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Abstract

This work improves and extends results of [35] on time-discretization ansatzes for power-law models ($p \leq 2$). New analytical results and techniques from [9] lead to improved convergence rates for a broader range of admissible p's. Then, optimally converging stabilization strategies for the time-discretization are discussed and it is shown how the range of p's enlarges, for which strong solutions of the stabilized system exist.

Keywords: Non-Newtonian fluid flow, degenerate parabolic system, time-discretization, weak and strong solution, shear-dependent viscosity, error analysis, stabilization

Subject Classification: 76A05, 35K65, 35B65, 65M12, 65M15

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1. Introduction and Main Results

The motion of incompressible viscous fluids is well–described by the system[1]

$$\begin{aligned} \rho\, \partial_t \mathbf{u} - \operatorname{div}\, \mathbf{S} + \rho\, [\nabla \mathbf{u}]\mathbf{u} + \nabla \pi &= \rho\, \mathbf{f}\,, \\ \operatorname{div}\, \mathbf{u} &= 0\,, \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the velocity, $\mathbf{S}$ the extra stress tensor, π is the pressure, $\mathbf{f}$ the external body force, and ρ the density. If the fluid under consideration can be viewed as a *generalized Newtonian* fluid, then the extra stress tensor is given by

$$\mathbf{S} = \mu\, \big(1 + |\mathbf{D}|^2\big)^{\frac{p-2}{2}} \mathbf{D}\,, \tag{1.2}$$

where $\mathbf{D} \equiv \frac{1}{2}\big(\nabla \mathbf{u} + \nabla \mathbf{u}^\top\big)$ is the symmetric part of the velocity gradient. This model belongs to the class of power–law models, which are frequently used in the engineering literature (cf. Bird, Armstrong, Hassager [7] for a detailed discussion of power–law models including early models due to Ostwald, de Waele and Kincaid, Stern, Powell and Eyring). We also refer to Málek, Rajagopal, Růžička [33] for a discussion of such models. Let us only mention that most real fluids that can be modeled by a constitutive law of type (1.2) have a small exponent p, i.e. $p \in (1, 2]$.

First mathematical investigations of the system (1.1), (1.2) have been carried out by Ladyzhenskaya [19], [20], [21] (see also Lions [27] for a comparable proof of the same results). The papers of Málek, Nečas, Růžička [30] and Bellout, Bloom, Nečas [6] have been a starting point for many investigations and improvements of the previous results. We refer to Málek, Nečas, Rokyta, Růžička [29], Frehse, Málek, Steinhauer [10], [11], Málek, Nečas, Růžička [31], Málek, Rajagopal, Růžička [33], Růžička [36], Pokorný [34], Diening, Růžička [9] and Diening [8] for results concerning the existence of weak and strong solutions for the steady and the unsteady system; to Málek, Nečas [28], Ladyzhenskaya, Seregin [22], [23], [24], Málek, Pražák [32] and Amann [1], [2], [3] for results concerning the long time behaviour; to Kaplický, Málek, Stará [16], [17], [18], Seregin [38], [39], Ladyzhenskaya, Seregin [25], Fuchs, Seregin [13], [14], Diening [8], Friedländer, Pavlovič [12] and Guo, Zhu [15] for regularity properties of solutions. On the other hand there are only very few numerical investigations of flows of generalized Newtonian fluids (cf. Baranger, Najib, Sandri [5], Bao, Barrett [4], Layton [26], Prohl, Růžička [35] and Diening [8]).

In this paper we want to improve and extend the results of [35]. Due to the results and techniques developed in [9] it is possible to improve for the time-discretization of (1.1), (1.2) both the convergence rates for the error and the range of p's for which these results hold. Moreover, for a subsequent analysis of the space-discretization it is shown in [35] that the existence of strong solutions to the time-discretized system is essential. Thus we will also discuss different stabilization strategies for the time-discretization and show how the range of p's enlarges, for which strong solutions of the stabilized system exist.

Before formulating the main results, we will collect some notations and state the assumptions under which we will treat our problem. We assume that $\Omega = (0, L)^3$,

[1] Here and in the following we use the notation $[\nabla \mathbf{u}]\mathbf{w} = \big(w_j \frac{\partial u_i}{\partial x_j}\big)_{i=1,2,3}$, where the summation convention over repeated indices is used.

$L \in (0,\infty)$ is a cube in $\mathbb{R}^3$ and denote $\Gamma_j = \partial\Omega \cap \{x_j = 0\}$ and $\Gamma_{j+3} = \partial\Omega \cap \{x_j = L\}$, for $j = 1,2,3$. For $T \in (0,\infty)$, we denote by Q_T the time-space cylinder $I \times \Omega$, where $I = (0,T)$ is a time interval. Let $(X(\Omega), \|\cdot\|_{X(\Omega)})$ be a Banach space of scalar functions that are defined on Ω. Then $\mathbf{X}(\Omega) \equiv X^3(\Omega)$ (resp. $X^{3\times 3}(\Omega)$) represents the space of vector-valued (resp. tensor-valued) functions whose components belong to $X(\Omega)$. By $\boldsymbol{\mathcal{D}}(\Omega)$ we denote the space of smooth periodic functions with mean value zero. Let further $p,q > 1$ and $k > 0$. Then $(L^p(\Omega), \|\cdot\|_p)$, resp. $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, is used for the usual Lebesgue, resp. Sobolev, spaces of periodic functions with mean value zero. We shall further make frequent use of spaces of divergence free functions defined by

$$\begin{aligned} \boldsymbol{\mathcal{V}} &\equiv \{\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\Omega) : \ \operatorname{div} \boldsymbol{\psi} = 0\}, \\ \mathbf{H} &\equiv \text{the closure of } \boldsymbol{\mathcal{V}} \text{ with respect to the } \|\cdot\|_2\text{-norm}, \\ \mathbf{V}_p &\equiv \text{the closure of } \boldsymbol{\mathcal{V}} \text{ with respect to the } \|\nabla\cdot\|_p\text{-norm}. \end{aligned}$$

Moreover, we denote by $L^q(I;X)$ Bochner spaces which are equipped with the norm $\left(\int_I \|\cdot\|_X^q \, ds\right)^{1/q}$. We make frequent use of the discrete counterparts of these spaces. Let $I_k = \{t_m\}_{m=0}^M$ be a given net in an interval $I = [0, t_M]$ with a constant time-step size $k := t_m - t_{m-1}$. We denote by $d_t\mathbf{u}^m := k^{-1}(\mathbf{u}^m - \mathbf{u}^{m-1})$ the divided difference in time. By $l^p(I_k;X)$ we denote the space of functions $\{\phi^{m+1}\}_{m=0}^M$ with finite norm $\left(k\sum_{m=0}^M \|\phi^m\|_X^p\right)^{1/p}$. In the case $p = \infty$, functions $\{\phi^m\}_{m=0}^M$ need to satisfy the bound $\max_{0\le m\le M} \|\phi^m\|_X < \infty$.

Let $\mathbf{f}$ and $\mathbf{u}_0$ be a given external body force and a given initial velocity, respectively. Further assume that the extra stress tensor $\mathbf{S}$ is given by a potential, i.e. we assume the existence of a convex function $\Phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ which belongs to $C^2(\mathbb{R}_0^+)$ and satisfies $\Phi(0) = \Phi'(0) = 0$, and the existence of constants $C_1, C_2 > 0$ such that for some $p > 1$ and for all $r,s,m,n = 1,2,3$, $\mathbf{B}$, $\mathbf{D} \in \mathbb{R}^{3\times3}_{sym} \equiv \{\mathbf{D} \in \mathbb{R}^{3\times3}; D_{ij} = D_{ji},\ i,j = 1,2,3\}$

$$S_{rs}(\mathbf{D}) = \partial_{rs}\Phi(|\mathbf{D}|) \equiv \frac{\partial\Phi(|\mathbf{D}|)}{\partial D_{rs}}, \tag{1.3}$$

$$\partial_{ij}\partial_{kl}\Phi(|\mathbf{D}|)B_{ij}B_{kl} \ge C_1(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}|\mathbf{B}|^2, \tag{1.4}$$

$$\left|\partial_{rs}\partial_{mn}\Phi(|\mathbf{D}|)\right| \le C_2(1+|\mathbf{D}|^2)^{\frac{p-2}{2}}. \tag{1.5}$$

We are seeking solutions $\mathbf{u}$ of the system [2]

$$\begin{aligned} \partial_t\mathbf{u} - \operatorname{div}\, \mathbf{S}(\mathbf{D}(\mathbf{u})) + [\nabla\mathbf{u}]\mathbf{u} + \nabla\pi &= \mathbf{f}, \\ \operatorname{div}\, \mathbf{u} &= 0, \\ \mathbf{u}(0) &= \mathbf{u}_0, \end{aligned} \tag{1.6}$$

endowed with space-periodic boundary conditions

$$\mathbf{u}\big|_{\Gamma_j} = \mathbf{u}\big|_{\Gamma_{j+3}}, \tag{1.7}$$

for $j = 1,2,3$. We refer to (1.6), (1.7) under the assumption (1.3)–(1.5) on $\mathbf{S}$ as problem $(\mathrm{NS})_p$. The state of the art for the problem $(\mathrm{NS})_p$ can be found in the above

[2]Note that the system (1.1) should be appropriately non-dimensionalized. Since we are not interested in the dependence of our results on the resulting non-dimensional numbers, we set $\rho = 1$ in (1.1).

mentioned literature. We only want to mention that in [8], [9] it is proven that there exists a time interval $I \equiv [0, T^*]$, with T^* depending on the data $\mathbf{f}$, $\mathbf{u}_0$, on which the problem $(\mathrm{NS})_p$ possesses a *strong solution* $\mathbf{u}$, i.e. for all $1 < r < 6(p-1)$ there holds

$$\begin{aligned} \mathbf{u} &\in L^{\frac{5p-6}{2-p}}\big(I; \mathbf{W}^{2,\frac{3p}{p+1}}(\Omega)\big) \ \cap\ C(I; \mathbf{V}_r)\,, \\ \partial_t \mathbf{u} &\in L^{\frac{p(5p-6)}{(3p-2)(p-1)}}\big(I; \mathbf{W}^{1,\frac{3p}{p+1}}(\Omega)\big) \ \cap\ L^\infty\big(I; \mathbf{L}^2(\Omega)\big)\,, \\ \partial_t^2 \mathbf{u} &\in L^2\big(I; (\mathbf{V}_2)^*\big)\,. \end{aligned} \tag{1.8}$$

The problem $(\mathrm{NS})_p$ is approximated by a time-discretization by means of the implicit Euler scheme:

Algorithm. *Given a time-step size $k > 0$ and a corresponding net $I_k = \{t_m\}_{m=0}^M$. For $m \ge 1$ and $\mathbf{u}^{m-1}$ given from the previous step, compute an iterate $\mathbf{u}^m$ that solves*

$$\begin{aligned} d_t \mathbf{u}^m - \operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{u}^m)) + [\nabla \mathbf{u}^m]\mathbf{u}^m + \nabla \pi^m &= \mathbf{f}(t_m)\,, \\ \operatorname{div} \mathbf{u}^m &= 0\,, \\ \mathbf{u}^0 &= \mathbf{u}_0\,, \end{aligned} \tag{1.9}$$

endowed with space-periodic boundary conditions (1.7)*. We refer to* (1.9) *with boundary conditions* (1.7) *under the assumption* (1.3)–(1.5) *on* $\mathbf{S}$ *as problem* $(NS_k)_p$*.*

Note that in [35] the existence of a weak solution $\mathbf{u}^m \in l^\infty(I_k; \mathbf{H}) \cap l^p(I_k; \mathbf{V}_p)$ is proven for $p > 3/2$.

