^0 is defined in the infinite domain Ω. We emphasize the fact that all information concerning the perturbing pattern is contained in the second term of (3) (whose contribution is localized near the corner), and u_0 carries information corresponding to the bounded domain ω (whose influence does not reach the corner).

Our paper is divided as follows: after recalling some preliminary results on singularities, we present in section 3 the method of *matched asymptotic expansion*, with the construction of the terms and matching conditions, and the validation of the expansion by remainder estimates. Section 4 is devoted to the *multi-scale approach*, where the terms are derived as well as optimal error estimates (sections 3 and 4 may be read independently). We conclude with a comparison of the latter two techniques, comments, and extensions.

2 Preliminary results

2.1 Singular functions: notations

From now on, we denote by λ the real number related to the opening angle α at the origin by:

$$\lambda = \frac{\pi}{\alpha}. \tag{4}$$

For any integer $p \in \mathbb{Z}^*$, we introduce the solutions of the Laplace equation with Dirichlet boundary condition in the sector of opening α, parametrized in polar coordinates by $(0;+\infty)_\rho \times (0;\alpha)_\theta$:

$$\mathfrak{s}^{p\lambda}(\rho,\theta) \quad = \quad \rho^{p\lambda}\ \sin(p\lambda\theta). \tag{5}$$

The function $\mathfrak{s}^{p\lambda}$ satisfies the property of separation of variables and is called *singular function* associated with the Laplace-Dirichlet problem in the considered sector. We shall recall in the next sections the significance of such functions in the studies of regularity.

2.2 The limit problem

We have gathered in this section some preliminary results, which will be useful for the study of the Laplace-Dirichlet equation in the limit domain $\omega_0 = \omega$. We first introduce some notation to quantify the behavior near the origin in terms of singularities:

Definition 1 *We denote by* $\mathrm{H}^1_*(\omega)$ *and* $\mathrm{W}^s_0(\omega)$ *(*$s \in \mathbb{R}$*) the spaces of functions:*

$$\mathrm{H}^1_*(\omega) = \left\{ u \in \mathcal{D}'(\omega)\ ;\ \varphi u \in \mathrm{H}^1(\omega),\ \varphi u = 0 \textit{ on } \partial\omega,\ \forall \varphi \in C^\infty(\overline{\omega}) \textit{ with } O \notin \textit{supp}(\varphi) \right\}. \tag{6}$$

$$\mathrm{W}^s_0(\omega)\ =\ \left\{ u \in \mathrm{H}^1_*(\omega)\ ;\ r^{-s-1}\ u\ \in\ \mathrm{L}^2(\omega) \textit{ and } r^{-s}\ \nabla u \in \mathrm{L}^2(\omega) \right\}. \tag{7}$$

Definition 2 *For* $s \in \mathbb{R}$*, the function* $u : \omega \to \mathbb{R}$ *is said to be a* $\mathcal{O}_{r\to 0}(r^s)$ *iff there exist a neighborhood* V *of* O *in* $\mathbb{R}^2$ *and a constant* $C > 0$ *such that* $u \in \mathcal{C}^\infty(\omega \cap V)$ *and*

$$\forall m,n \in \mathbb{N}, \qquad \left| r^m \frac{\partial^{m+n} u}{\partial r^m \partial\theta^n} \right| \ \leqslant\ C\ r^s \textit{ in } \omega \cap V. \tag{8}$$

The above definitions are connected by the following result, see [7]: if $u \in \mathrm{H}^1_*(\omega)$ and $\Delta u = 0$ in ω then

$$\Big(u \in \mathrm{W}^s_0(\omega) \Longleftrightarrow u = \mathcal{O}_{r\to 0}(r^s)\Big) \qquad \text{if } s \notin \lambda\mathbb{Z}^*. \tag{9}$$

We can now state the main two lemmas we shall use in the sequel; they concern the expansion into singular-regular parts of the solutions of the Laplace-Dirichlet problem in ω.

Lemma 1 *For any data $f \in \mathrm{L}^2(\omega)$ with compact support in ω, and any finite sequence $(a_{p\lambda})_{p\in\mathbb{N}}$ of real numbers (i.e. $a_{p\lambda} = 0$ except for a finite number of terms), there exists a unique function $u \in \mathrm{H}^1_*(\omega)$ such that:*

$$-\Delta u = f \text{ in } \omega \quad \text{and} \quad u - \sum_{p=1}^{+\infty} a_{p\lambda}\ \mathfrak{s}^{-p\lambda} = \mathcal{O}_{r\to 0}(1). \tag{10}$$

PROOF. We set $v = \sum_p a_{p\lambda}\ \mathfrak{s}^{-p\lambda}$, which obviously satisfies

$$v \in \mathrm{H}^1_*(\omega), \quad \Delta v = 0 \text{ in } \omega, \quad \text{and} \quad v|_{\partial\omega} \in \mathcal{C}^\infty(\partial\omega). \tag{11}$$

Hence, the problem to find w such that $-\Delta w = f$ in ω and $w = -v$ on $\partial\omega$ admits a unique variational solution $w \in \mathrm{H}^1(\omega)$. Moreover, since $\mathrm{H}^1_0(\omega) \subset \mathrm{W}^0_0(\omega)$ (use an angular Poincar´e inequality), by localization near point O, w is a $\mathcal{O}_{r\to 0}(1)$ thanks to (9); the function $u = w + v$ meets then the requirements. ■

Roughly speaking, the last lemma states that if we know the singular part of a function and its laplacian, then this function is uniquely defined. On the other hand, every solution of the Laplace-Dirichlet equation can be expanded near the corner point O in terms of the singular functions:

Lemma 2 *Let V be a neighborhood of O in $\mathbb{R}^2$. For any $u \in \mathrm{H}^1_*(\omega)$ such that:*

$$\Delta u = 0 \text{ in } V \cap \omega \quad \text{and} \quad \exists s \in \mathbb{R}\ ;\ u = \mathcal{O}_{r\to 0}(r^{-s}). \tag{12}$$

There exist a unique finite sequence $(a_{p\lambda})_{p\in\mathbb{N}^*}$ and a unique sequence $(b_{p\lambda})_{p\in\mathbb{N}^*}$ (generically infinite) such that for all $N \in \mathbb{N}$:

$$u(r,\theta) = \sum_{p=1}^{+\infty} a_{p\lambda}\ \mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{N} b_{p\lambda}\ \mathfrak{s}^{p\lambda}(r,\theta) + \mathcal{O}_{r\to 0}(r^{(N+1)\lambda}). \tag{13}$$

PROOF. One can prove this lemma using the Mellin transform, see [7]. In the particular case we are interested in, an argument based on separation of variables via angular Fourier series can also lead to the result. ■

The series $\sum a_{p\lambda}\ \mathfrak{s}^{-p\lambda}(r,\theta)$ and $\sum b_{p\lambda}\ \mathfrak{s}^{p\lambda}(r,\theta)$ are called the *singular* and *regular* parts of u, respectively. Note that the regular part only belongs to $\mathrm{H}^1(\omega)$, and does not fit the standard denomination of *regular* used for instance in [4] – that (more) regular part is actually contained in the $\mathcal{O}_{r\to 0}(r^{(N+1)\lambda})$.

2.3 Solutions in the pattern domain - behavior at infi nity

Solutions of the limit problem, which have just been investigated in the previous section, will give a good representation of the solution u_ε in the domain ω_ε, away from the corner point O. On the other hand, the geometries of domains ω and ω_ε differ near that point and the asymptotic expansion of u_ε with respect to ε will be given by terms defined in the pattern domain Ω. The tools needed for their construction are stated hereafter.

As we shall see further, the behavior at infinity of Laplace-Dirichlet solutions in Ω is the key for building and validating the asymptotic expansion of u_ε. Hence, we give similar definitions as in the previous section, $r \to 0$ being replaced with $R \to +\infty$.

