Analysis of a dual-primal FETI algorithm for edge elements on boundary layer meshes in two dimensions

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Abstract

FETI methods are among the most heavily tested domain decomposition methods. The purpose of this thesis is to analyze a dual-primal FETI method for *hp* edge element approximations in two dimensions on geometrically refined meshes. These meshes are highly anisotropic, where the aspect ratio grows exponentially with the polynomial degree. The primal constraints are here averages over subdomain edges. We prove that the condition number of our algorithm grows only polylogarithmically with the polynomial degree and is independent of the aspect ratio of the mesh and of potentially large jumps of the coefficients.

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1 Introduction

In this thesis, boundary value problems of the type

$$\begin{aligned} L\mathbf{u} := \mathbf{curl}\,(a\,\mathrm{curl}\,\mathbf{u}) + A\mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{t} &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

are considered, with Ω a bounded polygonal domain in $\mathbb{R}^2$ and $\mathbf{t}$ its unit tangent. For the definition of the curl operators see Subsection 2.4. Problems of such a type are derived for instance from reformulations of Maxwell's equations, see [10], Subsection 3.1 for a detailed discussion. It is well known that solutions of boundary value problems have corner singularities. In addition, boundary layers may also arise when small parameters are present. To achieve an exponential rate of convergence of *hp* finite element methods, meshes that are geometrically refined towards corners and edges are employed. These meshes are highly anisotropic and therefore many standard results for *hp* finite element discretizations do not hold. In this thesis, a dual-primal FETI method for the solution of the finite element approximation on these type of meshes is proposed and analyzed. It turns out that the condition number of the method grows only polylogarithmically with the polynomial degree k and is independent of possibly large aspect ratios of the mesh. This thesis is a generalization of [19] to the problem 1 for *hp* finite element approximations on anisotropic meshes.

1.1 Overview

In Subsection 1.2, the problem treated in this thesis will be introduced and its variational formulation be derived. In Section 2, the spaces needed for the variational formulation of the problem are introduced, together with some basic properties that will be used throughout the thesis. The construction of boundary layer meshes and the *hp* discretization of the problem on such meshes is then given in Section 3; this section is ended with a first important result: the discrete Friedrichs' inequality on anisotropic meshes, which will be used in order to prove an important decomposition result, see Subsection 4.4. Section 4 then introduces the dual-primal FETI method. In Subsection 4.2 we describe the method, whereas in Subsection 4.3 the condition number bound is proved. Finally, in Subsection 4.4, a decomposition result for geometrically refined meshes is proven. This last result is the second important result of this work.

1.2 Description of the problem

We consider problem (1). The coefficient matrix A is a symmetric, uniformly positive definite matrix-valued function with entries $A_{ij} \in L^\infty(\Omega)$, $1 \leqslant i,j \leqslant 2$, and $a \in L^\infty(\Omega)$ is a positive function bounded away from zero.

By multiplying (1) with a test function $\mathbf{v}$, integrating over Ω and using Green's formula (6) we obtain the variational formulation:

Find $\mathbf{u} \in V$, such that

$$\int_\Omega a\,\mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v}\,dx + \int_\Omega A\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx \qquad \forall\,\mathbf{v} \in V, \tag{2}$$

where V is a curl conforming space with vanishing tangential component along $\partial\Omega$. Such a space is $H_0(\mathrm{curl}\,,\Omega)$ and will be introduced in Subsection 2.4. For any $\mathcal{D} \subset \Omega$ we can define the bilinear form

$$a_\mathcal{D}(\mathbf{u},\mathbf{v}) := \int_\mathcal{D} (a\;\mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v} + A\mathbf{u}\cdot\mathbf{v})\;dx, \quad \mathbf{u},\mathbf{v} \in V \tag{3}$$

and the linear functional

$$l_{\mathcal{D}}(\mathbf{v}) := \int_{\mathcal{D}} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \mathbf{v} \in V. \tag{4}$$

The variational problem (2) can now be rewritten as:

Find $\mathbf{u} \in V$, such that

$$a_{\Omega}(\mathbf{u}, \mathbf{v}) = l_{\Omega}(\mathbf{v}), \qquad \forall\, \mathbf{v} \in V. \tag{5}$$

2 Function spaces

In this section, we introduce some function spaces that will be used throughout this thesis. They are presented together with some basic properties and well-known results that will be employed in this work. For a detailed discussion of these spaces we refer to [1], [4] and [14].

2.1 L^p-spaces

In the following, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded and open set with Lipschitz continuous boundary.

Definition 2.1 *The boundary $\partial\Omega$ is Lipschitz continuous if there exists a finite number of open sets $\mathcal{O}_i$, $i = 1, \dots, m$ that cover $\partial\Omega$, such that, for every i, the intersection $\partial\Omega \cap \mathcal{O}_i$ is the graph of a Lipschitz continuous function and $\Omega \cap \mathcal{O}_i$ lies on one side of this graph.*

$L^p(\Omega)$ is the space of Lebesgue measurable functions u with $||u||_{L^p(\Omega)} = ||u||_{0,p,\Omega} < \infty$ where the norms are given by

$$\begin{aligned} ||u||_{0,p,\Omega} &= \left(\int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}}, \\ ||u||_{0,\infty,\Omega} &= \operatorname*{ess\,sup}_{\Omega} |u|. \end{aligned}$$

$L^p(\Omega)$ is a Banach space and for $p = 2$ it is in addition a Hilbert space with the inner product

$$(u, v)_{0,\Omega} = \int_{\Omega} uv \, dx.$$

For simplicity, the norm $||\cdot||_{0,2,\Omega}$ will be denoted by $||\cdot||_{0,\Omega}$. If $p > 1$ and q satisfies $\frac{1}{p} + \frac{1}{q} = 1$, then the dual space of $L^p(\Omega)$ can be identified with $L^q(\Omega)$ in a natural way.

2.2 Sobolev spaces

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ be a multi-index. The norm of α and the α-derivatives for a sufficient smooth function u are defined as

$$\begin{aligned} |\alpha| &= \alpha_1 + \cdots + \alpha_n, \\ D^{\alpha} u &= \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u. \end{aligned}$$

To define the Sobolev spaces, weak derivatives need to be introduced:

Definition 2.2 *A function $u : \Omega \to \mathbb{R}$ is called k times weak differentiable if, for all $|\alpha| \leqslant k$, there exist functions $v_{\alpha} : \Omega \to \mathbb{R}$ such that*

$$\int_{\Omega} u D^{\alpha} \varphi \, dx = (-1)^{|\alpha|} \int_{\Omega} v_{\alpha} \varphi \, dx, \qquad \forall \varphi \in C_0^{\infty}(\Omega).$$

The function v_{α} is then called the α-th weak derivative of u.

The space $C_0^\infty(\Omega)$ consists of functions in $C^\infty(\Omega)$ with compact support in Ω. The Sobolev spaces $W^{k,p}(\Omega)$ are then defined as the space of functions $u \in L^p(\Omega)$ for which the weak derivatives up to order k exist and belong to $L^p(\Omega)$. The norm on $W^{k,p}(\Omega)$ is given by

$$||u||^p_{W^{k,p}(\Omega)} = ||u||^p_{k,p,\Omega} = \sum_{|\alpha|\leqslant k} ||D^\alpha u||^p_{0,p,\Omega}.$$

With an abuse of notation D^α is in the following understood as the α-th weak derivative. A seminorm on $W^{k,p}(\Omega)$ is given by

$$|u|^p_{k,p,\Omega} = \sum_{|\alpha|=k} ||D^\alpha u||^p_{0,p,\Omega}.$$

The Sobolev spaces are Banach spaces and for $p = 2$ they are Hilbert spaces with the inner product

$$(u, v)_{k,\Omega} = \sum_{|\alpha|\leqslant k} (D^\alpha u, D^\alpha v)_{0,\Omega},$$

and are denoted by $H^k(\Omega) := W^{k,2}(\Omega)$. If s is a non-negative real number and not an integer, the Sobolev spaces $W^{s,p}(\Omega)$ are defined in the following way:

Definition 2.3 *Let $\Omega \subset \mathbb{R}^n$ and let $s > 0$ be a non-integer number. Write s as $s = \lfloor s \rfloor + \sigma$ with $\sigma \in (0, 1)$. Then $u \in W^{s,p}(\Omega)$ if and only if $u \in W^{\lfloor s \rfloor,p}(\Omega)$ and for the seminorm it holds $|u|_{s,p,\Omega} < \infty$, where*

$$\begin{aligned} |u|^p_{s,p,\Omega} &= \sum_{|\alpha|=\lfloor s \rfloor} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+\sigma p}}\, dx\, dy, \\ |u|_{s,\infty,\Omega} &= \sum_{|\alpha|=\lfloor s \rfloor} \operatorname{ess\,sup}_{x\neq y\in\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\sigma}. \end{aligned}$$

As before $H^s(\Omega) := W^{s,2}(\Omega)$. The spaces $W_0^{s,p}(\Omega)$ and $H_0^s(\Omega)$ are defined as the closure of $C_0^\infty(\Omega)$ with respect to the $||\cdot||_{s,p,\Omega}$-norm and $||\cdot||_{s,\Omega}$-norm, respectively. One can prove that $W_0^{s,p}(\Omega)$ is a proper subspace of $W^{s,p}(\Omega)$ if and only if $s > \frac{1}{2}$. The spaces with a negative s are defined by duality. Hence,

$$\begin{aligned} W^{-s,p}(\Omega) &:= \left(W_0^{s,\frac{p}{p-1}}(\Omega)\right)', \\ H^{-s}(\Omega) &:= (H_0^s(\Omega))'. \end{aligned}$$

2.3 Trace spaces

In this subsection, Sobolev spaces on a set $\Sigma \subseteq \partial\Omega$ are defined. Similar as by the last subsection, the spaces $W^{s,p}(\Sigma)$, $s \geqslant 0$, are defined as the space consisting of functions on Σ such that

$$||u||^p_{W^{s,p}(\Sigma)} = ||u||^p_{W^{\lfloor s \rfloor,p}(\Sigma)} + |u|^p_{W^{s,p}(\Sigma)} < \infty,$$

where the norm and seminorm on the right hand side are defined as in the previous subsection with the obvious changes. Denote by γ_0 the operator that maps a function in $C(\Omega)$ into its boundary values in $C(\partial\Omega)$. The following theorem can be found in [9], Section 1.5.2.

Theorem 2.4 (Trace theorem) *Assume that Ω is Lipschitz continuous and $s > \frac{1}{2}$. Then the mapping γ_0 has a continuous extension to an operator*

$$H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega),$$

Remark 2.5 *With the same assumptions as in Theorem 2.4, there exists a continuous lifting operator* $\mathcal{R}_0 : H^{s-\frac{1}{2}}(\partial\Omega) \to H^s(\Omega)$, *such that* $\gamma_0(\mathcal{R}_0 u) = u$, $u \in H^{s-\frac{1}{2}}(\partial\Omega)$.

Hence, by mapping from Ω to $\partial\Omega$, we have a loss of regularity of order $\frac{1}{2}$.

The extensions by zero to $\partial\Omega$ of functions in $H^{\frac{1}{2}}(\Sigma)$ do not in general belong to $H^{\frac{1}{2}}(\partial\Omega)$. The space $H_{00}^{\frac{1}{2}}(\Sigma)$ is therefore defined as

$$H_{00}^{\frac{1}{2}}(\Sigma) := \{u \in H^{\frac{1}{2}}(\Sigma) | \mathcal{E}u \in H^{\frac{1}{2}}(\partial\Omega)\},$$

where $\mathcal{E}u$ is the extension by zero of u to $\partial\Omega$. The corresponding norm is defined by

$$||u||_{\frac{1}{2},00,\Sigma} := ||\mathcal{E}u||_{\frac{1}{2},\partial\Omega}$$

2.4 The space $H(\mathrm{curl}\,, \Omega)$ in two dimensions

Let $\mathbf{u}$ be a two-dimensional vector field and q be a scalar function. The vector and scalar curl operators are defined, respectively, by

$$\mathbf{curl}\, q = \left(\frac{\partial q}{\partial x_2}, -\frac{\partial q}{\partial x_1}\right),$$

and

$$\mathrm{curl}\, \mathbf{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}.$$

$H(\mathrm{curl}\,, \Omega)$ is then defined as

$$H(\mathrm{curl}\,, \Omega) := \{\mathbf{u} \in (L^2(\Omega))^2 | \mathrm{curl}\, \mathbf{u} \in L^2(\Omega)\}.$$

It is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_{\mathrm{curl}\,,\Omega} = \int_\Omega \mathbf{u} \cdot \mathbf{v}\, dx + \int_\Omega \mathrm{curl}\, \mathbf{u}\, \mathrm{curl}\, \mathbf{v}\, dx,$$

and the associated norm

$$||\mathbf{u}||^2_{\mathrm{curl}\,,\Omega} = ||\mathbf{u}||^2_{0,\Omega} + ||\mathrm{curl}\, \mathbf{u}||^2_{0,\Omega}.$$

Let $\mathbf{t}$ and $\mathbf{n}$ denote the unit tangent and unit normal vector, respectively, on the boundary $\partial\Omega$. As in the previous subsection, a trace operator γ_t can be defined, mapping vectors $\mathbf{u}$ on Ω onto their tangential component on the boundary, $\gamma_t(\mathbf{u}) = \mathbf{u} \cdot \mathbf{t}$. A similar trace theorem as in the previous subsection holds as well, see [8], Theorem I.2.11:

Theorem 2.6 *Let* Ω *be Lipschitz continuous. Then the operator* γ_t *can be extended continuously to an operator* $\gamma_t : H(\mathrm{curl}\,, \Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$.

