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‡RiskLab and Seminar for Applied Mathematics

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Abstract

The price of an American style contract on assets driven by a class of Markov processes containing, in particular, Lévy processes of pure jump type with infinite jump activity is expressed as solution of a parabolic variational integro-differential inequality (PIDI). A Galerkin discretization in logarithmic price using a wavelet basis is presented with compression of the moment matrix of the jump part of the price process' Dynkin operator. Iterative solution with wavelet preconditioning for the large matrix LCPs at each time-step is proposed and its efficiency demonstrated by numerical experiments for various jump-diffusion and pure jump models. Failure of the smooth pasting principle is observed for American put contracts on various pure jump price processes.

Keywords: Lévy processes, integro-differential operators, variational inequalities, Galerkin discretization, bi-orthogonal wavelet basis, wavelet preconditioning

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1 Introduction

In asset pricing, models beyond the classical Black-Scholes (B-S) have been proposed for the stochastic dynamics of the risky asset: we mention only stochastic volatility models and 'stochastic clocks'. The former lead to multivariate generalizations of the B-S equation with stochastic volatility (e.g. [25]) whereas the latter lead to so-called jump-diffusion price processes: the Wiener process in the B-S model is replaced by a Lévy process (see e.g. [38, 3, 21, 37, 15, 10, 11] and [7, 45] and the references there for background information on Lévy processes). Such processes allow more flexible price dynamics than B-S models (see e.g. [21]). The inclusion of jumps into the asset price dynamics has been investigated for several years – let us mention here only [2] and the references there.

Originally, jumps in risky assets' log-returns have been modelled as finite intensity processes, i.e in any finite time interval only a finite number of large jumps occur (e.g. [40, 41]). In the early 90ies, however, processes with infinite jump intensity and no diffusion component have been proposed as models for the log-returns. We mention here the Variance Gamma (VG) [38, 37], the extended Koponen family [11] (also referred to as KoBoL, temperated or truncated tempered stable processes or truncated Lévy flights in physics and later used in [15] under the name of CGMY-model), the Normal Inverse Gaussian process [4] and the Hyperbolic processes [22, 23]. All these processes, together with the B-S model, are Markov processes of Lévy type, or Lévy processes for short. Since their introduction, empirical evidence for their superiority over B-S in modelling observed returns has been gathered (e.g. [23, 15]).

For pricing European Vanilla contracts on assets with Lévy price processes, the translation invariance of the process' infinitesimal generator implied by stationarity and explicitly available characteristic functions allow to apply Fourier-Laplace transformations for the numerical pricing (e.g. [16]).

For American style contracts on assets with Lévy price processes, the analytical tool of Wiener-Hopf factorization allows, at least for infinite horizon problems, to derive semi-analytical solutions [12]. In the finite horizon case, even in the B-S setting, explicit analytical pricing formulas are not available. Using Carr's randomization method [14] which ends up with the same algorithm as in the analytical method of lines, one can discretize the time period $[0, T]$ into subperiods of length $\Delta t \to 0$ and approximations of the exercise boundary and of the rational price can be derived analogously to the perpetual case by backwards induction and by using the Wiener-Hopf factorization [13, Chapter 5]. Exact Wiener-Hopf factors are, except for some particular cases, not known explicitly and are in general difficult to compute. Approximate Wiener-Hopf factors for a large class of Lévy processes have been derived in [13]. These approximate factors are of the same form as the factors in the Gaussian case where closed form expressions for the exercise boundary and for the rational price had already been derived by Merton in 1973 [40]. These approximate, analytical methods, however, are not directly applicable to time-dependent or local volatility models where stationarity and, hence, translation invariance are absent.

The purpose of the present paper is the analysis and implementation of fast, convergent deterministic pricing schemes for American style contracts on assets driven by a class of Markov processes which contains, in particular, Lévy processes. Our approach is based on a multilevel finite element solution of the parabolic variational inequality formally associated with the optimal stopping problem for these processes. This inequality involves the Dynkin operator of the semigroup generated by the price process which, for the processes under consideration, is an integro-differential operator with nonintegrable kernel stemming from the process' jump measure. For infinite jump intensity, non-integrable, hypersingular integrals arise which must be interpreted in the sense of distributions. We account for this with a variational framework that regularizes non-integrable kernels corresponding to price processes of infinite jump intensity.

We discretize the variational integro-differential inequality by 'Canadization', or backward Euler, in time and by a piecewise linear, continuous wavelet Finite Element basis in the (logarithmic) price variable. This basis has two advantages: i) it allows to 'compress' the dense and ill-conditioned moment matrices due to the nonlocal infinitesimal generator of the process to sparse, well-conditioned ones while not affecting the accuracy of the computed prices and ii) the wavelet basis allows to precondition the iterative solver for the associated Linear Complementarity Problems (LCPs) in each time step. As we show, this preconditioning works for vanishing diffusion component as well as for all jump intensities Y between 0 and 2.
The resulting algorithm allows the deterministic pricing of American style contracts on assets for which the log price process is a general Markovian jump-diffusion price process that may exhibit infinite jump activity and has possibly nonstationary increments as, e.g., in local volatility models. It moreover allows for general, non-monotonic and non-smooth pay-off functions, in particular for digital and compound options with an American style early exercise feature. As we show, it is convergent, i.e. as in the simplest Finite Difference schemes for the B-S equations, the computed prices will, upon mesh refinement, approach the exact prices determined by the model.

As is well known, e.g [13, Chapter 6.3], the smooth pasting condition for the price of an American Put in the B-S setting may not hold for a pure jump process. Our approach does not impose any pasting condition a-priori and, indeed, our numerical experiments demonstrate failure of the smooth pasting condition for certain pure jump processes.

The outline of the paper is as follows: in Section 2, we present the admissible price processes. Section 3 contains the formulation of the American style pricing problem and the derivation of the price as solution of a parabolic integro-differential inequality. Section 4 discusses the discretization of the inequality – here, we show that the moment matrix due to the nonlocal part of the parabolic integrodifferential operator can be compressed to an approximate, sparse matrix while still having bounded condition number. Some a-priori error bounds for the numerical solution are also stated and we have included in the Appendix B a convergence proof which justifies our method and shows that the computed prices converge, as $h, k \to 0$, to the exact prices delivered by the model. In Section 5, we address the numerical solution of the large linear complementarity problems in each time-step. We develop a fixed point iteration in wavelet basis and prove that it converges at a rate independent of the discretization parameter. We also give a generalization of Cryer's algorithm in the wavelet basis to locate the exercise boundary. Section 6 contains numerical results obtained with our approach. They indicate in particular the failure of the smooth fit principle for certain pure jump, bounded variation price processes.

2 Price Processes

2.1 Lévy processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t<\infty}, \mathbf{P})$ be a filtered probability space satisfying the usual hypothesis. Let $X = (X_t)_{0\le t<\infty}$ with $X_0 = 0$ a.s. be a Lévy process, i.e., a process with stationary, independent increments that is stochastically continuous [7, 45].

The Lévy-Khintchine formula describes explicitly a Lévy process in terms of its Fourier transform $E_{\mathbb{Q}}[e^{-iuX_t}]$ under a chosen equivalent martingale measure $\mathbb{Q}$:

$$E_{\mathbb{Q}}[e^{-iuX_t}] = e^{-t\psi(u)} \tag{2.1}$$

for some function ψ called the Lévy exponent of X. It has the following representation

$$\psi(u) = \frac{\sigma^2}{2}u^2 + i\alpha u + \int_{|x|<1}(1-e^{-iux}-iux)\nu_{\mathbb{Q}}(\mathrm{d}x) + \int_{|x|\geq 1}(1-e^{-iux})\nu_{\mathbb{Q}}(\mathrm{d}x) \tag{2.2}$$

for some $\sigma, \alpha \in \mathbb{R}$ and for a measure $\nu_{\mathbb{Q}}$ on $\mathbb{R}\backslash\{0\}$ satisfying

$$\int_{\mathbb{R}} \min(1, x^2)\nu_{\mathbb{Q}}(\mathrm{d}x) < \infty. \tag{2.3}$$

The Lévy measure $\nu_{\mathbb{Q}}(\mathrm{d}x)$ measures the arrival rate of jumps of size x. The Lévy-triple $(\sigma, \alpha, \nu_{\mathbb{Q}})$ completely determines X_t and the characteristic exponent ψ is related to the symbol of the nonlocal operator $\mathcal{L} = \mathcal{L}_X^{\mathbb{Q}}$ which is the infinitesimal generator of the transition semigroup of X_t under the chosen equivalent martingale measure $\mathbb{Q}$ [45, 7]

$$\mathcal{L}[\varphi](x) := -\frac{\sigma^2}{2}\varphi''(x) + \left(\frac{\sigma^2}{2} - r\right)\varphi'(x) + \mathcal{A}_{\text{jump}}[\varphi](x), \tag{2.4}$$

with the integral operator $\mathcal{A}_{\text{jump}}$ given by

$$\mathcal{A}_{\text{jump}}[\varphi](x) = -\int_{\mathbb{R}} \left(\varphi(x+y) - \varphi(x) - (e^y - 1)\varphi'(x)\right)\nu_{\mathbb{Q}}(\mathrm{d}y). \tag{2.5}$$

We assume here that the equivalent martingale measure $\mathbb{Q}$ has been chosen by some procedure, we refer to [19, 20, 24, 17] and the references therein for various results in this direction. Let $\mu(\mathrm{d}x, \mathrm{d}t)$ denote the integer valued random measure (the *jump measure*) that counts the number of jumps of X_t in space-time. By stationarity of Lévy processes, the compensator of the measure $\mu(\mathrm{d}x, \mathrm{d}t)$ has the form $\nu_{\mathbb{Q}}(\mathrm{d}x) \times \mathrm{d}t$, with $\mathrm{d}t$ being the Lebesgue measure. In the following we will assume that the Lévy measure $\nu_{\mathbb{Q}}(\mathrm{d}x)$ has a density $k_{\mathbb{Q}}$, the *Lévy kernel*, with respect to Lebesgue measure so that $\nu_{\mathbb{Q}}(\mathrm{d}x) = k_{\mathbb{Q}}(x)\mathrm{d}x$ and we will drop the subscript $\mathbb{Q}$. The Lévy density $k(\cdot)$ describes the activity of jumps in the sense that jumps of sizes in the set A occur according to a Poisson process with parameter $\int_A k(x)dx$. In our analysis and numerical treatment we need the following assumptions on the Lévy density k:

(A1) the characteristic function ψ_0 of the pure jump part of the Lévy process satisfies: there exist constants c, $C_+ > 0$ and $Y < 2$ such that

$$|\psi_0(\xi) - ic\xi| \leq C_+(1 + |\xi|^2)^{Y/2} \quad \forall \xi \in \mathbb{R}. \tag{2.6}$$

(A2) (exponential decay) there are constants $C > 0$, $G > 0$ and $M > 1$ such that

$$\forall\, |x| > 1 : \quad k(x) \leq C \begin{cases} e^{-G|x|} & \text{if } x < 0, \\ e^{-M|x|} & \text{if } x > 0. \end{cases} \tag{2.7}$$

(A3) (smoothness)

$$\forall\, \alpha \in \mathbb{N}_0\ \exists\, C(\alpha) > 0 : \quad \forall\, x \neq 0\ :\ |k^{(\alpha)}(x)| \leq C(\alpha)|x|^{-(1+Y+\alpha)_+}. \tag{2.8}$$

If $\sigma = 0$ we assume $0 \leq Y < 2$ and in addition

(A4) (boundedness from below): there exists $C_- > 0$ such that

$$\forall\, 0 < |z| < 1 : \quad k(z) + k(-z) \geq C_- \frac{1}{|z|^{1+Y}}. \tag{2.9}$$

Remark 2.1 Assumption (A3) is required in the analysis of the wavelet compression of the moment matrix of $k(x)$; however, it only needs to be satisfied for a finite range of α. If the Lévy process is of finite activity we assume $\sigma > 0$ and that $k(x)$ satisfies (A1)–(A3) with $Y < 0$.

2.2 Markov-Processes

The numerical method developed below applies also when the log-price process X_t is a Markov process with increments $X_t - X_s$ that are no longer independent of X_s for $s < t$ [39]. Instead of the translation invariant infinitesimal generator $\mathcal{L}$ one can consider a more general operator

$$\mathcal{L}[\varphi](x) := -\frac{\sigma(x)^2}{2}\varphi''(x) + \left(\frac{\sigma(x)^2}{2} - r\right)\varphi'(x) + \mathcal{A}_{\text{jump}}[\varphi](x), \tag{2.10}$$

with

$$\mathcal{A}_{\text{jump}}[\varphi](x) = -\int_{\mathbb{R}} \left(\varphi(x+y) - \varphi(x) - (e^y - 1)\varphi'(x)\right) k(x,y)\mathrm{d}y.$$

Unless explicitly stated otherwise, we assume $0 < \sigma_0 \leq \sigma(x) \leq \sigma_1$ and that $k(x,y)$ satisfies the assumptions (A1)–(A2) uniformly with respect to x and, in place of (A3), the following Calderón-Zygmund estimates: for all $\beta, \gamma \in \mathbb{N}_0$, there holds

$$|\partial_x^\gamma \partial_y^\beta k(x, x-y)| \leq C(\gamma, \beta)|x-y|^{-(1+Y+\beta+\gamma)}. \tag{2.11}$$

In the case $\sigma = 0$, we require (A4) for $k(x,z)$ uniformly with respect to x.

2.3 Examples

Practically all price processes used in Lévy market models have densities which satisfy (A1)–(A3). Let us mention here a few of them.

In the classical Merton model [41] it is assumed that $X_t = \sigma W_t + \sum_{i=1}^{N_t} Y_i$ where $\{Y_i\}_i$ are i.i.d. with normal distribution function f_M with mean μ_M and variance σ_M and N_t being a Poisson process of intensity λ. Merton's jump-diffusion process is of finite activity with Lévy density $k(x) = \lambda f_M(x)$, $f_M(x) = (\sqrt{2\pi}\sigma_M)^{-1}\exp\left(-(x-\mu_M)^2/(2\sigma_M^2)\right)$ satisfying (A1)–(A3) with $Y = -\infty$. Another finite activity Lévy process has been proposed by Kou [34] with $f_{Kou}(x) = p_+ M \exp(-Mx)\chi_{\{x>0\}} + p_- G \exp(Gx)\chi_{\{x<0\}}$, $p_+ + p_- = 1$ satisfying (A1)–(A3) for $Y = -1$.

We also mention here the extended Koponen family [11] (also referred to as KoBoL, tempered or truncated tempered stable processes or truncated Lévy flights in physics and later used in [15] under the name of CGMY-model). The CGMY process can have both finite or infinite activity and finite or infinite variation. Specifically, the Lévy density of the CGMY process is given by

$$k_{CGMY}(x) = C \begin{cases} \dfrac{e^{-G|x|}}{|x|^{1+Y}} & \text{if} \quad x < 0 \\ \dfrac{e^{-M|x|}}{|x|^{1+Y}} & \text{if} \quad x > 0, \end{cases} \tag{2.12}$$

where $C > 0$, $G, M > 0$ and $Y < 2$. The case $Y = 0$ is the special case of the variance gamma process [38, 37]. The parameter C is related to the overall level of activity, G and M control the exponential rate of decay at $\mp\infty$ of the Lévy density and lead to skewed jump distributions if they are unequal. If $G = M$ one obtains Lévy processes from the Koponen family [32]. Empirical evidence however indicates that the probability density functions of returns are almost symmetric at the origin but that the left tails are fatter than the right ones, accounting for different distribution of large negative resp. positive price jumps which can be aacounted for in (2.12) by $G < M$, The parameter Y is related to the jump intensity of X_t. The density (2.12) satisfies (A1)–(A4). The characteristic function of the CGMY process is available in closed form [13, 15].

