

Boundary Element Methods for Maxwell Transmission Problems in Lipschitz Domains*

A. Buffa, R. Hiptmair[†], T. von Petersdorff[‡] and C. Schwab

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Seminar für Angewandte Mathematik
Eidgenössische Technische Hochschule
CH-8092 Zürich
Switzerland

*Research performed while T. von Petersdorff was visiting the Seminar für Angewandte Mathematik, ETH Zurich

[†]Mathematisches Institut, Universität Tübingen, D-72076 Tübingen, Germany

[‡]Tobias von Petersdorff, Department of Mathematics, University of Maryland, MD 20742, USA

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Dedicated to Professor Wolfgang Wendland to his 65th birthday

Abstract

We consider the Maxwell equations in a domain with Lipschitz boundary and the boundary integral operator A occurring in the Calderón projector. We prove an inf-sup condition for A using a Hodge decomposition. We apply this to two types of boundary value problems: the exterior scattering problem by a perfectly conducting body, and the dielectric problem with two different materials in the interior and exterior domain. In both cases we obtain an equivalent boundary equation which has a unique solution. We then consider Galerkin discretizations with Raviart-Thomas spaces. We show that these spaces have discrete Hodge decompositions which are in some sense close to the continuous Hodge decomposition. This property allows us to prove quasioptimal convergence of the resulting boundary element methods.

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*Mathematisches Institut, Universität Tübingen, D-72076 Tübingen, Germany

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A. Buffa¹, R. Hiptmair², T. von Petersdorff³, C. Schwab⁴

¹ Istituto di Analisi Numerica del C.N.R., Via Ferrata 1, 27100 Pavia, Italy

² Mathematisches Institut, Universität Tübingen, D-72076 Tübingen, Germany

³ Department of Mathematics, University of Maryland, College Park, MD 20742, USA

⁴ Seminar für Angewandte Mathematik, ETH-Zentrum, 8092 Zürich, Switzerland

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on the occasion of his 65th birthday*

Summary We consider the Maxwell equations in a domain with Lipschitz boundary and the boundary integral operator A occurring in the Calderón projector. We prove an inf-sup condition for A using a Hodge decomposition. We apply this to two types of boundary value problems: the exterior scattering problem by a perfectly conducting body, and the dielectric problem with two different materials in the interior and exterior domain. In both cases we obtain an equivalent boundary equation which has a unique solution. We then consider Galerkin discretizations with Raviart-Thomas spaces. We show that these spaces have discrete Hodge decompositions which are in some sense close to the continuous Hodge decomposition. This property allows us to prove quasioptimal convergence of the resulting boundary element methods.

1 Introduction

Time harmonic Maxwell equations occur in many applications and often involve unbounded domains, making a boundary integral formulation desirable. However, compared with other elliptic problems such as the Helmholtz equation the Maxwell equations pose two additional difficulties: boundary conditions are formulated in terms of tangential components of electric and magnetic fields and the operators (both in

the domain and on the boundary) are indefinite since the electric and magnetic field energies occur with opposite signs in the Lagrangian governing electromagnetic phenomena. Therefore, the general formalism for obtaining strongly elliptic boundary integral equations [20] is not directly applicable.

The first difficulty requires spaces so that traces and bilinear forms are continuous, leading in a natural way to variational formulations in $H(\mathbf{curl}, \Omega)$ and $H^{-1/2}(\text{div}, \Gamma)$. For smooth boundaries several approaches [3], [1], [26], [19] yielded quasioptimally convergent boundary element methods. Some of these approaches do not generalize to polyhedra or Lipschitz domains. There, the discretization must be chosen carefully. More precisely, in [17] M. Costabel proved that $H_0(\text{div}, \Omega) \cap H(\mathbf{curl}, \Omega)$ is a closed subspace of $H^1(\Omega) \cap H_0(\text{div}, \Omega)$ as soon as Ω is a bounded polyhedral domain with reentrant corners or edges (a similar result holds for exterior domains). This makes the approximation of the electric and magnetic field by standard finite elements infeasible and suggests that likewise standard, continuous boundary elements should not be used to solve the related integral equations [26].