Remark 2.7 *Under the same assumptions as in Theorem 2.6, there exists a continuous lifting operator* $\mathcal{R}_t : H^{-\frac{1}{2}}(\partial\Omega) \to H(\mathrm{curl}\,, \Omega)$ *such that* $\gamma_t(\mathcal{R}_t u) = u$, $u \in H^{-\frac{1}{2}}(\partial\Omega)$.

For $\mathbf{u} \in H(\mathrm{curl}\,, \Omega)$ and $q \in H^1(\Omega)$ there holds the Green formula:

$$\int_\Omega \mathrm{curl}\, \mathbf{u}\, q\, dx - \int_\Omega \mathbf{u} \cdot \mathbf{curl}\, q\, dx = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{t}\, q\, dS. \tag{6}$$

The following subspaces are well defined:

$$\begin{aligned} H_0(\mathrm{curl}\,, \Omega) &= \{\mathbf{u} \in H(\mathrm{curl}\,, \Omega) | \mathbf{u} \cdot \mathbf{t} = 0 \text{ on } \partial\Omega\}, \\ H(\mathrm{curl}\,_0, \Omega) &= \{\mathbf{u} \in H(\mathrm{curl}\,, \Omega) | \mathrm{curl}\, \mathbf{u} = 0\}, \\ H_0(\mathrm{curl}\,_0, \Omega) &= \{\mathbf{u} \in H_0(\mathrm{curl}\,, \Omega) | \mathrm{curl}\, \mathbf{u} = 0\}. \end{aligned}$$

2.5 Helmholtz decompositions

The Helmholtz decompositions are orthogonal decompositions of the spaces $H(\mathrm{curl}\,,\Omega)$ and $H_0(\mathrm{curl}\,,\Omega)$. The decompositions are based on the following decomposition result for $(L^2(\Omega))^n$:

Lemma 2.8 *The space $(L^2(\Omega))^n$ allows the following orthogonal decompositions:*

$$\begin{aligned} (L^2(\Omega))^n &= H(\mathrm{div}_0,\Omega) \oplus \mathbf{grad} H_0^1(\Omega), \\ (L^2(\Omega))^n &= H_0(\mathrm{div}_0,\Omega) \oplus \mathbf{grad} H^1(\Omega) \end{aligned}$$

where div *denotes the divergence operator and*

$$\begin{aligned} H(\mathrm{div}_0,\Omega) &= \{\mathbf{u} \in (L^2(\Omega))^2 | \mathrm{div}\mathbf{u} = 0\}, \\ H_0(\mathrm{div}_0,\Omega) &= \{\mathbf{u} \in H(\mathrm{div}_0,\Omega) | \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Proof See Proposition 1 on page 215 of [6].
Since $H(\mathrm{curl}\,,\Omega) \subset (L^2(\Omega))^2$ and $\mathbf{grad} H^1(\Omega) \subset H(\mathrm{curl}\,,\Omega)$, Lemma 2.8 gives rise to the following decompositions of $H(\mathrm{curl}\,,\Omega)$ and $H_0(\mathrm{curl}\,,\Omega)$:

Proposition 2.9 (Helmholtz decomposition of the curl-spaces) *The* curl *graph spaces allow the following orthogonal decompositions:*

$$\begin{aligned} H(\mathrm{curl}\,,\Omega) &= \mathbf{grad} H^1(\Omega) \oplus H^\perp(\mathrm{curl}\,,\Omega), \\ H_0(\mathrm{curl}\,,\Omega) &= \mathbf{grad} H_0^1(\Omega) \oplus H_0^\perp(\mathrm{curl}\,,\Omega), \end{aligned}$$

where

$$\begin{aligned} H^\perp(\mathrm{curl}\,,\Omega) &= H_0(\mathrm{div}_0,\Omega) \cap H(\mathrm{curl}\,,\Omega), \\ H_0^\perp(\mathrm{curl}\,,\Omega) &= H(\mathrm{div}_0,\Omega) \cap H_0(\mathrm{curl}\,,\Omega). \end{aligned}$$

Remark 2.10 *For a simply connected Ω it holds*

$$H(\mathrm{curl}\,_0,\Omega) = \mathbf{grad} H^1(\Omega),$$

and

$$H_0(\mathrm{curl}\,_0,\Omega) = \mathbf{grad} H_0^1(\Omega).$$

Hence, in this case, $H(\mathrm{curl}\,,\Omega)$ and $H_0(\mathrm{curl}\,,\Omega)$ can be decomposed into the kernel of the curl *operator and its orthogonal complement.*

For a simply connected $\Omega \subset \mathbb{R}^2$, the following deRahm diagram holds

$$H^1(\Omega)/\mathbb{R} \xrightarrow{\mathbf{grad}} H(\mathrm{curl}\,,\Omega) \xrightarrow{\mathrm{curl}} L^2(\Omega). \tag{7}$$

The section is ended with

Theorem 2.11 (Friedrichs' inequality for $H^\perp(\mathrm{curl}\,,\Omega)$) *If Ω is simply connected, the following inequality holds for $u \in H^\perp(\mathrm{curl}\,,\Omega)$:*

$$||u||_{0,\Omega} \leqslant C H_\Omega ||\mathrm{curl}\, u||_{0,\Omega}.$$

The same inequality holds for the space $H_0^\perp(\mathrm{curl}\,,\Omega)$.

See, e.g., [10], Theorem 2.6.

3 Discretization of the problem

3.1 *hp*-finite element approximation

We recall that we want to solve the variational problem (5):

Find $\mathbf{u} \in V$, such that

$$a_\Omega(\mathbf{u}, \mathbf{v}) = l_\Omega(\mathbf{v}), \qquad \forall \mathbf{v} \in V. \tag{8}$$

V is a Hilbert space and $a_\Omega(\cdot,\cdot)$ the bilinear form on $V \times V$ given by (3). The right hand side $l_\Omega(\cdot)$ is the continuous linear form on V given by (4). In the problem discussed in this thesis, the space V is $H_0(\mathrm{curl}\,,\Omega)$. The aim of this section is to introduce discrete curl -conforming spaces to solve the variational problem (8) with finite element approximations on a given triangulation.

For iterative substructuring methods, which are considered in this work, we start from an initial mesh $\mathcal{T}_H$ where the elements of $\mathcal{T}_H$ are then refined to obtain a triangulation $\mathcal{T}$. The refinement procedure used in the present work will be discussed in the next subsection. The mesh $\mathcal{T}_H$ is also called the coarse mesh and the elements of $\mathcal{T}_H$ are called substructures or macroelements and are denoted by Ω_i throughout this thesis. The local interfaces are defined as $\Gamma_i := \partial\Omega_i \backslash \partial\Omega$. The interface is then defined as the union $\Gamma = \bigcup_{i=1}^{N} \Gamma_i$. The coarse mesh is assumed to be shape-regular and the fine mesh $\mathcal{T}$ is assumed to be regular, i.e. it contains no hanging nodes.

After introducing a triangulation of the domain Ω, local finite elements spaces $X \subset H_0(\mathrm{curl}\,,\Omega)$ can be defined and the variational problem (8) can now be approximated as

Find $\mathbf{u} \in X$, such that

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \qquad \forall \mathbf{v} \in X. \tag{9}$$

By introducing a basis for the finite element space X, the problem (9) can be written as:

Find $\mathbf{u} \in X$, s.t.

$$A\mathbf{u} = \mathbf{l}. \tag{10}$$

The matrix A is called the (global) stiffness matrix and is the representation of the bilinear form $a_\Omega(\cdot,\cdot)$ in terms of the given basis while the right hand side $\mathbf{l}$ is called the load vector and is the representation of the linear form $l_\Omega(\cdot)$ in terms of the given basis.

3.2 Geometric boundary layer meshes

In order to resolve boundary layers and/or singularities, geometrically refined meshes can be used. They are determined by a mesh grading factor $\sigma \in (0, 1)$ and by a refinement level n. A two-dimensional boundary layer mesh $\mathcal{T}_{\mathrm{bl}}^{n,\sigma}$ is constructed by refining an initial shape-regular triangulation $\mathcal{T}_H$ into edge or corner patches. In the following the refinement will be given for the reference square $\hat{K} = (-1, 1)^2$. For an arbitrary element $\Omega_i \in \mathcal{T}_H$ the patches can be obtained by using the affine mapping $F_{\Omega_i} : \hat{K} \to \Omega_i$.

An *edge patch* is given by an anisotropic triangulation of the form

$$\mathcal{T}_e := \{I \times K_y | K_y \in \mathcal{T}_y\},$$

where I denotes the interval $(-1, 1)$ and $\mathcal{T}_y$ is a mesh of I, geometrically refined towards for example $y = -1$ with grading factor $\sigma \in (0, 1)$ and n layers, see Figure 1, left.

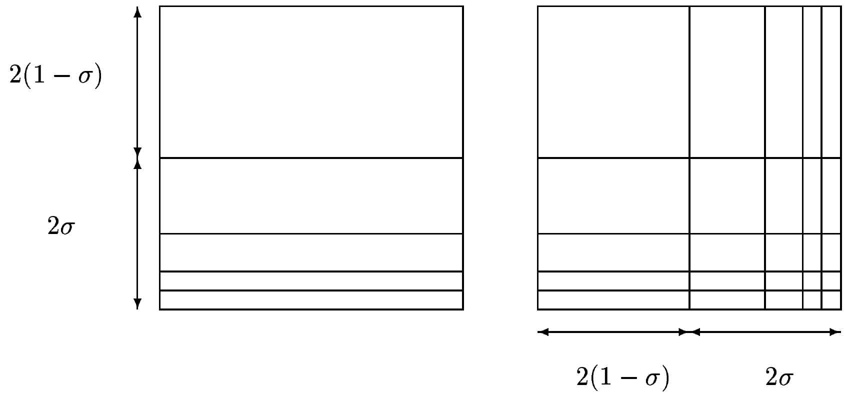

Figure 1: Edge and corner patches for $\hat{K}$ with $\sigma = 0.5$ and $n = 4$.

A *corner patch* is then given by an anisotropic triangulation of the form

$$\mathcal{T}_c := \{K_x \times K_y | K_x \in \mathcal{T}_x,\ K_y \in \mathcal{T}_y\},$$

where $\mathcal{T}_x$ and $\mathcal{T}_y$ are meshes of I geometrically refined towards one vertex with a grading factor $\sigma \in (0,1)$ and n layers, see Figure 1, right.

The number of elements in an edge or a corner patch is $O(n)$ and $O(n^2)$, respectively. The thinnest layer has a width proportional to σ^n. We note that every element K of $\mathcal{T}_e$ and $\mathcal{T}_c$ can, after a possible translation and rotation, be written in the form $(0, h_x) \times (0, h_y)$. The aspect ratio of K is then the maximum of the ratios h_x/h_y and h_y/h_x. The aspect ratio of the mesh is therefore proportional to σ^{-n}. For further properties of boundary layer meshes it is referred to [19].

3.3 Gauss-Lobatto nodes

The aim of this subsection is to provide a nodal basis for spectral finite element functions on an element $K \in \mathcal{T}_{\text{bl}}^{n,\sigma}$. As in the previous subsection, the nodes will be defined for the reference square $\hat{K}$ and for an arbitrary element $K \in \mathcal{T}_{\text{bl}}^{n,\sigma}$ they are obtained by the mapping $F_K : \hat{K} \to K$.