Further examples are the Normal Inverse Gaussian process ($Y = 0$) [4] and the generalized hyperbolic motion [22, 23] which satisfies the assumptions with $Y = 1$.

3 American Option Pricing

Our purpose is the valuation of American-style options, i.e., options that can be exercised at any time up to the expiration date T, on an underlying with price process X_t. Most listed stock options, including those on European exchanges, are American-style options. The early exercise feature makes their valuation more complex than that of European-style options in a Black-Scholes world. It also implies higher prices for an American-style option than for a European contract with the same price process. We formulate the pricing problem of an American-style contract as optimal stopping problem for X_t and express prices as solutions of parabolic variational integro-differential inequalities. To accomodate general pay-off functions which may grow polynomially at infinity, these inequalities are set in Sobolev spaces with exponential weights. Their numerical solution is prepared by a localization of admissible log-returns from $(-\infty, \infty)$ to a finite domain $\Omega_R = (-R, R)$ with sufficiently large $R > 0$ which is the basis for numerical solution methods in the next sections.

3.1 Optimal stopping problem

Consider the price $f(S_t, t)$ of an American option with expiry date (maturity) T when the risk-neutral dynamics of the risky asset S_t are given by

$$S_t = S_0 e^{(r+c-\sigma^2/2)t+X_t}. \tag{3.1}$$

Here X_t is a Lévy process of the form $X_t = \sigma W_t + Y_t$, with W_t denoting the Brownian motion and Y_t being a quadratic pure jump Lévy process independent of W_t as in Section 2. The parameter c in (3.1) is determined so that the mean rate of return on the asset is risk-neutrally r, i.e. that $e^{-ct} = E_{\mathbb{Q}}[e^{Y_t}]$. Let $g(S)$ denote a pay-off function of the option (conditions on g shall be discussed below). The problem of optimal exercising is equivalent to an optimal stopping problem for S_t and the value $f(S_t, t)$ of the contract is given by

$$f(S_t, t) = \sup_{t \le \tau \le T} E_{\mathbb{Q}}[e^{-r(\tau-t)} g(S_\tau)|\mathcal{F}_t], \tag{3.2}$$

where the supremum is taken over all stopping times τ on the probability space generated by the asset price process. Equation (3.2) means that the owner chooses the optimal exercise policy to maximize the expected discounted pay-off.

Remark 3.1 For the American put $g(S) = (K - S)_+$, with $K > 0$ being the strike price, and for each t there exists a critical value S_t^* such that for all $S_t \le S_t^*$ the value of the American put option is the value of immediate exercise, i.e., $f(t, S_t) = g(S_t)$, while for $S_t > S_t^*$ the value exceeds the immediate exercise value. The curve S_t^* is referred to as the critical exercise boundary, the region $\mathcal{C} = \{(t, S)|S > S_t^*\}$ is called the continuation region and the complement $\mathcal{E}$ of $\mathcal{C}$ is the exercise region. For the detailed study of the free boundary problem for the American put in the B-S case, see e.g. [47], Karatzas and Shreve (1998) [31], Musiela and Rutkowski (1997) [42].

Remark 3.2 By (3.1), (2.1)–(2.2) and by $E_{\mathbb{Q}}[S_t] < \infty$ we obtain that $E_{\mathbb{Q}}[e^{X_t}] = e^{-t\psi(i)} < \infty$, with ψ being the Lévy exponent in (2.2). As a consequence, the Lévy density k has to satisfy both the integrability condition (2.3) and $\int_{|x|\ge 1} e^x k(x) \mathrm{d}x < \infty$. For the CGMY-model (2.12) these integrability conditions imply $Y < 2$ and $M > 1$.

3.2 Parabolic integro-differential inequality

For sufficiently regular g and for $\sigma \neq 0$ the price f in (3.2) is known to satisfy the following parabolic integro-differential inequality (see, e.g. [6])

$$
\begin{aligned}
&\frac{\partial f}{\partial t}(t,S) + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2}(t,S) + rS\frac{\partial f}{\partial S}(t,S) - rf(t,S)\\
&\quad + \int_{\mathbb{R}}[f(t,Se^x) - f(t,S) - S\frac{\partial f}{\partial S}(t,S)(e^x-1)]\nu(\mathrm{d}x) \leq 0\\
&f(t,S) \geq g(S)\\
&(f(t,S)-g(S))\Big(\frac{\partial f}{\partial t}(t,S) + \frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2}(t,S) + rS\frac{\partial f}{\partial S}(t,S) - rf(t,S)\\
&\quad + \int_{\mathbb{R}}[f(t,Se^x) - f(t,S) - S\frac{\partial f}{\partial S}(t,S)(e^x-1)]\nu(\mathrm{d}x)\Big) = 0\\
&f(T,S) = g(S).
\end{aligned}
\tag{3.3}
$$

We also mention [12] for theoretical results on the relation between the solution of the free boundary value problem and that of the optimal stopping for the case of the perpetual American put, i.e., when $T = \infty$.

For numerical treatment, we change to logarithmic price $x = \ln(S) \in \mathbb{R}$ and time to maturity $\tau = T - t$ and introduce $u(\tau,x) = f(T-\tau, e^x)$. If we denote by $\psi(x) = g(e^x)$, the resulting parabolic integro-differential inequality for the value function u reads

$$\frac{\partial u}{\partial \tau} + \mathcal{A}_{\mathrm{B-S}}[u] + \mathcal{A}_{\mathrm{jump}}[u] \geq 0 \quad \text{in } (0,T)\times\mathbb{R} \tag{3.4}$$

$$u(\tau,x) \geq \psi(x) \quad \text{a.e. in } [0,T]\times\mathbb{R} \tag{3.5}$$

$$(u(\tau,x)-\psi(x))\Big(\frac{\partial u}{\partial \tau} + \mathcal{A}_{\mathrm{B-S}}u + \mathcal{A}_{\mathrm{jump}}[u]\Big) = 0 \quad \text{in } (0,T)\times\mathbb{R} \tag{3.6}$$

$$u(0,x) = \psi(x), \tag{3.7}$$

where the infinitesimal generator (or Dynkin operator) of the transition semi-group of X_t is given by

$$\mathcal{A} = \mathcal{A}_{\mathrm{B-S}} + \mathcal{A}_{\mathrm{jump}} \tag{3.8}$$

with

$$
\begin{aligned}
\mathcal{A}_{\mathrm{B-S}}[\varphi] &= -\frac{\sigma^2}{2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + (\frac{\sigma^2}{2} - r)\frac{\mathrm{d}\varphi}{\mathrm{d}x} + r\varphi,\\
\mathcal{A}_{\mathrm{jump}}[\varphi] &= -\int_{\mathbb{R}}\Big(\varphi(x+y) - \varphi(x) - (e^y-1)\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\Big)k(y)\mathrm{d}y.
\end{aligned}
$$

Remark 3.3 Unless explicitly stated otherwise, we assume in the following that the price process has a non-zero diffusion component, i.e. $\sigma \neq 0$.

3.3 Variational formulation

Of particular interest will be American Put contracts where the pay-off is $g(S) = (K-S)_+$. The pay-off in log-price variable $x = \log(S)$ is given by

$$\psi(x) = (K - e^x)_+. \tag{3.9}$$

We note in passing that all our results apply to more general pay-off functions $\psi(x)$ with polynomial growth as $|x| \to \infty$ as well.

Our pricing algorithm will be based on a Galerkin discretization of (3.4)–(3.7) in the logarithmic price $x = \log(S)$. This Galerkin discretization will be based on a variational formulation of (3.4)–(3.7). Since in logarithmic returns the pay-off ψ may grow exponentially as $|x| \to \infty$, we use Sobolev spaces with exponential weights, see also [29] for this technique in the Brownian case: for ν_-, $\nu_+ > 0$ define

$$\eta(x) := \begin{cases} \nu_- x & \text{if } x < 0 \\ \nu_+ x & \text{if } x > 0. \end{cases} \tag{3.10}$$

The weighted Sobolev spaces with exponent η are given by

$$H^j_\eta(\mathbb{R}) := \{v \in L^1_{\text{loc}}(\mathbb{R}) \mid \frac{\mathrm{d}^k v}{\mathrm{d}x^k} e^\eta \in L^2(\mathbb{R}) \ \ \forall\, k = 0, 1, \ldots, j\}.$$

We introduce the bilinear form $a^\eta(\cdot,\cdot)$ corresponding to the space operator $\mathcal{A}$: for $\varphi, \phi \in C^\infty_0(\mathbb{R})$ we define

$$\begin{aligned} a^\eta(\varphi,\phi) &= a^\eta_{\mathrm{B-S}}(\varphi,\phi) + a^\eta_{\text{jump}}(\varphi,\phi) := \int_{\mathbb{R}} \mathcal{A}[\varphi](x)\phi(x)e^{2\eta(x)}\mathrm{d}x \\ &= -\frac{\sigma^2}{2}\int_{\mathbb{R}} \left(\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(x) - \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\right)\phi(x)e^{2\eta(x)}\mathrm{d}x - r\int_{\mathbb{R}}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x) - \varphi(x)\right)\phi(x)e^{2\eta(x)}\mathrm{d}x \\ &\quad - \int_{\mathbb{R}}\int_{\mathbb{R}}\left\{\varphi(x+y) - \varphi(x) - \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)(e^y - 1)\right\}k(y)\phi(x)e^{2\eta(x)}\mathrm{d}y\mathrm{d}x. \end{aligned} \tag{3.11}$$

The following theorem implies the well-posedness of the integro-differential inequality (3.4) - (3.7) if the price u is sought in the weighted spaces $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$. Its proof is given in Appendix A.

Theorem 3.4 *Assume that the Lévy density $k(y)$ satisfies the assumptions (A1)–(A3) and that the exponent η in (3.10) satisfies $\nu_+ < \nu_-$ and $\int_{\mathbb{R}} e^{-\eta(y)}|y|k(y)\chi_{\{|y|\geq 1\}}\mathrm{d}y < +\infty$. Then $a^\eta(\cdot,\cdot)$ can be extended continuously to a bounded bilinear form on $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$. Moreover, $a^\eta(\cdot,\cdot)$ is coercive on $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$. More precisely, there exist α_η, $\beta_\eta > 0$ and $C_\eta > 0$ such that*

$$|a^\eta(\varphi,\phi)| \leq C_\eta \|\varphi\|_{H^1_\eta(\mathbb{R})}\|\phi\|_{H^1_\eta(\mathbb{R})} \quad \forall\, \varphi,\phi \in H^1_\eta(\mathbb{R}) \tag{3.12}$$

$$a^\eta(\varphi,\varphi) \geq \alpha_\eta\|\varphi\|^2_{H^1_\eta(\mathbb{R})} - \beta_\eta\|\varphi\|^2_{L^2_\eta(\mathbb{R})} \quad \forall\, \varphi \in H^1_\eta(\mathbb{R}). \tag{3.13}$$

In what follows we identify the bilinear form in (3.11) with its extension to $H^1_\eta(\mathbb{R}) \times H^1_\eta(\mathbb{R})$. Note that $\psi \in H^1_\eta(\mathbb{R})$ for all $\nu_-, \nu_+ > 0$.

Admissible solutions for the variational formulation of (3.4) –(3.7) will be sought in the convex cone

$$\mathcal{K}_\psi := \{v \in H^1_\eta(\mathbb{R}) \mid v \geq \psi \quad \text{a.e. } x\}.$$

The variational formulation of the parabolic integro-differential inequality (3.4)–(3.7) reads: Find $u \in L^2(0,T;H^1_\eta(\mathbb{R}))$, $\dfrac{\partial u}{\partial \tau} \in L^2(0,T;L^2_\eta(\mathbb{R}))$ such that $u(\tau,\cdot) \in \mathcal{K}_\psi$ almost everywhere in $(0,T)$ and such that for all $v \in H^1_\eta(\mathbb{R}) \cap \mathcal{K}_\psi$

$$(\frac{\partial u}{\partial \tau}, v - u)_{L^2_\eta(\mathbb{R})} + a^\eta(u, v-u) \geq 0 \quad \text{a.e. in } (0,T), \tag{3.14}$$

$$u(0,\cdot) = \psi. \tag{3.15}$$

If the pay-off ψ is such that $\mathcal{A}\psi \in L^2_\eta$, by (3.12) and (3.13) the variational inequality (3.14)–(3.15) admits a unique variational solution u (see [26], Chapter 6, Section 2) and it holds

$$u, \frac{\partial u}{\partial t} \in L^2(0,T;H^1_\eta) \cap L^\infty(0,T;L^2_\eta). \tag{3.16}$$

Since $u \in H^1_\eta(\mathbb{R})$ with η as in (3.10), u decays exponentially at $+\infty$. Heuristically, at $-\infty$ the exact boundary condition is $\psi(-\infty) = K$.

3.4 Localization

For the numerical solution, we localize (3.14)–(3.15) to a bounded domain $\Omega_R = (-R, R)$, $R > 0$ being the truncation parameter, and impose homogeneous essential boundary conditions at $\pm R$. To prepare the localization of (3.14)–(3.15), we introduce the *excess to pay-off function*

$$U = u - \psi \in \mathcal{K}_0 := \{v \in H^1_\eta(\mathbb{R}) \mid v \geq 0 \quad \text{a.e. } x\}. \tag{3.17}$$

We also restrict the range of admissible prices to a bounded domain $\Omega_R = (-R, R)$, with $R > 0$ sufficiently large. Then, in the triple $V \hookrightarrow H \cong H^* \hookrightarrow V^*$ with $V := H^1_0(\Omega_R)$, $H := L^2(\Omega_R)$, solve for $U^R(\tau, \cdot) \in \mathcal{K}_0$, $U^R \in L^2(0,T;V)$, $\dfrac{\partial U^R}{\partial \tau} \in L^2(0,T;H)$ such that

$$\left(\frac{\mathrm{d}U^R}{d\tau}, v - U^R\right)_{L^2(\Omega_R)} + a(U^R, v - U^R) \geq -a(\psi, v - U^R) \tag{3.18}$$
$$\text{a.e. in } (0,T), \quad \forall\, v \in V \cap \mathcal{K}_0$$
$$U^R(0,\cdot) = 0 \tag{3.19}$$

where the bilinear form $a(\varphi, \psi)$ is, for all $\varphi, \phi \in C_0^\infty(\Omega_R)$, given by

$$\begin{aligned}
a(\varphi,\phi) = &-\frac{\sigma^2}{2}\int_{\Omega_R}\left(\frac{\mathrm{d}^2\varphi}{dx^2}(x) - \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\right)\phi(x)\mathrm{d}x \\
&- r\int_{\Omega_R}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x) - \varphi(x)\right)\phi(x)\mathrm{d}x \\
&- \int_{\Omega_R}\int_{\mathbb{R}}\left\{\varphi(x+y) - \varphi(x) - \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)(e^y - 1)\right\}k(y)\phi(x)\mathrm{d}y\mathrm{d}x \\
= &\underbrace{\frac{\sigma^2}{2}(\varphi',\phi')_{L^2(\Omega_R)} + \left(\frac{\sigma^2}{2} - r\right)(\varphi',\phi)_{L^2(\Omega_R)} + r(\varphi,\phi)_{L^2(\Omega_R)}}_{a_{\mathrm{B-S}}(\varphi,\phi)} \\
&\underbrace{- \int_{\Omega_R}\int_{\Omega_R}\{\varphi(x) - \varphi(y) - \varphi'(y)(e^{x-y} - 1)\}\phi(y)k(x-y)\mathrm{d}y\mathrm{d}x}_{a_{\mathrm{jump}}(\varphi,\phi)}.
\end{aligned}$$

Since the latter form of $a(\varphi, \phi)$ is continuous on $V \times V$, we may extend $a(\cdot,\cdot)$ to $V \times V$. From now on $a(\cdot,\cdot)$ shall denote this extension. To simplify the notation, we drop in what follows the superscript R from U^R, i.e., we denote by $U = U^R$.

