

Dual-Primal FETI algorithms for edge element
approximations: Two-dimensional h and p
finite elements on shape-regular meshes¹

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Dual-Primal FETI algorithms for edge element approximations:
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Abstract

A family of dual-primal FETI methods for edge element approximations in two dimensions is proposed and analyzed. The primal constraints are here averages over subdomain edges. It is shown that the condition number of the corresponding method is independent of the number of substructures and grows only polylogarithmically with the number of unknowns associated with individual substructures. The estimate is also independent of the jumps of both of the coefficients of the original problem. Numerical results validating our theoretical bounds are given.

Keywords: Edge elements, Maxwell's equations, finite elements, spectral elements, domain decomposition, FETI, preconditioners, heterogeneous coefficients

Subject Classification: 65F10, 65N22, 65N30, 65N55

1. Introduction. In this paper, we consider the boundary value problem

$$\begin{aligned} L\mathbf{u} &:= \mathbf{curl}(a \mathbf{curl} \mathbf{u}) + A \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{t} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

with Ω a bounded polygonal domain in \mathbb{R}^2 . The domain Ω has unit diameter and \mathbf{t} is its unit tangent. We have

$$\mathbf{curl} v := \begin{bmatrix} \frac{\partial v}{\partial x_2} & -\frac{\partial v}{\partial x_1} \end{bmatrix}^T, \quad \mathbf{curl} \mathbf{u} := \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2};$$

see, e.g., [14]. The coefficient matrix A is a symmetric, uniformly positive definite matrix-valued function with entries $A_{ij} \in L^\infty(\Omega)$, $1 \leq i, j \leq 2$, and $a \in L^\infty(\Omega)$ is a positive function bounded away from zero.

The weak formulation of problem (1.1) requires the introduction of the Hilbert space $H(\mathbf{curl}; \Omega)$, defined by

$$H(\mathbf{curl}; \Omega) := \{ \mathbf{v} \in (L^2(\Omega))^2 \mid \mathbf{curl} \mathbf{v} \in L^2(\Omega) \}.$$

The space $H(\mathbf{curl}; \Omega)$ is equipped with the following inner product and graph norm,

$$(\mathbf{u}, \mathbf{v})_{\mathbf{curl}} := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} \mathbf{curl} \mathbf{u} \mathbf{curl} \mathbf{v} \, dx, \quad \|\mathbf{u}\|_{\mathbf{curl}}^2 := (\mathbf{u}, \mathbf{u})_{\mathbf{curl}},$$

and the tangential component $\mathbf{u} \cdot \mathbf{t}$, of a vector $\mathbf{u} \in H(\mathbf{curl}; \Omega)$ on the boundary $\partial\Omega$, belongs to the space $H^{-\frac{1}{2}}(\partial\Omega)$; see [4, 14]. The subspace of vectors in $H(\mathbf{curl}; \Omega)$ with vanishing tangential component on $\partial\Omega$ is denoted by $H_0(\mathbf{curl}; \Omega)$.

For any $\mathcal{D} \subset \Omega$, we define the bilinear form

$$a_{\mathcal{D}}(\mathbf{u}, \mathbf{v}) := \int_{\mathcal{D}} (a \mathbf{curl} \mathbf{u} \mathbf{curl} \mathbf{v} + A \mathbf{u} \cdot \mathbf{v}) \, dx, \quad \mathbf{u}, \mathbf{v} \in H(\mathbf{curl}; \Omega). \tag{1.2}$$

The variational formulation of Equation (1.1) is:

Find $\mathbf{u} \in H_0(\mathbf{curl}; \Omega)$ such that

$$a_{\Omega}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \mathbf{v} \in H_0(\mathbf{curl}; \Omega). \tag{1.3}$$

The purpose of this work is to construct and analyze a dual-primal FETI (FETI-DP) preconditioner for the finite and spectral element approximation of Problem (1.3). Neumann-Neumann (NN) and FETI algorithms are particular domain decomposition (DD) methods of iterative substructuring type: they rely on a nonoverlapping partition into subdomains. They are among the most popular and heavily tested DD methods and are now employed for the solution of huge problems on parallel architectures; see, e.g., [11, 6, 5, 25, 3]. The rate of convergence is often independent of possibly large jumps of the coefficients.

FETI methods rely on the reformulation of the original algebraic problem into an equivalent saddle point problem, involving discontinuous functions across the subdomain boundaries and a continuity constraint for the solution; see Equation (3.3). In the original one level FETI methods, completely discontinuous vectors are employed: the elimination of the primal variable thus requires the solution of local (generally

singular) Neumann problems and an equation for the Lagrange multipliers is then obtained. A preconditioner for this equation is then constructed by solving local Dirichlet problems on the subdomains, after eliminating the components belonging to a suitable coarse space, constructed from local subdomain kernels or suitable functions (constants for the Laplace equation, rigid body modes for linear elasticity, for example).

The more recently developed FETI-DP methods employ a smaller space for the solution, where a certain number of degrees of freedom or linear functionals are continuous across the subdomains. These so-called *primal constraints* ensure that a nonsingular global problem needs to be solved in order to obtain an equation for the Lagrange multipliers: this step requires the solution of modified nonsingular Neumann problems and the solution of a coarse problem the size of which equals the number of primal constraints. As before, a preconditioner is constructed by solving local Dirichlet problems. FETI-DP algorithms present considerable advantages: the same code can now be employed for a wider class of problems, much less dense coarse matrices need to be inverted, they do not require the characterization of the kernels of local Neumann operators or the introduction of a scaling matrix for the construction of the coarse component of the preconditioner, and they may start the conjugate gradient iteration from an arbitrary initial guess. For these reasons, they have now almost completely replaced one level FETI methods for large scale computations. Connections between NN and FETI methods are being investigated; see [9].

The motivation of this work lies in the fact that no iterative substructuring methods (and in particular no NN or FETI preconditioners) that are robust with respect to the number of unknowns, the number of subdomains, and large jumps of the coefficients are presently available for edge element approximations of three dimensional problems.

Some methods are available for two dimensional approximations:

In [24], a domain decomposition preconditioner was proposed, which is based on a standard coarse space and local spaces associated to the subdomain edges. NN preconditioners with standard coarse spaces were studied in [18]. One level FETI methods were developed in [20, 16], thanks to the introduction of suitable local functions which are the analog of constants and rigid body modes for the Laplace equation and linear elasticity, respectively. These functions were then employed to construct a Balancing NN method in [19]. Standard coarse spaces however are not in general suitable for quasioptimal preconditioners in three dimensions and the search for suitable local functions in three dimensions for Balancing NN and one level FETI methods has produced no results so far. For these reasons we believe that FETI-DP algorithms will turn out to be less hard to devise for three dimensional problems. The scope of this work is then to begin to understand a good set of primal constraints in *two dimensions*, which have not been available so far. It turns out that the natural choice of edge averages on the subdomain edges leads to a robust preconditioner.

For the analysis, we employ the tools developed in [24]. We note that, more general tools were later devised in [26], which consisted in decomposition results for trace functions in $H(\text{curl}; \Omega)$ in two dimensions or $H(\text{div}; \Omega)$ and stable curl/divergence free extensions from the subdomain boundaries. Here, we have chosen to employ the results in [24], since we have in mind extensions to anisotropic meshes which are often needed for problems in conductor materials with high jumps in the conductivity. The work in [21, 23, 22] for scalar problems showed that when dealing with highly anisotropic meshes the analysis cannot employ trace norms or stable extensions, since

the latter are not available in this case. The approach in [24] which does not rely on trace norms or stable extensions, but only on results for scalar problems (see Lemma 4.1 and its proof in section 5) appears more promising for edge element approximations on anisotropic meshes. This generalization is left to a future work.