The proper choice of spaces and trace operators for Maxwell's equations in polyhedra and Lipschitz domains was only recently understood [8], [9], [11]. Based on this functional setting, several types of integral equation formulations are possible and attention has been devoted to the electric field integral equation (EFIE), i.e., to the exterior scattering problem by a perfectly conducting body. In this context, methods proposed for regular domains have been very recently generalized to non-smooth ones. In [1] the authors proposed the direct use of the Hodge decomposition, requiring C^1 boundary elements. In [10], the authors introduce additional variables to obtain a mixed method using standard $H^1, H^{1/2}, H^{-1/2}$ spaces and provide quasi optimal convergence in arbitrary polyhedral domain. Besides the inconvenience of additional variables the convergence rate is limited by the index s^* of the regularity of an auxiliary problem which is not directly related to the regularity of the Maxwell solution. Engineers have been using instead a simpler variational formulation set in $H^{-1/2}(\text{div}, \Gamma)$ and Raviart-Thomas boundary elements. This 'natural' formulation has been first analyzed by Bendali [3] for C^∞ closed manifolds. Recently, Hiptmair and Schwab [24] obtained stability of the discrete problem and sub-optimal convergence rates for polyhedral domains (the sub-optimality stemming from the "artificial" regularity parameter s^* .) In [7] the same ideas are applied to Lipschitz (non orientable) screens and the method is proved to converge with

optimal asymptotic rate (i.e., the convergence rate is bounded only by the regularity of the solution and the degree of the polynomials).

Beyond the electric field integral equation, to our knowledge, no theory seems to exist for integral equations related to the Maxwell equations on Lipschitz domains, neither for the continuous or discretized setting. We refer to [12] for an overview over boundary value problems of electromagnetics and boundary integral equations for the smooth case. In the present paper, we present boundary integral equation formulations for electromagnetic transmission problems. To this end, we give a systematic analysis of the operator A appearing in the Calderón projector $P = \frac{1}{2}I + A$ associated to Maxwell equations under minimal regularity assumptions on the boundary Γ . We proceed similarly as in [28] for the Helmholtz equation and prove a symmetry property and an inf-sup property for A . For electromagnetics the operator A has the structure $A = \begin{pmatrix} M & C \\ C & M \end{pmatrix}$ where C is the integral operator arising in the EFIE for the perfect conductor problem. Because of the symmetry, the inf-sup condition for A rests on an inf-sup condition of the compactly perturbed operator C and on a compactness property of the operator M . To establish these results is the main objective of this paper. This implies in particular the unique solvability of the boundary integral equations. At a discrete level, we use the general setting introduced in [13] and we prove quasi optimal convergence of Galerkin boundary element methods using Raviart-Thomas type boundary element spaces. Finally, applications of the developed theory are presented. Namely, the dielectric interface problem and the direct method for the perfect conductor. In both applications, the well-posedness of the continuous and discrete problems are shown to be an elementary consequence of the previous analysis.

The paper is organized as follows: In sections 2 and 3 we state the Maxwell equations and the appropriate spaces and trace mappings. We also introduce a Hodge decomposition on the boundary, define the integral operators and prove the continuous inf-sup condition for A . In section 4 we show discrete inf-sup condition and quasioptimal convergence provided a discrete Hodge decomposition which is in some sense close to the continuous Hodge decomposition. We show that this is satisfied e.g. for Raviart-Thomas spaces. In section 5 the results are applied to the perfect conductor problem, and in section 6 we treat the dielectric problem.

2 Maxwell equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain with boundary $\Gamma = \partial\Omega$. The complement of Ω will be denoted by $\Omega^c := \mathbb{R}^3 \setminus \overline{\Omega}$. The unit normal vector \mathbf{n} on the boundary Γ points from Ω to Ω^c . Let ε and μ be the dielectric permittivity and magnetic permeability respectively. We suppose that ε and μ are positive constants in \mathbb{R}^3 .

By \mathbf{E} and \mathbf{H} we denote the electric and magnetic field, and suppose they satisfy the linear Maxwell equations:

$$\mathbf{curl} \mathbf{E} - i\omega\mu\mathbf{H} = 0, \quad \mathbf{curl} \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0 \quad (2.1)$$

both in Ω and Ω^c .

We define $k := \omega\sqrt{\mu\varepsilon}$. Both in Ω and Ω^c , the system (2.1) can be reduced to a second order equation for the electric field only which reads:

$$\mathbf{curl} \mathbf{curl} \mathbf{u} - k^2\mathbf{u} = 0 \quad \text{in } \Omega \cup \Omega^c, \quad (2.2)$$

where the function \mathbf{u} is the electric field \mathbf{E} , the magnetic flux density \mathbf{H} is given by $\mathbf{H} = \frac{1}{i\omega\mu} \mathbf{curl} \mathbf{u}$. Further, we impose standard Silver Müller radiation condition at infinity for the exterior domain Ω^c (see, e.g., [26], [12]):

$$\left| \mathbf{curl} \mathbf{u}(\mathbf{r}) \times \frac{\mathbf{r}}{|\mathbf{r}|} - ik\mathbf{u} \right| = o\left(\frac{1}{|\mathbf{r}|}\right) \quad |\mathbf{r}| \rightarrow \infty. \quad (2.3)$$

We will say that \mathbf{u} is a *Maxwell solution in Ω* if \mathbf{u} satisfies (2.2) in Ω and that \mathbf{u} is a *Maxwell solution in Ω^c* if it satisfies (2.2) in Ω^c and (2.3) at infinity.

