Sparse Wavelet Methods for Option Pricing under Stochastic Volatility[1]

N. Hilber, A.-M. Matache[2] and C. Schwab

Research Report No. 2004-07
September 2004

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

[1]Research was supported by Credit Suisse Group, Swiss Re and UBS AG through RiskLab, Switzerland
[2]RiskLab and Seminar for Applied Mathematics, ETH-Zentrum CH-8092 Zürich, Switzerland

Sparse Wavelet Methods for Option Pricing under Stochastic Volatility[1]

N. Hilber, A.-M. Matache[2] and C. Schwab

Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

Research Report No. 2004-07 September 2004

Abstract

Prices of European plain vanilla as well as barrier and compound options on one risky asset in a Black-Scholes market with stochastic volatility are expressed as solution of degenerate parabolic partial differential equations with two spatial variables: the spot price S and the volatility process variable y.
We present and analyze a pricing algorithm based on sparse wavelet space discretizations of order $p \geq 1$ in the spot price S or the log-returns $x = \log S$ and in y, the volatility driving process, and on hp-discontinuous Galerkin time-stepping with geometric step size reduction towards maturity T.
Wavelet preconditioners adapted to the volatility models for a GMRES solver allow to price contracts at all maturities $0 < t \leq T$ and all spot prices for a given strike K in essentially[3] $O(N)$ memory and work with accuracy of essentially $O(N^{-p})$ in essentially $O(N)$ operations, a performance comparable to that of the best FFT-based pricing methods for constant volatility models.

Keywords: Stochastic volatility models, degenerate parabolic partial differential equations, weighted Sobolev spaces, discontinuous Galerkin time stepping, compound options, sparse tensor products, hyperbolic cross approximations

---

[1]Research was supported by Credit Suisse Group, Swiss Re and UBS AG through RiskLab, Switzerland
[2]RiskLab and Seminar for Applied Mathematics, ETH-Zentrum CH-8092 Zürich, Switzerland
[3]Throughout, “essentially” means up to powers of $\log N$ and $|\log h|$, respectively

1 Introduction

The pricing of options by means of partial differential equations has become standard practice in quantitative finance, either by means of explicit solution formulas or by numerical methods.
In the standard Black-Scholes model the volatility σ of the risky asset is assumed to be constant or a function of time and explicit pricing formulas are available for European Vanillas, see [3]. Such models are generally too crude to match observed log-return prices well.
A more flexible class of models assumes that the volatility is a stochastic process. A widely used model is the Heston model in which the square of the volatility follows a familiar square-root process (used by Cox, Ingersoll and Ross (1985)). Alternatively, consider the volatility $\sigma_t = f(Y_t)$ as a function of a mean reverting Ornstein-Uhlenbeck process Y_t. While for European Vanilla options closed form pricing formulas are available [12], other types of contracts, as e.g., barriers, exotics or compound-style options have to be priced numerically.
The no arbitrage principle leads to a parabolic partial differential equation (PDE) for the value function $V(t, S, y)$ which depends on the time t, the spot price S_t of the underlying asset, and on the volatility process $y(t)$. These parabolic pricing PDEs have degenerate coefficients in the S variable, and, depending on the volatility model, possibly also in the y variable.
Numerical solution requires truncation of the PDE to a bounded computational domain. In our analysis, the localization is justified by a variational analysis in weighted Sobolev spaces following [1].
We emphasize that we neither advocate nor discourage the use of stochastic volatility processes – the goal of the present paper is to show design and analysis of pricing algorithms for stochastic volatility that have complexity comparable to the best FFT-based methods for Black-Scholes [5].

The paper is organized as follows: in Section 2 we introduce the pricing problem in terms of parabolic partial differential equations. Several models of volatility are considered, as well as non-vanilla contracts as e.g. compound options are discussed. The variational formulation (set in weighted Sobolev spaces) of these PDEs is performed in Section 3, where we also give an analysis of the domain truncation error. In Section 4 we describe the discretization of the problem, both in space and time. We briefly review the wavelet multiresolution analysis in terms of full and sparse tensor product spaces, and we consider hp-dG time-stepping with geometric step size reduction towards maturity to discretize in time. We also show how one can construct optimal diagonal and block diagonal preconditioners based on wavelet norm equivalences in weighted Sobolev spaces. Numerical experiments are presented in Section 5. There we solve several pricing PDEs and, in particular, we price compound options numerically.

2 Pricing European Vanillas with Stochastic Volatility

In "pure" stochastic volatility models the asset price $(S_t)_{t\geq 0}$ satisfies the stochastic differential equation (SDE)

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma_t S_t \mathrm{d}W_t, \tag{2.1}$$

where, contrary to the log-normal model [3], the volatility σ_t is supposed to be non-deterministic. The process $(\sigma_t)_{t\geq 0}$ has to be positive and is called the volatility process, $\mu S_t \mathrm{d}t$ is the drift term and W_t is a Brownian motion.
Evidence that the volatility process is not perfectly correlated with the Brownian motion $(W_t)_{t\geq 0}$ suggests (see e.g. [10] and the references therein) to model $(\sigma_t)_t$ to have an independent random component of its own, which is a different approach from models based on implied deterministic volatility (or local volatility) in which $\sigma_t = \sigma(t, S_t)$ with $\sigma(t, S)$ being a deterministic positive function.

2.1 Models of volatility

We discuss two volatility models, namely a mean-reverting Ornstein-Uhlenbeck process and a Cox-Ingersoll-Ross process.

2.1.1 Mean-Reverting Ornstein-Uhlenbeck (OU)

One assumes that σ_t is a function of a mean reverting Ornstein-Uhlenbeck process, i.e.

$$\sigma_t = f(Y_t), \qquad \mathrm{d}Y_t = \alpha(m - Y_t)\mathrm{d}t + \beta \mathrm{d}\hat{Z}_t, \tag{2.2}$$

with α, β, m positive constants and $\hat{Z}_t$ being a Brownian motion. The parameter α is called the rate of mean reversion, and the ratio $\frac{\beta^2}{\alpha}$ is the limit of the variance of Y_t as $t \to \infty$. The Brownian motion $\hat{Z}_t$ may be correlated with W_t, that is, it can be written as a linear combination of W_t and an independent Brownian motion Z_t

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t, \tag{2.3}$$

with a correlation factor $\rho \in [-1, 1]$.
Considering a European derivative with underlying asset S_t, maturity T and pay-off function $h(S_T)$, its price at time t depends on t, on the price of the underlying asset S_t and on Y_t.
Denote by $U(t, S_t, Y_t)$ the price of the derivative at time t and by $r(t)$ the risk-neutral interest rate. Then it can be shown [10, Chapter 2] – by the no arbitrage principle and

the two dimensional Itô's formula – that there exists a function γ such that the option value U satisfies the following parabolic partial differential equation:

$$\frac{\partial U}{\partial t}+\frac{1}{2}f(y)^2S^2\frac{\partial^2 U}{\partial S^2}+r(t)\left(S\frac{\partial U}{\partial S}-U\right)+\rho\beta S f(y)\frac{\partial^2 U}{\partial S\partial y}$$
$$+\frac{1}{2}\beta^2\frac{\partial^2 U}{\partial y^2}+\alpha(m-y)\frac{\partial U}{\partial y}-\beta\Lambda(t,S,y)\frac{\partial U}{\partial y}=0 \text{ in } (0,T)\times(0,\infty)\times\mathbb{R} \tag{2.4}$$

together with the terminal condition at maturity

$$U(T,S,y)=h(S), \tag{2.5}$$

and with Λ being given by

$$\Lambda(t,S,y)=\rho\frac{\mu-r(t)}{f(y)}+\sqrt{1-\rho^2}\gamma(t,S,y). \tag{2.6}$$

The so-called risk premium factor γ can be any bounded, measurable function. In the perfectly correlated case ($|\rho|=1$) the function γ does not appear, and we are in a Black-Scholes market which is complete. If $|\rho|<1$, the second source of randomness that drives the volatility renders the market incomplete. The risk premium factor γ can be viewed as parameterizing the set of equivalent martingale measures.
The first line in (2.4) is the standard Black-Scholes operator with volatility $\sigma=f(y)$, $\rho\beta f(y)\frac{\partial^2 U}{\partial S\partial y}$ is due to the correlation between W_t and Z_t, $\frac{1}{2}\beta^2\frac{\partial^2 U}{\partial y^2}+\alpha(m-y)\frac{\partial U}{\partial y}$ is the mean-reverting Ornstein-Uhlenbeck part and $\beta\Lambda(t,S,y)\frac{\partial U}{\partial y}$ is the premium for the market price of volatility risk (see also [10]).
There are several possible choices for the function f, depending on the volatility model. For a mean reverting Ornstein-Uhlenbeck process the models $f(y)=|y|$ (Stein and Stein) and $f(y)=e^y$ (Scott) are two examples. Unless explicitly stated otherwise, we will focus on the degenerate case when $f(y)=|y|$. We treat in the following only the case of constant interest rate $r(t)\equiv r$ and assume that $\gamma\equiv 0$. The assumption that $\gamma=0$ is not an essential limitation, parameterizations of γ of the form $\gamma=\lambda y$, with $\lambda\in\mathbb{R}$ lead to a similar PDE in (2.4) with $\tilde{\alpha}=\alpha+\beta\lambda\sqrt{1-\rho^2}$ and $\tilde{m}=\alpha m/\tilde{\alpha}$.
Furthermore, we consider two types of terminal conditions, namely

$$h(S)=(S-K)_+, \tag{2.7}$$

and

$$h(S)=(K-S)_+ \tag{2.8}$$

corresponding to European call and put options, respectively; $K>0$ denotes the so-called strike or exercise price.

2.1.2 Cox–Ingersoll–Ross (CIR) process

In the Heston model [12], the volatility σ_t is given as $\sigma_t = \sqrt{Y_t}$, where the driving process Y_t satisfies

$$\mathrm{d}Y_t = \kappa(m - Y_t)\mathrm{d}t + \sigma\sqrt{Y_t}\mathrm{d}\hat{Z}_t. \tag{2.9}$$

The stochastic dynamics of the risky asset S_t are given by the SDE

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{Y_t}S_t\mathrm{d}W_t. \tag{2.10}$$

Any European contingent claim U satisfies the following PDE

$$\frac{\partial U}{\partial t} + \frac{1}{2}yS^2\frac{\partial^2 U}{\partial S^2} + r\left(S\frac{\partial U}{\partial S} - U\right) + \rho\sigma yS\frac{\partial^2 U}{\partial y\partial S} \tag{2.11}$$

$$+\frac{1}{2}\sigma^2 y\frac{\partial^2 U}{\partial y^2} + \left(\kappa(m-y) - \lambda(t,S,y)\right)\frac{\partial U}{\partial y} = 0 \quad \text{in} \quad (0,T)\times(0,\infty)^2$$

$$U(T,S,y) = h(S) \quad \text{in} \quad (0,\infty)^2. \tag{2.12}$$

The term $\lambda(t,S,y)$ represents the price of volatility risk. A possible choice of λ is given by considering consumption-based models, which generate a risk-premium proportional to y, i.e. $\lambda(t,S,y) = \lambda y$.

The pricing of options within stochastic volatility models for general pay-offs or contracts requires numerical solutions of the generalized Black-Scholes equations (2.4), (2.11). Standard discretizations applied in this context face a) an increase in the spatial dimension $S \to (S,y)$ as compared to standard Black-Scholes setting and b) degenerate coefficients at zero which cause serious ill-conditioning problems.

2.2 Change of variables and unknown

In order to get rid of the degeneracy with respect to S one can switch to logarithmic price $x = \ln(S)$. The transformed PDE problems have coefficients which are constant with respect to the log-price variable x. We mention however that this effect does not arise in case of more sophisticated dynamic models of S_t, as e.g., constant elasticity of variance (CEV) models [7, 8]

$$dS_t = \mu S_t dt + \alpha S_t^\kappa dW_t, \qquad \kappa \in (0,1).$$

Numerical solution of (2.4)–(2.5) and (2.11)–(2.12) requires restriction to a bounded computational domain. Based on the observation due to [1] that the price U remains bounded for all y, we expect $U(t,S,y)e^{-\omega y^2/2}$, $\omega > 0$, to decay exponentially as $|y| \to \infty$. Likewise, the excess to discounted pay-off $U(t,S,y) - e^{-r(T-t)}h(Se^{r(T-t)})$ decays to zero as $S \to \infty$ and $S \to 0$. We therefore derive the equivalent PDEs after these transformations and truncate these to a bounded domain.

2.2.1 OU-model in real-price

We switch to time to maturity $t \to T - t$ and perform the following transformation in (2.4)

$$\bar{U}(t, S, y) := U(T - t, S, y)e^{-\omega y^2/2}, \tag{2.13}$$

with the parameter $\omega > 0$ to be fixed later.
With the notations in (2.13), $\bar{U}$ solves the following parabolic partial differential equation

$$\begin{aligned} \frac{\partial \bar{U}}{\partial t} + \mathcal{L}^{\mathrm{OU}}_{\mathrm{real},\rho}\bar{U} &= 0 \text{ in } (0,T) \times \mathbb{R}_+ \times \mathbb{R} & (2.14)\\ \bar{U}(0,\cdot) &= U_0 \text{ in } \mathbb{R}_+ \times \mathbb{R}, & (2.15) \end{aligned}$$

where the operator $\mathcal{L}^{\mathrm{OU}}_{\mathrm{real},\rho}$ takes the form

$$\begin{aligned} \mathcal{L}^{\mathrm{OU}}_{\mathrm{real},\rho}U &= -\frac{1}{2}S^2 f^2(y)\frac{\partial^2 U}{\partial S^2} - \frac{1}{2}\beta^2\frac{\partial^2 U}{\partial y^2} - \rho\beta S f(y)\frac{\partial^2 U}{\partial S \partial y} - \left[\omega\rho\beta y f(y) + r\right] S\frac{\partial U}{\partial S} \\ &\quad -\left[(\omega\beta^2 - \alpha)y - \beta\rho\frac{\mu - r}{f(y)} + \alpha m\right]\frac{\partial U}{\partial y} \qquad (2.16)\\ &\quad -\left[\left(\frac{1}{2}\omega^2\beta^2 - \alpha\omega\right)y^2 + \left(\alpha\omega m - \omega\beta\rho\frac{\mu - r}{f(y)}\right)y + \frac{1}{2}\omega\beta^2 - r\right]U, \end{aligned}$$

and the initial condition U_0 is given by $U_0 = h(S)e^{-\omega y^2/2}$.
To prepare the truncation in the S coordinate and to justify the artificial essential boundary conditions, we calculate instead the excess to pay-off value
$u := \bar{U} - e^{-rt}h(Se^{rt})e^{-\omega y^2/2}$ as solution of the following parabolic PDE

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}^{\mathrm{OU}}_{\mathrm{real},\rho}u &= F \text{ in } (0,T) \times \mathbb{R}_+ \times \mathbb{R} & (2.17)\\ u(0,\cdot) &= 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}, & (2.18) \end{aligned}$$

where the right hand side F is given by

$$F = e^{-rt}e^{-\omega y^2/2}\left(\frac{1}{2}S^2(f(y))^2\frac{\partial^2}{\partial S^2}h\right)(Se^{rt}).$$

2.2.2 OU–model in log–price

The degeneracy of the coefficients with respect to S can be removed by switching to log-price $x = \ln(S)$. We aim again to transform the resulting PDE (set now on $(\mathbb{R}\times\mathbb{R})\times(0,T]$) into an equivalent one which can be localized to a bounded domain and proceed as in [1].

We change to time to maturity $t \to T-t$ and to log-price $x = \ln(S)$ and obtain from (2.4) the following parabolic PDE for $u(t,x,y) := e^{-\omega y^2/2}[U(T-t,e^x,y) - e^{-r(T-t)}h(e^{x+r(T-t)})]$

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}^{\mathrm{OU}}_{\log,\rho} u &= F \text{ in } (0,T) \times \mathbb{R}^2 & (2.19) \\ u(0,\cdot) &= 0 \text{ in } \mathbb{R}^2, & (2.20) \end{aligned}$$

where $\mathcal{L}^{\mathrm{OU}}_{\log,\rho}$ is given by

$$\begin{aligned} \mathcal{L}^{\mathrm{OU}}_{\log,\rho} u &= -\frac{1}{2}(f(y))^2\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) - \frac{1}{2}\beta^2 \frac{\partial^2 u}{\partial y^2} - \rho\beta f(y)\frac{\partial^2 u}{\partial x \partial y} \\ &\quad -\big[\omega\rho\beta y f(y) + r\big]\frac{\partial u}{\partial x} - \left[(\omega\beta^2 - \alpha)y - \beta\rho\frac{\mu - r}{f(y)} + \alpha m\right]\frac{\partial u}{\partial y} \qquad (2.21) \\ &\quad -\left[\left(\frac{1}{2}\beta^2\omega^2 - \alpha\omega\right)y^2 + \left(\alpha\omega m - \omega\beta\rho\frac{\mu - r}{f(y)}\right)y + \frac{1}{2}\omega\beta^2 - r\right]u. \end{aligned}$$

The right hand side F takes in case of put or call contracts with strike K the form

$$F(t,y) = \frac{1}{2}Ke^{-rt}f(y)^2 e^{-\omega y^2/2}\delta_{\ln(K)-rt}. \qquad (2.22)$$

Remark 2.1 The case $r \neq 0$ can be reduced to the case $r = 0$ via the transformation

$$u(t,x,y) = e^{-rt}v(t, x+rt, y). \qquad (2.23)$$

2.2.3 Heston-model

Performing the same changes of variables as before (i.e. changing to time to maturity $t \to T-t$ and to logarithmic price $x = \ln(S)$) and writing $u(t,x,y) := e^{-\omega y^2/2}[U(T-t,e^x,y) - e^{-r(T-t)}h(e^{x+r(T-t)})]$, $\widetilde{\lambda}(t,x,y) = \lambda(T-t,e^x,y)$ in (2.11)–(2.12) leads to

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}^{\mathrm{H}}_{\log} u &= F \quad \text{in } (0,T) \times \mathbb{R} \times (0,\infty) & (2.24) \\ u(0,x,y) &= 0 \quad \text{in } \mathbb{R} \times (0,\infty), & (2.25) \end{aligned}$$

with the operator $\mathcal{L}^{\mathrm{H}}_{\log}$ defined as follows

$$\begin{aligned} \mathcal{L}^{\mathrm{H}}_{\log} u &= -\frac{1}{2}y\left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right) - \big(\omega\rho\sigma y + r\big)\frac{\partial u}{\partial x} - \frac{1}{2}\sigma^2 y \frac{\partial^2 u}{\partial y^2} \\ &\quad -\rho\sigma y\frac{\partial^2 u}{\partial y \partial x} - \Big[\omega\sigma^2 y^2 + \kappa\big(m - y\big) - \widetilde{\lambda}(t,x,y)\big)\Big]\frac{\partial u}{\partial y} \qquad (2.26) \\ &\quad -\left[\frac{1}{2}\omega\sigma^2 y(\omega y^2 + 1) + \omega y\big(\kappa(m-y) - \widetilde{\lambda}(t,x,y)\big) - r\right]u. \end{aligned}$$

