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Abstract

In this note we study the existence of finite point-sets in the plane which have a prescribed set of diameters. We recall a classical result concerning an upper bound for the number of diameters occuring in an n-point set in the plane and we describe the sets (called 'complete') for which this upper bound is attained. We also give an algorithm for the construction of all complete sets. The final section is then devoted to the question of enlarging an arbitrary finite plane set to a 'complete' one.

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1 Introduction

Let $n \geq 3$ be an integer and S be a set of n points, say $A_1, A_2, \ldots, A_n$, in the plane.

Definition 1.1 *A diameter of the set S is any of the segments $[A_iA_j]$, such that $i \neq j$ and its length is the largest possible among all segments joining two points in S:*

$$|A_iA_j| = \max\{|A_kA_l| \ \mid k, l \in \{1, \ldots, n\},\, k \neq l\} \tag{1.0}$$

This length will be denoted by d_S.

To set up some further notations, $d(S)$ will be in the following the number of diameters of the set S, while $\mathrm{Card}(S)$ will denote the cardinality of the same set. To S we shall frequently associate also a graph, denoted Γ_S, by saying that A_i and A_j are adjacent if and only if $[A_iA_j]$ is a diameter of S.

For the study of the diameters of a finite plane set S, the following result is essential.

Lemma 1.2 *The intersection of any two diameters of the set S is nonempty.*

Proof. As the result is well-known (see [2]), we only sketch the proof. Let $[A_iA_j]$ and $[A_kA_l]$ be two diameters of S. We may assume that i, j, k, l are distinct. It is easy to see that the convex envelope of the set $\{A_i, A_j, A_k, A_l\}$ is then a quadrilateral. So it remains to rule out the situation when the convex envelope reads $A_iA_jA_kA_l$. Indeed, if this were the case, since $[A_iA_k] \cap [A_jA_l] \neq \emptyset$, the triangle inequality would ensure that at least one of the lengths $|A_iA_k|$, $|A_jA_l|$ is larger than d_S. But this would contradict the defining maximality property of d_S. $\diamond$

As a consequence, we can easily deduce also the following known result (see [2], [3]).

Theorem 1.3 *If S is a finite plane set, then $d(S) \leq \mathrm{Card}(S)$.*

Proof. We use here induction on $n = \mathrm{Card}(S)$. The case $n = 3$ being trivial, we may assume the claim true for all plane sets containing at most $n-1$ points and we let S be an arbitrary n-point plane set. Now, if there exists a point $A_j \in S$ whose degree in Γ_S is at least 3, then we can find 3 diameters emerging from A_j, that we denote by $[A_jA_i], [A_jA_k]$ and $[A_jA_l]$. If, say, $|A_iA_l| = \max\{|A_iA_l|, |A_iA_k|, |A_kA_l|\}$, then it is easily seen that the line joining A_j and A_k separates the points A_i and A_l. This implies that there are no halflines emerging from A_k and intersecting both segments $[A_jA_i]$ and $[A_jA_l]$, other than $[A_kA_j$. This in turn ensures, due to Lemma 1.2, that there exists exactly one diameter to which A_k belongs, namely $[A_kA_j]$. In other words, the degree of A_k in Γ_S is equal to 1. Applying the induction hypothesis to $S \setminus A_k$, we are done. It follows that the degrees of all vertices of Γ_S are not larger than 2, which of course implies that Γ_S contains at most n edges, concluding the proof. $\diamond$

2 Complete sets

Theorem 1.3 enables us to give the following

Definition 2.1 *Suppose S is a finite plane set. Then*

1. *The positive integer $i(S) := \mathrm{Card}(S) - d(S)$ will be called the* defect index *of S.*
2. *S will be called* complete *provided $i(S) = 0$.*

Examples of complete sets S can be easily constructed. It suffices for instance to choose the vertices P_1, P_2, P_3 of an equilateral triangle $\mathcal{T}$, as well as another $n-3$ points on the arc of measure $\pi/3$ joining two of the vertices (say P_2, P_3) of $\mathcal{T}$ and lying outside of $\mathcal{T}$. However, such a configuration is not 'democratic' for $n \geq 4$, since P_1 belongs to $n-1$ diameters, while other points in S belong to only one diameter.

Definition 2.2 *A complete set S is called* democratic *if each point in S belongs to exactly two diameters.*

The problem we address next is the existence of a democratic set S of n points.

Theorem 2.3 *There exists a democratic set S of n points iff n is odd.*

Proof. If n is odd, the set S of all vertices of an n-regular polygon is obviously democratic. Conversely, suppose that a democratic set S containing n points exists. Since the degrees of all vertices of Γ_S equal 2, we can decompose Γ_S as a disjoint union of k cycles $\{\mathcal{C}(j)\}_{1 \leq j \leq k}$. We label the vertices of $\mathcal{C}_j$ from $A_{p(j-1)+1}$ to $A_{p(j)}$, where $p(0) := 0$ and $p(k) := n$. Let us further denote by d the line joining the points labeled A_1 and A_2.
Case 1. $k = 1$ (one cycle)
Suppose that n is even. Lemma 1.2 ensures then that the line d separates the points labeled A_j and A_{j+1} for all $j \in \{3, 4, \ldots, n-1\}$. This means that, denoting by H_1 and H_2 the open halfplanes with joint boundary d, $\{A_3, A_5, \ldots, A_{n-1}\} \subset H_1$ and $\{A_4, A_6, \ldots, A_n\} \subset H_2$. It follows that $[A_1\, A_n] \subset \overline{H_2}$ and $[A_2\, A_3] \subset \overline{H_1}$ are two disjoint diameters of S, contradicting Lemma 1.2.
Case 2. $k > 1$ (at least two cycles)
If one of $p(1)$, $p(2) - p(1)$ is even, we reach again a contradiction of Lemma 1.2 by arguing as in the first case on the corresponding cycle ($\mathcal{C}_1$ or $\mathcal{C}_2$, respectively). It follows therefore, that $p(1)$ is odd and $p(2)$ is even. As in the first case, we deduce that $\{A_{p(1)+1}, A_{p(1)+3}, \ldots, A_{p(2)}\} \subset H_1$ and $\{A_{p(1)+2}, \ldots, A_{p(2)-1}\} \subset H_2$. But then $[A_1\, A_2]$ and $[A_{p(1)+1}\, A_{p(2)}]$ are two disjoint diameters, contradicting once again Lemma 1.2. $\diamond$

The way we have ruled out the second case in the previous proof ensures that

Corollary 2.4 *The graph Γ_S associated to a democratic set S consists of exactly one cycle.*

As we have already mentioned, the set S of all $n = 2k+1$ vertices of a regular polygon is democratic. However, the converse is not true. To see this, it suffices to choose, for some $0 < x < 1/\sqrt{3}$, the five-element set $S = \{(0,1), (\pm x, 0), (\pm(1+x^2)^{1/2}, (3/4 - x^2/4 - x(1+x^2)^{1/2})^{1/2})\}$. Nevertheless, the vertices of a democratic set S still enjoy some properties regarding the convex hull as well as distances between them, similar to those of a regular polygon.

Theorem 2.5 *Let S be a democratic set, containing $n = 2k+1$ points. Label the vertices of Γ_S in such a way that the cycle reads $[A_1\, A_2\, A_3\, \ldots A_n]$. Then*

1. *The polygon $[A_1\, A_3\, A_5\, \ldots\, A_{2k+1}\, A_2\, A_4\, \ldots\, A_{2k}]$ is convex.*

2. *The following inequalities concerning the diagonals' lengths hold:*

$$|A_1\, A_3| < |A_1\, A_5| < |A_1\, A_7| < \ldots < |A_1\, A_{2k+1}|$$

$$|A_1\, A_{2k}| < |A_1\, A_{2k-2}| < |A_1\, A_{2k-4}| < \ldots < |A_1\, A_2|. \tag{2.1}$$

Similar inequalities hold for the diagonals emerging from A_i for all $i \in \{2, 3, \ldots, 2k+1\}$.