The family of Legendre polynomials $L_n(x)$ on $(-1,1)$ is defined by the following recursion:

$$L_0(x) = 1, \qquad L_1(x) = x,$$
$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x), \quad \text{for } n \geqslant 1.$$

$L_n(x)$ is a polynomial of degree n and it holds $L_n(1) = 1$ for all $n \geqslant 0$. Furthermore they form an orthogonal family, i.e.

$$\int_{-1}^{1} L_n(x) L_m(x)\, dx = \delta_{nm} \frac{2}{2n+1}.$$

The set of Gauss-Lobatto points are now defined by $GLL(k) = \{\xi_i | 1 \leqslant i \leqslant k\}$ where the ξ_i's are the distinct and real zeros of $(1-x^2)L'_{k-1}(x)$. For the reference square $\hat{K} = (-1,1)^2$ the Gauss-Lobatto nodes are given by $GLL(k_1, k_2) = \{\xi_{ij} = (\xi_i, \xi_j) | 1 \leqslant i \leqslant k_1,\ 1 \leqslant j \leqslant k_2\}$, where the ξ_i's and ξ_j's are defined as above. Let $\mathbb{Q}_{k_1,k_2}(\hat{K})$ denote the set of polynomials on $\hat{K}$ which are of order k_1 in the first and of order k_2 in the second variable. Given the nodes $GLL(k_1, k_2)$, the basis functions on $\mathbb{Q}_{k_1,k_2}$ are given by the tensor product of k_1-th order and k_2-th order Lagrange interpolating polynomials on $GLL(k_1)$ and $GLL(k_2)$, respectively, defined by $\hat{l}_i(\xi_j) = \delta_{ij}$. A scalar function $u(x,y) \in \mathbb{Q}_{k_1,k_2}$ on $\hat{K}$ can then be written as

$$u(x,y) = \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} u(\xi_i,\xi_j)\hat{l}_i(x)\hat{l}_j(y).$$

This defines an interpolation operator I^{k_1,k_2} on the reference element

$$I^{k_1,k_2}u(x,y) := \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} u(\xi_i,\xi_j)\hat{l}_i(x)\hat{l}_j(y).$$

The nodes $GLL(k,k)$ are simply denoted by $GLL(k)^2$.

3.4 Nédélec elements in $H(\mathrm{curl}\,,\Omega)$

To solve the discrete problem (9), it is necessary to define curl-conforming finite element spaces. Such spaces can be chosen as the Nédélec spaces. They were originally introduced in [16] and extended to the three-dimensional case in [15]. The Nédélec elements are sometimes also called edge elements. For the reference square $\hat{K}$ the Nédélec elements of order k are defined as

$$ND_k(\hat{K}) = \mathbb{Q}_{k-1,k}(\hat{K}) \times \mathbb{Q}_{k,k-1}(\hat{K}), \qquad k \geqslant 1,$$

with basis functions associated to the Gauss-Lobatto nodes on $\hat{K}$. These spaces are defined in such a way that for every $\mathbf{u} \in ND_k(\hat{K})$

$$\mathrm{curl}\,\mathbf{u} \in \mathbb{Q}_{k-1,k-1},$$

and

$$\mathbf{u}\cdot\mathbf{t}_e \in \mathbb{Q}_{k-1}(e),$$

for any edge e of $\hat{K}$ where $\mathbf{t}_e$ is the unit tangent vector of e. The associated degrees of freedom are given by

$$\int_e (\mathbf{w}\cdot\mathbf{t}_e)p\, ds, \qquad \forall p \in \mathbb{Q}_{k-1}(e),\ e \subset \partial\hat{K}, \quad k \geqslant 1, \tag{11}$$

$$\int_{\hat{K}} \mathbf{w}\cdot\mathbf{p}\, d\hat{\mathbf{x}}, \qquad \forall \mathbf{p} \in \mathbb{Q}_{k-1,k-2} \times \mathbb{Q}_{k-2,k-1}, \quad k > 1. \tag{12}$$

In the case $k = 1$ the space $ND_k(\hat{K})$ has the form

$$ND_1(\hat{K}) = \left\{ \mathbf{u} = \begin{pmatrix} a_1 + b_1 x_2 \\ a_2 + b_2 x_1 \end{pmatrix} | a_i, b_i \text{ constant for } i = 1,2 \right\}.$$

Its curl and tangential component are constant along each edge. The degrees of freedom consist only of the edge moments (11).

For an arbitrary element $K \in \mathcal{T}_{\mathrm{bl}}^{n,\sigma}$, the Nédélec elements $ND_k(K)$ are obtained by using the affine mapping $F_K : \hat{K} \to K$. The following result is well known

Lemma 3.1 *A vector function $\mathbf{u} : \Omega \to \mathbb{R}^2$ belongs to $H(\mathrm{curl}\,,\Omega)$ if and only if the restriction of $\mathbf{u}$ to every $K \in \mathcal{T}_{\mathrm{bl}}^{n,\sigma}$ belongs to $H(\mathrm{curl}\,,K)$, and, for each common edge $\overline{e} = \overline{K_1} \cap \overline{K_2}$, $K_1, K_2 \subset \mathcal{T}_{\mathrm{bl}}^{n,\sigma}$, we have*

$$\mathbf{u}\cdot\mathbf{t}_{K_1} = \mathbf{u}\cdot\mathbf{t}_{K_2}$$

on e with, e.g., $\mathbf{t} = \mathbf{t}_{K_1} = -\mathbf{t}_{K_2}$.

Given the local spaces $ND_k(K)$, the global ones, conforming in $H(\mathrm{curl}\,,\Omega)$ and with respect to a given triangulation $\mathcal{T}$ are then defined by

$$ND_k(\Omega,\mathcal{T}) = \{\mathbf{u} \in H(\mathrm{curl}\,,\Omega)|\mathbf{u}|_K \in ND_k(K),\; K \in \mathcal{T}\}, \tag{13}$$

i.e. they preserve the continuity of the tangential components along the element edges.

To obtain elements that are conforming to $H_0(\mathrm{curl}\,,\Omega)$ one can proceed in the same way just using now the local spaces

$$ND_{0,k}(K) = \{\mathbf{u} \in ND_k(K)|\mathbf{u}\cdot\mathbf{t}_e = 0,\; e \subset \partial K \cap \partial\Omega\}\,.$$

The resulting global space is then denoted by $ND_{0,k}(\Omega,\mathcal{T})$.

The Nédélec interpolant $\hat{\Pi}_{ND} : H^{\frac{1}{2}+\varepsilon}(\hat{K}) \to ND_k(\hat{K})$ on the reference square is defined by

$$\int_e (\mathbf{w} - \hat{\Pi}_{ND}\mathbf{w})\cdot\mathbf{t}_e p\; ds = 0, \qquad \forall p \in \mathbb{Q}_{k-1}(e),\; e \subset \partial\hat{K}, \quad k \geqslant 1, \tag{14}$$

$$\int_{\hat{K}} (\mathbf{w} - \hat{\Pi}_{ND}\mathbf{w})\cdot\mathbf{p}\; d\hat{\mathbf{x}} = 0, \qquad \forall \mathbf{p} \in \mathbb{Q}_{k-1,k-2} \times \mathbb{Q}_{k-2,k-1}, \quad k > 1, \tag{15}$$

and has the properties

Lemma 3.2 (Unisolvency and diagonality) *The degrees of freedom (11) and (12) are unisolvent and the interpolant $\hat{\Pi}_{ND}$ is diagonal, i.e. for $\mathbf{w} = (w_1, w_2)$,*

$$\hat{\Pi}_{ND}(w_i\hat{\mathbf{e}}_i) = (\hat{\Pi}_{ND}\mathbf{w})_i\hat{\mathbf{e}}_i, \qquad i = 1,2, \tag{16}$$

where $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2)$ denote the standard basis vectors.

Proof The proof follows the one of [5] where the unisolvency and diagonality is proven for a tetrahedron in three dimensions.

The number of degrees of freedom is $4k+2k(k-1)$ which is equal to $\dim ND_k = 2k(k+1)$. To prove unisolvency one has to show that if $\mathbf{w}$ has all its degrees of freedom equals to zero then $\mathbf{w} \equiv 0$ on $\hat{K}$.

¿From (11) it follows that

$$\int_{\hat{e}_i} w_i p\; ds = 0, \qquad \forall p \in \mathbb{Q}_{k-1}(\hat{e}_i), \quad i = 1,2,$$

for every edge $\hat{e}_i$ of $\hat{K}$ parallel to $\hat{\mathbf{e}}_i$. Choosing $p = w_i\,|_{\hat{e}_i}$ leads to

$$w_i\,|_{\hat{e}_i} \equiv 0. \tag{17}$$

¿From (12) it follows by choosing $\mathbf{p} = (\partial^2_{x_2} w_1, \partial^2_{x_1} w_2)$ and integrating by parts

$$\begin{aligned} 0 &= \int_{\hat{K}} \mathbf{w}\cdot\mathbf{p}\; d\hat{\mathbf{x}} \\ &= \int_{\hat{K}} w_1\partial^2_{x_2} w_1\; d\hat{\mathbf{x}} + \int_{\hat{K}} w_2\partial^2_{x_1} w_2\; d\hat{\mathbf{x}} \\ &= \int_{\hat{e}_1} [w_1\partial_{x_2} w_1]^1_{-1}\; ds + \int_{\hat{e}_2} [w_2\partial_{x_1} w_2]^1_{-1}\; ds \\ &\quad - \int_{\hat{K}} (\partial_{x_2} w_1)^2\; d\hat{\mathbf{x}} - \int_{\hat{K}} (\partial_{x_1} w_2)^2\; d\hat{\mathbf{x}}. \end{aligned}$$

Since the boundary terms are zero due to (17) one obtains

$$\partial_{x_1} w_2 \equiv 0, \qquad \partial_{x_2} w_1 \equiv 0. \tag{18}$$

This means that w_1 is a polynomial in x_1 and w_2 is a polynomial in x_2. Hence it follows that $\mathbf{w} \in \mathbb{Q}_{k-1,0} \times \mathbb{Q}_{0,k-1}$. This allows $\mathbf{p}$ to be chosen as $\mathbf{p} = \mathbf{w}$ in (12) and leads to the final result $\mathbf{w} \equiv 0$, i.e. the degrees of freedom (11) and (12) are unisolvent.

Now set $\hat{\Pi}_{ND}(w_i \hat{\mathbf{e}}_i) = \varphi_i$ with $\varphi_i \in \mathbb{Q}_{k-1,k} \times \mathbb{Q}_{k,k-1}$. For $\varphi_i = (\varphi_{i1}, \varphi_{i2})$ and $j \neq i$ it follows from (11) and (14)

$$\int_{\hat{e}_j} (\varphi_{ij} \hat{\mathbf{e}}_j) \cdot \mathbf{t}_{\hat{e}_j} p \, ds = \int_{\hat{e}_j} \varphi_i \cdot \mathbf{t}_{\hat{e}_j} p \, ds = \int_{\hat{e}_j} \hat{\Pi}_{ND}(w_i \hat{\mathbf{e}}_i) \cdot \mathbf{t}_{\hat{e}_j} p \, ds = \int_{\hat{e}_j} w_i \hat{\mathbf{e}}_i \cdot \mathbf{t}_{\hat{e}_j} p \, ds = 0,$$

$\forall p \in \mathbb{Q}_{k-1}(\hat{e}_j)$, and from (12) and (15)

$$\int_{\hat{K}} (\varphi_{ij} \hat{\mathbf{e}}_j) \cdot \mathbf{p} \, d\hat{\mathbf{x}} = \int_{\hat{K}} \hat{\Pi}_{ND}(\mathbf{w} \cdot \hat{\mathbf{e}}_i) \hat{\mathbf{e}}_j p_j \, d\hat{\mathbf{x}} = 0, \quad \forall \mathbf{p} \in \mathbb{Q}_{k-1,k-2} \times \mathbb{Q}_{k-2,k-1}.$$

This implies that φ_{ij} for $j \neq i$ verifies $2k + k(k-1)$ constraints which are independent since (11) and (12) are unisolvent degrees of freedom. Since $\varphi_{ij} \in \mathbb{Q}_{k-1,k}$ or $\mathbb{Q}_{k,k-1}$, respectively and the dimension of each one of these spaces is equal to $k(k+1)$, i.e. to the number of independent constraints, $\varphi_{ij} \equiv 0$ for all $j \neq i$. Therefore $\hat{\Pi}_{ND}$ is diagonal. □

3.5 Discrete Helmholtz decomposition

In Proposition 2.9 it has been shown that the space $H(\mathrm{curl}, \Omega)$ allows a decomposition into the kernel of the curl operator and its orthogonal complement. As the following proposition will show, a similar decomposition holds for the discrete curl-conforming Nédélec spaces as well. Let

$$\begin{aligned} S_k(\Omega, \mathcal{T}) &:= \{ q \in H^1(\Omega) | q \,|_K \in \mathbb{Q}_{k,k} \forall K \subset \mathcal{T} \}, \\ S_{0,k}(\Omega, \mathcal{T}) &:= \{ q \in H_0^1(\Omega) | q \,|_K \in \mathbb{Q}_{k,k} \forall K \subset \mathcal{T} \}, \end{aligned}$$

be the $H^1(\Omega)$-conforming and $H_0^1(\Omega)$-conforming spaces, respectively, with continuity across the interface.