For put and call contracts the right hand side F takes the form

$$F(t,y) = \frac{1}{2}e^{-\omega y^2/2}Ke^{-rt}y\delta_{\ln(K)-rt}. \qquad (2.27)$$

2.3 Compound options

2.3.1 OU-model

Let V^c denote the value of a European derivative (without loss of generality assume it is a call option) providing the right at its maturity T' to acquire, with exercise price K', another European option V on an asset S_t with dynamics described by (2.1), maturity $T > T'$ and exercise price K. With stochastic volatility process of S_t being as in (2.2) we write V^c as a function of (t, S, y). The function $V^c(t, S, y)$ solves the parabolic PDE (2.4) together with the terminal condition $V^c(T', S, y) = U(T', S, y)_+$. The usual transformations yield for $v(t, x, y) = e^{-\omega y^2/2}[V^c(T'-t, e^x, y) - \big(U(T'-t, e^x, y) - K'e^{-r(T'-t)}\big)_+]$ the PDE

$$\begin{aligned} \frac{\partial v}{\partial t} + \mathcal{L}_{\log}^{\mathrm{OU}} v &= F \quad \text{in} \quad (0,T) \times \mathbb{R}^2 & (2.28) \\ v(0,x,y) &= 0 \quad \text{in} \quad \mathbb{R}^2, & (2.29) \end{aligned}$$

where the operator $\mathcal{L}_{\log}^{\mathrm{OU}}$ is as in (2.21) and the right hand side F is

$$\begin{aligned} F = &\frac{1}{2} e^{-\omega y^2/2} f^2(y) \frac{1}{\sqrt{1+(x^{*\prime}(y))^2}} \frac{\partial U}{\partial x}\big(T'-t, x^*, y\big) \delta_{x^*}(\mathrm{d}x) \\ &- \frac{1}{2}\beta^2 e^{-\omega y^2/2} \frac{x^{*\prime}(y)}{\sqrt{1+(x^{*\prime}(y))^2}} \frac{\partial U}{\partial y}\big(T'-t, x^*, y\big)\delta_{x^*} \qquad (2.30) \\ &+ \rho\beta \frac{1}{\sqrt{1+(x^{*\prime}(y))^2}} f(y) \frac{\partial U}{\partial y}\big(T'-t, x^*, y\big)\delta_{x^*} \end{aligned}$$

with $x^*(T'-t, y)$ being such that $U(T'-t, e^{x^*(T'-t,y)}, y) = K'$ and U solves the PDE (2.4)–(2.5).

2.3.2 Heston model

Similar considerations apply also to e.g. the Heston model. We assume again that the option under consideration is a call option. Performing the same transformations as in Section 2.3.1 above, the transformed PDE for $v(t, x, y)$ is

$$\begin{aligned} \frac{\partial v}{\partial t} + \mathcal{L}_{\log}^{\mathrm{H}} v &= F \quad \text{in} \quad (0,T) \times \mathbb{R}^2 & (2.31) \\ v(0,x,y) &= 0 \quad \text{in} \quad \mathbb{R}^2 & (2.32) \end{aligned}$$

where the operator $\mathcal{L}^{\mathrm{H}}_{\log}$ is as in (2.26) and the right hand side is given by

$$\begin{aligned} F = & \frac{1}{2} e^{-\omega y^2/2} y \frac{1}{\sqrt{1+(x^{*\prime}(y))^2}} \frac{\partial U}{\partial x}\big(T'-t, x^*, y\big)\delta_{x^*}(\mathrm{d}x) \\ & - \frac{1}{2}\sigma^2 y e^{-\omega y^2/2} \frac{x^{*\prime}(y)}{\sqrt{1+(x^{*\prime}(y))^2}} \frac{\partial U}{\partial y}\big(T'-t, x^*, y\big)\delta_{x^*} \\ & + \rho\sigma \frac{1}{\sqrt{1+(x^{*\prime}(y))^2}} y \frac{\partial U}{\partial y}\big(T'-t, x^*, y\big)\delta_{x^*} \end{aligned} \tag{2.33}$$

with $x^*(T'-t, y)$ being such that $U(T'-t, e^{x^*(T'-t,y)}, y) = K'$. In (2.33) the function U is the solution of (2.11)–(2.12).

3 Variational Formulation

We recapitulate shortly the variational formulation of abstract parabolic problems and summarize the assumptions under which existence and uniqueness of the (weak) solution are ensured.

3.1 Abstract parabolic setting

Given Hilbert spaces $V \overset{d}{\hookrightarrow} H \cong H' \overset{d}{\hookrightarrow} V^*$ with dense injection and $F \in L^2(0,T;V^*)$, find $u \in L^2(J;V) \cap H^1(J;V^*)$ such that

$$\begin{aligned} \big(\tfrac{\mathrm{d}}{\mathrm{d}t} u(t,\cdot), v\big) + a(u(t,\cdot), v) &= \langle F(t), v\rangle_{V^* \times V} \quad \forall\, v \in V && (3.1) \\ u(0,\cdot) &= 0. && (3.2) \end{aligned}$$

The derivative $\frac{d}{dt}$ in (3.1) is understood in weak sense, and $\langle \cdot,\cdot\rangle_{V^*\times V}$ denotes the usual duality pairing in $V^* \times V$. The following general result for the existence of weak solutions of the parabolic problem (3.1)–(3.2) holds [13].

Proposition 3.1 *Assume that the bilinear form $a(\cdot,\cdot)$ in (3.1)*

(i) is continuous, i.e. there exists a constant $M > 0$ such that

$$|a(u,v)| \le M \|u\|_V \|v\|_V \qquad \forall\, u, v \in V,$$

(ii) satisfies the **Gårding inequality**, *i.e. there exist constants c_1, $c_2 > 0$ such that*

$$a(v,v) \ge c_1 \|v\|_V^2 - c_2 \|v\|_H^2 \quad \forall\, v \in V. \tag{3.3}$$

Then (3.1)–(3.2) admits a unique weak solution $u \in L^2(J;V) \cap H^1(J;V^*)$ and there holds

$$\|u\|_{L^2(J;V)} + \|u\|_{H^1(J;V^*)} \leq C\|F\|_{L^2(J;V^*)}. \tag{3.4}$$

Moreover, (3.4) implies that $u \in C^0([0,T];H)$ and

$$\|u\|_{C^0([0,T];H)} \leq C\left(\|u\|_{L^2(J;V)} + \|u\|_{H^1(J;V^*)}\right).$$

To describe decay of functions at ∞ we use weighted Sobolev spaces. We start by proving existence and uniqueness of solutions of (2.17)–(2.18), (2.19)–(2.20) in such weighted Sobolev spaces.

3.2 Variational formulation in log–price

To fix ideas, let us consider first the mean-reverting OU-model of Stein-Stein. If $\rho \neq 0$, the risk premium becomes singular as $y \to 0$ due to the presence of the excess return to risk ratio $\frac{\mu - r}{f(y)}$. Consequently, the price of the option does not depend on volatility near 0, i.e., $\frac{\partial U}{\partial y} \sim 0$ as $y \to 0$ and one has to solve two independent boundary value problems in $(0,T) \times \mathbb{R} \times \mathbb{R}_-$ and $(0,T) \times \mathbb{R} \times \mathbb{R}_+$ and to impose Dirichlet boundary conditions at $y = 0$.

3.2.1 Uncorrelated case ($\rho = 0$)

In the following, we assume in (2.19) $\rho = 0$ and $f(y) = |y|$. Let $\varphi, \psi \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$, $\varphi, \psi \geq 0$ and denote by

$$L^2_{\varphi,\psi}(\mathbb{R}^2) := \{v \mid v(x,y)\varphi(x)\psi(y) \in L^2(\mathbb{R}^2)\}. \tag{3.5}$$

Furthermore, let $V^{\varphi,\psi}$ be the weighted Sobolev space defined by

$$V^{\varphi,\psi} := \left\{v \mid \left(\sqrt{1+y^2}v, y\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right) \in \left(L^2_{\varphi,\psi}(\mathbb{R}^2)\right)^3\right\}, \tag{3.6}$$

and let $(V^{\varphi,\psi})^*$ denote its dual. The space $V^{\varphi,\psi}$ is equipped with the norm

$$\|u\|_{V^{\varphi,\psi}} := \left(\|u\|^2_{L^2_{\varphi,\psi}} + \|yu\|^2_{L^2_{\varphi,\psi}} + \left\|y\frac{\partial u}{\partial x}\right\|^2_{L^2_{\varphi,\psi}} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}}\right)^{\frac{1}{2}}. \tag{3.7}$$

We need the following result:

Lemma 3.2 *Let $\psi(y) := e^{\overline{\mu}y^2/2}$, $\overline{\mu} > 0$, be a weighting function. Then the semi-norm*

$$|u|_{V^{\varphi,\psi}} := \left(\|u\|^2_{L^2_{\varphi,\psi}} + \left\|y\frac{\partial u}{\partial x}\right\|^2_{L^2_{\varphi,\psi}} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}}\right)^{\frac{1}{2}} \tag{3.8}$$

is a norm on $V^{\varphi,\psi}$.

Proof. We have to estimate the term $\|yu\|_{L^2_{\varphi,\psi}}$ in (3.7). We have $\psi'(y) = \overline{\mu} y \psi(y)$ and thus

$$\begin{aligned}
\|yu\|^2_{L^2_{\varphi,\psi}} &= \int_{\mathbb{R}^2} y^2 u^2 \varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y = \frac{1}{\overline{\mu}} \int_{\mathbb{R}^2} y u^2 \varphi^2(x)\psi(y)\psi'(y)\mathrm{d}x\mathrm{d}y \\
&= -\frac{1}{2\overline{\mu}} \int_{\mathbb{R}^2} \frac{\partial}{\partial y}\big(yu^2\big)\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \\
&= -\frac{1}{2\overline{\mu}} \int_{\mathbb{R}^2} u^2\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y - \frac{1}{\overline{\mu}} \int_{\mathbb{R}^2} yu\frac{\partial u}{\partial y}\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \\
&\leq \frac{1}{\overline{\mu}}\left(\frac{\overline{\mu}}{2}\|yu\|^2_{L^2_{\varphi,\psi}} + \frac{1}{2\overline{\mu}}\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}}\right),
\end{aligned}$$

where in the last inequality we have used the Cauchy-inequality with $\varepsilon = \frac{\overline{\mu}}{2}$. We conclude

$$\|yu\|^2_{L^2_{\varphi,\psi}} \leq \frac{1}{\overline{\mu}^2}\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}}$$

and the assertion follows. □

Consider the operator $\mathcal{L}^{\mathrm{OU}}_{\log,0}$ given in (2.21). We associate to $\mathcal{L}^{\mathrm{OU}}_{\log,0}$ the bilinear form $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$ which is given by

$$a^{\varphi,\psi}_{\log,0}(u,v) := \int_{\mathbb{R}^2} (\mathcal{L}^{\mathrm{OU}}_{\log,0}u)(x,y)v(x,y)\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y, \qquad u,v \in C_0^\infty(\mathbb{R}^2). \tag{3.9}$$

With standard density arguments it can be shown that $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$ can be extended continuously to a bilinear form on $V^{\varphi,\psi} \times V^{\varphi,\psi}$ which we denote again by $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$.

We arrive at the following variational formulation of (2.17)–(2.18) when $\rho = 0$: Given $F \in (V^{\varphi,\psi})^*$, find $u \in L^2\big(0,T;V^{\varphi,\psi}\big) \cap H^1\big(0,T;\big(V^{\varphi,\psi}\big)^*\big)$ such that

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot),v\big)_{L^2_{\varphi,\psi}(\mathbb{R}^2)} + a^{\varphi,\psi}_{\log,0}\big(u(t,\cdot),v\big) = \langle F,v\rangle_{(V^{\varphi,\psi})^*\times V^{\varphi,\psi}}, \qquad \forall v \in V^{\varphi,\psi} \tag{3.10}$$

$$u(0,\cdot) = 0, \tag{3.11}$$

where $\langle F,v\rangle$ stands for

$$\langle F,v\rangle = \tfrac{K}{2}\varphi^2\big(\ln(K)\big)\int_{\mathbb{R}} y^2 v\big(\ln(K),y\big)\psi^2(y)e^{-\frac{1}{2}\omega y^2}\mathrm{d}y, \tag{3.12}$$

and the bilinear form $a^{\varphi,\psi}_{\log,0} : V^{\varphi,\psi} \times V^{\varphi,\psi} \to \mathbb{R}$ is given in Appendix A.1.

Theorem 3.3 *Let $\varphi(x) = e^{\nu|x|}$ and $\psi(y) = e^{\overline{\mu}y^2/2}$ with $\nu,\overline{\mu} > 0$. Assuming $\omega < \frac{2\alpha}{\beta^2}$, then there exist $\nu_0 > 0$ and $\overline{\mu}_0 > 0$ such that for all $\nu \in [0,\nu_0)$ and $\overline{\mu} \in [0,\overline{\mu}_0)$ the bilinear* *form $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$ satisfies a Gårding inequality in $V^{\varphi,\psi}\times V^{\varphi,\psi}$. Precisely, there exist positive constants $C,c>0$ such that for all $v\in V^{\varphi,\psi}$ there holds*

$$a^{\varphi,\psi}_{\log,0}(v,v)\geq C\|v\|^2_{V^{\varphi,\psi}}-c\|v\|^2_{L^2_{\varphi,\psi}(\mathbb{R}^2)}. \tag{3.13}$$

As a direct consequence of (3.13), we obtain the following existence and uniqueness result: there exists a unique weak solution $u\in L^2(0,T;V^{\varphi,\psi})\cap H^1(0,T;\left(V^{\varphi,\psi}\right)^*)$ of (3.10)–(3.12).

Proof. The proof of (3.13) is given in Appendix B. □

3.2.2 Correlated case ($\rho\neq 0$)

In (2.19) assume $f(y)=|y|$ and denote by $Q_\pm$ the half–planes $Q_\pm:=\mathbb{R}\times\mathbb{R}_\pm$. Similarly to the uncorrelated case, we introduce Sobolev spaces of functions with exponential decay at infinity

$$V^{\varphi,\psi}_{0,\pm}:=\overline{C^\infty_0(Q_\pm)}^{\|\cdot\|_{V^{\varphi,\psi}_{0,\pm}}} \tag{3.14}$$

where the norm $\|\cdot\|^2_{V^{\varphi,\psi}_{0,\pm}}$ is given by

$$\|v\|^2_{V^{\varphi,\psi}_{0,\pm}}=\|\sqrt{1+y^2}v\|^2_{L^2_{\varphi,\psi}(Q_\pm)}+\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)}+\left\||y|\frac{\partial v}{\partial x}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)}. \tag{3.15}$$

We denote by $u^\pm$ the solution of the parabolic PDE

$$\frac{\partial u^\pm}{\partial t}+\mathcal{L}^{\mathrm{OU}}_{\log,\rho}u^\pm=F \text{ in } (0,T)\times Q_\pm \tag{3.16}$$

$$u^\pm(0,x,y)=0 \text{ in } Q_\pm \tag{3.17}$$

$$u^\pm(t,x,0)=0 \text{ in } (0,T)\times\mathbb{R}, \tag{3.18}$$

where the operator $\mathcal{L}^{\mathrm{OU}}_{\log,\rho}$ and the right hand side F are given in (2.21) and (2.22), respectively. We associate to the infinitesimal generator $\mathcal{L}^{\mathrm{OU}}_{\log,\rho}$ the weighted bilinear forms $a^{\varphi,\psi}_{\log,\rho,\pm}(\cdot,\cdot)$ which are defined as

$$a^{\varphi,\psi}_{\log,\rho,\pm}(u,v):=\int_{Q_\pm}\left(\mathcal{L}^{\mathrm{OU}}_{\log,\rho}u\right)(x,y)v(x,y)\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\quad\forall\, u,v\in C^\infty_0(Q_\pm). \tag{3.19}$$

An explicit presentation of the bilinear forms $a^{\varphi,\psi}_{\log,\rho,\pm}(\cdot,\cdot)$ is given in Appendix A.1. Here again $a^{\varphi,\psi}_{\log,\rho,\pm}(\cdot,\cdot)$ can be continuously extended to bilinear forms on $V^{\varphi,\psi}_{0,\pm}\times V^{\varphi,\psi}_{0,\pm}$.

Theorem 3.4 *Assume that*

$$\beta>\frac{4|\rho||\mu-r|}{1-\rho^2} \tag{3.20}$$

and that $\omega = 2\eta\alpha/\beta^2$ with $\eta \in (0,1)$. Let $\varphi(x) = e^{\nu|x|}$ and $\psi(y) = e^{\bar{\mu}y^2/2}$. Then there exist $\nu_0 > 0$ and $\bar{\mu}_0 > 0$ sufficiently small such that for all $\nu \in [0,\nu_0)$, $\bar{\mu} \in [0,\bar{\mu}_0)$ the bilinear form $a^{\varphi,\psi}_{\log,\rho,\pm}(\cdot,\cdot)$ satisfies a Gårding inequality in $V^{\varphi,\psi}_{0,\pm} \times V^{\varphi,\psi}_{0,\pm}$. More precisely, for $0 < \eta_1 \le \eta \le \eta_2 < 1$ there exist positive constants $C, c > 0$ such that for all $\nu \in [0,\nu_0)$, $\bar{\mu} \in [0,\bar{\mu}_0)$ there holds

$$a^{\varphi,\psi}_{\log,\rho,\pm}(v,v) \ge C\|v\|^2_{V^{\varphi,\psi}_{0,\pm}} - c\|v\|^2_{L^2_{\varphi,\psi}(Q_\pm)}. \tag{3.21}$$

Consequently, the variational formulation of (3.16)–(3.18):

Given $F \in (V^{\varphi,\psi}_{0,\pm})^*$, find $u^\pm \in L^2\big(0,T;V^{\varphi,\psi}_{0,\pm}\big) \cap H^1\big(0,T;(V^{\varphi,\psi}_{0,\pm})^*\big)$ such that

$$\Big(\tfrac{\mathrm{d}}{\mathrm{d}t}u^\pm(t,\cdot),v\Big)_{L^2_{\varphi,\psi}(Q_\pm)} + a^{\varphi,\psi}_{\log,\rho,\pm}\big(u^\pm(t,\cdot),v\big) = \langle F,v\rangle_{(V^{\varphi,\psi}_{0,\pm})^*\times V^{\varphi,\psi}_{0,\pm}} \quad \forall v \in V^{\varphi,\psi}_{0,\pm} \tag{3.22}$$

$$u^\pm(0,x,y) = 0 \tag{3.23}$$

$$u^\pm(t,x,0) = 0 \tag{3.24}$$

admits a unique weak solution.