This paper is organized as follows:

in section 2, we introduce our discrete problems, the subdomain partition, and local and global finite element spaces. In section 3, we introduce our FETI-DP algorithms. Condition number bounds are given in section 4. First we give the technical tools necessary to prove them in subsection 4.1: these are the decomposition Lemma 4.1 and the abstract framework for the analysis of FETI-DP methods originally proposed in [10]. Our main result is the stability property in Lemma 4.6 in subsection 4.2. Lemma 4.1 is proven in [24] for h approximations and we provide a proof for the case of spectral elements in section 5. A practical implementation of our algorithm is given in section 6 and some numerical results in section 7.

2. Discrete spaces. In this paper, we consider both h version finite elements and spectral elements. We discretize this problem using edge elements, which are also known as Nédélec elements; see [15]. These are vector finite elements that only ensure the continuity of the tangential component across the elements, as is physically required for the electric and magnetic fields, solutions of Maxwell's equations. We refer to [14] for a general introduction of approximations of electromagnetic problems, the Sobolev space $H(\text{curl}; \Omega)$, and edge elements.

2.1. Triangulations and subdomain partitions. We introduce a shape-regular triangulation $\mathcal{T} = \mathcal{T}_h$ of the domain Ω , made of affinely mapped quadrilaterals. In particular, if $\widehat{Q} = (-1, 1)$ is a reference square, for each element $K \in \mathcal{T}$, there exists an affine mapping $F_K : \widehat{Q} \rightarrow K$, such that K is the image of \widehat{Q} . Here we only consider quadrilateral meshes for simplicity but note that our results are equally valid for h approximations on triangular meshes.

Let $\mathcal{E} = \mathcal{E}_h$ be the set of edges of \mathcal{T} . For every edge $e \in \mathcal{E}$, we fix a direction, given by a unit vector \mathbf{t}_e , tangent to e . The length of the edge e is denoted by $|e|$.

We next consider a non overlapping partition of the domain Ω ,

$$\mathcal{F}_H = \left\{ \Omega_i \mid 1 \leq i \leq N, \quad \bigcup_{i=1}^N \overline{\Omega}_i = \overline{\Omega} \right\},$$

such that each Ω_i is connected. The elements of \mathcal{F}_H are called *subdomains* or *substructures*. For the h version, we take the substructures Ω_i as unions of fine elements. We denote the diameter of Ω_i by H_i and define H as the maximum of the diameters of the subdomains:

$$H := \max_{1 \leq i \leq N} \{H_i\}.$$

In this case $h < H$. For the p version we take $\mathcal{F}_H = \mathcal{T}_h$ and thus $H = h$.

We always assume that the substructures are images of a reference square under sufficiently regular maps, which effectively means that their aspect ratios remain uniformly bounded. In addition, we assume that the ratio of the diameters of two adjacent subregions is bounded away from zero and infinity. Further assumptions, necessary for the analysis but not for the definition of the algorithms, are made at the beginning of section 4.1.

We define the *edges* of the partition as the interior E_{ij} of the intersections

$$\overline{E}_{ij} := \partial\Omega_i \cap \partial\Omega_j, \quad i \neq j, \quad |E_{ij}| > 0,$$

where $|E_{ij}|$ denotes the measure of E_{ij} and \overline{E}_{ij} its closure. We note that $E_{ji} = E_{ij}$. We introduce a unit vector $\mathbf{t}_{E_{ij}}$ that is tangent to E_{ij} . Let \mathcal{E}_H be the set of edges of \mathcal{F}_H , and let the interface Γ be the union of the edges of \mathcal{F}_H , or, equivalently the parts of the subdomain boundaries that do not belong to $\partial\Omega$:

$$\Gamma := \bigcup_{i=1}^N \partial\Omega_i \setminus \partial\Omega.$$

For every subdomain Ω_i , let \mathcal{I}_i be the set of indices j , such that E_{ij} is an edge of Ω_i :

$$\mathcal{I}_i := \{j \mid E_{ij} \subset \partial\Omega_i, E_{ij} \in \mathcal{E}_H\}.$$

Our assumptions on the partition \mathcal{F}_H ensure that the the number of edges $|\mathcal{I}_i|$ is uniformly bounded.

We assume that the coefficients a and A are constant in each substructure Ω_i and denote them by a_i and A_i , respectively. We also assume that

$$0 < \beta_i |\mathbf{x}|^2 \leq \mathbf{x}^t A_i \mathbf{x} \leq \gamma_i |\mathbf{x}|^2, \quad \mathbf{x} \in \mathbb{R}^2, \quad (2.1)$$

for $i = 1, \dots, N$, where $|\cdot|$ denotes the standard Euclidean norm.

2.2. Edge element functions. We next define the local spaces

$$\mathbf{H}_*(\text{curl}; \Omega_i) := \{\mathbf{u}_i \in \mathbf{H}(\text{curl}; \Omega_i) \mid \mathbf{u}_i \cdot \mathbf{t} = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\}$$

and the following polynomial spaces on the reference square, for $k \geq 1$,

$$\mathcal{R}_k(\widehat{Q}) = \mathbb{Q}_{k-1, k}(\widehat{Q}) \otimes \mathbb{Q}_{k, k-1}(\widehat{Q}),$$

with $\mathbb{Q}_{k_1, k_2}(\widehat{Q})$ the space of polynomials of degree k_i in the i -th variable. On an affinely mapped element $K \in \mathcal{T}$, we take

$$\mathcal{R}_k(K) = \{\mathbf{u} = J_{F_K}^{-T} \widehat{\mathbf{u}} \mid \widehat{\mathbf{u}} \in \mathcal{R}(\widehat{Q})\}, \quad (2.2)$$

with J_{F_K} the Jacobian of the transformation F_K . We note that the tangential component of a vector in $\mathcal{R}_k(K)$ is a function of \mathbb{Q}_{k-1} over each edge of K .

For the h version, we employ the lowest-order Nédélec finite element spaces, originally introduced in [15], defined on each subdomain Ω_i .

$$X_i = X^h(\Omega_i) := \{\mathbf{u} \in \mathbf{H}_*(\text{curl}; \Omega_i) \mid \mathbf{u}|_K \in \mathcal{R}_1(K), K \in \mathcal{T}_h, K \subset \Omega_i\}.$$

Higher polynomial degrees can also be considered and our results and bounds will remain valid with constants that depend on the polynomial degree. See, e.g., [14] for more details. Functions in X_i have a constant tangential component over the fine edges in \mathcal{E} . The degrees of freedom for X_i are the constant values of the tangential component on the fine edges in \mathcal{E} contained in $\overline{\Omega}_i$.

For spectral elements we choose

$$X_i = X^k(\Omega_i) := \mathcal{R}_k(\Omega_i) \cap \mathbf{H}_*(\text{curl}; \Omega_i)$$

and basis functions associated to the (mapped) Gauss-Lobatto nodes on Ω_i . The corresponding degrees of freedom are the values at these nodes. We refer to [2, 8, 13, 14] for more details on spectral element approximations of electromagnetic problems.

We next consider the product space

$$X = X(\Omega) := \prod_{i=1}^N X_i \subset \prod_{i=1}^N \mathbf{H}_*(\text{curl}; \Omega_i),$$

which consists of vectors that have in general a discontinuous tangential component along the subdomain edges. The discrete solution is sought in the conforming space

$$\widehat{X} := X \cap \mathbf{H}_0(\text{curl}; \Omega),$$

of vectors with a continuous tangential components along the edges in \mathcal{E}_H .

We now introduce some trace spaces consisting of tangential components on the boundaries of the substructures. A scalar function u , defined on $\partial\Omega_i \setminus \partial\Omega$, belongs to W_i if and only if there exists $\mathbf{u} \in X_i$ such that, for each edge,

$$u|_{E_{ij}} = \mathbf{u} \cdot \mathbf{t}_{E_{ij}}, \quad E_{ij} \in \mathcal{E}_H, \quad j \in \mathcal{I}(i).$$

For h approximations these are piecewise constant (or piecewise polynomial of degree $k-1$ if higher order Nédélec elements are considered) along the edges E_{ij} . For spectral approximations they are polynomials of degree $k-1$ on each edge E_{ij} . We will employ the product space of functions defined on Γ , $W := \prod_i W_i$, and its continuous subspace \widehat{W} consisting of tangential traces of vector in \widehat{X} .