Proof. 1. The case $n = 3$ being trivial, we assume $n \geq 5$. To prove the convexity of $[A_1\, A_3\, A_5\, \ldots\, A_{2k+1}\, A_2\, A_4\, \ldots\, A_{2k}]$, it suffices, due to symmetry reasons, to show that the line passing through the vertices A_1 and A_3 does not separate $S \setminus \{A_1, A_3\}$. Suppose this is not true, so that one can find two nonempty sets $S_1 \subset H_1$, $S_2 \subset H_2$ such that $S_1 \cup S_2 = S \setminus \{A_1, A_3\}$, where H_1 and H_2 are the two parts in which the line $A_1\, A_3$ splits the plane. Say, for instance, that $A_2 \in S_1$. Since $[A_1\, A_2] \cap [A_3\, A_4] \neq \emptyset$, it follows $A_4 \in S_1$, too. This implies that one can find a diameter $[A_j\, A_{j+1}]$ with $j \in \{4, 5, \ldots, 2k\}$ such that $A_j \in S_1$ and $A_{j+1} \in S_2$, otherwise S_2 would be empty. On the other hand, if we denote by $\mathcal{D}(X, r)$ the open disc of center X and radius r, it also follows that $A_j, A_{j+1} \in S \setminus \{A_1, A_2, A_3\} \subset \mathcal{M} := \mathcal{D}(A_1, d_S) \cap \mathcal{D}(A_2, d_S)$. Denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the two parts in which $[A_1\, A_3]$ splits $\mathcal{M}$. If, say, $S_1 \subset \mathcal{M}_1$ and $S_2 \subset \mathcal{M}_2$, we deduce that $[A_j\, A_{j+1}] \cap [A_1\, A_3] \neq \emptyset$. But $[A_j\, A_{j+1}] \cap [A_1\, A_2] \neq \emptyset$ and $[A_j\, A_{j+1}] \cap [A_2\, A_3] \neq \emptyset$ as well, in view of Lemma 1.2. Since a segment can not cut all the sides of a triangle without passsing through its vertices, we have reached the contradiction that finishes the proof.
2. We prove only the first chain of inequalities in (2.1), as the other can be proved similarly. Let us consider the diagonals $[A_1 A_{2j-1}]$ and $[A_1 A_{2j+1}]$ for some $j \in \{2, 3, \ldots, k\}$. As a consequence of i), the quadrilateral $[A_1\, A_{2j-1}\, A_{2j+1}\, A_{2j}]$ is convex and let us denote by O the intersection of its diagonals. Upon summing the triangle inequalities in $\Delta A_1 O A_{2j-1}$ and $\Delta A_{2j+1} O A_{2j}$ respectively, we get, noting that $|A_{2j-1}\, A_{2j}| = |A_{2j}\, A_{2j+1}| = d_S$, the desired inequality $|A_1\, A_{2j-1}| < |A_1\, A_{2j+1}|$. ◇

Now we turn to the study of an arbitrary complete set S. We shall see next that to each such set one can canonically associate a democratic subset, which is, moreover, unique in some natural sense.

Theorem 2.6 *If S is a complete set, then there exists at most one subset of S, denoted by $\tilde{S}$, such that $\tilde{S}$ is democratic and $d_S = d_{\tilde{S}}$.*

Proof. Suppose that there exist two democratic sets $\tilde{S}_1$ and $\tilde{S}_2$ such that $\tilde{S}_1, \tilde{S}_2 \subset S$ and $d_S = d_{\tilde{S}_1} = d_{\tilde{S}_2}$. The last equality ensures that each point in $\tilde{S}_1 \cup \tilde{S}_2$ belongs to at least two diameters of the set $\tilde{S}_1 \cup \tilde{S}_2$. Theorem 1.3 implies then that each point in $\tilde{S}_1 \cup \tilde{S}_2$ belongs to exactly two diameters of the same set, that is, $\tilde{S}_1 \cup \tilde{S}_2$ is democratic. The graph $\Gamma_{\tilde{S}_1 \cup \tilde{S}_2}$ associated to it, and containing all the diameters of $\tilde{S}_1$ and $\tilde{S}_2$ among its edges, consists then of only one cycle. As the diameters of $\tilde{S}_1$ build also a cycle in $\Gamma_{\tilde{S}_1 \cup \tilde{S}_2}$, it follows that $\tilde{S}_1 = \tilde{S}_1 \cup \tilde{S}_2$ and similarly $\tilde{S}_2 = \tilde{S}_1 \cup \tilde{S}_2$, concluding the proof. ◇

The existence of a democratic subset $\tilde{S} \subset S$ as in Theorem 2.6 can be proved constructively. The result reads as follows.

Theorem 2.7 *('Democratization by killing the poor')*
Let S be a complete set and denote by S_1 the set of all points in S belonging to exactly one diameter. Then $S \setminus S_1$ is democratic.
The following algorithm renders $S \setminus S_1$ in a finite number of steps.

$\tilde{S} := S$;
  while $(\exists P \in \tilde{S}, \deg_{\Gamma_{\tilde{S}}}(P) \neq 2)$
    choose $P \in \tilde{S}$ such that $\deg_{\Gamma_{\tilde{S}}}(P) = k \geq 3$;
    label the vertices adjacent to P, clockwise, by $Q_1, Q_2, \ldots, Q_k$;
    $\tilde{S} := \tilde{S} \setminus \{Q_2, \ldots, Q_{k-1}\}$;
  end;

Proof. To prove the first assertion, we use induction on $n = \mathrm{Card}(S)$. If $n = 3$, the claim is trivially true. We may therefore suppose $n \geq 4$ and the claim true for all complete sets S

containing at most $n-1$ points. We may assume also $S_1 \neq \emptyset$, otherwise there's again nothing to prove. We deduce that there exists $P \in S$ such that $k := \deg_{\Gamma_S}(P) \geq 3$. Then, obviously, the points adjacent to P lie on $\mathcal{C}(P, d_S)$, the circle of center P and radius d_S. More precisely, if we label them $Q_1, Q_2, \ldots, Q_k$ clockwise, then the angle $\measuredangle Q_1 P Q_k$ has measure no larger than $\pi/3$. Using an argument that already appeared in Theorem 1.3, it is easily seen that there are no diameters emerging from $Q_2, \ldots, Q_{k-1}$ other than those adjacent to P. We consider then the set $S' := S \setminus \{Q_2, \ldots, Q_{k-1}\}$ and the graph $\Gamma_{S'}$ associated to it. We note that through passing from Γ_S to $\Gamma_{S'}$, the degrees of all vertices of $\Gamma_{S'}$ have been preserved, except for $\deg_{\Gamma_S}(P) = k$, which has been lowered to $\deg_{\Gamma_{S'}}(P) = 2$. We may therefore apply the induction assumption to S' and conclude the proof of the first claim. As for the algorithm, it is straigthforward to see that every passage through the 'while' loop strictly lowers the number of points of degree different from 2, as long as such points exist. We shall only remark, for the sake of a better understanding of the underlying structure of S, that $\deg(Q_1), \deg(Q_k) \geq 2$. Indeed, we can rephrase the first assertion already proven, by saying that each point in S of degree at least 2 in Γ_S is also an element of the democratic set $S \setminus S_1$. It follows that the point P, subject to the procedure of a 'while' loop, must be adjacent to at least two points from $S \setminus S_1$ (its neighbours on the cycle $\Gamma_{S \setminus S_1}$) and having therefore degree at least 2 in Γ_S. These two points must be Q_1 and Q_k, since $Q_2, \ldots, Q_{k-1}$ have for sure degree 1 in Γ_S. ◇