The proof of Theorem 3.4 is given in Appendix C.

3.3 Variational formulation in real-price

Due to the presence of the excess return to risk ratio $\frac{\mu-r}{f(y)}$ which becomes singular as $y \to 0$ if $\rho \neq 0$ we distinguish again between $\rho = 0$ and $\rho \neq 0$. The variational analysis is similar to the log-price setting, i.e., based on weighted, degenerate Sobolev spaces. Gårding-inequalities as in Theorems 3.3, 3.4 follow by similar arguments. The difference is that the decay of the solution of (2.17)–(2.18) with respect to S is algebraic, and not exponential as in the log-price case. This circumstance is not surprising since we have shown that the decay in $x = \ln(S)$ is exponential, one expects an algebraical decay in S. In [1] it was proven that the the bilinear form in real price satisfies a Gårding inequality, both for the uncorrelated an correlated case. In the correlated case, the result in [1] holds under restrictive conditions on ω and ρ. We remove the above mentioned restrictions on ω and ρ.

3.3.1 Uncorrelated case ($\rho = 0$)

Let $f(y) = |y|$ and consider the problem (see also (2.17)–(2.18))

$$\frac{\partial u}{\partial t} + \mathcal{L}^{\mathrm{OU}}_{\mathrm{real},0}u = F \text{ in } (0,T)\times\mathbb{R}_+\times\mathbb{R}, \tag{3.25}$$

$$u(0,\cdot) = 0 \text{ in } \mathbb{R}_+\times\mathbb{R}. \tag{3.26}$$

Let $\Omega := \mathbb{R}_+ \times \mathbb{R}$ and denote by $W^{\varphi,\psi}$ the weighted Sobolev space

$$W^{\varphi,\psi} = \left\{ v \mid \left(\sqrt{1+y^2}v, \frac{\partial v}{\partial y}, S|y|\frac{\partial v}{\partial S} \right) \in \left(L^2_{\varphi,\psi}(\Omega) \right)^3 \right\}, \tag{3.27}$$

where $\varphi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}_+)$, $\psi \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R})$ and

$$L^2_{\varphi,\psi}(\Omega) := \{ v \mid v(S,y)\varphi(S)\psi(y) \in L^2(\Omega) \}.$$

We equip $W^{\varphi,\psi}$ with the norm

$$\|v\|^2_{W^{\varphi,\psi}} = \|\sqrt{1+y^2}v\|^2_{L^2_{\varphi,\psi}(\Omega)} + \left\| \frac{\partial v}{\partial y} \right\|^2_{L^2_{\varphi,\psi}(\Omega)} + \left\| S|y|\frac{\partial v}{\partial y} \right\|^2_{L^2_{\varphi,\psi}(\Omega)}. \tag{3.28}$$

We now multiply the PDE (3.25) by the test function $v(S,y)\varphi(S)\psi(y)$, $v(S,y) \in C_0^\infty(\Omega)$ and integrate by parts to get the following variational formulation: Given $F \in (W^{\varphi,\psi})^*$, find $u \in L^2\big(0,T; W^{\varphi,\psi}\big) \cap H^1\big(0,T; (W^{\varphi,\psi})^*\big)$ such that

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot), v\big)_{L^2_{\varphi,\psi}(\Omega)} + a^{\varphi,\psi}_{\mathrm{real},0}\big(u(t,\cdot), v\big) = \langle F, v \rangle_{(W^{\varphi,\psi})^* \times W^{\varphi,\psi}} \qquad \forall\, v \in W^{\varphi,\psi} \tag{3.29}$$

$$u(0,\cdot) = 0. \tag{3.30}$$

The expression of the bilinear form $a^{\varphi,\psi}_{\mathrm{real},0}(\cdot,\cdot)$ is given in Appendix A.2. For weighting exponents ω in (2.13) satisfying $\omega < 2\frac{\alpha}{\beta^2}$ it can be proved that $a^{\varphi,\psi}_{\mathrm{real},0}(\cdot,\cdot)$ satisfies a Gårding inequality in $W^{\varphi,\psi} \times W^{\varphi,\psi}$.

Theorem 3.5 *Let $\varphi(S) = S^\nu$ and $\psi(y) = e^{\overline{\mu}y^2/2}$ with $\nu, \overline{\mu} > 0$. Assume $\omega = 2\eta\frac{\alpha}{\beta^2}$, $\eta \in (0,1)$, then there exist $\nu_0 > 0$ and $\overline{\mu}_0 > 0$ such that $\forall \nu \in [0,\nu_0)$ and $\forall \overline{\mu} \in [0, \overline{\mu}_0)$ the bilinear form $a^{\varphi,\psi}_{\mathrm{real},0}(\cdot,\cdot)$ satisfies a Gårding inequality in $W^{\varphi,\psi} \times W^{\varphi,\psi}$: For $0 < \eta_1 \le \eta \le \eta_2 < 1$ there exist positive constants $C, c > 0$ such that*

$$a^{\varphi,\psi}_{\mathrm{real},0}(v,v) \ge C\|v\|^2_{W^{\varphi,\psi}} - c\|v\|^2_{L^2_{\varphi,\psi}(\Omega)} \quad \forall\, v \in W^{\varphi,\psi}. \tag{3.31}$$

Consequently, there exists a unique weak solution $u \in L^2\big(0,T; W^{\varphi,\psi}\big) \cap H^1\big(0,T; (W^{\varphi,\psi})^*\big)$ of (3.29)–(3.30).

Proof. In the proof, the same techniques as in Theorem 3.3 are applicable, therefore we omit the details. □

3.3.2 Correlated case ($\rho \neq 0$)

Denote by $\Omega_\pm$ the quarter–planes $\Omega_\pm := \mathbb{R}_+ \times \mathbb{R}_\pm$. We consider two independent boundary value problems

$$\frac{\partial u^\pm}{\partial t} + \mathcal{L}^{\mathrm{OU}}_{\mathrm{real},\rho} u^\pm = F \text{ in } (0,T) \times \Omega_\pm, \tag{3.32}$$

$$u^\pm(0,S,y) = 0 \text{ in } \Omega_\pm, \tag{3.33}$$

$$u^\pm(t,S,0) = 0 \text{ in } (0,T) \times \mathbb{R}_+. \tag{3.34}$$

Consider the weighted Sobolev space

$$W_{0,\pm}^{\varphi,\psi} := \overline{C_0^\infty(\Omega_\pm)}^{\|\cdot\|_{W_{0,\pm}^{\varphi,\psi}}} \tag{3.35}$$

where the closure is w.r.t the norm

$$\|v\|^2_{W_{0,\pm}^{\varphi,\psi}} = \|\sqrt{1+y^2}v\|^2_{L^2_{\varphi,\psi}(\Omega_\pm)} + \left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(\Omega_\pm)} + \left\|S|y|\frac{\partial v}{\partial x}\right\|^2_{L^2_{\varphi,\psi}(\Omega_\pm)}. \tag{3.36}$$

The weighted L^2 space is defined as

$$L^2_{\varphi,\psi}(\Omega_\pm) := \{v \mid v(S,y)\varphi(S)\psi^\pm(y) \in L^2(\Omega_\pm)\}$$

with $\varphi \in W^{1,\infty}_{\text{loc}}(\mathbb{R}_+)$, $\psi^\pm \in W^{1,\infty}_{\text{loc}}(\mathbb{R}_\pm)$. To the infinitesimal generator $\mathcal{L}^{\text{OU}}_{\text{real},\rho}$ we associate the weighted bilinear forms $a^{\varphi,\psi}_{\text{real},\rho,\pm}(\cdot,\cdot)$ defined as

$$a^{\varphi,\psi}_{\text{real},\rho,\pm}(u,v) := \int_{\Omega_\pm} (\mathcal{L}^{\text{OU}}_{\text{real},\rho}u)(S,y)v(S,y)(\varphi(S))^2(\psi^\pm(y))^2 \mathrm{d}S\mathrm{d}y \qquad u,v \in C_0^\infty(\Omega_\pm). \tag{3.37}$$

For the explicit representation of the bilinear form $a^{\varphi,\psi}_{\text{real},\rho,\pm}(\cdot,\cdot)$ we refer to Appendix A.2. Analogous to the correlated case in the log–price setting the bilinear forms $a^{\varphi,\psi}_{\text{real},\rho,\pm}(\cdot,\cdot)$ satisfy a Gårding inequality.

Theorem 3.6 *Assume that the parameter β satisfies the condition (3.20) and let $\omega = 2\eta\alpha/\beta^2$, $\eta \in (0,1)$. Let $\varphi(S) = S^\nu$ and $\psi(y) = e^{\overline{\mu}y^2/2}$. Then there exist $\nu_0 > 0$ and $\overline{\mu}_0 > 0$ such that for all $\nu \in [0,\nu_0)$ and all $\overline{\mu} \in [0,\overline{\mu}_0)$ the bilinear form $a^{\varphi,\psi}_{\text{real},\rho,\pm}(\cdot,\cdot)$ satisfies a Gårding inequality in $W_{0,\pm}^{\varphi,\psi} \times W_{0,\pm}^{\varphi,\psi}$: For $0 < \eta_1 \le \eta \le \eta_2 < 1$ there exist positive constants $C, c > 0$ such that*

$$a^{\varphi,\psi}_{\text{real},\rho,\pm}(v,v) \ge C\|v\|^2_{W_{0,\pm}^{\varphi,\psi}} - c\|v\|^2_{L^2_{\varphi,\psi}(\Omega_\pm)} \quad \forall\, v \in W_{0,\pm}^{\varphi,\psi}. \tag{3.38}$$

The variational formulation of (3.32)-(3.34):
Given $F \in (W_{0,\pm}^{\varphi,\psi})^*$, find $u^\pm \in L^2\big(0,T;W_{0,\pm}^{\varphi,\psi}\big) \cap H^1\big(0,T;(W_{0,\pm}^{\varphi,\psi})^*\big)$ such that

$$\begin{aligned}
\left(\tfrac{\mathrm{d}}{\mathrm{d}t}u^\pm(t,\cdot),v\right)_{L^2_{\varphi,\psi}(\Omega_\pm)} + a^{\varphi,\psi}_{\text{real},\rho,\pm}\big(u^\pm(t,\cdot),v\big) &= \langle F,v\rangle_{(W_{0,\pm}^{\varphi,\psi})^*\times W_{0,\pm}^{\varphi,\psi}}\ \forall\, v \in W_{0,\pm}^{\varphi,\psi} & (3.39)\\
u^\pm(0,\cdot) &= 0 & (3.40)\\
u^\pm(t,S,0) &= 0 & (3.41)
\end{aligned}$$

admits then a unique weak solution.

Proof. The proof of the Gårding inequality (3.38) follows as in Theorem 3.4, therefore we omit it here. □

3.4 Variational formulation for the Heston model

We assume without loss of generality in (2.11) that the price of volatility risk $\lambda \equiv 0$. Let $\Omega = \mathbb{R}\times(0,\infty)$, $\varphi, \psi \in W^{1,\infty}_{\text{loc}}(\Omega)$, $\varphi, \psi \geq 0$ and denote by $L^2_{\varphi,\psi}(\Omega) := \{v | v(x,y)\varphi(x)\psi(y) \in L^2(\Omega)\}$. Similarly to (3.6) consider the weighted Sobolev space

$$V^{\varphi,\psi} := \left\{ v \mid \left(v, \sqrt{y}\frac{\partial v}{\partial x}, \sqrt{y}\frac{\partial v}{\partial y} \right) \in \left(L^2_{\varphi,\psi}(\Omega)\right)^3 \right\}$$

and let $(V^{\varphi,\psi})^*$ be the dual of $V^{\varphi,\psi}$.
Multiply (2.24) by $v(x,y)\varphi^2(x)\psi^2(y)$, $v \in C_0^\infty(\Omega)$, and integrate by parts to get the variational formulation:
Given $F \in (V^{\varphi,\psi})^*$, find $u \in L^2\big(0,T;V^{\varphi,\psi}\big) \cap H^1\big(0,T;(V^{\varphi,\psi})^*\big)$ such that

$$\big(\tfrac{\mathrm{d}}{\mathrm{d}t}u(t,\cdot), v\big)_{L^2_{\varphi,\psi}} + a^{\varphi,\psi}_{\text{log,H}}\big(u(t,\cdot),v\big) = \langle F, v\rangle_{(V^{\varphi,\psi})^*\times V^{\varphi,\psi}} \qquad \forall\, v \in V^{\varphi,\psi} \tag{3.42}$$

$$u(0,\cdot) = 0. \tag{3.43}$$

The bilinear form $a^{\varphi,\psi}_{\text{log,H}}(\cdot,\cdot)$ is given by

$$\begin{aligned} a^{\varphi,\psi}_{\text{log,H}}(u,v) \;:=\; & \frac{1}{2}\int_\Omega y\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\varphi^2\psi^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\sigma^2\int_\Omega y\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\varphi^2\psi^2\mathrm{d}x\mathrm{d}y \\ & + \int_\Omega y\frac{\partial u}{\partial x}v\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}x}\psi^2\mathrm{d}x\mathrm{d}y - \int_\Omega\left(-\frac{1}{2}y + \omega\rho\sigma y^2 + r\right)\frac{\partial u}{\partial x}v\varphi^2\psi^2\mathrm{d}x\mathrm{d}y \\ & + \int_\Omega\left(\frac{1}{2}\sigma^2 - \omega\sigma^2y^2 - \kappa(m-y)\right)\frac{\partial u}{\partial y}v\varphi^2\psi^2\mathrm{d}x\mathrm{d}y + \sigma^2\int_\Omega y\frac{\partial u}{\partial y}v\varphi^2\psi\frac{\mathrm{d}\psi}{\mathrm{d}y}\mathrm{d}x\mathrm{d}y \\ & + \rho\sigma\int_\Omega y\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\varphi^2\psi^2\mathrm{d}x\mathrm{d}y + 2\rho\sigma\int_\Omega y\frac{\partial u}{\partial y}v\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}x}\psi^2\mathrm{d}x\mathrm{d}y \\ & - \int_\Omega\left(\frac{1}{2}\omega\sigma^2y(\omega y^2+1) + \omega y\kappa(m-y) - r\right)uv\varphi^2\psi^2\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.44}$$

3.5 Localization

In Theorems 3.3, 3.4, 3.5, 3.6 we have proved existence and uniqueness of solutions in appropriate weighted Sobolev spaces. In particular, such solutions are exponentially decaying at infinity both in x and y, or have algebraic decay when $S \to \infty$. For numerical solution it is essential to localize the corresponding PDEs to bounded domains and to impose artificial boundary conditions. We prove next that the truncation error decays exponential with respect to the size of the domain. The decay of solutions at infinity and the well posedness in weighted spaces are crucial in the truncation error estimates below. Exemplarily, we derive these estimates for the model problem (3.10)–(3.12), but the same techniques can be applied to all other cases as well.

For numerical solution we truncate the parabolic PDE (2.19) to a bounded computational domain $\Omega_R := \Omega_1 \times \Omega_2 = (-R_1, R_1) \times (-R_2, R_2)$ with $R_1, R_2 > 0$ sufficiently large and solve (2.19) in $J \times \Omega_R$ (where $J = (0,T)$) with homogenous Dirichlet boundary conditions on $\partial\Omega_R$

$$
\begin{aligned}
\frac{\partial u_R}{\partial t} + \mathcal{L}_{\log,R} u_R &= F|_{\Omega_R} \text{ in } J \times \Omega_R && (3.45)\\
u_R(t,\cdot)|_{\partial\Omega_R} &= 0 \text{ on } \partial\Omega_R && (3.46)\\
u_R(0,\cdot) &= 0 \text{ in } \Omega_R, && (3.47)
\end{aligned}
$$

where $\mathcal{L}_{\log,R}$ denotes the restriction of $\mathcal{L}^{\mathrm{OU}}_{\log,0}$ to Ω_R and F is as in (2.22). Let

$$
V := \left\{ v \mid \left(\sqrt{1+y^2}v, y\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \in \left(L^2(\Omega_R)\right)^3, \; v|_{\partial\Omega_R} = 0 \right\} \tag{3.48}
$$

and denote by V^* the dual of V. For a variational formulation of the truncated PDE (3.45)–(3.47) we denote by $a_{\log,R}(\cdot,\cdot) : V \times V \to \mathbb{R}$ the bilinear form induced by $\mathcal{L}_{\log,R}$

$$
a_{\log,R}(u,v) := \langle \mathcal{L}_{\log,R} u, v \rangle_{V^* \times V} \qquad \forall u, v \in V. \tag{3.49}
$$

The variational formulation of (3.45)–(3.47) reads: Given $F \in V^*$, find $u_R \in L^2\big(J; V\big) \cap H^1(J; V^*)$ such that

$$
\begin{aligned}
(\tfrac{\mathrm{d}}{\mathrm{d}t} u_R(t,\cdot), v)_{L^2(\Omega_R)} + a_{\log,R}(u_R(t,\cdot), v) &= \langle F, v \rangle_{V^* \times V} \quad \forall\, v \in V && (3.50)\\
u_R(0,\cdot) &= 0.
\end{aligned}
$$

The bilinear form $a_{\log,R}(\cdot,\cdot)$ can be alternatively interpreted as the restriction of $a^{1,1}_{\log,0}(\cdot,\cdot)$ to $V \times V$ and for convenience we give its expression below

$$
\begin{aligned}
a_{\log,R}(u,v) = {} & \frac{1}{2}\int_{\Omega_R} y^2 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\Omega_R} y^2 \frac{\partial u}{\partial x} v \mathrm{d}x\mathrm{d}y \\
& + \frac{1}{2}\beta^2 \int_{\Omega_R} \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \mathrm{d}x\mathrm{d}y - (\omega\beta^2 - \alpha)\int_{\Omega_R} y \frac{\partial u}{\partial y} v \mathrm{d}x\mathrm{d}y \\
& - \alpha m \int_{\Omega_R} \frac{\partial u}{\partial y} v \mathrm{d}x\mathrm{d}y - \left(\frac{1}{2}\beta^2\omega^2 - \alpha\omega\right)\int_{\Omega_R} y^2 u v \mathrm{d}x\mathrm{d}y \qquad (3.51)\\
& - \alpha\omega m \int_{\Omega_R} y u v \mathrm{d}x\mathrm{d}y - \frac{1}{2}\omega\beta^2 \int_{\Omega_R} u v \mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

By Theorem 3.3, the bilinear form $a_{\log,R}$ obtained as restriction of the bilinear form $a^{1,1}_{\log,0}(\cdot,\cdot)$ to $V \times V$ is continuous and satisfies a Gårding inequality. Note that the transformation $v_R = e^{-\lambda t} u_R$ leads to the following parabolic PDE for v_R

$$
\frac{\partial v_R}{\partial t} + (\mathcal{L}_{\log,R} + \lambda \cdot \mathrm{id}) v_R = e^{-\lambda t} F|_{\Omega_R} \text{ in } J \times \Omega_R,
$$

where $\mathcal{L}_{\log,R} + \lambda \cdot \mathrm{id}$ is, for all $\lambda > c$ as in (3.13), coercive.