Definition 3 *We denote by* $\mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ *the set of functions:*

$$\mathrm{H}^1_{0,\mathrm{loc}}(\Omega) \;=\; \big\{U \in \mathcal{D}'(\Omega)\;;\; \varphi U \in \mathrm{H}^1_0(\Omega),\ \forall \varphi \in \mathcal{C}^\infty(\overline{\Omega}) \textit{ with compact support}\big\} \tag{14}$$

and for $s \in \mathbb{R}$,

$$\mathrm{W}^s_\infty(\Omega) \;=\; \big\{U \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)\;;\; (1+|R|)^{-s-1}\, U \in \mathrm{L}^2(\Omega) \textit{ and } (1+|R|)^{-s}\, \nabla U \in \mathrm{L}^2(\Omega)\big\}\,. \tag{15}$$

In the following, we shall say that V is a neighborhood of infinity iff there exists a ball B_R of radius R such that:

$$^{c}B_R \;\subset V. \tag{16}$$

Definition 4 *A function* $U : \Omega \to \mathbb{R}$ *is said to be a* $\mathcal{O}_{R\to+\infty}(R^s)$ *(*$s \in \mathbb{R}$*) iff there exists a neighborhood* V *of infinity such that* $U \in \mathcal{C}^\infty(\Omega \cap V)$ *and*

$$\forall m, n \in \mathbb{N}, \qquad \left| R^m \frac{\partial^{m+n} U}{\partial R^m \partial \theta^n}(R,\theta)\right| \leqslant\; C\,R^s, \textit{ in } \Omega \cap V. \tag{17}$$

Using the conformal mapping $z \mapsto z^{-1}$, one can deduce from (9) the following result: if $U \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ and $\Delta U = 0$ in a neighborhood of $R = \infty$ then

$$\Big(U \in \mathrm{W}^s_\infty(\Omega) \Longleftrightarrow U = \mathcal{O}_{R\to+\infty}(R^s)\Big) \qquad\qquad \text{if } s \notin \lambda\mathbb{Z}^*. \tag{18}$$

The following two lemmas are the counterparts of lemmas 1 and 2.

Lemma 3 *For any* $F \in \mathrm{L}^2(\Omega)$ *with compact support in* $\overline{\Omega}$ *and any finite sequence* $(A_{p\lambda})_{p\in\mathbb{N}^*}$ *of real numbers, there exists a unique* $U \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ *such that*

$$\Delta U = -F \textit{ in } \Omega \quad \textit{and} \quad U - \sum_{p=1}^{+\infty} A_{p\lambda}\; \mathfrak{s}^{p\lambda} = \mathcal{O}_{R\to+\infty}(1). \tag{19}$$

PROOF. It is very similar to lemma 1, the suitable variational space being $\mathrm{W}^0_\infty(\Omega)$. ■

Lemma 4 *For any function* $U \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ *such that*

$$\Delta U = 0 \textit{ in a neighborhood of } R = +\infty \quad \textit{and} \quad \exists s \in \mathbb{R}, \quad U = \mathcal{O}_{R\to+\infty}(R^s), \tag{20}$$

there exist a unique finite sequence $(A_{p\lambda})_{p\in\mathbb{N}^*}$ *and a unique sequence* $(B_{p\lambda})_{p\in\mathbb{N}^*}$ *(generically infinite) such that for any* $N \in \mathbb{N}$

$$U(R,\theta) = \sum_{p=1}^{+\infty} A_{p\lambda}\; \mathfrak{s}^{p\lambda}(R,\theta) \;+\; \sum_{p=1}^{N} B_{p\lambda}\; \mathfrak{s}^{-p\lambda}(R,\theta) \;+\; \mathcal{O}_{R\to+\infty}(R^{-(N+1)\lambda}), \tag{21}$$

The series $\sum A_{p\lambda}\; \mathfrak{s}^{p\lambda}$ *and* $\sum B_{p\lambda}\; \mathfrak{s}^{-p\lambda}$ *are called the* growing part *and* non-growing part *of* U*, respectively.*

3 Matching of asymptotic expansions

3.1 Formal derivation of the asymptotic expansions

We will represent the solution u_ε as a formal series in each zone of interest, that is the *corner expansion* near the origin O and the *outer expansion* away from O. Respectively, it reads:

$$u^\varepsilon(r,\theta) \;\simeq\; \sum_{i=-\infty}^{+\infty} \varepsilon^{i\lambda}\; U^{i\lambda}(\tfrac{r}{\varepsilon},\theta) \quad \text{ and } \quad u^\varepsilon(r,\theta) \;\simeq\; \sum_{i=-\infty}^{+\infty} \varepsilon^{i\lambda}\; u^{i\lambda}(r,\theta). \tag{22}$$

(we shall see in section 4 that it is natural to write expansions into powers of ε^λ). Since the H^1-norm of u^ε is uniformly bounded with respect to ε, we know that all the $u^{i\lambda}$ and $U^{i\lambda}$ for $i<0$ are just zero. Moreover, it is clear that the terms of the asymptotic expansions must satisfy:

$$\begin{cases} -\Delta u^0 \;=\; f \text{ in } \omega \;\text{ and } u^0=0 \text{ on } \partial\omega, \\ \forall i>0,\; \Delta u^{i\lambda} \;=\; 0 \quad \text{for } i>0 \;\text{ in } \omega \;\text{ and } u^{i\lambda}=0 \text{ on } \partial\omega, \\ \forall i\geqslant 0,\; \Delta U^{i\lambda} \;=\; 0 \text{ and } U^{i\lambda}=0 \text{ on } \partial\Omega. \end{cases} \tag{23}$$

Now we need to ensure the matching of the two formal series in the transition zone:

$$\varepsilon \ll r \ll 1. \tag{24}$$

To do so, we expand the terms $u^{i\lambda}$ and $U^{i\lambda}$. Thanks to lemmas 2 and 4 – note that $r\ll 1$ and $\frac{r}{\varepsilon}\gg 1$ – theses expansions read:

$$\begin{cases} u^{i\lambda} = \displaystyle\sum_{p=1}^{+\infty}\Big(a^{i\lambda}_{p\lambda}\,\mathfrak{s}^{-p\lambda}(r,\theta) \;+\; b^{i\lambda}_{p\lambda}\,\mathfrak{s}^{p\lambda}(r,\theta)\Big), \\ U^{i\lambda} = \displaystyle\sum_{p=1}^{+\infty}\Big(A^{i\lambda}_{p\lambda}\,\mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta) \;+\; B^{i\lambda}_{p\lambda}\,\mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta)\Big). \end{cases} \tag{25}$$

We use the homogeneity of the functions $\mathfrak{s}^{p\lambda}$ and transform the fast variable $\frac{r}{\varepsilon}$ into the slow one r. Ensuring the equality of the two formal series (22), we get:

$$\begin{cases} \displaystyle\sum_i \Big(\varepsilon^{i\lambda}\sum_p\Big(a^{i\lambda}_{p\lambda}\mathfrak{s}^{-p\lambda}(r,\theta) \;+\; b^{i\lambda}_{p\lambda}\mathfrak{s}^{p\lambda}(r,\theta)\Big)\Big) \\ \qquad = \displaystyle\sum_i \Big(\varepsilon^{i\lambda}\sum_p\Big(\varepsilon^{-p\lambda}\,A^{i\lambda}_{p\lambda}\,\mathfrak{s}^{p\lambda}(r,\theta) \;+\; \varepsilon^{p\lambda}\,B^{i\lambda}_{p\lambda}\,\mathfrak{s}^{-p\lambda}(r,\theta)\Big)\Big) \\ \qquad = \displaystyle\sum_i \Big(\varepsilon^{i\lambda}\sum_p\Big(A^{(i+p)\lambda}_{p\lambda}\,\mathfrak{s}^{p\lambda}(r,\theta) \;+\; B^{(i-p)\lambda}_{p\lambda}\,\mathfrak{s}^{-p\lambda}(r,\theta)\Big)\Big). \end{cases} \tag{26}$$

Identifying the terms of the two series leads to (with $b^{(i-p)\lambda}_{p\lambda} = B^{(i-p)\lambda}_{p\lambda} \;=\; 0$ for $p>i$):

$$A^{i\lambda}_{p\lambda} \;=\; b^{(i-p)\lambda}_{p\lambda} \text{ and } a^{i\lambda}_{p\lambda} \;=\; B^{(i-p)\lambda}_{p\lambda} \tag{27}$$

i. e.

$$\begin{cases} a^{i\lambda}_{p\lambda} \;=\; B^{(i-p)\lambda}_{p\lambda} \text{ if } p\leqslant i \text{ and } a^{i\lambda}_{p\lambda} \;=\; 0 \text{ if } p>i, \\ A^{i\lambda}_{p\lambda} \;=\; b^{(i-p)\lambda}_{p\lambda} \text{ if } p\leqslant i \text{ and } A^{i\lambda}_{p\lambda} \;=\; 0 \text{ if } p>i. \end{cases} \tag{28}$$

Remark 1 *Here, we have choosen to derive the matching relations without any knowledge of the matched asymptotic technique. However, one can derive the relations* (28) *using the Van Dyke principle, see* [11].