Definition 2.8 *The democratic set $\tilde{S} = S \setminus S_1$ associated to a complete set S by Theorem 2.7 will be refered to as the* democratic core *of S.*

From the previous proof we deduce also

Corollary 2.9 *Let S be complete and $\tilde{S}$ be its democratic core. For each triple of consecutive points O, P, R on the cycle $\Gamma_{\tilde{S}}$ draw the arc $\mathcal{A}_P$ of $\mathcal{C}(P, d_S)$ joining O and R and lying inside the angle $\measuredangle OPR$. Then*

$$S_1 \subset \bigcup_{P \in \tilde{S}} \mathcal{A}_P.$$

Moreover, a converse, which will help us describe all complete sets, is also true.

Proposition 2.10 *Let $\tilde{S}$ be a democratic set. For each triple of consecutive vertices O, P, R of the cycle $\Gamma_{\tilde{S}}$ draw the arc $\mathcal{A}_P$ of $\mathcal{C}(P, d_S)$ joining O and R and lying inside the angle $\measuredangle OPR$. If S_1 is an arbitrary finite part of $\bigcup_{P \in \tilde{S}} \mathcal{A}_P$, then $S := \tilde{S} \cup S_1$ is complete.*

Proof. It is enough to check that if $Q_1, Q_2 \in \bigcup_{P \in \tilde{S}} \mathcal{A}_P$, then $d_{\tilde{S} \cup \{Q_1, Q_2\}} = d_{\tilde{S}}$. This amounts to proving that $|Q_1 Q_2|, |TQ_1|, |TQ_2| \leq d_{\tilde{S}}$ $\forall T \in \tilde{S}$. Let us prove only $|TQ_1| \leq d_{\tilde{S}}$ $\forall T \in \tilde{S}$, the proofs of the other two inequalities being similar. Suppose therefore $Q_1 \in \mathcal{A}_P$ for some $P \in \tilde{S}$, adjacent on the cycle $\Gamma_{\tilde{S}}$ to O and R. We may suppose $T \neq P$. Since the points of $\tilde{S}$ are the vertices of a convex polygon in which O and R are consecutive, it follows that either $[ORTP]$ or $[ORPT]$ is a convex quadrilateral. In both cases we deduce that the line TP does not cross $[OR]$, therefore it does not meet $\mathcal{A}_P$ either. This in turn yields $|TQ_1| \leq \max\{|TO|, |TR|\} \leq d_{\tilde{S}}$, as claimed. ◇

Let us now collect, for later use, some almost obvious properties of complete sets. We leave the proof to the reader.

Proposition 2.11 *Let S be complete. Due to Theorem 2.5, the points of S are the vertices of a convex polygon $\mathcal{P} = [P_1 P_2 \ldots P_n]$. Then*

1. *For each $1 \leq i \leq n$ there exists a unique vertex P_j of $\mathcal{P}$ such that $[P_j P_i]$ and $[P_j P_{i+1}]$ are diameters of S ($P_{n+1} := P_1$).*

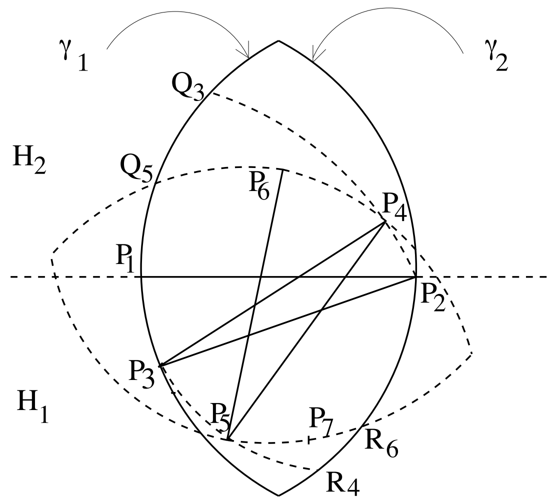

Figure 1: Recursive construction of a democratic set S.

2. *If $[P_jP_i]$ and $[P_jP_k]$ are two diameters of S, with $i < k$, then $[P_jP_l]$ is also a diameter of S for all l satisfying either $l \in]i,k[$ if $j \notin]i,k[$, or $l \notin]i,k[$ if $j \in]i,k[$.*

In view of Corollary 2.9 and Proposition 2.10, the construction of complete sets reduces to the one of democratic sets.

The next results describes how this can be achieved, in a recursive way. Since a democratic set containing three points is trivially an equilateral triangle, we are primarily interested in the case $\mathrm{Card}(S) \geq 5$.

Proposition 2.12 *Let P_1 and P_2 be two points in the plane and $n \geq 5$ an odd integer. Let d denote the distance between P_1 and P_2 and e the closed set $\mathcal{D}(P_1,d) \cap \mathcal{D}(P_2,D)$. We introduce further notations, $\{A,B\} := \mathcal{C}(P_1,d) \cap \mathcal{C}(P_2,d)$, $\gamma_i := \mathcal{C}(P_i,d) \cap \partial e$ and H_1, H_2 the half-planes in which the line P_1P_2 divides the plane. We choose an arbitrary point $P_3 \in H_1$ on $\gamma_1 \setminus \{P_1,A,B\}$ and we construct the points P_k for $3 \leq k \leq n-2$ recursively, as follows. $\mathcal{C}(P_k,d) \cap e$ consists of one single arc Q_kR_k, with $Q_k \in \gamma_1$ and $R_k \in \gamma_2$. We arbitrarily choose then*

$$P_{k+1} \in \begin{cases} P_{k-1}R_k & \text{if } k \text{ is odd} \\ P_{k-1}Q_k & \text{if } k \text{ is even} \end{cases}$$

Finally, $P_n := R_{n-1}$. Then $S := \{P_1,P_2,\ldots,P_n\}$ is a democratic set of diameter d. (see Figure 1)

Proof. We start by proving, using induction on $k \in \{2,3,\ldots,n\}$, the first claim concerning the existence of exactly one arc Q_kR_k and therefore of P_{k+1}, too. What we shall actually check, will be that $P_k \in H_1 \cap e$ if k is odd, and $P_k \in H_2 \cap e$ if k is even. Sice the cases $k = 1,2,3$ are trivial, suppose the claim true for all integers up to k and let us prove it for $k+1$. Say k is even (the case k odd can be treated analogously). Then $P_k \in \mathrm{Int}(e) \cap H_2$, which ensures $|P_kP_1|, |P_kP_2| < d$. As $e \cap H_1$ is convex, $\max_{X \in e \cap H_1} |P_kX|$ is attained for $X \in \{A,P_1,P_2\}$ and is larger than d, because $P_{k-1} \neq A$ lies also in $e \cap H_1$. Now, since the line P_1X doesn't meet $\gamma_2 \cap H_1$, it follows that there exists a unique $Q_k \in \gamma_2 \cap H_1$ such that $|P_kQ_k| = d$. Similarly, there is exactly one point $R_k \in \gamma_1 \cap H_1$ satisfying $|P_kR_k| = d$. So, the whole arc Q_kR_k containing

P_{k-1} is lying in $e \cap H_1$ and the construction of P_{k+1} is therefore possible.
We postpone the proof of the claim concerning the completeness of the set S, as this will follow directly from a later result (see Corollary 3.4). $\diamond$

3 Extensible sets

One further natural question we could raise now is the following. If the set S has $n \geq 3$ points and nonvanishing defect index, is it possible to enlarge S to a complete set $\bar{S}$, preserving also the length of the diameters ($d_S = d_{\bar{S}}$)? The main purpose of this section is to solve this extensibility problem. First, let us set up some terminology.