We can prove now that the localization error can be controlled by an exponentially decaying function. Precisely, let u denote the solution of (3.10)–(3.11) and let u_R be the solution of (3.50). Denote by $\widetilde{u}_R$ the zero extension of u_R to $\mathbb{R}^2$ and let $e_R := \widetilde{u}_R - u$ be the localization error. Our energy error estimates below are given in the domain $\Omega_{R/2} := (-R_1/2, R_1/2) \times (-R_2/2, R_2/2)$.

Theorem 3.7 *Let $\phi = \phi(x,y) \in C_0^\infty(\Omega_R)$ be a cut-off function with the following properties*

$$\phi \geq 0, \quad \phi \equiv 1 \text{ on } \Omega_{R/2} \text{ and } \|\nabla \phi\|_{L^\infty(\Omega_R)} \leq C$$

for some constant $C > 0$ independent of $R_1, R_2 \geq 1$. Then, there exist constants $c = c(T)$, $\alpha > 0$ independent of R_1, R_2 such that the following error estimate holds

$$\|\phi e_R(t,\cdot)\|^2_{L^2(\Omega_R)} + \int_0^t \|\phi e_R(s,\cdot)\|_V^2 \mathrm{d}s \leq c e^{-\alpha(R_1+R_2)}. \tag{3.52}$$

Proof. Inserting $v = u(t,\cdot)$ in (3.10)–(3.11) and integrating from 0 to t yields the a-priori estimate

$$\|u(t)\|^2_{L^2_{\varphi,\psi}} + \int_0^t \|u(s)\|^2_{V^{\varphi,\psi}} \mathrm{d}s \leq c\|F\|^2_{(V^{\varphi,\psi})^*} \qquad \forall t \in J \tag{3.53}$$

for some constant $c = c(T) > 0$ independent of R_1, R_2. Here we exploited the Gårding inequality (3.13) satisfied by $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$. Likewise it holds

$$\|\widetilde{u}_R(t)\|^2_{L^2_{\varphi,\psi}} + \int_0^t \|\widetilde{u}_R(s)\|^2_{V^{\varphi,\psi}} \mathrm{d}s \leq c\|F\|^2_{(V^{\varphi,\psi})^*} \qquad \forall t \in J \tag{3.54}$$

with the same constant $c = c(T)$ as in (3.53). In particular, c is independent of R_1, R_2. On the other hand, e_R satisfies the Galerkin orthogonality

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} e_R(t), v\right)_{L^2(\mathbb{R}^2)} + a^{1,1}_{\log,0}(e_R(t), v) = 0 \qquad \forall v \in V. \tag{3.55}$$

Inserting $v = \phi^2(x,y) e_R(t,x,y)$ in (3.55) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi e_R(t)\|^2_{L^2(\Omega_R)} + a_{\log,R}(\phi e_R(t), \phi e_R(t)) = \rho_R(t), \tag{3.56}$$

where the residual $\rho_R(t)$ is given by $\rho_R(t) := a_{\log,R}(\phi e_R(t), \phi e_R(t)) - a^{1,1}_{\log,0}(e_R(t), \phi^2 e_R(t))$, i.e.,

$$\begin{aligned} \rho_R(t) \quad = \quad & \frac{1}{2}\int_{\mathbb{R}^2} y^2 \left(\frac{\partial \phi}{\partial x} e_R(t)\right)^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^2} y^2 \phi \frac{\partial \phi}{\partial x} e_R^2(t) \mathrm{d}x\mathrm{d}y \\ & + \frac{1}{2}\beta^2 \int_{\mathbb{R}^2} y^2 \left(\frac{\partial \phi}{\partial y} e_R(t)\right)^2 \mathrm{d}x\mathrm{d}y - (\alpha\beta^2 - \alpha)\int_{\mathbb{R}^2} y\phi \frac{\partial \phi}{\partial y} e_R^2(t)\mathrm{d}x\mathrm{d}y \\ & - \alpha m \int_{\mathbb{R}^2} \phi \frac{\partial \phi}{\partial y} e_R^2(t) \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.57}$$

The integral terms in the expression of the residual $\rho_R(t)$ are supported by $\Omega_R \setminus \Omega_{R/2}$ and can estimated as follows

$$\begin{aligned}
|\rho_R(t)| \quad &\leq \quad \frac{1}{2}\int_{\mathbb{R}^2} y^2|\nabla\phi|^2|e_R(t)|^2 \mathrm{d}x\mathrm{d}y + \frac{1}{2}\int_{\mathbb{R}^2} y^2\phi|\nabla\phi||e_R(t)|^2 \mathrm{d}x\mathrm{d}y \\
&\quad + \frac{1}{2}\beta^2 \int_{\mathbb{R}^2} y^2|\nabla\phi|^2|e_R(t)|^2 \mathrm{d}x\mathrm{d}y + |\alpha\beta^2 - \alpha| \int_{\mathbb{R}^2} y\phi|\nabla\phi||e_R(t)|^2 \mathrm{d}x\mathrm{d}y \\
&\quad + |\alpha m| \int_{\mathbb{R}^2} \phi|\nabla\phi||e_R(t)|^2 \mathrm{d}x\mathrm{d}y \\
&\leq \quad C\int_{\Omega_R\setminus\Omega_{R/2}} |e_R(t,x,y)|^2 e^{2\nu|x|}e^{\overline{\mu}y^2}e^{-2\nu|x|}e^{-\overline{\mu}y^2}\mathrm{d}x\mathrm{d}y \\
&\leq \quad C\|e_R(t)\|^2_{V^{\varphi,\psi}} \int_{\Omega_R\setminus\Omega_{R/2}} e^{-2\nu|x|}e^{\overline{\mu}y^2}\mathrm{d}x\mathrm{d}y \\
&\leq \quad Ce^{-\alpha(R_1+R_2)}\|e_R(t)\|^2_{V^{\varphi,\psi}}.
\end{aligned}$$

Integrating from 0 to t in (3.56) and using the previous estimates on ρ_R we obtain

$$\frac{1}{2}\|\phi e_R(t)\|^2_{L^2(\Omega_R)} + \int_0^t a_{\log,R}(\phi e_R(s), \phi e_R(s))\mathrm{d}s \leq Ce^{-\alpha(R_1+R_2)}\int_0^t \|e_R(s)\|^2_{V^{\varphi,\psi}}\mathrm{d}s.$$

Using the a-priori estimates (3.53)–(3.54) and the Gårding inequality for the bilinear form $a_{\log,R}$ completes the proof. □

4 Discretization

4.1 Space discretization

In Section 3 we introduced weighted Sobolev spaces and supplied a variational analysis for the pricing problems (2.17)–(2.18),(2.19)–(2.20). The Sobolev spaces involved in our analysis account in particular for solution behaviour at infinity and are a qualitative tool for the justification of the truncation of the pricing problem to a finite domain and in the derivation of guaranteed error bounds for the truncation error. figure[ht] The variational formulations (3.10)–(3.11), (3.22)–(3.24), (3.29)–(3.30) and (3.39)–(3.41) do all fit into the abstract setting (3.1)–(3.2). In particular, the same holds for their localized variants. We aim now to solve numerically the truncated PDEs. To this end, we discretize them by a Finite Element Method (FEM) in space.
The abstract FE semi-discretization of (3.1)–(3.2) reads: for a given finite dimensional subspace $V_h \subset V$ of $\dim V_h = N < \infty$, find $u_h \in L^2(J; V_h) \cap H^1(J; (V_h)^*)$ such that

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}u_h(t,\cdot), v_h\right) + a(u_h(t,\cdot), v_h) = \langle F(t), v_h\rangle_{V^*\times V} \quad \forall\, v_h \in V_h \tag{4.1}$$

$$u_h(0,\cdot) = 0. \tag{4.2}$$

The FE formulation (4.1)–(4.2) is equivalent to a finite system of ODEs. Indeed, by fixing a basis $\mathcal{B} := \{\Phi_j\}_{j=1}^N$ of V_h, (4.1)–(4.2) is equivalent to a system of ODEs for the coefficient vector $\underline{U_h}$ of u_h with respect to the basis $\mathcal{B}$: find $\underline{U_h} \in \mathbb{R}^N$ such that

$$\mathbf{M}\dot{\underline{U}}_h + \mathbf{A}\underline{U_h} = \underline{F},$$

where $\mathbf{M}$ is the mass matrix and $\mathbf{A}$ is the moment or stiffness matrix

$$\mathbf{M} = \Big((\Phi_i, \Phi_j)_{L^2} \Big)_{1 \le i,j \le N}, \qquad \mathbf{A} = \Big(a(\Phi_j, \Phi_i) \Big)_{1 \le i,j \le N}. \tag{4.3}$$

4.2 Sparse Tensor Product Spaces

4.2.1 Wavelet Finite Elements in $\mathbb{R}$

We describe wavelet Finite Elements in the interval $I = (0,1)$. Define the mesh $\mathcal{T}^\ell$ given by the nodes $k2^{-\ell}$, $k = 0, \ldots, 2^\ell$, with mesh-width h_ℓ. Let V_ℓ be the space of piecewise linears on the mesh $\mathcal{T}^\ell$ (higher order polynomials of degree $p > 1$ are also possible, see [6]) which vanish at endpoints 0,1. We write $N^\ell := \dim V_\ell$, $N^{-1} := 0$ and $M^\ell := N^\ell - N^{\ell-1}$. We employ a wavelet basis ψ_k^ℓ, $k = 1, \ldots, M^\ell$, $\ell = 0, 1, 2, \ldots$ of V_ℓ with the properties:

$$V_L = \operatorname{span}\{\psi_k^\ell \mid 0 \le \ell \le L;\ 1 \le k \le M^\ell\}, \qquad \operatorname{diam}\,(\operatorname{supp}\psi_k^\ell) \le C 2^{-\ell}. \tag{4.4}$$

Any function $v \in V_L$ has the representation

$$v = \sum_{\ell=0}^{L} \sum_{k=0}^{M^\ell} v_{k,\ell} \psi_k^\ell \tag{4.5}$$

with the coefficients $v_{k,\ell} = (v, \tilde{\psi}_k^\ell)$, where the $\tilde{\psi}_k^\ell$ are the so-called dual wavelets. For $v \in L^2(I)$ one obtains the series

$$v = \sum_{\ell=0}^{\infty} \sum_{k=0}^{M^\ell} v_{k,\ell} \psi_k^\ell \tag{4.6}$$

which converges in $L^2(I)$ and in $H_0^1(I)$. Moreover there holds the *norm equivalence*

$$c_1 \|v\|_{H^s}^2 \le \sum_{\ell=0}^{\infty} \sum_{k=1}^{M^\ell} 2^{2\ell s} |v_{k,\ell}|^2 \le c_2 \|v\|_{H^s}^2, \quad 0 \le s \le 1. \tag{4.7}$$

For $v \in L^2(I)$ we define a projection $P_L : L^2(I) \to V_L$ by truncating (4.6):

$$P_L v := \sum_{\ell=0}^{L} \sum_{k=0}^{M^\ell} v_{k,\ell} \psi_k^\ell, \qquad P_{-1} := 0. \tag{4.8}$$

This projection satisfies the *approximation property* (Jackson-type estimate)

$$\|u - P_L u\|_{H^s(I)} \leq c 2^{-(t-s)L} \|u\|_{H^t(I)}, \quad 0 \leq s \leq 1,\ s \leq t \leq p+1. \tag{4.9}$$

The increment or detail spaces W_ℓ are defined by

$$W_\ell := \operatorname{span}\{\psi_k^\ell \mid 1 \leq k \leq M^\ell\},\ \ell = 1,2,3,\dots, \quad W_0 := V_0. \tag{4.10}$$

Then

$$V_\ell = V_{\ell-1} \oplus W_\ell \text{ for } \ell \geq 1, \quad \text{and} \quad V_\ell = \bigoplus_{j=0}^{\ell} W_j,\ \ell \geq 0, \tag{4.11}$$

and $Q_\ell := P_\ell - P_{\ell-1}$ is a projection from $L^2(I)$ to W_ℓ.
We give an example of wavelets vanishing at the endpoints 0,1 which we will use in Section 5. For $\ell \geq 0$ let $\mathcal{T}^\ell$ be the mesh given by the nodes $x_k^\ell := k2^{-\ell-1}$, $k = 0,\dots,2^{\ell+1}$. Then we have $N^\ell = 2^{\ell+1} - 1$ and $M^\ell = 2^\ell$.
We define the wavelets ψ_k^ℓ for level $\ell = 0,\dots$, $k = 1,\dots,M^\ell$: for $\ell = 0$ the function ψ_1^0 takes the value 1 at the node $x_1^0 = \frac{1}{2}$. For $\ell \geq 1$ we let $c_\ell := 2^{\ell/2}$. Then the left boundary wavelet ψ_1^ℓ has values $\psi_1^\ell(x_1^\ell) = 2c_\ell$, $\psi_1^\ell(x_2^\ell) = -c_\ell$ and zero at all other nodes. The right boundary wavelet $\psi_{M^\ell}^\ell$ has values $\psi_{M^\ell}^\ell(x_{N^\ell}^\ell) = 2c_\ell$, $\psi_{M^\ell}^\ell(x_{N^\ell-1}^\ell) = -c_\ell$ and zero at all other nodes. The interior wavelets ψ_k^ℓ with $1 < k < M^\ell$ have values $\psi_k^\ell(x_{2k-2}^\ell) = -c_\ell$, $\psi_k^\ell(x_{2k-1}^\ell) = 2c_\ell$, $\psi_k^\ell(x_{2k}^\ell) = -c_\ell$ and zero at all other nodes.

4.2.2 Sparse Tensor Products

In this section we introduce tensor product spaces and tensor product matrices in an abstract setting, see also [4, 16]. To this end, assume that $\{V_L^\kappa\}_{L=0}^\infty$, $\kappa = 1,2$ are two dense hierarchic sequences of finite dimensional subspaces of the Hilbert spaces V^κ

$$V_0^\kappa \subset V_1^\kappa \subset \cdots \subset V_L^\kappa \subset \cdots \subset V^\kappa.$$

To employ this hierarchy in the context of wavelet Galerkin Finite Element Methods we also assume that V_L^κ admits a wavelet basis as in (4.4). The full tensor product space $V_{\mathbf{L}} := V_L^1 \otimes V_L^2$ is defined by

$$V_{\mathbf{L}} = \bigoplus_{0 \leq \ell, \ell' \leq L} W_\ell^1 \otimes W_{\ell'}^2.$$

and we define the *sparse tensor product* space $\hat{V}_{\mathbf{L}}$ at level L as being given by

$$\hat{V}_{\mathbf{L}} = \bigoplus_{0 \leq \ell + \ell' \leq L} W_\ell^1 \otimes W_{\ell'}^2, \tag{4.12}$$

where the increment spaces W_ℓ^κ, $\kappa = 1,2$, are as in (4.10), see also Figure 1 below.

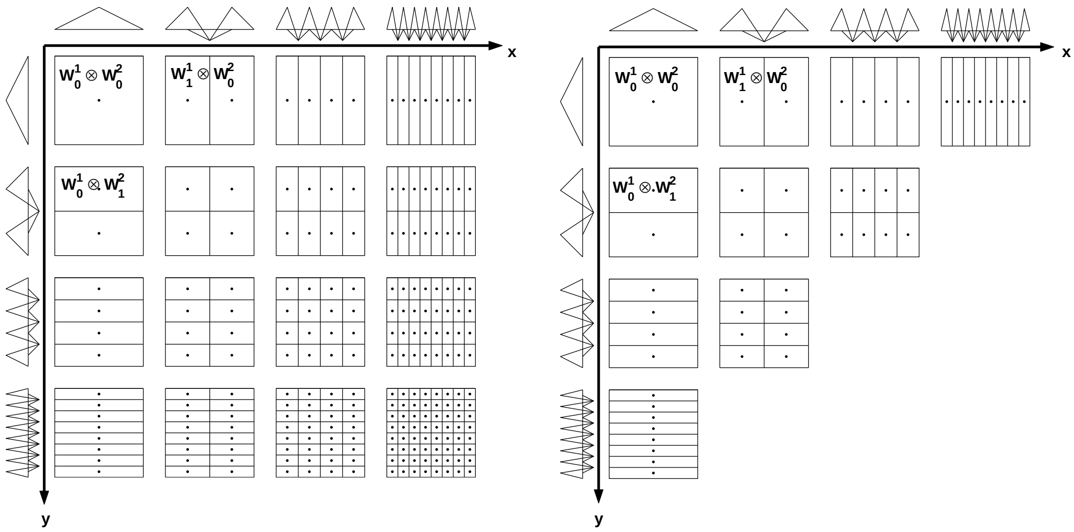

Figure 1: Schematic of full tensor product space $V_{\mathbf{L}}$ (left) and tensor products $W^1_\ell \otimes W^2_{\ell'}$ of the increments, right the sparse tensor product space $\hat{V}_{\mathbf{L}}$. For both spaces the bi-orthogonal spline wavelets are displayed.