The scalar functions in the spaces W_i and W are uniquely defined by the degrees of freedom of the spaces X_i and X involving the tangential components along edges in \mathcal{E}_H . Throughout this paper, we will use the following notations: we denote a generic vector function in X_i using a bold letter with the superscript (i) , e.g., $\mathbf{u}^{(i)}$, and employ the same notation for the corresponding column vector of degrees of freedom. Its tangential component $u^{(i)}$ is an element of W_i and is defined by

$$u^{(i)}|_{E_{ij}} := \mathbf{u}^{(i)} \cdot \mathbf{t}_{E_{ij}} \quad E_{ij} \in \mathcal{E}_H, \quad j \in \mathcal{I}(i).$$

It is uniquely determined by the degrees of freedom $\mathbf{u}^{(i)}$ involving the tangential component along $\partial\Omega_i \setminus \partial\Omega$. We use the same notation $u^{(i)}$ for the column vector of these tangential degrees of freedom and the same notation for the spaces of functions X_i and W_i and for the corresponding spaces of degrees of freedom. Similarly for global functions in X and W .

We remark that a vector \mathbf{u} belongs to the continuous space \widehat{X} (and consequently its tangential component to \widehat{W}) if

$$u^{(i)}|_{E_{ij}} = u^{(j)}|_{E_{ij}}, \quad E_{ij} \in \mathcal{E}_H. \quad (2.3)$$

Finally, for $i = 1, \dots, N$, we define the extensions into the interior of the Ω_i

$$\mathcal{H}_i : W_i \longrightarrow X_i,$$

that are discrete harmonic with respect to the bilinear forms $a_{\Omega_i}(\cdot, \cdot)$. We recall that $\mathbf{u}^{(i)} = \mathcal{H}_i u^{(i)}$ minimizes the energy $a_{\Omega_i}(\mathbf{u}^{(i)}, \mathbf{u}^{(i)})$ among all the vectors of X_i with tangential component equal to $u^{(i)}$ on $\partial\Omega_i \setminus \partial\Omega$. We will refer to \mathcal{H}_i as the *Maxwell* discrete harmonic extension.

2.3. Continuous finite and spectral element spaces. In the following we will also need the standard finite and spectral element spaces of scalar, continuous, piecewise polynomial functions. With

$$H_*^1(\Omega_i) := \{\phi \in H^1(\Omega_i) \mid \phi = 0 \text{ on } \partial\Omega \cap \partial\Omega_i\},$$

we define, for the h version, the space of continuous piecewise bilinear functions

$$Q_i = Q^h(\Omega_i) := \{\phi \in H_*^1(\Omega_i) \mid \phi \in \mathbb{Q}_{1,1}(K), K \in \mathcal{T}_h, K \subset \Omega_i\}.$$

For spectral elements, we employ

$$Q_i = Q^k(\Omega_i) := \mathbb{Q}_{k,k}(\Omega_i) \cap H^1(\Omega_i).$$

We note that in both cases $\text{grad } Q_i \subset X_i$.

Discrete harmonic functions in Q_i will be referred to as *Laplace* discrete harmonic in the following.

3. Dual-primal FETI methods. In this section, we introduce a dual-primal FETI method for the solution of the linear system arising from the edge element discretization of problem (1.3). In section 6, we give a practical implementation of the algorithm. Throughout the paper, we denote the Euclidean scalar product in l^2 by $\langle \cdot, \cdot \rangle$. We recall that dual-primal FETI methods were originally introduced in [5]. The first theoretical result was given in [12] for two dimensional problems and then later in [10] for three dimensions.

We first assemble the local stiffness matrices, relative to the bilinear forms $a_{\Omega_i}(\cdot, \cdot)$, and the local load vectors. The degrees of freedom that belong only to one substructure can be eliminated in parallel by block Gaussian elimination. We note that these are degrees of freedom associated to edges or nodes in the interior of the substructures, on $\partial\Omega$, and, in case polynomial spaces with $k > 0$ are employed they also consist of values of the normal component on the subdomain boundaries. We are then left with the degrees of freedom involving the *tangential* component along the substructure boundaries. Let $f^{(i)}$ be the resulting right hand sides and $S^{(i)}$ the Schur complement matrices

$$S^{(i)} : W_i \longrightarrow W_i,$$

relative to the tangential degrees of freedom on $\partial\Omega_i \setminus \partial\Omega$.

We recall that the local Schur complements satisfy the following property

$$|u^{(i)}|_{S^{(i)}}^2 := \langle u^{(i)}, S^{(i)}u^{(i)} \rangle = a_{\Omega_i}(\mathcal{H}_i u^{(i)}, \mathcal{H}_i u^{(i)}); \quad (3.1)$$

see, e.g., [17, 18]. Since the local bilinear forms are positive definite, so are the local Schur complements $S^{(i)}$.

We write

$$u := \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(N)} \end{bmatrix} \in W, \quad S := \text{diag}\{S^{(1)}, \dots, S^{(N)}\}, \quad f := \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(N)} \end{bmatrix}.$$

The solution $u \in W$ to the discrete problem can then be found by minimizing the energy

$$\frac{1}{2} \langle u, Su \rangle - \langle f, u \rangle$$

subject to the constraint that u is continuous, i.e., it belongs to \widehat{W} .

For dual-primal FETI methods we work in a subspace $\widetilde{W} \subset W$ of functions satisfying a certain number of continuity constraints. We have

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta.$$

Here the primal space $\widehat{W}_\Pi \subset \widehat{W}$ consists of continuous functions determined by degrees of freedom associated to the substructures. We choose a space of constant functions on the subdomain edges.

$$\widehat{W}_\Pi = \widehat{W}^H := \{u \in \widehat{W} \mid u|_{E_{ij}} \in \mathbb{Q}_0, E_{ij} \in \mathcal{E}_H\}. \quad (3.2)$$

The degrees of freedom (*primal variables*) associated to this space are the averages of tangential components over the subdomain edges:

$$\bar{u}_{E_{ij}} = \frac{\int_{E_{ij}} u \, ds}{|E_{ij}|} = \frac{\int_{E_{ij}} \mathbf{u} \cdot \mathbf{t}_{E_{ij}} \, ds}{|E_{ij}|}.$$

These are the same degrees of freedom associated to a standard coarse space in case the substructures are elements of a coarse mesh; see [24, 18] and section 4.1.

The dual space \widetilde{W}_Δ is the product space of spaces associated to the substructures

$$\widetilde{W}_\Delta := \prod_{i=1}^N \widetilde{W}_{\Delta,i}$$

of functions for which the functional given by the primal variables vanish:

$$\widetilde{W}_{\Delta,i} := \{u \in W_i \mid \bar{u}_{E_{ij}} = 0, j \in \mathcal{I}(i)\}.$$

Therefore, \widetilde{W} consists of functions that have a continuous average along the substructure edges, i.e., the averages are the same regardless of which substructure is considered for the calculation.

The primal degrees of freedom can then be eliminated together with the internal ones, at the expenses of solving one coarse problem. We are then left with a problem involving interface functions with vanishing mean value along the substructure edges and, consequently, in the dual space \widetilde{W}_Δ . Let $\tilde{S} : \widetilde{W}_\Delta \rightarrow \widetilde{W}_\Delta$ be the corresponding Schur complement and \tilde{f}_Δ the corresponding load vector. We then look for $u_\Delta \in \widetilde{W}_\Delta$, such that

$$\frac{1}{2} \langle u_\Delta, \tilde{S}u_\Delta \rangle - \langle \tilde{f}_\Delta, u_\Delta \rangle \longrightarrow \min$$

subject to the constraint that u_Δ is continuous. The continuity constraint is expressed by the equation

$$B_\Delta u_\Delta = 0,$$

where B_Δ is constructed from $\{0, 1, -1\}$ and evaluates the difference between all the corresponding tangential degrees of freedom on Γ ; cf. (2.3). We employ the same matrix as in our previous paper [20] and then enforce redundant conditions. The matrix B_Δ has the following block structure:

$$B_\Delta = [B_\Delta^{(1)} \quad B_\Delta^{(2)} \quad \dots \quad B_\Delta^{(N)}],$$

where each block corresponds to a substructure.

