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Abstract

We consider acoustic scattering at a non-smooth penetrable object and coupled boundary element finite element schemes for its numerical simulation. Straightforward coupling approaches are haunted by instabilities at wave numbers related to interior resonances, the so-called spurious resonances.

A remedy is offered by adopting the idea underlying the widely used combined field integral equations. We apply it in the form of modified trace operators. These will also feature regularizing operators to offset the lack of compactness of the double-layer potential integral operators on non-smooth surfaces. Calderón projectors can be defined based on the modified trace operators. Thus, Costabel's approach to the symmetric coupling of domain variational formulations and boundary integral equations carries over.

The modified traces guarantee uniqueness of solutions of the coupled problem, whereas regularization ensures coercivity. From this we immediately conclude asymptotic quasi-optimality of a combined finite element and boundary element Galerkin discretization.

1 Introduction

Let $\Omega^- \subset \mathbb{R}^3$ denote volume occupied by a inhomogeneous bounded object [1]. Plane time harmonic sound waves described by a pressure amplitude U^{i} propagate in the exterior homogeneous air region $\Omega^+ := \mathbb{R}^3 \setminus \bar{\Omega}^-$, hit the object and get scattered.

As explained in [12, Sect. 2.1], a suitably scaled pressure amplitude U of the resulting sound field will satisfy the homogeneous Helmholtz equation

$$-\Delta U - \kappa^2 n(\mathbf{x}) U = 0 \quad \text{in } \Omega^- \cup \Omega^+ , \tag{1}$$

plus suitable radiation boundary conditions at ∞. The refractive index n belongs to $L^\infty\left(\mathbb{R}^3\right)$. It is allowed to vary spatially inside Ω^-, but is equal to 1 in Ω^+. Furthermore we assume the *wave number* κ to be positive and real valued.

The numerical simulation of this acoustic scattering problem is faced with the unbounded domain Ω^+. Many different strategies have been devised to cope with this challenge: one could truncate Ω^+ and use standard finite elements in conjunction with absorbing boundary conditions [20]. An alternative is provided by infinite finite elements in Ω^+ [3, 1] or the method of fundamental solutions [18].

However, in this article we will restrict ourselves to another possibility, namely boundary integral equation methods, which reduce the problem in Ω^+ to equations on the bounded surface $\Gamma := \partial\Omega^-$. Boundary integral equations come in different varieties, among them direct and indirect methods [19, Ch. 8].

Useful integral equations remain elusive for boundary value problems with non-constant coefficients. This is the case inside Ω^- and, therefore, we are forced to use a classical spatial discretization like the finite element method to discretize (1) in Ω^-. This entails linking the weak variational formulation of (1) with boundary integral equations on Γ.

In short, coupled problems are derived by expressing the Dirichlet-to-Neumann map of the exterior problem by means of boundary integral operators. This can be done in many ways. Yet, in many cases, in particular with so-called indirect formulations, the resulting operator lacks structural properties of the Dirichlet-to-Neumann map. This is blatantly obvious in the case of second order elliptic problems [23]. If structure is not preserved, both theoretical analysis becomes much more difficult, and the linear systems of equations obtained through Galerkin boundary element discretization are adversely affected.

For second order elliptic problems Costabel [13] discovered that the so-called direct boundary integral equations provide a remedy. The key concept is that of the Calderón projector acting on the Cauchy data of the problem. For details and theoretical examinations we refer to [10, sect. 4.5] and [16]. In short, the Calderón projector supplies two sets of boundary integral equations. Judiciously combining them yields a version of the Dirichlet-to-Neumann map that perfectly lends itself to a Galerkin discretization. The realisation of Costabel's idea is called the "symmetric coupling approach" to marrying finite elements and boundary elements. It has been applied to a wide range of transmission problems; see, among many others, [9, 21, 25, 22].

Unfortunately, for the acoustic scattering problem the direct symmetric coupling approach invariably leads to equations vulnerable to spurious resonances [17, 29]: if κ^2 agrees with a Dirichlet or Neumann eigenvalue (resonant frequency) of the Laplacian in Ω^-, then the integral equations may fail to possess a unique solution, though the overall scattering problem remains well-posed.

One way to deal with spurious resonances is the use of integral operators with modified kernels [32, 24]. Here we will restrict our attention to another remedy, namely the widely used combined field integral equations (CFIE). They owe their name to the typical complex linear combination of different boundary integral operators on the left hand side of the final boundary integral equation. In the case of indirect schemes this trick has independently been discovered by Brakhage and Werner [4], Leis [26], and Panich [28] in 1965. In 1971 Burton and Miller used the same idea to obtain direct boundary integral equations without spurious resonances [8]. Meanwhile, CFIEs have become the foundation for numerous numerical methods in direct and inverse acoustic and electromagnetic scattering [12, Ch. 3 & 6].

[1] We assume that Ω^- is a curvilinear Lipschitz-polyhedron in the sense of [15, Sect. 1].

We aim to pursue symmetric coupling based on CFIE. To do so we first have to identify related Calderón projectors. Secondly, we have to overcome a potential lack of coercivity of the coupled system due to the fact that the double-layer integral operators fail to be compact on non-smooth surfaces. Both problems are tackled by introducing modified trace operators. These are motivated by the regularization approach to CFIE developed in [6, 5, 7] based on ideas by Panich [28].

2 The Transmission Problem

We depart from a formulation of the scattering problem as a transmission problem. To do so we have to rely on the following continuous and surjective trace mappings [14, Lemma 3.2]

$$\begin{aligned} &\text{Dirichlet trace} \qquad \gamma_0^\pm : H^1_{\mathrm{loc}}\left(\Omega^\pm\right) \to H^{\frac{1}{2}}(\Gamma)\,, \\ &\text{Neumann trace} \qquad \gamma_1^\pm : H_{\mathrm{loc}}\left(\Delta, \Omega^\pm\right) \to H^{-\frac{1}{2}}(\Gamma)\,. \end{aligned}$$

We refer to [27, Ch. 3] for the definitions function spaces $H^1_{\mathrm{loc}}(\Omega^\pm)$, $H_{\mathrm{loc}}(\Delta, \Omega^\pm)$, $H^{\frac{1}{2}}(\Gamma)$, and $H^{-\frac{1}{2}}(\Gamma)$. The trace operators generalize the following pointwise restrictions of smooth functions $V \in C^\infty\left(\overline{\Omega^\pm}\right)$

$$(\gamma_0^\pm V)(\mathbf{x}) := V(\mathbf{x}) \quad \text{and} \quad (\gamma_1^\pm V)(\mathbf{x}) := \mathbf{grad}\, V(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}), \quad \mathbf{x} \in \Gamma.$$

Then the mathematical model for the acoustic scattering problem boils down to the following transmission problem for the Helmholtz equation, see [30, Sect. 2.9].

$$\begin{aligned} -\Delta U - \kappa^2 n(\mathbf{x}) U = f(\mathbf{x}) \quad \text{in } \Omega^-, \qquad & -\Delta U^{\mathrm{s}} - \kappa^2 U^{\mathrm{s}} = 0 \quad \text{in } \Omega^+, \\ \gamma_0^+ U^{\mathrm{s}} - \gamma_0^- U = g_0 \quad \text{on } \Gamma, \qquad & \gamma_1^+ U^{\mathrm{s}} - \gamma_1^- U = g_1 \quad \text{on } \Gamma, \\ \frac{\partial U^{\mathrm{s}}}{\partial r} - i\kappa U^{\mathrm{s}} = o(r^{-1}) \quad & \text{uniformly for } r := |\mathbf{x}| \to \infty, \end{aligned} \tag{2}$$

with the refractive index $n \in L^\infty(\Omega^-)$, the source term $f \in H^{-1}(\Omega^-)$ and the wave number $\kappa > 0$. In the case of excitation by an incident field U^{i} the generic jump data $g_0 \in H^{\frac{1}{2}}(\Gamma)$ and $g_1 \in H^{-\frac{1}{2}}(\Gamma)$ evaluate to the Dirichlet and Neumann data of U^{i} on the boundary Γ:

$$g_0 := -\gamma_0^+ U^{\mathrm{i}}, \qquad g_1 := -\gamma_1^+ U^{\mathrm{i}}\,.$$

It is known that the transmission problem (2) has a unique solution $u \in H_{\mathrm{loc}}\left(\Delta, \mathbb{R}^3\right)$ [30, Sect. 2.10].

Remark 2.1. Please note that inside Ω^- the field U in (2) refers to the total field, whereas in Ω^+ we write U^{s} for the scattered field. There the total field can be recovered through $U = U^{\mathrm{s}} + U^{\mathrm{i}}$.

3 Potentials and Boundary Integral Operators

In this section we define relevant boundary integral operators and review some of their properties. Only sketches of proofs will be given and the reader is referred to [30, 27, 14] for details. To begin with, let us fix some notations and notions: jumps of traces across Γ will be designated by

$$[\gamma_0 V]_\Gamma := \gamma_0^+ V - \gamma_0^- V\,, \qquad [\gamma_1 V]_\Gamma := \gamma_1^+ V - \gamma_1^- V,$$

and averages across Γ will be denoted by

$$\{\gamma_0 V\}_\Gamma := \tfrac{1}{2}\left(\gamma_0^+ V + \gamma_0^- V\right), \qquad \{\gamma_1 V\}_\Gamma := \tfrac{1}{2}\left(\gamma_1^+ V + \gamma_1^- V\right).$$

For a fixed wave number $\kappa > 0$ a distribution U on $\mathbb{R}^3$ is called a *radiating Helmholtz solution*, if

$$\Delta U + \kappa^2 U = 0 \quad \text{in } \Omega^- \cup \Omega^+, \qquad \lim_{r\to\infty} r\left(\frac{\partial U}{\partial r} - i\kappa U\right) = 0,$$

where the limit is assumed to hold uniformly in all directions. Based on the Helmholtz kernel

$$G_\kappa(z) := \frac{1}{4\pi}\frac{\exp(i\kappa z)}{z} \tag{3}$$

we can state the transmission formula for radiating Helmholtz solutions U [30, Thm. 3.1.6]

$$U = -\Psi^\kappa_{\mathrm{SL}}([\gamma_1 U]_\Gamma) + \Psi^\kappa_{\mathrm{DL}}([\gamma_0 U]_\Gamma)\,. \tag{4}$$

with the potentials

$$\begin{aligned}
&\text{single-layer potential} \qquad \Psi^\kappa_{\mathrm{SL}}(\vartheta)(\mathbf{x}) := \int_\Gamma G_\kappa(|\mathbf{x}-\mathbf{y}|)\,\vartheta(\mathbf{y})\,\mathrm{d}S(\mathbf{y}),\\
&\text{double-layer potential} \qquad \Psi^\kappa_{\mathrm{DL}}(v)(\mathbf{x}) := \int_\Gamma \frac{\partial G_\kappa(|\mathbf{x}-\mathbf{y}|)}{\partial \mathbf{n}(\mathbf{y})} v(\mathbf{y})\,\mathrm{d}S(\mathbf{y}).
\end{aligned}$$