Assume given continuous bilinear forms $a^\kappa(\cdot,\cdot) : V^\kappa \times V^\kappa \to \mathbb{R}$ and denote by $\mathbf{A}^\kappa$ the stiffness matrices defined as

$$A^\kappa_{(k,\ell),(k',\ell')} := a^\kappa(\psi^\kappa_{k',\ell'}, \psi^\kappa_{k,\ell}). \tag{4.13}$$

Let $\mathbf{A}^\kappa_{\ell,\ell'}$ denote the block matrix with entries $\{A^\kappa_{(k,\ell),(k',\ell')}\}_{k=1\ldots,M^\ell,k'=1,\ldots,M^{\ell'}}$. We use the same notation when we refer to the matrix of same size as $\mathbf{A}^\kappa$ but with zero entries except the block matrix $\mathbf{A}^\kappa_{\ell,\ell'}$. With this convention, $\mathbf{A}^\kappa$ can be written as

$$\mathbf{A}^\kappa = \sum_{0\le\ell,\ell'\le L} \mathbf{A}^\kappa_{\ell,\ell'}.$$

The full tensor product matrix is then defined as

$$\mathbf{A}^1 \otimes \mathbf{A}^2 = \sum_{\substack{0\le\ell_1,\ell'_1\le L\\ 0\le\ell_2,\ell'_2\le L}} \mathbf{A}^1_{\ell_1,\ell'_1} \otimes \mathbf{A}^2_{\ell_2,\ell'_2} \tag{4.14}$$

and the *sparse tensor product* matrix takes the form

$$\mathbf{A}^1 \hat{\otimes} \mathbf{A}^2 = \sum_{\substack{0\le\ell_1+\ell_2\le L\\ 0\le\ell'_1+\ell'_2\le L}} \mathbf{A}^1_{\ell_1,\ell'_1} \otimes \mathbf{A}^2_{\ell_2,\ell'_2}. \tag{4.15}$$

Let us give an example for a sparse tensor product matrix. We consider the bilinear forms $a^{1,1}_{\mathrm{real},\rho}(\cdot,\cdot)$ and $a^{1,1}_{\log,\rho}(\cdot,\cdot)$ defined in (3.37) and (3.19), respectively. The stiffness matrices $\mathbf{A}_{\mathrm{real}}$ of $a^{1,1}_{\mathrm{real},\rho}(\cdot,\cdot)$, $\mathbf{A}_{\log}$ of $a^{1,1}_{\log,\rho}(\cdot,\cdot)$ in the sparse tensor product space $\hat{V}_{\mathbf{L}}$ are given by

$$\left.\begin{array}{l}\mathbf{A}_{\mathrm{real}}\\ \\ \mathbf{A}_{\log}\end{array}\right\} = \mathbf{A} + \left\{\begin{array}{l} \frac{1}{2}\mathbf{S}^{(x)}_{x^2}\hat{\otimes}\mathbf{M}^{(y)}_{|y|^2} + \mathbf{B}^{(x)}_{x}\hat{\otimes}\big[\mathbf{M}^{(y)}_{|y|^2} - \omega\beta\rho\mathbf{M}^{(y)}_{y|y|} - \beta\rho\mathbf{B}^{(y)}_{|y|} - r\mathbf{M}^{(y)}_{1}\big] \\ \\ \frac{1}{2}\mathbf{S}^{(x)}_{1}\hat{\otimes}\mathbf{M}^{(y)}_{|y|^2} + \mathbf{B}^{(x)}_{1}\hat{\otimes}\big[\frac{1}{2}\mathbf{M}^{(y)}_{|y|^2} - \omega\beta\rho\mathbf{M}^{(y)}_{y|y|} - \beta\rho\mathbf{B}^{(y)}_{|y|} - r\mathbf{M}^{(y)}_{1}\big]\end{array}\right. \tag{4.16}$$

where the matrix $\mathbf{A}$ reads as

$$\begin{aligned}\mathbf{A} \;=\; & \mathbf{M}^{(x)}_{1}\hat{\otimes}\big[\tfrac{\beta^2}{2}\mathbf{S}^{(y)}_{1} - (\tfrac{\omega^2\beta^2}{2} - \alpha\omega)\mathbf{M}^{(y)}_{y^2} - \alpha\omega m\mathbf{M}^{(y)}_{y} + \omega\beta\rho(\mu - r)\mathbf{M}^{(y)}_{\frac{y}{|y|}} \\ & -(\tfrac{\omega\beta^2}{2} - r)\mathbf{M}^{(y)}_{1} - (\omega\beta^2 - \alpha)\mathbf{B}^{(y)}_{y} + \beta\rho(\mu - r)\mathbf{B}^{(y)}_{\frac{1}{|y|}} - \alpha m\mathbf{B}^{(y)}_{1}\big].\end{aligned}$$

In (4.16) we use the notation

$$\mathbf{S}_w := \big(\langle(\psi^{\ell}_{k})', (\psi^{\ell'}_{k'})'\rangle_w\big)_{(k,\ell),(k',\ell')}, \quad \mathbf{M}_w := \big(\langle\psi^{\ell}_{k}, \psi^{\ell'}_{k'}\rangle_w\big)_{(k,\ell),(k',\ell')},$$
$$\mathbf{B}_w := \big(\langle(\psi^{\ell'}_{k'})', \psi^{\ell}_{k}\rangle_w\big)_{(k,\ell),(k',\ell')},$$

with $\langle u,v\rangle_w := \int w(x)u(x)v(x)dx$ being the (1-dimensional) weighted L^2 scalar product.

The full tensor product space $V_{\mathbf{L}}$ has $O(2^{2L}) = O(h^{-2})$ degrees of freedom, whereas the sparse tensor product space $\hat{V}_{\mathbf{L}}$ has considerably smaller dimension $\hat{N}_L = \dim(\hat{V}_{\mathbf{L}}) = O(L2^L) = O(h^{-1}|\log h|)$. Thus, wavelet sparse tensor product discretization applied to univariate stochastic volatility models yields degrees of freedom of size $O(h^{-1}|\log h|)$. This should be compared to standard Finite Element discretizations applied to deterministic volatility models with $O(h^{-1})$ degrees of freedom. We emphasize that wavelet sparse tensor product discretizations can be easily extended to more than just $d = 2$ space dimensions. We mention the pricing of contracts on baskets of d assets and multiscale stochastic volatility models, see [11], [18] for details.

4.3 Time discretization

We exploit the time-analyticity of the semigroups generated by the diffusion process (S_t, Y_t) in (2.1),(2.2) by an exponentially convergent *hp* discontinuous Galerkin (dG) time discretization [15, 16, 18]. For a description of the well known low order θ-scheme and related convergence results see (von Petersdorff, Schwab 2003) [17].

4.3.1 *hp*-dG time stepping

We start from the space semi-discrete problem (4.1)-(4.2).

For $0 < T < \infty$ and $M \in \mathbb{N}$, let $\mathcal{M} = \{I_m\}_{m=1}^M$ be a partition of $J = (0,T)$ into M subintervals $I_m = (t_{m-1}, t_m)$, $1 \leq m \leq M$ with $0 = t_0 < t_1 < t_2 < \ldots < t_M = T$. Moreover, we define the time-partition $\mathcal{M}_{M,\sigma}$ which is geometrically refined towards maturity with M steps and grading factor $\sigma \in (0,1)$ by $t_m = T\sigma^{M-m}$.
For $u \in H^1(\mathcal{M}, V_h) := \{v \in L^2(J, V_h) : v|_{I_m} \in H^1(I_m, V_h), m = 1, 2, \ldots, M\}$, define the one-sided limits

$$u_m^+ := \lim_{s\to 0^+} u(t_m + s),\ 0 \leq m \leq M-1, \qquad u_m^- := \lim_{s\to 0^-} u(t_m - s),\ 1 \leq m \leq M,$$

and the jumps $[u]_m := u_m^+ - u_m^-$, $1 \leq m \leq M-1$. In addition, to each time interval I_m, a polynomial degree (approximation order) $r_m \geq 0$ with linear growth (slope μ) in the sense that $r_m = \lfloor \mu m \rfloor$ is associated. These numbers are stored in the degree vector $\underline{r} = \{r_m\}_{m=1}^M$. Then the following space being used for the discontinuous Galerkin method is introduced:

$$\mathcal{S}^{\underline{r}}(\mathcal{M}, V_h) := \{u \in L^2(J, V_h) : u|_{I_m} \in \mathcal{P}_{r_m}(I_m, V_h), m = 1, 2, \ldots, M\},$$

where $\mathcal{P}_{r_m}(I_m, V_h)$ denotes the space of polynomials of degree at most r_m on I_m taking values in V_h.
With these definitions, the fully discrete dG scheme reads as follows: Find $U_h^{dG} \in \mathcal{S}^{\underline{r}}(\mathcal{M}, V_h)$ such that

$$B_{dG}(U_h^{dG}, W) = (u_0, W_0^+) + \sum_{m=1}^M \int_{I_m} \langle F, W\rangle_{V^*\times V} \mathrm{d}t, \qquad \forall W \in \mathcal{S}^{\underline{r}}(\mathcal{M}, V_h), \tag{4.17}$$

where

$$B_{dG}(u,v) := \sum_{m=1}^M \int_{I_m} \big((u', v) + a(u,v)\big)\mathrm{d}t + \sum_{m=1}^{M-1} ([u]_m, v_m^+) + (u_0^+, v_0^+). \tag{4.18}$$

Now let $V_h = \hat{V}_{\mathbf{L}}$ with mesh-width $h_L = 2^{-L}$, $L > 0$, and choose the dG time stepping with uniform degree vector r. We further choose $\mu = 1$ and a geometric time-step sequence $\mathcal{M}_{M,\sigma}$ in $(0,T)$ with grading factor $\sigma \in (0,1)$ and $M = r$ time steps. Proceeding as in [16, Thm. 5.1], there exist constants C, $b > 0$ independent of M, L such that

$$\|u(T) - U_h^{dG}(T)\|_{L^2} \leq C\big(h_L^{4/3} + e^{-br}\big). \tag{4.19}$$

Essential ingredients to derive (4.19) are the regularity assumption

$$u(t,\cdot) \in \mathcal{H}^2 := \{v \in H_0^1(\Omega_R) \mid \mathcal{D}^\alpha v \in L^2(\Omega_R),\ 0 \leq \alpha_i \leq 2,\ i = 1,2\}, \quad t \in (0,T),$$

and the approximation property (4.9). If we choose $r = O(|\log h_L|)$ it follows that $\|u(T) - U_h^{dG}(T)\|_{L^2} \leq Ch_L^{4/3}$.

4.3.2 Approximate Solution of Linear Equations

If $r_m = r = O(|\log h_L|)$, the hp-dG time stepping (4.17) requires after decoupling in each of the r time steps the solution of $r+1$ linear systems of size $\hat{N}_L$. These systems are of type

$$\underbrace{(\lambda_{j+1}\mathbf{M} + \tfrac{k}{2}\mathbf{A})}_{=:\mathbf{B}}\mathbf{w}_j = \mathbf{f}_j, \qquad j = r, r-1, \ldots, 0 \tag{4.20}$$

with $\lambda_j \in \mathbb{C}$, see e.g. [16]. For a standard finite element basis, the matrix $\mathbf{B}$ is ill conditioned and efficient preconditioning is necessary. The wavelet basis which we proposed so far is, due to multilevel norm equivalences in standard **and** weighted Sobolev spaces, ideal to build optimal preconditioners.

Consider the weighted mass matrix

$$\mathbf{M}_w = \left(\frac{\int_0^1 w^2(x)\psi_k^\ell(x)\psi_{k'}^{\ell'}(x)\mathrm{d}x}{w(2^{-\ell}k)w(2^{-\ell'}k')} \right)_{(k,\ell),(k',\ell')}.$$

The wavelets ψ_k^ℓ and the weighting function w are assumed to satisfy the following assumptions

(A1) The wavelets have one vanishing moment: $\int_0^1 \psi_k^\ell(x)dx = 0$.

(A2) The wavelets belong to $W^{1,\infty}(0,1)$. Furthermore the wavelets with $0 \in \mathrm{supp}(\psi_k^\ell)$ have the following decay condition at $x = 0$:

$$|\psi_k^\ell(x)| \le C2^{\ell/2}(2^\ell x)^\beta, \quad |(\psi_k^\ell)'(x)| \le C2^{3\ell/2}(2^\ell x)^{\beta-1}, \quad x \in [0, 2^{-\ell}], \beta \in \mathbb{N}_0$$

for $\beta > -\frac{1}{2} - \alpha$.

(A3) The nonnegative weighting function $w(\cdot)$ belongs to $W^{1,\infty}(\varepsilon, 1)$ for every $\varepsilon > 0$ and satisfies

$$C_w^{-1} \le \frac{w(x)}{x^\alpha} \le C_w, \quad C_w^{-1} \le \frac{w'(x)}{x^{\alpha-1}} \le C_w,$$

for some $C_w > 0$ and $\alpha \in \mathbb{R}$ as in assumption (A2).

Under the assumptions (A1)-(A3) it has been proven in [2] that the (infinite) weighted mass matrix $\mathbf{M}_w$ is bounded in ℓ^2. A direct consequence of the ℓ^2 boundedness of $\mathbf{M}_w$ is the equivalence between the L_w^2 norm of a function

$$u = \sum_{\ell=l_0}^{\infty} \sum_k u_k^\ell \psi_k^\ell \in L_w^2\big([0,1]\big)$$

and the discrete ℓ_w^2 norm of its wavelet coefficients $u_k^\ell \in \mathbb{R}$, i.e., with

$$|||u_k^\ell|||_w^2 := \sum_\ell \sum_k w^2(2^{-\ell}k)|u_k^\ell|^2,$$

That is, for $u \in L^2_w(0,1)$ there holds

$$\|u\|_w^2 \approx |||u_k^\ell|||_w^2. \tag{4.21}$$

Analogous to (4.21) higher order weighted norm equivalences are valid. Denote by $H^1_{w,0}(0,1)$ the weighted Sobolev space

$$H^1_{w,0}(0,1) = \{u \in L^2(0,1) \mid wu' \in L^2(0,1), u(1) = 0\}.$$

For $u = \sum_{\ell=\ell_0}^{\infty} \sum_k u_k^\ell \psi_k^\ell \in H^1_{w,0}(0,1)$ holds the norm equivalence

$$\|u'\|_w^2 \approx \sum_\ell \sum_k 2^{2\ell} w^2(2^{-\ell}k)|u_k^\ell|^2. \tag{4.22}$$

We now define preconditioners for the weighted mass matrix
$\mathbf{M}_w = \left(\int_0^1 w^2(x)\psi_k^\ell(x)\psi_{k'}^{\ell'}(x)\mathrm{d}x\right)_{(k,\ell),(k',\ell')}$ and for the weighted stiffness matrix
$\mathbf{S}_w = \left(\int_0^1 w^2(x)(\psi_k^\ell)'(x)(\psi_{k'}^{\ell'})'(x)\mathrm{d}x\right)_{(k,\ell),(k',\ell')}$:

$$\left(\mathbf{C}^{\mathrm{I}}_{m,w}\right)_{(k,\ell),(k',\ell')} = \delta_{k,k'}\delta_{\ell,\ell'} w^2(2^{-\ell}k), \qquad \left(\mathbf{C}^{\mathrm{I}}_{s,w}\right)_{(k,\ell),(k',\ell')} = \delta_{k,k'}\delta_{\ell,\ell'} 2^{2\ell} w^2(2^{-\ell}k). \tag{4.23}$$

Then, by norm equivalences (4.21),(4.22), the quantities $\kappa_2((\mathbf{C}^{\mathrm{I}}_{m,w})^{-1/2}\mathbf{M}_w(\mathbf{C}^{\mathrm{I}}_{m,w})^{-1/2})$ and
$\kappa_2((\mathbf{C}^{\mathrm{I}}_{m,w})^{-1/2}\mathbf{S}_w(\mathbf{C}^{\mathrm{I}}_{m,w})^{-1/2})$ are bounded uniformly with respect to the multiresolution level L.
In addition to the diagonal preconditioners in (4.23) we define the preconditioners $\widetilde{\mathbf{C}}^{\mathrm{II}}_{s,w}$, $\mathbf{C}^{\mathrm{II}}_{s,w}$ and $\mathbf{C}^{\mathrm{II}}_{m,w}$ with entries given by

$$\left((\widetilde{\mathbf{C}}^{\mathrm{II}}_{s,w})_{(\ell,k),(\ell',k')}\right)^2 = \begin{cases} \langle(\psi_k^\ell)', (\psi_{k'}^{\ell'})'\rangle_w & \text{if } k = k', \ell = \ell' \\ \langle(\psi_k^\ell)', (\psi_{k'}^{\ell'})'\rangle_w & \text{if } k = k' = 1 \\ 0 & \text{else,} \end{cases} \tag{4.24}$$

$$\left((\mathbf{C}^{\mathrm{II}}_{s,w})_{(\ell,k),(\ell',k')}\right)^2 = \delta_{k,k'}\delta_{\ell,\ell'}\langle(\psi_k^\ell)', (\psi_{k'}^{\ell'})'\rangle_w, \tag{4.25}$$

$$\left((\mathbf{C}^{\mathrm{II}}_{m,w})_{(\ell,k),(\ell',k')}\right)^2 = \delta_{k,k'}\delta_{\ell,\ell'}\langle\psi_k^\ell, \psi_{k'}^{\ell'}\rangle_w. \tag{4.26}$$

We then define preconditioners $\mathbf{C}^\kappa_{\mathrm{real}}$ and $\mathbf{C}^\kappa_{\mathrm{log}}$, $\kappa = \mathrm{I}, \mathrm{II}$, for $\mathbf{A}_{\mathrm{real}}$ and $\mathbf{A}_{\mathrm{log}}$, respectively:

$$\left.\begin{matrix} \mathbf{C}^\kappa_{\mathrm{real}} \\ \\ \mathbf{C}^\kappa_{\mathrm{log}} \end{matrix}\right\} = \mathbf{C}^\kappa + \left\{\begin{matrix} \frac{1}{2}\mathbf{C}^\kappa_{s,x^2}\hat{\otimes}\mathbf{C}^\kappa_{m,|y|^2} \\ \\ \frac{1}{2}\mathbf{C}^\kappa_{s,1}\hat{\otimes}\mathbf{C}^\kappa_{m,|y|^2} \end{matrix}\right. , \tag{4.27}$$

where $\mathbf{C}^\kappa = \mathbf{C}^\kappa_{m,1}\hat{\otimes}\left[\frac{\beta^2}{2}\mathbf{C}^\kappa_{s,1} - (\frac{\omega^2\beta^2}{2} - \alpha\omega)\mathbf{C}^\kappa_{m,y^2} - \alpha\omega m\mathbf{C}^\kappa_{m,y} - (\frac{\omega\beta^2}{2} - r)\mathbf{C}^\kappa_{m,1}\right]$, and $\mathbf{C}^{\mathrm{II}}_{s,x^2} = \widetilde{\mathbf{C}}^{\mathrm{II}}_{s,x^2}$. Note that $\mathbf{C}^{\mathrm{I}}_{\mathrm{real}}$, $\mathbf{C}^{\mathrm{I}}_{\mathrm{log}}$ are diagonal matrices. The main difference between case I and II is that in case II the preconditioner for the weighted stiffness matrix $\mathbf{S}_{x^2}$ is not a diagonal matrix, but it has some additional entries corresponding to the wavelets ψ_k^ℓ with $0 \in \operatorname{supp} \psi_k^\ell$, i.e. the wavelets at the left boundary $x = 0$ are taken into account. Numerical experiments (see [2]) show that the condition numbers of the preconditioned stiffness matrices are substantially smaller in case II than in case I.