3.2 Definition of the asymptotic terms

For $i\in\mathbb{N}$, the functions $u^{i\lambda}$ and $U^{i\lambda}$ are defined inductively: the following algorithm defines step by step $u^{i\lambda}:\omega\to\mathbb{R}$, $U^{i\lambda}:\Omega\to\mathbb{R}$, $b^{i\lambda}=(b^{i\lambda}_{p\lambda})_{p\in\mathbb{N}^*}$, and $B^{i\lambda}=(B^{i\lambda}_{p\lambda})_{p\in\mathbb{N}^*}$ for $i\in\mathbb{N}$.

Step 0: $u^0\in H^1_*(\omega)$ is defined via lemma 1 as the unique function satisfying:

$$\Delta u^0 \;=\; -f, \qquad \text{in } \omega, \qquad u^0 \;=\; O_{r\to 0}(1), \qquad \text{on } \partial\omega. \tag{29}$$

Moreover, U^0 is chosen to be 0. Let b^0 and B^0 be the sequences of numbers defined by lemmas 2 and 4:

$$b^0 \;=\; (b^0_{p\lambda})_{p\in\mathbb{N}^*} \text{ and } B^0 \;=\; (B^0_{p\lambda})_{p\in\mathbb{N}^*}=0. \tag{30}$$

Step i: We denote by $a^{i\lambda}$ and $A^{i\lambda}$ the two finite sequences of real numbers such that:

$$\left\{\begin{array}{l} a^{i\lambda} \;=\; (a^{i\lambda}_{p\lambda})_{p\in\mathbb{N}^*} \text{ with } a^{i\lambda}_{p\lambda} \;=\; B^{(i-p)\lambda}_{p\lambda} \text{ if } 1\leqslant p\leqslant i \text{ and } a^{i\lambda}_{p\lambda} \;=\; 0 \text{ if } p>i,\\ A^{i\lambda} \;=\; (A^{i\lambda}_{p\lambda})_{p\in\mathbb{N}^*} \text{ with } A^{i\lambda}_{p\lambda} \;=\; b^{(i-p)\lambda}_{p\lambda} \text{ if } 1\leqslant p\leqslant i \text{ and } A^{i\lambda}_{p\lambda} \;=\; 0 \text{ if } p>i.\end{array}\right. \tag{31}$$

The functions $u^{i\lambda}$ and $U^{i\lambda}$ are defined via lemmas 1 and 3 as the unique solutions of:

$$\left\{\begin{array}{l} \text{Find } u^{i\lambda}\in \mathrm{H}^1_*(\omega) \text{ such that } \Delta u^{i\lambda} \;=\; 0 \text{ in } \omega \text{ and } u^{i\lambda}-\displaystyle\sum_{p=1}^{+\infty} a^{i\lambda}_{p\lambda}\; \mathfrak{s}^{-p\lambda} = O_{r\to 0}(1),\\ \text{Find } U^{i\lambda}\in H^1_{0,loc}(\Omega) \text{ such that } \Delta U^{i\lambda}=0 \text{ in } \Omega \text{ and } U^{i\lambda}-\displaystyle\sum_{p=1}^{+\infty} A^{i\lambda}_{p\lambda}\; \mathfrak{s}^{p\lambda} = O_{R\to +\infty}(1).\end{array}\right. \tag{32}$$

Finally, we define the sequences $b^{i\lambda}$ and $B^{i\lambda}$ associated to $u^{i\lambda}$ and $U^{i\lambda}$ in the lemmas 2 and 4:

$$b^{i\lambda} \;=\; (b^{i\lambda}_{p\lambda})_{p\in\mathbb{N}^*} \text{ and } B^{i\lambda} \;=\; (B^{i\lambda}_{p\lambda})_{p\in\mathbb{N}^*}. \tag{33}$$

3.3 Global error estimates

The main idea to prove error estimates is to define a global approximation $\widehat{u}^{\varepsilon}_{n\lambda}\in \mathrm{H}^1_0(\omega_\varepsilon)$ of u^ε:

$$\widehat{u}^{\varepsilon}_{n\lambda}(r,\theta) \;=\; \varphi\big(\tfrac{r}{\eta(\varepsilon)}\big)\sum_{i=0}^{n}\varepsilon^{i\lambda}\; u^{i\lambda}(r,\theta)+\; \Big(1-\varphi\big(\tfrac{r}{\eta(\varepsilon)}\big)\Big)\sum_{i=0}^{n}\varepsilon^{i\lambda}\; U^{i\lambda}(\tfrac{r}{\varepsilon},\theta), \tag{34}$$

where φ is a cut-off function with $\varphi(\rho)=0$ for $\rho<1$ and $\varphi(\rho)=1$ for $\rho>2$ and η is a regular function of ε such that:

$$\lim_{\varepsilon\to 0}\eta(\varepsilon)=0 \text{ and } \lim_{\varepsilon\to 0}\frac{\eta(\varepsilon)}{\varepsilon}=+\infty. \tag{35}$$

Theorem 1 *There exists a constant C such that:*

$$\|u_\varepsilon-\widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \;\leqslant\; C\;\Big[\Big(\eta(\varepsilon)\Big)^{(n+1)\lambda}+\Big(\frac{\varepsilon}{\eta(\varepsilon)}\Big)^{(n+1)\lambda}\Big]. \tag{36}$$

PROOF. First, we denote by $\widehat{e}^{\varepsilon}_{n\lambda}$ the error at step n and by $\mathcal{E}^{\varepsilon}_{n\lambda}$ the matching error of order n, i.e.

$$\widehat{e}^{\varepsilon}_{n\lambda}(r,\theta) \;=\; \widehat{u}^{\varepsilon}_{n\lambda}(r,\theta)-u_\varepsilon(r,\theta) \text{ and } \mathcal{E}^{\varepsilon}_{\lambda n}(r,\theta) \;=\; \sum_{i=0}^{n}\varepsilon^{i\lambda}\;\big[u^{i\lambda}(r,\theta)\;-\;U^{i\lambda}(\tfrac{r}{\varepsilon},\theta)\big]. \tag{37}$$

Of course, the matching error is small only in the intermediate region; we shall express the H^1-norm of $\widehat{e}^{\varepsilon}_{n\lambda}$ over ω_ε in terms of $\mathcal{E}^{\varepsilon}_{n\lambda}$ in this region. Using harmonicity of u_ε and of the $u^{i\lambda}$ and $U^{i\lambda}$, we obtain:

$$\Delta\widehat{e}^{\varepsilon}_{n\lambda}(r,\theta)=\tfrac{2}{\eta(\varepsilon)}\;[\nabla\varphi](\tfrac{r}{\eta(\varepsilon)})\;\nabla\mathcal{E}^{\varepsilon}_{n\lambda}(r,\theta)\;+\;\tfrac{1}{[\eta(\varepsilon)]^2}[\Delta\varphi](\tfrac{r}{\eta(\varepsilon)})\;\mathcal{E}^{\varepsilon}_{n\lambda}(r,\theta). \tag{38}$$