Proposition 3.3 (Helmholtz decomposition of $ND_k(\Omega, \mathcal{T})$) *The Nédélec spaces allow the following orthogonal decompositions*

$$\begin{aligned} ND_k(\Omega, \mathcal{T}) &= \mathbf{grad} S_k(\Omega, \mathcal{T}) \oplus ND_k^{\perp}(\Omega, \mathcal{T}), \\ ND_{0,k}(\Omega, \mathcal{T}) &= \mathbf{grad} S_{0,k}(\Omega, \mathcal{T}) \oplus ND_{0,k}^{\perp}(\Omega, \mathcal{T}), \end{aligned}$$

where

$$\begin{aligned} ND_k^{\perp}(\Omega, \mathcal{T}) &= \{ \mathbf{u} \in ND_k(\Omega, \mathcal{T}) | (\mathbf{u}, \mathbf{grad} p_k)_0 = 0 \, \forall \, p_k \in S_k(\Omega, \mathcal{T}) \}, \\ ND_{0,k}^{\perp}(\Omega, \mathcal{T}) &= \{ \mathbf{u} \in ND_{0,k}(\Omega, \mathcal{T}) | (\mathbf{u}, \mathbf{grad} p_k)_0 = 0 \, \forall \, p_k \in S_{0,k}(\Omega, \mathcal{T}) \}. \end{aligned}$$

In general, the spaces $ND_k^{\perp}(\Omega, \mathcal{T})$ and $ND_{0,k}^{\perp}(\Omega, \mathcal{T})$ are not included in $H^{\perp}(\mathrm{curl}, \Omega)$ and $H_0^{\perp}(\mathrm{curl}, \Omega)$, the analogous spaces of the continuous Helmholtz decompositions.

If Ω is simply connected, the deRahm diagram (7) can be adapted in the following way:

$$S_k(\Omega, \mathcal{T}) \xrightarrow{\mathbf{grad}} ND_k(\Omega, \mathcal{T}) \xrightarrow{\mathrm{curl}} Y_{k-1}(\Omega, \mathcal{T}), \tag{19}$$

where $Y_{k-1}(\Omega, \mathcal{T})$ is a $L^2(\Omega)$-conforming space of piecewise polynomials of degree $k-1$.

3.6 Discrete anisotropic Friedrichs' inequality

For some calculations in the following, a discrete version of the Friedrichs' inequality will be needed. As the present work deals with highly anisotropic boundary layer meshes, the inequality must hold for such meshes. Since no such result seems to exist in published form, a proof is added. The next paragraph will give some preparations for the proof, while in the following paragraph the theorem will be formulated and proven.

3.6.1 Anisotropic interpolation error

The aim of this paragraph is to derive an hp-interpolation error for the Nédélec interpolant defined by (14) and (15) for anisotropic meshes. This is done by mapping a p-interpolation error for the reference element $\hat{K}$ on an arbitrary (not necessarily shape regular) element K by using similar techniques as in [5].

Let $F_K : \hat{K} \to K$ denote the affine element mapping. Scalar functions map like

$$q \circ F_K = \hat{q}, \tag{20}$$

whereas vector fields transform like gradients, i.e.

$$\mathbf{v} \circ F_K = DF_K^{-T} \hat{\mathbf{v}}. \tag{21}$$

The local system of Cartesian coordinates on K is given by $\mathbf{x}^K = (x_1^K, x_2^K) = F_K(\hat{x}_1, \hat{x}_2)$. In general, the mapping F_K can be represented as $F_K \hat{\mathbf{x}} = DF_K \hat{\mathbf{x}} + \mathbf{c}_K$ and for the following calculation it is required that the matrix DF_K satisfies the assumption

Assumption 3.4 *For any $K \in \mathcal{T}_h$ there exists a diagonal scaling matrix*

$$H_K = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where d_1 and d_2 are the characteristic dimensions of K, and two matrices B_K and $\breve{B}_K$ which, together with their inverses, are bounded independently of K, such that

$$DF_K = H_K B_K = \breve{B}_K H_K. \tag{22}$$

It is immediate to prove that Assumption 3.4 holds for our meshes.

The following interpolation error estimate on the reference square has been proved by Monk [13].

Proposition 3.5 *Let $\hat{\mathbf{u}} \in (H^r(\hat{K}))^2$ and $r > \frac{1}{2}$. Then the following error estimate holds on the reference square*

$$||\hat{\mathbf{u}} - \hat{\Pi}_{ND} \hat{\mathbf{u}}||_{0,\hat{K}} \leqslant C k^{-(r-\frac{1}{2})} ||\hat{\mathbf{u}}||_{r,\hat{K}}.$$

Proof See [13], Theorem 3.5.

Choosing $r = 1$ in the above proposition yields

$$||\hat{\mathbf{u}} - \hat{\Pi}_{ND} \hat{\mathbf{u}}||_{0,\hat{K}} \leqslant C k^{-\frac{1}{2}} ||\hat{\mathbf{u}}||_{1,\hat{K}}. \tag{23}$$

With the last equation it is possible to prove the main result of the current paragraph

Theorem 3.6 *Let $K \in \mathcal{T}$ be an arbitrary element and let Assumption 3.4 be satisfied. There exists a positive constant C such that for all functions $\mathbf{u} \in (H^1(K))^2$ the following interpolation error estimate holds:*

$$||\mathbf{u} - \Pi_{ND} \mathbf{u}||_{0,K}^2 \leqslant C k^{-1} (||\mathbf{u}||_{0,K}^2 + \sum_{l=1}^{2} d_l^2 ||\partial_l \mathbf{u}||_{0,K}^2)$$

Before proving the theorem, there is a corollary of it which will be of use later:

Corollary 3.7 *Let $K \in \mathcal{T}$, $H_{\Omega_i} = 1$ and the assumptions of Theorem 3.6 be satisfied. Then*

$$||\mathbf{u} - \Pi_{ND}\mathbf{u}||_{0,K} \leqslant Ck^{-\frac{1}{2}}||\mathbf{u}||_{1,K}, \tag{24}$$

with C independent of the aspect ratio of K.

Proof Use that $\max_i d_i \leqslant 1$. □

Proof of Theorem 3.6 ¿From the definition of the L^2-norm and the transformation property (21) it follows that

$$\begin{aligned} ||\mathbf{u} - \Pi_{ND}\mathbf{u}||_{0,K}^2 &= \int_K |\mathbf{u} - \Pi_{ND}\mathbf{u}|^2 \, d\mathbf{x} \\ &= \int_{\hat{K}} |\mathbf{u} \circ F_K - (\Pi_{ND}\mathbf{u}) \circ F_K|^2 |\det DF_K| \, d\hat{\mathbf{x}} \\ &= |\det DF_K| \int_{\hat{K}} |DF_K^{-T}\hat{\mathbf{u}} - DF_K^{-T}\hat{\Pi}_{ND}\hat{\mathbf{u}}|^2 \, d\hat{\mathbf{x}}. \end{aligned}$$

By Assumption 3.4 use the first equation of (22) and the fact that B_K is bounded to obtain

$$||\mathbf{u} - \Pi_{ND}\mathbf{u}||_{0,K}^2 \leqslant C|\det DF_K| \int_{\hat{K}} |H_K^{-1}(\hat{\mathbf{u}} - \hat{\Pi}_{ND}\hat{\mathbf{u}})|^2 \, d\hat{\mathbf{x}}.$$

From the diagonality (16) of the interpolation operator $\hat{\Pi}_{ND}$ it follows for $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2)$ and $\hat{\mathbf{u}}_i = \hat{u}_i\hat{\mathbf{e}}_i$ that $(\hat{\Pi}_{ND}\hat{\mathbf{u}})_i\hat{\mathbf{e}}_i = \hat{\Pi}_{ND}\hat{\mathbf{u}}_i$ and therefore

$$\hat{\mathbf{u}}_i - \hat{\Pi}_{ND}\hat{\mathbf{u}}_i = (\hat{u}_i - (\hat{\Pi}_{ND}\hat{\mathbf{u}}_i)_i)\hat{\mathbf{e}}_i.$$

This fact can be used to obtain

$$\begin{aligned} ||\mathbf{u} - \Pi_{ND}\mathbf{u}||_{0,K}^2 &\leqslant C|\det DF_K| \sum_{i=1}^{2} \int_{\hat{K}} |H_K^{-1}(\hat{\mathbf{u}}_i - \hat{\Pi}_{ND}\hat{\mathbf{u}}_i)|^2 \, d\hat{\mathbf{x}} \\ &\leqslant C|\det DF_K| \sum_{i=1}^{2} \int_{\hat{K}} \frac{1}{(d_i)^2} |\hat{\mathbf{u}}_i - \hat{\Pi}_{ND}\hat{\mathbf{u}}_i|^2 \, d\hat{\mathbf{x}}. \end{aligned}$$

Using (23) yields

$$||\mathbf{u} - \Pi_{ND}\mathbf{u}||_{0,K}^2 \leqslant Ck^{-1} \sum_{i=1}^{2} \frac{|\det DF_K|}{(d_i)^2} \left(\int_{\hat{K}} |\hat{\mathbf{u}}_i|^2 \, d\hat{\mathbf{x}} + \int_{\hat{K}} |\hat{\nabla}\hat{\mathbf{u}}_i|^2 \, d\hat{\mathbf{x}} \right).$$

Now apply (21) on $\hat{\mathbf{u}}_i$:

$$\hat{\mathbf{u}}_i = DF_K^T(\mathbf{u}_i \circ F_K).$$

With the definition $\breve{\mathbf{u}}_i := \breve{B}_K^T\mathbf{u}_i$ and the factorizations (22) the gradient $\hat{\nabla}$ can be transformed as

$$\hat{\nabla}\hat{\mathbf{u}}_i = H_K\breve{B}_K^T\left((\nabla\mathbf{u}_i) \circ F_K\right) H_K B_K = H_K\left((\nabla\breve{\mathbf{u}}_i) \circ F_K\right) H_K B_K.$$

The interpolation error can then be estimated as

$$
\begin{aligned}
||\mathbf{u}-\Pi_{ND}\mathbf{u}||_{0,K}^2 &\leqslant Ck^{-1}\sum_{i=1}^{2}\frac{1}{(d_i)^2}\left(\int_K |H_K\check{\mathbf{u}}_i|^2\,d\mathbf{x}+\int_K |H_K(\nabla\check{\mathbf{u}}_i)H_KB_K|^2\,d\mathbf{x}\right)\\
&\leqslant Ck^{-1}\sum_{i=1}^{2}\frac{1}{(d_i)^2}\left(\int_K |d_i\check{\mathbf{u}}_i|^2\,d\mathbf{x}+\int_K |d_i(\nabla\check{\mathbf{u}}_i)H_K|^2\,d\mathbf{x}\right)\\
&= Ck^{-1}\sum_{i=1}^{2}\left(\int_K |\breve{B}_K^T\mathbf{u}_i|^2\,d\mathbf{x}+\sum_{l=1}^{2}\int_K |d_l\partial_l\breve{B}_K^T\mathbf{u}_i|^2\,d\mathbf{x}\right)\\
&\leqslant Ck^{-1}\sum_{i=1}^{2}\left(\int_K |u_i|^2\,d\mathbf{x}+\sum_{l=1}^{2}d_l^2\int_K |\partial_l u_i|^2\,d\mathbf{x}\right)\\
&= Ck^{-1}\left(||\mathbf{u}||_{0,K}^2+\sum_{l=1}^{2}d_l^2||\partial_l\mathbf{u}||_{0,K}^2\right).
\end{aligned}
$$

This ends the proof of the theorem. □

3.6.2 Friedrichs' inequality

The steps to prove the discrete Friedrichs' inequality follow those of [10], Theorem 7.18. To prove the anisotropic discrete Friedrichs' inequality, the following Lemma will be used:

Lemma 3.8 *Assume that the bounded and convex domain Ω_i with $H_{\Omega_i}=1$ has a Lipschitz boundary and is covered with an edge or corner patch. Assume also that $\mathbf{u}\in H_0^{\perp}(\mathrm{curl}\,,\Omega_i)$. Then the Nédélec interpolant satisfies the following L^2-bound.*

$$||\mathbf{u}-\Pi_{ND}\mathbf{u}||_{0,\Omega_i}\leqslant Ck^{-\frac{1}{2}}||\mathbf{curl}\mathbf{u}||_{0,\Omega_i},$$

with C independent of σ,n and p.