The potentials provide radiating Helmholtz solutions and continuous mappings [30, Thm. 3.1.16]

$$\begin{aligned}
\Psi^\kappa_{\mathrm{SL}} : H^{-\frac12}(\Gamma) &\to H^1_{\mathrm{loc}}(\mathbb{R}^3) \cap H_{\mathrm{loc}}\left(\Delta, \Omega^- \cup \Omega^+\right),\\
\Psi^\kappa_{\mathrm{DL}} : H^{\frac12}(\Gamma) &\to H_{\mathrm{loc}}\left(\Delta, \Omega^- \cup \Omega^+\right).
\end{aligned}$$

Applying the trace mappings yields the following four continuous boundary integral operators

$$\begin{aligned}
\mathsf{V}_\kappa &: H^{s-\frac12}(\Gamma) \to H^{s+\frac12}(\Gamma), & \mathsf{V}_\kappa &:= \{\gamma_0 \Psi^\kappa_{\mathrm{SL}}\}_\Gamma\,,\\
\mathsf{K}_\kappa &: H^{s+\frac12}(\Gamma) \to H^{s+\frac12}(\Gamma), & \mathsf{K}_\kappa &:= \{\gamma_0 \Psi^\kappa_{\mathrm{DL}}\}_\Gamma\,,\\
\mathsf{K}'_\kappa &: H^{s-\frac12}(\Gamma) \to H^{s-\frac12}(\Gamma), & \mathsf{K}'_\kappa &:= \{\gamma_1 \Psi^\kappa_{\mathrm{SL}}\}_\Gamma\,,\\
\mathsf{W}_\kappa &: H^{s+\frac12}(\Gamma) \to H^{s-\frac12}(\Gamma), & \mathsf{W}_\kappa &:= -\{\gamma_1 \Psi^\kappa_{\mathrm{DL}}\}_\Gamma\,,
\end{aligned}$$

for a scale of Sobolev spaces with $|s| < \frac12$, see [14, Thm. 1]. From the *jump relations* [30, Thm. 3.3.1]

$$\begin{array}{lllllll}
[\gamma_0 \Psi^\kappa_{\mathrm{SL}}(\vartheta)]_\Gamma &=& 0\,, & [\gamma_1 \Psi^\kappa_{\mathrm{SL}}(\vartheta)]_\Gamma &=& -\vartheta, & \forall \vartheta \in H^{-\frac12}(\Gamma)\,,\\
[\gamma_0 \Psi^\kappa_{\mathrm{DL}}(\varphi)]_\Gamma &=& \varphi\,, & [\gamma_1 \Psi^\kappa_{\mathrm{DL}}(\varphi)]_\Gamma &=& 0\,, & \forall \varphi \in H^{\frac12}(\Gamma)\,,
\end{array} \tag{5}$$

we can directly deduce the following four identities

$$\begin{array}{llllll}
\gamma_0^\pm \Psi^\kappa_{\mathrm{SL}} &=& \mathsf{V}_\kappa\,, & \gamma_1^\pm \Psi^\kappa_{\mathrm{SL}} &=& \mathsf{K}_\kappa \pm \frac12 \mathsf{Id}\,,\\
\gamma_0^\pm \Psi^\kappa_{\mathrm{DL}} &=& \mathsf{K}'_\kappa \mp \frac12 \mathsf{Id}\,, & \gamma_1^\pm \Psi^\kappa_{\mathrm{DL}} &=& -\mathsf{W}_\kappa\,.
\end{array} \tag{6}$$

In the sequel $(\cdot,\cdot)_\Gamma$ will stand for the $L^2(\Gamma)$-inner product

$$(\vartheta, \varphi)_\Gamma \quad := \quad \int_\Gamma \overline{\vartheta}\varphi\,\mathrm{d}S \qquad \vartheta, \varphi \in L^2(\Gamma)\,,$$

which can be extended to a duality pairing on $H^{-\frac12}(\Gamma) \times H^{\frac12}(\Gamma)$. Adjoints of operators with respect to $(\cdot,\cdot)_\Gamma$ will be tagged by $*$.

Crucial for any variational formulation based on boundary integral operators will be the following three lemmata, see [30, Lemma 3.9.8], [14, Thm. 2].

Lemma 3.1. *The following operators are compact*

$$\begin{aligned}
\mathsf{V}_\kappa - \mathsf{V}_0 &: H^{-\frac12}(\Gamma) \to H^{\frac12}(\Gamma)\,,\\
\mathsf{K}_\kappa - \mathsf{K}_0 &: H^{\frac12}(\Gamma) \to H^{\frac12}(\Gamma)\,,\\
\mathsf{K}'_\kappa - \mathsf{K}'_0 &: H^{-\frac12}(\Gamma) \to H^{\frac12}(\Gamma)\,,\\
\mathsf{W}_\kappa - \mathsf{W}_0 &: H^{\frac12}(\Gamma) \to H^{-\frac12}(\Gamma)\,.
\end{aligned}$$

The proof relies on the fact that both $\mathsf{V}_\kappa - \mathsf{V}_0$ and $\mathsf{K}_\kappa - \mathsf{K}_0$ turn out to be integral operators with continuous and bounded kernels, which ensures that they map into $H^1(\Gamma)$, which is compactly embedded in $H^{\frac{1}{2}}(\Gamma)$. Details can be found in [5, Sect. 2]. This result combined with the ellipticity of both V_0 and W_0 in $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$, respectively, yields the next lemma.

Lemma 3.2. *The operators V_κ and W_κ satisfy a generalized Gårding inequality in the sense that there exists a constant $\gamma > 0$ and compact operators*

$$\mathsf{T}_\mathsf{V} : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \qquad \mathsf{T}_\mathsf{W} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$

such that

$$\begin{aligned} \operatorname{Re}\left\{(\vartheta, (\mathsf{V}_\kappa + \mathsf{T}_\mathsf{V})(\vartheta))_\Gamma\right\} &\geq \gamma \|\vartheta\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \\ \operatorname{Re}\left\{((\mathsf{W}_\kappa + \mathsf{T}_\mathsf{W})(\varphi), \varphi)_\Gamma\right\} &\geq \gamma \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)} \end{aligned}$$

holds true for all $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ and $\varphi \in H^{\frac{1}{2}}(\Gamma)$.

Finally, K'_κ is the $(\cdot,\cdot)_\Gamma$-adjoint of K_κ up to a compact perturbation:

Lemma 3.3. *There exists a compact operator $\mathsf{T}_\mathsf{K} : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ such that*

$$(\mathsf{K}^*_\kappa(\vartheta), \varphi)_\Gamma = ((\mathsf{K}'_\kappa + \mathsf{T}_\mathsf{K})(\vartheta), \varphi)_\Gamma$$

holds true for all $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ and $\varphi \in H^{\frac{1}{2}}(\Gamma)$, where K^*_κ denotes the $L^2(\Gamma)$-adjoint of K_κ.

Proof. Following [30, Sect. 3.1] and [14], we recall the representations

$$\mathsf{K}_\kappa = \{\gamma_0\}_\Gamma \circ \mathcal{N}_\kappa \circ {\gamma_1}^* \quad , \quad \mathsf{K}'_\kappa = \{\gamma_1\}_\Gamma \circ \mathcal{N}_\kappa \circ {\gamma_0}^*,$$

where $\mathcal{N}_\kappa : H^{-1}_{\mathrm{comp}}(\mathbb{R}^3) \to H^1_{\mathrm{loc}}(\mathbb{R}^3)$ is the Newton potential for the Helmholtz kernel.

$${\mathsf{K}_\kappa}^* - \mathsf{K}'_\kappa = \{\gamma_1\}_\Gamma \circ (\mathcal{N}_\kappa - {\mathcal{N}_\kappa}^*) \circ {\gamma_0}^* .$$

Observe that

$$(\mathcal{N}_\kappa - {\mathcal{N}_\kappa}^*)(V)(\mathbf{x}) = \frac{i}{2\pi} \int_{\mathbb{R}^3} \frac{\sin(\kappa|\mathbf{x}-\mathbf{y}|)}{|\mathbf{x}-\mathbf{y}|} V(\mathbf{y})\,\mathrm{d}\mathbf{y}$$

is an integral operator with analytic kernel, which maps continuously $H^{-1}_{\mathrm{comp}}(\mathbb{R}^3) \mapsto H^s_{\mathrm{loc}}(\mathbb{R}^3)$ for any $s \in \mathbb{R}$. Thus, $\mathsf{K}^*_\kappa - \mathsf{K}'_\kappa : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1(\Gamma)$ is continuous and the compact embedding $H^1(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma)$ finishes the proof. □

4 Calderón Projectors

A crucial tool for the coupling of the variational equations on Ω^- and boundary integral equations on Γ are the two *Calderón projectors* [30, Sect. 3.6]

$$\mathsf{P}_\pm := \begin{bmatrix} \frac{1}{2}\mathsf{Id} \pm \mathsf{K}_\kappa & \mp \mathsf{V}_\kappa \\ \mp \mathsf{W}_\kappa & \frac{1}{2}\mathsf{Id} \mp \mathsf{K}'_\kappa \end{bmatrix} : H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma) .$$

They arise from applying the trace operators $\gamma_0^\pm$ and $\gamma_1^\pm$ to (4) and using (6). The operators P_+ and P_- obviously satisfy the identity

$$\mathsf{P}_+ + \mathsf{P}_- = \mathsf{Id}. \tag{7}$$

The Calderón projectors can be used to characterise pairs of functions in $H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ that are eligible as traces of Helmholtz solutions, see [33].

Theorem 4.1. *If and only if* $(\varphi, \vartheta) \in H^{\frac{1}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ *belongs to the range of* $\mathsf{P}_{\pm}$, *there is a Helmholtz solution* U *such that* $\varphi = \gamma_0^{\pm} U$ *and* $\vartheta = \gamma_1^{\pm} U$.

The theorem paves the way for establishing expressions for the exterior *Dirichlet-to-Neumann* map for the Helmholtz problem in Ω^+. This is the operator $\mathrm{DtN}_\kappa^+ : H^{\frac{1}{2}}(\Gamma) \mapsto H^{-\frac{1}{2}}(\Gamma)$ returning the Neumann traces of an exterior Helmholtz solution matching prescribed Dirichlet boundary conditions on Γ. Three different formulas can instantly be obtained from (4.1), at least formally, because the inverses of operators might not exist:

$$\mathrm{DtN}_\kappa^+ := \mathsf{V}_\kappa{}^{-1} \circ \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right), \tag{8}$$

$$\mathrm{DtN}_\kappa^+ := -\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa'\right)^{-1} \circ \mathsf{W}_\kappa, \tag{9}$$

$$\mathrm{DtN}_\kappa^+ := -\mathsf{W}_\kappa + \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa'\right) \circ \mathsf{V}_\kappa{}^{-1} \circ \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right). \tag{10}$$

Only the third formula reflects the essential symmetry of the boundary value problem in the case $\kappa = 0$. It will be the starting point for symmetric coupling.