Proposition 3.5 *Theorem 3.4 holds with a^η replaced by a and $H^1_\eta(\mathbb{R})$ replaced by $V = H^1_0(\Omega_R)$. Moreover, there exists a positive constant $C = C(R) > 0$ such that for all $\varphi \in H^1_0(\Omega_R)$*

$$a_{\mathrm{B-S}}(\varphi,\varphi) \geq C\sigma^2\|\varphi\|^2_{H^1(\Omega_R)}, \qquad a_{\mathrm{jump}}(\varphi,\varphi) \geq 0.$$

For a proof, see e.g. Remark 3.6 in [39]. Due to these properties, (3.18)–(3.19) admits a unique solution, see also [26]. For the pure jump case $\sigma = 0$, the form $a_{\text{jump}}(\cdot,\cdot)$ still has eigenvalues with positive real part, so that our solution algorithm below remains stable also in this limiting case.

3.5 Pure jump case $\sigma = 0$

If $\sigma = 0$, the natural space V for the variational inequality problem (3.18), (3.19) is not $H_0^1(\Omega_R)$ anymore, but the fractional Sobolev space $\tilde{H}^{Y/2}(\Omega_R)$ which is defined for $0 \le Y \le 2$ by

$$\tilde{H}^{Y/2}(\Omega_R) := \{u|_{\Omega_R} \,:\, u \in H^{Y/2}(\mathbb{R}) \text{ and } u|_{\mathbb{R}\setminus\Omega_R} = 0\}.$$

For $Y = 0$ we have $\tilde{H}^Y(\Omega_R) = L^2(\Omega_R)$, for $Y = 2$ we have $\tilde{H}^Y(\Omega_R) = H_0^1(\Omega_R)$. In the case $0 < Y < 2$, $Y \ne 1$ it holds $\tilde{H}^{Y/2}(\Omega_R) = H_0^{Y/2}(\Omega_R)$ which is the closure of $C_0^\infty(\Omega_R)$ with respect to the norm in $H^{Y/2}(\Omega_R)$:

$$\begin{aligned} \tilde{H}^{Y/2}(\Omega_R) &:= \overline{C_0^\infty(\Omega_R)}^{\|\cdot\|_{\tilde{H}^{Y/2}(\Omega_R)}} \\ \|v\|^2_{\tilde{H}^{Y/2}(\Omega_R)} &:= \|v\|^2_{L^2(\Omega_R)} + \int_{\Omega_R}\int_{\Omega_R} \frac{(v(x)-v(y))^2}{|x-y|^{1+Y}} \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.20}$$

For $Y = 1$ which occurs if X_t is a generalized hyperbolic or a Normal Inverse Gaussian process (e.g. [5, 21, 46]), using the norm (3.20) would give the space $H^{1/2}(\Omega_R)$ which is different from $\tilde{H}^{1/2}(\Omega_R)$: in fact $\tilde{H}^{1/2}(\Omega_R) = H_{00}^{1/2}(\Omega_R)$ and

$$\|v\|^2_{\tilde{H}^{1/2}(\Omega_R)} = \|v\|^2_{L^2(\Omega_R)} + \int_{\Omega_R}\int_{\Omega_R} \frac{|v(x)-v(y)|^2}{|x-y|^2} dx\, dy + \int_{\Omega_R} \frac{|v(x)|^2}{R^2 - x^2} dx.$$

Proposition 3.6 *Assume $\sigma = 0$, i.e., X_t is a pure jump Lévy process with Lévy density k satisfying (A1)–(A4) for some $0 < Y < 2$. Then there exist positive constants $\alpha = \alpha(R) > 0$ and $\beta = \beta(R) > 0$ such that*

$$\forall\, u \in V := \tilde{H}^{Y/2}(\Omega_R) \,:\, a(u,u) = a_{\text{jump}}(u,u) \ge \alpha \|u\|^2_{\tilde{H}^{Y/2}(\Omega_R)} - \beta \|u\|^2_{L^2(\Omega_R)}, \tag{3.21}$$

i.e., the bilinear form $a(\cdot,\cdot)$ satisfies a **Gårding inequality** *in $V = \tilde{H}^{Y/2}(\Omega_R)$.*

Proof. Without loss of generality, by (A2) and (A3) we may assume that $R = 1/2$ and, by density, that $u \in C_0^\infty(\Omega_R)$. Then $(u', u) = 0$. It holds that (see e.g. [39, Proposition 4.2])

$$a(u,u) = (\mathcal{A}_{\text{jump}}[u], u) = \frac{1}{2}\int_{\Omega_R}\int_{\Omega_R} \frac{1}{2}(k(x-y)+k(y-x))(u(x)-u(y))^2 \mathrm{d}y\mathrm{d}x. \tag{3.22}$$

Due to $R = 1/2$, $x, y \in \Omega_R$ implies $|x-y| < 1$ and we obtain from (A4) and by (3.20) that

$$a(u,u) \ge C_- \frac{1}{2}\int_{\Omega_R}\int_{\Omega_R} \frac{(u(x)-u(y))^2}{|x-y|^{1+Y}} \mathrm{d}y\mathrm{d}x \tag{3.23}$$

which implies (3.21).

□

By Propositions 3.5, 3.6 we obtain

Corollary 3.7 *If $\sigma \neq 0$, there exist $C_1, C_2 > 0$ and $C_3 \geq 0$ independent of σ such that*

$$\forall\, u \in H_0^1(\Omega_R) \;:\; a(u,u) \geq C_1\sigma^2\|u\|^2_{H^1(\Omega_R)} + C_2\|u\|^2_{\tilde{H}^{Y/2}(\Omega_R)} - C_3\|u\|^2_{L^2(\Omega_R)}. \tag{3.24}$$

Remark 3.8 Our approach is valid for both, the jump diffusion as well as the pure jump case. Both cases can be treated in a unified fashion, if the norm $\|\cdot\|_V$ is chosen to be

$$\|u\|_V^2 = \|\sigma u'\|^2_{L^2(\Omega_R)} + \|u\|^2_{\tilde{H}^{Y/2}(\Omega_R)} \tag{3.25}$$

and if the order ρ of the operator $\mathcal{A}$ in (3.8) is defined by

$$\rho := \begin{cases} 2 \text{ if } \sigma > 0, \\ Y \text{ else.} \end{cases} \tag{3.26}$$

Proposition 3.9 *Assume $\sigma = 0$ and let X_t be a Lévy process satisfying (A1)–(A4) with $Y \in [1,2)$. Then the bilinear form $a(\cdot,\cdot) = a_{\text{jump}}(\cdot,\cdot)$ is continuous on $V \times V$, with $V := H_0^{Y/2}(\Omega_R)$, if $1 < Y < 2$ and $V := H_{00}^{1/2}(\Omega_R)$ if $Y = 1$, i.e., there exists a constant $C > 0$ such that*

$$|a(u,v)| \leq C\|u\|_V\|v\|_V, \quad \forall\, u,v \in V. \tag{3.27}$$

Proof. With $\sigma = 0$ and with X_t being a Lévy process satisfying (A1)–(A4) there holds

$$\begin{aligned} \mathcal{A}[u](x) \;&= -\int_{\mathbb{R}} u''(x+y)k^{(-2)}(y)\mathrm{d}y \\ &+(c_1 - r)u'(x) + (c_0 + r)u(x), \quad \forall\, u \in C_0^\infty(\Omega_R). \end{aligned} \tag{3.28}$$

The integral kernel $k^{(-2)}$ in (3.28) is defined as a finite part integral

$$k^{(-2)}(x) = \text{p.f.} \int_0^x k(y)(x-y)\mathrm{d}y$$

and

$$c_1 = \text{p.f.} \int_{\mathbb{R}} (e^x - 1)k(x)\mathrm{d}x, \quad c_0 = \text{p.f.} \int_{\mathbb{R}} k(x)\mathrm{d}x. \tag{3.29}$$

For $Y \in (1,2)$, $\tilde{H}^{Y/2}(\Omega_R) = H_0^{Y/2}(\Omega_R)$ and (u', v) can be understood as duality between $H^{Y/2-1}(\Omega_R) \cong (H^{1-Y/2}(\Omega_R))^*$ and $H^{1-Y/2}(\Omega_R)$, since $H^{1-Y/2}(\Omega_R) \hookleftarrow H_0^{Y/2}(\Omega_R)$. As a consequence (3.27) holds, i.e., the bilinear form $a(\cdot,\cdot)$ is continuous on $V \times V$ with $V = H_0^{Y/2}(\Omega_R)$. If $Y = 1$, $\tilde{H}^{Y/2}(\Omega_R) = H_{00}^{1/2}(\Omega_R)$ and (3.27) still holds.

□

Remark 3.10 For $Y \in (0,1)$, the 'drift' term can be removed such that the transformed equation satisfies the continuity estimate (3.27) on $V = \tilde{H}^{Y/2}(\Omega_R)$, see also [39, Remark 3.3]. This transformation yields a parabolic inequality problem with time dependent right hand side which does not satisfy the regularity requirements formulated in the standard literature on variational inequalities [26]. Our numerical algorithm below, however, can handle all jump intensities $Y \in [0,2)$ equally well, *if the drift term has been removed.*

By (3.21)–(3.27) then, if $\mathcal{A}_{\text{jump}}[\psi] \in L^2(\Omega_R)$, there exists a unique solution

$$U^R \in L^2(0,T; \tilde{H}^{Y/2}(\Omega_R)) \cap L^2(0,T; L^2(\Omega_R))$$

of (3.18)–(3.19).

4 Discretization

Since closed form solutions of (3.14), (3.15) are not available in general, numerical solutions of the pricing problem are necessary. To this end, we discretize (3.18)–(3.19) by a Finite Element (FE) method in Ω_R and by the backward Euler scheme in time. This approach is closely related to the 'Canadization' of the problem.

Since $\mathcal{A}_{\text{jump}}$ is nonlocal and unbounded, its stiffness matrix is densely populated and ill-conditioned. Using a wavelet basis of the corresponding finite dimensional space, we show that the matrix for $\mathcal{A}_{\text{jump}}$ can be 'compressed', i.e. approximated by a sparse and well conditioned matrix without compromising accuracy.

4.1 Time stepping

Let $k = T/M$, with $M \in \mathbb{N}$ be a time step and denote by U^m, $m = 0, 1, \ldots, M$ the solution of the following backward Euler discretization of (3.18)–(3.19):
Find $U^{m+1} \in V \cap \mathcal{K}_0$, $m = 0, 1, \ldots, M-1$, such that

$$(\underline{\partial} U^m, v - U^{m+1})_{L^2(\Omega_R)} + a(U^{m+1}, v - U^{m+1}) \geq -a(\psi, v - U^{m+1}) \quad \text{a.e. in } (0,T), \quad \forall\, v \in V \cap \mathcal{K}_0 \tag{4.1}$$

$$U^0 = 0. \tag{4.2}$$

Here $\underline{\partial}$ is the finite difference operator $\underline{\partial} U^m := (U^{m+1} - U^m)/k$.

4.2 Space discretization

The sequence (4.1)–(4.2) of elliptic variational inequalities can be reduced to a sequence of finite dimensional Linear Complementarity Problems (LCPs) by restricting V in (4.1)–(4.2) to a finite dimensional subspace V_N. We use spaces V_N of continuous piecewise linear functions with respect to a equidistant subdivision $\mathcal{T} : -R = x_0 < x_1 < \cdots < x_{N+1} = R$ of the truncated (log) price-domain Ω_R:

$$V_N = \text{span}\left\{ v(x) \in V : v|_{(x_{i-1}, x_i)}, x_i \in \mathcal{T}, \text{ is linear} \right\}.$$

Then the Finite Element (FE) discretization of (4.1)–(4.2) reads: find $U_N^m : (0,T) \to V_N \cap \mathcal{K}_0$ such that

$$(\underline{\partial} U_N^m, v - U_N^{m+1})_{L^2(\Omega_R)} + a(U_N^{m+1}, v - U_N^{m+1}) \geq -a(\psi, v - U_N^{m+1}) \quad \text{a.e. in } (0,T), \quad \forall\, v \in V_N \cap \mathcal{K}_0 \tag{4.3}$$

$$U_N^0 = 0. \tag{4.4}$$

Remark 4.1 The discretization (4.3), (4.4) is based on the "parabolic" nature of the generalized B-S operator in (3.5). If $\sigma = 0$ and $Y \in [0,1)$, the operator $\partial_\tau + \mathcal{A}_{\text{jump}}$ is, in general, hyperbolic since the drift term is then dominant. In this case, the "parabolic" discretization (4.3), (4.4) may exhibit instabilities. However, then discontinuous wavelet approximations are admissible and the drift term $\partial_x u$ can be stably discretized by an upwinding Finite Volume Method (FVM). This will be not be elaborated here. Instead, we always assume that the drift term has been removed (cf. Remark 3.10).

4.3 Matrix LCPs

The sequence of finite dimensional variational inequalities (4.3)–(4.4) corresponds to a sequence of matrix linear complementarity problems which we now derive.

Let $\mathcal{B} = \{\Phi_j\}_{j=1}^N$ be a basis of V_N, i.e. $V_N = \mathrm{Span}\,\mathcal{B}$. Denote by $\mathbf{M}$ the mass matrix with respect to $\mathcal{B}$ and by $\mathbf{A}$ the stiffness matrix of $a(\cdot,\cdot)$ with respect to $\mathcal{B}$, i.e., $M_{i,j} = (\Phi_i, \Phi_j)_{L^2(\Omega_R)}$, $A_{i,j} = a(\Phi_j, \Phi_i)$. The matrix $\mathbf{A}$ is in general fully populated.
We further denote by $\underline{F}$ the load vector with components $F_j = -a(\psi, \Phi_j)$ and by $\underline{v}$ the coefficient vector of the FE function $v \in V_N$ with respect to $\mathcal{B}$.
Then coefficient vectors $\underline{v}$ of FE functions v in $V_N \cap \mathcal{K}_0$ are column vectors in $\mathbb{R}^N$ satisfying componentwise

$$\mathbf{C}\underline{v} \geq 0, \tag{4.5}$$

where $\mathbf{C}$ stands for the change of basis from $\mathcal{B}$ into the canonical 'hat' function basis $\{\Psi_j\}_{j=1}^N$ with $\Psi_j(x) = \max(0, 1 - |x - x_j|/h)$. Therefore (4.3)–(4.4) is a sequence of matrix LCPs:

Find $\underline{U}_N^m \in \underline{\mathcal{K}}_0 := \{\underline{v} \in \mathbb{R}^N \mid \mathbf{C}\underline{v} \geq 0\}$, $m = 0, 1, \ldots, M$, such that

$$(\underline{v} - \underline{U}_N^{m+1})^\top (\mathbf{M} + k\mathbf{A})\underline{U}_N^{m+1} \geq (\underline{v} - \underline{U}_N^{m+1})^\top (k\underline{F} + \mathbf{M}\underline{U}_N^m) \quad \forall \underline{v} \in \underline{\mathcal{K}}_0. \tag{4.6}$$

4.4 Wavelet basis

Rather than the classical "hat" Finite Element shape functions $\Psi_j(x) = \max(0, 1 - |x - x_j|/h)$, we choose as basis $\mathcal{B}$ of V_N biorthogonal spline wavelets with a larger support. These slightly more involved shape functions serve two purposes: first, in the wavelet basis the bilinear form $a(\cdot,\cdot)$ will correspond to a matrix where most elements are negligible, yielding an approximate bilinear form $\tilde{a}(\cdot,\cdot)$ and a "compressed", sparse matrix $\tilde{\mathbf{A}}$ with $O(N \log N)$ non-vanishing entries. The error introduced into the solution by this matrix compression is not larger than the error due to Galerkin discretization [43]. Second, the wavelet basis will also allow optimal preconditioning.