Since $\widehat{e}^{\varepsilon}_{n\lambda}$ belongs to $\mathrm{H}^1_0(\omega_\varepsilon)$, the Green formula leads to:

$$\left\{\begin{array}{rcl} \displaystyle\int_{\omega_\varepsilon}\big(\nabla\widehat{e}^{\varepsilon}_{n\lambda}\big)^2 &=& \tfrac{2}{\eta(\varepsilon)}\displaystyle\int_{\omega_\varepsilon}[\nabla\varphi](\tfrac{r}{\eta(\varepsilon)})\;\nabla\mathcal{E}^{\varepsilon}_{n\lambda}\;\widehat{e}^{\varepsilon}_{n\lambda}\;+\;\tfrac{1}{[\eta(\varepsilon)]^2}\displaystyle\int_{\omega_\varepsilon}[\Delta\varphi](\tfrac{r}{\eta(\varepsilon)})\;\mathcal{E}^{\varepsilon}_{n\lambda}\;\widehat{e}^{\varepsilon}_{n\lambda}\\ &\leqslant& \tfrac{C}{[\eta(\varepsilon)]^2}\;\Big[\|\mathcal{E}^{\varepsilon}_{n\lambda}\|_{\infty,\eta}+\eta(\varepsilon)\|\nabla\mathcal{E}^{\varepsilon}_{n\lambda}\|_{\infty,\eta}\Big]\|\widehat{e}^{\varepsilon}_{n\lambda}\|_{1,\eta},\end{array}\right. \tag{39}$$

with

$$\|u\|_{p,\eta} \;=\; \|u\|_{\mathrm{L}^p(\{(r,\theta)\in\omega\;;\;\eta(\varepsilon)\;\leqslant\; r\;\leqslant\; 2\eta(\varepsilon)]\})}. \tag{40}$$

Using a Poincar´e inequality on ω_ε (uniform with respect to ε), we get:

$$\|\widehat{e}^{\varepsilon}_{n\lambda}\|^2_{\mathrm{H}^1(\omega_\varepsilon)} \;\leqslant\; \frac{C}{(\eta(\varepsilon))^2}\;\Big[\|\mathcal{E}^{\varepsilon}_{n\lambda}\|_{\infty,\eta}+\eta(\varepsilon)\|\nabla\mathcal{E}^{\varepsilon}_{n\lambda}\|_{\infty,\eta}\Big]\times\|\widehat{e}^{\varepsilon}_{n\lambda}\|_{1,\eta}. \tag{41}$$

The conclusion follows from the following two lemmas (proved below). ■

Lemma 5 *There exists a constant C such that for all $u \in \mathrm{H}^1_0(\omega_\varepsilon)$, the norm $\|u\|_{1,\eta}$, defined by* (40) *can be estimated as follows*

$$\|u\|_{1,\eta} \;\leqslant\; C\,[\eta(\varepsilon)]^2\,\|u\|_{\mathrm{H}^1(\omega_\varepsilon)}. \tag{42}$$

Lemma 6 *There exists a constant C such that – for the definition of the norms, see* (40)

$$\|\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta} \;\leqslant\; C\,\Big[\Big(\eta(\varepsilon)\Big)^{(n+1)\lambda} + \Big(\frac{\varepsilon}{\eta(\varepsilon)}\Big)^{(n+1)\lambda}\Big], \tag{43}$$

$$\|\nabla\mathcal{E}^\varepsilon_{n\lambda}\|_{\infty,\eta} \;\leqslant\; C\,\frac{1}{\eta(\varepsilon)}\,\Big[\Big(\eta(\varepsilon)\Big)^{(n+1)\lambda} + \Big(\frac{\varepsilon}{\eta(\varepsilon)}\Big)^{(n+1)\lambda}\Big]. \tag{44}$$

One can optimize the estimate (36) by choosing the best η:

Corollary 1 *For $\eta = \varepsilon^{\frac{1}{2}}$, there exists a constant C such that:*

$$\|u^\varepsilon - \widehat{u}^\varepsilon_{n\lambda}\|_{H^1(\omega_\varepsilon)} \;\leqslant\; C\,\varepsilon^{\lambda\frac{n+1}{2}}. \tag{45}$$

PROOF OF LEMMA 5. For all $u \in \mathrm{H}^1_0(\omega_\varepsilon)$ and for all $r \in [\eta(\varepsilon); 2\eta(\varepsilon)]$,

$$\int_0^\alpha |u(r,\theta)|d\theta \;\leqslant\; \int_0^\alpha \Big[\int_0^\theta |\frac{\partial u}{\partial\theta}(r,\theta')|d\theta'\Big]d\theta \leqslant \alpha\int_0^\alpha |\frac{\partial u}{\partial\theta}(r,\theta)|\;r\;dr\;d\theta. \tag{46}$$

Hence, we have:

$$\int_{\eta(\varepsilon)}^{2\eta(\varepsilon)}\int_0^\alpha |u(r,\theta)|\;r\;dr\;d\theta \quad\leqslant\quad \alpha\int_{\eta(\varepsilon)}^{2\eta(\varepsilon)}\int_0^\alpha |\frac{\partial u}{\partial\theta}(r,\theta)|r\;dr\;d\theta\;d\theta \quad\leqslant\quad C\;\eta(\varepsilon)\;\|\nabla u\|_{1,\eta}. \tag{47}$$

We conclude using the Cauchy-Schwartz inequality that:

$$\|u\|_{1,\eta} \;\leqslant\; C\;\eta(\varepsilon)\;\|\nabla u\|_{1,\eta} \;\leqslant\; C\;[\eta(\varepsilon)]^2\;\|\nabla u\|_{2,\eta}. \tag{48}$$

■

PROOF OF LEMMA 6. We will give the proof of (43); the inequality (44) can be obtained using the same technique. The first step is to expand the $u^{i\lambda}$ and $U^{i\lambda}$ using (13) and (21). By definition of $u^{i\lambda}$ and $U^{i\lambda}$ -see (31) and (32)- the $a^{i\lambda}_{p\lambda}$ et $A^{i\lambda}_{p\lambda}$ are given by:

$$\left\{\begin{array}{rcl} a^{i\lambda}_{p\lambda} &=& B^{(i-p)\lambda}_{p\lambda} \text{ for } 1 \leqslant p \leqslant i \text{ and } a^{i\lambda}_{p\lambda} = 0 \text{ for } p > i, \\ A^{i\lambda}_{p\lambda} &=& b^{(i-p)\lambda}_{p\lambda} \text{ for } 1 \leqslant p \leqslant i \text{ and } A^{i\lambda}_{p\lambda} = 0 \text{ for } p > i. \end{array}\right. \tag{49}$$

One has:

$$\left\{\begin{array}{rcl} u^{i\lambda}(r,\theta) &=& \displaystyle\sum_{p=1}^{i} B^{(i-p)\lambda}_{p\lambda}\;\mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{n-i} b^{i\lambda}_{p\lambda}\;\mathfrak{s}^{p\lambda}(r,\theta) \;+\; O_{r\to 0}(r^{(n+1-i)\lambda}), \\ U^{i\lambda}(R,\theta) &=& \displaystyle\sum_{p=1}^{i} b^{(i-p)\lambda}_{p\lambda}\;\mathfrak{s}^{p\lambda}(R,\theta) + \sum_{p=1}^{n-i} B^{i\lambda}_{p\lambda}\;\mathfrak{s}^{-p\lambda}(R,\theta) \;+\; O_{R\to+\infty}(R^{-(n+1-i)\lambda}). \end{array}\right. \tag{50}$$

As $\eta(\varepsilon)$ tends to 0 and $\eta(\varepsilon)/\varepsilon$ tends to ∞ when ε tends to 0, one has for $\eta(\varepsilon) \leqslant r \leqslant 2\eta(\varepsilon)$:

$$\left\{\begin{array}{l} \Big|u^{i\lambda}(r,\theta) - \displaystyle\sum_{p=1}^{i} B^{(i-p)\lambda}_{p\lambda}\;\mathfrak{s}^{-p\lambda}(r,\theta) - \sum_{p=1}^{n-i} b^{i\lambda}_{p\lambda}\;\mathfrak{s}^{p\lambda}(r,\theta)\Big| \;\leqslant\; C\;[\eta(\varepsilon)]^{(n+1-i)\lambda}, \\ \Big|U^{i\lambda}(\tfrac{r}{\varepsilon},\theta) \;-\; \displaystyle\sum_{p=1}^{i} b^{(i-p)\lambda}_{p\lambda}\;\mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta) + \sum_{p=1}^{n-i} B^{i\lambda}_{p\lambda}\;\mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta)\Big| \;\leqslant\; C\;[\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1-i)\lambda}. \end{array}\right. \tag{51}$$