We obtain the saddle point problem

$$\begin{aligned} \tilde{S}u_\Delta + B_\Delta^T \lambda &= \tilde{f}_\Delta \\ B_\Delta u_\Delta &= 0 \end{aligned} \quad (3.3)$$

with $u_\Delta \in \widetilde{W}_\Delta$ and $\lambda \in V := \text{Range}(B_\Delta)$.

We note that \tilde{S} can be obtained from the restriction of S to the space \widetilde{W} , by eliminating the primal degrees of freedom. We have therefore the minimization property

$$\langle u_\Delta, \tilde{S}u_\Delta \rangle = \min \langle u, Su \rangle, \quad (3.4)$$

where the minimum is taken over all the functions $u = u_\Delta + w_\Pi$, $w_\Pi \in \widehat{W}_\Pi$. This property ensures that \tilde{S} is also positive definite.

Since the Schur complement \tilde{S} is invertible, an equation for λ can easily be found:

$$F\lambda = d, \quad (3.5)$$

with

$$F := B_\Delta \tilde{S}^{-1} B_\Delta^T, \quad d := B_\Delta \tilde{S}^{-1} \tilde{f}_\Delta. \quad (3.6)$$

In section 6, we provide explicit formulas for F and d . Once λ is found, the primal variables are given by

$$u_\Delta = \tilde{S}^{-1}(\tilde{f}_\Delta - B_\Delta^T \lambda) \in \widetilde{W}.$$

In order to define a preconditioner for (3.5), we need to define scaling matrices and functions defined on the subdomain boundaries. As in our previous work they are constructed with the coefficient A only. For each substructure, we define $\delta_i^\dagger \in W_i$, such that on the edge E_{ij} , $j \in \mathcal{I}(i)$

$$\delta_i^\dagger = \frac{\gamma_i^\chi}{\gamma_i^\chi + \gamma_j^\chi}, \quad (3.7)$$

for an arbitrary but fixed $\chi \in [1/2, +\infty)$; see (2.1). By direct calculation, we find

$$\gamma_i \delta_j^{\dagger 2} \leq \min(\gamma_i, \gamma_j). \quad (3.8)$$

For each substructure Ω_i , we next introduce a diagonal matrix $D_\Delta^{(i)} : V \rightarrow V$. The diagonal entry corresponding to the Lagrange multipliers that enforce the continuity along an edge E_{ij} is set equal to the (constant) value of δ_j^\dagger along E_{ji}

$$\delta_{ji}^\dagger := \delta_j^\dagger|_{E_{ji}} = \frac{\gamma_j^\chi}{\gamma_i^\chi + \gamma_j^\chi}.$$

We next define the scaled matrix

$$B_{D,\Delta} = [D_\Delta^{(1)} B_\Delta^{(1)} \quad D_\Delta^{(2)} B_\Delta^{(2)} \quad \dots \quad D_\Delta^{(N)} B_\Delta^{(N)}] : \widetilde{W}_\Delta \rightarrow V.$$

We solve the dual system (3.5) using the preconditioned conjugate gradient algorithm with the preconditioner

$$M^{-1} := B_{D,\Delta} S B_{D,\Delta}^T = \sum_{i=1}^N D_\Delta^{(i)} B_\Delta^{(i)} S^{(i)} B_\Delta^{(i)T} D_\Delta^{(i)}; \quad (3.9)$$

see [5, 12, 10].

4. Condition number bounds.

4.1. Technical Tools. The analysis of the FETI-DP methods presented here relies on a decomposition result. We first need to introduce coarse spaces on the subdomains. As is often customary in the analysis of iterative substructuring methods, we require that the substructures are elements of a shape-regular coarse mesh \mathcal{T}_H . This is always the case for spectral elements. We next define

$$X^H(\Omega_i) := \mathcal{R}_0(\Omega_i), \quad (4.1)$$

the lowest order edge element space on the coarse element Ω_i ; see (2.2). We note that the tangential traces of vectors in $X^H(\Omega_i)$ are restrictions of functions in the space \widehat{W}^H , defined in (3.2), to the boundary of Ω_i .

The following result can be found in [24, Lem. 4.2] for h approximations. The proof for the spectral element case is given in section 5. We need the scaled norm

$$\|\mathbf{u}\|_{\text{curl}, \Omega_i}^2 := \|\mathbf{u}\|_{L^2(\Omega_i)}^2 + H_i^2 \|\text{curl } \mathbf{u}\|_{L^2(\Omega_i)}^2, \quad \mathbf{u} \in X_i.$$

LEMMA 4.1. *Let Ω_i be a substructure. Then, for every $\mathbf{u} \in X_i$ there exists a unique decomposition*

$$\mathbf{u} = \mathbf{u}_H + \sum_{j \in \mathcal{I}(i)} \mathbf{u}_{ij} + \mathbf{u}^{int}, \quad (4.2)$$

such that,

1. \mathbf{u}_H is a coarse function in $X^H(\Omega_i)$;
2. $\mathbf{u}_{ij} = \nabla \phi_{ij}$, with $\phi_{ij} \in Q_i$ is a Laplace discrete harmonic function that vanishes on $\partial\Omega_i \setminus E_{ij}$;
3. \mathbf{u}^{int} has a vanishing tangential component on $\partial\Omega_i$.

In addition, for $j \in \mathcal{I}(i)$,

$$\int_{E_{ij}} (\mathbf{u} - \mathbf{u}_H) \cdot \mathbf{t}_{E_{ij}} ds = \int_{E_{ij}} \nabla \phi_{ij} \cdot \mathbf{t}_{E_{ij}} ds = 0, \quad (4.3)$$

and

$$\|\nabla \phi_{ij}\|_{L^2(\Omega_i)}^2 \leq C\omega^2 \|\mathbf{u}\|_{\text{curl}, \Omega_i}^2, \quad (4.4)$$

with $\omega = (1 + \log(H/h))$ for h approximations and $\omega = (1 + \log k)$ for spectral elements.

We note that bounds for the components \mathbf{u}_H and \mathbf{u}^{int} can also be found but they will not be necessary for the analysis in this paper.

The following result is a straightforward application of the existence of a stable finite element or a spectral element extension and of a trace theorem.

LEMMA 4.2. *Let Ω_i and Ω_j be two substructures that share an edge E_{ij} . Let $\phi^{(i)} \in Q_i$ and $\phi^{(j)} \in Q_j$ be two Laplace discrete harmonic functions that have a common trace on E_{ij} and vanish on $\partial\Omega_i \setminus E_{ij}$ and $\partial\Omega_j \setminus E_{ij}$, respectively. Then there exists a constant C , independent of h , k , H_i , and H_j , such that*

$$\|\nabla \phi^{(j)}\|_{L^2(\Omega_j)}^2 \leq C \|\nabla \phi^{(i)}\|_{L^2(\Omega_i)}^2.$$

We now recall an abstract framework for the analysis of FETI-DP algorithms, which was originally given in [10]. It turns out that condition number bounds rely on one stability estimate for the following jump operator

$$P_\Delta := B_{D,\Delta}^T B_\Delta : \widetilde{W} \longrightarrow \widetilde{W}.$$

We summarize the properties of P_Δ proven in [10, Sect. 6] in the following lemma.

LEMMA 4.3. *The operator P_Δ is a projection and preserves the jump of any function $w \in \widetilde{W}$, i.e.,*

$$B_\Delta P_\Delta w = B_\Delta w.$$

If $v := P_\Delta w$, for $w \in \widetilde{W}$, then on every edge E_{ij} of a substructure Ω_i , we have

$$v^{(i)} = \delta_j^\dagger(w^{(i)} - w^{(j)}). \quad (4.5)$$

Finally, $P_\Delta w = 0$, if $w \in \widehat{W}$.