For the matrices $\mathbf{B}_{\log} := \lambda\mathbf{M} + \frac{k}{2}\mathbf{A}_{\log}$ and $\mathbf{B}_{\text{real}} := \lambda\mathbf{M} + \frac{k}{2}\mathbf{A}_{\text{real}}$ of the fully discrete scheme we now consider the preconditioners

$$\mathbf{S}_{\log}^{\kappa} := (\operatorname{Re}(\lambda)\mathbf{I} + \tfrac{k}{2}\mathbf{C}_{\log}^{\kappa})^{1/2}, \qquad \mathbf{S}_{\text{real}}^{\kappa} := (\operatorname{Re}(\lambda)\mathbf{I} + \tfrac{k}{2}\mathbf{C}_{\text{real}}^{\kappa})^{1/2}, \qquad \kappa = \text{I}, \text{II}. \tag{4.28}$$

The preconditioned matrices $\hat{\mathbf{B}}_{\log}$ and $\hat{\mathbf{B}}_{\text{real}}$ are

$$\hat{\mathbf{B}}_{\log} := (\mathbf{S}_{\log}^{\kappa})^{-1}\mathbf{B}_{\log}(\mathbf{S}_{\log}^{\kappa})^{-1}, \qquad \hat{\mathbf{B}}_{\text{real}} := (\mathbf{S}_{\text{real}}^{\kappa})^{-1}\mathbf{B}_{\text{real}}(\mathbf{S}_{\text{real}}^{\kappa})^{-1}. \tag{4.29}$$

Proposition 4.1 *Let $\hat{\mathbf{B}}$ be any of the matrices in (4.29). Then*

$$\kappa(\hat{\mathbf{B}}) := \lambda_{\min}((\hat{\mathbf{B}} + \hat{\mathbf{B}}^H)/2))\|\hat{\mathbf{B}}\|^{-1} \le c < \infty$$

where the constant c is independent of λ, L and time-step size k.

Proof. We prove the result for the full tensor product and consider only $\hat{\mathbf{B}}_{\log}$. The proof for $\hat{\mathbf{B}}_{\text{real}}$ is analogous. From the norm equivalence (4.7) with $s = 0$ it follows that every $v \in V_{\mathbf{L}}$ with coefficient vector $\mathbf{v} = (v_k^\ell)$

$$C_1\|v\| \le \mathbf{v}^H\mathbf{M}\mathbf{v} \le C_2\|v\| \tag{4.30}$$

with constants C_1, C_2 independent of L. Assume w.l.o.g. that bilinear form $a_{\log}(\cdot,\cdot)$ satisfies $a_{\log}(v,v) \ge C\|v\|_V^2$. Then (4.7) with $s = 0, 1$, (4.21) and a tensor product argument imply

$$C_1\mathbf{v}^H\mathbf{C}_{\log}^{\text{I}}\mathbf{v} \le \mathbf{v}^H\mathbf{A}_{\log}\mathbf{v} \le C_2\mathbf{v}^H\mathbf{C}_{\log}^{\text{I}}\mathbf{v} \tag{4.31}$$

with constants C_1, C_2 independent of L. Since $\operatorname{Re}(\mathbf{x}^H i\operatorname{Im}(\lambda)\mathbf{M}\mathbf{x}) = 0$ we obtain from (4.30), (4.31) that

$$\operatorname{Re}(\mathbf{x}^H\mathbf{B}\mathbf{x}) \ge C\mathbf{x}^H\mathbf{S}^2\mathbf{x}, \qquad \forall \mathbf{x} \in \mathbb{C}^{\dim V_{\mathbf{L}}},$$

which reads with $\mathbf{y} = \mathbf{S}\mathbf{x}$ as

$$\operatorname{Re}(\mathbf{y}^H\hat{\mathbf{B}}\mathbf{y}) \ge C_3\|\mathbf{y}\|^2, \qquad \forall \mathbf{y} \in \mathbb{C}^{\dim V_{\mathbf{L}}}, \tag{4.32}$$

with C_3 independent of L. We have

$$|\mathbf{x}^H\mathbf{B}\mathbf{y}| \le |\lambda\mathbf{x}^H\mathbf{M}\mathbf{y}| + \tfrac{k}{2}|\mathbf{x}^H\mathbf{A}_{\log}\mathbf{y}| \le C|\lambda|\|\mathbf{x}\|\|\mathbf{y}\| + C\tfrac{k}{2}\|(\mathbf{C}_{\log}^{\text{I}})^{1/2}\mathbf{x}\|\|(\mathbf{C}_{\log}^{\text{I}})^{1/2}\mathbf{y}\|.$$

With $\mathbf{D} := |\lambda|\mathbf{I} + \frac{k}{2}\mathbf{C}^{\mathrm{I}}_{\log}$ we obtain

$$|\mathbf{x}^H\mathbf{B}\mathbf{y}| \leq C(\mathbf{x}^H\mathbf{D}\mathbf{x})^{1/2}(\mathbf{y}^H\mathbf{D}\mathbf{y})^{1/2}.$$

From $\mathrm{Re}(\lambda) \geq C|\lambda|r^{-2}$ (see [16, Lem. 5.5]) we deduce

$$|\mathbf{x}^H\mathbf{B}\mathbf{y}| \leq Cr^2(\mathbf{x}^H\mathbf{S}^2\mathbf{x})^{1/2}(\mathbf{y}^H\mathbf{S}^2\mathbf{y})^{1/2}$$

or

$$|\mathbf{x}^H\hat{\mathbf{B}}\mathbf{y}| \leq C_4r^2\|\mathbf{x}\|\|\mathbf{y}\|. \tag{4.33}$$

Inequalities (4.32) and (4.33) can be stated as

$$\lambda_{\min}(\hat{\mathbf{B}} + \hat{\mathbf{B}}^H) \geq C_3, \qquad \|\hat{\mathbf{B}}\| \leq C_4r^2.$$

The sparse tensor product matrices are obtained from the full tensor product matrices by deleting appropriate rows and columns. Then inequalities (4.34) are still valid (with the same constants). □

4.3.3 Complexity

We solve the systems (4.20) approximately with the incomplete GMRES iteration, yielding an approximation $\tilde{U}_L$ for u. For the linear system $\hat{\mathbf{B}}\hat{\mathbf{x}} = \hat{\mathbf{b}}$ let $\hat{\mathbf{x}}_j$ be the iterates obtained by the restarted GMRES(m_0). According to [9] it follows from Prop. 4.1 that the non-restarted GMRES method for the matrix $\hat{\mathbf{B}}$ yields iterates $\mathbf{x}_j = \mathbf{S}^{-1}\hat{\mathbf{x}}_j$ and residuals $\mathbf{r}_j = \mathbf{S}\hat{\mathbf{b}} - \mathbf{B}\mathbf{x}_j$ satisfying

$$\|\mathbf{r}_j\| \leq \left(1 - \frac{C_3^2}{r^4C_4^2}\right)^{j/2}\|\mathbf{r}_0\|.$$

This together with (4.19) implies that $O(|\log h|^5)$ GMRES iterations are sufficient to achieve an accuracy $O(\hat{N}^{-p})$ at maturity T of the L_2-error $\|u(T) - \tilde{U}_L(T)\|_{L^2}$. For the derivation of this result, see [16, Sec. 5].

The iterative solution of the linear systems with GMRES requires the matrix-vector product of the form $(\sum_i \mathbf{X}_i^{(x)}\hat{\otimes}\mathbf{Y}_i^{(y)})(\mathbf{x}\hat{\otimes}\mathbf{y})$, compare with (4.16). Each of those products can be performed in $C\hat{N}_L$ operations by an iterative scheme (see [16, Sec. 5.6]). We obtain

Theorem 4.2 *The fully discrete Galerkin scheme with uniform order $r = O(|\log h_L|)$, geometric time step sequence $\mathcal{M}_{r,\sigma}$ in $J = (0,T)$ and sparse grids in space with mesh-width $h_L = 2^{-L}$ yields an approximation $\tilde{U}_L(T)$ with accuracy $\|u(T) - \tilde{U}_L(T)\|_{L^2} \leq Ch_L^{4/3}$ which can be computed with at most $Ch_L^{-1}|\log h_L|^8$ operations.*

Proof. For each of the $O(r)$ time steps we solve $r+1$ linear systems. Each of these systems can be solved in $n = O(|\log h_L|^5)$ GMRES iterations. Each GMRES iteration involves a matrix-vector product using $C\hat{N}_L$ operations. Hence the total number of iterations is bounded by $Cr^2n\hat{N}_L$ with $r = O(|\log h_L|)$ and $\hat{N}_L = O(h_L^{-1}|\log h_L|)$. □

5 Numerical experiments

In all computations we use the wavelets described in Section 4.2.1, normalized such that $\|\psi_k^\ell\|_{L^2} = 1$. Note that these wavelets do not belong to $W^{1,\infty}(0,1)$ (Assumption (A2)). Nevertheless, numerical experiments show that they seem to have the property described in Prop. 4.1.

5.1 A model problem

For the PDEs we solve here numerically no explicit solution formulas are available in general. To verify our pricing methodology and to investigate the impact of sparse grid on the accuracy of the computed prices, we introduce a model problem for which the solution is explicitly known.

For $\Omega = (0,2) \times \big(-\frac{1}{2}, \frac{1}{2}\big)$, $Q_T = \Omega \times (0,T)$, $T > 0$ and $\Sigma_T = \partial\Omega \times (0,T)$ let us consider the following model problem

$$\left\{\begin{array}{rcll} u_t - \dfrac{1}{2}x^2|y|^2u_{xx} - \dfrac{1}{2}\beta^2 u_{yy} - \rho\beta x|y|u_{yx} - r\big(xu_x - u\big) - \alpha(m-y)u_y &=& f & \text{in } Q_T \\ u &=& 0 & \text{on } \Sigma_T \\ u(\cdot,0) &=& u_0 & \text{in } \Omega, \end{array}\right.$$

We set the exact solution u to $u(t,x,y) = e^{-t(x+y)}x\sin(\frac{\pi}{2}x)(\frac{1}{2}-y)\cos(\pi y)$ and adapt the right hand side $f(t,x,y)$ such that the PDE is satisfied. We consider the constants $\alpha = 1$, $\beta = \frac{1}{\sqrt{2}}$, $\rho = -0.5$, $r = 0.05$ and $m = 0.2$. The variational formulation of the above problem is the one in (3.39), with the bilinear from $a^{1,1}_{\mathrm{real},\rho}(\cdot,\cdot)$ in (3.19) with $\mu = r$ and $\omega = 0$ (i.e. we do not apply the variable transformations described in Section 2.2 to this problem). The stiffness matrix of the fully discrete problem is (instead of hp-dG time stepping we use the θ-scheme to discretize in time for this example)

$$\begin{aligned} \mathbf{K} \;:=\; & \tfrac{\theta}{2}\mathbf{S}^{(x)}_{x^2} \otimes \mathbf{M}^{(y)}_{|y|^2} + \theta\mathbf{B}^{(x)}_{x} \otimes \Big[\mathbf{M}^{(y)}_{|y|^2} - r\mathbf{M}^{(y)}_{1} - \beta\rho\mathbf{B}^{(y)}_{|y|}\Big] \\ & +\theta\mathbf{M}^{(x)}_{1} \otimes \Big[\big(\tfrac{1}{k\theta} + r\big)\mathbf{M}^{(y)}_{1} + \tfrac{\beta^2}{2}\mathbf{S}^{(y)}_{1} + \alpha\mathbf{B}^{(y)}_{y-m}\Big]. \end{aligned}$$

We only consider the preconditioner of type II

$$\big(\mathbf{C}^{\mathrm{II}}\big)^2 \;:=\; \tfrac{\theta}{2}\widetilde{\mathbf{C}}^{\mathrm{II}}_{s,x^2} \otimes \mathbf{C}^{\mathrm{II}}_{m,|y|^2} + \theta\mathbf{C}^{\mathrm{II}}_{m,1} \otimes \Big[\big(\tfrac{1}{k\theta} + r\big)\mathbf{C}^{\mathrm{II}}_{m,1} + \tfrac{\beta^2}{2}\mathbf{C}^{\mathrm{II}}_{s,1}\Big].$$

We choose $\theta = \frac{1}{2}$ and time step $k = 10^{-2}$ in the θ-scheme and the levels $\mathbf{L} = (\ell,\ell)$, $\ell = 4,\ldots 8$, for the approximation on the full grid (yielding linear systems of size $N := (2^{\ell+1}-1)^2$), and the levels $\hat{\mathbf{L}} = (\ell,\ell)$, $\ell = 5,\ldots,9$, for the approximation on the sparse grid (which produces linear systems of size $\hat{N} := \ell 2^{\ell+1}+1$). In all cases, the linear systems are solved with the restarted GMRES(200) to a relative residuum $\varepsilon = 10^{-10}$. For each level ℓ and time $t = 0.1$ we compute the functional (the maximal nodal error)

$$\ell_u(u_{\mathrm{FE}}) = \max_{\mathcal{T}_{\mathbf{h}}} \left| u|_{\mathcal{T}_{\mathbf{h}}}(0.1,\cdot) - u_{\mathrm{FE}}(0.1,\cdot) \right|,$$

where u is the exact solution and u_{FE} is the Finite Element solution, and plot the values of ℓ_u against N and $\hat{N}$, see Figure 2.

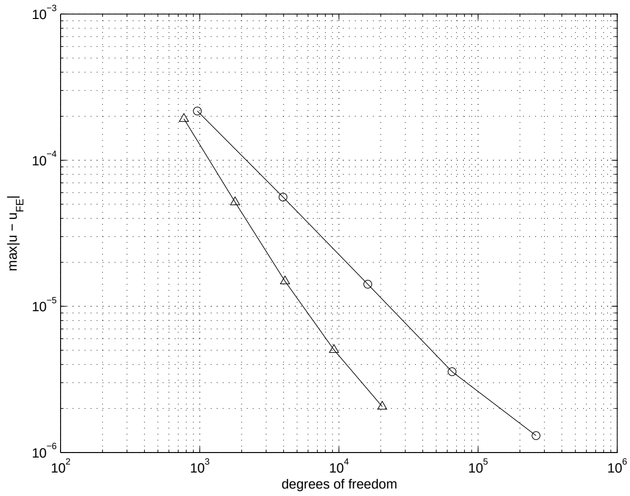

Figure 2: The maximal nodal error of the FE–solution at maturity $T = 0.1$ on the full grid (○–marker), and on the sparse grid (▽–marker).

Another result in this direction is the following. For the Heston model (see (2.11)-(2.12) for the pricing PDE, and (3.42)-(3.43) for its variational formulation) a closed form solution in terms of integrals is available [12]. We use this availability to compare the exact values to the values stemming from numerical simulation. To this end, we consider a European call option with the model parameters $K = 1$, $\kappa = 2.5$, $\lambda = 0$, $m = 0.06$, $\rho = -0.5$, $r = 0$, $\sigma = 0.5$ and $T = 0.5$. To get the numerical solution, we choose $\Omega_R = (0,8) \times (0,3.2)$. In Tables 1 and 2, we denote by u_{ex} the exact value of the option at the point (S,y) and by $u_{(l,l)}$, $\hat{u}_{(l,l)}$ the approximated value obtained by using level $\mathbf{L} = (l,l)$ in the spatial discretization on the full grid and the sparse grid, respectively. The superscripts θ, hpdG refer to the θ-scheme respectively to hp-dG time-stepping. For the θ-scheme, we choose $\theta = \frac{1}{2}$ and the time step $k = 0.01$. In the hp-dG case, the geometric partition $\mathcal{M}_{M,\sigma}$ for the time interval $J = (0,T)$ is set to $M = 5$ and $\sigma = 0.3$, i.e. $0 = t_0 < t_1 < \ldots < t_M = T$ with $t_m = T\sigma^{M-m}$, $m = 1,\ldots,M$. We assume the corresponding polynomial degree vector $\underline{r}$ to be $\underline{r} = (0,1,1,1,2)$. To solve the linear systems, we use restarted GMRES(200) and iterate to a relative residual of 10^{-7}.

The values obtained on the sparse grid are less accurate than the ones obtained on the full grid, independent of the method of time discretization. For example, the relative error of $u^{\theta}_{(8,8)}$ at the point $(S, y) = (1, 0.2)$ is approximatively $3.9 \cdot 10^{-4}$, whereas at the same point the relative error of $\hat{u}^{\theta}_{(8,8)}$ is about $5.5 \cdot 10^{-3}$. We obtain solutions on the sparse grid about 20 times faster than on the full grid with comparable accuracy.
We remark that with our choices of the time discretization parameters, the hp-dG time-stepping yields less accurate values than the θ-scheme. On the other hand, at comparable accuracy hp-dG time-stepping was about 50% faster than the θ-scheme.

(S, y)	(1,0.2)	(1,0.4)	(1.5,0.4)
u_{ex}	0.102052	0.138267	0.532889
$u^{\theta}_{(5,5)}$	0.101093	0.137553	0.532635
$u^{\theta}_{(6,6)}$	0.101792	0.138090	0.532825
$u^{\theta}_{(7,7)}$	0.101972	0.138226	0.532873
$u^{\theta}_{(8,8)}$	0.102012	0.138244	0.532885
$u^{\text{hpdG}}_{(5,5)}$	0.101057	0.137505	0.532672
$u^{\text{hpdG}}_{(6,6)}$	0.101758	0.138043	0.532863
$u^{\text{hpdG}}_{(7,7)}$	0.101939	0.138180	0.532911
$u^{\text{hpdG}}_{(8,8)}$	0.101990	0.138214	0.532923

Table 1: Values of a Call option in the Heston model: full grid.