To define the spline wavelets, we consider dyadic partitions $\mathcal{T}_L$ of Ω_R into $N + 1 = 2^L$ subintervals of equal size. We set $N = N_L$ and denote V_N by V_L to indicate the dependence on the subdivision level L.
We use piecewise linear, continuous biorthogonal wavelets ψ_j^l that in the interior of Ω_R have values $0, \ldots 0, -1, 2, -1, 0, \ldots, 0$. In the case of Dirichlet conditions the values are $0, 2, -1, 0, \ldots, 0$ (and similarly at the right boundary), see Figure 1.
The support of wavelet ψ_j^l is denoted by $S_j^l := \mathrm{supp}\,\psi_j^l$. It has diameter bounded by $C\,2^{-l}$. Wavelets ψ_j^l with $\bar{S}_j^l \cap \partial\Omega_R = \emptyset$ have vanishing moments up to order 1, i.e.,

$$(\psi_j^l, 1) = (\psi_j^l, x) = 0. \tag{4.7}$$

The boundary wavelets do not have any vanishing moments. The functions ψ_j^l for $l \geq l_0$ are obtained by scaling and translation of the generating wavelets ψ_j^1, $j = 0, 1, 2$ shown in Figure 1. Any $v \in V_L$ has the representation

$$v(x) = \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_j^l(x)$$

with $v_j^l = (v, \tilde{\psi}_j^l)$ where $\tilde{\psi}_j^l$ are the so-called dual wavelets (note that in our Galerkin scheme these dual basis functions never enter explicitly).

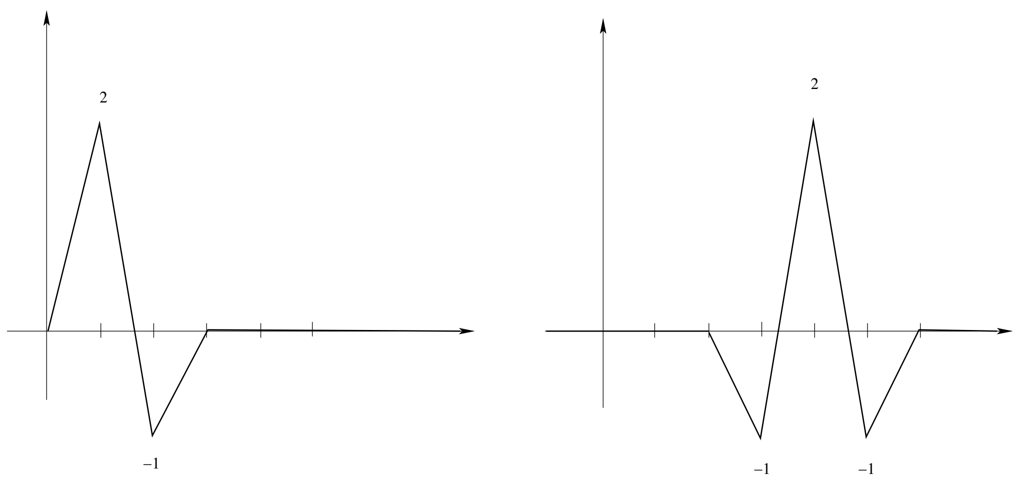

Figure 1: Generating wavelets: interior wavelets (right) and boundary wavelets for Dirichlet boundary conditions (left).

Any $v \in V$ can be written as infinite series

$$v(x) = \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} v_j^l \psi_j^l(x)$$

with $v_j^l = (v, \tilde{\psi}_j^l)$ which converges in $\tilde{H}^\theta(\Omega_R) := \{v|_{\Omega_R} \mid v \in H^\theta(\mathbb{R}),\ v|_{\mathbb{R}\setminus\Omega_R} = 0\}$ for $0 \le \theta \le 1$. For preconditioning our LCP solver, we exploit the **norm equivalence**

$$\forall v \in \tilde{H}^\theta(\Omega_R): \quad c_1 \|v\|_{\tilde{H}^\theta}^2 \le \sum_{l=0}^{\infty} \sum_{j=1}^{M^l} |v_l^j|^2 2^{2l\theta} \le c_2 \|v\|_{\tilde{H}^\theta}^2, \quad 0 \le \theta \le 1. \tag{4.8}$$

4.5 Matrix compression

The bilinear form a on $V_L \times V_L$ in the wavelet basis corresponds to a matrix $\mathbf{A}$ with elements $A_{(l,j),(l',j')} = a(\psi_j^l, \psi_{j'}^{l'})$. The density of $\mathcal{A}_{\text{jump}}$ is assumed to satisfy (A1), (A2) uniformly with respect to x and the Calderón-Zygmund type estimates (2.11); such densities arise for Lévy processes, but also for the more general 'homogeneous diffusions with jumps' X_t in the sense of Definition III.2.18 in [28], see also Section 2.2. Relation (2.11) implies with (4.7) the decay of the matrix elements with increasing distance of the supports of corresponding wavelets. To be specific, we define the compressed, sparse matrix $\tilde{\mathbf{A}}$ by setting certain small matrix elements in $\mathbf{A}$ to zero: with $A_{(j,l),(j',l')} = a(\psi_j^l, \psi_{j'}^{l'})$, we set

$$\tilde{A}_{(j,l),(j',l')} := \begin{cases} A_{(j,l),(j',l')} & \text{if } \operatorname{dist}(S_j^l, S_{j'}^{l'}) \leq \delta_{l,l'} \text{ or } S_j^l \cap \partial\Omega_R \neq \emptyset \\ 0 & \text{otherwise,} \end{cases} \tag{4.9}$$

where $S_j^l := \operatorname{supp}(\psi_j^l)$. Here the truncation parameters $\delta_{l,l'}$ are given by

$$\delta_{l,l'} := \kappa \max\{2^{-L+\hat{\alpha}(2L-l-l')}, 2^{-l}, 2^{-l'}\} \tag{4.10}$$

where $\kappa > 0$ and $\hat{\alpha} > 0$ are parameters. Their meaning is as follows: indexing of the wavelet basis ψ_j^l by levels l implies a block structure of the matrix $\mathbf{A}$. The compressed matrix $\tilde{\mathbf{A}}$ obtained from (4.9) retains only diagonals of each block $\mathbf{A}_{l,l'}$ resulting in the typical 'finger-band' structure of wavelet-compressed stiffness matrices (see [39] and the references there).

In (4.10), the parameter κ governs the bandwidth in the largest block $\tilde{\mathbf{A}}_{L,L}$ of $\tilde{\mathbf{A}}$ which is fixed independently of L while $\hat{\alpha}$ governs the growth of this bandwidth in the blocks $\tilde{\mathbf{A}}_{l,l'}$ with $l + l' < 2L$, see Figure 2. If the truncation parameters $\hat{\alpha}$ and κ are suitably chosen, the

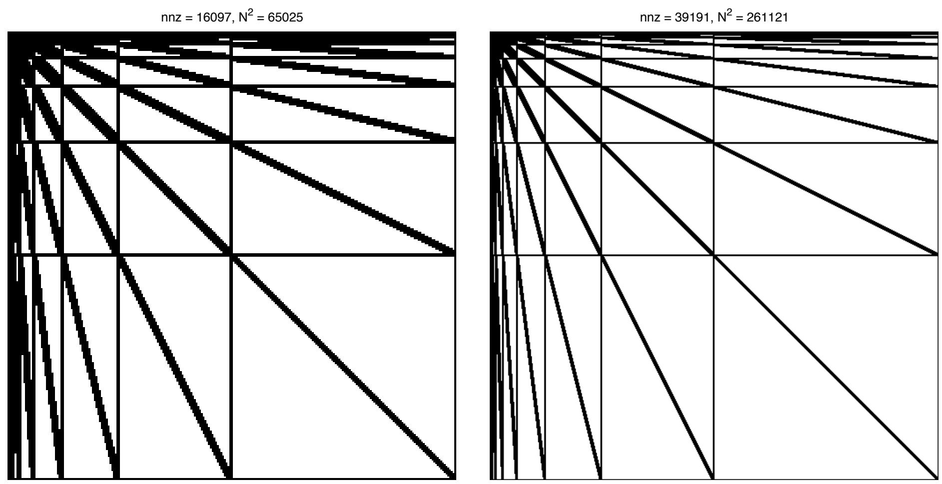

Figure 2: Sparsity pattern of the compressed matrix $\tilde{A}$ in wavelet basis; compression parameters: $\kappa = 1.0$, $\hat{\alpha} = 0.8$; CGMY parameters: $C = 1.0$, $Y = 1.4$, $G = 0.4$, $M = 1.6$; $L = 7$ (left) and $L = 8$ (right).

corresponding perturbation in the bilinear forms is small.

Remark 4.2 We considered only piecewise linear wavelets of degree $p = 1$. Results analogous to Proposition B.1 are also available for wavelets of degree $p > 1$ and we refer to [43] for details. We finally remark that a more refined criterion than (4.9) allows compression to $O(N)$ nonzero entries in certain cases while keeping (B.8).

Remark 4.3 For $Y < 0$, i.e. for finite intensity jump processes, we assume that X_t has a diffusion component so that $\sigma > 0$ then. The norm in V is then the $H^1(\Omega_R)$-norm. Compressing $\mathbf{A}$ with respect to this norm allows to reduce the number of nonzero entries in $\mathbf{A}$ due to the jump-measure to $O(N^\beta)$ with $0 < \beta < 1$ while preserving accuracy of the scheme [39, Section 5.3.3].

Using the compressed matrix $\tilde{\mathbf{A}}$ in place of $\mathbf{A}$ in (4.3) gives instead of (4.6) the sequence of perturbed LCPs

$$(\underline{v} - \underline{\tilde{U}}_N^{m+1})^\top (\mathbf{M} + k\tilde{\mathbf{A}})\underline{\tilde{U}}_N^{m+1} \geq (\underline{v} - \underline{\tilde{U}}_N^{m+1})^\top (k\underline{F} + \mathbf{M}\underline{\tilde{U}}_N^m) \quad \forall \in \underline{v} \in \underline{\mathcal{K}}_0 \tag{4.11}$$

with corresponding solution vectors $\underline{\tilde{U}}_N^m$, $m = 0, ..., M$.

We denote by $\tilde{u}_N^m(x) = \sum_{j=1}^N (\underline{\tilde{U}}_N^m)_j^l \psi_j^l(x) \in V_L$ the corresponding price functions.

4.6 Convergence

The approximate prices $\tilde{u}_N^m$ of (3.14), (3.15) obtained from the matrix LCPs (4.11) contain two errors: the *discretization error* obtained by passing from the continuous problem (3.14), (3.15) to the matrix LCPs (4.6) and the error introduced by the matrix compression (4.9) yielding the perturbed matrix LCPs (4.11) [1]. To separate these *numerical errors* from, for example, *modelling errors* such as unsuitable choices for the price process X_t, it is essential to quantify numerical errors and to prove *convergence of the computed prices*, i.e. to show that the numerical errors become negligible as the meshwidth $h = 2R/(N+1)$ and the size $k = T/M$ of the time step tend to zero.

The computed prices $\tilde{u}_N^m$ converge indeed, as $M, N \to \infty$, to the exact prices (3.2) of the model: if the truncation parameters $\kappa, R > 0$ are fixed sufficiently large independent of M and N and if $\hat{\alpha} > 4/(4+Y)$ and, moreover, if the exact prices u are, as function of x and t, sufficiently regular [2], we have for certain $0 < \gamma \le 1$ and for $\max\{1, Y\}/2 < s \le 2$ the error estimate

$$\max_m \|u^m - \tilde{u}_N^m\|_{L^2(\Omega_R)} + \left(\sum_{m=1}^M k \|u^m - \tilde{u}_N^m\|_V^2 \right)^{1/2} \le C(k^\gamma + \sigma h^{s-1} + h^{\min(s/2, s-Y/2)}). \quad (4.12)$$

Here, the norm $\| \circ \|_V$ is as in (3.25) and the constant C depends on the exact prices $u(x,t)$ and its derivatives. The bound (4.12) indicates convergence as time-stepsize k and mesh width $h = 2R/N$ in the log-price variable tend to zero. The *rate of convergence*, i.e. the precise values of γ and s in (4.12), depend on the smoothness of the exact solution $U = u - \psi$ and, in particular, on the validity of the smooth pasting condition. Since the proof of (4.12) is not needed for the pricing algorithm, we give it in Appendix B.

5 Pricing Algorithm

In the error bound (4.12), we assumed that the LCPs (4.11) are solved exactly in each time step. If the meshwidth h is small, the size N of these matrix LCPs is large and standard solution methods like PSOR (projected SOR) [18] and PSSOR (projected symmetrized SOR) are not suitable, since their rate of convergence depends on N. Unlike in the BS case, in the Lévy case symmetry of the matrix $\tilde{\mathbf{A}}$ can not be achieved by transformations, since accurate modelling of log-returns requires asymmetric tails of the Lévy densities.

Our solution algorithm is described first in an abstract framework which is also applicable to other models as e.g. BS models with stochastic volatility. It relies on a fixed point (outer) iteration where in each step a projection onto the convex cone $\underline{\mathcal{K}}_0$ has to be realized (inner iteration). Owing to the norm equivalence (4.8) of the wavelet basis, the outer fixed point iteration applied to (4.11) converges at a rate independent of the dimension N of the FE discretization. In each outer iteration, one must realize the V (or an equivalent) projection

[1] Strictly speaking, (3.14), (3.15) already contain yet another source of error due to the truncation of the log-price range from $(-\infty, \infty)$ to Ω_R. This *truncation error* is small if R is sufficiently large; see [39], Sect. 4, for more on this. Further, we ignore here effects due to *roundoff error* assuming that all calculations are done in double precision float point arithmetic with a mantissa of about 15 decimals as used e.g. in MATLAB

[2] more precisely, if $u_t \in C^\gamma([0,T]; L^2(\Omega_R))$ with some $0 < \gamma \le 1$ and if $u \in C^0((0,T]; \mathcal{H}^s(\Omega_R))$ with $1/2 < s \le 2$, cf. Appendix B

$P_{\underline{\mathcal{K}}_0}$ onto $\underline{\mathcal{K}}_0$. We realize this projection based on a wavelet generalization of the classical Cryer algorithm [18]. Since in the pure jump case V is a fractional order Sobolev space, the wavelet basis is essential here.