From (51) and triangular inequalities, we obtain:

$$\|\mathcal{E}^\varepsilon_{n\lambda}(r,\theta) - S\|_{\infty,\eta} \;\leqslant\; C \sum_{i=0}^{n} \varepsilon^{i\lambda}\; [\eta(\varepsilon)]^{(n+1-i)\lambda} \;+\; C \sum_{i=0}^{n} \varepsilon^{i\lambda}\; [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1-i)\lambda} \tag{52}$$

where S is given by:

$$\begin{aligned} S \;\;=\;\; & \sum_{i=0}^{n} \varepsilon^{i\lambda} \Big(\sum_{p=1}^{i} B_{p\lambda}^{(i-p)\lambda}\; \mathfrak{s}^{-p\lambda}(r,\theta) + \sum_{p=1}^{n-i} b_{p\lambda}^{i\lambda}\; \mathfrak{s}^{p\lambda}(r,\theta)\Big) \\ & - \sum_{i=0}^{n} \varepsilon^{i\lambda} \Big(\sum_{p=1}^{i} b_{p\lambda}^{(i-p)\lambda}\; \mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta) + \sum_{p=1}^{n-i} B_{p\lambda}^{i\lambda}\; \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta)\Big). \end{aligned} \tag{53}$$

One can now estimate the right hand side of (52):

$$\|\mathcal{E}^\varepsilon_{n\lambda}(r,\theta) - S\|_{\infty,\eta} \;\leqslant\; C \sum_{i=0}^{n} [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{i\lambda}\; [\eta(\varepsilon)]^{(n+1)\lambda} \;+\; C \sum_{i=0}^{n} \eta(\varepsilon)^{i\lambda}\; [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1)\lambda}. \tag{54}$$

Hence, one has:

$$\|\mathcal{E}^\varepsilon_{n\lambda}(r,\theta) - S\|_{\infty,\eta} \;\leqslant\; C\, \Big[[\eta(\varepsilon)]^{(n+1)\lambda} + [\tfrac{\varepsilon}{\eta(\varepsilon)}]^{(n+1)\lambda} \Big]. \tag{55}$$

Now, we will show that $S = 0$. By definition (see (5)) $\mathfrak{s}^{p\lambda}$ satisfies:

$$\mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta) \;=\; \varepsilon^{p\lambda}\; \mathfrak{s}^{-p\lambda}(r,\theta) \text{ and } \mathfrak{s}^{p\lambda}(r,\theta) \;=\; \varepsilon^{p\lambda}\; \mathfrak{s}^{p\lambda}(\tfrac{r}{\varepsilon},\theta). \tag{56}$$

Therefore, S is given by:

$$\begin{aligned} S \;\;=\;\; & \Big(\sum_{i=0}^{n}\sum_{p=1}^{i} \varepsilon^{(i-p)\lambda} B_{p\lambda}^{(i-p)\lambda}\; \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta)\Big) \\ & + \Big(\sum_{i=0}^{n}\sum_{p=1}^{n} \varepsilon^{i\lambda} b_{p\lambda}^{i\lambda}\; \mathfrak{s}^{p\lambda}(r,\theta)\Big) \\ & - \Big(\sum_{i=0}^{n}\sum_{p=1}^{i} \varepsilon^{(i-p)\lambda} b_{p\lambda}^{(i-p)\lambda}\; \mathfrak{s}^{p\lambda}(r,\theta)\Big) \\ & - \Big(\sum_{i=0}^{n}\sum_{p=1}^{n} \varepsilon^{i\lambda} B_{p\lambda}^{i\lambda}\; \mathfrak{s}^{-p\lambda}(\tfrac{r}{\varepsilon},\theta)\Big). \end{aligned} \tag{57}$$

The change of variables $i-p \mapsto i$ in the second and third terms leads to $S = 0$. The conclusion comes from (55). ∎

3.4 Local error estimates

In this paragraph B_r will denote the ball of radius r and of center O. Starting from the global error estimates obtained in (45), it is easy to get estimates far from the corner and near the corner:

Theorem 2 *For any $r_0 > 0$, there exists $C > 0$ such that:*

$$\Big\| u_\varepsilon(r,\theta) - \sum_{i=0}^{n} \varepsilon^{i\lambda}\; u^{i\lambda}(r,\theta) \Big\|_{\mathrm{H}^1(\omega\setminus B_{r_0})} \;=\; O(\varepsilon^{(n+1)\lambda}). \tag{58}$$

For any $R_0 > 0$, there exists $C > 0$ such that:

$$\Big\| u_\varepsilon(\varepsilon R,\theta) - \sum_{i=0}^{n} \varepsilon^{i\lambda}\; U^{i\lambda}(R,\theta) \Big\|_{\mathrm{H}^1(\Omega\cap B_{R_0})} \;=\; O(\varepsilon^{(n+1)\lambda}). \tag{59}$$

PROOF. To prove (58), we remark that, for ε small enough, the only contribution comes from the terms $u^{i\lambda}$:

$$\widehat{u}^{\varepsilon}_{n\lambda} \;=\; \sum_{i=1}^{n} \varepsilon^{i\lambda}\; u^{i\lambda} \text{ in } \omega_\varepsilon \setminus B_{r_0}. \tag{60}$$

Consequently,

$$\begin{aligned} \|u_\varepsilon - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus B_{r_0})} &\leqslant \|u_\varepsilon - \widehat{u}^{\varepsilon}_{(2n+2)\lambda}\|_{\mathrm{H}^1(\omega\setminus B_{r_0})} \;+\; \|\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus B_{r_0})} \\ &\leqslant \|u_\varepsilon - \widehat{u}^{\varepsilon}_{(2n+2)\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \;+\; \|\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega\setminus B_{r_0})}. \end{aligned} \tag{61}$$

On the other hand, it follows from (60)

$$\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda} \;=\; \sum_{i=n+1}^{2n+2} \varepsilon^{i\lambda}\; u^{i\lambda}, \tag{62}$$

and, as the $u^{i\lambda}$ do not depend on ε:

$$\|\widehat{u}^{\varepsilon}_{(2n+2)\lambda} - \widehat{u}^{\varepsilon}_{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \;\leqslant\; C\,\varepsilon^{(n+1)\lambda} \tag{63}$$

Due to (45), one finally has:

$$\|u_\varepsilon - \widehat{u}^{\varepsilon}_{(2n+2)\lambda}\|_{\mathrm{H}^1(\omega\setminus B_{r_0})} \;\leqslant\; C\,\varepsilon^{(n+1)\lambda}. \tag{64}$$

The estimate (58) follows from (60), (61), (63) and (64). Estimate (59) can be proved using the same technique; a scaling is needed ($R = r/\varepsilon$) to recover a domain independent on ε. ■

Remark 2 Due to estimates (58) and (59), the outer and corner expansions are unique. Moreover, as the remainders are of the same orders as the first neglected term in the outer and corner expansions, these estimates are optimal. The outer and corner expansions can be seen as Taylor expansions of the exact solution expressed in the (r,θ) or $(r/\varepsilon,\theta)$ coordinates. □

4 Multiscale technique

4.1 Introduction

The technique of *multiscale expansion* consists in building a global approximation of the solution in the domain ω_ε. The expansion is composed of two different types of terms: the ones involving the original variable x, and the *profiles* appearing in the scaled variable $\frac{x}{\varepsilon}$. They are superposed via cut-off functions:

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon}) \sum_{i=0}^{n} \varepsilon^{i\lambda} v^{i\lambda}(x) + \psi(x) \sum_{i=0}^{n} \varepsilon^{i\lambda} V^{i\lambda}(\tfrac{x}{\varepsilon}) + o(\varepsilon^{n\lambda}), \tag{65}$$

where the functions χ and ψ are smooth and radial, satisfying

$$\begin{aligned} &\chi(X) = 1 \text{ for } |X| > R^* \quad \text{and} \quad \chi(X) = 0 \text{ for } |X| < \tfrac{R^*}{2}; \\ &\psi(x) = 1 \text{ for } |x| < \tfrac{r^*}{2} \quad \text{and} \quad \psi(x) = 0 \text{ for } |x| > r^*. \end{aligned} \tag{66}$$

Obviously, the first sum in (65) has its support away from the corner point and, conversely, the second brings a contribution only in a neighborhood of the corner. The two sums have to be taken into account simultaneously only in the intermediate region $\operatorname{supp}\chi(\frac{\cdot}{\varepsilon}) \cap \operatorname{supp}\psi$.