The first main result in this paper is the following:

Theorem 1.10. *Let $\mathbf{u}_0 \in \mathbf{W}^{2,2}(\Omega) \cap \mathbf{V}_p$, $\mathbf{f} \in C(I; \mathbf{W}^{1,2}(\Omega))$, $\partial_t \mathbf{f} \in C(I; \mathbf{L}^2(\Omega))$ be given. Let $\mathbf{u}$ be a strong solution of the problem $(NS)_p$ for $p \in (\frac{11+\sqrt{21}}{10}, 2] \approx (1.5583, 2]$ satisfying* (1.8)*. Suppose that $\mathbf{u}^m$ is a weak solution of problem $(NS_k)_p$ satisfying* (3.5) *and $t_M \le T^*$. Then for all*

$$\alpha < \alpha_0(p) := \frac{5p-6}{4(p-1)} \tag{1.11}$$

there exists a constant c that only depends on $\mathbf{u}_0, \mathbf{f}, \Omega, T^*$, and α but not on the time-step size k, such that the following error estimate is valid, provided that the time-step size is chosen sufficiently small, i.e. $k \le k_0(p, T^*)$,

$$\max_{0 \le m \le M} \|\mathbf{u}(t_m) - \mathbf{u}^m\|_2^2 + k \sum_{m=0}^M \big\|\mathbf{D}\big(\mathbf{u}(t_m) - \mathbf{u}^m\big)\big\|_p^2 \le c\, k^{2\alpha}\,. \tag{1.12}$$

Remark 1.13. Theorem 1.10 improves Theorem 1.1 in [35] considerably both with respect to the convergence rate $\alpha(p)$ and with respect to the range of admissible p's. In [35] it is proven that for $p \in (\frac{3+\sqrt{29}}{5}, 2] \approx (1.6769, 2]$ estimate (1.12) holds with $\alpha(p) = \frac{5p-6}{2p}$. This improvement is possible due to the new regularity (1.8) of the solution $\mathbf{u}$ of the problem $(\mathrm{NS})_p$, in particular the $C(I; \mathbf{W}^{1,r}(\Omega))$–information, $1 < r < 6(p-1)$, is crucial.

In [35] it is shown that one needs strong solutions $\mathbf{u}^m$ of the problem $(\mathrm{NS}_k)_p$ in order to show appropriate error estimates for the spatial discretization of (1.9). The existence of strong solutions $\mathbf{u}^m$ of the problem $(\mathrm{NS}_k)_p$ is proven in [35] for $p \in (9/5, 2]$.

Now, even with Theorem 1.10 at hand it is not possible to enlarge the range for which strong solutions $\mathbf{u}^m$ of the problem $(\mathrm{NS}_k)_p$ exist. However, the regularity of $\mathbf{u}^m$ can be improved to the discrete analogue of (1.8) (cf. (1.18)). One possibility to enlarge the range for p for which strong solutions $\mathbf{u}^m$ of a time-discretization of problem $(\mathrm{NS})_p$ exist, is to stabilize (1.9) with $-k^{\alpha_0(p)}\Delta\mathbf{u}^m$. This way the order of convergence is not reduced and it is possible to extract better regularity properties for $\mathbf{u}^m$ (in negative powers of k) from the apriori estimates. However, our problem under consideration is already in the main part nonlinear and thus there is no need to restrict ourselves to a linear stabilization. As long as computational costs are not increased significantly one can also stabilize (1.9) by a term similar to our main term $-\operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{u}^m))$, namely $-k^{\alpha_0(p)}\operatorname{div}\big((1+|\mathbf{D}(\mathbf{u}^m)|^2)^{\frac{q-2}{2}}\mathbf{D}(\mathbf{u}^m)\big)$, with $2\le q$ chosen appropriately. Since for $q=2$ both possibilities coincide we will immediately use the second possibility for the stabilization.

Our stabilized time-discretization of the problem $(\mathrm{NS})_p$, which we denote by $(\mathrm{NSS}_k)_p$, reads as follows:

Algorithm. *Given a time-step size $k>0$, and a corresponding net $I_k=\{t_m\}_{m=0}^M$. For $m\ge 1$ and $\mathbf{u}^{m-1}$ given from the previous step compute an iterate $\mathbf{u}^m$ that solves for some $q\ge 2$ and $\beta>0$*

$$
\begin{aligned}
d_t\mathbf{u}^m-\operatorname{div}\mathbf{S}(\mathbf{D}(\mathbf{u}^m))-k^\beta\operatorname{div}\mathbf{T}_q(\mathbf{D}(\mathbf{u}^m))+[\nabla\mathbf{u}^m]\mathbf{u}^m+\nabla\pi^m&=\mathbf{f}(t_m)\,,\\
\operatorname{div}\mathbf{u}^m&=0\,,\\
\mathbf{u}^0&=\mathbf{u}_0\,,
\end{aligned}
\tag{1.14}
$$

endowed with space-periodic boundary conditions (1.7)*. We used the notation*[3]

$$
\mathbf{T}_q(\mathbf{D}(\mathbf{u}^m))\equiv\big(1+|\mathbf{D}(\mathbf{u}^m)|^2\big)^{\frac{q-2}{2}}\mathbf{D}(\mathbf{u}^m)\,,\qquad q\ge 2\,. \tag{1.15}
$$

Now, the second main result in this paper is the following:

Theorem 1.16. *Let $q\ge 2$, $\mathbf{u}_0\in\mathbf{W}^{2,\frac{6(q-1)}{2q-1}}(\Omega)\cap\mathbf{V}_q$ and $\mathbf{f}\in C(I;\mathbf{W}^{1,2}(\Omega))$, $\partial_t\mathbf{f}\in C(I;\mathbf{L}^2(\Omega))$ be given. Let $\mathbf{u}$ be a strong solution of the problem $(NS)_p$ for $p\in(\frac{11+\sqrt{21}}{10},2]\approx(1.5583,2]$. Then the problem $(NSS_k)_p$ with $\beta:=\alpha_0(p)$, where $\alpha_0(p)$ is given by* (1.11)*, and q satisfying*

$$
\frac{-9p^3+44p^2-55p+18}{2p(3p-4)}<q<\frac{7p-6}{2} \tag{1.17}
$$

possesses a strong solution $\mathbf{u}^m$ as long as $t_M\le T^*$. This solution satisfies for all $1<r<6(p-1)$

$$
\begin{aligned}
\mathbf{u}^m&\in l^{\frac{5p-6}{2-p}}\big(I_k;\mathbf{W}^{2,\frac{3p}{p+1}}(\Omega)\big)\;\cap\;l^\infty\big(I_k;\mathbf{V}_r(\Omega)\big)\,,\\
d_t\mathbf{u}^m&\in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}\big(I_k;\mathbf{W}^{1,\frac{3p}{p+1}}(\Omega)\big)\;\cap\;l^\infty\big(I_k;\mathbf{L}^2(\Omega)\big)\,.
\end{aligned}
\tag{1.18}
$$

Moreover, for all $\alpha<\alpha_0(p)$ there exists a constant c that only depends on $\mathbf{u}_0,\mathbf{f},\Omega,T^*$, and α but not on the time-step size k, such that the following error estimate holds,

[3] Note, that $\mathbf{T}_q$ satisfies the assumptions (1.3)–(1.5) with $p=q$.

provided that the time-step size is chosen sufficiently small, i.e. $k \le k_0(p, T^*)$,

$$\max_{0\le m\le M} \|\mathbf{u}(t_m) - \mathbf{u}^m\|_2^2 + k \sum_{m=0}^{M} \big\|\mathbf{D}\big(\mathbf{u}(t_m) - \mathbf{u}^m\big)\big\|_p^2 \qquad + k^{\alpha_0(p)}\, k \sum_{m=0}^{M} \big\|\mathbf{D}\big(\mathbf{u}(t_m) - \mathbf{u}^m\big)\big\|_q^q \le c\, k^{2\alpha} . \tag{1.19}$$

Remark 1.20. According to Theorem 1.16 it is possible, for the range of p's considered, to find a q such that the stabilization with $-k^{\alpha_0(p)} \operatorname{div} \mathbf{T}_q(\mathbf{D}(\mathbf{u}^m))$ does not effect the convergence rate if compared with the convergence rate of the non-stabilized system $(\mathrm{NS}_k)_p$. Note that the choice $q = 2$ is possible as long as the lower bound in (1.17) is smaller than 2, i.e. $1.6955 \lesssim p \le 2$.

The remainder of the paper is organized as follows. In Section 2 we collect some useful results for the extra stress tensor $\mathbf{S}$ satisfying (1.3)–(1.5) and related quantities. Section 3 is devoted to the proof of Theorem 1.10, while Theorem 1.16 is proven in Section 4. We would like to mention that following the arguments becomes much easier if on replaces the strict inequalities $r < 6(p-1)$ and $\alpha < \frac{5p-6}{4(p-1)}$ by $r = 6(p-1)$ and $\alpha = \frac{5p-6}{4(p-1)}$.

2. Preliminaries

We start with a lemma that collects some consequences of the assumption (1.3)–(1.5). The proof can be found for example in [29, Lemma 5.1.19, Lemma 5.1.35] and [35, Lemma 2.8].

Lemma 2.1. *Suppose that* Φ *and* $\mathbf{S}$ *satisfy* (1.3)–(1.5) *for some* $p > 1$. *Then there are constants* $c = c(p)$ *such that for all* $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$

$$\mathbf{S}(\mathbf{A}) \cdot \mathbf{A} \ge c \begin{cases} \big(1 + |\mathbf{A}|^2\big)^{\frac{p-2}{2}} \big|\mathbf{A}\big|^2 , \\ \big(|\mathbf{A}|^{p-1} - 1\big)\big|\mathbf{A}\big| , \end{cases} \tag{2.2}$$

$$|\mathbf{S}(\mathbf{A})| \le c \big(1 + |\mathbf{A}|^2\big)^{\frac{p-1}{2}} , \tag{2.3}$$

$$\Phi(|\mathbf{A}|) \ge c \big(|\mathbf{A}|^p - 1\big) . \tag{2.4}$$

In the case $p \in (1, 2]$ *we additionally have*

$$\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) \cdot \big(\mathbf{A} - \mathbf{B}\big) \ge c \big|\mathbf{A} - \mathbf{B}\big|^2 \big(1 + \big|\mathbf{B}\big| + \big|\mathbf{A} - \mathbf{B}\big|\big)^{p-2} , \tag{2.5}$$

$$\big|\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big| \le c \big|\mathbf{A} - \mathbf{B}\big| \big(1 + \big|\mathbf{B}\big| + \big|\mathbf{A} - \mathbf{B}\big|\big)^{p-2} , \tag{2.6}$$

while in the case $p \ge 2$ *we have*

$$\big(\mathbf{S}(\mathbf{A}) - \mathbf{S}(\mathbf{B})\big) \cdot \big(\mathbf{A} - \mathbf{B}\big) \ge c \big|\mathbf{A} - \mathbf{B}\big|^2 \big(1 + \big|\mathbf{A} - \mathbf{B}\big|\big)^{p-2} . \tag{2.7}$$

We would like to stress once more that the stabilizing term $\mathbf{T}_q(\mathbf{D})$ defined in (1.15) satisfies the assumptions of the lemma with $p = q$.

Remark 2.8. From (2.5) one easily deduces that for $r \in [1,\infty)$ and $p \in (1,2]$ it holds

$$\begin{aligned}\int_\Omega \big(\mathbf{S}(\mathbf{D}(\mathbf{u})) - \mathbf{S}(\mathbf{D}(\mathbf{v}))\big)\cdot \mathbf{D}(\mathbf{u}-\mathbf{v})\,dx \\ \geq c\,\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|^2_{\frac{2r}{2-p+r}}\big(1+\|\mathbf{D}(\mathbf{u})\|_r + \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_r\big)^{p-2},\end{aligned} \tag{2.9}$$

while for $q \geq 2$ it holds (cf. (2.7))

$$\int_\Omega \big(\mathbf{T}_q(\mathbf{D}(\mathbf{u})) - \mathbf{T}_q(\mathbf{D}(\mathbf{v}))\big)\cdot \mathbf{D}(\mathbf{u}-\mathbf{v})\,dx \geq c\,\big(\|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_q^q + \|\mathbf{D}(\mathbf{u}-\mathbf{v})\|_2^2\big)\,. \tag{2.10}$$

Let us introduce some notation for terms which arise from $\mathbf{S}$ and $\mathbf{T}_q$ when we test equation (1.6) with $-\Delta\mathbf{u}$ or with $\partial_t^2\mathbf{u}$. Namely, we set for $p>1$

$$\begin{aligned}\mathcal{I}_p(\mathbf{u}) &= \int_\Omega \big(1+|\mathbf{D}(\mathbf{u})|^2\big)^{\frac{p-2}{2}}\big|\mathbf{D}(\nabla\mathbf{u})\big|^2\,dx\,,\\ \mathcal{J}_p(\mathbf{u}) &= \int_\Omega \big(1+|\mathbf{D}(\mathbf{u})|^2\big)^{\frac{p-2}{2}}\big|\mathbf{D}(\partial_t\mathbf{u})\big|^2\,dx\,.\end{aligned}$$