Proof Corollary 3.7 yields

$$||\mathbf{u}-\Pi_{ND}\mathbf{u}||_{0,K}^2\leqslant Ck^{-1}||\mathbf{u}||_{1,K}^2.$$

Summing over the elements $K\subset\Omega_i$ and using that $\max h_K\leqslant H_{\Omega_i}=1$ leads to

$$||\mathbf{u}-\Pi_{ND}\mathbf{u}||_{0,\Omega_i}^2\leqslant Ck^{-1}||\mathbf{u}||_{1,\Omega_i}^2.$$

For the norm on the right hand side it can be used that the space $H_0^{\perp}(\mathrm{curl}\,,\Omega_i)$ is continuously embedded in $H^1(\Omega_i)$; see e.g. [2], Theorem 2.12. We therefore have

$$||\mathbf{u}||_{1,\Omega_i}\leqslant C(||\mathbf{u}||_{0,\Omega_i}+||\mathbf{curl}\,\mathbf{u}||_{0,\Omega_i}).$$

By the continuous Friedrichs' inequality $||\mathbf{u}||_{0,\Omega_i}\leqslant C||\mathrm{curl}\,\mathbf{u}||_{0,\Omega_i}$ the lemma follows. □

Theorem 3.9 (Anisotropic discrete Friedrichs' inequality, *hp*-version) *Assume that the bounded and convex domain Ω_i has a Lipschitz boundary and is covered with an edge or corner patch. Let Φ_k be a discrete divergence-free edge element function of degree k in $ND_{0,k}^{\perp}(\Omega_i,\mathcal{T})$. Then there exist constants C and C', both independent of σ,n and k, such that*

$$||\Phi_k||_{0,\Omega_i}\leqslant CH_i(1+k^{-1/2})||\mathrm{curl}\,\Phi_k||_{0,\Omega_i}\leqslant C'H_i||\mathrm{curl}\,\Phi_k||_{0,\Omega_i}.$$

Proof The second inequality follows trivially from the first, since the coefficient is a function that decreases with k.

The first inequality is proven in several steps. We assume that Ω_i has unit diameter since the explicit dependence on H_i is found by a scaling argument.

Define $v \in H_0^1(\Omega_i)$ as the solution of the generalized Dirichlet problem

$$(\mathbf{grad}v, \mathbf{grad}q)_0 = (\Phi_k, \mathbf{grad}q)_0 \quad \forall q \in H_0^1(\Omega_i).$$

Then, $\mathbf{w} := \Phi_k - \mathbf{grad}v \in H_0^{\perp}(\mathrm{curl}\,, \Omega_i)$ and thus satisfies

$$\mathrm{curl}\,\mathbf{w} = \mathrm{curl}\,\Phi_k, \quad \mathrm{div}\mathbf{w} = 0, \quad \mathbf{w} \cdot \mathbf{t}\,|_{\partial\Omega_i} = 0.$$

By Proposition 3.7 of [2] it follows that $\mathbf{w} \in H^{\frac{1}{2}+\varepsilon}$, for an $\varepsilon > 0$. This means that, thanks to Theorem 2.4, the edge moments are defined and therefore $\Pi_{ND}\mathbf{w}$ is defined. Since Φ_k is in the Nédélec space, its interpolant is defined, and therefore $\Pi_{ND}(\mathbf{grad}v)$ is defined. The appropriate version of the commuting diagram property shows that there is a piecewise polynomial v_k such that

$$\Pi_{ND}(\mathbf{grad}v) = \mathbf{grad}v_k,$$

and therefore $\Phi_k = \Pi_{ND}\Phi_k = \Pi_{ND}\mathbf{w} + \mathbf{grad}v_k$. We have $(\Phi_k, \mathbf{grad}q_k)_0 = 0$ for all $q_k \in H_0^1(\Omega_i)$ and therefore for $q_k{=}v_k$. This gives that $(\Phi_k, \Phi_k)_0 = (\Pi_{ND}\mathbf{w}, \Phi_k)_0 + (\mathbf{grad}v_k, \Phi_k)_0 = (\Pi_{ND}\mathbf{w}, \Phi_k)_0$ and an application of the Cauchy-Schwarz inequality gives

$$||\Phi_k||_{0,\Omega_i} \leqslant ||\Pi_{ND}\mathbf{w}||_{0,\Omega_i}.$$

We next use Lemma 3.8 and the triangle inequality to show

$$\begin{aligned} ||\Pi_{ND}\mathbf{w}||_{0,\Omega_i} &\leqslant ||\mathbf{w}||_{0,\Omega_i} + ||\mathbf{w} - \Pi_{ND}\mathbf{w}||_{0,\Omega_i} \\ &\leqslant C(1 + k^{-1/2})||\mathrm{curl}\,\mathbf{w}||_{0,\Omega_i}. \end{aligned}$$

In the last step we have used again the continuous Friedrichs' inequality. The proof is completed by recalling that $\mathrm{curl}\,\mathbf{w} = \mathrm{curl}\,\Phi_k$. □

4 A dual-primal FETI method

In this section, a dual-primal FETI method for the solution of the linear system (10) is introduced. Dual-primal FETI methods were originally introduced in [7] and the first theoretical results for two dimensions were given in [12]. A dual-primal FETI method for (1) has already been developed in [18] for h and p approximations on isotropic meshes in two dimensions.

4.1 Definition and notations

Let $\mathcal{T}_H$ denote the coarse triangulation into subdomains Ω_i and $\mathcal{T} = \mathcal{T}_{\mathrm{bl}}^{n,\sigma}$ its refinement into edge and corner patches. The set of internal edges of $\mathcal{T}_H$ is denoted by $\mathcal{E}_H$. The set $\mathcal{I}(i)$ consists of indices j such that Ω_j shares an edge with Ω_i. The dimension $|\mathcal{I}(i)|$ is uniformly bounded. The edge shared by two adjacent subdomains Ω_i and Ω_j is denoted by E_{ij}.

The coarse space X_H is defined by

$$X_H(\Omega) := ND_{0,1}(\Omega, \mathcal{T}_H) \subset H_0(\mathrm{curl}\,, \Omega_i).$$

The local spaces defined on each subdomain Ω_i are given by

$$X_i := \{\mathbf{u} \in ND_k(\Omega_i, \mathcal{T}) | \mathbf{u} \cdot \mathbf{t} = 0 \text{ on } \partial\Omega_i \cap \partial\Omega\} \subset H(\mathrm{curl}\,, \Omega_i).$$

Next, the product space

$$X = X(\Omega) := \prod_{i=1}^{N} X_i,$$

is considered, which consists of vectors that have in general a discontinuous tangential component along the subdomain edges. The subspace of vectors with a continuous tangential component along the edges is given by

$$\widehat{X} := X \cap H_0(\mathrm{curl}\,, \Omega).$$

Now, we introduce some trace spaces consisting of the tangential components on the boundaries of the subdomains. A scalar function u, defined on $\partial\Omega_i \backslash \partial\Omega$, belongs to W_i if and only if there exists a vector $\mathbf{u} \in X_i$ such that, for each edge,

$$u\,|_{E_{ij}} = \mathbf{u} \cdot \mathbf{t}_{E_{ij}}, \qquad E_{ij} \in \mathcal{E}_H, \quad j \in \mathcal{I}(i).$$

These are piecewise polynomials of degree $k-1$ along the edges E_{ij}. The product space of functions defined on the interface Γ is defined by $W := \prod_i W_i$ and its continuous subspace $\widehat{W}$ consists of tangential traces of vectors in $\widehat{X}$.

Throughout this section, the following notations are used. A vector function in X_i is denoted by a bold letter with superscript (i), e.g. $\mathbf{u}^{(i)}$, and the same notation is employed for the corresponding column vector of degrees of freedom. Its tangential component $u^{(i)}$ is an element of W_i and is defined by

$$u^{(i)} := \mathbf{u}^{(i)} \cdot \mathbf{t}_{E_{ij}}, \qquad E_{ij} \in \mathcal{E}_H, \quad j \in \mathcal{I}(i).$$

It is uniquely defined by the degrees of freedom $\mathbf{u}^{(i)}$ involving the tangential component along $\partial\Omega_i \backslash \partial\Omega$. The same notation $u^{(i)}$ is used for the column vector of these tangential degrees of freedom. It is remarked that a vector $\mathbf{u}$ belongs to the continuous space $\widehat{X}$ (and consequently its tangential component to $\widehat{W}$) if

$$u^{(i)}\,|_{E_{ij}} = u^{(j)}\,|_{E_{ij}}, \qquad E_{ij} \in \mathcal{E}_H. \tag{25}$$

Given a function $u^{(i)} \in W_i$ it can be extended in a unique way into the interior of Ω_i such that it is discrete harmonic with respect to the bilinear form $a_{\Omega_i}(\cdot, \cdot)$. The extension operator is denoted by

$$\mathcal{H}_i^M : W_i \longrightarrow X_i,$$

and is referred to as the *Maxwell discrete harmonic extension*. We recall that $\mathbf{u}^{(i)} = \mathcal{H}_i^M u^{(i)}$ minimizes the energy $a_{\Omega_i}(\mathbf{u}^{(i)}, \mathbf{u}^{(i)})$ among all the vectors of X_i with tangential component equal to $u^{(i)}$ on $\partial\Omega_i \backslash \partial\Omega$.

4.2 Description of the method

We start from the linear system (10)

$$A\mathbf{u} = \mathbf{l}, \tag{26}$$

where the stiffness matrix A and the load vector $\mathbf{l}$ are obtained by assembling the local stiffness matrices $A^{(i)}$, relative to the bilinear form $a_{\Omega_i}(\cdot, \cdot)$, and the local load vectors $l^{(i)}$, where the Ω_i's are elements of the coarse triangulation $\mathcal{T}_H$. The local stiffness matrices satisfy the assumption

$$0 < \beta_i |\mathbf{x}|^2 \leqslant \mathbf{x}^T A_i \mathbf{x} \leqslant \gamma_i |\mathbf{x}|^2, \qquad \mathbf{x} \in \mathbb{R}^2, \tag{27}$$

for $i = 1, \dots, N$. The local stiffness matrices and local load vectors can be written in the form

$$A^{(i)} = \begin{pmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{pmatrix}, \qquad \mathbf{l}^{(i)} = \begin{pmatrix} l_I^{(i)} \\ l_\Gamma^{(i)} \end{pmatrix}.$$

Hence, the local degrees of freedom are divided into degrees of freedom associated to the interior of Ω_i and to the tangential component on the interface Γ_i. By block Gaussian elimination, the internal degrees of freedom can be eliminated and the system (26) is reduced to an equivalent system involving u, the vector consisting of the components associated to the interface. We introduce

$$S = \begin{pmatrix} S^{(1)} & & \\ & \ddots & \\ & & S^{(N)} \end{pmatrix}, \qquad u = \begin{pmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{pmatrix}, \qquad f = \begin{pmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{pmatrix},$$

with $S^{(i)} = A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Gamma}^{(i)}$ and $f^{(i)} = l_\Gamma^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}l_I^{(i)}$. The solution to the discrete problem can then be found by minimizing the energy

$$\frac{1}{2}(u, Su) - (f, u),$$

subject to the constraint that u is continuous, i.e. belongs to $\widehat{W}$. In all FETI-DP algorithms, only a relatively small number of continuity constraints across the interface are enforced in each step. This means we work in a subspace $\widetilde{W} \subset W$ of functions satisfying a certain number of continuity constraints. Here, only constraints on the degrees of freedom associated to the averages of the tangential components over the subdomain edges are enforced in each iteration, i.e. they belong to the so-called primal degrees of freedom (primal variables). Hence, the primal space $\widehat{W}_\Pi \subset \widehat{W}$ is a space of constant functions on the subdomain edges:

$$\widehat{W}_\Pi := \{u \in \widehat{W} | u\,|_{E_{ij}} \in \mathbb{Q}_0, E_{ij} \in \mathcal{E}_H\}.$$

The space $\widetilde{W}$ can then be decomposed into the sum

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta,$$

with the primal space $\widehat{W}_\Pi$ and the dual space $\widetilde{W}_\Delta$ given by

$$\widetilde{W}_\Delta := \prod_{i=1}^{N} \widetilde{W}_{\Delta,i},$$

where the spaces $\widetilde{W}_{\Delta,i}$ are spaces of functions associated to the corresponding substructures for which the functional given by the primal variables vanish:

$$\widetilde{W}_{\Delta,i} := \{u \in W_i | \bar{u}_{E_{ij}} := \frac{1}{|E_{ij}|}\int_{E_{ij}} \mathbf{u} \cdot \mathbf{t}_{E_{ij}}\, ds = 0,\ j \in \mathcal{I}(i)\}.$$