Remark 4.2. If the incident wave U^{i} can be extended to an interior Helmholtz solution, which is evidently the case, when U^{i} is a plane wave or generated by a sound source compactly supported in Ω^+, then, by (4) and (5), its traces on Γ will fulfill

$$\begin{bmatrix} \gamma_0 U^{\mathrm{i}} \\ \gamma_1 U^{\mathrm{i}} \end{bmatrix} = \mathsf{P}_- \begin{bmatrix} \gamma_0 U^{\mathrm{i}} \\ \gamma_0 U^{\mathrm{i}} \end{bmatrix} \quad \Leftrightarrow \quad \mathsf{P}_+ \begin{bmatrix} \gamma_0 U^{\mathrm{i}} \\ \gamma_1 U^{\mathrm{i}} \end{bmatrix} = 0\,. \tag{11}$$

For the same reasons, the scattered field U^{s} satisfies

$$\begin{bmatrix} \gamma_0^+ U^{\mathrm{s}} \\ \gamma_1^+ U^{\mathrm{s}} \end{bmatrix} = \mathsf{P}_+ \begin{bmatrix} \gamma_0^+ U^{\mathrm{s}} \\ \gamma_1^+ U^{\mathrm{s}} \end{bmatrix}. \tag{12}$$

Using that the total field in Ω^+ is given by $U = U^{\mathrm{s}} + U^{\mathrm{i}}$, we can eliminate U^{s} from (11) and (12) and end up with

$$\begin{bmatrix} \gamma_0^+ U \\ \gamma_1^+ U \end{bmatrix} = \mathsf{P}_+ \begin{bmatrix} \gamma_0^+ U \\ \gamma_1^+ U \end{bmatrix} - \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} \tag{13}$$

As above Dirichlet-to-Neumann maps for the total field can be constructed from this relationship.

5 Classical Symmetric Coupling

For the sake of completeness we will review the classical approach to the coupling of boundary integral equations and variational formulation in Ω^- due to Costabel [13]. First, integration by parts shows that a solution U of problem (2) will fulfill

$$\mathsf{a}(U, V) - \left(\gamma_1^- U, \gamma_0^- V\right)_\Gamma = \mathsf{f}(v) \quad \forall v \in H^1\left(\Omega^-\right), \tag{14}$$

where we have used the abbreviations

$$\mathsf{a}(U, V) := \int_{\Omega^-} \mathbf{grad}\overline{U} \cdot \mathbf{grad} V - \kappa^2 n(\mathbf{x})\, \overline{U}\, V \,\mathrm{d}\mathbf{x}\,, \qquad U, V \in H^1\left(\Omega^-\right),$$

$$\mathsf{f}(V) := \int_{\Omega^-} \overline{f}\, V \,\mathrm{d}\mathbf{x}\,, \qquad V \in H^1\left(\Omega^-\right).$$

Lemma 5.1. *The sesqui-linear form* a *satisfies a generalized Gårding inequality in the sense that there exists a constant* $\gamma > 0$ *and a compact sesqui-linear form*

$$\mathsf{k} : H^1\left(\Omega^-\right) \times H^1\left(\Omega^-\right) \to \mathbb{C}$$

such that

$$\mathrm{Re}\left\{\mathsf{a}(U, U) + \mathsf{k}(U, U)\right\} \geq \gamma \left\|U\right\|^2_{H^1(\Omega^-)}$$

holds true for all $u \in H^1(\Omega^-)$.

Proof. The lemma is a straightforward consequence of the compact embedding $H^1(\Omega^-) \hookrightarrow L^2(\Omega^-)$. □

The variational problem associated with the classical symmetric coupling approach emerges by employing the transmission conditions of (2) and using the Dirichlet-to-Neumann map (10) to express $\gamma_1^- U$ in (14). In order to avoid the operator products occurring in (10) we also introduce $\gamma_1^+ U^s$ as the new variable

$$\vartheta \ := \ \left(\mathsf{V}_\kappa{}^{-1} \circ \left(\mathsf{K}_\kappa - \tfrac{1}{2} Id\right)\right)\left(\gamma_0^- U + g_0\right) \in H^{-\frac{1}{2}}(\Gamma) \ .$$

Thus, we end up with: find $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ there holds

$$\begin{aligned} \mathsf{a}(U,V) + \left(\mathsf{W}_\kappa\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma - \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right)(\vartheta), \gamma_0^- V\right)_\Gamma &= \widetilde{f}(V) \ , \\ \left(\varphi, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa\right)\left(\gamma_0^- U\right)\right)_\Gamma + \left(\varphi, \mathsf{V}_\kappa(\vartheta)\right)_\Gamma &= \widetilde{g}(\varphi) \ , \end{aligned} \tag{15}$$

where

$$\begin{aligned} \widetilde{f}(V) &:= \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma - \left(\mathsf{W}_\kappa(g_0), \gamma_0^- V\right)_\Gamma \ , \\ \widetilde{g}(\varphi) &:= \left(\varphi, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)(g_0)\right)_\Gamma \ . \end{aligned}$$

Using the lemmata of the previous section it is not difficult to verify that the bilinear form associated with (15) satisfies a Gårding inequality. Unfortunately, this is no safeguard against spurious resonances:

Assume the resonance case, that is, κ^2 is a Dirichlet eigenvalue of $-\Delta$ in Ω^-. Then we can find $U \in H^1(\Omega^-) \setminus \{0\}$ such that

$$\Delta U + \kappa^2 U = 0 \quad \text{in } \Omega^- \quad \text{and} \quad U = 0 \quad \text{on } \Gamma \ .$$

Since $\gamma_0^- U = 0$,by Thm. 4.1 we have that

$$\begin{bmatrix} 0 \\ \gamma_1^- U \end{bmatrix} = \mathsf{P}_- \begin{bmatrix} 0 \\ \gamma_1^- U \end{bmatrix} = \begin{bmatrix} \mathsf{V}_\kappa\left(\gamma_1^- U\right) \\ \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right)\left(\gamma_1^- U\right) \end{bmatrix} ,$$

which means that $(0, \gamma_1^- U)$ provides a solution of (15) in the case $\widetilde{f} = \widetilde{g} = 0$.

Even in the resonance case, the right hand side of (15) will be consistent and the variational problem still has solutions (U, ϑ), whose first component will still be unique. Alas, this is little comfort as far as numerical solution procedures are concerned: firstly, inevitable perturbations introduced by discretization will destroy the consistency of the right hand side. Secondly, whenever κ^2 is merely close to an interior resonant frequency, the resulting linear systems of equations may not be useless, but will be extremely ill-conditioned: see the profound analysis of the impact of spurious resonances in the case of electromagnetic scattering given in [11].

So, from a numerical point of view suppressing spurious resonances is essential for the efficacy of methods based on boundary integral equations.

Remark 5.2. Under the assumptions made in Remark 4.2 we may use a symmetric Dirichlet-to-Neumann map derived from (13). This will lead to a coupled variational problem of the form (15) with much simpler right hand sides $\widetilde{f}(V) = \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma$ and $\widetilde{g}(\varphi) = -(\varphi, g_0)_\Gamma$.

6 Transformed Traces

As pointed out at the end of the previous section, the existence of spurious resonances is directly linked to the fact that for certain κ there are non-trivial interior Helmholtz solutions U that satisfy $\gamma_0^- U = 0$. We know that there are Robin-type (mixed) boundary conditions that ensure the unique solvability of the corresponding boundary value problem for $-\Delta U - \kappa^2 U = 0$ in Ω^-. Note that we can rely on two Robin-type boundary operators to state the transmission conditions of (2) as long as we can recover the conventional Dirichlet and Neumann trace from them. In fact, this idea can serve as the starting point for the derivation of all CFIEs. Here, it motivates the introduction of the following generic trace transformation operator.

$$\mathcal{T} := \begin{bmatrix} \mathsf{A} & \mathsf{B} \\ \mathsf{C} & \mathsf{D} \end{bmatrix} : H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)\ . \tag{16}$$

We demand that the interior homogeneous "Dirichlet problem" for $-\Delta U - \kappa^2 U = 0$ and the modified traces has a unique solution for every κ. In light of Thm. 4.1 this amounts to the following assumption:

Assumption 6.1. *The trace transformation operator $\mathcal{T}$ satisfies*

$$\mathrm{Range}\,(\mathcal{T} \circ \mathsf{P}_-) \cap \left(\{0\} \times H^{-\frac{1}{2}}(\Gamma)\right) \quad = \quad \{0\}\ .$$

Then one can use $\mathcal{T}$, build associated Calderón projectors for the modified traces, derive symmetrically coupled variational problems and check their properties. Here, we would like to skip this tedious process of creative discovery [2] and present the final finding on what is required for $\mathcal{T}$:

Assumption 6.2. *The blocks of the transformation operator $\mathcal{T}$ from* (16) *are assumed to possess the following properties*

1. $\mathcal{T} : H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ *is bijective,*
2. $\mathsf{A} : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ *is bounded and bijective,*
3. $\mathsf{B} : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ *is compact,*
4. $\mathsf{C} : H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ *is compact,*
5. $\mathsf{D} : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ *is bounded and bijective.*

The first requirement enables us to retrieve the conventional Dirichlet and Neumann trace from their transformed counterparts. This is essential, because it is these traces that will invariably occur in (14) so that we have to resort to them in one way or another when pursuing the coupling of (14) with boundary integral equations. Switching back and forth between conventional and transformed traces employs the following splitting of the trace transformation operator

$$\mathcal{T} = \mathcal{R} + \mathcal{S}, \qquad \mathcal{R} := \begin{bmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{D} \end{bmatrix}, \qquad \mathcal{S} := \begin{bmatrix} 0 & \mathsf{B} \\ \mathsf{C} & 0 \end{bmatrix}. \tag{17}$$

Based on it we define the following generalized Calderón projectors

$$\mathcal{P}_{\pm} \quad := \quad \mathcal{R}^{-1} \circ (\mathcal{T} \circ \mathsf{P}_{\pm} - \mathcal{S})\,. \tag{18}$$

Note that they are meant to act on conventional traces. Let us make the transformed exterior Calderón projector more explicit: an elementary computation yields

$$\mathcal{P}_{+} \quad = \quad \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{bmatrix} \tag{19}$$

where the entries of the operator matrix are given by

$$\mathbb{A} \quad := \quad \tfrac{1}{2}\mathsf{Id} + \mathsf{K}_\kappa - \mathsf{A}^{-1} \circ \mathsf{B} \circ \mathsf{W}_\kappa, \tag{20}$$

$$\mathbb{B} \quad := \quad -\mathsf{A}^{-1} \circ \mathsf{B} \circ \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right) - \mathsf{V}_\kappa, \tag{21}$$

$$\mathbb{C} \quad := \quad \mathsf{D}^{-1} \circ \mathsf{C} \circ \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right) - \mathsf{W}_\kappa, \tag{22}$$

$$\mathbb{D} \quad := \quad \tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa - \mathsf{D}^{-1} \circ \mathsf{C} \circ \mathsf{V}_\kappa\ . \tag{23}$$

An analogue of Thm. 4.1 still holds for the transformed Calderón projectors.