The following fundamental result can be found in [10, Th. 1]. It employs the norms

$$|v|_S^2 := \langle v, Sv \rangle = \sum_{i=1}^N \langle v^{(i)}, S^{(i)} v^{(i)} \rangle, \quad |v|_{\widetilde{S}}^2 := \langle v, \widetilde{S}v \rangle. \quad (4.6)$$

THEOREM 4.4. *Let C_{P_Δ} be such that*

$$|P_\Delta w_\Delta|_S^2 \leq C_{P_\Delta} |w_\Delta|_{\widetilde{S}}^2, \quad w_\Delta \in \widetilde{W}_\Delta. \quad (4.7)$$

Then, if \widetilde{S} and M^{-1} are invertible,

$$\langle M\lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq C_{P_\Delta} \langle M\lambda, \lambda \rangle, \quad \lambda \in V. \quad (4.8)$$

4.2. Main results. We now present two lemmas. The first one is trivial for our approximations and ensures that the Schur complement \widetilde{S} and the preconditioner M^{-1} are invertible. The second provides a key stability estimate in order to bound the largest eigenvalue of the preconditioned operator $M^{-1}F$. Our main result is given in Theorem 4.7.

LEMMA 4.5. *The Schur complement \widetilde{S} and the preconditioner M^{-1} are invertible.*

Proof. The result for \widetilde{S} is an immediate consequence of the fact that the local bilinear forms $a_{\Omega_i}(\cdot, \cdot)$ are positive definite. Indeed the Schur complement S is invertible and so is \widetilde{S} thanks to (3.4).

In order to prove the invertibility of M^{-1} , we assume that there is a $\lambda = B_\Delta w_\Delta$, $w_\Delta \in \widetilde{W}$, such that

$$0 = M^{-1}\lambda = B_{D,\Delta} S B_{D,\Delta}^T B_\Delta w_\Delta.$$

This implies

$$0 = \langle \lambda, M^{-1}\lambda \rangle = |P_\Delta w_\Delta|_S^2.$$

Since the local Schur complements $S^{(i)}$ are invertible, this implies $P_\Delta w_\Delta = 0$. Lemma 4.3 then implies

$$\lambda = B_\Delta w_\Delta = B_\Delta P_\Delta w_\Delta = 0.$$

□

LEMMA 4.6. *There is a constant C , such that, for $w_\Delta \in \widetilde{W}_\Delta$,*

$$|P_\Delta w_\Delta|_S^2 \leq C \eta \omega^2 |w_\Delta|_S^2$$

where ω is the same as in Lemma 4.1 and

$$\eta := \max_{1 \leq i \leq N} \frac{\gamma_i}{\beta_i} \left(1 + \frac{H_i^2 \gamma_i}{a_i} \right).$$

Proof. Using the minimization property in (3.4), we consider the element $w = w_\Delta + w_\Pi$, $w_\Pi \in \widetilde{W}_\Pi$ such that

$$|w_\Delta|_S^2 = |w|_S^2. \quad (4.9)$$

We note that, since w_Π is continuous,

$$v := P_\Delta w_\Delta = P_\Delta w.$$

We then need to calculate

$$|P_\Delta w|_S^2 = \sum_{i=1}^N |v^{(i)}|_{S^{(i)}}^2 = \sum_{i=1}^N a_{\Omega_i} (\mathcal{H}_i v^{(i)}, \mathcal{H}_i v^{(i)}).$$

On an edge E_{ij} of a substructure Ω_i , we employ the representation in (4.5). We recall that the function δ_j^\dagger is constant along an edge E_{ij} and δ_{ji}^\dagger is this value. We then decompose $v^{(i)}$ into contributions supported on single edges:

$$v^{(i)} = \sum_{j \in \mathcal{I}(i)} \theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - w^{(j)}), \quad (4.10)$$

where $\theta_{E_{ij}} \in W_i$ is identically one on E_{ij} and vanishes on $\partial\Omega_i \setminus E_{ij}$. We consider each contribution in this sum separately. Since, in addition, w is an element of \widetilde{W} , its average $\bar{w}_{E_{ij}}$ is the same whether it is calculated using $w^{(i)}$ or $w^{(j)}$. We can therefore write

$$\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - w^{(j)}) = \theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - \bar{w}_{E_{ij}}) - \theta_{E_{ij}} \delta_{ji}^\dagger (w^{(j)} - \bar{w}_{E_{ij}}). \quad (4.11)$$

We consider the two terms in (4.11) separately:

In order to bound the first, we employ the decomposition in Lemma 4.1 for the vector $\mathbf{u} := \mathcal{H}_i w^{(i)}$. We recall that the tangential component of $\mathbf{u}_{ij} = \nabla \phi_{ij}$ vanishes on $\partial\Omega_i \setminus E_{ij}$ and, thanks to (4.3), it is equal to $\theta_{E_{ij}} (w^{(i)} - w^{(j)})$. Using (3.8), the minimizing property of the Maxwell discrete harmonic extension in (3.1), (2.1), and (4.4), we find

$$\begin{aligned} |\theta_{E_{ij}} \delta_{ji}^\dagger (w^{(i)} - \bar{w}_{E_{ij}})|_{S^{(i)}}^2 &\leq \gamma_i \|\nabla \phi_{ij}\|_{L^2(\Omega_i)}^2 \\ &\leq C \gamma_i \omega^2 (\|\mathbf{u}\|_{L^2(\Omega_i)}^2 + H_i^2 \|\operatorname{curl} \mathbf{u}\|_{L^2(\Omega_i)}^2) \\ &\leq C \eta \omega^2 a_{\Omega_i} (\mathcal{H}_i w^{(i)}, \mathcal{H}_i w^{(i)}) = C \eta \omega^2 |w^{(i)}|_{S^{(i)}}^2. \end{aligned} \quad (4.12)$$

We then consider the second term in (4.11). The vector

$$\mathbf{u}^{(i)} := \mathcal{H}_i(\theta_{E_{ij}}(w^{(j)} - \bar{w}_{E_{ij}}))$$

can be decomposed according to Lemma 4.1, into the sum of two contributions $\mathbf{u}_{ij} = \nabla\phi_{ij}$ and \mathbf{u}^{int} . We next apply Lemma 4.1 to the function $\mathcal{H}_j w^{(j)}$ and obtain

$$\mathbf{u}^{(j)} := \mathcal{H}_j w^{(j)} = \mathbf{u}_H + \sum_{k \in \mathcal{I}(j)} \mathbf{u}_{jk} + \tilde{\mathbf{u}}^{int}.$$

We note that the functions $\mathbf{u}_{ji} = \nabla\phi_{ji}$ and $\mathbf{u}_{ij} = \nabla\phi_{ij}$ have the same tangential component along the common edge E_{ij} , which is equal to $\theta_{E_{ij}}(w^{(j)} - \bar{w}_{E_{ij}})$. Using (3.8), the minimizing property of the Maxwell discrete harmonic extension, Lemma 4.2, (2.1), and (4.4), we find

$$\begin{aligned} |\theta_{E_{ij}} \delta_{ji}^\dagger(w^{(j)} - \bar{w}_{E_{ij}})|_{S^{(i)}}^2 &\leq \gamma_j \|\nabla\phi_{ij}\|_{L^2(\Omega_i)}^2 \\ &\leq C\gamma_j \|\nabla\phi_{ji}\|_{L^2(\Omega_j)}^2 \\ &\leq C\gamma_j \omega^2 (\|\mathbf{u}^{(j)}\|_{L^2(\Omega_j)}^2 + H_j^2 \|\operatorname{curl} \mathbf{u}^{(j)}\|_{L^2(\Omega_j)}^2) \\ &\leq C\eta\omega^2 a_{\Omega_j}(\mathcal{H}_j w^{(j)}, \mathcal{H}_j w^{(j)}) = C\eta\omega^2 |w^{(j)}|_{S^{(j)}}^2. \end{aligned} \tag{4.13}$$

Combining (4.10), (4.12), and (4.13), and summing over the edges E_{ij} , we finally find

$$|v^{(i)}|_{S^{(i)}}^2 \leq C\eta\omega^2 |w^{(i)}|_{S^{(i)}}^2 + C\eta\omega^2 \sum_{j \in \mathcal{I}(i)} |w^{(j)}|_{S^{(j)}}^2.$$

The proof is then concluded by summing over the substructures Ω_i and using (4.9). \square

By combining Lemmas 4.6 and 4.5, and Theorem 4.4, we obtain our final result.