In order to obtain boundary value problems with a unique solution suitable boundary conditions have to be added. We will discuss two cases later: In section 5 we will consider a Dirichlet problem in Ω^c , and in section 6 we will consider a problem with different material constants in Ω and Ω^c with certain interface conditions on Γ .

3 Definition and main properties of boundary integral operators

3.1 Spaces and traces

We present function spaces in which variational boundary integral equations on nonsmooth domains are set. Spaces on the boundary Γ

shall be defined in terms of the following spaces in the domain Ω :

$$\begin{aligned}\mathbf{H}(\mathbf{curl}, \Omega) &= \{\mathbf{u} \in L^2(\Omega)^3 : \mathbf{curl} \mathbf{u} \in L^2(\Omega)^3\}; \\ \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c) &= \{\mathbf{u} \in L^2_{\text{loc}}(\Omega^c)^3 : \mathbf{curl} \mathbf{u} \in L^2_{\text{loc}}(\Omega^c)^3\}; \\ \mathbf{H}(\mathbf{curl} \mathbf{curl}, \Omega) &= \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) \mid \mathbf{curl} \mathbf{curl} \mathbf{u} \in L^2(\Omega)^3\}; \\ \mathbf{H}_{\text{loc}}(\mathbf{curl} \mathbf{curl}, \Omega^c) &= \{\mathbf{u} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c) \mid \mathbf{curl} \mathbf{curl} \mathbf{u} \in L^2_{\text{loc}}(\Omega^c)^3\}.\end{aligned}$$

If $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ is a Maxwell solution, then obviously $\mathbf{u} \in \mathbf{H}(\mathbf{curl} \mathbf{curl}, \Omega)$. If $\mathbf{u} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c)$ is a Maxwell solution, then obviously $\mathbf{u} \in \mathbf{H}_{\text{loc}}(\mathbf{curl} \mathbf{curl}, \Omega^c)$.

We shall use standard Sobolev spaces, $H^s(\Gamma)$, $s \in [-1, 1]$ (with the standard notation $H^0(\Gamma) = L^2(\Gamma)$) of complex-valued functions on the boundary Γ endowed with standard norms $\|\cdot\|_s$. Moreover, we denote by γ the standard trace operator mapping $\gamma : H^{s+1/2}(\Omega) \rightarrow H^s(\Gamma)$, $u \mapsto u|_{\Gamma}$, $s \in (0, 1)$, continuously.

Finally, we define spaces of complex valued tangential vector fields as:

$$V_{\pi}^s = (\mathbf{n} \times H^s(\Gamma)^3) \times \mathbf{n} \quad s \in [0, 1] \quad (3.1)$$

endowed with the induced operator norms $\|\cdot\|_{V_{\pi}^s}$. We will be mainly concerned with the space $V_{\pi}^{1/2}$ and in this case we drop the superscript, $V_{\pi} = V_{\pi}^{1/2}$ (compare with the notation adopted in [11].) Moreover, we denote by V'_{π} its dual space with V_{π}^0 as pivot space and by $\langle \cdot, \cdot \rangle_{V'_{\pi}, V_{\pi}}$ the corresponding duality pairing. Finally we shall make use of first order differential operators defined on Γ (we refer to [11] for precise definitions).

Definition 3.1 For $\mathbf{u} \in C^{\infty}(\bar{\Omega})$ we define the traces

$$\gamma_D \mathbf{u} := (\mathbf{n} \times \mathbf{u})|_{\Gamma} \quad \text{and} \quad \gamma_N \mathbf{u} := k^{-1} (\mathbf{n} \times \mathbf{curl} \mathbf{u})|_{\Gamma}.$$

Let

$$\mathbf{X} := \{\boldsymbol{\lambda} \in V'_{\pi} \mid \text{div}_{\Gamma} \boldsymbol{\lambda} \in H^{-1/2}(\Gamma)\}$$

endowed with the graph norm

$$\|\boldsymbol{\lambda}\|_{\mathbf{X}} = \|\boldsymbol{\lambda}\|_{V'_{\pi}} + \|\text{div}_{\Gamma} \boldsymbol{\lambda}\|_{-1/2}.$$

The following theorems have been proved in [11].