5.1 Outer iteration

5.1.1 Fixed point iteration

We describe an iterative solution algorithm for an abstract elliptic variational inequality set in a Hilbert space $(\mathcal{V}, (\cdot,\cdot)_\mathcal{V})$. Let $\|\cdot\|_\mathcal{V}$ denote the corresponding norm ($\|v\|_\mathcal{V} = (v,v)_\mathcal{V}^{1/2}$) and let $\mathcal{K} \subset \mathcal{V}$ be a closed, convex cone in $\mathcal{V}$. Without loss of generality it can be assumed that $0 \in \mathcal{K}$. Let $u \in \mathcal{K}$ be the solution of the following variational inequality

$$\text{Find } \; u \in \mathcal{K} \quad : \quad b(u, v-u) \geq l(v-u) \quad \forall\, v \in \mathcal{K}. \tag{5.1}$$

The bilinear form $b : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is assumed continous and coercive and the linear form $l : \mathcal{V} \to \mathbb{R}$ is continuous with respect to $\|\cdot\|_\mathcal{V}$, i.e. there exist constants $C > 0$ and $\alpha > 0$ such that for all $v, w \in \mathcal{V}$

$$b(v,v) \geq \alpha \|v\|_\mathcal{V}^2, \quad |b(v,w)| \leq C\|v\|_\mathcal{V}\|w\|_\mathcal{V}, \quad |l(v)| \leq C\|v\|_\mathcal{V}.$$

Let $\langle\cdot,\cdot\rangle_\mathcal{V}$ be an inner product on $\mathcal{V}$ equivalent to $(\cdot,\cdot)_\mathcal{V}$. Since for each $v \in \mathcal{V}$ it holds that $b(v,\cdot) \in \mathcal{V}^*$, Riesz' theorem applies and there exists $B : \mathcal{V} \to \mathcal{V}^*$ and $b_l \in \mathcal{V}$ such that

$$b(v,w) = \langle Bv, w\rangle_\mathcal{V}, \quad l(v) = \langle b_l, v\rangle_\mathcal{V} \qquad \forall\, v, w \in \mathcal{V}.$$

The variational inequality (5.1) translates into

$$\text{Find } \; u \in \mathcal{K} \quad : \quad \langle Bu, v-u\rangle_\mathcal{V} \geq \langle b_l, v-u\rangle_\mathcal{V} \quad \forall\, v \in \mathcal{K}. \tag{5.2}$$

Denote by $|||\cdot|||_\mathcal{V}$ the norm corresponding to $\langle\cdot,\cdot\rangle_\mathcal{V}$. By our assumptions on $b(\cdot,\cdot)$, there are $C_1 > 0$, $C_2 > 0$ such that for all $v, w \in \mathcal{V}$ it holds

$$|\langle Bv, w\rangle_\mathcal{V}| \leq C_1 |||v|||_\mathcal{V} |||w|||_\mathcal{V}, \quad \langle Bv, v\rangle_\mathcal{V} \geq C_2 |||v|||_\mathcal{V}^2. \tag{5.3}$$

Let us denote by $P_\mathcal{K}$ the $\langle\cdot,\cdot\rangle_\mathcal{V}$ projection onto the convex set $\mathcal{K}$. Solving (5.2) is equivalent to solving the fix-point problem [36]

$$u = \mathcal{S}u := P_\mathcal{K}(u - \rho(Bu - b_l)), \quad \rho > 0 \tag{5.4}$$

We solve (5.4) iteratively:

$$\text{Given } u_0 \in \mathcal{V} \text{ define } u_{n+1} := P_\mathcal{K}(u_n - \rho(Bu_n - b_l)) \quad \forall\, n \geq 0.$$

Then $u_n \to u$ as $n \to \infty$ provided that $0 < \rho < 2C_2/(C_1)^2$, since in this range of ρ the operator $\mathcal{S}$ is non-expanding. The optimal choice is $\rho_{\text{opt}} = C_2/(C_1)^2$, for which $|||\mathcal{S}u_1 - \mathcal{S}u_2|||_\mathcal{V} \leq q|||u_1 - u_2|||_\mathcal{V}$, with $q := (1 - (C_2)^2/(C_1)^2)^{1/2} < 1$. Note that the rate of convergence of the fix-point iteration depends only on the constants C_1, C_2 in (5.3).

5.1.2 Discretization

We apply the fix-point iteration (5.4) to $b(\cdot,\cdot) = (\cdot,\cdot)_{L^2(\Omega_R)} + k\tilde{a}(\cdot,\cdot)$. For clarity of exposition we continue with the description of the FE discretization of (5.1) in the abstract framework of the previous section and explain in Section 5.1.3 how this applies to (4.1)–(4.2).

Let $\mathcal{V}_N = \mathrm{Span}\,\{\Phi_i\}_{i=1}^N \subset \mathcal{V}$ be a finite dimensional subspace of $\mathcal{V}$ of dimension $\dim \mathcal{V}_N = N$. Let $\mathcal{K}_N := \mathcal{K} \cap \mathcal{V}_N$ and let u_N be the solution of the following variational inequality

$$\text{Find } \; u_N \in \mathcal{K}_N : \quad \langle Bu_N, v - u_N\rangle_{\mathcal{V}} \geq \langle b_l, v - u_N\rangle_{\mathcal{V}} \quad \forall\, v \in \mathcal{K}_N. \tag{5.5}$$

Again, (5.5) is equivalent to the following fix-point iteration

$$\text{Given } u_{0,N} \in \mathcal{V}_N \text{ define } u_{n+1,N} := P_{\mathcal{K}_N}(u_{n,N} - \rho(Bu_{n,N} - b_l)) \quad \forall\, n \geq 0 \tag{5.6}$$

where $P_{\mathcal{K}_N}$ denotes here the $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ projection onto $\mathcal{K}_N$

$$\text{Given } v \in \mathcal{V}, \; \text{ find } \; P_{\mathcal{K}_N} v \in \mathcal{K}_N \text{ such that } \langle P_{\mathcal{K}_N} v, w - v\rangle_{\mathcal{V}} \geq \langle v, w - v\rangle_{\mathcal{V}} \quad \forall\, w \in \mathcal{K}_N.$$

Let $\mathbf{H}$ denote the 'mass' matrix of $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ in the basis $\mathcal{B}$, $\mathbf{B}$ the 'stiffness' matrix of the bilinear form $b(\cdot,\cdot)$ in the basis $\mathcal{B}$ and $\underline{l}$ the 'load' vector, i.e.

$$H_{i,j} := \langle \Phi_j, \Phi_i\rangle_{\mathcal{V}}, \quad B_{i,j} = b(\Phi_j, \Phi_i), \quad l_i = l(\Phi_i) \qquad 1 \leq i,j \leq N.$$

The fix-point iteration (5.6) is equivalent to:
Find $\underline{u}_{n+1} \in \underline{\mathcal{K}}_N := \{\underline{v} \in \mathbb{R}^N \;:\; v := \sum_{i=1}^N v_i \Phi_i \in \mathcal{K}_N\}$ such that

$$\underline{u}_{n+1}^\top \mathbf{H}(\underline{v} - \underline{u}_{n+1}) \geq (\mathbf{H}\underline{u}_n - \rho(\mathbf{B}\underline{u}_n - \underline{l}))^\top (\underline{v} - \underline{u}_{n+1}) \quad \forall\, \underline{v} \in \underline{\mathcal{K}}_N, \tag{5.7}$$

where $\underline{v}$ is the coefficient vector of v with respect to the basis $\{\Phi_i\}_{i=1}^N$.
Let us denote by $(\cdot,\cdot)_{\mathbf{H}}$ the scalar product $(\underline{v}, \underline{w})_{\mathbf{H}} = \underline{v}^\top \mathbf{H} \underline{w}$ induced by the matrix $\mathbf{H}$. Then (5.7) can be written as:
Find $\underline{u}_{n+1} \in \underline{\mathcal{K}}_N$ such that

$$(\underline{u}_{n+1}, \underline{v} - \underline{u}_{n+1})_{\mathbf{H}} \geq (\mathbf{H}\underline{u}_n - \rho(\mathbf{B}\underline{u}_n - \underline{l}))^\top (\underline{v} - \underline{u}_{n+1}) \quad \forall\, \underline{v} \in \underline{\mathcal{K}}_N \tag{5.8}$$

which is the fixed point iteration applied to the bilinear form $b_N(\cdot,\cdot) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and the linear form $l_N : \mathbb{R}^N \to \mathbb{R}$ given by

$$b_N(\underline{v}, \underline{w}) = \underline{v}^\top \mathbf{B}\, \underline{w}, \qquad l_N(\underline{v}) = \underline{v}^\top \underline{l}.$$

The constants $C_{1,N}$, $C_{2,N}$ that enter into the choice of the relaxation parameter $0 < \rho < 2C_{2,N}/(C_{1,N})^2$ and that determine the rate of convergence of the fix-point iteration (5.8) are $\|\mathbf{H}^{-1/2}\mathbf{B}\mathbf{H}^{-1/2}\|_2$ and $\lambda_{\min}((\mathbf{H}^{-1/2}(\mathbf{B} + \mathbf{B}^\top)\mathbf{H}^{-1/2})$.

5.1.3 Application to (4.1)–(4.2)

The choice of the equivalent inner product $\langle\cdot,\cdot\rangle_{\mathcal{V}}$ and of the matrix $\mathbf{H}$ in (5.8) will be used for preconditioning the fixed point iteration.
Denote by $\mathbf{A}^{\mathrm{E}} := \mathbf{M} + k\tilde{\mathbf{A}}$. For a standard finite element basis and $\sigma \neq 0$, $\mathbf{A}^{\mathrm{E}}$ has a condition number of order h^{-2} for small h and fixed k. For the matrix $\mathbf{A}^{\mathrm{E}}$ in wavelet basis one can achieve uniformly bounded condition number, so that the number of iterations (5.7) is essentially independent of the number of unknowns N in log price.

Proposition 5.1 *Assume that $0 \le Y < 2$ and fix κ in (B.6) sufficiently large, but independent of L. Then the quantities*

$$\|\hat{\mathbf{A}}^{\mathrm{E}}\|_2 := \|(\mathbf{H}^{\mathrm{E}})^{-1/2}\mathbf{A}^{\mathrm{E}}(\mathbf{H}^{\mathrm{E}})^{-1/2}\|_2, \quad \lambda_{\min}((\mathbf{H}^{\mathrm{E}})^{-1/2}(\mathbf{A}^{\mathrm{E}} + (\mathbf{A}^{\mathrm{E}})^\top)(\mathbf{H}^{\mathrm{E}})^{-1/2}) \tag{5.9}$$

where $\mathbf{H}^{\mathrm{E}}$ is a diagonal matrix with entries $H^{\mathrm{E}}_{(j,l),(j,l)} = 1 + k2^{2l}$, are bounded from above and below, respectively, in L and k. In particular, the fixed point iteration (5.8) with $\mathbf{B} = \mathbf{A}^{\mathrm{E}}$ and $\mathbf{H} = \mathbf{H}^{\mathrm{E}}$ converges with rate $q < 1$ independent of k and L.

Proof. Define $\hat{\mathbf{A}}^{\mathrm{E}} := (\mathbf{H}^{\mathrm{E}})^{-1/2}\mathbf{A}^{\mathrm{E}}(\mathbf{H}^{\mathrm{E}})^{-1/2}$. By the norm equivalences (4.8) and the consistency condition (B.6) for sufficiently large κ in (4.10) it holds

$$C_1\|x\|_{\ell_2}^2 \le x^\top \mathbf{M} x, \quad x^\top \mathbf{M} y \le C_2 \|x\|_{\ell_2}\|y\|_{\ell_2}$$
$$C_3\|\mathbf{D}x\|_{\ell_2}^2 \le x^\top \tilde{\mathbf{A}} x, \quad x^\top \tilde{\mathbf{A}} y \le C_4 \|\mathbf{D}x\|_{\ell_2}\|\mathbf{D}y\|_{\ell_2}$$

with $\mathbf{D}$ being the diagonal matrix with entries $D_{(j,l),(j,l)} = 2^l$ and with constants C_j independent of L. It follows that there exist some constants C_5 and $C_6 > 0$ independent of L such that

$$C_5 x^\top(\mathrm{I} + k\mathbf{D})x \le x^\top \hat{\mathbf{A}}^{\mathrm{E}} x$$
$$x^\top \hat{\mathbf{A}}^{\mathrm{E}} y \le C_6[\|x\|_{\ell_2}\|y\|_{\ell_2} + k\|\mathbf{D}x\|_{\ell_2}\|\mathbf{D}y\|_{\ell_2}] \le C_6\|(\mathrm{I}+k\mathbf{D})x\|_{\ell_2}\|(\mathrm{I}+k\mathbf{D})y\|_{\ell_2}$$

which completes the proof.

□

The fix-point iteration (5.8) applied to (4.3)–(4.4) reads:

For $m = 0, 1, \dots, T/M - 1$ do:

For $n = 0, 1, 2, \dots$ until convergence:

Find $\underline{U}^{m+1}_{n+1,N} \in \underline{\mathcal{K}}_0$ such that

$$\underline{U}^{m+1}_{n+1,N}{}^\top \mathbf{H}^{\mathrm{E}}(\underline{v} - \underline{U}^{m+1}_{n+1,N}) \ge (\mathbf{H}^{\mathrm{E}}\underline{U}^{m+1}_{n,N} - \rho\mathbf{A}^{\mathrm{E}}\underline{U}^{m+1}_{n,N} + \rho(\mathbf{M}\underline{U}^m_N + k\underline{F}))^\top(\underline{v} - \underline{U}^{m+1}_{n+1,N}) \quad \forall \underline{v} \in \underline{\mathcal{K}}_0$$

Next n

Set $\underline{U}^{m+1}_N := \underline{U}^{m+1}_{n,N}$

Next m

$L \setminus \rho$	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
6	31	28	25	23	20	18	17	15	14	15	18	20	24
7	31	28	25	23	20	18	17	15	14	15	18	20	24
8	32	28	25	24	22	20	19	17	17	17	18	25	24
9	29	34	28	28	26	22	20	22	17	20	24	33	28

Table 1: Performance of the outer fix point iteration for $\sigma = 0.2$. The CGMY parameters are $C = 1.0$, $G = 1.4$, $M = 2.5$ and $Y = 1.4$.

Remark 5.2 In the pure jump unbounded variation case, i.e., when $\sigma = 0$ and $Y \in [1, 2)$ the bilinear form is continuous on $\tilde{H}^{Y/2}(\Omega_R) \times \tilde{H}^{Y/2}(\Omega_R)$ and satisfies the **Gårding inequality** (3.21), see also Proposition 3.6 and Proposition 3.9. By the norm equivalences (4.8) we define in this case $\mathbf{H}^{\mathrm{E}}$ as being the diagonal matrix with entries $H^{\mathrm{E}}_{(j,l),(j,l)} = 1 + k2^{Yl}$ and the proof of Proposition 5.1 holds verbatim. When $Y \in [0, 1]$ we applied the same numerical scheme and the perfomance of our solution algorithm turns out to be the same as reported for the case when $\sigma > 0$ or $\sigma = 0$ and $Y \in (1, 2)$. With this choice of $\mathbf{H}^{\mathrm{E}}$, the number of iterations (5.7) per time step is independent of the number N of mesh points in log price *uniformly* with respect to σ and k.

Remark 5.3 In the context of Remark 3.8, we emphasize that the choice of $\mathbf{H}^{\mathrm{E}}_{(j,l),(j,l)} = 1 + k(\sigma 2^{2l} + C2^{Yl})$ in (5.7) renders the number of iterations in (5.1.3) independent of N, k and σ. This choice, therefore, gives a preconditioner for the LCP which is robust with respect to a vanishing diffusion component in X_t.

In Table 1 we study the rate of convergence of the outer iteration in dependence on ρ and L (the level of the FE discretization). More precisely, for a fixed time $t = 0.5$ and a fixed time step $k = 0.01$ we count the number n of outer iterations needed for $\|\underline{U}_{n+1,N} - \underline{U}_{n,N}\|_{\mathbf{H}^{\mathrm{E}}}$ to fall below a given tolerance $tol = 10^{-8}$ as the number $N_L = 2^L$ of degrees of freedom increases. In this case, $\sigma = 0.2$ and the $\mathbf{H}^{E}$ projection corresponds, by the wavelet norm equivalences, to the H^1 projection onto the convex cone of admissible solutions $\underline{\mathcal{K}}_0$

$L \setminus \rho$	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.20
6	24	21	19	17	16	16	13	13	13	12	15	19	23
7	28	25	23	21	19	18	17	16	17	20	26	33	45
8	30	27	24	23	22	20	19	18	22	27	32	40	59
9	32	28	27	26	22	21	20	20	26	31	41	55	83

Table 2: Performance of the outer fix point iteration for a pure jump Lévy process, i.e. $\sigma = 0.0$. The CGMY parameters are $C = 1.0$, $G = 8.8$, $M = 9.2$ and $Y = 1.6$.

In Table 2 we repeat this experiment for a pure jump process (i.e. $\sigma = 0$) with $Y = 1.6$. The diagonal preconditioning matrix $\mathbf{H}^{E}$ is now given by $H^{\mathrm{E}}_{(j,l),(j,l)} = 1 + k2^{Yl}$, i.e. does not correspond to the Laplace matrix anymore. We observe that the number of outer iterations for e.g. $\rho = 1.0$ is independent of σ and of the choice of the discretization level parameter L.