Definition 3.1 *Suppose S is a finite plane set. Then*

1. *If each point of S belongs to at least one diameter, the set S will be called* weakly complete.
2. *If there exists a weakly complete set $\overline{S}$ such that $S \subset \overline{S}$ and $d_S = d_{\overline{S}}$, S will be called* weakly extensible *and $\overline{S}$ will be refered to as a* weak extension *of S.*
3. *If there exists a complete set $\overline{S}$ such that $S \subset \overline{S}$ and $d_S = d_{\overline{S}}$, S will be called* strongly extensible *and $\overline{S}$ will be refered to as a* strong extension *of S.*

Using this definitions, we shall see next that finding a positive answer to the question above amounts to bringing all the points of the set S on diameters, by adding some new points.

Theorem 3.2 *A set S is strongly extensible iff it is weakly extensible.*

Proof. It is of course enough to prove that a weakly complete set S is strongly extensible. We prove this by induction on $i(S)$. As the claim is trivial if $i(S) = 0$, let us suppose that S has nonvanishing defect index. Let further S_1 denote the set of all vertices of degree 1 in Γ_S. Denote by Γ_S^2 the subgraph of Γ_S, consisting of all $v(\Gamma_S^2)$ vertices of degree at least 2 in Γ_S and the edges between them.
Case 1. $\Gamma_S^2 \neq \emptyset$
Then there exists at least one vertex of Γ_S^2, denoted by Q, which has degree at most 1 in Γ_S^2. Indeed, if all vertices of Γ_S^2 had degree at least 2, we would use the weak completeness and we would conclude that the number of edges in Γ_S is at least $\mathrm{Card}(S_1) + v(\Gamma_S^2) = \mathrm{Card}(S)$, meaning that S is complete. So the point Q has degree at least 2 in Γ_S and at most one of the vertices adjacent to Q has degree at least 2 in Γ_S. We label the vertices adjacent to Q by $P_1, P_2, \ldots, P_k$, $k \geq 2$ in such a way that $P_2, \ldots P_{k-1}$ lie inside the angle $\measuredangle P_1QP_k$. It follows that at least one of the vertices P_1 or P_k (say, P_1) has degree 1. We construct next the closed plane set e_{P_1Q} defined by $e_{P_1Q} := \mathcal{D}(P_1, d(S)) \cap \mathcal{D}(Q, d(S)) = e^1_{P_1Q} \cup e^2_{P_1Q}$, where $\mathcal{D}(X, r)$ denotes the closed disc of center X and radius r and $e^1_{P_1Q}, e^2_{P_1Q}$ are the closures of the two halves in which $[P_1Q]$ splits e_{P_1Q}. As $[P_1Q]$ is a diameter, all points of S must lie in e_{P_1Q}. Due to the choice of P_1, we may suppose that all points in S adjacent to Q are in $e^1_{P_1Q}$. This means that there are no points in S lying on the boundary of $e^2_{P_1Q}$, except for P_1 and Q. We choose next a point $R \in S$ such that the center N on $\partial e^2_{P_1Q}$ of the circle passing through R and Q with radius d_S is as close to P_1 as possible (see Figure 2). Of course, $N \notin S$ and we claim that $S \cup \{N\}$ has diameter d_S, $\mathrm{Card}(S) + 1$ points and at least two diameters more than S, namely $[QN]$ and $[NR]$. In fact, all we have to check is that $d_{S\cup\{N\}} = d_S$. Noting that e^1_{NQ} and e^2_{NQ} are two closed plane sets, with diameter d_S, the equality above follows immediately from the defining property of R, which ensures $S \subset e_{P_1Q} \cap e_{QN}$. Applying the induction assumption to the set $S \cup \{N\}$, for which $i(S \cup \{N\}) = \mathrm{Card}(S \cup \{N\}) - d(S \cup \{N\}) \leq \mathrm{Card}(S) - d(S) - 1 = i(S) - 1$ holds, we are done.

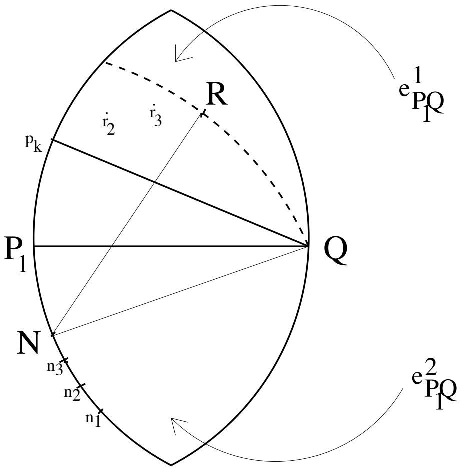

Figure 2: Construction of a new point N, based on the geometry of S.

Case 2. $\Gamma_S^2 = \emptyset$
This case is now trivial, as we can simply repeat the previous argument after having chosen Q to be an arbitrary point in S. $\diamond$

Due to Theorem 3.2, a weakly (or strongly) extensible set S will be called in the following *extensible*. For later use, we make the following remark, whose validity follows from the proof of Theorem 3.2.

Remark 3.3 *If S is weakly complete, then there exists a strong extension $\tilde{S}$ of S such that $\tilde{S} = \cup_{j=0}^{\infty} \mathcal{F}_i$, where $\mathcal{F}_0 := S$ and*

$$\mathcal{F}_j := \{P \in \tilde{S} \setminus \bigcup_{i=0}^{j-1} \mathcal{F}_i \mid P \text{ is adjacent in } \Gamma_{\tilde{S}} \text{ to at least 2 points of } \bigcup_{i=0}^{j-1} \mathcal{F}_i\},\ \forall j \geq 1.$$

As a second consequence of the proof of Theorem 3.2, we get next a sharp estimate for the cardinality of the strong extension that we have inductively constructed. More precisely,

Corollary 3.4 *If S is weakly complete, then there exists a strong extension $\bar{S}$ of S, containing exactly $2 \cdot \mathrm{Card}(S) - d(S)$ points.*

Proof. Within the setting of the previous proof, it is enough to check that $d(S \cup \{N\}) = d(S) + 2$, which amounts to proving that there are exactly two points in S (one of them being, of course, Q), lying on the dashed arc in Figure 2, that is, on $\mathcal{C}(N, d_S) \cap e_{P_1Q}$. Suppose that there are at least three such points, Q, R_1 and R_2, labeled in such a way that R_1 is between Q and R_2. We repeat here an argument we have already employed in Theorem 1.3. More precisely, since from $R_1 \in S \cup \{N\}$ emerges at least one diameter (with both ends in S), this must cross all other diameters, in particular $[NR]$ and $[NQ]$. It follows that R_1 is adjacent to exactly one diameter, namely $[NR_1]$. This in turn implies $N \in S$, contradicting therefore the defining property of P_1 which ensured $N \notin S$. The proof is complete. $\diamond$