(S, y)	(1,0.2)	(1,0.4)	(1.5,0.4)
u_{ex}	0.102052	0.138267	0.532889
$\hat{u}^{\theta}_{(5,5)}$	0.076182	0.134385	0.531107
$\hat{u}^{\theta}_{(6,6)}$	0.091135	0.139603	0.532686
$\hat{u}^{\theta}_{(7,7)}$	0.100682	0.139215	0.532779
$\hat{u}^{\theta}_{(8,8)}$	0.102614	0.138095	0.532857
$\hat{u}^{\text{hpdG}}_{(5,5)}$	0.076171	0.134348	0.531152
$\hat{u}^{\text{hpdG}}_{(6,6)}$	0.091105	0.139536	0.532733
$\hat{u}^{\text{hpdG}}_{(7,7)}$	0.100673	0.139259	0.532816
$\hat{u}^{\text{hpdG}}_{(8,8)}$	0.102741	0.138357	0.532896

Table 2: Values of a Call option in the Heston model: sparse grid.

5.2 European vanilla put options

Let us now return to the problem of numerical pricing of option contracts. We shall first solve for the price of a European put in an uncorrelated case ($\rho = 0$) using the real price S of the underlying asset as well as the log-transformed price $x = \log(S)$ and compare the results. Then we shall consider a correlated case ($\rho \neq 0$). We consider in all cases the Stein-Stein model $f(y) = |y|$ as the function of the Ornstein-Uhlenbeck process.

5.2.1 An uncorrelated case

The following parameters are taken $K = 20$, $\alpha = 1$, $\beta = \frac{1}{\sqrt{2}}$, $\rho = 0$,$m = 0.2$, $r = 0.05$ and $T = 1$. We want to approximate the value of the option U in the domain $\overline{\Omega} = (0, 150) \times (-1.5, 1.5)$. For the real price we compute the approximative solution in the domain $\Omega_R = (0, 600) \times (-4, 4)$ and restrict this solution to the smaller domain $\overline{\Omega}$. The reason for solving in the larger domain Ω_R is that the homogenous Dirichlet boundary conditions (3.46) at $y = \pm 4$ and at $S = 600$ are not satisfied by U, producing a boundary layer at $y = \pm 4$. The solution U in $\overline{\Omega}$ is not affected by these artificial boundary conditions.

To discretize the variational formulation (3.45)-(3.47) in Ω_R, we take $\mathbf{L} = (8, 6)$ for the full tensor product space, i.e. 8 levels in x-coordinate direction and 6 levels in y-coordinate direction are used. For the θ-scheme we choose $\theta = \frac{1}{2}$ (Crank-Nicholson) and time step $k = 0.01$ leading to a linear system with 64897 unknowns which has to be solved $M = 100$–times. To solve the linear systems we use the preconditioner of type II and apply the restarted GMRES(200), which iterates until the relative residuum is smaller than $\varepsilon = 10^{-8}$ in each time step. In Figure 3 we plot the numerical solution U in the domain $\overline{\Omega}$. To

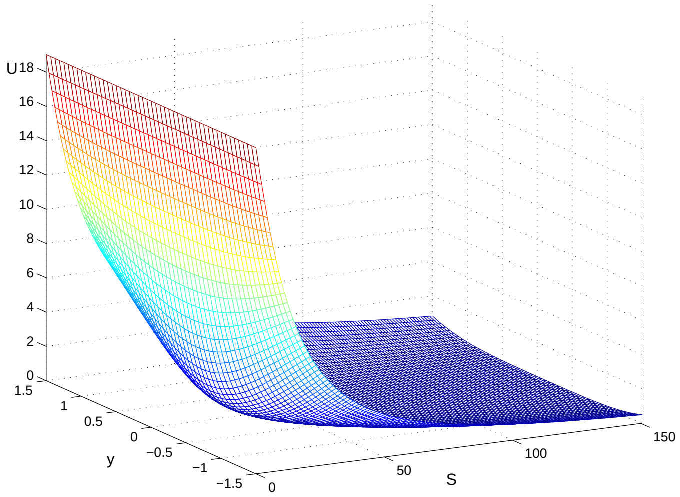

Figure 3: The approximated solution of U restricted to the domain $\overline{\Omega}$ at $T = 1$.

compare the approximated solution U obtained in the real–price setting, we perform the same simulation changing to log–price of the underlying asset. We use the computational domain $\Omega = (-7, 7) \times (-4, 4)$. We choose the same discretization parameters as in the real price setting, and again the preconditioner of type II. Figure 4 shows the option value obtained by solving the problem in real-price (top) and in log-price (bottom).

5.2.2 A correlated case

We solve for a European put in a correlated case with the following parameters $K = 20$, $\alpha = 1$, $\beta = \frac{1}{\sqrt{2}}$, $\mu = 0.15$, $\rho = -0.5$, $m = 0.2$, $r = 0.05$, and $T = 1$. The computational domain is $\Omega_R = (0, 600) \times (0, 4)$, again larger than the domain $\overline{\Omega} = (0, 150) \times (0, 1.5)$ of interest. The discretization parameters are as in the uncorrelated case. The stiffness

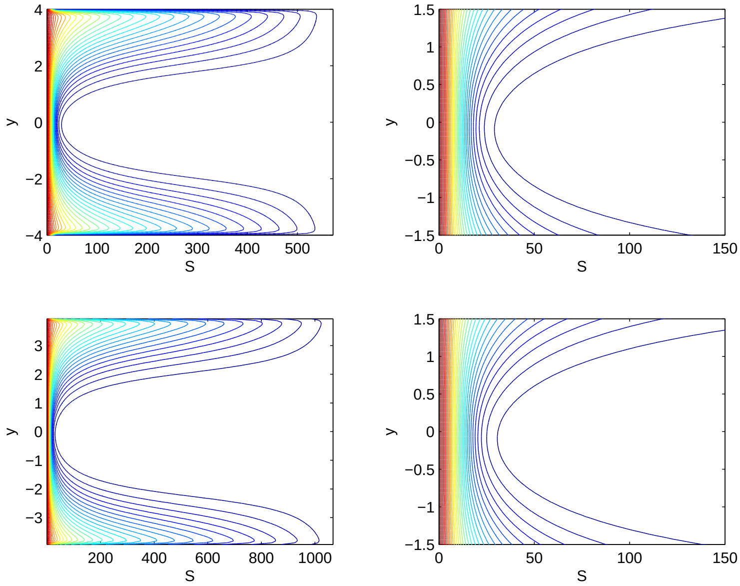

Figure 4: Option prices of European vanilla put at $T = 1$, $\rho = 0$. On the left top, the solution in $(0, 600) \times (-4, 4)$ obtained by solving the real-price equation, right the same solution restricted to $(0, 150) \times (-1.5, 1.5)$. On the left bottom the solution in the domain is $(e^{-7}, e^{7}) \times (-4, 4)$ obtained by solving the log-price equation is displayed. On the right bottom, again the solution restricted to $(0, 150) \times (-1.5, 1.5)$. In both cases, the boundary layer produced by the artificial boundary condition can be observed.

matrix $\mathbf{K}_{\text{real}}$ of the arising linear system is preconditioned with the preconditioner of type II. In Figure 5, we show the solution at $T = 1$.

5.3 Compound options

We show that our numerical scheme does also apply for options which are more sophisticated than simple European options. As an example we consider a compound option, also called “option on an option”, i.e. compound options have underlying assets which are themselves options: for example, a call on another call.

We now price a call compound option for the case that the underlying option is also a call. We choose the following parameters: $\alpha = 1$, $\beta = \frac{1}{\sqrt{2}}$, $\rho = 0$ (i.e. uncorrelated), $r = 0.05$, $m = 0.2$, $K = 20$, $K^c = 12$, $T = 0.5$ and $T^c = 0.4$. The discretization parameters are the restricted domain $\Omega_R = (-7, 7) \times (-4, 4)$, time step $k = 0.01$ and $\theta = \frac{1}{2}$ for the θ–scheme. Furthermore, the levels $\mathbf{L} = (7, 6)$ for the full tensor product space are taken, leading to 32385 unknowns in each time step.

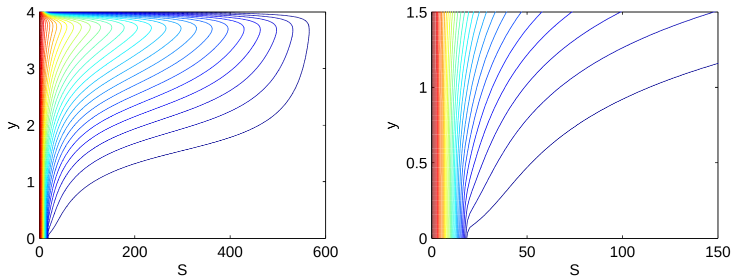

Figure 5: Option prices of European vanilla put at $T = 1$, $\rho \neq 0$. Left the solution in the domain $(0, 600) \times (0, 4)$, where the boundary layer at $y = 4$ generated by the Dirichlet boundary conditions can be seen. Right the same solution, restricted to $(0, 150) \times (0, 1.5)$

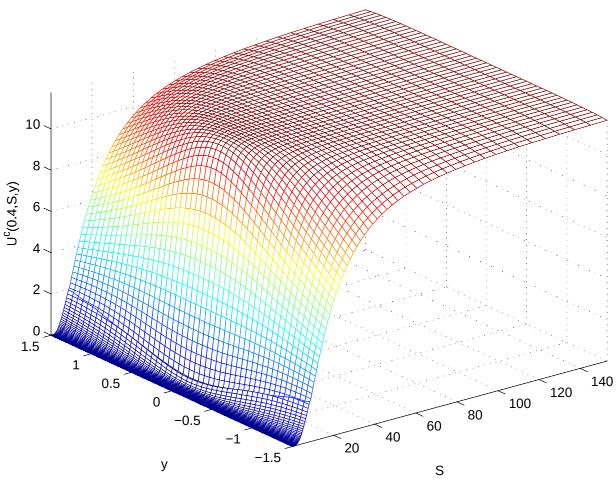

Figure 6: The value U^c of a compound option (call–call) with strike price $K^c = 12$.

A The bilinear forms

We give here the explicit expressions of the bilinear forms $a^{\varphi,\psi}_{\mathrm{real},0}$, $a^{\varphi,\psi}_{\mathrm{real},\rho,\pm}$, $a^{\varphi,\psi}_{\log,0}$ and $a^{\varphi,\psi}_{\log,\rho,\pm}$.

A.1 Bilinear forms $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$, $a^{\varphi,\psi}_{\log,\rho,\pm}(\cdot,\cdot)$

Integration by parts in (3.19) gives

$$
\begin{aligned}
a^{\varphi,\psi}_{\log,\rho,\pm}(u,v) \;=\; & \tfrac{1}{2}\int_{Q_\pm} y^2 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y + \int_{Q_\pm} y^2\frac{\partial u}{\partial x} v\varphi(x)\frac{\mathrm{d}\varphi}{\mathrm{d}x}\psi^2(y)\mathrm{d}x\mathrm{d}y \qquad \text{(A.1)}\\
& +\tfrac{1}{2}\int_{Q_\pm} y^2\frac{\partial u}{\partial x} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y + \tfrac{1}{2}\beta^2\int_{Q_\pm}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& +\beta^2\int_{Q_\pm}\frac{\partial u}{\partial y} v\varphi^2(x)\psi(y)\frac{\mathrm{d}\psi}{\mathrm{d}y}\mathrm{d}x\mathrm{d}y + \rho\beta\int_{Q_\pm}|y|\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& +2\rho\beta\int_{Q_\pm}|y|\frac{\partial u}{\partial y} v\varphi(x)\frac{\mathrm{d}\varphi}{\mathrm{d}x}\psi^2(y)\mathrm{d}x\mathrm{d}y - \omega\rho\beta\int_{Q_\pm} y|y|\frac{\partial u}{\partial x} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& -r\int_{Q_\pm}\frac{\partial u}{\partial x} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y - (\omega\beta^2-\alpha)\int_{Q_\pm} y\frac{\partial u}{\partial y} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& -\alpha m\int_{Q_\pm}\frac{\partial u}{\partial y} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y + \beta\rho(\mu-r)\int_{Q_\pm}\frac{1}{|y|}\frac{\partial u}{\partial y} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& -\alpha\omega m\int_{Q_\pm} yuv\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& -\left(\tfrac{1}{2}\omega^2\beta^2-\alpha\omega\right)\int_{Q_\pm} y^2uv\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& +\omega\beta\rho(\mu-r)\int_{Q_\pm}\frac{y}{|y|}uv\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\\
& -\left(\tfrac{1}{2}\omega\beta^2-r\right)\int_{Q_\pm} uv\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

In the case when $\rho = 0$ (uncorrelated variants) the bilinear form $a^{\varphi,\psi}_{\log,0}(\cdot,\cdot)$ is given by (A.1) in which the terms containing the factor ρ are dropped and the integration domain is $\mathbb{R}^2$ instead of $Q_\pm$.

A.2 Bilinear form $a^{\varphi,\psi}_{\mathrm{real},\rho,\pm}(\cdot,\cdot)$

Again with an integration by parts we get from (3.37)

$$\begin{aligned}
a^{\varphi,\psi}_{\mathrm{real},\rho,\pm}(u,v) = {} & \tfrac{1}{2}\int_{\Omega_\pm} y^2 S^2 \frac{\partial u}{\partial S}\frac{\partial v}{\partial S}\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y + \int_{\Omega_\pm} y^2 S^2 \frac{\partial u}{\partial S} v\varphi(S)\frac{\mathrm{d}\varphi}{\mathrm{d}S}\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& + \int_{\Omega_\pm} y^2 S \frac{\partial u}{\partial S} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y + \tfrac{1}{2}\beta^2\int_{\Omega_\pm} \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& + \beta^2\int_{\Omega_\pm} \frac{\partial u}{\partial y} v\varphi^2(S)\psi(y)\frac{\mathrm{d}\psi}{\mathrm{d}y}\mathrm{d}S\mathrm{d}y + \rho\beta\int_{\Omega_\pm} |y| S\frac{\partial u}{\partial S}\frac{\partial v}{\partial y}\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& + 2\rho\beta\int_{\Omega_\pm} |y| S\frac{\partial u}{\partial S} v\varphi^2(S)\psi(y)\frac{\mathrm{d}\psi}{\mathrm{d}y}\mathrm{d}S\mathrm{d}y \qquad \text{(A.2)} \\
& + \rho\beta\int_{\Omega_\pm} \mathrm{sign}(y) S\frac{\partial u}{\partial S} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& - \omega\rho\beta\int_{\Omega_\pm} y|y| S\frac{\partial u}{\partial S} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y - r\int_{\Omega_\pm} S\frac{\partial u}{\partial S} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& - (\omega\beta^2 - \alpha)\int_{\Omega_\pm} y\frac{\partial u}{\partial y} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& - \alpha m\int_{\Omega_\pm} \frac{\partial u}{\partial y} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& + \beta\rho(\mu - r)\int_{\Omega_\pm} \frac{1}{|y|}\frac{\partial u}{\partial y} v\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& - \alpha\omega m\int_{\Omega_\pm} yuv\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& - \left(\tfrac{1}{2}\omega^2\beta^2 - \alpha\omega\right)\int_{\Omega_\pm} y^2 uv\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& + \omega\beta\rho(\mu - r)\int_{\Omega_\pm} \frac{y}{|y|} uv\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y \\
& - \left(\tfrac{1}{2}\omega\beta^2 - r\right)\int_{\Omega_\pm} uv\varphi^2(S)\psi^2(y)\mathrm{d}S\mathrm{d}y.
\end{aligned}$$

B Proof of Theorem 3.3

Proof. With $u = v$, integration by parts and the definition of the $L^2_{\varphi,\psi}$-norm we obtain from (A.1):

$$\begin{aligned} a^{\varphi,\psi}_{\log,0}(u,u) &= \frac{1}{2}\left\|y\frac{\partial u}{\partial x}\right\|^2_{L^2_{\varphi,\psi}} + \int_{\mathbb{R}^2} y^2\frac{\partial u}{\partial x}u\varphi^2\psi^2\frac{\varphi'}{\varphi}\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int_{\mathbb{R}^2} y^2u^2\varphi^2\psi^2\frac{\varphi'}{\varphi}\mathrm{d}x\mathrm{d}y \\ &+\frac{\beta^2}{2}\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}} + \beta^2\int_{\mathbb{R}^2} y\frac{\partial u}{\partial y}u\varphi^2\psi^2\frac{\psi'}{y\psi}\mathrm{d}x\mathrm{d}y + \frac{\omega\beta^2-\alpha}{2}\|u\|^2_{L^2_{\varphi,\psi}} \\ &+(\omega\beta^2-\alpha)\int_{\mathbb{R}^2} y^2u^2\varphi^2\psi^2\frac{\psi'}{y\psi}\mathrm{d}x\mathrm{d}y - \alpha m\int_{\mathbb{R}^2}\frac{\partial u}{\partial y}u\varphi^2\psi^2\mathrm{d}x\mathrm{d}y \\ &-\left(\frac{\beta^2\omega^2}{2}-\alpha\omega\right)\|yu\|^2_{L^2_{\varphi,\psi}} - \alpha\omega m\int_{\mathbb{R}^2} yu^2\varphi^2\psi^2\mathrm{d}x\mathrm{d}y - \frac{\omega\beta^2}{2}\|u\|^2_{L^2_{\varphi,\psi}} \end{aligned}$$

We estimate the integrals using the Cauchy-inequality:

$$\begin{aligned} \int_{\mathbb{R}^2} y^2\frac{\partial u}{\partial x}u\varphi^2\psi^2\frac{\varphi'}{\varphi}\mathrm{d}x\mathrm{d}y &\le \left\|\frac{\varphi'}{\varphi}\right\|_{L^\infty}\left(\varepsilon_1\left\|y\frac{\partial u}{\partial x}\right\|^2_{L^2_{\varphi,\psi}} + \frac{1}{4\varepsilon_1}\|yu\|^2_{L^2_{\varphi,\psi}}\right), \\ \int_{\mathbb{R}^2} y^2u^2\varphi^2\psi^2\frac{\varphi'}{\varphi}\mathrm{d}x\mathrm{d}y &\le \left\|\frac{\varphi'}{\varphi}\right\|_{L^\infty}\|yu\|^2_{L^2_{\varphi,\psi}}, \\ \int_{\mathbb{R}^2} y\frac{\partial u}{\partial y}u\varphi^2\psi^2\frac{\psi'}{y\psi}\mathrm{d}x\mathrm{d}y &\le \left\|\frac{\psi'}{y\psi}\right\|_{L^\infty}\left(\varepsilon_2\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}} + \frac{1}{4\varepsilon_2}\|yu\|^2_{L^2_{\varphi,\psi}}\right), \\ \int_{\mathbb{R}^2} y^2u^2\varphi^2\psi^2\frac{\psi'}{y\psi}\mathrm{d}x\mathrm{d}y &\le \left\|\frac{\psi'}{y\psi}\right\|_{L^\infty}\|yu\|^2_{L^2_{\varphi,\psi}}, \\ \int_{\mathbb{R}^2}\frac{\partial u}{\partial y}u\varphi^2\psi^2\mathrm{d}x\mathrm{d}y &\le \varepsilon_3\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}} + \frac{1}{4\varepsilon_3}\|u\|^2_{L^2_{\varphi,\psi}}, \\ \int_{\mathbb{R}^2} yu^2\varphi^2\psi^2\mathrm{d}x\mathrm{d}y &\le \varepsilon_4\|yu\|^2_{L^2_{\varphi,\psi}} + \frac{1}{4\varepsilon_4}\|u\|^2_{L^2_{\varphi,\psi}}. \end{aligned}$$