[2]We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work – R. Feynman

Lemma 6.3. *If and only if U is an exterior/interior radiating Helmholtz solution we have*

$$\mathcal{P}_\pm \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} = \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} .$$

Proof. As $\mathcal{T}$ is one-to-one, we immediately conclude from Thm. 4.1 that U is an exterior/interior radiating Helmholtz solution if and only if

$$\mathsf{P}_\pm \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} = \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix}$$

$$\Updownarrow$$

$$(\mathcal{T} \circ \mathsf{P}_\pm) \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} = \mathcal{T} \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} = (\mathcal{R} + \mathcal{S}) \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} = \mathcal{R}^{-1} \circ (\mathcal{T} \circ \mathsf{P}_\pm - \mathcal{S}) \begin{bmatrix} \gamma_0^\pm U \\ \gamma_1^\pm U \end{bmatrix} .$$

□

Now, the same formal manipulations as in Sect. 5 yield the following operator expression for the Dirichlet-to-Neumann map

$$\mathrm{DtN}_\kappa^+ := \mathbb{C} + \mathbb{D} \circ \mathbb{B}^{-1} \circ (\mathsf{Id} - \mathbb{A}) , \tag{24}$$

which maps exterior Dirichlet traces of radiating Helmholtz solutions U to exterior Neumann traces.

Remark 6.4. Again, if the incident wave U^{i} can be extended to an interior Helholtz solution, then we can apply the trace transformation operator to (13) and we end up with

$$\mathcal{T} \begin{bmatrix} \gamma_0^+ U \\ \gamma_1^+ U \end{bmatrix} = (\mathcal{T} \circ \mathsf{P}_+) \begin{bmatrix} \gamma_0^+ U \\ \gamma_1^+ U \end{bmatrix} - \mathcal{T} \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} . \tag{25}$$

Using the operator splitting (17) and definition (18) of the generalized Calderón projector we can eliminate the trace transformation operator $\mathcal{T}$ from the left hand side of (25) and we obtain

$$\begin{bmatrix} \gamma_0^+ U \\ \gamma_1^+ U \end{bmatrix} = \mathcal{P}_+ \begin{bmatrix} \gamma_0^+ U \\ \gamma_1^+ U \end{bmatrix} - \left(\mathcal{R}^{-1} \circ \mathcal{T}\right) \begin{bmatrix} g_0 \\ g_1 \end{bmatrix} . \tag{26}$$

As above this relationship can be used to construct new Dirchlet-to-Neumann maps for the total field.

We end this section with an easily verifyable criterion telling us when Ass. 6.1 is satisfied:

Lemma 6.5. *If the following is equivalent*

$$\mathrm{Im}\left\{ \left(\vartheta, \left(\mathsf{A}^{-1} \circ \mathsf{B}\right)(\vartheta)\right)_\Gamma \right\} = 0 \quad \Leftrightarrow \quad \vartheta = 0 ,$$

then

$$\mathrm{Range}\,(\mathcal{T} \circ \mathsf{P}_-) \cap \left(\{0\} \times H^{-\frac{1}{2}}(\Gamma)\right) = \{0\} .$$

Proof. If $\xi \in H^{-\frac{1}{2}}(\Gamma)$ satisfies

$$\begin{bmatrix} 0 \\ \xi \end{bmatrix} \in \mathrm{Range}\,(\mathcal{T} \circ \mathsf{P}_-) ,$$

then there exists $\vartheta \in H^{\frac{1}{2}}(\Gamma)$ and $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ such that

$$\begin{bmatrix} 0 \\ \xi \end{bmatrix} = (\mathcal{T} \circ \mathsf{P}_-) \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix}.$$

Taking the transformed interior traces of the function

$$U(\mathbf{x}) := -\Psi^{\kappa}_{\mathrm{DL}}(\vartheta)(\mathbf{x}) + \Psi^{\kappa}_{\mathrm{SL}}(\varphi)(\mathbf{x}), \qquad \mathbf{x} \in \Omega^-$$

gives us the following set of equations

$$\mathcal{T} \begin{bmatrix} \gamma_0^- U \\ \gamma_1^- U \end{bmatrix} = (\mathcal{T} \circ \mathsf{P}_-) \begin{bmatrix} \vartheta \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ \xi \end{bmatrix}. \tag{27}$$

Thus U is a solution to the boundary value problem

$$\Delta U + \kappa^2 U = 0 \quad \text{in } \Omega^-, \tag{28}$$
$$\mathsf{A}\left(\gamma_0^- U\right) + \mathsf{B}\left(\gamma_1^- U\right) = 0 \quad \text{on } \Gamma. \tag{29}$$

Recalling equation (14) and using that A is bijective, we obtain

$$\mathsf{a}(U,U) - \left(\gamma_1^- U, \gamma_0^- U\right)_\Gamma = \mathsf{a}(U,U) + \left(\gamma_1^- U, (\mathsf{A}^{-1} \circ \mathsf{B})\left(\gamma_1^- U\right)\right)_\Gamma = 0 .$$

Since, $\mathsf{a}(U,U) \in \mathbb{R}$, taking the imaginary part, we get

$$\operatorname{Im}\left\{\left(\gamma_1^- U, \left(\mathsf{A}^{-1} \circ \mathsf{B}\right)\left(\gamma_1^- U\right)\right)_\Gamma\right\} = 0.$$

Thus, the assumption of the lemma implies $\gamma_1^- U = 0$, and via (29) we conclude $\gamma_0^- U = 0$. Eventually, (27) shows that $\xi = 0$. □

7 Stabilized Coupling

Parallel to the approach in Sect. 5, we use equation (14) in combination with the transformed Dirichlet-to-Neumann map (24) and introduce the new variable

$$\vartheta := -\left(\mathbb{B}^{-1} \circ (\mathsf{Id} - \mathbb{A})\right)\left(\gamma_0^- U + g_0\right) \in H^{-\frac{1}{2}}(\Gamma) . \tag{30}$$

If U solves the Helmholtz transmission problem (2), then $\gamma_0^- U + g_0 = \gamma_0^+ U^{\mathrm{s}}$, and we learn from Lemma 6.3 and (19) that actually $\vartheta = -\gamma_1^+ U^{\mathrm{s}}$. As in the case of classical coupling, ϑ will supply the exterior Neumann trace of the scattered field.

Thus we arrive at the following *regularized variational formulation*: find $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ there holds

$$\begin{aligned} \mathsf{a}(U,V) - \left(\mathbb{C}\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma + \left(\mathbb{D}(\vartheta), \gamma_0^- V\right)_\Gamma &= \widehat{f}(V) , \\ \left(\varphi, (\mathbb{A} - \mathsf{Id})\left(\gamma_0^- U\right)\right)_\Gamma - (\varphi, \mathbb{B}(\vartheta))_\Gamma &= \widehat{g}(\varphi) , \end{aligned} \tag{31}$$

where

$$\widehat{f}(V) := \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma + \left(\mathbb{C}(g_0), \gamma_0^- V\right)_\Gamma , \tag{32}$$
$$\widehat{g}(V) := (\varphi, (\mathsf{Id} - \mathbb{A})(g_0))_\Gamma . \tag{33}$$

We first investigate the $H^1(\Omega^-) \times H^{-\frac{1}{2}}(\Gamma)$-coercivity of the sesqui-linear form underlying (31). From assumption 6.2 it is immediate that the operators $\mathsf{A}^{-1} \circ \mathsf{B}$, $\mathsf{D}^{-1} \circ \mathsf{C}$ are compact. This plays a key role in the proofs of the following two lemmata.

Lemma 7.1. *There exists a constant $\gamma > 0$ and compact operators*

$$\mathsf{T}_{\mathbb{B}} : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \qquad \mathsf{T}_{\mathbb{C}} : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$

such that

$$\begin{aligned} -\operatorname{Re}\{(\vartheta, (\mathbb{B} + \mathsf{T}_{\mathbb{B}})(\vartheta))_\Gamma\} &\geq \gamma \|\vartheta\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \\ -\operatorname{Re}\{(\varphi, (\mathbb{C} + \mathsf{T}_{\mathbb{C}})(\varphi))_\Gamma\} &\geq \gamma \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)}, \end{aligned}$$

for all $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ and $\varphi \in H^{\frac{1}{2}}(\Gamma)$.

Proof. Using (21) and (22) a straightforward application of Lemma 3.2 yields

$$\begin{aligned} &\operatorname{Re}\left\{-(\vartheta, \mathbb{B}(\vartheta))_\Gamma - \left(\vartheta, \left(\mathsf{A}^{-1} \circ \mathsf{B} \circ \left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right)\right)(\vartheta)\right)_\Gamma + (\vartheta, \mathsf{T}_\mathsf{V}(\vartheta))_\Gamma\right\} \\ &\quad = \operatorname{Re}\{(\vartheta, (\mathsf{V}_\kappa + \mathsf{T}_\mathsf{V})(\vartheta))_\Gamma\} \geq \gamma \|\vartheta\|^2_{H^{-\frac{1}{2}}(\Gamma)}, \\ &\operatorname{Re}\left\{-(\mathbb{C}(\varphi), \varphi)_\Gamma + \left(\left(\mathsf{D}^{-1} \circ \mathsf{C} \circ \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)\right)(\varphi), \varphi\right)_\Gamma + (\mathsf{T}_\mathsf{W}(\varphi), \varphi)_\Gamma\right\} \\ &\quad = \operatorname{Re}\{((\mathsf{W}_\kappa + \mathsf{T}_\mathsf{W})(\varphi), \varphi)_\Gamma\} \geq \gamma \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)}, \end{aligned}$$

for all $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$, $\varphi \in H^{\frac{1}{2}}(\Gamma)$. □

Lemma 7.2. *There exist compact operators*

$$\mathsf{T}_{\mathbb{A}} : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \qquad \mathsf{T}_{\mathbb{D}} : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$$

such that

$$(\vartheta, (\mathbb{A} - \mathsf{Id} + \mathsf{T}_{\mathbb{A}})(\varphi))_\Gamma + ((\mathbb{D} + \mathsf{T}_{\mathbb{D}})(\vartheta), \varphi)_\Gamma = 0$$

for all $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$, $\varphi \in H^{\frac{1}{2}}(\Gamma)$.