The discrete analogue for $\mathcal{J}_p(\mathbf{u})$ for a function defined on a net I_k reads as follows

$$\mathcal{K}_p(\mathbf{u}^m) = \int_\Omega \big(1+|\mathbf{D}(\mathbf{u}^m)|^2 + |\mathbf{D}(\mathbf{u}^{m-1})|^2\big)^{\frac{p-2}{2}}\big|\mathbf{D}(d_t\mathbf{u}^m)\big|^2\,dx\,.$$

Lemma 2.11. *Let $\mathbf{u} \in C^1(I;\mathbf{C}^2(\Omega))$ be a space periodic function with mean value zero and $p \in (1,2]$. Then there exists a constant c depending only on Ω, p, such that for $s \in [1,\infty)$*

$$\|\nabla\mathbf{u}\|^2_{\frac{6s}{6-3p+s}} + \|\nabla^2\mathbf{u}\|^2_{\frac{2s}{2-p+s}} \leq c\,\mathcal{I}_p(\mathbf{u})\big(1+\|\nabla\mathbf{u}\|_s\big)^{2-p}\,, \tag{2.12}$$

$$\|\nabla\mathbf{u}\|^p_{3p} + \|\nabla^2\mathbf{u}\|^p_{\frac{3p}{p+1}} \leq c\,\big(1+\mathcal{I}_p(\mathbf{u})\big)\,, \tag{2.13}$$

$$\|\partial_t\mathbf{u}\|^2_{\frac{6s}{6-3p+s}} + \|\nabla\partial_t\mathbf{u}\|^2_{\frac{2s}{2-p+s}} \leq c\,\mathcal{J}_p(\mathbf{u})(1+\|\nabla\mathbf{u}\|_s)^{2-p}\,, \tag{2.14}$$

$$\|\partial_t\mathbf{u}\|^p_{3p} + \|\nabla\partial_t\mathbf{u}\|^p_{\frac{3p}{p+1}} \leq c\,\big(1+\mathcal{I}_p(\mathbf{u})\big)^{\frac{2-p}{2}}\mathcal{J}_p(\mathbf{u})^{\frac{p}{2}} \tag{2.15}$$

$$\leq c\,\big(1+\mathcal{I}_p(\mathbf{u})+\mathcal{J}_p(\mathbf{u})\big)\,. \tag{2.16}$$

Moreover, for $1 \leq r < 6(p-1)$ it holds

$$\sup_{t\in I}\|\nabla\mathbf{u}\|_r^p \leq c\,\Big(1+\int_I \mathcal{I}_p(\mathbf{u})^{\frac{5p-6}{2-p}} + \mathcal{J}_p(\mathbf{u})\,dt\Big)\,. \tag{2.17}$$

Proof : The assertions (2.12)–(2.16) can be immediately deduced from the proofs of [29, Lemma 5.3.24], [37, Lemma 2.1], [35, Lemma 2.7]. Note that in these papers mostly the additive inequalities are stated only. However, in the proofs also the multiplicative inequalities are contained. In order to prove (2.17) we raise the first inequality in (2.15) to the power $\gamma > 0$, integrate over I, and obtain with the help of Young's inequality

$$\begin{aligned}\int_I \|\partial_t\nabla\mathbf{u}\|^{p\gamma}_{\frac{3p}{p+1}}\,dt &\leq c\int_I \big(1+\mathcal{I}_p(\mathbf{u})\big)^{\gamma\frac{2-p}{2}}\mathcal{J}_p(\mathbf{u})^{\gamma\frac{p}{2}}\,dt\\ &\leq c\int_I \big(1+\mathcal{I}_p(\mathbf{u})\big)^{\frac{5p-6}{2-p}} + \mathcal{J}_p(\mathbf{u})\,dt\end{aligned} \tag{2.18}$$

if $\gamma = \frac{5p-6}{(3p-2)(p-1)}$. Furthermore, (2.13) raised to the power $\frac{5p-6}{2-p}$ and integrated over I implies that

$$\int_I \|\nabla^2 \mathbf{u}\|_{\frac{3p}{p+1}}^{p\,\frac{5p-6}{2-p}} \, dt \le c \int_I \big(1 + \mathcal{I}_p(\mathbf{u})\big)^{\frac{5p-6}{2-p}} \, dt \,. \tag{2.19}$$

Now the statement follows from a general parabolic embedding result (cf. [8], [9]). A special case of these results can be found in [35, Lemma 2.3]. ■

Since $\mathcal{K}_p(\mathbf{u})$ is the discrete version of $\mathcal{J}_p(\mathbf{u})$ we immediately obtain in the same way:

Lemma 2.20. *Let $\mathbf{u} \in l^\infty(I_k; \mathbf{C}^2(\Omega))$ be a space periodic function with mean value zero and $p \in (1,2]$. Then there exists a constant c depending only on Ω, p, such that for $s \in [1,\infty)$*

$$\|d_t \mathbf{u}^m\|^2_{\frac{6s}{6-3p+s}} + \|d_t \nabla \mathbf{u}^m\|^2_{\frac{2s}{2-p+s}} \le c\,\mathcal{K}_p(\mathbf{u}^m)(1 + \|\nabla \mathbf{u}^m\|_s + \|\nabla \mathbf{u}^{m-1}\|_s)^{2-p} \,, \tag{2.21}$$

$$\|d_t \mathbf{u}^m\|^p_{3p} + \|d_t \nabla \mathbf{u}^m\|^p_{\frac{3p}{p+1}} \le c\,\big(1 + \mathcal{I}_p(\mathbf{u}^m) + \mathcal{I}_p(\mathbf{u}^{m-1})\big)^{\frac{2-p}{2}} \mathcal{K}_p(\mathbf{u}^m)^{\frac{p}{2}} \tag{2.22}$$

$$\le c\,\big(1 + \mathcal{I}_p(\mathbf{u}^m) + \mathcal{I}_p(\mathbf{u}^{m-1}) + \mathcal{K}_p(\mathbf{u}^m)\big) \,. \tag{2.23}$$

Moreover, for $1 \le r < 6(p-1)$ it holds

$$\max_{0 \le m \le M} \|\nabla \mathbf{u}^m\|^p_r \le c \Big(1 + k \sum_{m=0}^{M} \big(\mathcal{I}_p(\mathbf{u}^m)^{\frac{5p-6}{2-p}} + \mathcal{I}_p(\mathbf{u}^{m-1})^{\frac{5p-6}{2-p}} + \mathcal{K}_p(\mathbf{u}^m)\big)\Big) \,. \tag{2.24}$$

For the stabilization we also need the analogue of some of the above assertions for $q \ge 2$.

Lemma 2.25. *Let $\mathbf{u} \in \mathbf{C}^2(\Omega)$ and $\mathbf{u}^m \in l^\infty(I_k; \mathbf{C}^2(\Omega))$ be space periodic functions with mean value zero and $q \ge 2$. Then there exists a constant c depending only on Ω, q, such that*

$$\|\nabla \mathbf{u}\|^q_{3q} + \|\nabla^2 \mathbf{u}\|^2_2 \le c\,\mathcal{I}_q(\mathbf{u}) \,, \tag{2.26}$$

$$\|\nabla d_t \mathbf{u}^m\|^2_2 \le c\,\mathcal{K}_q(\mathbf{u}^m) \,. \tag{2.27}$$

Proof : The first assertion can be found in [29, Lemma 5.3.24]. The second estimate follows directly from the definition of $\mathcal{K}_q(\mathbf{u}^m)$, $q \ge 2$ and Korn's inequality. ■

3. Proof of Theorem 1.10

The existence of local in time *strong* solutions for large data of the problem $(\mathrm{NS})_p$ is ensured by the following proposition.

Proposition 3.1. *Let $\mathbf{u}_0 \in \mathbf{W}^{2,2}(\Omega) \cap \mathbf{V}_p$, $\mathbf{f} \in C(I; \mathbf{W}^{1,2}(\Omega))$, and $\partial_t \mathbf{f} \in C(I; \mathbf{L}^2(\Omega))$ be given. Then there exists a $T^* > 0$, such that a strong solution $\mathbf{u}$ of the problem $(NS)_p$ exists on $I = [0, T^*]$ whenever $p > \frac{7}{5}$. This solution satisfies*

$$\operatorname*{esssup}_{s \in I} \|\partial_t \mathbf{u}(s)\|_2^2 + \int_0^{T^*} \mathcal{I}_p(\mathbf{u})^{\frac{5p-6}{2-p}} + \mathcal{J}_p(\mathbf{u}) \, dt \le c(\mathbf{f}, \mathbf{u}_0) \,. \tag{3.2}$$

In particular we have that for $1 < r < 6(p-1)$

$$
\begin{aligned}
\mathbf{u} &\in L^{\frac{5p-6}{2-p}}\big(I;\mathbf{W}^{2,\frac{3p}{p+1}}(\Omega)\big) \,\cap\, C(I;\mathbf{V}_r)\,,\\
\partial_t\mathbf{u} &\in L^{\frac{p(5p-6)}{(3p-2)(p-1)}}\big(I;\mathbf{W}^{1,\frac{3p}{p+1}}(\Omega)\big) \,\cap\, L^\infty\big(I;\mathbf{L}^2(\Omega)\big)\,,\\
\partial_t^2\mathbf{u} &\in L^2\big(I;(\mathbf{V}_2)^*\big)\,.
\end{aligned}
\tag{3.3}
$$

Proof : The proof for $p \in (7/5, 2]$ can be found in [8], [9]. The case $p > 5/3$ is covered in [29]. To get the full regularity stated in (3.2) one additionally needs to test (1.6) with $\partial_t^2\mathbf{u}$ and follow the procedure from [8], [9]. This however is straightforward and we will use similar ideas later on. Thus we skip the details here. The regularity stated in (3.3) can be easily deduced from (3.2) and Lemma 2.11. ■

Note that the strategy employed in the proof of Proposition 3.1 to ensure the existence of strong solutions is not applicable in the discrete case. However, the existence of *weak* solutions to the problem $(\mathrm{NS}_k)_p$ is ensured in [35, Lemma 4.1], which we recall.

Lemma 3.4. *Let* $\mathbf{u}_0$ *and* $\mathbf{f}$ *satisfy the same assumptions as in Proposition 3.1. Then there exists a weak solution* $\mathbf{u}^m$ *of the problem* $(NS_k)_p$ *satisfying*

$$
\max_{0\le m\le M}\|\mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M}\|\mathbf{D}(\mathbf{u}^m)\|_p^p \le c(\mathbf{f},\mathbf{u}_0)\,,
\tag{3.5}
$$

whenever $p > 3/2$.

In order to verify Theorem 1.10 we have to deal with two problems. Namely that the discrete solution $\mathbf{u}^m$ of the problem $(\mathrm{NS}_k)_p$ is only weak and secondly that the information about $\partial_t^2\mathbf{u}$ is also weak. Thus we introduce an auxiliary problem to split these problems subsequently. We follow the procedure introduced in [35] and consider the following auxiliary problem:

Algorithm. *Suppose that* $\mathbf{u}$ *is a strong solution to the problem* $(NS)_p$ *with the properties stated in Proposition 3.1. Then determine* $\mathbf{U}^m$, $m = 0,\dots,M$, *that solves*

$$
\begin{aligned}
d_t\mathbf{U}^m - \operatorname{div}\,\mathbf{S}(\mathbf{D}(\mathbf{U}^m)) + [\nabla\mathbf{U}^m]\mathbf{u}(t_m) + \nabla\Pi^m &= \mathbf{f}(t_m)\,,\\
\operatorname{div}\,\mathbf{U}^m &= 0\,,\\
\mathbf{U}^0 &= \mathbf{u}_0\,,
\end{aligned}
\tag{3.6}
$$

endowed with space-periodic boundary conditions (1.7).

We have linearized the convective term around the continuous solution $\mathbf{u}(t_m)$, for which we have good regularity properties. The hope is that $\mathbf{U}^m$ inherits the regularity from $\mathbf{u}$. In fact this is the case at the expense of restricting ourselves to a smaller range of p's.