Collecting the terms yields

$$\begin{aligned} a^{\varphi,\psi}_{\log,0}(u,u) &\ge \left(\frac{1}{2}-\varepsilon_1\left\|\frac{\varphi'}{\varphi}\right\|_{L^\infty}\right)\left\|y\frac{\partial u}{\partial x}\right\|^2_{L^2_{\varphi,\psi}} + \beta^2\left(\frac{1}{2}-\varepsilon_2\left\|\frac{\psi'}{y\psi}\right\|_{L^\infty} - \alpha|m|\varepsilon_3\right)\left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}} \\ &+\Bigg[-\frac{1}{4\varepsilon_1}\left\|\frac{\varphi'}{\varphi}\right\|_{L^\infty} - \frac{1}{2}\left\|\frac{\varphi'}{\varphi}\right\|_{L^\infty} - \frac{\beta^2}{4\varepsilon_2}\left\|\frac{\psi'}{y\psi}\right\|_{L^\infty} - (\omega\beta^2+\alpha)\left\|\frac{\psi'}{y\psi}\right\|_{L^\infty} \\ &\quad-\left(\frac{\beta^2\omega^2}{2}-\alpha\omega\right) - \varepsilon_4\alpha\omega|m|\Bigg]\|yu\|^2_{L^2_{\varphi,\psi}} \\ &-\left(\frac{\alpha}{2}+\alpha|m|\frac{1}{4\varepsilon_3}-\alpha\omega|m|\frac{1}{4\varepsilon_4}\right)\|u\|^2_{L^2_{\varphi,\psi}}. \end{aligned}$$

Let $\nu := \|\frac{\varphi'}{\varphi}\|_{L^\infty}$ and $\overline{\mu} := \|\frac{\psi'}{y\psi}\|_{L^\infty}$. Choose $\varepsilon_1 = \frac{1}{4\nu}$, $\varepsilon_2 = \frac{1}{4\overline{\mu}}$, $\varepsilon_3 = \frac{1}{16\alpha|m|}$ and let $\omega = \eta\frac{2\alpha}{\beta^2}$ for some $\eta \in (0,1)$. Then the coefficient in front of $\|yu\|^2_{L^2_{\varphi,\psi}}$ is given by:

$$2\left(\frac{\alpha}{\beta}\right)^2\eta(1-\eta) - \alpha\overline{\mu}(2\eta+1) - \nu^2 - \frac{\nu}{2} - \beta^2\overline{\mu}^2 - 2\left(\frac{\alpha}{\beta}\right)^2\varepsilon_4(1-\eta)|m| \geq 0$$

for $\nu, \overline{\mu}, \varepsilon_4$ sufficiently small. Hence we have

$$a^{\varphi,\psi}_{\log,0}(u,u) \geq C_1\left(\left\|y\frac{\partial u}{\partial x}\right\|^2_{L^2_{\varphi,\psi}} + \left\|\frac{\partial u}{\partial y}\right\|^2_{L^2_{\varphi,\psi}}\right) - C_2\|u\|^2_{L^2_{\varphi,\psi}}$$

for some constants C_1, C_2. With $c > C_2$ we conclude

$$a^{\varphi,\psi}_{\log,0}(u,u) + c\|u\|^2_{L^2_{\varphi,\psi}} \geq \min\{C_1, c - C_2\}|u|_{V^{\varphi,\psi}},$$

and the Proposition follows with $C = \min\{C_1, c - C_2\}$.

□

C Proof of Theorem 3.4

Proof. Let $\omega = 2\alpha\eta/\beta^2$ with $\eta \in (0,1)$. Taking $v = u$ and using $\frac{\partial v}{\partial x}v = \frac{1}{2}\frac{\partial}{\partial x}(v^2)$, $\frac{\partial v}{\partial y}v = \frac{1}{2}\frac{\partial}{\partial y}(v^2)$ in $(A.1)$ yields

$$
\begin{aligned}
a^{\varphi,\psi}_{\log,\rho,\pm}(v,v) \;=\;& \frac{1}{2}\left\|y\frac{\partial v}{\partial x}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)} + \int_{Q_\pm} y^2\frac{\partial v}{\partial x}v\varphi^2(x)\psi^2(y)\frac{\varphi'(x)}{\varphi(x)}\mathrm{d}x\mathrm{d}y \\
& -\frac{1}{2}\int_{Q_\pm} y^2v^2\varphi^2(x)\psi^2(y)\frac{\varphi'(x)}{\varphi(x)}\mathrm{d}x\mathrm{d}y + \frac{1}{2}\beta^2\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)} \\
& +\beta^2\int_{Q_\pm} y\frac{\partial v}{\partial y}v\varphi^2(x)\psi^2(y)\frac{\psi'(y)}{y\psi(y)}\mathrm{d}x\mathrm{d}y \\
& +\rho\beta\int_{Q_\pm} |y|\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \\
& -\rho\beta\int_{Q_\pm} \mathrm{sign}(y)v^2\varphi^2(x)\psi^2(y)\frac{\varphi'(x)}{\varphi(x)}\mathrm{d}x\mathrm{d}y \\
& -2\rho\beta\int_{Q_\pm} y|y|v^2\varphi^2(x)\psi^2(y)\frac{\psi'(y)}{y\psi(y)}\mathrm{d}x\mathrm{d}y \\
& +2\frac{\alpha}{\beta}\eta\rho\int_{Q_\pm} y|y|v^2\varphi^2(x)\psi^2(y)\frac{\varphi'(x)}{\varphi(x)}\mathrm{d}x\mathrm{d}y + r\int_{Q_\pm} v^2\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \\
& +\frac{1}{2}\alpha(2\eta-1)\|v\|^2_{L^2_{\varphi,\psi}(Q_\pm)} + \alpha(2\eta-1)\int_{Q_\pm} y^2v^2\varphi^2(x)\psi^2(y)\frac{\psi'(y)}{y\psi(y)}\mathrm{d}x\mathrm{d}y \\
& -\alpha m\int_{Q_\pm} \frac{\partial v}{\partial y}v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \\
& +\beta\rho(\mu-r)\int_{Q_\pm} \frac{1}{|y|}\frac{\partial v}{\partial y}v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \\
& -2\left(\frac{\alpha}{\beta}\right)^2\eta m\int_{Q_\pm} yv^2\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y + 2\left(\frac{\alpha}{\beta}\right)^2\eta(1-\eta)\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)} \\
& +2\frac{\alpha}{\beta}\rho(\mu-r)\eta\int_{Q_\pm} \frac{y}{|y|}v^2\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y + (r-\alpha\eta)\|v\|^2_{L^2_{\varphi,\psi}(Q_\pm)}.
\end{aligned}
$$

With $\varphi(x) = e^{\nu|x|}$ and $\psi(y) = e^{\overline{u}y^2/2}$, $\nu, \overline{\mu} > 0$ to be fixed later and by applying the Cauchy–Schwartz inequality we obtain the estimates for the above integrals

$$
\begin{aligned}
\left|\int_{Q_\pm} y^2\frac{\partial v}{\partial x}v\varphi^2(x)\psi^2(y)\frac{\varphi'(x)}{\varphi(x)}\mathrm{d}x\mathrm{d}y\right| &\le \nu\left\|y\frac{\partial v}{\partial x}\right\|_{L^2_{\varphi,\psi}(Q_\pm)}\|yv\|_{L^2_{\varphi,\psi}(Q_\pm)} \\
&\le \frac{1}{2}\zeta_1\left\|y\frac{\partial v}{\partial x}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)} + \frac{\nu^2}{2\zeta_1}\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)}
\end{aligned}
$$

for arbitrary $\zeta_1 > 0$. Likewise,

$$\left|\rho\beta\int_{Q_\pm}|y|\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\right| \le |\rho|\beta\left\|y\frac{\partial v}{\partial x}\right\|_{L^2_{\varphi,\psi}(Q_\pm)}\left\|\frac{\partial v}{\partial y}\right\|_{L^2_{\varphi,\psi}(Q_\pm)}$$

$$\le \frac{1}{2}\zeta_2\left\|y\frac{\partial v}{\partial x}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)} + \frac{\rho^2\beta^2}{2\zeta_2}\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)}$$

for all $\zeta_2 > 0$ and

$$2|\rho|\beta\left|\int_{Q_\pm}y|y|v^2\varphi^2(x)\psi^2(y)\frac{\psi'(y)}{y\psi(y)}\frac{\varphi'(x)}{\varphi(x)}\mathrm{d}x\mathrm{d}y\right| \le 2|\rho|\beta\bar{\mu}\nu\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)}.$$

Furthermore,

$$\beta^2\left|\int_{Q_\pm}y\frac{\partial v}{\partial y}v\varphi^2(x)\psi^2(y)\frac{\psi'(y)}{y\psi(y)}\mathrm{d}x\mathrm{d}y\right| \le \beta^2\bar{\mu}\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)}\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)}$$

$$\le \frac{1}{2}\beta^2\zeta_3\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)} + \frac{\beta^2\bar{\mu}^2}{2\zeta_3}\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)}$$

for arbitrary $\zeta_3 > 0$ and

$$2\alpha\eta\left|\int_{Q_\pm}y^2v^2\varphi^2(x)\psi^2(y)\frac{\psi'(y)}{y\psi(y)}\mathrm{d}x\mathrm{d}y\right| \le 2\alpha\eta\bar{\mu}\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)}.$$

Finally,

$$\alpha|m|\left|\int_{Q_\pm}\frac{\partial v}{\partial y}v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\right| \le \alpha|m|\left\|\frac{\partial v}{\partial y}\right\|_{L^2_{\varphi,\psi}(Q_\pm)}\|v\|_{L^2_{\varphi,\psi}(Q_\pm)}$$

$$\le \frac{1}{2}\beta^2\zeta_4\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q_\pm)} + \frac{m^2}{2\zeta_4}\left(\frac{\alpha}{\beta}\right)^2\|v\|^2_{L^2_{\varphi,\psi}(Q_\pm)}$$

$$2\left(\frac{\alpha}{\beta}\right)^2|m|\eta\left|\int_{Q_\pm}yv^2\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y\right| \le 2\left(\frac{\alpha}{\beta}\right)^2|m|\eta\varepsilon\|yv\|^2_{L^2_{\varphi,\psi}(Q_\pm)} +$$

$$2\left(\frac{\alpha}{\beta}\right)^2|m|\eta\frac{1}{4\varepsilon}\|v\|^2_{L^2_{\varphi,\psi}(Q_\pm)}$$

hold for any $\zeta_4 > 0$ and $\varepsilon > 0$. Next, Hardy's inequality

$$\left\|\frac{v}{y}\right\|_{L^2(Q_\pm)} \le 2\left\|\frac{\partial v}{\partial y}\right\|_{L^2(Q_\pm)}$$

implies

$$\left\|\frac{v}{y}\right\|_{L^2_{\varphi,\psi}(Q\pm)} \leq 2\left(\left\|\frac{\partial v}{\partial y}\right\|_{L^2_{\varphi,\psi}(Q\pm)} + \bar{\mu}\|yv\|_{L^2_{\varphi,\psi}(Q\pm)}\right). \tag{C.1}$$

By (C.1) and by the Cauchy-Schwartz inequality we deduce that

$$\beta|\rho||\mu - r|\int_{Q_\pm} \frac{1}{|y|}\frac{\partial v}{\partial y} v\varphi^2(x)\psi^2(y)\mathrm{d}x\mathrm{d}y \leq \left(2\beta|\rho||\mu - r| + \frac{1}{2}\beta^2\zeta_5\right)\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q\pm)} + \frac{2\rho^2(\mu - r)^2\bar{\mu}^2}{\zeta_5}\|yv\|^2_{L^2_{\varphi,\psi}(Q\pm)}$$

holds for any $\zeta_5 > 0$. Summing all these estimates we obtain

$$a^{\varphi,\psi}_{\log,\rho,\pm}(v,v) \geq \alpha_1\left\|y\frac{\partial v}{\partial x}\right\|^2_{L^2_{\varphi,\psi}(Q\pm)} + \alpha_2\|yv\|^2_{L^2_{\varphi,\psi}(Q\pm)} + \alpha_3\left\|\frac{\partial v}{\partial y}\right\|^2_{L^2_{\varphi,\psi}(Q\pm)} + \alpha_4\|v\|^2_{L^2_{\varphi,\psi}(Q\pm)}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{1}{2}(1 - \zeta_1 - \zeta_2)\\
\alpha_2 &= -\frac{\nu^2}{2\zeta_1} - \frac{\nu}{2} - 2|\rho|\beta\bar{\mu}\nu - 2|\rho|\frac{\alpha}{\beta}\eta\nu - \beta^2\bar{\mu}^2\frac{1}{2\zeta_3} + 2\left(\frac{\alpha}{\beta}\right)^2\eta(1-\eta) - \alpha\bar{\mu}\\
&\quad - 2\left(\frac{\alpha}{\beta}\right)^2 m\eta\varepsilon - 2\frac{\rho^2(\mu - r)^2\bar{\mu}^2}{\zeta_5} - 2\alpha\eta\bar{\mu}\\
\alpha_3 &= \frac{1}{2}\beta^2(1 - \zeta_3 - \zeta_4 - \zeta_5) - \frac{(\rho\beta)^2}{2\zeta_2} - 2\beta|\rho||\mu - r|\\
\alpha_4 &= -|\rho|\beta\nu - \frac{(\alpha|m|)^2}{2\zeta_4\beta^2} - \frac{1}{2}\alpha - 2\left(\frac{\alpha}{\beta}\right)^2|m|\eta\frac{1}{4\varepsilon} - 2\frac{\alpha}{\beta}|\rho||\mu - r|\eta + r(1-\nu).
\end{aligned}$$

In order to ensure $\alpha_3 > 0$, the parameter β needs to satisfy the following inequality $\beta > \frac{4|\rho|\mu}{1-\rho^2}$. If (3.20) holds, we can find $\zeta_3, \zeta_4, \zeta_5$ sufficiently small and $\zeta_2 \in (0,1)$ sufficiently close to 1 such that $\alpha_3 > 0$. If ζ_1 is next chosen sufficiently small then $\alpha_1 > 0$. Finally, with $\zeta_1, \zeta_2, \zeta_3, \zeta_4$ and ζ_5 fixed and for $0 < \eta_1 < \eta_2 < 1$ there exists a constant $c_0 = c_0(\eta_1, \eta_2) > 0$ and ν_0, $\bar{\mu}_0$ and $\varepsilon > 0$ small enough such that $\alpha_2 > c_0$ for all $\eta_1 \leq \eta \leq \eta_2$ and all $\nu \in [0, \nu_0)$, $\bar{\mu} \in [0, \bar{\mu}_0)$.

□

References

[1] Y. Achdou and N. Tchou: Variational analysis for the Black and Scholes equation with stochastic volatility. ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 36, N°3 (2002) 373–395.

[2] S. Beuchler, R. Schneider and C. Schwab: Multiresolution weighted norm equivalences and applications. Numerische Mathematik **98** (2004) 67–97.

[3] F. Black and M. Scholes: The pricing of options and corporate liabilities. Journal of Political Economy **81** (1973) 637–654.

[4] H.J. Bungartz and M. Griebel: Sparse Grids. Acta Numerica **13** (2004) 147–269.

[5] P. Carr and D.B. Madan: Option Valuation Using the Fast Fourier Transform. Journal of Computational Finance, Vol. 2, N°4 (1999) 61–73.

[6] A. Cohen: Numerical Analysis of Wavelet Methods. Studies in Mathematics and its Applications, Vol. **32**, Elsevier publ. (2003).

[7] J.C. Cox and S.A. Ross: The Valuation of Options for Alternative Stochastic Processes. Journal of Financial Economics **3** (1976) 145–166.

[8] F. Delbaen and H. Shirakawa: A Note on Option Pricing for the Constant Elasticity of Variance Model. Asia-Pacific Financial Markets **9** (2002) 85–99.

[9] S.C. Eisenstat, H.C. Elman and M.H. Schultz: Variational iterative methods for nonsymmetric systems of linear equations. SIAM Journal on Numerical Analysis **20** (1983) 345–357.

[10] J.-P. Fouque, G. Papanicolaou and R. Sircar: Derivatives in financial markets with stochastic volatility. Cambridge University Press, Cambridge (2000).

[11] J.-P. Fouque, G. Papanicolaou, R. Sircar and K. Solna: Multiscale stochastic volatility asymptotics. Multiscale Modeling & Simulation **2** (2003) 22–42.

[12] S.L. Heston: A closed form solution for options with stochastic volatility with applications to bonds and currency options. Review of Financial studies **6** (1993) 327–343.

[13] J.-L. Lions and E. Magenes: Problèmes aux limites non homogènes et applications. Vol. I and II, Dunod, Paris (1968).

[14] A.-M. Matache, T. von Petersdorff and C. Schwab: Fast Deterministic Pricing of Options on Lévy Driven Assets. ESAIM: Mathematical Modelling and Numerical Analysis, Vol. 38, N°1 (2004) 37–71.

[15] D. Schötzau and C. Schwab: Time discretization of parabolic problems by the *hp*-version of the discontinuous Galerkin finite element method. SIAM Journal on Numerical Analysis **38**, (2000) 837–875.

[16] T. von Petersdorff and C. Schwab: Numerical solution of parabolic equations in high dimensions. ESAIM: Mathematical Modelling and Numerical Analysis **38** N°1 (2004) 93–127.

[17] T. von Petersdorff and C. Schwab: Wavelet-discretizations of parabolic integro-differential equations, SIAM Journal on Numerical Analysis **41** (2003) 159–180.

[18] T. Wihler, A.-M. Matache and C. Schwab: Fast Numerical Solution of Parabolic Integro-Differential Equations with Applications in Finance, Reseach report No. 1954 IMA University of Minnesota (2004).

[19] P. Wilmott, S. Howison and J. Dewynne: The Mathematics of Financial Derivatives. Cambridge University Press, Cambridge (1995).