Proof. We begin with an application of lemma 3.3 and obtain

$$\begin{aligned} &\left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right)(\vartheta), \varphi\right)_\Gamma + \left(\vartheta, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)(\varphi)\right)_\Gamma \\ &\quad = \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}^*_\kappa + \mathsf{T}_\mathsf{K}\right)(\vartheta), \varphi\right)_\Gamma - \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}^*_\kappa\right)(\vartheta), \varphi\right)_\Gamma \end{aligned}$$

for all $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$, $\varphi \in H^{\frac{1}{2}}(\Gamma)$. Using this result we finally arrive at the following equation

$$\begin{aligned} &(\vartheta, (\mathbb{A} - \mathsf{Id})(\varphi))_\Gamma + (\mathbb{D}(\vartheta), \varphi)_\Gamma \\ &\quad = \left(\left(\mathsf{T}_\mathsf{K} - \mathsf{D}^{-1} \circ \mathsf{C} \circ \mathsf{K}_\kappa\right)(\vartheta), \varphi\right)_\Gamma - \left(\vartheta, \left(\mathsf{A}^{-1} \circ \mathsf{B} \circ \mathsf{W}_\kappa\right)(\varphi)\right)_\Gamma, \end{aligned}$$

which holds for arbitrary $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$, $\varphi \in H^{\frac{1}{2}}(\Gamma)$. □

Summing up, from these lemmata and Lemma 5.1 we conclude that the sesqui-linear form of the regularised variational problem (31) satisfies a Gårding inequality in $H^1(\Omega^-) \times H^{-\frac{1}{2}}(\Gamma)$. It remains to establish uniqueness of solutions, which amounts to confirming that (31) is really immune to spurious resonances.

Theorem 7.3. *Solutions to the regularised variational problem* (31) *are unique.*

Proof. In order to establish uniqueness of solutions of (31) we consider the case $\widehat{f} = \widehat{g} = 0$: seek $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ there holds

$$\begin{aligned} \mathsf{a}(U, V) - \left(\mathbb{C}\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma + \left(\mathbb{D}(\vartheta), \gamma_0^- V\right)_\Gamma &= 0, \\ \left(\varphi, (\mathbb{A} - \mathsf{Id})\left(\gamma_0^- U\right)\right)_\Gamma - (\varphi, \mathbb{B}(\vartheta))_\Gamma &= 0. \end{aligned}$$

Using integration by parts, we obtain

$$\Delta U + \kappa^2 n(\mathbf{x}) U \quad = \quad 0 \qquad \text{in } \Omega^- \ ,$$

and from $\mathsf{a}(U,V) = \left(\gamma_1^- U, \gamma_0^- V\right)_\Gamma$ the identities

$$\begin{bmatrix} \gamma_0^- U \\ \gamma_1^- U \end{bmatrix} = \mathcal{P}_+ \begin{bmatrix} \gamma_0^- U \\ -\vartheta \end{bmatrix} . \tag{34}$$

By the definition of $\mathcal{P}_+$ and (7)

$$\begin{aligned} \mathcal{P}_+ &= \mathcal{R}^{-1} \circ (\mathcal{T} \circ \mathsf{P}_+ - \mathcal{S}) = \mathcal{R}^{-1} \circ (\mathcal{T} \circ (\mathsf{Id} - \mathsf{P}_-) - \mathcal{S}) = \mathcal{R}^{-1} \circ (\mathcal{R} + \mathcal{S} - \mathcal{T} \circ \mathsf{P}_- - \mathcal{S}) \\ &= \mathsf{Id} - \mathcal{R}^{-1} \circ \mathcal{T} \circ \mathsf{P}_- \ , \end{aligned}$$

and we infer

$$\mathcal{T} \circ \mathsf{P}_- = \mathcal{R} \circ (\mathsf{Id} - \mathcal{P}_+) \ .$$

Together with (34) this identity confirms

$$(\mathcal{T} \circ \mathsf{P}_-) \begin{bmatrix} \gamma_0^- U \\ -\vartheta \end{bmatrix} \quad = \quad -\mathcal{R} \begin{bmatrix} 0 \\ \gamma_1^- U + \vartheta \end{bmatrix} \in \left(\{0\} \times H^{\frac{1}{2}}(\Gamma)\right) \ ,$$

and by Ass. 6.1 and Ass. 6.2, **5.**, we conclude that

$$\gamma_1^- U \quad = \quad -\vartheta \ .$$

From this and (34) we directly obtain

$$\begin{bmatrix} \gamma_0^- U \\ \gamma_1^- U \end{bmatrix} \quad = \quad \mathsf{P}_+ \begin{bmatrix} \gamma_0^- U \\ -\vartheta \end{bmatrix} \ .$$

Hence, setting

$$W(\mathbf{x}) \quad := \quad \begin{cases} U(\mathbf{x}) \, , & \mathbf{x} \in \Omega^- \\ \Psi_{\mathrm{DL}}^{\kappa}\left(\gamma_0^- U\right)(\mathbf{x}) - \Psi_{\mathrm{SL}}^{\kappa}(-\vartheta)(\mathbf{x}) \, , & \mathbf{x} \in \Omega^+ \end{cases}$$

provides us with a solution to the Helmholtz transmission problem with zero right hand side. and thus uniqueness of solutions to the Helmholtz transmission problem finishes the proof. ☐

Eventually, the existence of solutions to the variational problem (31) follows from Thm. 7.3 and a Fredholm argument, see for instance [27, Thm. 2.33].

Finally, the arguments in the proof of Thm. 7.3 have also confirmed that we really get information about the solution of the Helmholtz transmission problem from (31):

Corollary 7.4. *If* $(W, \vartheta) \in H^1(\Omega^-) \times H^{-\frac{1}{2}}(\Gamma)$ *solves* (31)*, then* $W = U$ *and* $\vartheta = -\gamma_1^+ U^s$ *with* (U, U^s) *solving* (2)*.*

8 Regularisation operators

In this section we present a rather simple specimen of a trace transformation operator $\mathcal{T}$, which satisfies all the assumptions 6.2 and 6.1. Its main ingredient is a *regularising operator*

$$\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma) \ ,$$

which satisfies the following assumption.

Assumption 8.1. *We suppose that*

1. $\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ *is compact, and*

2. $(\vartheta, \mathsf{M}(\vartheta))_\Gamma > 0$ *for all* $\vartheta \in H^{-\frac{1}{2}}(\Gamma) \setminus \{0\}$.

Various examples of such operators are discussed in [7]. Below we will present a concrete representative. Then, for $\eta \in \mathbb{R} \setminus \{0\}$ we chose the following trace transformation operators

$$\mathcal{T}_1 := \begin{bmatrix} \mathsf{Id} & i\eta\mathsf{M} \\ i\eta & \mathsf{Id} \end{bmatrix}, \qquad \mathcal{T}_2 := \begin{bmatrix} \mathsf{Id} & i\eta\mathsf{M} \\ 0 & \mathsf{Id} \end{bmatrix} \tag{35}$$

Now, we have to verify the assumptions 6.1 and 6.2. We note that Ass. 6.1 can instantly be concluded from Ass. 8.1, **2.**, and Lemma 6.5. Items **2.** through **5.** of Ass. 6.2 are evident appealing to Ass. 8.1, **1.**. It is also obvious that $\mathcal{T}_2$ is bijective with

$$\mathcal{T}_2^{-1} = \begin{bmatrix} \mathsf{Id} & -i\eta\mathsf{M} \\ 0 & \mathsf{Id} \end{bmatrix}.$$

It remains to establish that $\mathcal{T}_1$ is bijective, too. Key will be the following lemma.

Lemma 8.2. *For* $\zeta \in \mathbb{R}_+$ *or* $\zeta \in i\mathbb{R}$ *the following operators are bijective*

$$\mathsf{Id} + \zeta\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \qquad \mathsf{Id} + \zeta\mathsf{M} : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma).$$

Proof. We verify that the operators have trivial kernel. In the first case we find that $(\mathsf{Id} + \zeta\mathsf{M})(\vartheta) = 0$ implies

$$(\vartheta, \varphi)_\Gamma + \overline{\zeta}\,(\mathsf{M}(\vartheta), \varphi)_\Gamma = 0,$$

which holds true for all $\varphi \in H^{\frac{1}{2}}(\Gamma)$. We choose $\varphi := \mathsf{M}(\vartheta)$ and we obtain

$$(\vartheta, \mathsf{M}(\vartheta))_\Gamma + \overline{\zeta}\,\|\mathsf{M}(\vartheta)\|^2_{L^2(\Gamma)} = 0.$$

For either $\zeta > 0$ or $\zeta \in i\mathbb{R}$ Ass. 8.1, **2.**, implies

$$(\vartheta, \mathsf{M}(\vartheta))_\Gamma = 0 \quad \Leftrightarrow \quad \vartheta = 0.$$

Thanks to Ass. 8.1, **1.**, we have a Fredholm alternative argument [27, Thm. 2.27] at our disposal and conclude that the operator $\mathsf{Id} + \zeta\mathsf{M} : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ is surjective from the fact that it is injective.

In the $H^{\frac{1}{2}}(\Gamma)$-setting $(\mathsf{Id} + \zeta\mathsf{M})(\varphi) = 0$ is equivalent to

$$(\vartheta, \varphi)_\Gamma + \zeta\,(\vartheta, \mathsf{M}(\varphi))_\Gamma = 0 \quad \forall \vartheta \in H^{-\frac{1}{2}}(\Gamma).$$

The same reasoning as above also settles this case. □

The lemma tells us that the formal inverse

$$\mathcal{T}_1^{-1} = \left(\mathsf{Id} + \eta^2\mathsf{M}\right)^{-1} \circ \begin{bmatrix} \mathsf{Id} & -i\eta\mathsf{M} \\ -i\eta & \mathsf{Id} \end{bmatrix}$$

is well defined, which implies Ass. 6.2, **1.**, for $\mathcal{T}_1$.