Theorem 3.2 The operators γ_D and γ_N are linear and continuous from $C^{\infty}(\bar{\Omega})$ to V_{π}^0 and they can be extended to linear and continuous operators from $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathbf{curl} \mathbf{curl}, \Omega)$, respectively, to \mathbf{X} . Moreover, they admit linear and continuous right inverses.

For $\mathbf{u} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c)$, $\mathbf{v} \in \mathbf{H}_{\text{loc}}(\mathbf{curl curl}, \Omega^c)$ we define $\gamma_D^c \mathbf{u}$ and $\gamma_N^c \mathbf{v}$ in the same way and the same mapping properties hold true. We set then:

$$\mathbf{H}_0(\mathbf{curl}, \Omega) = \{\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega) : \gamma_D \mathbf{u} = 0\}.$$

Clearly, $\mathbf{H}_0(\mathbf{curl}, \Omega)$ is a closed subspace of $\mathbf{H}(\mathbf{curl}, \Omega)$. We further need:

Theorem 3.3 *The operator $\times \mathbf{n} : V_\pi^0 \rightarrow V_\pi^0$ associated with the mapping $\mathbf{u} \mapsto \mathbf{u} \times \mathbf{n}$ can be extended to a linear and continuous isomorphism between \mathbf{X} and its dual.*

In the variational formulation of boundary integral equations we shall need the mapping $b : \mathbf{X} \times \mathbf{X} \rightarrow \mathbb{C}$ defined by

$$b(\mathbf{v}, \mathbf{w}) = \int_\Gamma \mathbf{v} \cdot (\mathbf{w} \times \mathbf{n}). \quad (3.2)$$

This defines a duality pairing for the space \mathbf{X} , i.e., b is continuous and $\mathbf{v} \mapsto b(\mathbf{v}, \cdot)$ maps \mathbf{X} onto its dual. Note that b is antisymmetric, i.e., $b(\mathbf{v}, \mathbf{w}) = -b(\mathbf{w}, \mathbf{v})$.

Finally, we need the following Hodge decomposition defined on the space \mathbf{X} . This theorem has been proved in [11] in the case of a simply connected manifold Γ , and can be extended to general topology by means of the results in [5], [6].

Theorem 3.4 *Define*

$$\mathbf{W} := \{\boldsymbol{\lambda} \in \mathbf{X} : \text{div}_\Gamma \boldsymbol{\lambda} = 0\}$$

and

$$\mathbf{V} := \{\boldsymbol{\lambda} \in \mathbf{X} : \int_\Gamma \boldsymbol{\lambda} \cdot \mathbf{w} = 0 \ \forall \mathbf{w} \in \mathbf{W} \cap V_\pi^0\}.$$

Then there holds the decomposition

$$\mathbf{X} = \mathbf{W} \oplus \mathbf{V}$$

where \oplus denotes a direct sum which is orthogonal in the following sense:

$$\forall \boldsymbol{\lambda} \in \mathbf{V}, \ \mathbf{w} \in \mathbf{W} \cap V_\pi^0 \quad \int_\Gamma \boldsymbol{\lambda} \cdot \mathbf{w} = 0.$$

The space \mathbf{W} can be decomposed as follows:

$$\mathbf{W} = \mathbf{W}_0 \oplus \mathbb{H}, \quad \mathbf{W}_0 = \mathbf{curl}_\Gamma H^{\frac{1}{2}}(\Gamma); \quad \dim\{\mathbb{H}\} = 2N_e \quad (3.3)$$

where \oplus denotes a direct sum and N_e is the first Betti number associated with the domain Ω . Moreover, if $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v} \in \mathbf{V}$, $\mathbf{w} \in \mathbf{W}$, we have the following norm equivalence:

$$\begin{aligned} c_1(\|\mathbf{v}\|_{\mathbf{X}} + \|\mathbf{w}\|_{\mathbf{X}}) &\leq \|\mathbf{u}\|_{\mathbf{X}} \leq \|\mathbf{v}\|_{\mathbf{X}} + \|\mathbf{w}\|_{\mathbf{X}} \\ \|\operatorname{div}_\Gamma \mathbf{u}\|_{-1/2} &\leq \|\mathbf{v}\|_{\mathbf{X}} \leq c_2 \|\operatorname{div}_\Gamma \mathbf{u}\|_{-1/2} \end{aligned} \quad (3.4)$$

where the constants c_1 , c_2 depend only on the geometry.