5.2 Realization of $P_{\mathcal{K}_N}$

The early exercise feature of american style contracts is taken into account by the projection $P_{\mathcal{K}_N}$ of the approximate solution onto the admissible prices. Here, this projection is realized by the variational inequality (5.7). Note that $\mathbf{H}^{\mathrm{E}}$ is symmetric and, if the diagonal wavelet preconditioner is used, possibly diagonal. To compute the $\mathbf{H}^{\mathrm{E}}$ projection onto the convex cone $\underline{\mathcal{K}}_0$ we use a generalization of the Cryer algorithm [18].

5.2.1 Generalized Cryer algorithm

Let $\mathbf{H} \in \mathbb{R}^{N \times N}$ be any symmetric positive definite matrix and let $\mathbf{C} \in \mathbb{R}^{N \times N}$ be any invertible matrix. Specific choices of $\mathbf{H}$ and $\mathbf{C}$ will be given below. Consider the minimization problem

$$\min_{\underline{u}} \underline{u}^\top \mathbf{H} \underline{u} - 2\underline{f}^\top \underline{u} \qquad \text{subject to } \mathbf{C}\underline{u} \geq 0 \text{ element-wise,} \tag{5.10}$$

which corresponds to the LCP

$$\begin{aligned}
&\text{Find } \underline{u} \in \mathbb{R}^N \text{ such that } \mathbf{C}\underline{u} \geq 0 \\
&\mathbf{C}^{-\top}(\mathbf{H}\underline{u} - \underline{f}) \geq 0 \\
&\underline{u}^\top(\mathbf{H}\underline{u} - \underline{f}) = 0.
\end{aligned}$$

We use the following

Algorithm 5.4 *Choose $\omega \in (0, 2)$. Set $\underline{s}_i := \mathbf{C}^{-1}\underline{e}_i$, $i = 1, \ldots, N$.*
0) *Choose a starting vector $\underline{u}^0$ with $\mathbf{C}\underline{u}^0 \geq 0$.*
1) *For $k = 1, \ldots$ do (until convergence):*
- ***1.1)*** *Set $\underline{u}_0^k := \underline{u}^{k-1}$.*
- ***1.2)*** *For $i = 1, \ldots, N$ do:*
  - ***1.2.1)*** *Set $\underline{r}_i^k = (\underline{s}_i^\top \mathbf{H}\underline{s}_i)^{-1}\underline{s}_i^\top \left[\underline{f} - \mathbf{H}\underline{u}_{i-1}^k\right] \underline{s}_i$.*
  - ***1.2.2)*** *Choose the maximal $\tilde{\omega}_i^k$ with $\tilde{\omega}_i^k \leq \omega$, such that $\mathbf{C}(\underline{u}_{i-1}^k + \tilde{\omega}_i^k \underline{r}_i^k) \geq 0$.*
  - ***1.2.3)*** *Set $\underline{u}_i^k = \underline{u}_{i-1}^k + \tilde{\omega}_i^k \underline{r}_i^k$.*
- ***1.3)*** *Next i.*
- ***1.4)*** *Set $\underline{u}^k := \underline{u}_N^k$.*

2) *Next k.*

Lemma 5.5 *Under the above assumptions, for any starting vector $\underline{u}^0 \in \mathbb{R}^N$, the sequence of iterates generated by Algorithm 5.4 converges to the solution $\underline{u}$ of (5.10).*

Proof. Let $G(\underline{u}) = \underline{u}^\top \mathbf{H}\underline{u} - 2\underline{f}^\top \underline{u}$. Using symmetry of $\mathbf{H}$, one has

$$G(\underline{u}) - G(\underline{v}) = (\underline{u} - \underline{v})^\top \mathbf{H}(\underline{u} - \underline{v}) + 2(\underline{u} - \underline{v})^\top(\mathbf{H}\underline{v} - \underline{f}).$$

With

$$(\underline{r}_i^k)^\top(\mathbf{H}\underline{u}_{i-1}^k - \underline{f}) = -\frac{\underline{s}_i^\top \mathbf{H}\underline{s}_i}{\underline{s}_i^\top \underline{s}_i}(\underline{r}_i^k)^\top \underline{r}_i^k,$$

we get

$$G(\underline{u}_i^k) - G(\underline{u}_{i-1}^k) = (\tilde{\omega}_i^k)^2 (\underline{r}_i^k)^\top \mathbf{H}\underline{r}_i^k - 2\tilde{\omega}_i^k \frac{\underline{s}_i^\top \mathbf{H}\underline{s}_i}{\underline{s}_i^\top \underline{s}_i}(\underline{r}_i^k)^\top \underline{r}_i^k. \tag{5.11}$$

Since $\underline{r}_i^k$ is a multiple of $\underline{s}_i$, equation (5.11) reads

$$G(\underline{u}_i^k) - G(\underline{u}_{i-1}^k) = -\tilde{\omega}_i^k(2 - \tilde{\omega}_i^k)(\underline{r}_i^k)^\top \mathbf{H}\underline{r}_i^k \leq 0,$$

the last inequality due to $\tilde{\omega}_i^k \in [0, \omega] \subset [0, 2)$.
Hence, the sequence $\{G(\underline{u}_i^k)\}$ is monotonically decreasing. By positivity of $\mathbf{H}$, G is strictly convex and bounded below. It follows, that $G(\underline{u}_i^k) \to G$ from above.
Using

$$(\underline{r}_i^k)^\top \mathbf{H}\underline{r}_i^k = \frac{1}{(\tilde{\omega}_i^k)^2}(\underline{u}_i^k - \underline{u}_{i-1}^k)^\top \mathbf{H}(\underline{u}_i^k - \underline{u}_{i-1}^k)$$

in case $\tilde{\omega}_i^k \neq 0$, we infer

$$G(\underline{u}_i^k) - G(\underline{u}_{i-1}^k) \leq \left(1 - \frac{2}{\omega}\right)(\underline{u}_i^k - \underline{u}_{i-1}^k)^\top \mathbf{H}(\underline{u}_i^k - \underline{u}_{i-1}^k) \leq -c\|\underline{u}_i^k - \underline{u}_{i-1}^k\|^2$$

with some $c > 0$, i. e.

$$\|\underline{u}_i^k - \underline{u}_{i-1}^k\|^2 \leq C\left[G(\underline{u}_{i-1}^k) - G(\underline{u}_i^k)\right].$$

Hence,

$$\|\underline{u}_i^k - \underline{u}_{i-1}^k\| \to 0. \tag{5.12}$$

Since G is convex and bounded from below, and since $G(\underline{u}_i^k) \to G$ from above, the sequence $(\underline{u}_i^k)$ is bounded and has a limit point $\underline{u}$. We will show now, that $\underline{u}$ is the (unique) solution to the minimization problem (5.10). Then it follows, that the whole sequence converges to $\underline{u}$.
Let

$$K = \{\underline{v} \in \mathbb{R}^N \,:\, \mathbf{C}\underline{v} \geq 0\}.$$

The constraint set K is the mapped positive orthant $K_0 = \{\underline{v} \in \mathbb{R}^N \,:\, \underline{v} \geq 0\}$, i.e. $K = \mathbf{C}^{-1}K_0$. Let $\underline{x} \in \mathbb{R}^N$ and let $\underline{h}_i$, $i = 1, \ldots, n$, be a set of directions. If $G'(\underline{x})\underline{h}_i$, the directional derivative of G at $\underline{x}$ in the direction $\underline{h}_i$, is non-negative for all $i = 1, \ldots, n$, then for every set of non-negative numbers α_i, $i = 1, \ldots, n$, it holds $G(\underline{x} + \sum \alpha_i \underline{h}_i) \geq G(\underline{x})$, or, equivalently, $\underline{x}$ is the minimizer of G over the cone spanned by the directions $\underline{h}_i$.
To see this, set $\underline{h} := \sum \alpha_i \underline{h}_i$ and note, that by linearity of G' we have $G'(\underline{x})\underline{h} \geq 0$. Since G, restricted to the line $\{\underline{x} + \alpha\underline{h}, \alpha \in \mathbb{R}\}$, is a convex funtion in α, the claim follows.
We call a direction $\underline{d} \in \mathbb{R}^N$ *pointing outwards of* K *at* x, if for every $\varepsilon > 0$ the point $\underline{x} + \varepsilon d \notin K$. (Note, that for $\underline{x} \in K$ by convexity of K the fact $\underline{x} + \varepsilon_0 d \notin K$ implies $\underline{x} + \varepsilon d \notin K$ for every $\varepsilon \geq \varepsilon_0$.)
Set $S = \{\pm\underline{s}_i, i = 1, \ldots, N\}$ and consider the set

$$S(\underline{u}) := \{\underline{s} \in S \,:\, \underline{s} \text{ is not pointing outwards of } K \text{ at } \underline{u}\}.$$

We claim now $G'(\underline{u})\underline{s} \geq 0$ for $\underline{s} \in S(\underline{u})$: Assume, to the contrary, that $G'(\underline{u})\underline{s}_i = -\varepsilon$ for some $\underline{s}_i \in S(\underline{u})$ and $\varepsilon > 0$. For $\underline{u}_{i-1}^k$ sufficiently close to $\underline{u}$, we have

$$\underline{u}_i^k - \underline{u}_{i-1}^k = \tilde{\omega}_i^k r_i^k = \frac{\tilde{\omega}_i^k}{\underline{s}_i^T \mathbf{H} \underline{s}_i}\left(-G'(\underline{u}_{i-1}^k)\underline{s}_i\right)\underline{s}_i \geq \frac{\tilde{\omega}_i^k}{\underline{s}_i^T \mathbf{H} \underline{s}_i}\frac{\varepsilon}{2}\underline{s}_i,$$

where $\geq$ is to be understood in the sense of direction $\underline{s}_i$. Since $\tilde{\omega}_{i-1}^k$ is chosen maximal (but $\leq \omega$), this contradicts (5.12). Hence, $\underline{u}$ is the minimizer of G over the convex cone $CS(\underline{u})$ generated by $S(\underline{u})$.
With

$$K \subset CS(\underline{u}),$$

the lemma follows.

□

There seem to be no convergence rates available for Cryer's algorithm. Multilevel techniques [33] promise log-linear efficiency, but in the parabolic examples considered below, the direct application of Cryer's algorithm turned out to be superior, since the initial values taken from the previous steps are very close to the respective solutions. In our experiments, the number of inner iterations was usually of the same size as the number of iterations in the outer (Stampacchia) iteration.

6 Numerical Results

We present numerical experiments to illustrate the flexibility of our code with respect to the choice of the price process and to the discretization parameters. European contracts have been dealt with in [39], so we focus here on american style contracts. In all experiments below the level L for the wavelet resolution is $L = 11$.

In the first example shown on Figure 4 we take $K = 500$, $T = 0.5$, $r = 0.4$, $\sigma = 0.2$ and the following CGMY parameters: $C = 1$, $G = 1.4$, $M = 2.5$ and $Y = 1.4$. We compare the value of the American put option with that of the European one for the same set of parameters. The early exercise boundary of the American contract is plotted on the right.
In Figure 5 we display the prices of American put options with respect to different time horizons $T = 0.1, 0.25, 0.5$. The price process is a pure jump CGMY Lévy process with parameters $C = 1.0$, $G = 8.8$, $M = 9.2$ and $Y = 1.6$. In Figure 6 we take VG as price process, i.e., $\sigma = 0.0$ and $Y = 0.0$.
In [12] it is shown that in the case of perpetual American put, i.e., for $T = \infty$ (equivalently, for the stationary variant of (3.4)–(3.7)), the principle of smooth fit may fail in the case of infinite intensity pure jump Lévy processes. From the results of [12] it follows in particular that for the family of CGMY Lévy processes, the principle of smooth fit may fail if $Y \in (0, 1)$, but also that the smooth fit condition always holds for $Y \in [1, 2)$. We emphasize that we do not rely on the principle of smooth fit within our numerical scheme. Our numerical experiments also reveal that in the case $Y \in [1, 2)$ the smooth pasting condition holds (see Figures 4, 5) whereas it fails for the case of the VG process in Figure 6. For CGMY Lévy processes of bounded variation, i.e., when $Y < 1$, the sign of the 'drift' term (see [12], Section 7) is essential for the smooth pasting condition to hold or to fail. Precisely, if the sign of the coefficient $\mu := c_1 - r$ ($c_1 := \int_{\mathbb{R}} (e^x - 1)k(x)dx$) in front of u' is $\mu \geq 0$ then the smooth pasting condition holds, but if $\mu < 0$ there is no smooth pasting, only continuous fit. We illustrate this effect in Figure 3 for $Y = 0.2$.

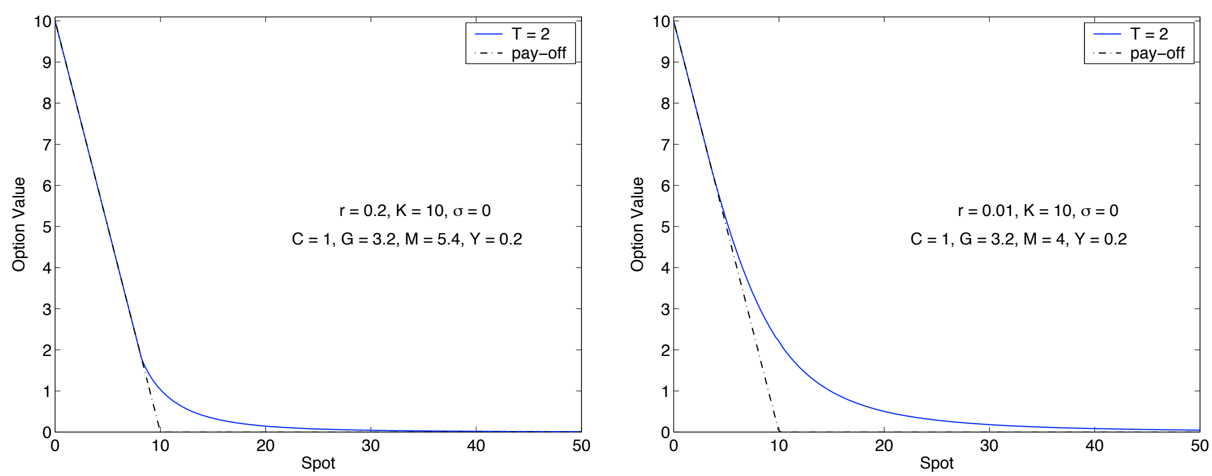

Figure 3: CGMY Lévy processes of bounded variation with alternating sign of the 'drift' term: $\mu = -0.28$ continuous fit (left) and $\mu = 0.09$ smooth fit (right).

Studying sensitivity of prices with respect to parameters is of great importance and we illustrate this feature in Figure 7, where we fix all parameters except Y and study how the exercise boundary depends on Y.
Our last example concerns an American butterfly option. Its pay-off is given by

$$(S - K_1)_+ - 2(S - (K_1 + K_2)/2)_+ + (S - K_2)_+$$

and is constructed by holding a long position in two calls with strikes at K_1 and K_2 and a short position in two calls struck at $(K_1 + K_2)/2$. Note that the pay-off function is not convex anymore, a feature that is often exploited to speed up convergence of certain algorithms [3]

[3] In [9] an algorithm for evaluating the American put option within the classical Black-Scholes framework

We emphasize that our implementation does not rely on any topological assumption on the free boundary as e.g., graph-like and monotone. We plot in Figure 8 the option value of an American butterfly option with $K_1 = 3$ and $K_2 = 10$ and the early exercise boundary for this case.