A particularly convenient regularising operator has been presented in [5]: there, $\mathsf{M} : H^{-1}(\Gamma) \to H^1(\Gamma)$ is implicitly defined by

$$(\mathbf{grad}_\Gamma \mathsf{M}(p), \mathbf{grad}_\Gamma q)_\Gamma + (\mathsf{M}(p), q)_\Gamma = (p, q)_\Gamma \tag{36}$$

for all $q \in H^1(\Gamma)$. It is an easy exercise to verify Ass. 8.1 for this M, see [5, Sect. 4.2]. For later use we define the following sesqui-linear form

$$\mathsf{b}(p, q) := (\mathbf{grad}_\Gamma p, \mathbf{grad}_\Gamma q)_\Gamma + (p, q)_\Gamma, \qquad p, q \in H^1(\Gamma), \tag{37}$$

which allows us to restate definition (36) as

$$\mathsf{b}(\mathsf{M}(p), q) = (p, q)_\Gamma \quad \forall q \in H^1(\Gamma). \tag{38}$$

9 Mixed Regularised Variational Formulations

Using the two trace transformation operators we obtain two variational formulations which are free from spurious resonances. However, from the point of view of boundary element discretization, they are not yet useful, because they still contain products of (non-local) operators that elude a straightforward Galerkin discretization. To get rid of the operator products, we rely on the usual trick and introduce extra unknown functions. We discuss the resulting variational problems for the trace transformation operators $\mathcal{T}_1$ and $\mathcal{T}_2$ from (35) and M given by (36):

Case $\mathcal{T} = \mathcal{T}_1$: find $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$

$$\begin{aligned} \mathsf{a}(U,V) - \left(\left(i\eta\left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right) - \mathsf{W}_\kappa\right)\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma & \\ + \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa - i\eta\mathsf{V}_\kappa\right)(\vartheta), \gamma_0^- V\right)_\Gamma &= f_1(V) \;, \\ \left(\varphi, \left(i\eta\mathsf{M}\circ\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right) + \mathsf{V}_\kappa\right)(\vartheta)\right)_\Gamma + \left(\varphi, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} - i\eta\mathsf{M}\circ\mathsf{W}_\kappa\right)\left(\gamma_0^- U\right)\right)_\Gamma &= g_1(\varphi) \;, \end{aligned} \tag{39}$$

where the right hand sides are given by

$$\begin{aligned} f_1(V) &:= \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma + \left(\left(i\eta\left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right) - \mathsf{W}_\kappa\right)(g_0), \gamma_0^- V\right)_\Gamma \;, \\ g_1(\varphi) &:= \left(\varphi, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa + i\eta\mathsf{M}\circ\mathsf{W}_\kappa\right)(g_0)\right)_\Gamma \;. \end{aligned}$$

Case $\mathcal{T} = \mathcal{T}_2$: find $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$

$$\begin{aligned} \mathsf{a}(U,V) + \left(\mathsf{W}_\kappa\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma + \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right)\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma &= f_2(V) \;, \\ \left(\varphi, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} - i\eta\mathsf{M}\circ\mathsf{W}_\kappa\right)\left(\gamma_0^- U\right)\right)_\Gamma + \left(\varphi, \left(i\eta\mathsf{M}\circ\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right) + \mathsf{V}_\kappa\right)(\vartheta)\right)_\Gamma &= g_2(\varphi) \;, \end{aligned} \tag{40}$$

where the right hand sides are given by

$$\begin{aligned} f_2(V) &:= \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma - \left(\mathsf{W}_\kappa(g_0), \gamma_0^- V\right)_\Gamma \;, \\ g_2(\varphi) &:= \left(\varphi, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa + i\eta\mathsf{M}\circ\mathsf{W}_\kappa\right)(g_0)\right)_\Gamma \;. \end{aligned}$$

Both regularised variational formulations contain the same operator products, namely

$$\begin{aligned} -\mathbb{B} &= \mathsf{V}_\kappa + i\eta\mathsf{M}\circ\left(\tfrac{1}{2}\mathsf{Id} + \mathsf{K}'_\kappa\right) \;, \\ \mathbb{A} - \mathsf{Id} &= \mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id} - i\eta\mathsf{M}\circ\mathsf{W}_\kappa \;. \end{aligned}$$

This suggests that we introduce the new variable

$$p := \left(\mathsf{M}\circ\left(\tfrac{1}{2} + \mathsf{K}'_\kappa\right)\right)(\vartheta) - \left(\mathsf{M}\circ\mathsf{W}_\kappa\right)\left(\gamma_0^- U + g_0\right) \in H^1(\Gamma) \;, \tag{41}$$

which converts (31) into the following two variational problems. The first arises from using $\mathcal{T}_1$: find $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\begin{aligned} \mathsf{a}(U,V) + i\eta\left(\left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma + \left(\mathsf{W}_\kappa\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma & \\ + \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}'_\kappa\right)(\vartheta), \gamma_0^- V\right)_\Gamma + i\eta\left(\mathsf{V}_\kappa(\vartheta), \gamma_0^- V\right)_\Gamma &= f_1(V) \;, \\ \left(\varphi, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)\left(\gamma_0^- U\right)\right)_\Gamma + \left(\varphi, \mathsf{V}_\kappa(\vartheta)\right)_\Gamma + i\eta\left(\varphi, p\right)_\Gamma &= g_1(\varphi) \;, \\ \left(\mathsf{W}_\kappa\left(\gamma_0^- U\right), q\right)_\Gamma - \left(\left(\mathsf{K}'_\kappa + \tfrac{1}{2}\mathsf{Id}\right)(\vartheta), q\right)_\Gamma + \mathsf{b}(p,q) &= h_1(q) \;, \end{aligned} \tag{42}$$

with right hand sides

$$\begin{aligned} f_1(V) &:= \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma - i\eta\left(\left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)(g_0), \gamma_0^- V\right)_\Gamma - \left(\mathsf{W}_\kappa(g_0), \gamma_0^- V\right)_\Gamma \;, \\ g_1(\varphi) &:= \left(\varphi, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa\right)(g_0)\right)_\Gamma \;, \\ h_1(q) &:= -\left(\mathsf{W}_\kappa(g_0), q\right)_\Gamma \;. \end{aligned}$$

The second arises from using $\mathcal{T}_2$: find $U \in H^1(\Omega^-)$, $\vartheta \in H^{-\frac{1}{2}}(\Gamma)$ and $p \in H^1(\Gamma)$ such that for all $V \in H^1(\Omega^-)$, $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ and $q \in H^1(\Gamma)$ there holds

$$\begin{aligned} \mathsf{a}(U,V) + \left(\mathsf{W}_\kappa\left(\gamma_0^- U\right), \gamma_0^- V\right)_\Gamma + \left(\left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa'\right)(\vartheta), \gamma_0^- V\right)_\Gamma &= f_2(V) \ , \\ \left(\varphi, \left(\mathsf{K}_\kappa - \tfrac{1}{2}\mathsf{Id}\right)\left(\gamma_0^- U\right)\right)_\Gamma + \left(\varphi, \mathsf{V}_\kappa(\vartheta)\right)_\Gamma + i\eta\left(\varphi, p\right)_\Gamma &= g_2(V) \ , \\ \left(\mathsf{W}_\kappa\left(\gamma_0^- U\right), q\right)_\Gamma - \left(\left(\mathsf{K}_\kappa' + \tfrac{1}{2}\mathsf{Id}\right)(\vartheta), q\right)_\Gamma + \mathsf{b}(p,q) &= h_2(q) . \end{aligned} \tag{43}$$

with right hand sides

$$\begin{aligned} f_2(V) &:= \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma - \left(\mathsf{W}_\kappa(g_0), \gamma_0^- V\right)_\Gamma \ , \\ g_2(\varphi) &:= \left(\varphi, \left(\tfrac{1}{2}\mathsf{Id} - \mathsf{K}_\kappa\right)(g_0)\right)_\Gamma \ , \\ h_2(q) &:= -\left(\mathsf{W}_\kappa(g_0), q\right)_\Gamma \ . \end{aligned}$$

In order to settle the issue of existence and uniqueness of solutions of (42) and (43) we first observe that by the very definition of M in (36) and (41) the first two components of any solution (U, ϑ, p) of (42) and (43) will also solve (39) and (40), respectively. Since these are special cases of (31) and both $\mathcal{T}_1$ and $\mathcal{T}_2$ are valid trace transformation operators, Thm. 7.3 yields uniqueness.

Next, it follows directly from the compact embeddings $H^1(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma)$ and $H^{-\frac{1}{2}}(\Gamma) \hookrightarrow H^{-1}(\Gamma)$ that all new off diagonal terms are compact sesqui-linear forms. Since b is $H^1(\Gamma)$-elliptic, we obtain that the sesqui-linear forms for both variational formulations satisfy a generalized Gårding inequality.

Again, a Fredholm argument ensures the existence of solutions from the the uniqueness result. The statement of Cor. 7.4 directly carries over to the (U, ϑ)-components of (39) and (40). Thus we have obtained two well-posed variational formulations which yield weak solutions to the Helmholtz transmission problem 2 and which are also amenable to standard Galerkin discretizations.

We finish this section by an important observation: (41) can be recast into

$$p = \left(\mathsf{M} \circ \left(\tfrac{1}{2} + \mathsf{K}_\kappa'\right)\right)(\vartheta) - \left(\mathsf{M} \circ \mathsf{W}_\kappa\right)\left(\gamma_0^- U + g_0\right) \ .$$

At second glance, we realize that $p = 0$, if (U, ϑ) solve (39) and (40), respectively. This directly follows from Cor. 7.4, Thm. 4.1 and the definition of the exterior Calderón projector P_+. In short, p is a "dummy variable".

Remark 9.1. Under the assumptions made in remark 4.2 we can derive a Dirichlet-to-Neumann map from (26), to obtain coupled variational problems of the form (42) and (43) with much simpler right hand sides

$$\begin{aligned} f_1(V) &= \mathsf{f}(V) + i\eta\left(g_0, \gamma_0^- V\right)_\Gamma - \left(g_1, \gamma_0^- V\right)_\Gamma \ , & f_2(V) &= \mathsf{f}(V) - \left(g_1, \gamma_0^- V\right)_\Gamma \ , \\ g_1(\varphi) &= (\varphi, g_0)_\Gamma \ , & g_2(\varphi) &= +(\varphi, g_0)_\Gamma \ , \\ h_1(q) &= -(g_1, q)_\Gamma \ , & h_2(q) &= -(g_1, q)_\Gamma \ . \end{aligned}$$

The solution U in Ω^- will remain the same.

10 Galerkin Discretization

With operator products removed, the Galerkin discretization of the variational problems (39) and (40) is easily achieved by restricting them to finite element subspaces $\mathcal{V}_h$ of $H^1(\Omega^-)$ and boundary element subspaces Θ_h and $\mathcal{Q}_h$ of $H^{-\frac{1}{2}}(\Gamma)$ and $H^1(\Gamma)$, respectively. A powerful theorem about the Galerkin approximation of coercive variational problems, see [31] and [34], will then yield the *asymptotic quasi-optimality* of the Galerkin solutions: assuming a minimal resolution of $\mathcal{V}_h$, Θ_h, and $\mathcal{Q}_h$, existence and uniqueness of discrete solutions $(U_h, \vartheta_h, p_h) \in \mathcal{V}_h \times \Theta_h \times \mathcal{Q}_h$ of (39) and (40) is guaranteed and we have the a priori error estimate

$$\begin{aligned} &\|U - U_h\|_{H^1(\Omega^-)} + \|\vartheta - \vartheta_h\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\qquad \leq \gamma\left(\inf_{V_h \in \mathcal{V}} \|U - V_h\|_{H^1(\Omega^-)} + \inf_{\varphi_h \in \Theta_h} \|\vartheta - \varphi_h\|_{H^{-\frac{1}{2}}(\Gamma)}\right) \ , \end{aligned} \tag{44}$$

where the constant $\gamma > 0$ does not depend on the discrete trial spaces.