Finally there holds $\mathbf{V} \hookrightarrow V_\pi^0$ with compact injection.

Proof We need only to prove the last statement. To this end, we note that by construction the space \mathbf{V} can be characterized (see [11]) as the space $\nabla_\Gamma \mathcal{H}(\Gamma)$ where $\mathcal{H}(\Gamma) := \{u \in H^1(\Gamma) : \Delta_\Gamma u \in H^{-1/2}(\Gamma)\}$. Then we have for $\mathbf{v} \in \mathbf{V}$ that $\|\mathbf{v}\|_{V_\pi^0} \leq C \|\operatorname{div}_\Gamma \mathbf{v}\|_{-1, \Gamma}$. \square

Remark 3.5 The space \mathbb{H} is composed of the direct sum of the tangential traces of the Neumann fields (see [2] for a suitable definition) associated with Ω and with Ω^c .

In Theorem 3.2, the existence of an extension operator is stated. We shall need to use a ‘‘minimal energy’’ (in a proper sense) extension operator for some choices of $\mathbf{u} \in \mathbf{X}$. The existence of such an operator is the purpose of the following lemma.

Lemma 3.6 For any $\mathbf{u} \in \mathbf{X}$, there exists an extension $\mathbf{U} \in \mathbf{H}(\operatorname{div}, \Omega) \cap \mathbf{H}(\operatorname{curl}, \Omega)$ such that $\operatorname{div} \mathbf{U} = 0$. Moreover, if $\mathbf{u} \in V_\pi^0$, then $\mathbf{U}|_\Gamma \cdot \mathbf{n} \in L^2(\Gamma)$ and

$$\|\mathbf{U}|_\Gamma \cdot \mathbf{n}\|_0 \leq C \|\mathbf{u}\|_{V_\pi^0}. \quad (3.5)$$

Finally, for any $\mathbf{u} \in \mathbf{W}_0$ the extension \mathbf{U} can be chosen to satisfy $\operatorname{curl} \mathbf{U} = 0$ and $\operatorname{div} \mathbf{U} = 0$.

Proof Since $\mathbf{H}_0(\operatorname{curl}, \Omega)$ is a closed subspace of $\mathbf{H}(\operatorname{curl}, \Omega)$, given $\mathbf{u} \in \mathbf{X}$ we construct the extension $\mathbf{U} \in \mathbf{H}(\operatorname{curl}, \Omega)$ to be $\mathbf{H}(\operatorname{curl})$ -orthogonal to $\mathbf{H}_0(\operatorname{curl}, \Omega)$. This means to solve the problem:

Find $\mathbf{U} \in \mathbf{H}(\operatorname{curl}, \Omega)$ such that $\gamma_D \mathbf{U} = \mathbf{u}$ and

$$\int_\Omega (\operatorname{curl} \mathbf{U} \cdot \operatorname{curl} \bar{\mathbf{v}} + \mathbf{U} \cdot \bar{\mathbf{v}}) = 0 \quad \forall \mathbf{v} \in \mathbf{H}_0(\operatorname{curl}, \Omega).$$

Obviously, $\operatorname{div}(\mathbf{U}) = 0$ by construction. Using the regularity result [16] we deduce that if $\mathbf{u} \in V_\pi^0$, then $\mathbf{U}|_\Gamma \cdot \mathbf{n} \in L^2(\Gamma)$ and (3.5) is straightforward.

If $\mathbf{u} \in \mathbf{W}_0$, the previous construction does not lead to the desired, curl free extension. We need to use explicitly the fact that $\mathbf{u} = \operatorname{curl}_\Gamma p$ for some $p \in H^{1/2}(\Gamma)$. Denote by $P \in H^1(\Omega)$ the harmonic extension of p . Then ∇P verifies $\operatorname{div} \nabla P = 0$, $\operatorname{curl} \nabla P = 0$, and $\gamma_D \nabla P = \operatorname{curl}_\Gamma p$. \square

3.2 First Green formula

From [11], we know that

$$\forall \mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega) : \int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl} \mathbf{v} = b(\gamma_D \mathbf{v}, \gamma_D \mathbf{u}). \quad (3.6)$$

We define the bilinear form $\Phi_{\Omega} : \mathbf{H}(\mathbf{curl}, \Omega) \times \mathbf{H}(\mathbf{curl}, \Omega) \rightarrow \mathbb{C}$ by

$$\Phi_{\Omega}(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \left(k^{-1} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} - k \mathbf{u} \cdot \mathbf{v} \right). \quad (3.7)$$

For $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega^c)$ we define correspondingly $\Phi_{\Omega^c}(\mathbf{u}, \mathbf{v})$.