THEOREM 4.7. *The condition number of the preconditioned system $M^{-1}F$ satisfies*

$$\kappa(M^{-1}F) \leq C\eta(1 + \log(H/h))^2$$

for finite element approximations and

$$\kappa(M^{-1}F) \leq C\eta(1 + \log k)^2$$

for spectral element approximations. Here, η is defined in Lemma 4.6.

5. Proof of Lemma 4.1. As already mentioned, the proof of Lemma 4.1 for the case of finite elements is given in [24, Lemma 4.2]. In this section we provide a proof for the case of spectral elements. The proof follows that of [24, Lemma 4.2] and is given here for completeness. It employs suitable orthogonal decomposition of edge element functions into gradients of scalar functions and *discrete curl free* functions.

Let $X_i^0 \subset X^k(\Omega_i)$ be the subspace of vectors with vanishing tangential component of $\partial\Omega_i$. If $Q_i^0 \subset Q_i$ is the subspace of functions that vanish on $\partial\Omega_i$, then $\operatorname{grad} Q_i^0 \subset X_i^0$ and the following orthogonal decomposition is well defined

$$X_i^0 = \operatorname{grad} Q_i^0 \oplus X_i^{0,\perp}. \tag{5.1}$$

Proofs of the following fundamental result for a substructure of unit diameter can be found in [7, Th. 7.18] and in [13, Sect. 4]. The case of a subdomain of diameter H_i can

be treated by a scaling argument. We recall that its proof employs an interpolation operator on the edge element space.

LEMMA 5.1. *Let $\mathbf{u} \in X_i^{0,\perp}$. Then there is a constant, independent of H_i and k , such that*

$$\|\mathbf{u}\|_{L^2(\Omega_i)} \leq CH_i \|\operatorname{curl} \mathbf{u}\|_{L^2(\Omega_i)}.$$

We also need a decomposition result for scalar, spectral element functions. It is a classical result that is available in the literature in various forms. Since we did not find it in exactly the form that we need, we have included a proof which employs the tools in [1] in order to facilitate the reader.

LEMMA 5.2. *Let $\psi_H \in \mathbb{Q}_{1,1}(\Omega_i)$ and, for $j \in \mathcal{I}(i)$, $\psi_{ij} \in Q^k(\Omega_i)$ be a Laplace discrete harmonic function that vanishes on $\partial\Omega_i \setminus E_{ij}$. If*

$$\psi := \psi_H + \sum_{j \in \mathcal{I}(i)} \psi_{ij},$$

then

$$|\psi_{ij}|_{H^1(\Omega_i)}^2 \leq C(1 + \log k)^2 |\psi|_{H^1(\Omega_i)}^2,$$

with a constant that is independent of k and H_i .

Proof. We consider the case of a substructure of unit diameter. The more general case $H_i < 1$ can be treated by a scaling argument. The function ψ_{ij} belongs to $H_{00}^{1/2}(E_{ij})$, the subspace of $H^{1/2}(\partial\Omega)$ of functions that vanish on $\partial\Omega_i \setminus E_{ij}$; see, e.g., [1, Sect. 2] for the definition of these spaces and the corresponding norms. Using the stable extension in [1, Th. 7.5], we find

$$|\psi_{ij}|_{H^1(\Omega_i)}^2 \leq C \|\psi_{ij}\|_{H^{1/2}(\partial\Omega_i)}^2 \leq C \|\psi_{ij}\|_{H_{00}^{1/2}(E_{ij})}^2,$$

and, using [1, Th. 6.6],

$$\|\psi_{ij}\|_{H_{00}^{1/2}(E_{ij})}^2 \leq \|\psi_{ij}\|_{H^{1/2}(E_{ij})}^2 + C(1 + \log k) \|\psi_{ij}\|_{L^\infty(E_{ij})}^2.$$

Combining these two inequalities yields

$$|\psi_{ij}|_{H^1(\Omega_i)}^2 \leq C(1 + \log k) \|\psi - \psi_H\|_{L^\infty(E_{ij})}^2 + \|\psi - \psi_H\|_{H^{1/2}(E_{ij})}^2. \quad (5.2)$$

We note that ψ_H is the nodal interpolant of ψ on the linear space $\mathbb{Q}_{1,1}$ and therefore the inverse inequality in [1, Th. 6.2] can be employed. We obtain

$$(1 + \log k) \|\psi - \psi_H\|_{L^\infty(E_{ij})}^2 \leq C(1 + \log k)^2 \|\psi\|_{H^{1/2}(E_{ij})}^2 \quad (5.3)$$

and

$$\|\psi - \psi_H\|_{H^{1/2}(E_{ij})}^2 \leq C(1 + \log k) \|\psi\|_{H^{1/2}(E_{ij})}^2. \quad (5.4)$$

Combining (5.2), (5.3), (5.4), and a trace estimate, we find

$$|\psi_{ij}|_{H^1(\Omega_i)}^2 \leq C(1 + \log k)^2 \|\psi\|_{H^1(\Omega_i)}^2.$$

We note that if we add a constant to ψ , the left hand side does not change. A quotient type argument then allows to replace the full norm with the seminorm on the left hand side. \square

We recall that the coarse space $X^H(\Omega_i)$ was defined in (4.1). We now introduce the coarse interpolant

$$\rho_H : X^k(\Omega_i) \longrightarrow X^H(\Omega_i).$$

Here, $\rho_H \mathbf{u}$ is the unique vector that satisfies

$$\int_{E_{ij}} (\rho_H \mathbf{u} - \mathbf{u}) \cdot \mathbf{t}_{E_{ij}} ds = 0, \quad j \in \mathcal{I}(i). \quad (5.5)$$

We also define $X_{ij} \subset X^k(\Omega_i)$ as the space of functions $\nabla \phi_{ij}$, where $\phi_{ij} \in Q^k(\Omega_i)$ is Laplace discrete harmonic and vanishes on $\partial\Omega_i \setminus E_{ij}$.

We are now ready to give a proof of Lemma 4.1. It is immediate to see that, for the substructure Ω_i and for $j \in \mathcal{I}(i)$, $l \in \mathcal{I}(i)$, $j \neq l$,

$$X^H(\Omega_i) \cap X_i^0 = X^H(\Omega_i) \cap X_{ij} = X_{ij} \cap X_i^0 = X_{ij} \cap X_{il} = \{0\}.$$

Counting the degrees of freedom, we see that

$$X^k(\Omega_i) = X^H(\Omega_i) \oplus \sum_{j \in \mathcal{I}(i)} X_{ij} \oplus X_i^0 \quad (5.6)$$

is a direct sum. We have therefore proved the existence and the uniqueness of the decomposition (4.2).

The first equality in (4.3) is a consequence of the fact that the tangential component of \mathbf{u}_i and of \mathbf{u}_{il} , for $l \neq j$, vanishes on the edge E_{ij} . The second one comes from the fact that ϕ_{ij} vanishes at the end points of E_{ij} .