Therefore, $\widetilde{W}$ consists of functions that have a continuous average along the substructure edges, i.e., the averages are the same regardless of which substructure is considered for the calculation. The primal degrees of freedom are then eliminated together with the internal ones, at the expense of solving one coarse problem. This means that the problem reduces to one involving interface functions with vanishing mean value along the subdomain edges. Let $\widetilde{S} : \widetilde{W}_\Delta \longrightarrow \widetilde{W}_\Delta$ and $\tilde{f}_\Delta$ be the corresponding Schur complement and right hand side, respectively. The minimization property

$$(u_\Delta, \widetilde{S}u_\Delta) = \min(u, Su) \qquad (28)$$

holds, where the minimum is taken over all the functions $u = u_\Delta + w_\Pi$, $w_\Pi \in \widehat{W}_\Pi$. This property ensures that $\widetilde{S}$ is also positive definite. The reduced problem is

$$\begin{cases} \min & \frac{1}{2}(u_\Delta, \widetilde{S} u_\Delta) - (\tilde{f}_\Delta, u_\Delta), \\ \text{s.t.} & B_\Delta u_\Delta = 0, \end{cases} \tag{29}$$

where the equation $B_\Delta u_\Delta = 0$ expresses the continuity constraint. B_Δ is constructed from $\{0, -1, 1\}$ and evaluates the difference between all the corresponding tangential degrees of freedom on Γ, c.f. (25). The same matrix as in [17] is employed. The matrix B_Δ has the following block structure:

$$B_\Delta = [B_\Delta^{(1)}, B_\Delta^{(2)}, \ldots, B_\Delta^{(N)}],$$

where each block corresponds to a subdomain. By introducing a set of Lagrange multipliers $\lambda \in V := \mathrm{range}(B_\Delta)$ to enforce the continuity constraints, a saddle point formulation of (29) can be obtained:

$$\begin{aligned} \widetilde{S} u_\Delta + B_\Delta^T \lambda &= \tilde{f}_\Delta, \\ B_\Delta u_\Delta &= 0, \end{aligned}$$

with $u_\Delta \in \widetilde{W}_\Delta$. Since the Schur complement $\widetilde{S}$ is invertible, an equation for λ can easily be found:

$$F\lambda = d, \tag{30}$$

with

$$F := B_\Delta \widetilde{S}^{-1} B_\Delta^T, \quad d := B_\Delta \widetilde{S}^{-1} \tilde{f}_\Delta. \tag{31}$$

Once λ is found, the primal variables are obtained by back solve:

$$u_\Delta = \widetilde{S}^{-1}(\tilde{f}_\Delta - B_\Delta^T \lambda).$$

In order to define a preconditioner for (30), we need to define scaling matrices and functions defined on the subdomain boundaries. For each substructure, $\delta_i^\dagger \in W_i$ are defined such that on the edge E_{ij}, $j \in \mathcal{I}(i)$

$$\delta_i^\dagger = \frac{\gamma_i^\chi}{\gamma_i^\chi + \gamma_j^\chi},$$

for an arbitrary but fixed $\chi \in [1/2, \infty)$, see (27). By an elementary argument it can be proven that

$$\gamma_i (\delta_j^\dagger)^2 \leqslant \min(\gamma_i, \gamma_j). \tag{32}$$

For each substructure Ω_i, we introduce a diagonal matrix $D_\Delta^{(i)} : V \longrightarrow V$. The diagonal entry corresponding to the Lagrange multipliers that enforce the continuity along an edge E_{ij} is set equal to the (constant) value of $\delta_j^\dagger$ along E_{ji}

$$\delta_{ji}^\dagger := \delta_j^\dagger \mid_{E_{ji}} = \frac{\gamma_j^\chi}{\gamma_i^\chi + \gamma_j^\chi}.$$

Next the scaled matrix

$$B_{D,\Delta} = [D_\Delta^{(1)} B_\Delta^{(1)}, \, D_\Delta^{(2)} B_\Delta^{(2)}, \, \ldots, \, D_\Delta^{(N)} B_\Delta^{(N)}] : \widetilde{W}_\Delta \longrightarrow V.$$

is defined. The dual system (30) is then solved using the preconditioned conjugate gradient algorithm with the preconditioner

$$M^{-1} := B_{D,\Delta} S B_{D,\Delta}^T = \sum_{i=1}^N D_\Delta^{(i)} B_\Delta^{(i)} S^{(i)} B_\Delta^{(i)T} D_\Delta^{(i)}. \tag{33}$$

A method for an efficient implementation of a preconditioned dual-primal FETI algorithm on shape regular meshes is given in [18], Section 6. The implementation aspects do not rely on the mesh type and are therefore applicable to the boundary layer meshes discussed in this work.

4.3 Condition number bounds

4.3.1 Auxiliary results

The analysis of the dual-primal FETI method presented here relies on a decomposition result for $\mathbf{u} \in X_i$ which will be stated in the next lemma. We note that the tangential traces of functions in X_H are restrictions of functions in the space W_Π to the boundary of Ω_i. The following result can be found in [20] for h-approximations and in [18] for p-approximations. The proof for the hp-version on anisotropic meshes is given in subsection 4.4. Let $X_H(\Omega_i)$ denote the space $X_H(\Omega)$ restricted to the subdomain Ω_i and

$$Q_i := S_k(\Omega_i, \mathcal{T}) \cap \{\phi \in H^1(\Omega_i) | \phi = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}.$$

Lemma 4.1 *Let Ω_i be a substructure. Then, for every $\mathbf{u} \in X_i$ there exists a unique decomposition*

$$\mathbf{u} = \mathbf{u}_H + \sum_{j \in \mathcal{I}(i)} \mathbf{u}_{ij} + \mathbf{u}^{\text{int}},$$

such that

1. *$\mathbf{u}_H \in X_H(\Omega_i)$ is a coarse function;*
2. *$\mathbf{u}_{ij} = \nabla \phi_{ij}$, with $\phi_{ij} \in Q_i$ a Laplace discrete harmonic function that vanishes on $\partial\Omega_i \backslash E_{ij}$;*
3. *$\mathbf{u}^{\text{int}} \in X_i$ with vanishing tangential component on $\partial\Omega_i$.*

In addition, for $j \in \mathcal{I}(i)$,

$$\int_{E_{ij}} (\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{t}_{E_{ij}} \, ds = \int_{E_{ij}} \nabla \phi_{ij} \cdot \mathbf{t}_{E_{ij}} \, ds = 0, \tag{34}$$

and

$$||\nabla \phi_{ij}||^2_{0,\Omega_i} \leqslant C\omega^2 ||\mathbf{u}||^2_{\text{curl},\Omega_i}, \tag{35}$$

with $\omega = (1-\sigma)^{-4}(1 + \log k)$ and C independent of H_i, σ, n, k, and therefore of the aspect ratio.

The following result can be found in [19], Corollary 7.10:

Lemma 4.2 *Let Ω_i and Ω_j be two adjacent subdomains that share an edge E_{ij}. Let $\phi \in S_k(\Omega, \mathcal{T})$ be a piecewise Laplace discrete harmonic function that is identically zero at all nodal points in $\Gamma \backslash E_{ij}$. Then there exists a constant C such that*

$$||\nabla \phi||^2_{0,\Omega_j} \leqslant C(1-\sigma)^{-1} ||\nabla \phi||^2_{0,\Omega_i}. \tag{36}$$

In the analysis of the FETI-DP method it turns out that the condition number bounds rely on one stability estimate for the jump operator

$$P_\Delta := B^T_{D,\Delta} B_\Delta : \widetilde{W} \longrightarrow \widetilde{W};$$

see e.g. [11]. It has the following properties (see [11] and also Lemma 4.3 of [18]).

Lemma 4.3 *The operator P_Δ is a projection and preserves the jump of any function $w \in \widetilde{W}$, i.e.*

$$B_\Delta P_\Delta w = B_\Delta w.$$

If $v := P_\Delta w$, for $w \in \widetilde{W}$, then on every edge E_{ij} of a substructure Ω_i, it holds

$$v^{(i)} = \delta_j^\dagger (w^{(i)} - w^{(j)}).$$

Finally, $P_\Delta w = 0$, if $w \in \widehat{W}$.

The following fundamental result can be found in [11], Theorem 1:

Theorem 4.4 *Let C_{P_Δ} be such that*

$$|P_\Delta w_\Delta|_S^2 \leqslant C_{P_\Delta} |w_\Delta|_{\widetilde{S}}^2, \qquad w_\Delta \in \widetilde{W}_\Delta.$$

Then, if $\widetilde{S}$ and M^{-1} are invertible,

$$(M\lambda, \lambda) \leqslant (F\lambda, \lambda) \leqslant C_{P_\Delta}(M\lambda, \lambda), \quad \lambda \in V.$$

The norms used in the theorem are given by

$$|v|_S^2 := (v, Sv) = \sum_{i=1}^N (v^{(i)}, S^{(i)} v^{(i)}), \quad |v|_{\widetilde{S}}^2 := (v, \widetilde{S} v).$$

4.3.2 Main results

The aim of this paragraph is to prove the assumptions of Theorem 4.4. The first lemma ensures that $\widetilde{S}$ and M^{-1} are invertible while the second provides the stability estimate of the jump operator.

Lemma 4.5 *The Schur complement $\widetilde{S}$ and the preconditioner M^{-1} are invertible.*

Proof see e.g. [18], Lemma 4.5.

Lemma 4.6 *There is a constant C, such that, for $w_\Delta \in \widetilde{W}_\Delta$,*

$$|P_\Delta w_\Delta|_S^2 \leqslant C\eta(1-\sigma)^{-9}(1+\log k)^2 |w_\Delta|_{\widetilde{S}}^2,$$

where

$$\eta := \max_i \frac{\gamma_i}{\beta_i}\left(1 + \frac{H_i^2 \gamma_i}{a_i}\right).$$

Proof Using the minimization property in (28), the element $w = w_\Delta + w_\Pi$, $w_\Pi \in \widehat{W}_\Pi$ is considered such that

$$|w_\Delta|_{\widetilde{S}}^2 = |w|_S^2. \tag{37}$$

We note that, since w_Π is continuous,

$$v := P_\Delta w_\Delta = P_\Delta w.$$

The aim is to calculate

$$|P_\Delta w|_S^2 = \sum_{i=1}^N |v^{(i)}|_{S^{(i)}}^2 = \sum_{i=1}^N a_{\Omega_i} (\mathcal{H}_i^M v^{(i)}, \mathcal{H}_i^M v^{(i)}).$$

On an edge E_{ij} of a substructure Ω_i, the representation in Lemma 4.3 is employed.

It is recalled that the function $\delta_j^\dagger$ is constant along an edge E_{ij} and $\delta_{ji}^\dagger$ is this value. $v^{(i)}$ is then decomposed into contributions supported on single edges:

$$v^{(i)} = \sum_{j\in\mathcal{I}(i)} \theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - w^{(j)}),$$

where $\theta_{E_{ij}} \in W_i$ is a cut-off function that is identically one on E_{ij} and vanishes on $\partial\Omega_i \setminus E_{ij}$. Each contribution in this sum is considered separately. Since, in addition, w is an element of $\widetilde{W}$, its average $\bar{w}_{E_{ij}}$ is the same whether it is calculated using $w^{(i)}$ or $w^{(j)}$. Therefore it can be written as

$$\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - w^{(j)}) = \underbrace{\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - \bar{w}_{E_{ij}})}_{\mathrm{I}} - \underbrace{\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(j)} - \bar{w}_{E_{ij}})}_{\mathrm{II}}. \qquad (38)$$

The two terms in (38) are considered separately:

In order to bound I, the decomposition in Lemma 4.1 is employed for the vector $\mathbf{u} := \mathcal{H}_i^M w^{(i)}$. It is recalled that the tangential component of $\mathbf{u}_{ij} = \nabla\phi_{ij}$ vanishes on $\partial\Omega_i \setminus E_{ij}$ and, thanks to (34), it is equal to $\theta_{E_{ij}}(w^{(i)} - \bar{w}_{E_{ij}})$. Using (32), the minimizing property

$$|u^{(i)}|_{S^{(i)}}^2 := (u^{(i)}, S^{(i)} u^{(i)}) = a_{\Omega_i}(\mathcal{H}_i^M u^{(i)}, \mathcal{H}_i^M u^{(i)})$$

of the Maxwell discrete harmonic extension, (27) and (35), one obtains

$$\begin{aligned}
|\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - \bar{w}_{E_{ij}})|_{S^{(i)}}^2 &\leqslant \gamma_i ||\nabla\phi_{ij}||_{0,\Omega_i}^2 \\
&\leqslant C\gamma_i\omega^2(||\mathbf{u}||_{0,\Omega_i}^2 + H_i^2 ||\mathrm{curl}\,\mathbf{u}||_{0,\Omega_i}^2) \\
&\leqslant C\,\eta\,\omega^2\, a_{\Omega_i}(\mathcal{H}_i^M w^{(i)}, \mathcal{H}_i^M w^{(i)}) = C\,\eta\,\omega^2\, |w^{(i)}|_{S^{(i)}}^2.
\end{aligned}$$

Then II in (38) is considered. The vector

$$\mathbf{u}^{(i)} := \mathcal{H}_i^M(\theta_{E_{ij}}(w^{(j)} - \bar{w}_{E_{ij}}))$$

can be decomposed according to Lemma 4.1, into the sum of two contributions $\mathbf{u}_{ij} = \nabla\phi_{ij}$ and $\mathbf{u}^{\mathrm{int}}$. Next, Lemma 4.1 is applied to the function $\mathcal{H}_j^M w^{(j)}$ to obtain

$$\mathbf{u}^{(j)} := \mathcal{H}_j^M w^{(j)} = \mathbf{u}_H + \sum_{k\in\mathcal{I}(j)} \mathbf{u}_{jk} + \widetilde{\mathbf{u}}^{\mathrm{int}}.$$