Proposition 3.7. *Let* $\mathbf{u}_0$ *and* $\mathbf{f}$ *satisfy the same assumptions as in Proposition 3.1. Let* $\mathbf{u}$ *defined on* $I = [0,T^*]$ *be the strong solution ensured by Proposition 3.1, and let* $t_M < T^*$. *Then there exists a strong solution* $\mathbf{U}^m$ *of the problem* (3.6) *whenever* $p \in (\frac{11+\sqrt{21}}{10}, 2]$. *This solution satisfies*

$$
\max_{0\le m\le M}\|d_t\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M}\Big(\mathcal{I}_p(\mathbf{U}^m)^{\frac{5p-6}{2-p}} + \mathcal{K}_p(\mathbf{U}^m)\Big) \le c(\mathbf{f},\mathbf{u}_0)\,.
\tag{3.8}
$$

In particular we have that for all $1 < r < 6(p-1)$ *it holds*

$$\begin{aligned} \mathbf{U}^m &\in l^{\frac{5p-6}{2-p}}\big(I_k; \mathbf{W}^{2,\frac{3p}{p+1}}(\Omega)\big) \ \cap \ l^\infty(I_k; \mathbf{V}_r)\,, \\ d_t\mathbf{U}^m &\in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}\big(I_k; \mathbf{W}^{1,\frac{3p}{p+1}}(\Omega)\big) \ \cap \ l^\infty\big(I_k; \mathbf{L}^2(\Omega)\big)\,. \end{aligned} \tag{3.9}$$

PROOF : The existence of a strong solution $\mathbf{U}^m$ of (3.6) follows from the regularity in (3.9) using the Galerkin approach. The regularity (3.9) is an immediate consequence of (3.8), (2.13), (2.22) and (2.24). Thus we shall only derive these estimates. For all the missing details in the following computations we refer to [29, Section 5.3].

First of all we test the weak formulation of (3.6), which reads for all $\boldsymbol{\varphi} \in \mathbf{V}_p$

$$\big(d_t\mathbf{U}^m, \boldsymbol{\varphi}\big) + \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^m)), \mathbf{D}(\boldsymbol{\varphi})\big) + \big([\nabla\mathbf{U}^m]\mathbf{u}(t_m), \boldsymbol{\varphi}\big) = \big(\mathbf{f}(t_m), \boldsymbol{\varphi}\big)\,, \tag{3.10}$$

with $\mathbf{U}^m$ and sum up over all iteration steps to obtain

$$\max_{0\le m\le M} \|\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M} \|\mathbf{D}(\mathbf{U}^m)\|_p^p \le c(\mathbf{f}, \mathbf{u}_0)\,, \tag{3.11}$$

where we used the skew symmetry of the linearized convective term. This estimate implies

$$\mathbf{U}^m \in l^\infty\big(I_k; \mathbf{L}^2(\Omega)\big) \ \cap \ l^p(I_k; \mathbf{V}_p)\,. \tag{3.12}$$

The next step is to use in (3.11) $-\Delta\mathbf{U}^m$ as a test function. Again we use the skew symmetry of the linearized convective term, the properties of $\mathbf{S}$ (cf. (1.3)–(1.5)), the definition of $\mathcal{I}_p(\mathbf{U}^m)$ and obtain, after summation up to level $N \in \{1, \dots, M\}$,

$$\|\nabla\mathbf{U}^N\|_2^2 + k\sum_{m=0}^{N} \mathcal{I}_p(\mathbf{U}^m) \le c(\mathbf{f}, \mathbf{u}_0)\Big(1 + k\sum_{m=0}^{N}\int_\Omega |\nabla\mathbf{u}(t_m)|\,|\nabla\mathbf{U}^m|^2\,dx\Big)\,. \tag{3.13}$$

The last term can, for $1 < r < 6(p-1)$, $\alpha \in (0,1)$, be estimated by

$$\|\nabla\mathbf{u}(t_m)\|_r \|\nabla\mathbf{U}^m\|_{2r'}^2 \le c(\mathbf{f}, \mathbf{u}_0)\|\nabla\mathbf{U}^m\|_{2r'}^2 = c(\mathbf{f}, \mathbf{u}_0)\|\nabla\mathbf{U}^m\|_{2r'}^{2(\alpha+1-\alpha)}\,, \tag{3.14}$$

where r' is the dual exponent to r and where we used $\mathbf{u} \in C(I; \mathbf{V}_r)$. Now, we interpolate $L^{2r'}(\Omega)$ between $L^2(\Omega)$ and $L^{3p}(\Omega)$ resp. $L^p(\Omega)$ and $L^{3p}(\Omega)$, which gives

$$\begin{aligned} \|\nabla\mathbf{U}^m\|_{2r'} &\le \|\nabla\mathbf{U}^m\|_2^{\frac{r(3p-2)-3p}{r(3p-2)}} \|\nabla\mathbf{U}^m\|_{3p}^{\frac{3p}{r(3p-2)}}\,, \\ \|\nabla\mathbf{U}^m\|_{2r'} &\le \|\nabla\mathbf{U}^m\|_p^{\frac{1}{4}\frac{r(3p-2)-3p}{r}} \|\nabla\mathbf{U}^m\|_{3p}^{\frac{3}{4}\frac{r(2-p)+p}{r}}\,. \end{aligned} \tag{3.15}$$

Using also (2.13) the right-hand side of (3.14) can be estimated by

$$c\left(1 + \|\nabla\mathbf{U}^m\|_2^2\right)^{Q_1} \|\nabla\mathbf{U}^m\|_p^{p\,Q_2}\,\mathcal{I}_p(\mathbf{U}^m)^{Q_3}\,, \tag{3.16}$$

where

$$Q_1 = (1-\alpha)\frac{r(3p-2)-3p}{r(3p-2)}\,, \qquad Q_2 = \alpha\frac{1}{2p}\frac{r(3p-2)-3p}{r}\,,$$

$$Q_3 = (1-\alpha)\frac{2}{p}\frac{3p}{r(3p-2)} + \alpha\frac{3}{2p}\frac{r(2-p)+p}{r}\,.$$

Young's inequality together with the requirements

$$Q_2 \cdot \delta = \frac{1}{1+\varepsilon}\,, \quad Q_3 \cdot \delta' = 1\,, \quad \frac{1}{\delta} + \frac{1}{\delta'} = 1$$

for any prescribed $\varepsilon > 0$ yields

$$1 + \|\nabla \mathbf{U}^N\|_2^2 + k \sum_{m=0}^{N} \mathcal{I}_p(\mathbf{U}^m) \le c(\mathbf{f},\mathbf{u}_0)\Big(1 + k \sum_{m=0}^{N} \|\nabla \mathbf{U}^m\|_p^{\frac{p}{1+\varepsilon}} \big(1 + \|\nabla \mathbf{U}^m\|_2^2\big)^{\lambda_\varepsilon(r)}\Big)\,,$$

where

$$\lambda_\varepsilon(r) \searrow \lambda = \frac{2(p-1)(2-p)}{3p^2-5p+1} \qquad \text{for } \varepsilon \searrow 0, r \nearrow 6(p-1)\,.$$

In view of (3.12) we have to check whether $\lambda < 1$, which is the case for $p \in (\frac{11+\sqrt{21}}{10}, 2]$. Therefore we can employ discrete Gronwall's lemma and obtain

$$\max_{0\le m\le M} \|\nabla \mathbf{U}^m\|_2^2 + k \sum_{m=0}^{M} \mathcal{I}_p(\mathbf{U}^m) \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{3.17}$$

Now we use $d_t\mathbf{U}^m$ as a test function in (3.10), sum up through $m = 0, \dots, N$, and obtain

$$\begin{aligned} k \sum_{m=0}^{N} \|d_t \mathbf{U}^m\|_2^2 + \sum_{m=0}^{N} \int_\Omega \mathbf{S}\big(\mathbf{D}(\mathbf{U}^m)\big) \cdot \mathbf{D}(\mathbf{U}^m - \mathbf{U}^{m-1})\, dx \\ \le c(\mathbf{f},\mathbf{u}_0)\Big(1 + k \sum_{m=0}^{N} \|[\nabla \mathbf{U}^m]\mathbf{u}(t_m)\|_2^2\Big)\,. \end{aligned} \tag{3.18}$$

Property (1.3) and the convexity of $\Phi(\cdot)$ yield

$$\int_\Omega \mathbf{S}\big(\mathbf{D}(\mathbf{U}^m)\big) \cdot \mathbf{D}(\mathbf{U}^m - \mathbf{U}^{m-1})\, dx \ge c\,\big(\Phi(|\mathbf{D}(\mathbf{U}^m)|) - \Phi(|\mathbf{D}(\mathbf{U}^{m-1})|)\big). \tag{3.19}$$

To bound the convective term we employ $\mathbf{u} \in C(I;\mathbf{W}^{1,r}(\Omega))$, $1 < r < 6(p-1)$, the embedding $\mathbf{W}^{1,r}(\Omega) \hookrightarrow \mathbf{L}^\infty(\Omega)$, which is possible for $p > 3/2$, and (3.17) to obtain

$$k \sum_{m=0}^{N} \|[\nabla \mathbf{U}^m]\mathbf{u}(t_m)\|_2^2 \le c\,k \sum_{m=0}^{N} \|\mathbf{u}(t_m)\|_\infty^2 \|\nabla \mathbf{U}^m\|_2^2 \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{3.20}$$

Using also (2.4) and Korn's inequality we thus derived

$$k \sum_{m=0}^{N} \|d_t \mathbf{U}^m\|_2^2 + \max_{0\le m\le M} \|\nabla \mathbf{U}^m\|_p^p \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{3.21}$$

Now one would like to use $d_t^2\mathbf{U}^m$ as a test function in (3.10). However this works only for $p > 5/3$, which would be a further restriction for the range of p's, which is not desirable. Thus we shall use $-d_t^2\mathbf{U}^m$ and $-\Delta\mathbf{U}^m$ "almost pointwise in time" simultaneously as test functions. Here "almost pointwise in time" means that we also estimate the term coming from the discrete time derivative, which has the advantage that one can take powers of the resulting equation. Let us now be more precise. Firstly, we have to introduce $\mathbf{U}^{-1}$. For that we set for all $\boldsymbol{\varphi} \in \mathbf{V}_p$

$$\frac{1}{k}\big(\mathbf{U}^0 - \mathbf{U}^{-1}, \boldsymbol{\varphi}\big) + \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^0)), \mathbf{D}(\boldsymbol{\varphi})\big) + ([\nabla \mathbf{U}^0]\mathbf{U}^0, \boldsymbol{\varphi}) = (\mathbf{f}(0), \boldsymbol{\varphi})\,.$$

Using $\mathbf{U}^0 = \mathbf{u}_0$, $p \le 2$ and the assumption on $\mathbf{u}_0$ we obtain

$$\|d_t \mathbf{U}^0\|_2^2 \le c\,\|\mathbf{f}(0)\|_2^2 + \|[\nabla \mathbf{u}_0]\mathbf{u}_0\|_2^2 + \|\operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{u}_0))\|_2^2 \le c(\mathbf{f}, \mathbf{u}_0)\,. \tag{3.22}$$

Now we can take the discrete time derivative of the weak formulation (3.10), use $d_t \mathbf{U}^m$ as a test function, and sum up to obtain

$$\begin{aligned} &\|d_t \mathbf{U}^N\|_2^2 + k^{-1} \sum_{m=0}^{N} \int_\Omega \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^m)) - \mathbf{S}(\mathbf{D}(\mathbf{U}^{m-1}))\big) \cdot \mathbf{D}(\mathbf{U}^m - \mathbf{U}^{m-1})\,dx \\ &\le c(\mathbf{f}, \mathbf{u}_0) \Big(1 + k \sum_{m=0}^{N} \Big| \int_\Omega [\nabla \mathbf{U}^m] d_t \mathbf{u}(t_{m-1}) \cdot d_t \mathbf{U}^m\,dx \Big| \Big)\,, \end{aligned} \tag{3.23}$$

where we used (3.22). From the formula $d_t \mathbf{u}(t_m) = k^{-1} \int_{t_{m-1}}^{t_m} \partial_t \mathbf{u}(s)\,ds$ and $(3.3)_2$ we deduce

$$\|d_t \mathbf{u}(t_m)\|_2 \le \operatorname*{ess\,sup}_I \|\partial_t \mathbf{u}\|_2 \le c(\mathbf{f}, \mathbf{u}_0)\,, \tag{3.24}$$

and thus we can bound the last term in (3.23) by

$$\begin{aligned} \|d_t \mathbf{u}(t_{m-1})\|_2 \big\| |\nabla \mathbf{U}^m|\,|d_t \mathbf{U}^m| \big\|_2 &\le c(\mathbf{f}, \mathbf{u}_0)\, \|\nabla \mathbf{U}^m\|_{3p}\, \|d_t \mathbf{U}^m\|_{\frac{6p}{3p-2}} \\ &\le c(\mathbf{f}, \mathbf{u}_0) \big(1 + \mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{p}}\, \|d_t \mathbf{U}^m\|_{\frac{6p}{3p-2}}\,, \end{aligned} \tag{3.25}$$

where we also used (2.13). From (2.5) and the definition of $\mathcal{K}_p(\mathbf{U}^m)$ we obtain that

$$k^{-1} \int_\Omega \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^m)) - \mathbf{S}(\mathbf{D}(\mathbf{U}^{m-1}))\big) \cdot \mathbf{D}(\mathbf{U}^m - \mathbf{U}^{m-1})\,dx \ge c\,k\,\mathcal{K}_p(\mathbf{U}^m)\,. \tag{3.26}$$