We are then left with the proof of the stability property (4.4). Since the decomposition is unique, thanks to (5.5), we find $\mathbf{u}_H = \rho_H \mathbf{u}$. We now decompose each term into a gradient of a scalar function and a remainder. Since the coarse space $X^H(\Omega_i)$ is $\mathcal{R}_k(\Omega_i)$, we can write

$$\mathbf{u}_H = \nabla \phi_H + \alpha \begin{bmatrix} y - y_i \\ x_i - x \end{bmatrix} =: \nabla \phi_H + \mathbf{u}_H^\perp,$$

with $\phi_H \in \mathbb{Q}_{1,1}$ is bilinear and (x_i, y_i) is the center of gravity of Ω_i . By direct calculation, we find that this is an L^2 orthogonal decomposition and that

$$\|\mathbf{u}_H^\perp\|_{L^2(\Omega_i)} \leq CH_i \|\text{curl } \mathbf{u}_H^\perp\|_{L^2(\Omega_i)}. \quad (5.7)$$

For the term $\mathbf{u}_i \subset X_i^0$, we employ the orthogonal decomposition in (5.1) and find

$$\mathbf{u}_i = \nabla \phi_i + \mathbf{u}_i^\perp.$$

Finally, by definition, $\mathbf{u}_{ij} = \nabla \phi_{ij}$, for each edge E_{ij} . We then group the gradient terms and the remainders and set

$$\phi := \phi_H + \sum_{j \in \mathcal{I}(i)} \phi_{ij} + \phi_i, \quad \mathbf{u}^\perp := \mathbf{u}_H^\perp + \mathbf{u}_i^\perp.$$

We have therefore the decomposition

$$\mathbf{u} = \nabla\phi + \mathbf{u}^\perp. \quad (5.8)$$

We need to bound the $\nabla\phi_{ij}$ in terms of \mathbf{u} . Since ϕ_H and the $\{\phi_{il}\}$ are Laplace discrete harmonic, we can apply Lemma 5.2 and find

$$|\phi_{ij}|_{H^1(\Omega_i)}^2 \leq C(1 + \log k)^2 \left| \phi_H + \sum_{l \in \mathcal{I}(i)} \phi_{il} \right|_{H^1(\Omega_i)}^2 \leq C(1 + \log k)^2 |\phi|_{H^1(\Omega_i)}^2. \quad (5.9)$$

For the last step we also have used that fact that ϕ_i vanishes on $\partial\Omega_i$ and is thus orthogonal to Laplace discrete harmonic functions.

The last step is to bound $\nabla\phi$ in terms of \mathbf{u} . We first note that, using (5.7) and Lemma 5.1, we obtain

$$\|\mathbf{u}^\perp\|_{L^2(\Omega_i)}^2 \leq CH_i^2 (\|\operatorname{curl} \mathbf{u}_H^\perp\|_{L^2(\Omega_i)}^2 + \|\operatorname{curl} \mathbf{u}_i^\perp\|_{L^2(\Omega_i)}^2).$$

Since $\operatorname{curl} \mathbf{u}_H^\perp$ is constant and $\operatorname{curl} \mathbf{u}_i^\perp$ has a vanishing mean value on Ω_i , these two functions are L^2 orthogonal and thus

$$\|\mathbf{u}^\perp\|_{L^2(\Omega_i)}^2 \leq C_\perp H_i^2 \|\operatorname{curl} \mathbf{u}^\perp\|_{L^2(\Omega_i)}^2. \quad (5.10)$$

Using (5.8), (5.10), and the Young's inequality, we find

$$\begin{aligned} \|\mathbf{u}\|_{\operatorname{curl}, \Omega_i}^2 &= |\phi|_{H^1(\Omega_i)}^2 + \|\mathbf{u}^\perp\|_{\operatorname{curl}, \Omega_i}^2 + 2 \int_{\Omega_i} \nabla\phi \cdot \mathbf{u}^\perp dx \\ &\geq (1 - \epsilon) |\phi|_{H^1(\Omega_i)}^2 + (1 + (1 - \epsilon^{-1}) C_\perp H_i^2) \|\operatorname{curl} \mathbf{u}^\perp\|_{L^2(\Omega_i)}^2, \end{aligned} \quad (5.11)$$

for $\epsilon \in (0, 1)$. The choice $\epsilon = C_\perp H_i^2 / (C_\perp H_i^2 + 1)$ ensures

$$|\phi|_{H^1(\Omega_i)}^2 \leq C \|\mathbf{u}\|_{\operatorname{curl}, \Omega_i}^2,$$

which, combined with (5.9), concludes the proof.

6. Implementation aspects. In this section, we describe how we can efficiently implement the preconditioned algorithm described in this paper. Indeed, we need to construct the matrix F and the vector d (see (3.5) and (3.6)) and the preconditioner M^{-1} in (3.9).

In principle, a change of basis should be performed and the degrees of freedom of \tilde{X} partitioned into I (interior to the substructures), Π (common averages along the subdomain edges), and Δ . However, such change of basis is not trivial or advisable. Since the basis functions associated to the Δ block are not local in general, this would spoil the sparsity of certain matrices. In practice we will work with full vectors in the original product space consisting of all the degrees of freedom on Γ , satisfying no continuity constraint. We will then make sure that these degrees of freedom belong to the dual space \widetilde{W}_Δ , i.e., the averages along all the subdomain edges vanish. For this reason, as already pointed out in section 3, the matrix $B_\Delta = B$ is the same as that of the one level FETI method in [20]: it is constructed from $\{0, 1, -1\}$ and evaluates the difference between all the corresponding tangential degrees of freedom on Γ .

We then consider an initial vector of Lagrange multipliers λ_0 . We note that since we work with the matrix B which acts on the whole space W , in order to ensure that $\lambda_0 \in V = \operatorname{Range}(B_\Delta)$, we need to choose $\lambda_0 = Bu_0$, with $u_0 \in \widetilde{W}$.

We work with the matrix $K : X \rightarrow X$ which acts on the product space and is block diagonal; each block $K^{(i)}$ corresponds to a substructure Ω_i and is the representation of the local bilinear form $a_{\Omega_i}(\cdot, \cdot)$. We also work with global vectors $\mathbf{u} \in X$ and the load vector, still denoted by \mathbf{f} , which represents the linear functional

$$\int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, dx, \quad \mathbf{w} \in X. \quad (6.1)$$

In addition, if $w \in W$ is a vector of degrees of freedom on Γ , let

$$\tilde{R}^T : W \rightarrow X$$

be the extension by zero from Γ into the whole of Ω .

We can then write the system for the solution $\mathbf{u} \in X$ as

$$\begin{aligned} K\mathbf{u} + C^T\mu + (B\tilde{R})^T\lambda &= \mathbf{f} \\ C\mathbf{u} &= 0 \\ (B\tilde{R})\mathbf{u} &= 0 \end{aligned} \quad (6.2)$$

Here $C\mathbf{u} = 0$ imposes the constraint to a vector $\mathbf{u} \in X$ that it have vanishing averages along the subdomain edges and μ is a vector of Lagrange multipliers associated to these constraints. We note that the last condition imposes then redundant constraints. An equation for λ is obtained by eliminating \mathbf{u} and μ . We obtain

$$\mathbf{u} = K^{-1}(I - C^T(CK^{-1}C^T)^{-1}CK^{-1})(\mathbf{f} - (B\tilde{R})^T\lambda) =: H(\mathbf{f} - (B\tilde{R})^T\lambda)$$

and thus

$$B\tilde{R}H\tilde{R}^TB^T\lambda = B\tilde{R}H\mathbf{f}.$$

We finally find

$$\begin{aligned} F &= B\tilde{R}H\tilde{R}^TB^T \\ d &= B\tilde{R}H\mathbf{f}. \end{aligned}$$

We note that $\tilde{R}H\tilde{R}^T$ gives an expression for \tilde{S}^{-1} . In addition, the application of H to a vector requires *two* applications of K^{-1} (and then the solution of two Neumann problems on each substructure) and *one* application of $(CK^{-1}C^T)^{-1}$. If we partition

$$C = [C^{(1)} \ C^{(2)} \ \dots \ C^{(N)}],$$

with each block corresponding to a substructure, we can write

$$F_0 := CK^{-1}C^T = \sum_{i=1}^N C^{(i)}K^{(i)-1}C^{(i)T}.$$

Since the number of constraints (and thus of non zeros columns in $C^{(i)T}$) is equal to the number of edges of Ω_i , we need to apply $K^{(i)-1}$ only to these nonzero columns in order to calculate F_0 . The matrix F_0 is then factored once and for all and its inversion provides a coarse problem, the size of which is equal to the number of edges of the subdomain partition. Finally, by construction, the operator H always returns a vector in the kernel of C , which therefore has vanishing mean value on the subdomain

TABLE 7.1

FETI-DP method. Estimated condition number and number of CG iterations necessary to obtain a relative preconditioned residual less than 10^{-12} (in parentheses), versus H/h and n . Case of $a = 1$, $b = 1$. The asterisks denote the cases for which we had not enough memory to run the corresponding algorithm.