We note that the functions $\mathbf{u}_{ji} = \nabla\phi_{ji}$ and $\mathbf{u}_{ij} = \nabla\phi_{ij}$ have the same tangential component along the common edge E_{ij}, which is equal to $\theta_{E_{ij}}(w^{(j)} - \bar{w}_{E_{ij}})$. Using (32), the minimizing property of the Maxwell discrete harmonic extension, Lemma 4.2, (27) and (35), one finds

$$\begin{aligned}
|\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(j)} - \bar{w}_{E_{ij}})|_{S^{(i)}}^2 &\leqslant \gamma_j ||\nabla\phi_{ij}||_{0,\Omega_i}^2 \\
&\leqslant C(1-\sigma)^{-1}\gamma_j ||\nabla\phi_{ji}||_{0,\Omega_j}^2 \\
&\leqslant C(1-\sigma)^{-1}\gamma_j\omega^2(||\mathbf{u}^{(j)}||_{0,\Omega_j}^2 + H_j^2 ||\mathrm{curl}\,\mathbf{u}^{(j)}||_{0,\Omega_j}^2) \\
&\leqslant C\,(1-\sigma)^{-1}\,\eta\,\omega^2\, a_{\Omega_j}(\mathcal{H}_j^M w^{(j)}, \mathcal{H}_j^M w^{(j)}) \\
&= C\,(1-\sigma)^{-1}\,\eta\,\omega^2\, |w^{(j)}|_{S^{(j)}}^2.
\end{aligned}$$

Combining (38), the bounds for the terms I and II and summing over the edges E_{ij}, one finally finds

$$|v^{(i)}|_{S^{(i)}}^2 \leqslant C\,\eta\,\omega^2\, |w^{(i)}|_{S^{(i)}}^2 + C\,(1-\sigma)^{-1}\,\eta\,\omega^2 \sum_{j\in\mathcal{I}(i)} |w^{(j)}|_{S^{(j)}}^2.$$

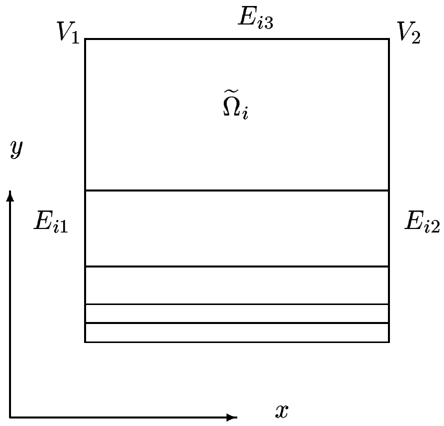

Figure 2: Edge patch $\Omega_i = (0,1)^2$ used in the proof of Lemma 4.11.

Summing over the substructures Ω_i yields

$$\begin{aligned} |P_\Delta w| &= \sum_{i=1}^{N} |v^{(i)}|^2_{S^{(i)}} \\ &\leqslant C\,\eta\,\omega^2\,|w|^2_S + C\,(1-\sigma)^{-1}\,\eta\,\omega^2\,M|w|^2_S \\ &\leqslant C\,(1-\sigma)^{-1}\,\eta\,\omega^2\,|w|^2_S. \end{aligned}$$

M is the maximum number of adjacent subdomains, which is bounded. The proof is then concluded by using (37). □

Combining Lemmas 4.5 and 4.6 and Theorem 4.4 leads to the final result:

Theorem 4.7 *The condition number of the preconditioned system $M^{-1}F$ satisfies*

$$\kappa(M^{-1}F) \leqslant C\,\eta\,(1-\sigma)^{-9}\,(1+\log k)^2.$$

4.4 Proof of Lemma 4.1

The proof follows that of [18] for the p-version and is adapted to an hp-version on boundary layer meshes. It deals with orthogonal decompositions of edge element functions into gradients of scalar functions and discrete curl free functions. Let

$$X_{0,i} := ND_{0,k}(\Omega_i, \mathcal{T}) \subset X_i$$

be the local space of vectors with vanishing tangential component on the whole boundary $\partial\Omega_i$ and

$$Q_{0,i} := S_{0,k}(\Omega_i, \mathcal{T}) \subset Q_i$$

be the subspace of functions that vanish on the boundary $\partial\Omega_i$. Then Proposition 3.3 implies that $X_{0,i}$ can be decomposed into

$$X_{0,i} = \mathbf{grad} Q_{0,i} \oplus X^{\perp}_{0,i}. \tag{39}$$

The discrete Friedrichs' inequality, Theorem 3.9, for $\mathbf{u} \in X^{\perp}_{0,i}$ yields

$$||\mathbf{u}||_{0,\Omega_i} \leqslant CH_i||\mathrm{curl}\,\mathbf{u}||_{0,\Omega_i}, \tag{40}$$

with a constant that is independent of the aspect ratio of $\mathcal{T}$. For the proof of the following Lemma 4.9 we will need the following auxiliary result that can be found in [3], Lemma 6.6:

Lemma 4.8 *Let I denote the interval $(0,1)$ and u be a polynomial of degree k with $u(0) = u(1) = 0$. Then, for all $k > 0$,*

$$||u||^2_{\frac{1}{2},00,I} \leqslant |u|^2_{\frac{1}{2},I} + C(1+\log k)||u||^2_{\infty,I}.$$

The following lemma will be used in the proof of Lemma 4.1:

Lemma 4.9 (edge patch) *Let Ω_i be a subdomain refined as an edge patch. Let $\phi_H \in \mathbb{Q}_{1,1}(\Omega_i)$ and, for $j \in \mathcal{I}(i)$, $\phi_{ij} \in Q_i$ be a Laplace discrete harmonic function that vanishes on $\partial\Omega_i \setminus E_{ij}$. If*

$$\phi := \phi_H + \sum_{j\in\mathcal{I}(i)} \phi_{ij},$$

then

$$|\phi_{ij}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-4}(1+\log k)^2|\phi|^2_{1,\Omega_i}.$$

Proof For simplicity we assume that Ω_i is the unit square, i.e. $\Omega_i = (0,1)^2$. The case of a more general Ω_i can be treated with a scaling argument. The proof is then performed in two steps. We first prove the lemma for the edge E_{i3}, see Figure 2. In the second step we prove it for the remaining edges E_{i1} and E_{i2}. As the first step deals with Laplace discrete harmonic extensions we have to recall a well-known property. Let ω_i be a piecewise linear function defined on $\partial\Omega_i$ and $\nu_i \in Q_i$ such that $\nu_i\,|_{\partial\Omega_i} = \omega_i$. The minimization property says that Laplace discrete harmonic functions minimize the energy among all functions with the same boundary values:

$$|\mathcal{H}_i(\omega_i)|_{1,\Omega_i} \leqslant |\nu_i|_{1,\Omega_i}. \tag{41}$$

Step 1 Let $\widetilde{\Omega}_i$ denote the unique element of the edge patch which contains the two vertices of Ω_i that do not lie on $\partial\Omega$, see Figure 2. Let $\widetilde{\phi}_{i3}$ denote the function that is Laplace discrete harmonic in $\widetilde{\Omega}_i$ with boundary values

$$\widetilde{\phi}_{i3}\,|_{\partial\widetilde{\Omega}_i} = \begin{cases} \phi_{i3} & \text{on } E_{i3}, \\ 0 & \text{on } \partial\widetilde{\Omega}_i \setminus E_{i3}, \end{cases}$$

and is extended by zero to the rest of Ω_i. Since $\widetilde{\phi}_{i3}$ is a Laplace discrete harmonic bilinear function in $\widetilde{\Omega}_i$, we can equivalently work with the norms on $\widetilde{\Omega}_i$ and on its boundary. Thus, a scaling argument yields

$$|\widetilde{\phi}_{i3}|^2_{1,\widetilde{\Omega}_i} \leqslant C(1-\sigma)^{-2}|\widetilde{\phi}_{i3}|^2_{\frac{1}{2},00,E_{i3}}. \tag{42}$$

¿From the minimizing property (41),(42), and Lemma 4.8 it follows that

$$\begin{aligned} |\phi_{i3}|^2_{1,\Omega_i} &\leqslant |\widetilde{\phi}_{i3}|^2_{1,\widetilde{\Omega}_i} \\ &\leqslant C(1-\sigma)^{-2}|\widetilde{\phi}_{i3}|^2_{\frac{1}{2},00,E_{i3}} \\ &\leqslant C(1-\sigma)^{-2}(|\widetilde{\phi}_{i3}|^2_{\frac{1}{2},E_{i3}} + (1+\log k)||\widetilde{\phi}_{i3}||^2_{\infty,E_{i3}}) \\ &\leqslant C(1-\sigma)^{-2}(|\phi-\phi_H|^2_{\frac{1}{2},E_{i3}} + (1+\log k)||\phi||^2_{\infty,\widetilde{\Omega}_i}) \\ &\leqslant C(1-\sigma)^{-2}(|\phi|^2_{\frac{1}{2},E_{i3}} + |\phi_H|^2_{\frac{1}{2},E_{i3}} + (1+\log k)||\phi||^2_{\infty,\widetilde{\Omega}_i}). \end{aligned} \tag{43}$$

For the first term in (43) we use the trace Theorem 2.4 for $H^1(\Omega_i)$. This leads to

$$|\phi|^2_{\frac{1}{2},E_{i3}} \leqslant C||\phi||^2_{1,\Omega_i}. \tag{44}$$

The second term $|\phi_H|^2_{\frac{1}{2},E_{i3}}$ can be bounded directly: Since $\phi_H(x) = \phi(V_1)+(\phi(V_2)-\phi(V_1))x$ on E_{i3}, then

$$\begin{aligned}
|\phi_H|^2_{\frac{1}{2},E_{i3}} &= \int_{E_{i3}}\int_{E_{i3}} \frac{|\phi_H(x)-\phi_H(y)|^2}{|x-y|^2}\,dxdy \\
&= \int_{E_{i3}}\int_{E_{i3}} \frac{|(\phi(V_2)-\phi(V_1))(x-y)|^2}{|(x-y)|^2}\,dxdy \\
&= |\phi(V_2)-\phi(V_1)|^2 \\
&\leqslant C||\phi||^2_{\infty,\tilde{\Omega}_i}.
\end{aligned}$$

From [19], Lemma 7.6 it follows that

$$||\phi||^2_{\infty,\tilde{\Omega}_i} \leqslant C(1-\sigma)^{-2}(1+\log k)||\phi||^2_{1,\tilde{\Omega}_i}$$

Therefore it holds

$$|\phi_H|^2_{\frac{1}{2},E_{i3}} \leqslant C(1-\sigma)^{-2}(1+\log k)||\phi||^2_{1,\Omega_i}. \tag{45}$$

Inequality (43) together with (44), (45) and Lemma 7.6 of [19] gives

$$|\phi_{i3}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-4}(1+\log k)^2||\phi||^2_{1,\Omega_i}.$$

We note that if a constant is added to ϕ_{ij}, the left hand side does not change. A Poincaré inequality then allows to replace the full norm with the seminorm on the right hand side to obtain

$$|\phi_{i3}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-4}(1+\log k)^2|\phi|^2_{1,\Omega_i}. \tag{46}$$

Step 2 The main result used in this step is a result from [19], Lemma 7.2: Let $u \in Q_i$ be a piecewise bilinear function on Ω_i. Given $\theta \in W^{1,\infty}(\Omega_i)$, with $||\theta||_{\infty,\Omega_i} \leqslant C$, then

$$|\mathcal{H}_i(\theta u)|^2_{1,\Omega_i} \leqslant C(|u|^2_{1,\Omega_i} + ||\nabla\theta||^2_{\infty,\Omega_i}||u||^2_{0,\Omega_i}) \tag{47}$$

Let $\breve{\phi}$ be the sum

$$\breve{\phi} = \phi_{i1} + \phi_{i2},$$

and $\theta^H_{E_{i1}}$ be the linear function in x with value 1 on E_{i1} and 0 on E_{i2}, i.e.

$$\theta^H_{E_{i1}}(x) = 1-x.$$

Then it holds $\phi_{i1} = \mathcal{H}_i(\theta^H_{E_{i1}}\breve{\phi})$ and therefore it follows by using (47)

$$|\phi_{i1}|^2_{1,\Omega_i} \leqslant C(||\nabla\breve{\phi}||^2_{0,\Omega_i} + ||\nabla\theta^H_{E_{i1}}||^2_{\infty,\Omega_i}||\breve{\phi}||^2_{0,\Omega_i}).$$

Since it holds

$$||\nabla\theta^H_{E_{i1}}||_{\infty,\Omega_i} \leqslant C,$$

it follows

$$|\phi_{i1}|^2_{1,\Omega_i} \leqslant C(||\nabla\breve{\phi}||^2_{0,\Omega_i} + ||\breve{\phi}||^2_{0,\Omega_i}). \tag{48}$$

As $\breve{\phi}$ vanishes on a part of $\partial\Omega_i$, a Poincaré inequality yields

$$||\breve{\phi}||^2_{0,\Omega_i} \leqslant ||\nabla\breve{\phi}||^2_{0,\Omega_i}. \tag{49}$$

Therefore it remains to bound the first term in (48):

$$\begin{aligned}
||\nabla\breve{\phi}||^2_{0,\Omega_i} &= |\phi-\phi_{i3}-\phi_H|^2_{1,\Omega_i} \\
&\leqslant |\phi|^2_{1,\Omega_i} + |\phi_{i3}|^2_{1,\Omega_i} + |\phi_H|^2_{1,\Omega_i}.
\end{aligned} \tag{50}$$

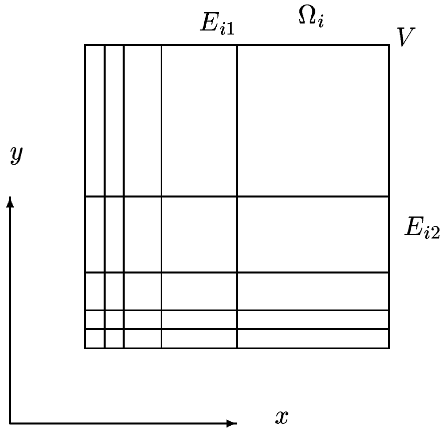

Figure 3: Corner patch $\Omega_i = (0,1)^2$ used in the proof of Lemma 4.10.