Alltogether we therefore derived

$$\max_{0 \le m \le M} \|d_t \mathbf{U}^m\|_2^2 + k \sum_{m=0}^{M} \mathcal{K}_p(\mathbf{U}^m) \le c \Big(1 + k \sum_{m=0}^{M} \big(1 + \mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{p}}\, \|d_t \mathbf{U}^m\|_{\frac{6p}{3p-2}}\Big)\,. \tag{3.27}$$

Using $-\Delta \mathbf{U}^m$ as a test function in (3.10), where also the term with the discrete time derivative is estimated, yields for $1 < r < 6(p-1)$ (cf. (3.13), (3.14))

$$\begin{aligned} 1 + \mathcal{I}_p(\mathbf{U}^m) &\le c\,\big(1 + \|\nabla \mathbf{U}^m\|_{2r'}^2 + \|d_t \mathbf{U}^m\|_{\frac{4}{p}} \|\nabla^2 \mathbf{U}^m\|_{\frac{4}{4-p}}\big) \\ &\le c\,\Big(1 + c_\varepsilon \|\nabla \mathbf{U}^m\|_2^2 + \varepsilon\,\mathcal{I}_p(\mathbf{U}^m) + \|d_t \mathbf{U}^m\|_{\frac{4}{p}} \big(1 + \mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{2}}\Big) \\ &\le c\,\Big(c_\varepsilon + \varepsilon\,\mathcal{I}_p(\mathbf{U}^m) + \|d_t \mathbf{U}^m\|_{\frac{4}{p}} \big(1 + \mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{2}}\Big)\,, \end{aligned} \tag{3.28}$$

where we used the interpolation of $L^{2r'}(\Omega)$ between $L^2(\Omega)$ and $L^{\frac{12}{8-3p}}(\Omega)$, which is possible for $p > 3/2$, $\mathbf{U}^m \in l^\infty(I_k; \mathbf{W}^{1,2}(\Omega))$ and (2.12) with $s = 2$. For ε sufficiently small we can absorb the term $c\,\varepsilon\,\mathcal{I}_p(\mathbf{U}^m)$ into the left-hand side of (3.28). Thus we get

$$\big(1 + \mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{2}} \le c\,\big(1 + \|d_t \mathbf{U}^m\|_{\frac{4}{p}}\big)\,. \tag{3.29}$$

Interpolating $L^{\frac{4}{p}}(\Omega)$ between $L^2(\Omega)$ and $L^{\frac{12}{8-3p}}(\Omega)$ and using (2.21) with $s = 2$ and (3.17) we obtain

$$\|d_t\mathbf{U}^m\|_{\frac{4}{p}} \leq c\,\|d_t\mathbf{U}^m\|_2^{1-\lambda}\mathcal{K}_p(\mathbf{U}^m)^{\frac{\lambda}{2}}\,, \tag{3.30}$$

with $\lambda = 3\frac{2-p}{3p-2}$. Inserting (3.30) into (3.29) and raising the result to the power 2γ we arrive at

$$\begin{aligned}\big(1+\mathcal{I}_p(\mathbf{U}^m)\big)^{\gamma} &\leq c\Big(1+\|d_t\mathbf{U}^m\|_2^{2\gamma(1-\lambda)}\mathcal{K}_p(\mathbf{U}^m)^{\gamma\,\lambda}\Big) \\ &\leq c\Big(1+\|d_t\mathbf{U}^m\|_2^{2\frac{(\gamma-1)(3p-2)}{3p-2-3\gamma(2-p)}}\,\|d_t\mathbf{U}^m\|_2^2+\mathcal{K}_p(\mathbf{U}^m)\Big)\,,\end{aligned} \tag{3.31}$$

where we used Young's inequality. In view of (3.21) and in preparation for discrete Gronwall's inequality we require that $\frac{(\gamma-1)(3p-2)}{3p-2-3\gamma(2-p)} < 1$, which gives $\gamma < \frac{3p-2}{2}$. For such γ we have proved

$$k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\gamma} \leq c\Big(1+k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m)+k\sum_{m=0}^{M}\|d_t\mathbf{U}^m\|_2^{2\frac{(\gamma-1)(3p-2)}{3p-2-3\gamma(2-p)}}\,\|d_t\mathbf{U}^m\|_2^2\Big)\,. \tag{3.32}$$

Adding now (3.32) and (3.27) we get for $\gamma < \frac{3p-2}{2}$

$$\begin{aligned}&\max_{0\leq m\leq M}\|d_t\mathbf{U}^m\|_2^2+k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m)+k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\gamma} \\ &\leq c\Big(1+k\sum_{m=0}^{M}\big(1+\mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{p}}\,\|d_t\mathbf{U}^m\|_{\frac{6p}{3p-2}}+k\sum_{m=0}^{M}\|d_t\mathbf{U}^m\|_2^{2\frac{(\gamma-1)(3p-2)}{3p-2-3\gamma(2-p)}}\,\|d_t\mathbf{U}^m\|_2^2\Big)\,.\end{aligned} \tag{3.33}$$

Now we proceed similar as in (3.30) and interpolate $L^{\frac{6p}{3p-2}}(\Omega)$ between $L^2(\Omega)$ and $L^{\frac{12}{8-3p}}(\Omega)$, use (2.21) with $s = 2$, $\mathbf{U}^m \in l^\infty(I_k;\mathbf{W}^{1,2}(\Omega))$ and apply Young's inequality to bound the second term on the right-hand side of (3.33) by

$$c\Big(c_\varepsilon+c_\varepsilon k\sum_{m=0}^{M}\|d_t\mathbf{U}^m\|_2^2+\varepsilon\,k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\frac{2}{p}}+\varepsilon\,k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m)\Big)\,.$$

For ε sufficiently small we can absorb the last two terms into the left-hand side if $\frac{2}{p} < \gamma < \frac{3p-2}{2}$, which holds for $p > \frac{1+\sqrt{13}}{3}$. Note that this requirement is less restrictive than $p > \frac{11+\sqrt{21}}{10}$. Thus we can apply discrete Gronwall's inequality and obtain for $\gamma < \frac{3p-2}{2}$

$$\max_{0\leq m\leq M}\|d_t\mathbf{U}^m\|_2^2+k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m)+k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\gamma} \leq c(\mathbf{f},\mathbf{u}_0)\,. \tag{3.34}$$

With this new information we can improve the exponent of $\mathcal{I}_p(\mathbf{U}^m)$ in the previous estimate. For that we go again into (3.28) and estimate the term coming from the discrete time derivative by

$$\|d_t\mathbf{U}^m\|_{\frac{3p}{2p-1}}\|\nabla^2\mathbf{U}^m\|_{\frac{3p}{p+1}} \leq c\,\|d_t\mathbf{U}^m\|_{\frac{3p}{2p-1}}\big(1+\mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{1}{p}}\,, \tag{3.35}$$

where we used (2.13). Thus we get instead of (3.29)

$$\big(1+\mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{p-1}{p}} \le c\,\big(1+\|d_t\mathbf{U}^m\|_{\frac{3p}{2p-1}}\big)\,. \tag{3.36}$$

Now we interpolate $L^{\frac{3p}{2p-1}}(\Omega)$ between $L^2(\Omega)$ and $L^{3p}(\Omega)$, use $d_t\mathbf{U}^m \in l^\infty(I_k;\mathbf{L}^2(\Omega))$ and (2.22) to arrive at

$$\big(1+\mathcal{I}_p(\mathbf{U}^m)\big)^{\frac{p-1}{p}} \le c\,\Big(1+\mathcal{K}_p(\mathbf{U}^m)^{\frac{\lambda}{2}}\big(1+\mathcal{I}_p(\mathbf{U}^m)+\mathcal{I}_p(\mathbf{U}^{m-1})\big)^{\lambda\,\frac{2-p}{2p}}\Big)\,, \tag{3.37}$$

with $\lambda = \frac{2-p}{3p-2}$. We raise this inequality to the power γ and apply Young's and get

$$\begin{aligned}\big(1+\mathcal{I}_p(\mathbf{U}^m)\big)^{\gamma\,\frac{p-1}{p}} &\le c\,\Big(1+\mathcal{K}_p(\mathbf{U}^m)^{\gamma\,\frac{\lambda}{2}}\big(1+\mathcal{I}_p(\mathbf{U}^m)+\mathcal{I}_p(\mathbf{U}^{m-1})\big)^{\gamma\,\lambda\,\frac{2-p}{2p}}\Big) \\ &\le c\,\Big(1+c_\varepsilon\mathcal{K}_p(\mathbf{U}^m)+\varepsilon\big(1+\mathcal{I}_p(\mathbf{U}^m)+\mathcal{I}_p(\mathbf{U}^{m-1})\big)^{\frac{2\gamma}{2-\gamma\lambda}\,\lambda\,\frac{2-p}{2p}}\Big)\,.\end{aligned} \tag{3.38}$$

We now require $\gamma\,\frac{p-1}{p} = \frac{2\gamma}{2-\gamma\lambda}\,\lambda\,\frac{2-p}{2p}$, which gives $\gamma = \frac{p}{p-1}\,\frac{5p-6}{2-p}$. With this γ and ε sufficiently small we can absorb the last term in (3.38) into the left-hand side after summation over all time steps. Thus we derived instead of (3.32)

$$k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\frac{5p-6}{2-p}} \le c\,\Big(1+k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m)\Big)\,, \tag{3.39}$$

and consequently

$$\begin{aligned}&\max_{0\le m\le M}\|d_t\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m) + k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\frac{5p-6}{2-p}} \\ &\le c(\mathbf{f},\mathbf{u}_0)\,\Big(1+k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\frac{1}{p}}\,\|d_t\mathbf{U}^m\|_{\frac{6p}{3p-2}}\Big)\,.\end{aligned} \tag{3.40}$$

Now we handle the right-hand side in the same way as before and obtain

$$\max_{0\le m\le M}\|d_t\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M}\mathcal{K}_p(\mathbf{U}^m) + k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m)^{\frac{5p-6}{2-p}} \le c(\mathbf{f},\mathbf{u}_0)\,, \tag{3.41}$$

which is nothing else than (3.8). The proof is complete. ■

Proposition 3.7 shows that the solution $\mathbf{U}^m$ of (3.6) has the same regularity properties as the solution $\mathbf{u}$ of the problem $(\mathrm{NS})_p$. Thus we can split the error into two parts, namely

$$\mathbf{u}(t_m)-\mathbf{u}^m = \big(\mathbf{u}(t_m)-\mathbf{U}^m\big)+\big(\mathbf{U}^m-\mathbf{u}^m\big) =: \mathbf{E}^m+\mathbf{e}^m\,. \tag{3.42}$$

Let us first discuss the error $\mathbf{E}^m$, where we can take advantage of the regularity properties proved before. The error $\mathbf{E}^m$ is governed by the following system, which holds for all $\boldsymbol{\varphi}\in\mathbf{V}_p$,

$$\begin{aligned}\big(d_t\mathbf{E}^m,\boldsymbol{\varphi}\big) + \big(\mathbf{S}\big(\mathbf{D}(\mathbf{u}(t_m))\big)-\mathbf{S}\big(\mathbf{D}(\mathbf{U}^m)\big),\mathbf{D}(\boldsymbol{\varphi})\big)& \\ +\big([\nabla\mathbf{E}^m]\,\mathbf{u}(t_m),\boldsymbol{\varphi}\big) &= (\mathbf{R}^m,\boldsymbol{\varphi})\,,\end{aligned} \tag{3.43}$$

supplemented with

$$\mathbf{R}^m \equiv d_t\mathbf{u}(t_m) - \partial_t\mathbf{u}(t_m) = \frac{1}{k}\int_{t_{m-1}}^{t_m} (s - t_{m-1})\partial_t^2\mathbf{u}(s)\, ds\,. \tag{3.44}$$