4.2 The construction of the terms

We first focus on the construction of the first terms and then we give the general algorithm which allows to build v^λ and V^λ arising in (65).

4.2.1 The first terms

Step 0. If $v^0 \in \mathrm{H}^1_0(\omega)$ denotes the solution of $-\Delta v^0 = f$, then v^0 seems to be a good starting point for the expansion. Nevertheless, it is defined on the domain ω, and not ω_ε. For this reason, we consider the truncated function $\tilde{v}^0 = \chi(\frac{x}{\varepsilon})v^0$. Besides, we set $V^0 = 0$.

Step 1. By construction, the laplacian of the remainder $r^0_\varepsilon = \tilde{v}^0 - v_\varepsilon$ satisfies

$$\Delta r^0_\varepsilon = \Delta[v^0(x)\chi(\tfrac{x}{\varepsilon})]\mathbb{I}_{\mathrm{supp}\,\nabla\chi(\cdot/\varepsilon)}, \tag{67}$$

whose leading term comes from the first singular term in the splitting into regular and singular parts of v^0 (see Lemma 2):

$$v^0(x) = \mathbf{b}^0_\lambda\ \mathfrak{s}^\lambda(x) + O_{|x|\to 0}(|x|^{2\lambda}). \tag{68}$$

Using the homogeneity of the singular function $\mathfrak{s}^\lambda$, the leading term in (67) is nothing but

$$\mathbf{b}^0_\lambda\ \varepsilon^\lambda\ \Delta[\mathfrak{s}^\lambda(\tfrac{x}{\varepsilon})\chi(\tfrac{x}{\varepsilon})]. \tag{69}$$

Thanks to Lemma 4, we are able to define a profile $V^\lambda \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega)$ such that

$$\Delta V^\lambda(X) = -\mathbf{b}^0_\lambda \Delta[\mathfrak{s}^\lambda(X)\chi(X)] \quad \text{in } \Omega \quad \text{and } V^\lambda(X) = O_{|X|\to+\infty}(1), \tag{70}$$

and a better start for the asymptotic expansion reads

$$v_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})v^0(x) + \psi(x)\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) + r^\lambda_\varepsilon(x). \tag{71}$$

We recover the beginning of expansion (65), provided we have set $v^\lambda = 0$.

Step 2. The remainder r^λ_ε satisfies

$$\Delta r^\lambda_\varepsilon = \Delta[v^0\chi(\tfrac{x}{\varepsilon})]\mathbb{I}_{\mathrm{supp}\,\nabla\chi(\cdot/\varepsilon)} + \mathbf{b}^0_\lambda\varepsilon^\lambda\Delta[\psi(x)V^\lambda(\tfrac{x}{\varepsilon})]. \tag{72}$$

To write the leading term of $\Delta r^\lambda_\varepsilon$, we expand v^0 into singular functions near the corner, as well as V^λ at infinity:

$$v^0(x) = \mathbf{b}^0_\lambda\mathfrak{s}^\lambda(x) + \mathbf{b}^0_{2\lambda}\mathfrak{s}^{2\lambda}(x) + O_{|x|\to 0}(|x|^{3\lambda}). \tag{73}$$

$$V^\lambda(X) = \mathbf{B}^\lambda_\lambda\mathfrak{s}^{-\lambda}(X) + O_{|X|\to+\infty}(|X|^{-2\lambda}). \tag{74}$$

Considering relation (70), the leading term in (72) is therefore

$$\mathbf{b}^0_{2\lambda}\varepsilon^{2\lambda}\Delta[\mathfrak{s}^{2\lambda}(\tfrac{x}{\varepsilon})\chi(\tfrac{x}{\varepsilon})] + \mathbf{B}^\lambda_\lambda\varepsilon^{2\lambda}\Delta[\psi(x)\mathfrak{s}^{-\lambda}(x)]. \tag{75}$$

Hence, we define $v^{2\lambda}$ and $V^{2\lambda}$ by

$$v^{2\lambda} \in \mathrm{H}^1_0(\omega) \text{ and } \Delta v^{2\lambda}(x) = -\mathbf{B}^\lambda_\lambda\Delta(\psi(x)s^{-\lambda}(x). \tag{76}$$

$$V^{2\lambda} \in \mathrm{H}^1_{0,\mathrm{loc}}(\Omega),\ \Delta V^{2\lambda}(X) = -\mathbf{b}^0_{2\lambda}\Delta[\mathfrak{s}^{2\lambda}(\tfrac{x}{\varepsilon})\chi(\tfrac{x}{\varepsilon})], \text{ and } V^{2\lambda}(X) = O_{|X|\to+\infty}(1). \tag{77}$$

Then, the beginning of the asymptotic expansion becomes:

$$v_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})\left[v^0(x) + \varepsilon^{2\lambda}v^{2\lambda}(x)\right] + \psi(x)\left[\varepsilon^\lambda V^\lambda(\tfrac{x}{\varepsilon}) + \varepsilon^{2\lambda}V^{2\lambda}(\tfrac{x}{\varepsilon})\right] + r^{2\lambda}_\varepsilon(x). \tag{78}$$

4.2.2 The general construction

Let us assume the asymptotic expansion built up to order $n-1$, i.e.

$$v_\varepsilon(x) = \chi(\tfrac{x}{\varepsilon})\sum_{i=0}^{n-1}\varepsilon^{i\lambda}v^{i\lambda}(x) + \psi(x)\sum_{i=1}^{n-1}\varepsilon^{i\lambda}V^{i\lambda}(\tfrac{x}{\varepsilon}) + r^{(n-1)\lambda}_\varepsilon(x) \tag{79}$$

For $i = 0, \ldots, n-1$, we expand the term $v^{i\lambda}$ into singular functions at the corner point (see lemma 2):

$$v^{i\lambda}(x) = \sum_{p=1}^{n-i} \mathbf{b}_{p\lambda}^{i\lambda} \mathfrak{s}^{p\lambda}(x) + O_{|x|\to 0}\big(|x|^{(n-i)\lambda}\big) \tag{80}$$

and we also expand the profiles $V^{i\lambda}$ into dual singular functions at infinity (see lemma 4):

$$V^{i\lambda}(X) = \sum_{p=1}^{n-i} \mathbf{B}_{p\lambda}^{i\lambda} \mathfrak{s}^{-p\lambda}(X) + O_{|X|\to\infty}\big(|X|^{-(n-i)\lambda}\big). \tag{81}$$

In the sums (80) and (81), only the term corresponding to $p = n - i$ has not yet been taken into account. The leading term of the remainder laplacian $\Delta r_\varepsilon^{(n-1)\lambda}$ is therefore

$$\Delta r_\varepsilon^{(n-1)\lambda} = \Delta \left[\sum_{i=0}^{n-1} \varepsilon^{i\lambda} \mathbf{b}_{(n-i)\lambda}^{i\lambda} \mathfrak{s}^{(n-i)\lambda}(x) \chi(\tfrac{x}{\varepsilon}) \right] + \Delta \left[\sum_{i=1}^{n-1} \varepsilon^{i\lambda} \mathbf{B}_{(n-i)\lambda}^{i\lambda} \mathfrak{s}^{-(n-i)\lambda}(\tfrac{x}{\varepsilon}) \psi(x) \right] \tag{82}$$

$$= \varepsilon^{n\lambda} \Delta \left[\sum_{i=0}^{n-1} \mathbf{b}_{(n-i)\lambda}^{i\lambda} \mathfrak{s}^{(n-i)\lambda}(\tfrac{x}{\varepsilon}) \chi(\tfrac{x}{\varepsilon}) \right] + \varepsilon^{n\lambda} \Delta \left[\sum_{i=1}^{n-1} \mathbf{B}_{(n-i)\lambda}^{i\lambda} \mathfrak{s}^{-(n-i)\lambda}(x) \psi(x) \right] \tag{83}$$