Using Corollary 3.4 we now complete the proof of Proposition 2.12. In those notations, we shall prove, again by induction on k ($3 \leq k \leq n-2$), that

$$|P_k P_i| \leq d \qquad \forall 1 \leq i < k. \tag{3.1}$$

We suppose therefore that (3.1) holds for all integers up to k, and let us prove it for $k+1$. We assume again w.l.g. k odd and we remark first that (3.1) implies the extensibility of the set $\{P_1, P_2, \ldots, P_k\}$, as a weakly complete set. Moreover, Corollary 3.4 secures the existence of a point Q such that $\{P_1, P_2, \ldots, P_k, Q\}$ is a complete set. It is easy to see that, on account of Proposition 2.7, there are only two possibilities, namely: either $|QP_{k-1}| = |QP_2| = d$, or $|QP_1| = |QP_k| = d$. In both cases it follows, via Theorem (2.5), that $[P_1P_3 \ldots P_2P_4 \ldots]$ is a convex polygon. Of course, since $e \cap H_1$ is a closed convex set of diameter d, $|P_{k+1}P_i| \leq d$, $\forall i$ odd, $1 \leq i \leq k$. Now, defining $f := \mathcal{D}(P_k, d) \cap \mathcal{D}(P_{k-1}, d)$ and denoting by G_1 and G_2 the half-planes in which the line P_kP_{k-1} splits the plane (say $P_1 \in G_1$), it follows from the convexity of $[P_1P_3 \ldots P_2P_4 \ldots]$ that $P_i \in f \cap G_2$, $\forall i$ even, $1 \leq i \leq k$. Since also $P_{k+1} \in f \cap G_2$, and f is a closed convex set of diameter d, we deduce $|P_{k+1}P_i| \leq d$, also for all i even, $1 \leq i \leq k$. ◇

We shall prove next that the bound we obtained in Corollary 3.4 for the cardinality of a strong extension of a weakly complete set is optimal.

Theorem 3.5 *Let S be weakly complete and $\overline{S}$ be a strong extension of S. Then it holds:*

$$\mathrm{Card}(\overline{S}) \geq 2 \cdot \mathrm{Card}(S) - d(S). \tag{3.2}$$

Proof. Let $\overline{S}_1$ be the set defined by $\overline{S}_1 := \{P \in \overline{S} \setminus S \mid \deg_{\Gamma_{\overline{S}}}(P) = 1\}$. It follows that $\overline{S}' := \overline{S} \setminus \overline{S}_1$ is again a strong extension of S. Hence, it suffices to prove the inequality (3.2) with $\overline{S}$ replaced by $\overline{S}'$. We claim that each $P \in \overline{S}' \setminus S$ belongs to at most two diameters in $\overline{S}'$. If this is the case, then $\mathrm{Card}(\overline{S}') = d(\overline{S}') \leq d(S) + 2 \cdot \mathrm{Card}(\overline{S}' \setminus S)$, which implies (3.2), since $\mathrm{Card}(\overline{S}' \setminus S) = \mathrm{Card}(\overline{S}') - \mathrm{Card}(S)$.
It remains therefore to prove the claim above. Suppose there exists $P \in \overline{S}' \setminus S$ belonging to three diameters of $\overline{S}' \setminus S$, say $[PA], [PB], [PC]$. Without loss of generality, we may suppose that $B \in \mathrm{Int}(\measuredangle APC)$. Consequently, there exists exactly one diameter of $\overline{S}'$ emerging from B, namely $[BP]$. Now, either $B \in S$, which entails $P \in S$ (S is weakly complete!), or $B \in \overline{S}' \setminus S$ which leads to $\deg_{\overline{S}}(B) = 1$. Both conclusions are absurd, due to $P \in \overline{S}' \setminus S$ and to the definition of $\overline{S}'$. The proof is complete. ◇

So far we have proved that weak completeness ensures strong extensibility and we have been also able to construct in this case a strong extension containing as few points as possible. However, if the finite plane set S is arbitrary, the existence of an extension (weak or strong) is no longer guaranteed (to see this, take S such that not all of its points lie on the boundary of its convex hull). The characterization of strongly extensible sets is the problem we shall be addressing next. First, we shall see that if the extension is possible, then it can be done by adding essentially not more points than in the weakly complete case. Then we shall use this result to derive the desired extensibility criteria.

Theorem 3.6 *Suppose that S is an extensible finite point plane set. Then there exists a strong extension $\overline{S}$ such that*

$$\mathrm{Card}(\overline{S}) \leq 2 \cdot \mathrm{Card}(S) - d(S). \tag{3.3}$$

Proof. For each weak extension S^* of S we can define on the set $S_0 := \{M \in S \mid deg_{\Gamma_S}(M) = 0\}$ at least one function f which associates to M a point $f(M) \in S^* \setminus S$ adjacent to M in Γ_{S^*}, that is, such that $[Mf(M)]$ is a diameter in S^*. Such pairs (S^*, f) will be called in the following *admissible*. We note that the range of f depends on the weak extension S^*, while the domain is S_0, a fixed subset of S. For each admissible pair (S^*, f) we define $S' := S \cup \mathrm{Range}(f)$ and we note that S' is weakly complete. If we invoke at this moment Corollary 3.4, for the set S', this will provide us with a strong extension $\overline{S}$ of S', therefore of S too, such that $\mathrm{Card}(\overline{S}) \le 2 \cdot \mathrm{Card}(S') - d(S') \le 2 \cdot \mathrm{Card}(S) - d(S) + \mathrm{Card}(S_0)$. But this inequality is by one large term ($\mathrm{Card}(S_0)$) weaker than what we actually need. Hence, at this moment Corollary 3.4 is not worth employing. Instead of using it, let us recursively define, for each admissible pair (S^*, f), the sets $(\mathcal{F}_j)_{j\ge 0}$ by $\mathcal{F}_0 := S$ and

$$\mathcal{F}_j := \{P \in \mathrm{Range}(f) \setminus \bigcup_{i=0}^{j-1} \mathcal{F}_i \mid P \text{ is adjacent in } \Gamma_{S^*} \text{ to at least 2 points from } \bigcup_{i=0}^{j-1} \mathcal{F}_i\},\ \forall j \ge 1.$$

We remark that the sets $\mathcal{F}_j$ are pairwise disjoint and empty for large j, namely for $j \ge \mathrm{Card}(S_0) + 1$. For later use, we also note that the very definition of the sets $(\mathcal{F}_j)_{j\ge 0}$ already provides us with a lower estimate for the number $d(\bigcup_{i\ge 0} \mathcal{F}_i)$ of diameters of S' having both ends in $\bigcup_{i\ge 0} \mathcal{F}_i$, that is,

$$d(\bigcup_{i\ge 0} \mathcal{F}_i) \ge d(S) + 2 \cdot \mathrm{Card}(\bigcup_{i\ge 0} \mathcal{F}_i). \tag{3.4}$$

Define further $\mathcal{E}(S^*, f) := \mathrm{Range}(f) \setminus \bigcup_{i\ge 0} \mathcal{F}_i$. Obviously, $\mathcal{E}(S^*, f)$ is a finite set, so let us choose now the pair (S^*, f) in such a way that $\mathrm{Card}(\mathcal{E}(S^*, f))$ is as small as possible. We claim that for this choice of (S^*, f), the set $\mathcal{E}(S^*, f)$ is empty. Indeed, suppose that $\mathcal{E}(S^*, f) \ne \emptyset$. We infer then from the definition of $\mathcal{E}(S^*, f)$ that each point $P = f(M) \in \mathcal{E}(S^*, f)$ is adjacent in Γ_{S^*} to exactly one vertex from $\bigcup_{i\ge 0} \mathcal{F}_i$, namely M. We choose now $P \in \mathcal{E}(S^*, f)$ and $A, B \in \bigcup_{i\ge 0} \mathcal{F}_i$ such that $[AB]$ is a diameter and the measure $0 < \alpha \le \pi/2$ of the angle between $[PM]$ and $[AB]$ is the smallest possible for all choices of $P \in \mathcal{E}(S^*, f)$ and $A, B \in \bigcup_{i\ge 0} \mathcal{F}_i$ with A, B adjacent (see Figure 3).