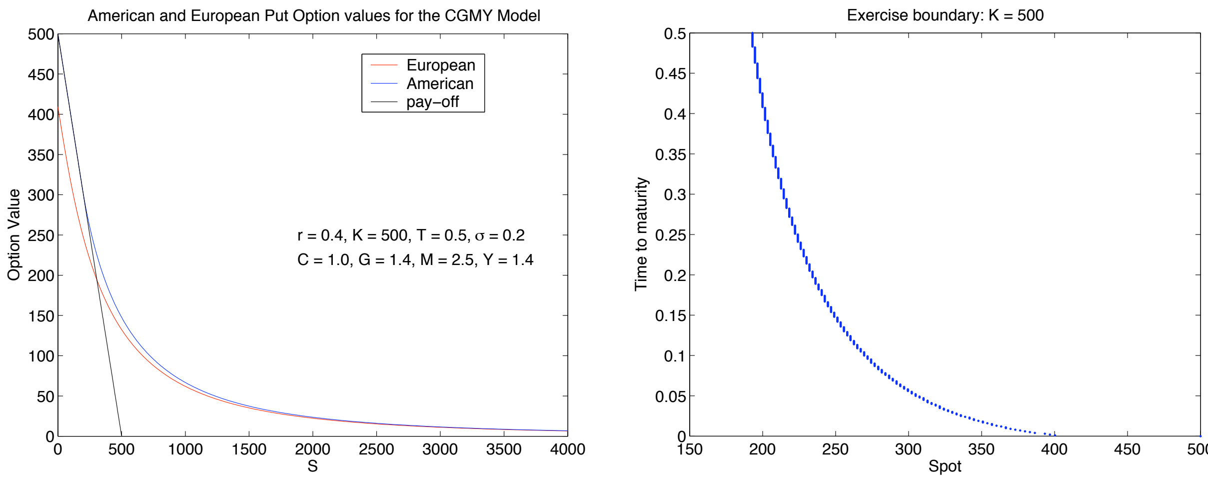

Figure 4: American vs. European and the exercise boundary

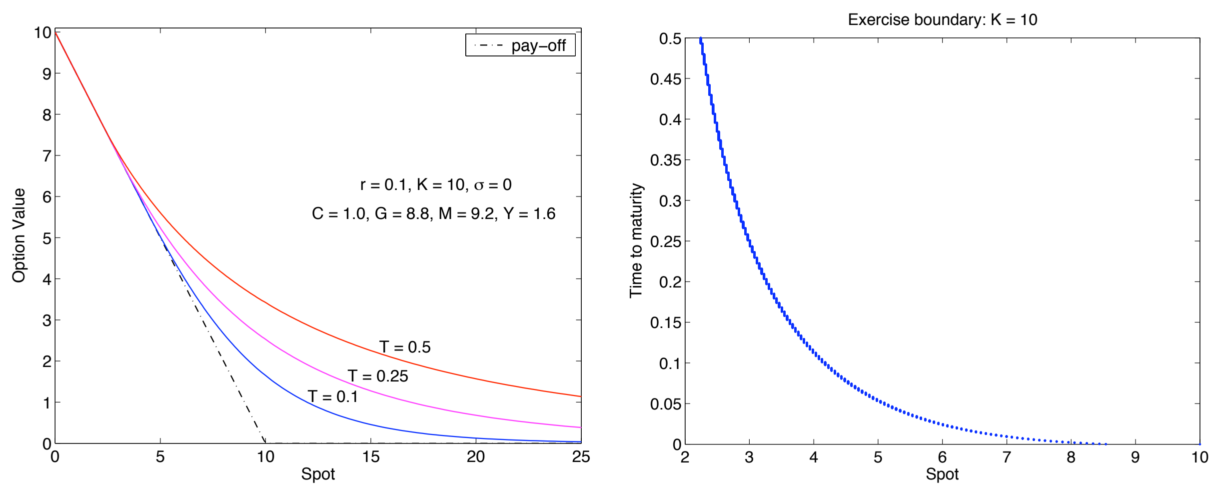

Figure 5: American Put option values for different maturities and for a pure jump Lévy process of CGMY type with $C = 1.0$, $G = 8.8$, $M = 9.2$ and $Y = 1.6$.

A Proof of Theorem 3.4

Here, we give the proof of Theorem 3.4. We split the bilinear form a^η into the following expressions

$$a^\eta = a_1^\eta + a_2^\eta + a_3^\eta$$

in linear computational complexity with respect to the number of grid points is proposed. The method relies however on the monotonicity of the exercise boundary and on the band structure of the matrix $\mathbf{M} + k\mathbf{A}$ in the LCP (4.6).

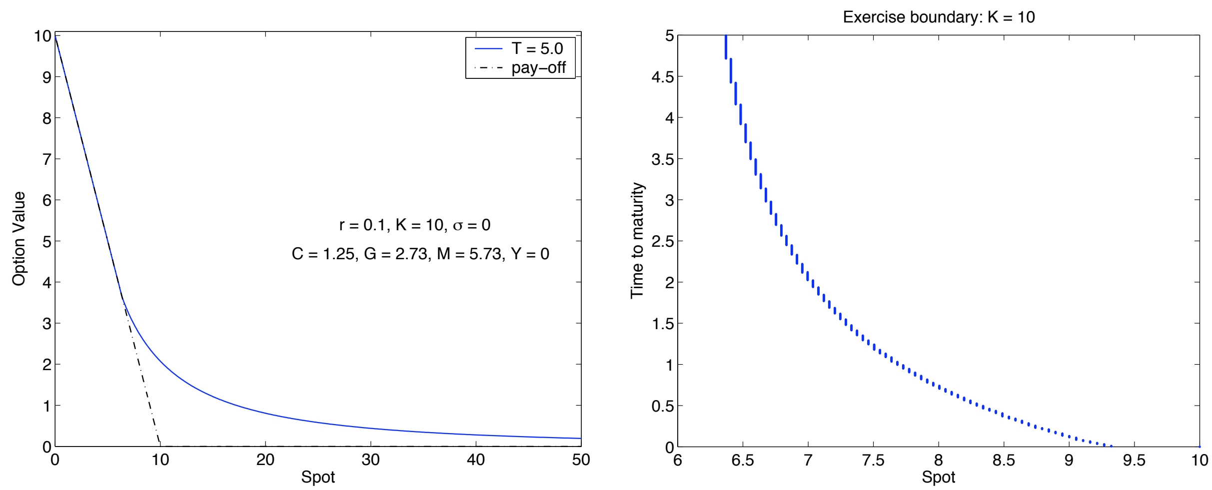

Figure 6: Failure of the smooth pasting condition for pure jump VG as price process.

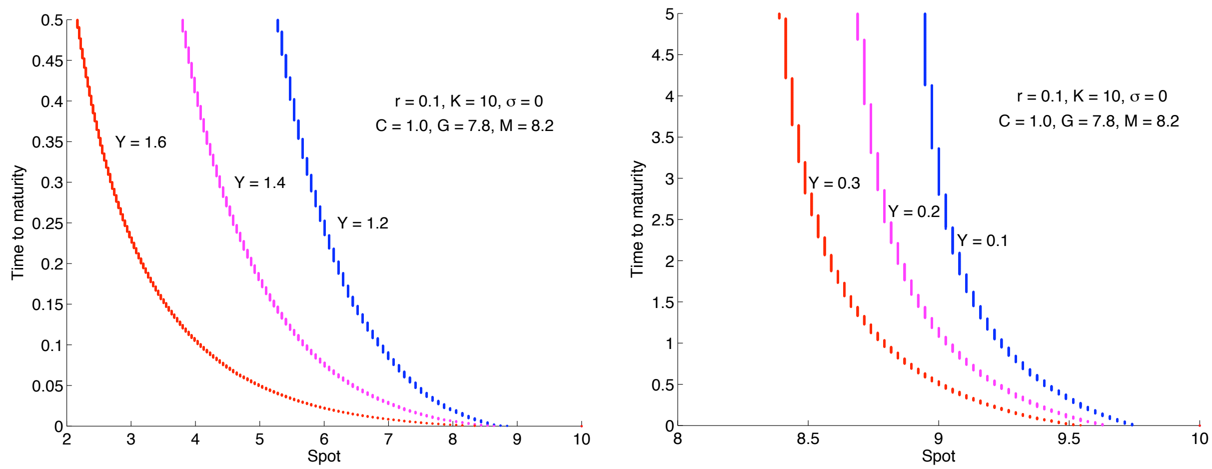

Figure 7: Sensitivity of the exercise boundary with respect to the jump intensity parameter Y in the CGMY model.

where

$$\begin{aligned} a_1^\eta(\varphi,\phi) &= \frac{\sigma^2}{2}\int_{\mathbb{R}} \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}(x)e^{2\eta(x)}\mathrm{d}x - r\int_{\mathbb{R}}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)-\varphi(x)\right)\phi(x)e^{2\eta(x)}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}}\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\phi(x)\left(\frac{\sigma^2}{2}\left(2\frac{\mathrm{d}\eta}{\mathrm{d}x}(x)+1\right)+\int_{\mathbb{R}}\left(e^y-1-y\chi_{\{|y|\le 1\}}(y)\right)k(y)\mathrm{d}y\right)e^{2\eta(x)}\mathrm{d}x \\ a_2^\eta(\varphi,\phi) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 \mathrm{d}\theta\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x+\theta y)\phi(x)e^{2\eta(x)}y\chi_{\{|y|\ge 1\}}(y)k(y)\mathrm{d}y\mathrm{d}x \\ a_3^\eta(\varphi,\phi) &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 \mathrm{d}\theta\int_0^\theta \mathrm{d}\theta'\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}(x+\theta' y)y^2\phi(x)e^{2\eta(x)}\chi_{\{|y|\le 1\}}(y)k(y)\mathrm{d}y\mathrm{d}x. \end{aligned}$$

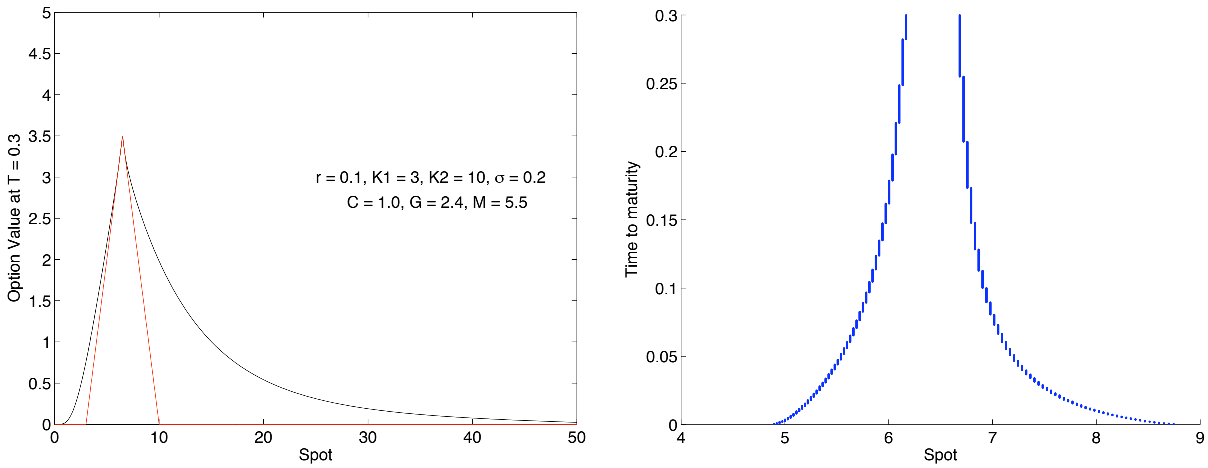

Figure 8: American butterfly: prices (left) and exercise boundary (right)

Clearly, for all $\varphi, \phi \in H^1_\eta(\mathbb{R})$ it holds

$$|a_1^\eta(\varphi,\phi)| \le C(\|\eta'\|_{L^\infty(\mathbb{R})}) \left\|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})} \left\|\frac{\mathrm{d}\phi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})}$$

$$a_1^\eta(\varphi,\varphi) \ge \frac{\sigma^2}{2}\left\|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\|^2_{L^2_\eta(\mathbb{R})} + r\|\varphi\|^2_{L^2_\eta(\mathbb{R})} - \left(\frac{\sigma^2}{2}\left(2\|\eta'\|_{L^\infty(\mathbb{R})}+1\right) + \left|\int_{\mathbb{R}}(e^y - 1 - y\chi_{\{|y|\le 1\}}(y))k(y)\mathrm{d}y\right|\right)\left\|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})}\|\varphi\|_{L^2_\eta(\mathbb{R})}.$$

If we insert $e^{\eta(x+\theta y)}e^{-\eta(x+\theta y)}$ in the definition of a_2^η and use the hypothesis on the weighting exponent η we obtain that

$$|a_2^\eta(\varphi,\phi)| \le C\left\|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})}\|\phi\|_{L^2_\eta(\mathbb{R})}.$$

The remaining part a_3^η needs a more careful inspection. It can be shown that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$$|a_3^\eta(\varphi,\phi)| \le \varepsilon\left\|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})}\left\|\frac{\mathrm{d}\phi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})} + C(\varepsilon)\left\|\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right\|_{L^2_\eta(\mathbb{R})}\|\phi\|_{L^2_\eta(\mathbb{R})}. \tag{A.1}$$

To prove (A.1) we write $a_3^\eta(\varphi,\phi)$ in the following form

$$\begin{aligned} a_3^\eta(\varphi,\phi) = &\int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 \mathrm{d}\theta \int_0^\theta \mathrm{d}\theta' \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x+\theta' y)\left(\frac{\mathrm{d}\phi}{\mathrm{d}x}(x) + 2\eta'(x)\phi(x)\right)e^{2\eta(x)}y^2\chi_{\{|y|\le\delta\}}(y)k(y)\mathrm{d}y\mathrm{d}x \\ &- \int_{\mathbb{R}}\int_{\mathbb{R}}\int_0^1 \mathrm{d}\theta \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x+\theta y)e^{\eta(x+\theta y)}\phi(x)e^{\eta(x)}e^{-\eta(x+\theta y)+\eta(x)}y\chi_{\{\delta\le|y|\le 1\}}(y)k(y)\mathrm{d}y\mathrm{d}x \\ &+ \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)\phi(x)e^{2\eta(x)}y\chi_{\{\delta\le|y|\le 1\}}(y)k(y)\mathrm{d}y\mathrm{d}x. \end{aligned}$$

Since $\int_{\{|y|\le 1\}} y^2 k(y)\mathrm{d}y < +\infty$, given $\varepsilon > 0$, one can choose $\delta = \delta(\varepsilon)$ sufficiently small such that (A.1) holds.

□

B Convergence Analysis

To obtain an error estimate for the computed prices $\tilde{u}^m$, we proceed as in [30]. We assume throughout that the drift term has been removed (cf. Remark 3.10).

We need to consider functions in V which have additional regularity and introduce for this purpose a scale of regularity spaces $\mathcal{H}^s(\Omega_R)$ which are defined as

$$\mathcal{H}^s(\Omega_R) = \tilde{H}^{\rho/2}(\Omega_R) \cap H^s(\Omega_R), \quad s > \rho/2 \tag{B.2}$$

where the order ρ of the operator $\mathcal{A}$ associated to X_t is as in (3.26).

In the compression step certain entries of the matrix $\mathbf{A}$ are replaced by zero, resulting in the compressed stiffness matrix $\tilde{\mathbf{A}}$. Both, $\mathbf{A}$ and $\tilde{\mathbf{A}}$, induce mappings from V_L to $(V_L)'$ which we denote by $\mathcal{A}^L$ and $\tilde{\mathcal{A}}^L$, respectively, and bilinear forms $a^L(\cdot,\cdot)$ and $\tilde{a}^L(\cdot,\cdot)$, respectively which are defined on $V \times V$ and given by

$$a^L(u,v) := \langle \mathcal{A}^L P_L u, P_L v\rangle, \quad \tilde{a}^L(u,v) := \langle \tilde{\mathcal{A}}^L P_L u, P_L v\rangle. \tag{B.3}$$

Here P_L denotes, for $v \in H := L^2(\Omega_R)$, the bounded projection $P_L : H \to V_L$ obtained by truncating the wavelet expansion of v in the log-price variable:

$$P_L v := \sum_{l=0}^{L} \sum_{j=1}^{M^l} v_j^l \psi_j^l(x). \tag{B.4}$$

The projection $P_L v$ approximates $v \in H$. More precisely, for $v \in \mathcal{H}^t$ holds the *approximation property*

$$\|v - P_L v\|_{\tilde{H}^s(\Omega_R)} \le C h_L^{t-s} \|v\|_{\mathcal{H}^t(\Omega_R)}, \qquad 0 \le s \le t \le 2. \tag{B.5}$$

Wavelet compression (4.10) of $\mathbf{A}$ implies only a small perturbation of the bilinear form $a^L(\cdot,\cdot)$.