The definitions for the next terms $v^{n\lambda}$ and $V^{n\lambda}$ follow: $v^{n\lambda} \in \mathrm{H}_0^1(\omega)$ solves

$$\Delta v^{n\lambda}(x) = -\Delta \left[\sum_{i=1}^{n-1} \mathbf{B}_{(n-i)\lambda}^{i\lambda} \mathfrak{s}^{-(n-i)\lambda}(x) \psi(x) \right], \tag{84}$$

and $V^{n\lambda} \in \mathrm{H}_{0,\mathrm{loc}}^1(\Omega)$ satisfies

$$\Delta V^{n\lambda}(X) = -\Delta \left[\sum_{i=0}^{n-1} \mathbf{b}_{(n-i)\lambda}^{i\lambda} \mathfrak{s}^{(n-i)\lambda}(X) \chi(X) \right] \text{ with } V^{n\lambda}(X) = O_{|X|\to+\infty}(1). \tag{85}$$

4.3 Optimal error estimate

Theorem 3 *The solution u_ε of problem* (2) *admits the following multiscale expansion*

$$u_\varepsilon = \chi(\tfrac{x}{\varepsilon}) \sum_{i=0}^{n} \varepsilon^{i\lambda} v^{i\lambda}(x) + \psi(x) \sum_{i=0}^{n} \varepsilon^{i\lambda} V^{i\lambda}(\tfrac{x}{\varepsilon}) + r_\varepsilon^{n\lambda}, \tag{86}$$

where the terms $v^{i\lambda}$ and $V^{i\lambda}$ do not depend on ε, and are defined in ω and Ω, respectively. Moreover, the remainder $r_\varepsilon^{n\lambda}$ satisfies the following estimate

$$\|r_\varepsilon^{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \le C\varepsilon^{(n+1)\lambda}. \tag{87}$$

PROOF. A basic idea to estimate the remainder consists in investigating the Laplace-Dirichlet problem it solves. By construction, $r_\varepsilon^{n\lambda}$ satisfies the homogeneous Dirichlet boundary condition on $\partial\omega_\varepsilon$. Moreover, its laplacian is given by (83) and has to be estimated in the L^2-norm. Thanks to the harmonicity of the singular functions $\mathfrak{s}^{p\lambda}$, we can write

$$\Delta\big(\mathfrak{s}^{(n+1-i)\lambda}(\tfrac{x}{\varepsilon})\chi(\tfrac{x}{\varepsilon})\big) = 2\varepsilon^{-2}\, \nabla\mathfrak{s}^{(n+1-i)\lambda}(\tfrac{x}{\varepsilon}) \cdot \nabla\chi(\tfrac{x}{\varepsilon}) + \varepsilon^{-2}\, \mathfrak{s}^{(n+1-i)\lambda}(\tfrac{x}{\varepsilon})\, \Delta\chi(\tfrac{x}{\varepsilon}), \tag{88}$$

with the following estimates

$$\|\nabla\mathfrak{s}^{(n+1-i)\lambda}(\tfrac{x}{\varepsilon}) \cdot \nabla\chi(\tfrac{x}{\varepsilon})\|_{\mathrm{L}^2(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^{-1}) \quad \text{and} \quad \|\mathfrak{s}^{(n+1-i)\lambda}(\tfrac{x}{\varepsilon})\Delta\chi(\tfrac{x}{\varepsilon})\|_{\mathrm{L}^2(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^{-1}). \tag{89}$$

In the same way,

$$\|\nabla\mathfrak{s}^{-(n+1-i)\lambda}(x) \cdot \nabla\psi(x)\|_{\mathrm{L}^2(\omega_\varepsilon)} = \mathcal{O}(1) \quad \text{and} \quad \|\mathfrak{s}^{-(n+1-i)\lambda}(x)\Delta\psi(x)\|_{\mathrm{L}^2(\omega_\varepsilon)} = \mathcal{O}(1). \tag{90}$$

Therefore, one has:

$$\|\Delta r_\varepsilon^{n\lambda}\|_{\mathrm{L}^2(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^{(n+1)\lambda-3}) + O(\varepsilon^{(n+1)\lambda}) = O(\varepsilon^{(n+1)\lambda-3}). \quad (91)$$

Using an apriori estimate on problem (2), we immediately obtain the bound

$$\|r_\varepsilon^{n\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} \leq C\varepsilon^{(n+1)\lambda-3}. \quad (92)$$

To get (87), we just need to write the asymptotic expansion at order $n+2$:

$$r_\varepsilon^{n\lambda} = r_\varepsilon^{(n+4)\lambda} + \chi(\tfrac{x}{\varepsilon}) \sum_{i=n+1}^{n+4} \varepsilon^{i\lambda} v^{i\lambda}(x) + \psi(x) \sum_{i=n+1}^{n+4} \varepsilon^{i\lambda} V^{i\lambda}(\tfrac{x}{\varepsilon}) \quad (93)$$

Indeed, thanks to (92), $\|r_\varepsilon^{(n+4)\lambda}\|_{\mathrm{H}^1(\omega_\varepsilon)} = \mathcal{O}(\varepsilon^{(n+5)\lambda-1}) = \mathcal{O}(\varepsilon^{(n+1)\lambda})$ since $\lambda > \frac{1}{2}$. The result will be proven as soon as we show the following energy estimates:

$$\|\chi(\tfrac{x}{\varepsilon})v^{i\lambda}(x)\|_{\mathrm{H}^1(\omega_\varepsilon)} = \mathcal{O}(1) \text{ and } \|\psi(x)V^{i\lambda}(\tfrac{x}{\varepsilon})\|_{\mathrm{H}^1(\omega_\varepsilon)} = \mathcal{O}(1). \quad (94)$$

As the left estimate of (94) is easier to obtain than the right one, we will just deal with the latter. We need to use the behavior at infinity of the profile $V^{i\lambda}$. Due to (18), one has:

$$V^{i\lambda} \;=\; O_{R\to\infty}(1) \Longrightarrow \nabla V^{i\lambda} \in \mathrm{L}^2(\Omega) \text{ and } (1+R)^{-1}\; V^{i\lambda} \in \mathrm{L}^2(\Omega). \quad (95)$$

Therfore, one has:

$$\int_{\omega_\varepsilon} \varepsilon^{-2}|\psi(x)\nabla V^{i\lambda}(\tfrac{x}{\varepsilon})|^2 \mathrm{d}x \;=\; \int_\Omega |\psi(\varepsilon X)\nabla V^{i\lambda}(X)|^2 \mathrm{d}X \;\leqslant\; \int_\Omega |\nabla V^{i\lambda}(X)|^2 \mathrm{d}X \;=\; O(1). \quad (96)$$

By the same way, we get:

$$\int_{\omega_\varepsilon} \big(|\psi(x)|^2 + |\nabla\psi(x)|^2\big)|V^{i\lambda}(\tfrac{x}{\varepsilon})|^2 \mathrm{d}x \;\leqslant\; C \int_{\{X\in\Omega\ /\ R\leqslant\frac{r^*}{\varepsilon}\}} \varepsilon^2\; |V^{i\lambda}(R,\theta)|^2 \mathrm{d}X \quad (97)$$

As $R\varepsilon \;\leqslant\; r^*$, one has:

$$\int_{\omega_\varepsilon} \big((\psi(x))^2 + (\nabla\psi(x))^2\big)|V^{i\lambda}(\tfrac{x}{\varepsilon})|^2 \mathrm{d}x \;\leqslant\; C' \int_\Omega \frac{|V^{i\lambda}(X)|^2}{(1+|R|)^2} \mathrm{d}X \;=\; O(1). \quad (98)$$

Estimates (94) follow. ■

5 Conclusion

5.1 Comparison of the two expansions

In section 3, starting from the outer and corner (matched) expansions, we were able to build a global asymptotic expansion for the solution u_ε of problem (2), see expression (34). Using the multiscale technique, we proved in section 4 another asymptotic expansion, which is also valid in the whole domain ω_ε. The global error estimates given in theorems 1 and 3 allow to compare these expansions.