From (3.44) and (3.3) we compute that

$$\|\mathbf{R}^m\|_2^2 \le c \sup_{s\in[t_{m-1},t_m]} \|\partial_t\mathbf{u}(s)\|_2^2\,, \tag{3.45}$$

$$\|\mathbf{R}^m\|_{(\mathbf{V}_2)^*}^2 \le c\, k \int_{t_{m-1}}^{t_m} \|\partial_t^2\mathbf{u}(s)\|_{(\mathbf{V}_2)^*}^2\, ds\,. \tag{3.46}$$

If we use $\mathbf{E}^m$ as a test function in (3.43) and sum over the number of iteration steps, we obtain, for $1 < r < 6(p-1)$,

$$\max_{0\le m\le M} \|\mathbf{E}^m\|_2^2 + k\sum_{m=0}^M \big(\|\mathbf{D}(\mathbf{E}^m)\|_{\frac{2r}{2-p+r}}^2 + \|\mathbf{D}(\mathbf{E}^m)\|_p^2\big) \le c(\mathbf{f},\mathbf{u}_0,r)\, k\sum_{m=0}^M (\mathbf{R}^m,\mathbf{E}^m)\,, \tag{3.47}$$

where we have used (2.9) and $\mathbf{u}(t_m), \mathbf{U}^m \in l^\infty(I_k;\mathbf{V}_r)$. We can bound the term on the right-hand side with the help of the embedding $\mathbf{W}^{1,\frac{2r}{2-p+r}}(\Omega) \hookrightarrow \mathbf{W}^{\frac{2r-6+3p}{2r},2}(\Omega)$ and the interpolation of $\mathbf{W}^{\frac{2r-6+3p}{2r},2}(\Omega)$ between $\mathbf{W}^{1,2}(\Omega)$ and $\mathbf{L}^2(\Omega)$ as follows

$$\begin{aligned}(\mathbf{R}^m,\mathbf{E}^m) &\le \|\mathbf{R}^m\|_{\mathbf{H}}^{1-\frac{2r-6+3p}{2r}} \|\mathbf{R}^m\|_{(\mathbf{V}_2)^*}^{\frac{2r-6+3p}{2r}} \|\mathbf{E}^m\|_{\mathbf{V}_{\frac{2r}{2-p+r}}} \\ &\le c(\mathbf{f},\mathbf{u}_0)\, \|\mathbf{R}^m\|_{(\mathbf{V}_2)^*}^{\frac{2r-6+3p}{r}} + \tfrac{1}{2}\, \|\mathbf{D}(\mathbf{E}^m)\|_{\frac{2r}{2-p+r}}^2\,,\end{aligned} \tag{3.48}$$

where we also used Korn's and Young's inequalities and (3.45). Now, we move the last term in (3.48) to the left-hand side of (3.47) and it remains to bound the first term in (3.48). Note, that

$$\frac{2r-6+3p}{2r} =: \widetilde{\alpha}(p,r) \nearrow \alpha_0(p) := \frac{5p-6}{4(p-1)}\,, \qquad \text{for } r \nearrow 6(p-1)\,. \tag{3.49}$$

From (3.46) and $(3.3)_3$ we derive

$$k\sum_{m=0}^M \|\mathbf{R}^m\|_{(\mathbf{V}_2)^*}^{2\widetilde{\alpha}(p,r)} \le c\, k^{2\widetilde{\alpha}(p,r)} \Big(\sum_{m=0}^M \int_{t_{m-1}}^{t_m} \|\partial_t^2\mathbf{u}(s)\|_{(\mathbf{V}_2)^*}^2\, ds\Big)^{2\widetilde{\alpha}(p,r)} \le c(\mathbf{f},\mathbf{u}_0)\, k^{2\widetilde{\alpha}(p,r)}\,,$$

which together with (3.47) yields

$$\max_{0\le m\le M} \|\mathbf{E}^m\|_2^2 + k\sum_{m=0}^M \|\mathbf{D}(\mathbf{E}^m)\|_p^2 \le c(\mathbf{f},\mathbf{u}_0,r)\, k^{2\widetilde{\alpha}(p,r)}\,, \tag{3.50}$$

with $\widetilde{\alpha}(p,r)$ defined in (3.49).

We still have to deal with the error $\mathbf{e}^m$, which is governed by the following system

$$(d_t\mathbf{e}^m,\boldsymbol{\varphi}) + \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^m)) - \mathbf{S}(\mathbf{D}(\mathbf{u}^m)),\mathbf{D}(\boldsymbol{\varphi})\big) = (\mathbf{r}^m,\boldsymbol{\varphi})\,, \tag{3.51}$$

which holds for all $\boldsymbol{\varphi} \in \mathbf{V}_p$, and where

$$\begin{aligned}-\mathbf{r}^m &= [\nabla\mathbf{U}^m]\mathbf{u}(t_m) - [\nabla\mathbf{u}^m]\mathbf{u}^m \\ &= [\nabla\mathbf{U}^m]\mathbf{E}^m + [\nabla\mathbf{U}^m]\mathbf{e}^m + [\nabla\mathbf{e}^m]\mathbf{u}^m\,.\end{aligned} \tag{3.52}$$

If we use in (3.51) the test function $\mathbf{e}^m$ and sum over all iteration steps, we get

$$
\begin{aligned}
&\max_{0\le m\le M} \|\mathbf{e}^m\|_2^2 + k\sum_{m=0}^{M} \frac{\|\mathbf{D}(\mathbf{e}^m)\|_p^2}{c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}} \\
&\le c\,k\sum_{m=0}^{M}\int_\Omega |\mathbf{E}^m|\,|\mathbf{e}^m|\,|\nabla \mathbf{U}^m|\,dx + c\,k\sum_{m=0}^{M}\int_\Omega |\mathbf{e}^m|^2\,|\nabla \mathbf{U}^m|\,dx \qquad (3.53)\\
&=: c\,k\sum_{m=0}^{M}\big(I_1^m + I_2^m\big)\,.
\end{aligned}
$$

For the lower bound of the elliptic term we used (2.9) with $r = p$ and the uniform bound for $\nabla \mathbf{U}^m \in l^\infty(I_k;\mathbf{L}^p(\Omega))$. Using Hölder's inequality and the interpolation inequality $\|\mathbf{v}\|_{2r'} \le \|\mathbf{v}\|_2^{1-\lambda}\|\nabla \mathbf{v}\|_p^{\lambda}$ with $\lambda = \frac{3p}{r(5p-6)}$ and that $\nabla \mathbf{U}^m \in l^\infty(I_k;\mathbf{L}^r(\Omega))$, $1 < r < 6(p-1)$, we find that

$$
\begin{aligned}
I_1^m &\le \|\nabla \mathbf{U}^m\|_r\|\mathbf{e}^m\|_{2r'}\|\mathbf{E}^m\|_{2r'} \qquad (3.54)\\
&\le c(\mathbf{f},\mathbf{u}_0)\|\mathbf{E}^m\|_2^{1-\lambda}\|\nabla \mathbf{E}^m\|_p^{\lambda}\|\mathbf{e}^m\|_2^{1-\lambda}\frac{\|\mathbf{D}(\mathbf{e}^m)\|_p^{\lambda}}{\big(c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}\big)^{\frac{\lambda}{2}}}\big(c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}\big)^{\frac{\lambda}{2}}\\
&\le c\,\|\mathbf{e}^m\|_2\|\mathbf{E}^m\|_2\big(c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}\big)^{\frac{\lambda}{2(1-\lambda)}} + \frac{1}{2}\frac{\|\mathbf{D}(\mathbf{e}^m)\|_p}{\big(c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}\big)^{\frac{1}{2}}}\|\mathbf{D}(\mathbf{E}^m)\|_p\\
&\le c\,\|\mathbf{E}^m\|_2^2 + c\,\big(c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{p}\big)^{\frac{2-p}{p}\frac{\lambda}{1-\lambda}}\|\mathbf{e}^m\|_2^2 + c\,\|\mathbf{D}(\mathbf{E}^m)\|_p^2 + \frac{1}{2}\frac{\|\mathbf{D}(\mathbf{e}^m)\|_p^2}{c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}}\,.
\end{aligned}
$$

The last term on the right-hand side is absorbed into the left-hand side of (3.53). For the first term and the third term in the last line of (3.54) we use estimate (3.50). The term I_2^m is treated analogously, replacing $\mathbf{E}^m$ by $\mathbf{e}^m$ and stopping the computations before the last line in (3.54). Thus we arrive at

$$
\begin{aligned}
&\max_{0\le m\le M} \|\mathbf{e}^m\|_2^2 + k\sum_{m=0}^{M} \frac{\|\mathbf{D}(\mathbf{e}^m)\|_p^2}{c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{2-p}} \\
&\qquad\le c\,k^{2\widetilde{\alpha}(p,r)} + k\sum_{m=0}^{M}\big(c+\|\mathbf{D}(\mathbf{e}^m)\|_p^{p}\big)^{\frac{2-p}{p}\frac{\lambda}{1-\lambda}}\|\mathbf{e}^m\|_2^2
\end{aligned}
\qquad (3.55)
$$

and we can use the discrete Gronwall's lemma whenever $\frac{2-p}{p}\frac{\lambda}{1-\lambda} < 1$, where $\lambda = \frac{3p}{r(5p-6)}$, $1 < r < 6(p-1)$. One easily computes that this requirement is equivalent to $p > \frac{11+\sqrt{21}}{10}$. After the application of Gronwall's lemma we obtain that the left-hand side of (3.55) is bounded by $c\,k^{2\widetilde{\alpha}(p,r)}$, with $\widetilde{\alpha}(p,r)$ given by (3.49). We can always choose r such that $2\widetilde{\alpha}(p,r) > 1$ and we readily obtain that

$$
\max_{0\le m\le M} \|\mathbf{D}(\mathbf{e}^m)\|_p^2 \le c
$$

and in turn we derive

$$\max_{0\le m\le M} \|\mathbf{e}^m\|_2^2 + k\sum_{m=0}^{M} \|\mathbf{D}(\mathbf{e}^m)\|_p^2 \le c(\mathbf{f},\mathbf{u}_0,r)\, k^{\widetilde{\alpha}(p,r)} . \tag{3.56}$$

Since the same estimates hold for $\mathbf{E}^m$ we have furnished the proof of Theorem 1.10.

4. Proof of Theorem 1.16

For the proof of the second part of Theorem 1.16, namely the error estimate (1.19), we follow the same strategy as in the proof of Theorem 1.10. In fact, the additional term $-k^\beta \operatorname{div}\big(\mathbf{T}_q(\mathbf{D}(\mathbf{u}^m))\big)$ is a coercive, strongly monotone operator and thus produces only positive terms. Thus we will be brief in this section and only point out the differences to Section 3.

The existence of a weak solution to problem $(\mathrm{NSS}_k)_p$ is ensured by the following

Lemma 4.1. *Let $\beta > 0$ and $q \ge 2$ be given, and let $\mathbf{u}_0$ and $\mathbf{f}$ satisfy the same assumptions as in Theorem 1.16. Then there exists a weak solution $\mathbf{u}^m$ of the problem $(NSS_k)_p$ satisfying*

$$\max_{0\le m\le M} \|\mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M} \|\mathbf{D}(\mathbf{u}^m)\|_p^p + k^\beta\, k\sum_{m=0}^{M} \|\mathbf{D}(\mathbf{u}^m)\|_q^q \le c(\mathbf{f},\mathbf{u}_0) . \tag{4.2}$$

Proof : Since $q \ge 2$, it follows from the energy estimate (4.2) that

$$-\operatorname{div}\, \mathbf{S}(\mathbf{D}(\cdot)) - k^\beta \operatorname{div}\, \mathbf{T}_q(\mathbf{D}(\cdot)) : \mathbf{V}_q \to \big(\mathbf{V}_q\big)^*$$

is a coercive, monotone operator. Moreover, $d_t\mathbf{u}^m$ belongs to $l^\infty(I_k;\mathbf{L}^2(\Omega))$, where the norms can depend on k, and thus we can view (1.14) as a steady system. The existence of weak solutions of the problem $(\mathrm{NSS}_k)_p$ now follows from the standard theory of monotone operators. ∎

As in Section 3 we introduce an auxiliary problem:

Algorithm. *Suppose that $\mathbf{u}$ is a strong solution to the problem $(NS)_p$ with the properties stated in Proposition 3.1. For $\beta > 0$ and $q \ge 2$ given, determine $\mathbf{U}^m$, $m = 0,\ldots,M$, that solves*

$$\begin{aligned} d_t\mathbf{U}^m - \operatorname{div}\, \mathbf{S}(\mathbf{D}(\mathbf{U}^m)) - k^\beta \operatorname{div}\, \mathbf{T}_q(\mathbf{D}(\mathbf{U}^m)) + [\nabla\mathbf{U}^m]\mathbf{u}(t_m) + \nabla\Pi^m &= \mathbf{f}(t_m) , \\ \operatorname{div}\, \mathbf{U}^m &= 0 , \\ \mathbf{U}^0 &= \mathbf{u}_0 , \end{aligned} \tag{4.3}$$

endowed with space-periodic boundary conditions (1.7).