Case1. $\{M, P\} \cap \{A, B\} \ne \emptyset$

We may assume $A = M$ and it follows $B \notin S$ (otherwise P wouldn't exist!). This shows that B is in the range of f, that is $B = f(Q) \in \bigcup_{i\ge 0} \mathcal{F}_i$, with $Q \ne M$. We define then the function $g : S_0 \to S' \setminus (S \cup \{P\})$ to be equal to f on $S_0 \setminus \{M\}$ and by $g(M) := f(Q) = B$. It follows that $(S' \setminus \{P\}, g)$ is an admissible pair and that $\mathcal{E}(S' \setminus \{P\}, g) \subset \mathcal{E}(S^*, f) \setminus \{P\}$, contradicting the minimality property of the pair (S^*, f).

Case 2. $\{M, P\} \cap \{A, B\} = \emptyset$

Let us denote by X the crossing point of the diameters $[PM]$ and $[AB]$. We may assume that the measure of the angle $\measuredangle MXB$ is not larger than $\pi/2$ and we claim that no points of $S' := S \cup \mathrm{Range}(f)$ lie then inside the angles $\measuredangle AXP$ and $\measuredangle MXB$. Indeed, if R were such a point, say $R \in \mathrm{Int}(\measuredangle AXP)$, then an arbitrary diameter $[RT]$ emerging from R (S' is weakly complete!) would satisfy, due to Lemma 1.2, $T \in \mathrm{Int}(\measuredangle MXB)$ (see Figure 3). If $R \in \mathrm{Range}(f)$ or $T \in \mathrm{Range}(f)$ (say, for instance, that the first holds), it would follow $R = f(U)$, with $U \in \mathrm{Int}(\measuredangle MXB)$, and we would contradict the minimality property of α, either with the angle between the diameters $[RU]$ and $[AB]$ if $R \in \mathcal{E}(S^*, f)$, or with the angle between $[PM]$ and $[RU]$ if $R \in \bigcup_{i\ge 0} \mathcal{F}_i$. We deduce that $R, T \in S \subset \bigcup_{i\ge 0} \mathcal{F}_i$ should hold, and in this case we would once again contradict the minimality of α with the angle between $[PM]$ and $[RT]$. The proof of the fact that $S' \cap (\mathrm{Int}(\measuredangle AXP) \cup \mathrm{Int}(\measuredangle MXB)) = \emptyset$ is therefore complete.

We remark now that inside the angle $\measuredangle AXP$ there exists exactly one point $\tilde{P}$ such that $|M\tilde{P}| = |B\tilde{P}| = d_S$. We consider then the set $S'' := (S' \setminus \{P\}) \cup \{\tilde{P}\}$ and we note that $d_{S''} = d_S$. Indeed, if $Q \in S' \setminus \{P\}$, then Q lies in $\mathrm{Int}(\measuredangle AXM) \cup \mathrm{Int}(\measuredangle PXB)$, which implies (say, $Q \in \mathrm{Int}(\measuredangle AXM)$, so that $[Q\tilde{P}PM]$ is a convex quadrilateral) $|Q\tilde{P}| + |MP| < |QP| + |M\tilde{P}|$ or,

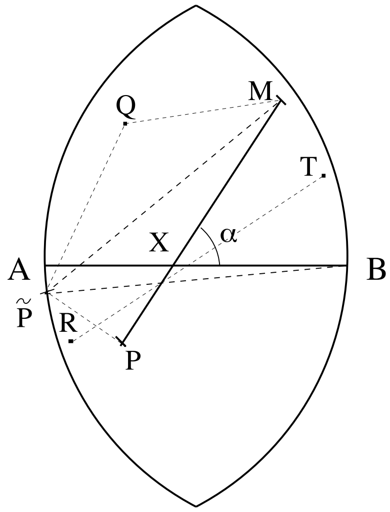

Figure 3: Replacing P by $\tilde{P}$ and the new diameter $[\tilde{P}B]$ gained thereby.

equivalently, $|Q\tilde{P}| < |QP| \leq d_S$ (see again Figure 3). Moreover, the set S'' is weakly complete and we can define $g : S_0 \to S'' \setminus \{P\}$ to be equal to f on $S_0 \setminus \{M\}$ and by $g(M) := \tilde{P}$. The pair (S'', g) is admissible and $\mathcal{E}(S'', g) \subset \mathcal{E}(S^*, f) \setminus \{P\}$, contradicting again the minimality property of the pair (S^*, f). We conclude that the set $\mathcal{E}(S^*, f)$ is empty. Hence (3.4) reads now

$$d(S') \geq d(S) + 2 \cdot \mathrm{Card}(\mathrm{Range}(f)). \tag{3.5}$$

Besides,

$$\mathrm{Card}(S') = \mathrm{Card}(S) + \mathrm{Card}(\mathrm{Range}(f)). \tag{3.6}$$

Since S' is weakly closed, there exists, cf. Corollary 3.4, a strong extension $\overline{S}$ of S', therefore of S too, such that

$$\mathrm{Card}(\overline{S}) \leq 2 \cdot \mathrm{Card}(S') - d(S') \tag{3.7}$$

Using (3.5) and (3.6) in (3.7) we obtain (3.3). The proof of the theorem is complete. ◇

Remark 3.7 *The previous result shows that if an arbitrary finite plane set S is extensible, then there exists a strong extension $\tilde{S}$ of S such that $\tilde{S} = \cup_{j=0}^{\infty} \mathcal{F}_i$, where $\mathcal{F}_0 := S$ and*

$$\mathcal{F}_j := \{P \in \tilde{S} \setminus \bigcup_{i=0}^{j-1} \mathcal{F}_i \mid P \textit{ is adjacent in } \Gamma_{\tilde{S}} \textit{ to at least 2 points from } \bigcup_{i=0}^{j-1} \mathcal{F}_i\},\ \forall j \geq 1.$$

(compare Remark 3.3)

Based only on Remark 3.7, we can give already at this stage an algorithm to check on whether an arbitrary finite plane set S containing n points is extensible or not. Let us briefly describe how we proceed. As long as $i(S) > 0$, we add to the set S a new point P_{n+k+1} (say that we have already constructed k new points $P_{n+1}, P_{n+2}, \ldots, P_{n+k}$) such that $S \cup \{P_{n+k+1}\}$ has at least two diameters more than S ($[P_{n+k+1}P_i]$ and $[P_{n+k+1}P_j]$, for some $1 \leq i < j \leq n+k$). Of

course there are already $2 \cdot \binom{n+k}{2}$ possibilities to be considered, and the algorithm should carefully examine all of them. Now, to ensure that we are on the way to a complete set, $S \cup \{P_{n+k+1}\}$ must preserve the diameter d_S. If, however, this is not the case, we pick a new pair (i, j) with $1 \leq i < j \leq n+k$ in order to construct the point P_{n+k+1}. If, moreover, no pair (i, j) provides us with a good candidate for P_{n+k+1}, we go backwards, remove all the points we have added to our set (that is, $P_{n+k}, P_{n+k-1}, \ldots$) up to the next one for which a new choice is still possible (say, P_{l+1}). We replace then P_{l+1} by a new point from the $2 \cdot \binom{n+l}{2}$ candidates at this level and restart the procedure. In this way, we either reach a complete set, or, after examining all possibilities, we are left with the original set S.
As we can see, the question as to whether an arbitrary set is extensible or not can be answered using a computational approach. However, the algorithm presented above is, in some respects, not optimal. It does not take into account for instance various geometrical properties of the point sets involved and this in turn leads to a very high complexity which restricts its applicability only to small values of $\mathrm{Card}(S)$.