Assume that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ is a Maxwell solution in Ω and $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$. Then

$$\Phi_{\Omega}(\mathbf{u}, \mathbf{v}) = b(\gamma_D \mathbf{v}, \gamma_N \mathbf{u}). \quad (3.8)$$

This relationship is a crucial tool for our analysis. Parallel to the approach in [15] and [28], it allows to shift parts of the investigation of the boundary integral operators to the domains Ω and Ω^c .

Remark 3.7 *There is the following symmetry between electric and magnetic field quantities: Assume that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ is a Maxwell solution in Ω and let $\tilde{\mathbf{u}} := k^{-1} \mathbf{curl} \mathbf{u}$. Then we have*

$$\begin{aligned} \tilde{\mathbf{u}} &= k^{-1} \mathbf{curl} \mathbf{u}, & \mathbf{u} &= k^{-1} \mathbf{curl} \tilde{\mathbf{u}} \\ \gamma_D \tilde{\mathbf{u}} &= \gamma_N \mathbf{u}, & \gamma_D \mathbf{u} &= \gamma_N \tilde{\mathbf{u}}. \end{aligned}$$

If additionally $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \Omega)$ is a Maxwell solution in Ω and $\tilde{\mathbf{v}} := k^{-1} \mathbf{curl} \mathbf{v}$ we then have using the first Green formula

$$\begin{aligned} b(\gamma_D \mathbf{u}, \gamma_N \mathbf{v}) &= b(\gamma_D \mathbf{v}, \gamma_N \mathbf{u}) = \Phi_{\Omega}(\mathbf{u}, \mathbf{v}) = -\Phi_{\Omega}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \\ &= -b(\gamma_N \mathbf{u}, \gamma_D \mathbf{v}) = -b(\gamma_N \mathbf{v}, \gamma_D \mathbf{u}) \end{aligned}$$

which is consistent with the antisymmetry of b .

3.3 Potentials

Let G_k denote the fundamental solution for the Helmholtz equation, namely

$$G_k(\mathbf{x}, \mathbf{y}) = \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}. \quad (3.9)$$

In the next sections we drop the subindex \cdot_k and set $G := G_k$ as we work at a fixed wave number k . We shall also use G_0 , the fundamental solution of Laplace's equation.

The single layer potential Ψ is given by

$$(\Psi v)(x) := \int_{\Gamma} G(x-y)v(y)dS_y, \quad x \in \Omega \cup \Omega^c \quad (3.10)$$

and we denote by Ψ_0 the single layer potential associated with the kernel G_0 . The single layer potential is applied also to tangent vectors and, since it is never misleading, we shall use the same notation. We shall need the following mapping and coercivity properties of the potentials:

Theorem 3.8 *The operators*

$$\begin{aligned} \Psi, \Psi_0 &: H^{-\frac{1}{2}+\sigma}(\Gamma) \rightarrow H^{1+\sigma}(\Omega) \times H_{loc}^{1+\sigma}(\Omega^c) \\ \Psi, \Psi_0 &: V'_\pi \rightarrow H^1(\Omega)^3 \times H_{loc}^1(\Omega^c)^3 \end{aligned}$$

are linear and continuous for any $\sigma \in [-\frac{1}{2}, \frac{1}{2}]$. Moreover, there holds:

$$\begin{aligned} \exists \alpha > 0 &: \langle \bar{u}, \gamma \Psi_0 u \rangle_{-1/2, 1/2} \geq \alpha \|u\|_{-1/2}^2, \quad \forall u \in H^{-1/2}(\Gamma); \\ \exists \alpha > 0 &: b(\bar{\lambda}, \gamma_D \Psi_0 \lambda) \geq \alpha \|\lambda\|_{V'_\pi}^2, \quad \forall \lambda \in V'_\pi. \end{aligned}$$

For the proof of this theorem in the scalar case we refer to the pioneering work [27] which was adapted for Lipschitz domains in [15]. For the vector case we refer to [10] or [23].