We have the analogue of Proposition 3.7, namely

Proposition 4.4. *Let $\beta > 0$ and $q \ge 2$ be given and let $\mathbf{u}_0$ and $\mathbf{f}$ satisfy the same assumptions as in Theorem 1.16. Let $\mathbf{u}$ defined on $I = [0, T^*]$ be the strong solution ensured by Proposition 3.1 and let $t_M < T^*$. Then there exists a strong solution $\mathbf{U}^m$*

of the problem (4.3) *whenever* $p \in (\frac{11+\sqrt{21}}{10}, 2]$. *This solution satisfies*

$$\begin{aligned}\max_{0\le m\le M} \|d_t\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M} \big(\mathcal{I}_p(\mathbf{U}^m)^{\frac{5p-6}{2-p}} + \mathcal{K}_p(\mathbf{U}^m)\big)\\ + k^\beta\, k\sum_{m=0}^{M}\mathcal{K}_q(\mathbf{U}^m) + k^{\frac{5p-6}{2-p}\,\beta}\, k\sum_{m=0}^{M}\mathcal{I}_q(\mathbf{U}^m)^{\frac{5p-6}{2-p}} \le c(\mathbf{f},\mathbf{u}_0)\,.\end{aligned} \tag{4.5}$$

In particular for all $1 < r < 6(p-1)$ *it holds*

$$\begin{aligned}\mathbf{U}^m &\in l^{\frac{5p-6}{2-p}}\big(I_k; \mathbf{W}^{2,\frac{3p}{p+1}}(\Omega)\big) \;\cap\; l^\infty(I_k;\mathbf{V}_r)\,,\\ d_t\mathbf{U}^m &\in l^{\frac{p(5p-6)}{(3p-2)(p-1)}}\big(I_k; \mathbf{W}^{1,\frac{3p}{p+1}}(\Omega)\big) \;\cap\; l^\infty\big(I_k;\mathbf{L}^2(\Omega)\big)\,.\end{aligned} \tag{4.6}$$

Proof : The proof follows exactly the lines of that one of Proposition 3.7. Note that the stabilization $-k^\beta \operatorname{div} \mathbf{T}_q(\mathbf{D}(\mathbf{U}^m))$ produces only positive terms when we use $\mathbf{U}^m$, $-\Delta\mathbf{U}^m$, $d_t\mathbf{U}^m$, and $d_t^2\mathbf{U}^m$ as test functions. Moreover, we will not use these additional positive terms coming from $-k^\beta \operatorname{div} \mathbf{T}_q(\mathbf{D}(\mathbf{U}^m))$ to handle the terms which appear on the right-hand sides, when using the above test functions, but we will only use the terms which are already present without the stabilization term. Thus we will be brief and only indicate the differences to the proof of Proposition 3.7. The weak formulation of (4.3) reads for all $\boldsymbol{\varphi} \in \mathbf{V}_q$

$$\begin{aligned}\big(d_t\mathbf{U}^m,\boldsymbol{\varphi}\big) + \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^m)),\mathbf{D}(\boldsymbol{\varphi})\big)\\ + k^\beta\big(\mathbf{T}_q(\mathbf{D}(\mathbf{U}^m)),\mathbf{D}(\boldsymbol{\varphi})\big) + \big([\nabla\mathbf{U}^m]\mathbf{u}(t_m),\boldsymbol{\varphi}\big) = \big(\mathbf{f}(t_m),\boldsymbol{\varphi}\big)\,.\end{aligned} \tag{4.7}$$

We use $\mathbf{U}^m$ as a test function in (4.7) and obtain after summation over all iteration steps (cf. (3.11))

$$\max_{0\le m\le M}\|\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M}\|\mathbf{D}(\mathbf{U}^m)\|_p^p + k^\beta\, k\sum_{m=0}^{M}\|\mathbf{D}(\mathbf{U}^m)\|_q^q \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{4.8}$$

Next, we use $-\Delta\mathbf{U}^m$ as a test function in (4.7) and obtain (cf. (3.13)–(3.17))

$$\max_{0\le m\le M}\|\nabla\mathbf{U}^m\|_2^2 + k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{U}^m) + k^\beta\, k\sum_{m=0}^{M}\mathcal{I}_q(\mathbf{U}^m) \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{4.9}$$

After that we use $d_t\mathbf{U}^m$ as a test function in (4.7) and arrive as in (3.18)–(3.21) at

$$k\sum_{m=0}^{N}\|d_t\mathbf{U}^m\|_2^2 + \max_{0\le m\le M}\|\nabla\mathbf{U}^m\|_p^p + k^\beta \max_{0\le m\le M}\|\nabla\mathbf{U}^m\|_q^q \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{4.10}$$

Now we use $-d_t^2\mathbf{U}^m$ and $-\Delta\mathbf{U}^m$ "almost pointwise in time" simultaneously as test functions. Firstly, we have to introduce $\mathbf{U}^{-1}$ and to verify that $d_t\mathbf{U}^0 \in \mathbf{L}^2(\Omega)$. For that we set for all $\boldsymbol{\varphi} \in \mathbf{V}_q$

$$\begin{aligned}\frac{1}{k}\big(\mathbf{U}^0 - \mathbf{U}^{-1},\boldsymbol{\varphi}\big) + \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^0)),\mathbf{D}(\boldsymbol{\varphi})\big)\\ + k^\beta\big(\mathbf{T}_q(\mathbf{D}(\mathbf{U}^0)),\mathbf{D}(\boldsymbol{\varphi})\big) + ([\nabla\mathbf{U}^0]\mathbf{U}^0,\boldsymbol{\varphi}) = (\mathbf{f}(0),\boldsymbol{\varphi})\,.\end{aligned}$$

Using $\mathbf{U}^0 = \mathbf{u}_0$, $p \le 2$, $\mathbf{u}_0 \in \mathbf{W}^{2,\frac{6(q-1)}{2q-1}}(\Omega)$ and Hölder's inequality we obtain

$$\begin{aligned}\|d_t \mathbf{U}^0\|_2^2 &\le c\,\|\mathbf{f}(0)\|_2^2 + \|[\nabla \mathbf{u}_0]\mathbf{u}_0\|_2^2 + \|\operatorname{div} \mathbf{S}(\mathbf{D}(\mathbf{u}_0))\|_2^2 + k^{2\beta}\|\operatorname{div} \mathbf{T}_q(\mathbf{D}(\mathbf{u}_0))\|_2^2 \\ &\le c(\mathbf{f}, \mathbf{u}_0)\,.\end{aligned}$$

Now we can take the discrete time derivative of the weak formulation (3.10), use $d_t\mathbf{U}^m$ as a test function, and proceed as in (3.23)–(3.34) to derive, for $\gamma < \frac{3p-2}{2}$,

$$\begin{aligned}\max_{0\le m\le M} \|d_t \mathbf{U}^m\|_2^2 + k\sum_{m=0}^M \mathcal{K}_p(\mathbf{U}^m) + k\sum_{m=0}^M \mathcal{I}_p(\mathbf{U}^m)^\gamma \\ + k^\beta\, k\sum_{m=0}^M \mathcal{K}_q(\mathbf{U}^m) + k^{\gamma\beta}\, k\sum_{m=0}^M \mathcal{I}_q(\mathbf{U}^m)^\gamma \le c(\mathbf{f}, \mathbf{u}_0)\,.\end{aligned} \tag{4.11}$$

Note that in (3.28), (3.29), (3.2), (3.32) one has to replace $\mathcal{I}_p(\mathbf{U}^m)$ by $\mathcal{I}_p(\mathbf{U}^m) + k^\beta \mathcal{I}_q(\mathbf{U}^m)$. Again one can improve the exponent of $\mathcal{I}_p(\mathbf{U}^m)$ and $k^\beta\mathcal{I}_q(\mathbf{U}^m)$ in the same way as in (3.34)–(3.41). The previous remark applies accordingly. Thus we derive

$$\begin{aligned}\max_{0\le m\le M} \|d_t \mathbf{U}^m\|_2^2 + k\sum_{m=0}^M \mathcal{K}_p(\mathbf{U}^m) + k\sum_{m=0}^M \mathcal{I}_p(\mathbf{U}^m)^{\frac{5p-6}{2-p}} \\ + k^\beta\, k\sum_{m=0}^M \mathcal{K}_q(\mathbf{U}^m) + k^{\frac{5p-6}{2-p}\beta}\, k\sum_{m=0}^M \mathcal{I}_q(\mathbf{U}^m)^{\frac{5p-6}{2-p}} \le c(\mathbf{f}, \mathbf{u}_0)\,.\end{aligned} \tag{4.12}$$

This finishes the proof. ∎

Regarding the regularity of solutions we are now in the same situation as in Section 3 and thus we can split again the error into two parts, namely

$$\mathbf{u}(t_m) - \mathbf{u}^m = \big(\mathbf{u}(t_m) - \mathbf{U}^m\big) + \big(\mathbf{U}^m - \mathbf{u}^m\big) =: \mathbf{E}^m + \mathbf{e}^m\,, \tag{4.13}$$

where $\mathbf{u}$ is a solution of the problem $(\mathrm{NS})_p$, $\mathbf{U}^m$ is a solution of (4.3) and $\mathbf{u}^m$ is a solution of the problem $(\mathrm{NSS}_k)_p$. Note that until now no restrictions other than $q \ge 2$ and $\beta > 0$ have been used. From now on we fix

$$\beta := \alpha_0(p) = \frac{5p-6}{4(p-1)}\,.$$

The error $\mathbf{E}^m$ is governed by the following system, which holds for all $\boldsymbol{\varphi} \in \mathbf{V}_q$,

$$\begin{aligned}&\big(d_t\mathbf{E}^m, \boldsymbol{\varphi}\big) + \big(\mathbf{S}(\mathbf{D}(\mathbf{u}(t_m))) - \mathbf{S}(\mathbf{D}(\mathbf{U}^m)), \mathbf{D}(\boldsymbol{\varphi})\big) + \big([\nabla \mathbf{E}^m]\,\mathbf{u}(t_m), \boldsymbol{\varphi}\big) \\ &+ k^{\alpha_0(p)}\big(\mathbf{T}_q(\mathbf{D}(\mathbf{u}(t_m))) - \mathbf{T}_q(\mathbf{D}(\mathbf{U}^m)), \mathbf{D}(\boldsymbol{\varphi})\big) = (\mathbf{R}^m, \boldsymbol{\varphi}) + k^{\alpha_0(p)}\big(\mathbf{T}_q(\mathbf{D}(\mathbf{u}(t_m))), \mathbf{D}(\boldsymbol{\varphi})\big)\,,\end{aligned} \tag{4.14}$$

where $\mathbf{R}^m$ is defined in (3.44). We use $\mathbf{E}^m$ as a test function in (4.14), sum over the number of iteration steps, and obtain, for $1 < r < 6(p-1)$,

$$\begin{aligned}\max_{0\le m\le M} \|\mathbf{E}^m\|_2^2 + k\sum_{m=0}^M \big(\|\mathbf{D}(\mathbf{E}^m)\|_{\frac{2r}{2-p+r}}^2 + \|\mathbf{D}(\mathbf{E}^m)\|_p^2\big) + k^{\alpha_0(p)}\, k\sum_{m=0}^M \|\mathbf{D}(\mathbf{E}^m)\|_q^q \\ \le c(\mathbf{f}, \mathbf{u}_0, r)\, k\sum_{m=0}^M \Big(\big(\mathbf{R}^m, \mathbf{E}^m\big) + k^{\alpha_0(p)}\,\big(\mathbf{T}_q\big(\mathbf{D}(\mathbf{u}(t_m))\big), \mathbf{D}(\mathbf{E}^m)\big)\Big)\,,\end{aligned} \tag{4.15}$$