In the following we show therefore how further geometrical information can be employed to refine this algorithm. The main purpose is of course to reduce its complexity, which is coming from the huge (although finite) number of choices for new points to be added to S at each step.

Definition 3.8 *An arbitrary finite plane set S is called* reduced *if $\deg_{\Gamma_S} P \leq 2$ for all $P \in S$.*

To each set S we then canonically associate a reduced set S_r as follows: we choose P such that $k := \deg_{\Gamma_S} P \geq 3$ and label the points adjacent to P in Γ_S clockwise, by $P_1, P_2, \ldots, P_k$. We remove then $P_2, \ldots, P_{k-1}$ from S and we repeat this operation as long as there exist points in S of degree at least 3 in Γ_S (compare Theorem 2.7). In the end, what we are left with is a reduced set, denoted by S_r and referred to as the *reduced core* of S. Note the consistency of this definition with the one of the democratic core associated to a complete set: the reduced core of a complete set coincides, due to Theorem 2.7, with its democratic core. In view of Corollary 2.9 and Proposition 2.10, it is straightforward to deduce

Proposition 3.9 *An arbitrary finite plane set S is extensible if and only if S_r is extensible.*

We shall also consider for an arbitrary finite plane set S the splitting of Γ_S in connected components. Note that if the set S is reduced and Γ_c is such a connected component, then Γ_c can be either a point, a chain or a cycle. In case Γ_c is a cycle, the question of extensibility is easily answered. Indeed, due to Corollary 2.4, if S is extensible, it must be already complete and this information is computationally available at a low cost.

The main tool we shall use to refine the extension algorithm is the following result.

Proposition 3.10 *Let S be an arbitrary plane finite set. If S is reduced, extensible, and no connected component of Γ_S is a cycle, then*

1. *We can label its points by $P_1, P_2, \ldots, P_n$ such that $\mathcal{P} := [P_1 P_2 \ldots P_n]$ is a convex polygon.*

2. *Denoting by Γ_c a connected component of Γ_S which contains at least one edge, Γ_c is a chain and it reads $\Gamma_c = [P_i P_j P_{i+1} P_{j+1} \ldots P_m]$ for some $1 \leq i, j \leq n$, where $P_{n+k} := P_k$ $\forall k \in \mathbb{N}^*$.*

3. *With Γ_c as in 2., let P^1_{n+1} and P^2_{n+1} be the points defined by $|P_{i-1}P^1_{n+1}| = |P_i P^1_{n+1}| = d_S$, $|P_{j-1}P^2_{n+1}| = |P_j P^2_{n+1}| = d_S$ and lying on the same side of the lines $P_i P_{i-1}$ and $P_j P_{j-1}$*

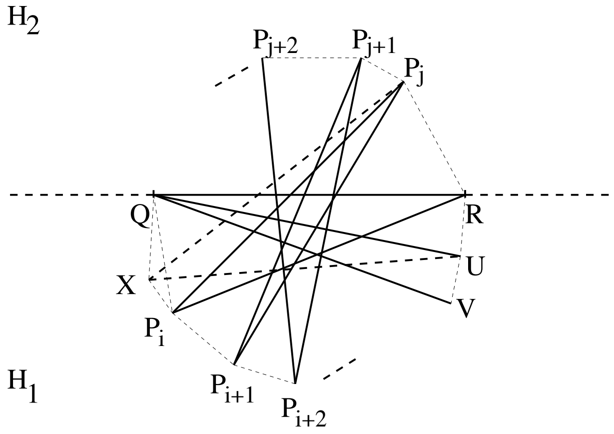

Figure 4: Replacing R by X, as in *Step 1.*

respectively, as $\mathcal{P}$. Then at least one of the sets $S \cup \{P^1_{n+1}\}, S \cup \{P^2_{n+1}\}$ is extensible and its defect index is strictly smaller than $i(S)$.

Proof. 1. follows directly from Theorem 2.5, while 2. is a consequence of the fact that S is reduced. To prove 3., let us first recall that, S being extensible, Remark 3.7 ensures the existence of a set

$$\tilde{S} \text{ strong extension of } S, \text{ s.t. } \tilde{S} = \bigcup_{j=0}^{\infty} \mathcal{F}_i, \quad \text{with} \quad \mathcal{F}_0 := S \quad \text{and}$$

$$\mathcal{F}_j := \{P \in \tilde{S} \setminus \bigcup_{i=0}^{j-1} \mathcal{F}_i \mid P \text{ is adjacent in } \Gamma_{\tilde{S}} \text{ to at least 2 points from } \bigcup_{i=0}^{j-1} \mathcal{F}_i\}, \ \forall j \geq 1. \quad (3.8)$$

To each such $\tilde{S}$ we associate the integer $\ell(\tilde{S}) := \sum_{P \in \tilde{S}} l(P)$, where $l(P) \in \mathbb{N}$ is well-defined for each $P \in \tilde{S}$ by $P \in \mathcal{F}_{l(P)}$. Now, of all $\tilde{S}$ satisfying (3.8), we choose one, denoted in the following also by $\tilde{S}$, such that $\ell(\tilde{S})$ is minimal. It is for this $\tilde{S}$ that we shall prove that $P^1_{n+1} \in \tilde{S}$ or $P^2_{n+1} \in \tilde{S}$ holds. To this end, let us consider Q, R to be vertices of $\mathcal{P}$ such that Q, P_i, P_{i+1} and R, P_j, P_{j+1} are consecutive vertices of $\mathcal{P}$. What we shall prove is that

$$\text{either} \quad Q \in S, R \notin S, |RQ| = |RP_i| = d_S, \quad \text{or} \quad Q \notin S, R \in S, |QR| = |QP_j| = d_S, \qquad (3.9)$$

that is, either $R = P^1_{n+1}$, or $Q = P^2_{n+1}$.
In the following, we denote by H_1 and H_2 the half-planes containing P_i and P_j respectively, in which the line QR splits the plane.
Step 1. $Q \in S$ or $R \in S$
Let us argue by contradiction and suppose that $Q \notin S$ and $R \notin S$. This implies that at least one of $[RQ], [RP_i]$ is a diameter, otherwise Lemma 1.2 would ensure $\deg_{\Gamma_{\tilde{S}}}(Q) = 1$ with $[QP_j]$ the only diameter emerging from Q, which would in turn clearly contradict $Q \in \tilde{S} \setminus S$. Similarly, one of $[QR], [QP_j]$ must be a diameter. Since the triangle inequality guarantees that $[QP_j]$ and $[RP_i]$ can not be simultaneously diameters, we conclude that $[QR]$ is a diameter. Further, it is straightforward to see that exactly one of $[RP_i], [QP_j]$ is a diameter. Indeed, if