We define a potential Ψ_E generated by an electric current $\mathbf{j} \in \mathbf{X}$ by

$$\Psi_E \mathbf{j} := k \Psi \mathbf{j} + k^{-1} \nabla \Psi \operatorname{div}_\Gamma \mathbf{j}. \quad (3.11)$$

This can also be written as $\Psi_E \mathbf{j} := k^{-1} \mathbf{curl} \mathbf{curl} \Psi \mathbf{j}$ because of the Helmholtz equation and the identity $-\Delta = \mathbf{curl} \mathbf{curl} - \nabla \operatorname{div}$. We define a ‘‘magnetic analogue’’ Ψ_M of Ψ_E generated by $\mathbf{m} \in \mathbf{X}$ as

$$\Psi_M \mathbf{m} := \mathbf{curl} \Psi \mathbf{m}. \quad (3.12)$$

These potentials are solutions of the Maxwell equations in Ω and Ω^c satisfying

$$k^{-1} \mathbf{curl} \Psi_E = \Psi_M, \quad k^{-1} \mathbf{curl} \Psi_M = \Psi_E. \quad (3.13)$$

This and the mapping properties of Ψ show that Ψ_E, Ψ_M are continuous mappings from \mathbf{X} to $\mathbf{H}(\mathbf{curl} \mathbf{curl}, \Omega) \times \mathbf{H}_{loc}(\mathbf{curl} \mathbf{curl}, \Omega^c)$. (Note that Ψ, Ψ_E, Ψ_M are defined as mappings to a pair of functions, one defined in Ω and the other defined in Ω^c ; hence equations (3.11), (3.12), (3.13) are to be understood separately on Ω and Ω^c .)

Therefore the traces $\gamma_D, \gamma_D^c, \gamma_N, \gamma_N^c$ can be applied to Ψ_E, Ψ_M and yield continuous mappings from \mathbf{X} to \mathbf{X} . Note that we have from (3.13)

$$\gamma_N \Psi_E = \gamma_D \Psi_M, \quad \gamma_N \Psi_M = \gamma_D \Psi_E \quad (3.14)$$

(and the same for the exterior traces).

In particular we are interested in the jumps $[\gamma_D] := \gamma_D - \gamma_D^c$ and $[\gamma_N] := \gamma_N - \gamma_N^c$. There hold the following **jump relations** for Ψ_E (see [10])

$$[\gamma_D] \Psi_E = 0 \quad [\gamma_N] \Psi_E = -I$$

implying with (3.14)

$$[\gamma_D] \Psi_M = -I \quad [\gamma_N] \Psi_M = 0$$

where I denotes the identity mapping.

This is related to the **representation formula** (see [10]): Assume that $u|_\Omega \in \mathbf{H}(\mathbf{curl}, \Omega)$ is a Maxwell solution in Ω , and that $u|_{\Omega^c} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c)$ is a Maxwell solution in Ω^c . Then we have with $\mathbf{j} := [\gamma_N] \mathbf{u}$, $\mathbf{m} := [\gamma_D] \mathbf{u}$ that on $\Omega \cup \Omega^c$

$$\mathbf{u} = -\Psi_E \mathbf{j} - \Psi_M \mathbf{m} \quad (3.15)$$

We define for $\mathbf{j}, \mathbf{m} \in \mathbf{X}$ the **boundary integral operators** by applying the symmetric parts $\{\gamma_D\} := \frac{1}{2}(\gamma_D + \gamma_D^c)$, $\{\gamma_N\} := \frac{1}{2}(\gamma_N + \gamma_N^c)$ of the traces to (3.15):

$$A \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix} := \begin{pmatrix} \{\gamma_D\} \\ \{\gamma_N\} \end{pmatrix} (-\Psi_E \mathbf{j} - \Psi_M \mathbf{m}) \quad (3.16)$$

With the operators

$$\begin{aligned} C &:= -\{\gamma_D\} \Psi_E = -\{\gamma_N\} \Psi_M, \\ M &:= -\{\gamma_N\} \Psi_E = -\{\gamma_D\} \Psi_M \end{aligned} \quad (3.17)$$

we have

$$A = \begin{pmatrix} M & C \\ C & M \end{pmatrix}.$$

The operator $A: \mathbf{X}^2 \rightarrow \mathbf{X}^2$ is continuous, mapping the space of Cauchy data to itself.

The **Calderón projector** P giving Cauchy data of a Maxwell solution in the interior domain Ω is given by $P = \frac{1}{2}I + A$. The corresponding projector for the exterior domain Ω^c is given by $P^c = I - P = \frac{1}{2}I - A$. Note that the range of P coincides with the kernel of P^c .