The standard choices for $\mathcal{V}_h$, Θ_h, and $\mathcal{Q}_h$ are based on a tetrahedral or quadrilateral mesh $\mathcal{M}$ of Ω^-, which yields a mesh $\mathcal{M}_\Gamma$ of Γ by plain restriction to Γ. Then we may pick

$$\begin{aligned} \mathcal{V}_h &:= \{V \in C^0(\Omega^-) :\ V_{|K} \in \mathcal{P}_k(K)\ \forall K \in \mathcal{M}\}\ , \\ \Theta_h &:= \{\varphi \in L^2(\Gamma) :\ \varphi_{|K} \in \mathcal{P}_{k-1}(K)\ \forall K \in \mathcal{M}_\Gamma\}\ , \\ \mathcal{Q}_h &:= \{q \in C^0(\Gamma) :\ q_{|K} \in \mathcal{P}_k(K)\ \forall K \in \mathcal{M}_\Gamma\}\ . \end{aligned} \tag{45}$$

Here, $\mathcal{P}_k(K)$ stands for the space of polynomials of degree $\leq k$ on the cell K. This refers to the total degree in the case of tetrahedra and the degree in each variable in the case of hexahedra.

Then, the usual best approximation estimates [30] for the h-version of finite elements and boundary elements give us

$$\begin{aligned} \inf_{V_h \in \mathcal{V}_h} \|U - V_h\|_{H^1(\Omega^-)} &\leq \gamma h^{\min\{s-1,k\}} \|U\|_{H^s(\Omega^-)}\ , \\ \inf_{\varphi_h \in \Theta_h} \|\vartheta - \varphi_h\|_{H^{-\frac{1}{2}}(\Gamma)} &\leq \gamma h^{\min\{s+1/2,k\}} \|\vartheta\|_{H^s(\Gamma)}\ , \end{aligned}$$

with constants depending on the shape regularity of $\mathcal{M}$ and $h > 0$ denoting the meshwidth of $\mathcal{M}$.

Remark 10.1. Why do we have to approximate the dummy variable p at all, though it vanishes and apparently the choice of $\mathcal{Q}_h$ does not affect the convergence of Galerkin solutions. The reason is that (44) is an asymptotic statement, whose proof also hinges on sufficiently good approximation properties of $\mathcal{Q}_h$. In the context of the h-version of finite elements and boundary elements, this means that the mesh has to be sufficiently fine to make (44) hold.

11 Numerical Experiments

The above theoretical developments are set in three dimensions, but they carry over verbatim to two dimensions, when replacing the kernel G_κ by

$$G_\kappa(z) := \frac{i}{4} H_0^{(1)}(kz)\ , \tag{46}$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order zero.

For the numerical experiments we considered the unit circle $\Omega_\circ^- := \{\mathbf{x} \in \mathbb{R}^2 :\ |\mathbf{x}| < 1\}$ and the unit square $\Omega_\Box^- := \{\mathbf{x} \in \mathbb{R}^2 :\ -1/2 < x_1, x_2 < 1/2\}$. For the former we find the two lowest interior resonant frequencies $\kappa_1 = 5.5201$ and $\kappa_2 = 11.7915$, which correspond to the second and fourth zero of the Bessel function $J_0(x)$. For the square we two lowest resonant frequencies are $\kappa_3 = 2\pi/\sqrt{2}$ and $\kappa_4 = 5\pi/\sqrt{2}$.

On each domain regular finite element meshes $\mathcal{M}_l$, $l \in \mathbb{N}$, consisting of quadrilaterals with straight edges were used. In the case of $\Omega_\circ^-$ the triangulation $\mathcal{M}_l$ is created by inscribing $\Omega_\circ^-$ a regular 2^{l+3}-gon and a centered unit square. The portions of the line segments from the center to the corners of the polygon are split into 2^l equal parts, whose endpoints are connected to form a quadrilateral mesh outside the unit square. This is extended by an orthogonal tensor product mesh inside the unit square. The mesh $\mathcal{M}_1$ is drawn in Fig. 1. The family of meshes arising from this construction will be quasi-uniform and shape-regular with meshwidth of $\mathcal{M}_l$ being proportional to $2^{-(l+1)}$.

On $\Omega_\Box^-$ the mesh $\mathcal{M}_l$ is a plain uniform orthogonal tensor product grid with meshwidth $h = 2^{-(l+1)}$.

We used mapped bilinear Lagrangian finite elements to build $\mathcal{V}_h$, piecewise constants on $\mathcal{M}_\Gamma$ for Θ_h, and linear surface elements for $\mathcal{Q}_h$, that is, the case $k = 1$ of (45). The finite element stiffness matrix was assembled using a four-point Gaussian quadrature rule on the reference element. All computations were done in MATLAB and a direct solver was used whenever we aimed to study discretization errors. As far as the stable regularized coupled schemes are concerned we consistently rely on the variants discussed in Sect. 9.

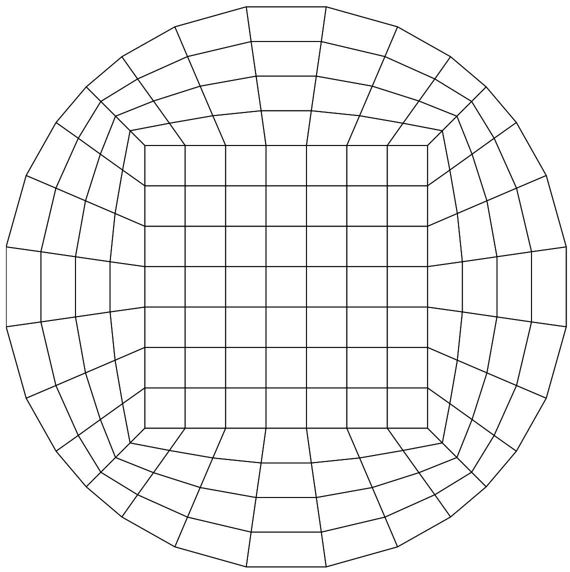

Figure 1: Quadrilateral mesh of the unit circle

In all the experiments we used $n(\mathbf{x}) = 1$ in Ω^- and excitation by incident plane waves. These will also provide the exact solutions. Please note that in this setting $\vartheta = 0$, because there is no scattered field.

If analytic solutions are known, we measure the discretization error in the interior total field in either the $H^1(\Omega^-)$ or the $L^2(\Omega^-)$-norm and the error in ϑ in either the $H^{-\frac{1}{2}}(\Gamma)$ or the $L^2(\Gamma)$-norm. To compute the norms on the domain Ω^-, we use a local four-point Gaussian quadrature rule. The fractional Sobolev norm is evaluated by means of the discrete single layer potential operator on that mesh. To evaluate the fourth norm we again rely on the same local four-point Gaussian quadrature rule.

In a first experiment, a plane incident wave $U^{\mathrm{i}}(\mathbf{x}) = \exp(-i\kappa\mathbf{d}\cdot\mathbf{x})$, $|\mathbf{d}| = 1$, is used, where the incident angle between the propagation direction $\mathbf{d}$ and the x-axis is $\pi/4$. We measure the discretization errors in different norms on the domain $\Omega^-_{\square}$ for the two frequencies κ_3 and κ_4 on a series of shape-regular meshes using the second regularized variational formulation (43) with a coupling parameter $\eta = 1$.

From our computations we obtained the following experimental convergence rates for the first experiment:

1. The discretization error in the U-component measured in the $H^1(\Omega^-)$-norm yields an algebraic convergence rate of ≈ 1 for both κ_3 and κ_4. The discretization error in the ϑ-component measured in the $H^{-\frac{1}{2}}(\Gamma)$-norm decays with a rate of 2 for both κ_3 or κ_4. This is much more than expected from the mere approximation properties of the trial spaces, but this may be due to the fact that $\vartheta = 0$.

2. The discretization error in the U-component measured in the $L^2(\Omega^-)$-norm and the ϑ-component in the $L^2(\Gamma)$-norm results in a convergence rate of 2 in the case of κ_3 and κ_4.

The second experiment, see figures 4 and 5, still asumes plane wave incident fields with the same incident angle. This time we measure the discretization errors on a series of shape-regular meshes of the unit circle $\Omega^-_{\circ}$ for the two frequencies κ_1 and κ_2.

The numerical results of our second experiment are similar to the results of our first experiment and can be summed up as follows:

1. The $H^1(\Omega^-)$-norm of discretization errors in the U-component decays algebraically with a rate of ≈ 1 in each case. As above, we observe a convergence rate of ≈ 2 for the $H^{-\frac{1}{2}}(\Gamma)$-norm of the ϑ-unknown.

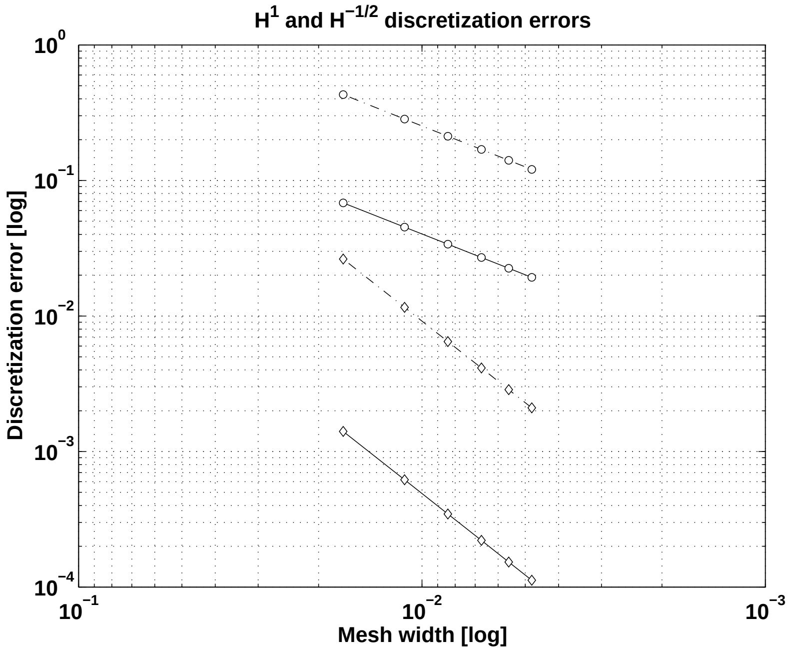

Figure 2: Discretization errors for κ_3 $(-)$ and κ_4 $(-\cdot)$ on the unit square $\Omega_\Box^-$. Errors in the total interior field U are labeled with $\circ$ and errors in the ϑ-component with $\diamond$.