H/h	32	16	8	4	2
n=32	-	1.529 (5)	2.212 (11)	1.777 (11)	1.309 (8)
n=64	1.801 (6)	2.950 (12)	2.446 (13)	1.806 (10)	1.312 (7)
n=128	3.827 (13)	3.278 (15)	2.484 (12)	1.819 (10)	1.314 (7)
n=192	4.154 (17)	3.329 (15)	2.496 (12)	1.816 (9)	*
n=256	4.265 (17)	3.337 (14)	2.500 (12)	*	*

edges. We note that, in case the subdomain partition coincides with a coarse mesh the coarse matrix F_0 has the same size and stencil as the coarse one for the Balancing NN method in [18], for which coarse degrees of freedom are also associated to the edges of a coarse mesh.

Concerning the preconditioner M^{-1} in (3.9), the local Schur complements $S^{(i)}$ are the same as those employed for the one level FETI method in [20] and are obtained from the local stiffness matrices in the standard way; see, e.g., [18, Eq. 3.3].

7. Numerical Results. We consider the same mesh, partitions, and coefficient distribution as in [20, Sect. 6] in order to allow a comparison with one level FETI. The domain $\Omega := (0, 1)^2$ is partitioned into two uniform meshes \mathcal{T}_h and \mathcal{T}_H . The fine triangulation is made of triangles, and the coarse one of squares that are unions of fine triangles. The substructures Ω_i are the elements of the coarse triangulation \mathcal{T}_H . The fine triangulation \mathcal{T}_h consists of $2 * n^2$ triangles, with $h = 1/n$. We choose

$$A = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}, \quad \mathbf{f} = (\exp(-x/3 + y^2), \quad -3 \cos(2x - 5y - 10))^T,$$

and use the value $\chi = 1/2$ for the definition of the scaling matrices $D_{\Delta}^{(i)}$; see (3.7). We consider a conjugate gradient (CG) algorithm and estimate the condition number of the preconditioned operator using the quantities provided by CG. Since, however, convergence is much faster here we employ a more restrictive stopping criterion than in [20] in order to obtain good condition number estimates: we stop the iteration when $\|z_k\|/\|f\|$ is less than 10^{-12} , instead of 10^{-6} . Here, z_k is the k -th preconditioned residual $M^{-1}(d - F\lambda_k)$. The estimated condition numbers here can then be compared with those in [20, Sect. 6], while in order to compare the iteration counts we need to consider the double of those in [20].

In Table 7.1, we show the estimated condition number and the number of iterations as functions of the dimensions of the fine and coarse meshes, for $a = b = 1$. For a fixed ratio H/h , the condition number and the number of iterations are quite insensitive to the dimension of the fine mesh and are consistent with a quadratic logarithmic growth; see Theorem 4.7. The condition numbers here can be compared with those in [20, Table 1]. Those for the FETI-DP method are generally slightly smaller than those for the one level method. This is related to the fact that a coarse problem of larger size is solved here. For the uniform partition into square substructures, we have one coarse function for each substructure for one level FETI and two for FETI-DP (four degrees of freedom for the four edges of each subdomain, shared by two substructures).

TABLE 7.2

FETI-DP method. Checkerboard distribution for $b: (b_1, b_2)$. Estimated condition number and number of CG iterations to obtain a relative preconditioned residual less than 10^{-12} (in parentheses), versus H/h and b_2 . Case of $n = 128$, $a = 1$, and $b_1 = 100$.

H/h	4	8	16
$b_2 = 1e-4$	3.777 (21)	5.395 (28)	7.633 (32)
$b_2 = 1e-3$	3.760 (20)	5.382 (27)	7.606 (30)
$b_2 = 1e-2$	3.713 (20)	5.308 (25)	7.504 (29)
$b_2 = 1e-1$	3.561 (18)	5.089 (23)	7.196 (27)
$b_2 = 1$	3.155 (16)	4.502 (20)	6.364 (25)
$b_2 = 1e+1$	2.355 (13)	3.338 (17)	4.692 (20)
$b_2 = 1e+2$	1.800 (10)	2.436 (13)	3.068 (15)
$b_2 = 1e+3$	2.298 (13)	3.059 (15)	3.798 (17)
$b_2 = 1e+4$	2.612 (14)	3.036 (16)	3.435 (17)
$b_2 = 1e+5$	2.203 (12)	2.630 (14)	2.918 (15)
$b_2 = 1e+6$	2.085 (12)	2.593 (13)	2.820 (14)

On the other hand, comparison of the iteration counts shows a faster convergence for the FETI-DP algorithm. This is related to the smaller condition number and also to the fact that there is basically no freedom for the initial guess of one level FETI methods (cf., e.g., the algorithm in [16, Pg. 100]): this may often give a quite high initial residual. An arbitrary initial guess can be employed for FETI-DP and the null vector employed here provides relatively small initial residual for our tests.

In Table 7.2, we show some results when the coefficient b has jumps across the interface. We consider a 4×4 checkerboard distribution, where b assumes two values, b_1 and b_2 . For a fixed value of $n = 128$, $b_1 = 100$, and $a = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and b_2 . A similar behaviour as in [20, Table 2] is observed here. For $b_2 = 100$, the coefficient b has a uniform distribution, and this corresponds to a local minimum for the condition number and the number of iterations. When b_2 decreases or increases, the condition number and the number of iterations normally increase, but they can still be bounded independently of b_2 . We note however that for some very large values of b_2 convergence may be faster than in the uniform case. We also remark that, when b_2 is large, the local ratio b_2/a is also large; see η in Theorem 4.7. In this case however our results remain good and the condition number even appears to be less sensitive to H/h . We remark that condition numbers and iteration counts are smaller than the corresponding ones in [20, Table 2] for one level FETI.

In Table 7.3, we show some results when the coefficient a has jumps. We consider the same 4×4 checkerboard distribution shown as for the previous tests. For a fixed value of $n = 128$, $a_1 = 0.01$, and $b = 1$, the estimated condition number and the number of iterations are shown as a function of H/h and a_2 . For $a_2 = 0.01$, the coefficient a has a uniform distribution. A slight increase in the number of iterations and the condition number may be observed for some larger or smaller values of a_2 and when H/h is large. As for the previous table, when a_2 is small, the local ratio b/a_2 is large and our results remain good. The condition numbers and the iteration counts are smaller than the corresponding ones in [20, Table 3].

TABLE 7.3

Checkerboard distribution for a : (a_1, a_2) . Estimated condition number and number of CG iterations to obtain a relative preconditioned residual less than 10^{-12} (in parentheses), versus H/h and a_2 . Case of $n = 128$, $b = 1$, and $a_1 = 0.01$.

H/h	4	8	16
a2=1.e-7	2.668 (15)	4.342 (20)	7.097 (26)
a2=1.e-6	2.285 (14)	3.665 (19)	6.024 (25)
a2=1.e-5	1.769 (12)	2.418 (16)	3.869 (21)
a2=1.e-4	1.764 (12)	2.294 (15)	2.814 (17)
a2=1.e-3	1.791 (12)	2.353 (15)	2.814 (17)
a2=1.e-2	1.813 (13)	2.447 (16)	3.071 (18)
a2=1.e-1	1.816 (12)	2.467 (15)	3.173 (18)
a2=1	1.808 (10)	2.466 (14)	3.182 (16)
a2=1.e+1	1.801 (9)	2.454 (12)	3.172 (14)
a2=1.e+2	1.791 (8)	2.438 (10)	3.164 (12)
a2=1.e+3	1.771 (7)	2.427 (9)	3.159 (11)

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