In Step 1 we have already found a bound for $|\phi_{i3}|^2_{1,\Omega_i}$, therefore it only remains to find a bound for $|\phi_H|^2_{1,\Omega_i}$:

$$\begin{aligned} |\phi_H|^2_{1,\Omega_i} &\leqslant C||\phi||_{\infty,\widetilde{\Omega}_i} \\ &\leqslant C(1-\sigma)^{-2}(1+\log k)||\phi||^2_{1,\Omega_i} \\ &= C(1-\sigma)^{-2}(1+\log k)|\phi|^2_{1,\Omega_i}. \end{aligned} \tag{51}$$

Here, Lemma 7.6 of [19] as well as a Poincaré inequality were used again. Combining (48) with (49),(50) and (51) gives

$$|\phi_{i1}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-4}(1+\log k)^2|\phi|^2_{1,\Omega_i}. \tag{52}$$

With similar arguments, the same bound can be found for $|\phi_{i2}|^2_{1,\Omega_i}$.

Hence, by comparing (46) with (52) we find the final result for the edge patch:

$$|\phi_{ij}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-4}(1+\log k)^2|\phi|^2_{1,\Omega_i}. \tag{53}$$

□

A similar lemma holds for a corner patch

Lemma 4.10 (corner patch) *Let Ω_i be a subdomain refined as a corner patch. Let $\phi_H \in \mathbb{Q}_{1,1}(\Omega_i)$ and, for $j \in \mathcal{I}(i)$, $\phi_{ij} \in Q_i$ be a Laplace discrete harmonic function that vanishes on $\partial\Omega_i \setminus E_{ij}$. If*

$$\phi := \phi_H + \sum_{j\in\mathcal{I}(i)} \phi_{ij},$$

then

$$|\phi_{ij}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-8}(1+\log k)^2|\phi|^2_{1,\Omega_i}.$$

Proof The proof for the corner patch relies on the result for the edge patch. It is carried out for the edge E_{i2}, see Figure 3. For simplicity assume that Ω_i is the unit square $(0,1)^2$. As usual a scaling argument gives the result for a more general Ω_i. Let $\widetilde{\Omega}_i$ denote the layer of points in Ω_i within a distance $(1-\sigma)$ from E_{i2}. Let $\theta_{E_{i2}}$ be the function

$$\theta_{E_{i2}} = \begin{cases} 0 & \Omega_i \setminus \widetilde{\Omega}_i, \\ (1-\sigma)^{-1}x - \frac{\sigma}{1-\sigma} & \widetilde{\Omega}_i. \end{cases}$$

We define $\widetilde{\phi} := \mathcal{H}_{\widetilde{\Omega}_i}(\theta_{E_{i2}}\phi)$, where, we recall, $\mathcal{H}_{\widetilde{\Omega}_i}$ denotes the Laplace discrete harmonic extension. It follows by using (47):

$$|\widetilde{\phi}|^2_{1,\widetilde{\Omega}_i} \leqslant C(|\phi|^2_{1,\widetilde{\Omega}_i} + (1-\sigma)^{-2}||\phi||^2_{0,\widetilde{\Omega}_i}). \tag{54}$$

We can decompose $\widetilde{\phi}$ on the dilated edge patch $\widetilde{\Omega}_i$ according to Lemma 4.9:

$$\widetilde{\phi} := \widetilde{\phi}_H + \sum_j \widetilde{\phi}_{ij}.$$

We note that ϕ_{i2} and $\widetilde{\phi}_{i2}$ both vanish outside E_{i2} and have the same value on this edge. Since $\widetilde{\Omega}_i$ can be obtained from an edge patch $\widehat{\Omega} = (0,1)^2$ by a dilation of $(1-\sigma)$ in x-direction, it follows from (41) and Lemma 4.9 together with a scaling argument

$$\begin{aligned} |\phi_{i2}|^2_{1,\Omega_i} &\leqslant |\widetilde{\phi}_{i2}|^2_{1,\widetilde{\Omega}_i} \\ &\leqslant C(1-\sigma)^{-6}(1+\log k)^2|\widetilde{\phi}|^2_{1,\widetilde{\Omega}_i}. \end{aligned} \tag{55}$$

Combining (55) with (54) and a Poincaré inequality leads to the final result

$$|\phi_{i2}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-8}(1+\log k)^2|\phi|^2_{1,\Omega_i}.$$

□

Comparing Lemma 4.9 with Lemma 4.10 leads to

Theorem 4.11 *Let Ω_i be a subdomain refined as an edge or corner patch. Let $\phi_H \in \mathbb{Q}_{1,1}(\Omega_i)$ and, for $j \in \mathcal{I}(i)$, $\phi_{ij} \in Q_i$ be a Laplace discrete harmonic function that vanishes on $\partial\Omega_i \setminus E_{ij}$. If*

$$\phi := \phi_H + \sum_{j\in\mathcal{I}(i)} \phi_{ij},$$

then

$$|\phi_{ij}|^2_{1,\Omega_i} \leqslant C(1-\sigma)^{-8}(1+\log k)^2|\phi|^2_{1,\Omega_i}.$$

Proof of Lemma 4.1 Once the discrete Friedrichs' inequality of Theorem 3.9 and the stability estimate of Theorem 4.11 are established, the proof follows that of [18], Lemma 4.1. It is given here for completeness. The coarse space $X^H(\Omega_i)$ was defined in Subsection 4.3. Now the coarse interpolant

$$\rho_H : X_i \longrightarrow X_H(\Omega_i)$$

is introduced. Here, $\rho_H \mathbf{u}$ is the unique vector that satisfies

$$\int_{E_{ij}} (\rho_H\mathbf{u} - \mathbf{u}) \cdot \mathbf{t}_{E_{ij}}\, ds = 0, \quad j \in \mathcal{I}(i). \tag{56}$$

We also define $X_{ij} \subset X_i$ as the space of functions $\nabla\phi_{ij}$, where $\phi_{ij} \in Q_i$ is Laplace discrete harmonic and vanishes on $\partial\Omega_i \setminus E_{ij}$.

It is immediate to see that, for the substructure Ω_i and for $j \in \mathcal{I}(i)$, $l \in \mathcal{I}(i)$, $j \neq l$,

$$X_H(\Omega_i) \cap X_{0,i} = X_H(\Omega_i) \cap X_{ij} = X_{ij} \cap X_{0,i} = X_{ij} \cap X_{il} = \{0\}.$$

Counting the degrees of freedom, it can be seen

$$X_i = X_H(\Omega_i) \oplus \sum_{j\in\mathcal{I}(i)} X_{ij} \oplus X_{0,i}$$

is a direct sum. We have therefore proved the existence and the uniqueness of the decomposition in Lemma 4.1.

The first equality in (34) is a consequence of the fact that the tangential component of $\mathbf{u}_i := \mathbf{u}^{\text{int}}$ and of $\mathbf{u}_{il}$, for $l \neq j$, vanishes on the edge E_{ij}. The second comes from the fact that ϕ_{ij} vanishes at the end points of E_{ij}.

We are then left with the proof of the stability property (35). Since the decomposition is unique, thanks to (56), we find $\mathbf{u}_H = \rho_H \mathbf{u}$. Now each term is decomposed into the gradient of a scalar function and a remainder. Since the coarse space $X_H(\Omega_i)$ is $ND_1(\Omega_i)$, we can write

$$\mathbf{u}_H = \nabla \phi_H + \alpha \begin{pmatrix} y - y_i \\ x_i - x \end{pmatrix} =: \nabla \phi_H + \mathbf{u}_H^{\perp},$$

where $\phi_H \in \mathbb{Q}_{1,1}(\Omega_i)$ is bilinear and (x_i, y_i) is the center of gravity of Ω_i. By direct calculation it can be found that this is an L^2 orthogonal decomposition and that

$$||\mathbf{u}_H^{\perp}||_{0,\Omega_i} \leqslant C H_i ||\text{curl}\, \mathbf{u}_H^{\perp}||_{0,\Omega_i}. \tag{57}$$

For the term $\mathbf{u}_i := \mathbf{u}^{\text{int}} \in X_{0,i}$, the orthogonal decomposition in (39) is employed and we find

$$\mathbf{u}_i = \nabla \phi_i + \mathbf{u}_i^{\perp}.$$

Finally, by definition, $\mathbf{u}_{ij} = \nabla \phi_{ij}$, for each edge E_{ij}. We then group the gradient terms and the remainders and set

$$\phi := \phi_H + \sum_{j \in \mathcal{I}(i)} \phi_{ij} + \phi_i, \qquad \mathbf{u}^{\perp} := \mathbf{u}_H^{\perp} + \mathbf{u}_i^{\perp}.$$

We therefore have the decomposition

$$\mathbf{u} = \nabla \phi + \mathbf{u}^{\perp}. \tag{58}$$

We need to bound the $\nabla \phi_{ij}$'s in terms of $\mathbf{u}$. Since ϕ_H and the $\{\phi_{il}\}$ are Laplace discrete harmonic, Theorem 4.11 can be used and we find

$$\begin{aligned} |\phi_{ij}|_{1,\Omega_i}^2 &\leqslant C(1-\sigma)^{-8}(1+\log k)^2 \left| \phi_H + \textstyle\sum_{l \in \mathcal{I}(i)} \phi_{il} \right|_{1,\Omega_i}^2 \\ &\leqslant C(1-\sigma)^{-8}(1+\log k)^2 |\phi|_{1,\Omega_i}^2. \end{aligned} \tag{59}$$

For the last step the fact has been used that ϕ_i vanishes on $\partial\Omega_i$ and is thus orthogonal to Laplace discrete harmonic functions.

The last step is to bound $\nabla \phi$ in terms of $\mathbf{u}$. We first note that, using (57) and (40), we obtain

$$||\mathbf{u}^{\perp}||_{0,\Omega_i}^2 \leqslant C H_i^2 (||\text{curl}\, \mathbf{u}_H^{\perp}||_{0,\Omega_i}^2 + ||\text{curl}\, \mathbf{u}_i^{\perp}||_{0,\Omega_i}^2).$$

Since $\text{curl}\, \mathbf{u}_H^{\perp}$ is constant and $\text{curl}\, \mathbf{u}_i^{\perp}$ has a vanishing mean value on Ω_i, these two functions are L^2 orthogonal and thus

$$||\mathbf{u}^{\perp}||_{0,\Omega_i}^2 \leqslant C_{\perp} H_i^2 ||\text{curl}\, \mathbf{u}^{\perp}||_{0,\Omega_i}^2. \tag{60}$$

Using (57), (60), and the Young's inequality, we find

$$\begin{aligned} ||\mathbf{u}||_{\text{curl},\Omega_i}^2 &= |\phi|_{1,\Omega_i}^2 + ||\mathbf{u}^{\perp}||_{\text{curl},\Omega_i}^2 + 2 \int_{\Omega_i} \nabla \phi \cdot \mathbf{u}^{\perp}\, dx \\ &\geqslant (1-\epsilon)|\phi|_{1,\Omega_i}^2 + (1 + (1-\epsilon^{-1})C_{\perp}) H_i^2 ||\text{curl}\, \mathbf{u}^{\perp}||_{0,\Omega_i}^2, \end{aligned}$$

for $\epsilon \in (0,1)$. The choice $\epsilon = C_{\perp}/(C_{\perp}+1)$ gives

$$|\phi|_{1,\Omega_i}^2 \leqslant (C_{\perp}+1) ||\mathbf{u}||_{\text{curl},\Omega_i}^2,$$

which, combined with (59), concludes the proof. □

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