where we have used (2.9), (2.10) and $\mathbf{u}(t_m), \mathbf{U}^m \in l^\infty(I_k; \mathbf{V}_r)$. The first term on the right-hand side can be bounded as in Section 3 by

$$c(\mathbf{f}, \mathbf{u}_0, r)\, k^{2\widetilde{\alpha}(p,r)}\,, \tag{4.16}$$

where $\widetilde{\alpha}(p,r)$ is defined in (3.49). For the second term on the right-hand side of (4.15) we use Hölder's and Young's inequalities and $\mathbf{u} \in C(I; \mathbf{V}_r)$, $1 < r < 6(p-1)$, to obtain

$$\begin{aligned} k^{\alpha_0(p)}\, k \sum_{m=0}^{M} \big|\big(\mathbf{T}_q\big(\mathbf{D}(\mathbf{u}(t_m))\big), \mathbf{D}(\mathbf{E}^m)\big)\big| \\ &\le c_\varepsilon\, k^{2\alpha_0(p)}\, k \sum_{m=0}^{M} \big(1 + \|\nabla \mathbf{u}(t_m)\|^{2(q-1)}_{(q-1)\frac{2r}{r+p-2}}\big) + \varepsilon\, k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{E}^m)\|^2_{\frac{2r}{2-p+r}} \\ &\le c_\varepsilon\, k^{2\alpha_0(p)} + \varepsilon\, k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{E}^m)\|^2_{\frac{2r}{2-p+r}}\,, \end{aligned} \tag{4.17}$$

provided that $(q-1)\frac{2r}{r+p-2} < 6(p-1)$, which is possible for $q < \frac{7p-6}{2}$, which is the upper bound for q appearing in Theorem 1.16. The second term is absorbed into the left-hand side of (4.15). Since $\widetilde{\alpha}(p,r) < \alpha_0(p)$ (cf. (3.49)) we have proven

$$\max_{0\le m\le M} \|\mathbf{E}^m\|_2^2 + k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{E}^m)\|_p^2 + k^{\alpha_0(p)}\, k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{E}^m)\|_q^q \le c(\mathbf{f}, \mathbf{u}_0, r)\, k^{2\widetilde{\alpha}(p,r)}\,, \tag{4.18}$$

with $\widetilde{\alpha}(p,r)$ defined in (3.49).

We still have to deal with the error $\mathbf{e}^m$, which is governed by the following system

$$\begin{aligned} (d_t \mathbf{e}^m, \boldsymbol{\varphi}) + \big(\mathbf{S}(\mathbf{D}(\mathbf{U}^m)) - \mathbf{S}(\mathbf{D}(\mathbf{u}^m)), \mathbf{D}(\boldsymbol{\varphi})\big) \\ + k^{\alpha_0(p)}\big(\mathbf{T}_q(\mathbf{D}(\mathbf{U}^m)) - \mathbf{T}_q(\mathbf{D}(\mathbf{u}^m)), \mathbf{D}(\boldsymbol{\varphi})\big) = (\mathbf{r}^m, \boldsymbol{\varphi})\,, \end{aligned} \tag{4.19}$$

which holds for all $\boldsymbol{\varphi} \in \mathbf{V}_q$, and where $\mathbf{r}^m$ is defined in (3.52). The new term in this equation compared to equation (3.51), when tested with $\mathbf{e}^m$, is positive and thus we can proceed exactly as in Section 3 (cf. (3.53)–(3.56)). We arrive at

$$\max_{0\le m\le M} \|\mathbf{e}^m\|_2^2 + k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{e}^m)\|_p^2 + k^{\alpha_0(p)}\, k \sum_{m=0}^{M} \|\mathbf{D}(\mathbf{e}^m)\|_q^q \le c(\mathbf{f}, \mathbf{u}_0, r)\, k^{\widetilde{\alpha}(p,r)}\,. \tag{4.20}$$

From (4.18), (4.20) and the definition of $\alpha_0(p)$ and $\widetilde{\alpha}(p,r)$ we immediately get for all $0 < \alpha < \alpha_0(p)$ and $2 \le q < \frac{7p-6}{2}$

$$\begin{aligned} \max_{0\le m\le M} \|\mathbf{u}(t_m) - \mathbf{u}^m\|_2^2 + k \sum_{m=0}^{M} \big\|\mathbf{D}\big(\mathbf{u}(t_m) - \mathbf{u}^m\big)\big\|_p^2 \\ + k^{\alpha_0(p)}\, k \sum_{m=0}^{M} \big\|\mathbf{D}\big(\mathbf{u}(t_m) - \mathbf{u}^m\big)\big\|_q^q \le c(\mathbf{f}, \mathbf{u}_0, \alpha)\, k^{2\alpha}\,, \end{aligned} \tag{4.21}$$

which is the second statement in Theorem 1.16. We still have to prove that $\mathbf{u}^m$ is a strong solution of the problem $(\mathrm{NSS}_k)_p$. Since $\mathbf{u} \in l^\infty(I_k; \mathbf{V}_r)$, $1 < r < 6(p-1)$, we immediately derive that

$$\begin{aligned}\max_{0\le m\le M}\|\nabla\mathbf{u}^m\|_p &\le c\,k^{\frac{2\alpha-1}{2}}\,,\\ \max_{0\le m\le M}\|\nabla\mathbf{u}^m\|_q &\le c\,k^{\frac{2\alpha-\alpha_0(p)}{q}}\,.\end{aligned}\tag{4.22}$$

Using the interpolation of $L^s(\Omega)$, $p<s<q$, between $L^p(\Omega)$ and $L^q(\Omega)$ we obtain, with $\lambda=\frac{p(q-s)}{s(q-p)}$,

$$\begin{aligned}\max_{0\le m\le M}\|\nabla\mathbf{u}^m\|_s &\le \max_{0\le m\le M}\|\nabla\mathbf{u}^m\|_q^{1-\lambda}\max_{0\le m\le M}\|\nabla\mathbf{u}^m\|_p^{\lambda}\\ &\le c\,k^{(1-\lambda)\frac{2\alpha-\alpha_0(p)}{q}}\,k^{\lambda\frac{2\alpha-1}{2}}\,,\\ &\le c\,,\end{aligned}\tag{4.23}$$

whenever $(1-\lambda)\,\frac{2\alpha-\alpha_0(p)}{q}+\lambda\,\frac{2\alpha-1}{2}\ge 0$, which is the case for

$$s<s_0(p,q):=\frac{p(2-p+q(3p-4))}{(3p-2)(p-1)}\,.\tag{4.24}$$

With this new information, namely $\mathbf{u}^m\in l^\infty(I_k;\mathbf{V}_s)$ with $1\le s<s_0(p,q)$, we can now show that $\mathbf{u}^m$ is a strong solution. For that we use $-\Delta\mathbf{u}^m$ as a test function in the weak formulation of (1.14) with $\beta=\alpha_0(p)$, which reads for all $\boldsymbol{\varphi}\in\mathbf{V}_q$

$$\begin{aligned}&\big(d_t\mathbf{u}^m,\boldsymbol{\varphi}\big)+\big(\mathbf{S}(\mathbf{D}(\mathbf{u}^m)),\mathbf{D}(\boldsymbol{\varphi})\big)\\ &\qquad+k^{\alpha_0(p)}\big(\mathbf{T}_q(\mathbf{D}(\mathbf{u}^m)),\mathbf{D}(\boldsymbol{\varphi})\big)+\big([\nabla\mathbf{u}^m]\mathbf{u}^m,\boldsymbol{\varphi}\big)=\big(\mathbf{f}(t_m),\boldsymbol{\varphi}\big)\,.\end{aligned}\tag{4.25}$$

We obtain (cf. (3.13))

$$\|\nabla\mathbf{u}^M\|_2^2+k\sum_{m=0}^{M}\mathcal{I}_p(\mathbf{u}^m)\le c(\mathbf{f},\mathbf{u}_0)\Big(1+k\sum_{m=0}^{M}\int_\Omega|\nabla\mathbf{u}^m|\,|\nabla\mathbf{u}^m|^2\,dx\Big)\,.\tag{4.26}$$

For $1<s<s_0(p,q)$, the last term can be estimated by

$$\begin{aligned}\|\nabla\mathbf{u}^m\|_s\|\nabla\mathbf{u}^m\|_{2s'}^2 &\le c\,\|\nabla\mathbf{u}^m\|_{2s'}^2\\ &\le c_\varepsilon\,\|\nabla\mathbf{u}^m\|_2^2+\varepsilon\,\mathcal{I}_p(\mathbf{u}^m)\,,\end{aligned}\tag{4.27}$$

where s' is the dual exponent to s, and where we used (4.23), the interpolation of $L^{2s'}(\Omega)$ between $L^2(\Omega)$ and $L^{\frac{6s}{6-3p+s}}(\Omega)$, and (2.12). This is of course only possible as long as $2s'<\frac{6s}{6-3p+s}$, which is equivalent to the requirement

$$s>\frac{9-3p}{2}\,.\tag{4.28}$$

This condition together with (4.24) gives a lower bound for q, namely

$$q>\frac{-9p^3+44p^2-55p+18}{2p(3p-4)}\,,\tag{4.29}$$

which together with $q\ge 2$ has to be compatible[4] with the upper bound $q<\frac{7p-6}{2}$. However for $p\in(\frac{11+\sqrt{21}}{10},2]$ there is no problem with these requirements. After the application of the discrete Gronwall's inequality we thus have proved for all q's satisfying

[4]Note that for $1.6955\lesssim p\le 2$ the requirement $q\ge 2$ is stronger than (4.29)

the requirements of Theorem 1.16 that

$$\max_{0\le m\le M} \|\nabla \mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M} \mathcal{I}_p(\mathbf{u}^m) \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{4.30}$$

Next, we take the discrete time derivative of (4.25) and use $d_t\mathbf{u}^m$ as a test function and obtain (cf. (3.23), (3.26))

$$\max_{0\le m\le M} \|d_t\mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M} \mathcal{K}_p(\mathbf{u}^m) \le c\left(1 + k\sum_{m=0}^{M}\int_\Omega |\nabla \mathbf{u}^m|\,|d_t\mathbf{u}^m|^2\,dx\right). \tag{4.31}$$

Since we have the same lower bounds for $d_t\mathbf{u}^m$ in terms of $\mathcal{K}_p(\mathbf{u}^m)$ as for $\nabla\mathbf{u}^m$ in terms of $\mathcal{I}_p(\mathbf{u}^m)$ we can proceed exactly as in (4.27) to obtain that the right-hand side of (4.31) can be estimated by

$$\|\nabla \mathbf{u}^m\|_s \|d_t\mathbf{u}^m\|_{2s'}^2 \le c\,\|d_t\mathbf{u}^m\|_{2s'}^2 \le c_\varepsilon\,\|d_t\mathbf{u}^m\|_2^2 + \varepsilon\,\mathcal{K}_p(\mathbf{u}^m)\,. \tag{4.32}$$

Discrete Gronwall's inequality now gives

$$\max_{0\le m\le M} \|d_t\mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M} \mathcal{K}_p(\mathbf{u}^m) \le c(\mathbf{f},\mathbf{u}_0)\,. \tag{4.33}$$

It remains to show that the exponent of $\mathcal{I}_p(\mathbf{u}^m)$ in (4.30) can be improved. For that we use again $-\Delta\mathbf{u}^m$ as a test function in (4.25) and obtain (cf. (3.28), (3.35))

$$1 + \mathcal{I}_p(\mathbf{u}^m) \le c\left(1 + c_\varepsilon\|\nabla \mathbf{u}^m\|_2^2 + \varepsilon\,\mathcal{I}_p(\mathbf{u}^m) + \|d_t\mathbf{u}^m\|_{\frac{3p}{2p-1}}\big(1+\mathcal{I}_p(\mathbf{u}^m)\big)^{\frac{1}{p}}\right). \tag{4.34}$$

Using (4.27) we arrive at (3.36) with $\mathbf{U}^m$ replaced by $\mathbf{u}^m$ and then we can proceed exactly as in Section 3 (cf. (3.37), (3.38)) to obtain (3.38), which in view of (4.33) delivers

$$\max_{0\le m\le M} \|d_t\mathbf{u}^m\|_2^2 + k\sum_{m=0}^{M} \mathcal{K}_p(\mathbf{u}^m) + k\sum_{m=0}^{M} \mathcal{I}_p(\mathbf{u}^m)^{\frac{5p-6}{2-p}} \le c(\mathbf{f},\mathbf{u}_0)\,.$$

This immediately implies (1.18) and the proof of Theorem 1.16 is complete.

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