none of them were diameters, it would follow from $\deg_{\Gamma_{\tilde{S}}}(R) \geq 2$ ($R \in \tilde{S} \setminus S$!), via Lemma 1.2, that $\deg_{\Gamma_{\tilde{S}}}(Q) = 1$, which would contradict once again $Q \in \tilde{S} \setminus S$.
Let us say, for instance, that $[RP_i]$ is a diameter. (The case when $[QP_j]$ is a diameter can be ruled out analogously.) Note first that $\deg_{\Gamma_{\tilde{S}}}(R)$ is then exactly 2 (otherwise again $\deg_{\Gamma_{\tilde{S}}}(Q) = 1$, absurd), with $[RP_i]$ and $[RQ]$ the only two diameters emerging from R (see Figure 4). This means, of course, that $l(R) = l(Q) + 1$. Now $Q \in \mathcal{F}_{l(Q)}$ is adjacent to at least two points in $\cup_{j=0}^{l(Q)-1} \mathcal{F}_i$ and this ensures, taking into account also the second claim of Proposition 2.11, the existence of two diameters $[QU]$ and $[QV]$, where P_jRUV are consecutive vertices of $\mathcal{P}$. Since, obviously, $\deg_{\Gamma_{\tilde{S}}}(U) = 1$, we conclude $U \in S$. We construct next a point X such that $|XP_j| = |XU| = d_{\tilde{S}}$ and X, R do not lie on the same side of the line going through P_j and U. The same argument we have used in Theorem 3.6 shows that $d_{(\tilde{S} \cup \{X\}) \setminus \{R\}} = d_{\tilde{S}}$. Moreover, by removing the point R from $\tilde{S}$ and adding then X to $\tilde{S} \setminus \{R\}$ we preserve the number of diameters (remove 2 of them and add 2 new ones). This means that $(\tilde{S} \cup \{X\}) \setminus \{R\}$ is still a complete set. We note further that $(\tilde{S} \cup \{X\}) \setminus \{R\}$ fulfills condition (3.8) if we remove, of course, R from $\mathcal{F}_{l(R)}$ and then add X to $\mathcal{F}_1$. Since $l(X) = 1$, we can then write $\ell((\tilde{S} \cup \{X\}) \setminus \{R\}) = \ell(\tilde{S}) - l(R) + l(X) = \ell(\tilde{S}) - l(Q) < \ell(\tilde{S})$. But this contradicts the definition of $\tilde{S}$, which involved the minimality of $\ell(\tilde{S})$. Hence the proof of the first step is complete.
Step 2. $\{Q, R\} \not\subset S$
We have to check that both Q and R can not be simultaneously elements of S. Suppose, again by contradiction, that $Q, R \in S$. Since P_j and R are consecutive vertices of $\mathcal{P}$, the first claim of Proposition 2.11 secures the existence of a point $W \in \tilde{S}$ such that $[WP_j]$ and $[WR]$ are both diameters. Then, necessarily, $W = P_i$, otherwise either $[WR] \cap [P_iP_j] = \emptyset$, contradicting Lemma 1.2, if $W \in H_1$, or $[QP_j]$ is a diameter, (use again *2.* from Proposition 2.11) contradicting S reduced, if $W \in H_2$. Further, if $W = P_i$, it follows $\deg_{\Gamma_S}(P_i) = 2$, again absurd, due to the fact that P_i is one end of the chain Γ_c and has therefore degree 1 in Γ_S. This contradiction concludes the proof of the second step.
Step 3. Proof of (3.9)
If, say, $Q \notin S$ and $R \in S$, let us consider again $W \in \tilde{S}$ such that $[WP_j]$ and $[WR]$ are both diameters. We claim that $W = Q$, which actually means $Q = P^2_{n+1}$. Now, $W \in H_1$ can not hold, since this would imply $[WR] \cap [P_iP_j] = \emptyset$ absurd in view of Lemma 1.2. Neither can $W = P_i$ hold, since this would imply again $\deg_{\Gamma_S}(P_i) = 2$. So $W \in H_2$. If $W = Q$ we are done, while if $W \neq Q$, Lemma 1.2 ensures that there are exactly two diameters emerging from Q (there are at least two, anyway, since $Q \notin S$!), namely $[QP_j]$ and $[QR]$. But this contradicts the earlier assumption $W \neq Q$, thereby ruling out this case.
If, reversely, $Q \in S$ and $R \notin S$, we argue similarly and we eventually draw a symmetric conclusion, namely that $[QP_j]$ and $[QR]$ are diameters of $\tilde{S}$, or, in other words, $R = P^1_{n+1}$. ⋄

As we claimed, we use now Proposition 3.10 to improve the extension algorithm by reducing its complexity. We start by replacing the arbitrary set S by its reduced core S_r, of cardinality, say, n. As long as $i(S_r) > 0$, we perform steps 1. and 2. as described in Proposition 3.10 (if already this is not possible, the set S is not extensible). We add then to the set S_r a new point, P_{n+k+1} (say that we have already added k new points $P_{n+1}, P_{n+2}, \ldots, P_{n+k}$), constructed as described in step 3., Proposition 3.10. We stress that there are only two possibilities to do this, which is a significant reduction, compared with the original algorithm, where already at this point $2 \cdot \binom{n+k}{2}$ chioces were admissible. So, either $P_{n+k+1} := P^1_{n+k+1}$ or $P_{n+k+1} := P^2_{n+k+1}$ (the superscript defined according to step 3., Proposition 3.10). We choose first $P_{n+k+1} := P^1_{n+k+1}$ and we check that $S_r \cup \{P^1_{n+k+1}\}$ has diameter d_S. If this is not the case, we replace P^1_{n+k+1}

by P^2_{n+k+1} and check again that this has not increased the diameter. If, however, the diameter has changed, we go backwards, remove all the points we have added to our set (that is, $P^2_{n+k}, P^2_{n+k-1}, \ldots$) up to the next one for which a new choice is still possible (say, P^1_{l+1}). We replace then P^1_{l+1} by P^2_{l+1} and restart the procedure. In this way, we either reach a complete set, or, after examining all possibilities, we are left with set S_r.

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No.	Authors	Title
03-05	M. Becheanu, R.A. Todor	On the Set of Diameters of Finite Point-Sets in the Plane
03-04	C. Schwab, R.A. Todor	Sparse finite elements for stochastic elliptic problems - higher order moments
03-03	R. Sperb	Bounds for the first eigenvalue of the elastically supported membrane on convex domains
03-02	F.M. Buchmann	Computing exit times with the Euler scheme
03-01	A. Toselli, X. Vasseur	Domain decomposition preconditioners of Neumann-Neumann type for *hp*-approximations on boundary layer meshes in three dimensions
02-26	M. Savelieva	Theoretical study of axisymmetrical triple flame
02-25	D. Schötzau, C. Schwab, A. Toselli	Mixed *hp*-DGFEM for incompressible flows III: Pressure stabilization
02-24	F.M. Buchmann, W.P. Petersen	A stochastically generated preconditioner for stable matrices
02-23	A.W. Rüegg, A. Schneebeli, R. Lauper	Generalized *hp*-FEM for Lattice Structures
02-22	L. Filippini, A. Toselli	*hp* Finite Element Approximations on Non-Matching Grids for the Stokes Problem
02-21	D. Schötzau, C. Schwab, A. Toselli	Mixed *hp*-DGFEM for incompressible flows II: Geometric edge meshes
02-20	A. Toselli, X. Vasseur	A numerical study on Neumann-Neumann and FETI methods for *hp*-approximations on geometrically refined boundary layer meshes in two dimensions
02-19	D. Schötzau, Th.P. Wihler	Exponential convergence of mixed *hp*-DGFEM for Stokes flow in polygons
02-18	P.-A. Nitsche	Sparse approximation of singularity functions
02-17	S.H. Christiansen	Uniformly stable preconditioned mixed boundary element method for low-frequency electromagnetic scattering
02-16	S.H. Christiansen	Mixed boundary element method for eddy current problems
02-15	A. Toselli, X. Vasseur	Neumann-Neumann and FETI preconditioners for *hp*-approximations on geometrically refined boundary layer meshes in two dimensions