Proposition B.1 *[43] Assume $\sigma > 0$. Then, for any $\delta > 0$ there exists κ in (4.10) sufficiently large such that for all $L > 0$ holds*

$$|a^L(u,v) - \widetilde{a}^L(u,v)| \le \delta \, \|u\|_{\tilde{H}^{Y/2}(\Omega_R)} \, \|v\|_{\tilde{H}^{Y/2}(\Omega_R)} \quad \forall\, u, v \in V. \tag{B.6}$$

If additionally $u \in \mathcal{H}^s(\Omega_R)$, $v \in \mathcal{H}^{s'}(\Omega_R)$ for some $Y/2 \le s, s' \le 2$, then for

$$\hat{\alpha} \ge \frac{4}{4+Y}, \tag{B.7}$$

holds

$$|a^L(u,v) - \widetilde{a}^L(u,v)| \le C h^{s+s'-Y} |\log h|^{\nu} \|u\|_{\mathcal{H}^s(\Omega_R)} \, \|v\|_{\mathcal{H}^{s'}(\Omega_R)} \tag{B.8}$$

with $\nu = 0$ if $Y/2 \le s, s' < 2$, $\nu = \frac{3}{2}$ if $s = 2$ or if $s' = 2$ and $\nu = 3$ if $s = s' = 2$.

If $\hat{\alpha} < 1$ in (4.10), then the number of nonzero entries in $\tilde{\mathbf{A}}$ is bounded by $CN_L \log N_L$. In the case $Y = 0$, $s = s' = 2$ (which corresponds to the VG process), $\hat{\alpha} = 1$ and the above result still holds with at most $CN_L(\log N_L)^2$ nonzero entries.

We see that for $1 \ge \hat{\alpha} \ge 4/(4+Y)$ in (B.7), the matrix $\mathbf{A}_{\text{jump}}$ can be compressed to a sparse matrix $\tilde{\mathbf{A}}$ with $O(N \log N)$ nonzero entries with small difference between the bilinear forms. We recall that the exact solution $u \in \mathcal{K}_0 = \{v \in \tilde{H}^{Y/2}(\Omega_R) : v \ge 0 \text{ a.e. } x \in \Omega_R\}$ satisfies

$$(\partial_t u, v-u) + a(u, v-u) \ge (f, v-u) \qquad \text{a.e. } t, \quad \forall\, v \in \mathcal{K}_0 \tag{B.9}$$

and that the sequence $\{U_h^m\}$ of approximate solutions $U_h^m \in V_L \cap \mathcal{K}_0$ satisfies

$$(\underline{\partial} U_h^m, v - U_h^{m+1}) + \tilde{a}^L(U_h^{m+1}, v - U_h^{m+1}) \geq (f^{m+1}, v - U_h^{m+1}) \quad \forall\, v \in V_L \cap \mathcal{K}_0. \tag{B.10}$$

We assume that the exact solution $u \in L^\infty(0,T; H^s(\Omega_R))$ for some $s > 1/2$ and set $e = u - U_h$ and $\eta = u(t) - I_L u(t)$ where $I_L u \in V_L \cap \mathcal{K}_0$ denotes the nodal interpolant of $u \in H^s(\Omega_R) \cap \mathcal{K}_0$ with $s > 1/2$, by piecewisewe linear, continuous functions. We have the identity

$$\begin{aligned}(\underline{\partial} e^m, e^{m+1}) + a(e^{m+1}, e^{m+1}) &= (\underline{\partial} e^m, \eta^{m+1}) + a(e^{m+1}, \eta^{m+1}) \\ &+ (\underline{\partial} u^m, I_L u^{m+1} - U_h^{m+1}) + a(u^{m+1}, I_L u^{m+1} - U_h^{m+1}) \\ &- (\underline{\partial} U_h^m, I_L u^{m+1} - U_h^{m+1}) - a(U_h^{m+1}, I_L u^{m+1} - U_h^{m+1}).\end{aligned} \tag{B.11}$$

Inserting $v = U_h^{m+1}$ and $t = t_{m+1}$ into (B.9), we get also

$$(u_t^{m+1}, U_h^{m+1} - u^{m+1}) + a(u^{m+1}, U_h^{m+1} - u^{m+1}) \geq (f^{m+1}, U_h^{m+1} - u^{m+1}) \tag{B.12}$$

and inserting $v = I_L u^{m+1} \in \mathcal{K}_0$ into (B.10) gives

$$\begin{aligned}&(\underline{\partial} U_h^m, I_L u^{m+1} - U_h^{m+1}) + a^L(U_h^{m+1}, I_L u^{m+1} - U_h^{m+1}) \\ &\qquad + (\tilde{a}^L - a^L)(U_h^{m+1}, I_L u^{m+1} - U_h^{m+1}) \geq (f^{m+1}, I_L u^{m+1} - U_h^{m+1}).\end{aligned} \tag{B.13}$$

Adding (B.12) and (B.13) to (B.11) and observing that on $V_L \times V_L$ holds $a(\cdot,\cdot) = a^L(\cdot,\cdot)$, we get the inequality

$$(\underline{\partial} e^m, e^{m+1}) + a(e^{m+1}, e^{m+1}) \leq (\tilde{a}^L - a^L)(U_h^{m+1}, I_L u^{m+1} - U_h^{m+1}) + \sum_{j=1}^{4} p_j^m \tag{B.14}$$

with the p_j^m given by

$$p_1^m := (\underline{\partial} e^m, \eta^{m+1}), \quad p_2^m := a(e^{m+1}, \eta^{m+1}),$$

$$p_3^m := -(\underline{\partial} u^m, \eta^{m+1}) - a(u^{m+1}, \eta^{m+1}) + (f^{m+1}, \eta^{m+1}),$$

and with

$$p_4^m := \left(u_t(t_{m+1}) - \underline{\partial} u^m, U_h^{m+1} - u^{m+1}\right).$$

Multiplying (B.14) by k and summing from $m = 0$ to $M - 1$ gives the bound

$$\frac{1}{2} \max_m \|e^m\|_0^2 + k \sum_{m=0}^{M-1} a(e^{m+1}, e^{m+1}) \leq \frac{1}{2}\|e^0\|_0^2 + \sum_{j=1}^{5} S_j \tag{B.15}$$

where the S_j are defined by

$$S_1 := k \sum_{m=0}^{M-1} (\underline{\partial} e^m, \eta^{m+1}), \quad S_2 := k \sum_{m=0}^{M-1} a(e^{m+1}, \eta^{m+1})$$

$$S_3 := k \sum_{m=0}^{M-1} -(\underline{\partial} u^m, \eta^{m+1}) - a(u^{m+1}, \eta^{m+1}) + (f^{m+1}, \eta^{m+1}),$$

$$S_4 := k \sum_{m=0}^{M-1} (u_t^{m+1} - \underline{\partial} u^m, U_h^{m+1} - u^{m+1})$$

and

$$S_5 := k \sum_{m=0}^{M-1} (\tilde{a}^L - a^L)(U_h^{m+1}, I_L u^{m+1} - U_h^{m+1}).$$

To estimate S_j for $j = 1, ..., 4$ we note that the nodal interpolant I_L satisfies the error estimate

$$\|v - I_L v\|_{\tilde{H}^s(\Omega_R)} \le C h_L^{t-s} \|v\|_{\mathcal{H}^t(\Omega_R)}, \qquad 0 \le s \le t,\ 1/2 < t \le 2. \tag{B.16}$$

For S_1 we use summation by parts to write

$$S_1 = -\sum_{m=0}^{M-1} k(e^m, \underline{\partial}\eta^m) + (e^M, \eta^M) - (e^0, \eta^0).$$

We have by (B.16) with $s = 0$ the bound

$$\|\eta_t\|_{L^2(0,T;L^2(\Omega_R))} \le C h^s \|u_t\|_{L^2(0,T;\mathcal{H}^s(\Omega_R))} \quad \text{for} \quad 1/2 < s \le 2$$

and estimate

$$\begin{aligned} \|\underline{\partial}\eta^m\|_{L^2(\Omega_R)} &= k^{-1}\|\eta^{m+1} - \eta^m\|_{L^2(\Omega_R)} \\ &\le k^{-1} \int_{t_m}^{t_{m+1}} \|\eta_t\|_{L^2(\Omega_R)} ds \le C k^{-1/2} h^s \|u_t\|_{L^2(t_m,t_{m+1};\mathcal{H}^s(\Omega_R))}. \end{aligned}$$

Thus, we obtain the bound

$$S_1 \le \frac{1}{8} k \sum_{m=0}^{M-1} a(e^{m+1}, e^{m+1}) + C h^{2s} \|u_t\|^2_{L^2(J,\mathcal{H}^s(\Omega_R))} + \frac{1}{8}\|e^M\|_0^2 + \|e^0\|_0^2 + C h^{2s} \|u\|^2_{L^\infty(J,\mathcal{H}^s(\Omega_R))}. \tag{B.17}$$

For S_2 we get for $0 < s \le 2$ that

$$S_2 \le \frac{1}{8} k \sum_{m=0}^{M-1} a(e^{m+1}, e^{m+1}) + C h^{2s-Y} \|u\|^2_{L^\infty(J,\mathcal{H}^s(\Omega_R))}. \tag{B.18}$$

For S_3 we find, upon integration by parts, that

$$S_3 \le C k \sum_{m=0}^{M-1} \left[\|\underline{\partial} u^m\|_0 + \|\mathcal{A} u^{m+1}\|_0 + \|f^{m+1}\|_0 \right] \|\eta^{m+1}\|_0 \le C h^s \|u\|_{\mathcal{H}^s(\Omega_R)}.$$

For S_4 we proceed as follows: recall that

$$S_4 = k \sum_{m=0}^{M-1} (\underline{\partial} u^m - u_t^{m+1}, e^{m+1}) \le k \sum_{m=0}^{M-1} \|\underline{\partial} u^m - u_t^{m+1}\|_0 \|e^{m+1}\|_0.$$

To estimate the first factor we distinguish two cases:
a)

$$\underline{\partial} u^m = k^{-1}(u^{m+1} - u^m) = k^{-1} \int_{t_m}^{t_{m+1}} u_t ds$$

gives

$$\|\partial u^m - u_t^{m+1}\|_0^2 = \|u_t^{m+1} - k^{-1}\int_{t_m}^{t_{m+1}} u_t ds\|_0^2.$$

If $u_t \in C^0([t_m, t_{m+1}]; L^2(\Omega_R))$, we get

$$\|\underline{\partial} u^m - u_t^{m+1}\|_0 \le \|u_t\|_{C^0([t_m,t_{m+1}];L^2(\Omega_R))}.$$

b) If $u \in C^{1,\gamma}([t_m, t_{m+1}]; L^2(\Omega_R))$ for some $0 < \gamma \le 1$, the elementary identity

$$u_t^{m+1} - k^{-1}\int_{t_m}^{t_{m+1}} u_t ds = u_t^{m+1} - u_t^m - \frac{1}{k}\int_{t_m}^{t_{m+1}} [u_t(s) - u_t^m] ds$$

gives the bound

$$\|\underline{\partial} u^m - u_t^{m+1}\|_0 \le k^\gamma \|u_t\|_{C^{0,\gamma}([t_m,t_{m+1}];L^2(\Omega_R))}$$

from where we get for any $\varepsilon > 0$ the bound

$$S_4 \le C\left[\frac{k^{2\gamma}}{2\varepsilon}\|u_t\|^2_{C^\gamma([t_m,t_{m+1}];L^2(\Omega_R))} + \frac{\varepsilon}{2}k\sum_{m=1}^{M} a(e^m, e^m)\right].$$

Choosing $\varepsilon > 0$ sufficiently small, the second term may be absorbed into the left hand side of (B.15).

To estimate S_5, we note that the forms $a^L(\cdot,\cdot)$ and $\tilde{a}^L(\cdot,\cdot)$ in (B.3) are defined on the whole space $V \times V$.

$$\begin{aligned}
|S_5| &\le k\sum_{m=0}^{M-1} |(a^L - \tilde{a}^L)(u^{m+1} + U_h^{m+1} - u^{m+1}, I_L u^{m+1} - U_h^{m+1})| \\
&\le k\sum_{m=0}^{M-1} |(a^L - \tilde{a}^L)(u^{m+1}, I_L u^{m+1} - U_h^{m+1})| + |(a^L - \tilde{a}^L)(e^{m+1}, I_L u^{m+1} - U_h^{m+1})| \\
&\le Ck\sum_{m=0}^{M-1} h^{s-Y/2}\|u^{m+1}\|_{\mathcal{H}^s(\Omega_R)}(\|(I - I_L)u^{m+1}\|_V + \|e^{m+1}\|_V) \\
&\qquad + |(a^L - \tilde{a}^L)(e^{m+1}, I_L u^{m+1} - U_h^{m+1})| \\
&\le Ck\sum_{m=0}^{M-1} h^{s-Y/2}\|u^{m+1}\|_{\mathcal{H}^s(\Omega_R)}(\|(I - I_L)u^{m+1}\|_V + \|e^{m+1}\|_V) \\
&\qquad + \delta\|e^{m+1}\|_V(\|(I - I_L)u^{m+1}\|_V + \|e^{m+1}\|_V) \\
&\le Ck\sum_{m=0}^{M-1} (1 + \frac{\delta}{2} + \frac{2}{\varepsilon})h^{2s-Y}\|u^{m+1}\|^2_{\mathcal{H}^s(\Omega_R)} + (\frac{3\delta}{2} + \frac{\varepsilon}{2})\|e^{m+1}\|_V^2 \\
&\le Ck\sum_{m=0}^{M-1} (1 + \frac{\delta}{2} + \frac{2}{\varepsilon})h^{2s-Y}\|u^{m+1}\|^2_{\mathcal{H}^s(\Omega_R)} + \sum_{m=0}^{M-1}(\frac{3\delta}{2} + \frac{\varepsilon}{2})Cka(e^{m+1}, e^{m+1}).
\end{aligned}$$

Inserting these bounds into (B.15) and fixing $\delta > 0$ and $\varepsilon > 0$ sufficiently small independent of L and M, we can absorb the term

$$C(\varepsilon + \delta)\sum_{m=0}^{M-1} ka(e^{m+1}, e^{m+1})$$

into the left hand side of (B.15). We have shown

Theorem B.2 *The sequence* $\{\tilde{u}^m\}_{m=0}^M$ *of approximate prices obtained from the exact solution of the matrix LCPs (4.11) with the matrix compression (4.9), (4.10) for sufficiently large* κ *and* $\hat{\alpha} > 4/(4+Y)$ *satisfies the error bound*

$$\begin{aligned}&\max_{1\le m\le M} \|e^m\|_0^2 + k\sum_{m=1}^{M} a(e^m, e^m)\\&\le C\left(\|e^0\|_0^2 + h^{2\min\{s/2, s-Y/2\}}\sum_{m=1}^{M} k\|u^m\|^2_{\mathcal{H}^s(\Omega_R)} + k^{2\gamma}\|u_t\|^2_{C^\gamma([0,T];L^2(\Omega_R))}\right)\end{aligned} \tag{B.19}$$

for $0 < \gamma \le 1$*,* $0 < Y \le 2$ *and for* $1/2 < s \le 2$*.*

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