Theorem 4 *The expansions* (34) *and* (86) *compare in the following way:*

- *the terms $u^{n\lambda}$ and $v^{n\lambda}$ coincide away from the corner point i. e. for $r \geqslant r^*$;*
- *the profiles $U^{n\lambda}$ and $V^{n\lambda}$ coincide in the corner region i. e. for $R \leqslant R^*/2$;*

More precisely, we have the identities

$$\left\{\begin{array}{rcrcl} v^{n\lambda}(x) & = & u^{n\lambda}(x) & - & \psi(x)\displaystyle\sum_{p=1}^{n} a_{p\lambda}^{n\lambda}\mathfrak{s}^{-p\lambda}(x), \\ V^{n\lambda}(X) & = & U^{n\lambda}(X) & - & \chi(X)\displaystyle\sum_{p=1}^{n} A_{p\lambda}^{n\lambda}\mathfrak{s}^{p\lambda}(X). \end{array}\right. \tag{99}$$

PROOF. The first two statements follow directly from the optimal estimates, via localization. To get formulas (99), we start from problem (85) which defines $V^{n\lambda}$. We set

$$\widetilde{U}^{n\lambda}(X) = V^{n\lambda}(X) + \chi(X)\sum_{i=0}^{n-1}\mathbf{b}_{(n-i)\lambda}^{i\lambda}\mathfrak{s}^{(n-i)\lambda}(X) \tag{100}$$

$$= V^{n\lambda}(X) + \chi(X)\sum_{p=1}^{n}\mathbf{b}_{p\lambda}^{(n-p)\lambda}\mathfrak{s}^{p\lambda}(X). \tag{101}$$

From the definition of $V^{n\lambda}$ – see (85), $\widetilde{U}^{n\lambda}$ satisfies $\Delta\widetilde{U}^{n\lambda} = 0$ in Ω. Hence, one has:

$$\left\{\begin{array}{l} \widetilde{U}^{n\lambda} - U^{n\lambda} \in C^{\infty}(\omega), \\ \Delta[\widetilde{U}^{n\lambda} - U^{n\lambda})] = 0 \text{ in } \omega, \\ \widetilde{U}^{n\lambda}(R,\theta) - U^{n\lambda}(R,\theta) = 0 \text{ for } R \leqslant R^*/2. \end{array}\right. \tag{102}$$

By unique continuation theorem, $U^{n\lambda} = \widetilde{U}^{n\lambda}$. Moreover, as $V^{n\lambda}$ is a $O_{R\to+\infty}(1)$, one has $A_{p\lambda}^{n\lambda} = \mathbf{b}_{p\lambda}^{(n-p)\lambda}$:

$$U^{n\lambda}(X) = V^{n\lambda}(X) + \chi(X)\sum_{p=1}^{n} A_{p\lambda}^{n\lambda}\mathfrak{s}^{p\lambda}(X). \tag{103}$$

The same argumentation can be done for $u^{n\lambda}$. ■

Remark 3 *As can be seen in* (101)*, another way to link the two expansions is the following:*

$$\left\{\begin{array}{rcrcl} u^{n\lambda}(x) & = & v^{n\lambda}(x) & + & \psi(x)\displaystyle\sum_{p=1}^{n}\mathbf{B}_{p\lambda}^{(n-p)\lambda}\mathfrak{s}^{-p\lambda}(x), \\ U^{n\lambda}(X) & = & V^{n\lambda}(X) & + & \chi(X)\displaystyle\sum_{p=1}^{n}\mathbf{b}_{p\lambda}^{(n-p)\lambda}\mathfrak{s}^{p\lambda}(X). \end{array}\right. \tag{104}$$

Moreover, as $A_{p\lambda}^{n\lambda} = \mathbf{b}_{p\lambda}^{(n-p)\lambda}$ and due to the matching condition (31)*, one has:*

$$\mathbf{B}_{p\lambda}^{i\lambda} = B_{p\lambda}^{i\lambda} \quad \textit{and} \quad \mathbf{b}_{p\lambda}^{i\lambda} = b_{p\lambda}^{i\lambda}, \qquad \forall i\in\mathbb{N} \textit{ and } \forall p\in\mathbb{N}^*. \tag{105}$$

Remark 4 The mechanism to switch from expansion (86) to expansion (34) consists in using the homogeneity of the singular functions $\mathfrak{s}^{p\lambda}$ to pass them from fast variables into slow variables:

$$\psi(x)\sum_{i=0}^{n}\varepsilon^{i\lambda}V^{i\lambda}(\tfrac{x}{\varepsilon}) = \psi(x)\sum_{i=0}^{n}\varepsilon^{i\lambda}\left[U^{i\lambda}(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon})\sum_{p=1}^{i}A_{p\lambda}^{i\lambda}\mathfrak{s}^{p\lambda}(\tfrac{x}{\varepsilon})\right] \tag{106}$$

$$= \psi(x)\sum_{i=0}^{n}\varepsilon^{i\lambda}U^{i\lambda}(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon})\psi(x)\sum_{i=0}^{n}\sum_{p=1}^{i}\varepsilon^{(i-p)\lambda}A_{p\lambda}^{i\lambda}\mathfrak{s}^{p\lambda}(x) \tag{107}$$

$$= \psi(x)\sum_{i=0}^{n}\varepsilon^{i\lambda}U^{i\lambda}(\tfrac{x}{\varepsilon}) - \chi(\tfrac{x}{\varepsilon})\psi(x)\sum_{j=0}^{n}\varepsilon^{j\lambda}\sum_{p=0}^{n-j}\underbrace{A_{p\lambda}^{(p+i)\lambda}}_{=b_{p\lambda}^{i\lambda}}\mathfrak{s}^{p\lambda}(x). \tag{108}$$

The second term involves the slow variable and will contribute to the terms $(u^{i\lambda})$ in the intermediate region.
□

Finally, it turns out that it is very easy to obtain one expansion from the other, via formulas (99). We emphasize however the particularities of each method:

- the matched asymptotic expansion builds outer and corner terms which are canonical, i.e. they do depend only of the domains ω and Ω, and not on cut-off functions, as it is the case for the multiscale technique;
- the multiscale technique gives a straightforward global approximation of the solution, with optimal estimates of the remainder, whereas more work is needed in the case of matched asymptotic expansions.

5.2 Concluding remarks

The obtained expansion of u_ε, cf. (22) or (65), allows for investigating the appearance of corner singularities in u_ε. Starting from (65) at order 0

$$u_\varepsilon \simeq \chi(\tfrac{x}{\varepsilon})u_0 + \psi(x)\varepsilon^{\frac{\pi}{\alpha}}U^0(\tfrac{x}{\varepsilon}), \tag{109}$$

it becomes clear that, pointwise, for $x \neq 0$, the value of $u_\varepsilon(x)$ is approximated by $u_0(x)$. The contribution of the profile is small: the term $\psi(x)\varepsilon^{\frac{\pi}{\alpha}}U^0(\frac{x}{\varepsilon})$ can be seen as a *corner layer*, allowing to match the local geometry near the corner (note that there is no exponential decay in this case). The corner singularities – contained in u_0 – are just "chopped off" for $\varepsilon > 0$ via the cut-off in the scaled variable $\chi(\frac{x}{\varepsilon})$. It is worth noticing that no regularity assumption have been made on the perturbing pattern Ω, which can also present corners (of different opening than α), or even cracks. The resulting singularities are just part of the profiles: the interaction between these singularities and those of the limit problem is completely contained in the profiles, which connect the local geometry around O with the plane sector of opening α at infinity.

The displayed techniques in the present model case apply to much more complex situations, either geometric as already mentioned, or concerning the operator. Even if it is a lot more technical to investigate elliptic problems such as Linear Elasticity or harmonic Maxwell Equations, such cases also enter the framework of the presented techniques, although extra difficulty may arise from the singularity study, cf [3]. Non-local perturbations of the domain can also be treated, see for example [6, 1].

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