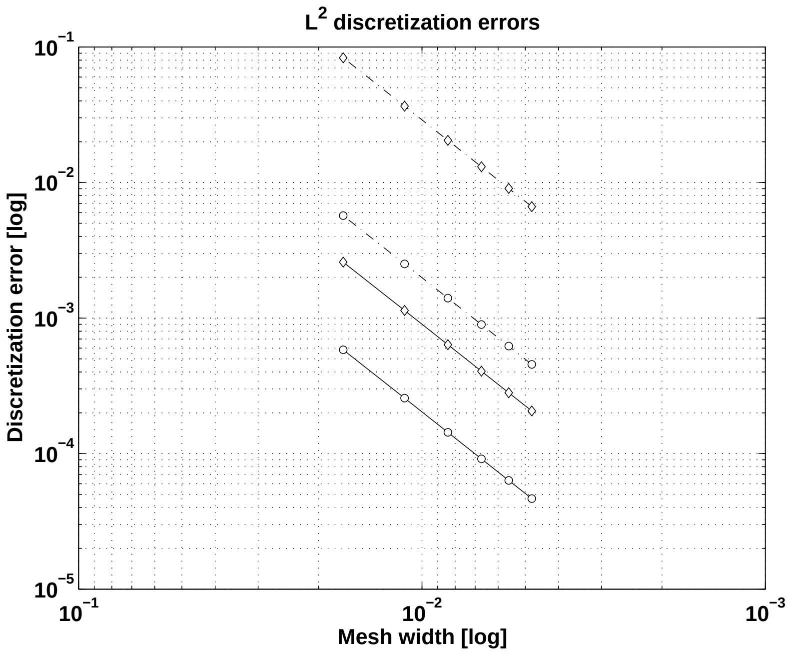

Figure 3: Discretization errors for κ_3 $(-)$ and κ_4 $(-\cdot)$ on the unit square $\Omega_\Box^-$. Errors in the total interior field U are labeled with $\circ$ and errors in the ϑ-component with $\diamond$.

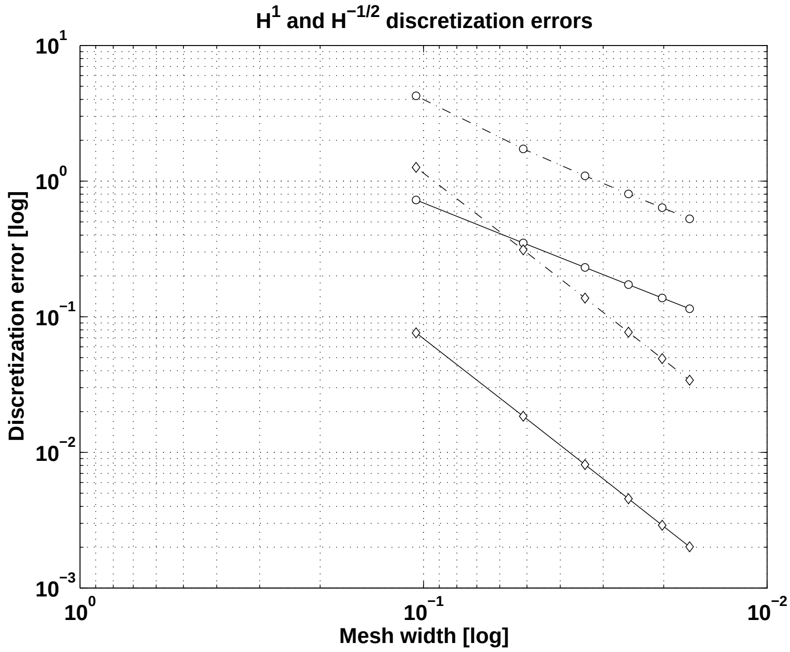

Figure 4: Discretization errors for κ_1 $(-)$ and κ_2 $(-\cdot)$ on the unit circle $\Omega_\circ^-$. Errors in the total interior field U are labeled with $\circ$ and errors in the ϑ-component with $\diamond$.

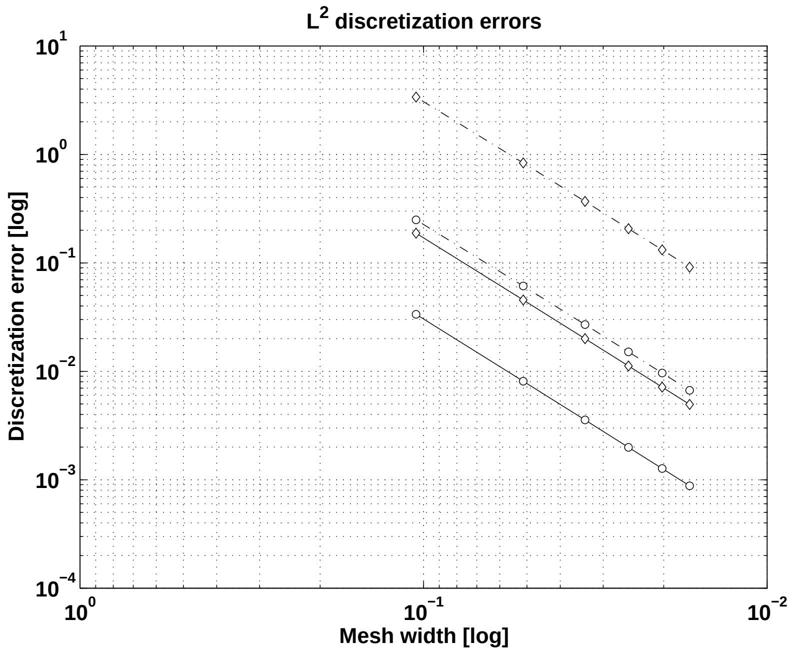

Figure 5: Discretization errors for κ_1 $(-)$ and κ_2 $(-\cdot)$ on the unit circle $\Omega_\circ^-$. Errors in the total interior field U are labeled with $\circ$ and errors in the ϑ-component with $\diamond$.

2. The discretization errors in U and ϑ measured in the $L^2(\Omega^-)$-norm and $L^2(\Gamma)$-norm, respectively, converge to zero with a rate of ≈ 2 for all wave numbers.

The third experiment, see figure 6, shows the dependency of the discretization error, measured in the $H^1(\Omega^-)$ and $H^{-\frac{1}{2}}(\Gamma)$-norms, on the wave number for a mesh of the domain $\Omega_\square^-$ with 14161 elements using the second regularized variational formulation (43) with a coupling parameter $\eta = 1$. This time the incident angle was choosen to be equal to $\pi/4$.

It is hardly surprising that the discretization error grows as κ increases, because this is already observed, if the Helmholtz equation is discretized by means of low order Lagrangian finite elements alone [2].

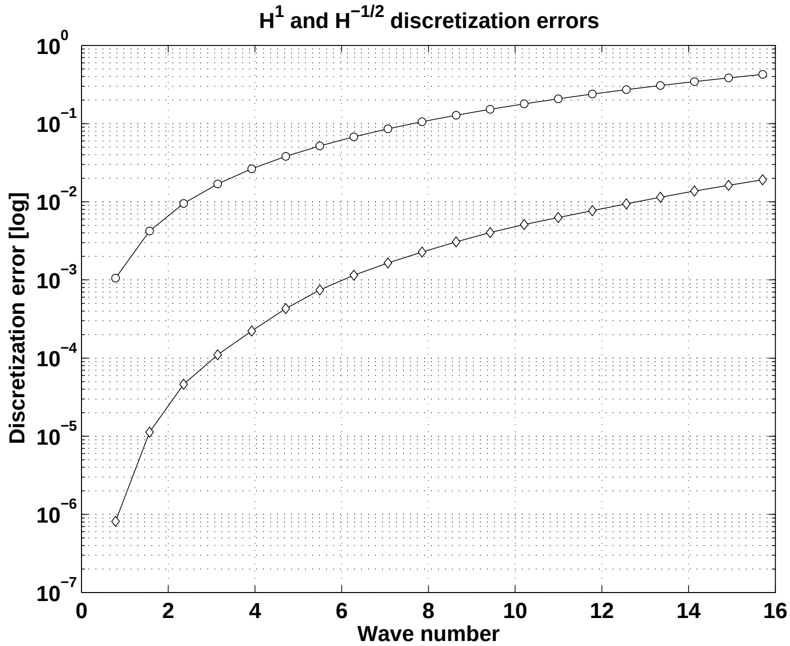

Figure 6: Discretization errors on the unit square $\Omega_\square^-$. Errors in the total interior field U are labeled with $\circ$ and errors in the ϑ-component with $\diamond$.

The fourth experiment, see figure 7, shows the dependency of the condition number of the entire system matrix on the wave number for

1. the symmetric FEM-BEM coupling (15) and
2. the second version of regularized FEM-BEM coupling (43)

in the neighbourhood of the resonant frequency κ_3 for a mesh of the domain $\Omega_\square^-$ with 14161 elements. In each case the extremal eigenvalues were computed by means of direct in inverse power iterations. Obviously, regularization manages to suppress the pronounced peak in the condition number of the symmtrically coupled problem.

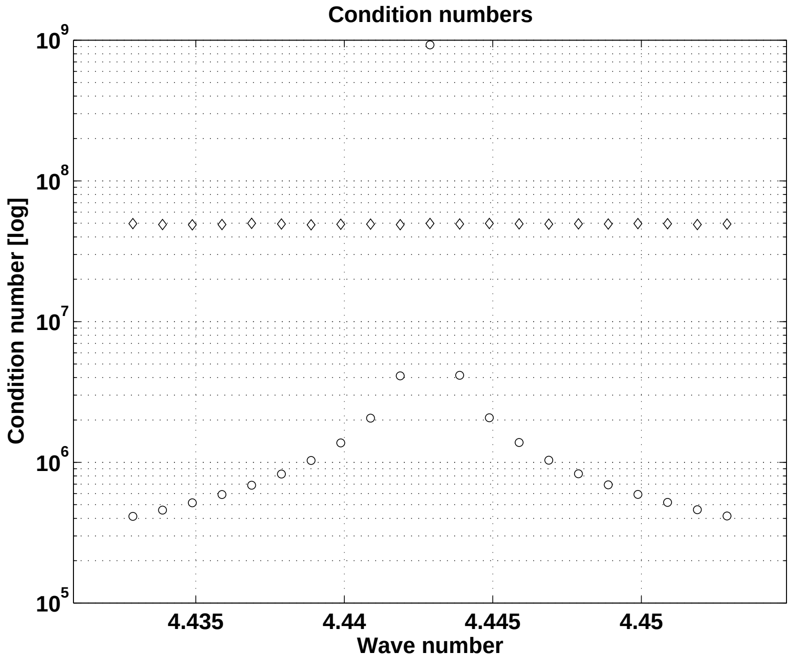

Figure 7: Condition numbers on the unit square $\Omega_{\square}^{-}$ close to the resonant frequency κ_3. Condition numbers of the matrix underlying the classical variational formulation are labelled with $\circ$, whereas condition numbers related to the matrix underlying the regularized variational formulation are labelled with $\diamond$.

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