3.4 Strong Ellipticity

We shall now establish strong ellipticity of the boundary integral operator A in (3.16). To this end, we define the following *antisymmetric* bilinear form $B: \mathbf{X}^2 \times \mathbf{X}^2 \rightarrow \mathbb{C}$ for two sets of Cauchy data:

$$B \left(\begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{m}} \\ \tilde{\mathbf{j}} \end{pmatrix} \right) := -b(\mathbf{m}, \tilde{\mathbf{j}}) + b(\tilde{\mathbf{m}}, \mathbf{j})$$

We first need to prove the following symmetry property of the operator A with respect to the antisymmetric form B .

Theorem 3.9 (Symmetry) *We have for all $\mathbf{m}, \mathbf{j}, \tilde{\mathbf{m}}, \tilde{\mathbf{j}} \in \mathbf{X}$*

$$B \left(A \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{m}} \\ \tilde{\mathbf{j}} \end{pmatrix} \right) = B \left(A \begin{pmatrix} \tilde{\mathbf{m}} \\ \tilde{\mathbf{j}} \end{pmatrix}, \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix} \right) \quad (3.18)$$

or equivalently

$$b(\tilde{\mathbf{m}}, C\mathbf{m}) = b(\mathbf{m}, C\tilde{\mathbf{m}}), \quad b(\tilde{\mathbf{m}}, M\mathbf{j}) = b(\mathbf{j}, M\tilde{\mathbf{m}})$$

For the proof of this theorem, we need the following lemma and a few notations. Let B_R be a ball of radius R sufficiently large such that $\bar{\Omega} \subset B_R$. We set $S_R := \partial B_R$ and denote \mathbf{n}_R the exterior unit normal vector to B_R .

Lemma 3.10 *Let $\mathbf{u}, \mathbf{v} \in \mathbf{H}_{loc}(\mathbf{curl}, \Omega^c)$ be two solutions of the exterior Maxwell problems. Then*

$$\int_{S_R} (\gamma_D \mathbf{u} \cdot (\gamma_N \mathbf{v} \times \mathbf{n}_R) - \gamma_D \mathbf{v} \cdot (\gamma_N \mathbf{u} \times \mathbf{n}_R)) \rightarrow 0 \quad R \rightarrow \infty. \quad (3.19)$$

Proof By abuse of notation, for any $\mathbf{u} \in \mathbf{H}_{loc}(\mathbf{curl}, \Omega^c)$ we will denote again by γ_D and by γ_N the Neumann and Dirichlet trace on S_R . Thanks to the Silver-Müller radiation condition, we have on S_R :

$$ik\mathbf{u} + \gamma_N \mathbf{u} = o\left(\frac{1}{R}\right) \quad R \rightarrow \infty,$$

which implies

$$ik\gamma_D(\mathbf{u}) - \gamma_N \mathbf{u} \times \mathbf{n}_R = o\left(\frac{1}{R}\right) \quad R \rightarrow \infty.$$

Then, we have:

$$\begin{aligned} & \int_{S_R} (\gamma_D \mathbf{u} \cdot (\gamma_N \mathbf{v} \times \mathbf{n}_R) - \gamma_D \mathbf{v} \cdot (\gamma_N \mathbf{u} \times \mathbf{n}_R)) \\ &= \int_{S_R} \gamma_D \mathbf{u} \cdot \frac{\gamma_D \mathbf{v}}{ik} - \gamma_D \mathbf{v} \cdot \frac{\gamma_D \mathbf{u}}{ik} + o(1) = o(1). \end{aligned}$$

□

Proof of Theorem 3.9. Using (3.8), for any \mathbf{v} such that $\mathbf{v}|_{\Omega^c} \in \mathbf{H}(\mathbf{curl}, \Omega^c)$ and $\mathbf{v}|_{\Omega} \in \mathbf{H}(\mathbf{curl}, \Omega)$, and for any \mathbf{u} solution of the Maxwell equations (2.2) in Ω and Ω^c , we have

$$\Phi_{\Omega}(\mathbf{u}, \mathbf{v}) + \Phi_{\Omega^c}(\mathbf{u}, \mathbf{v}) = b(\gamma_D \mathbf{v}, \gamma_N \mathbf{u}) - b(\gamma_D^c \mathbf{v}, \gamma_N^c \mathbf{u}) \quad (3.20)$$

and rearranging terms

$$\Phi_{\Omega}(\mathbf{u}, \mathbf{v}) + \Phi_{\Omega^c}(\mathbf{u}, \mathbf{v}) = b([\gamma_D] \mathbf{v}, \{\gamma_N\} \mathbf{u}) + b(\{\gamma_D\} \mathbf{v}, [\gamma_N] \mathbf{u}) \quad (3.21)$$

Let B_R be the ball defined above and $\Phi_{\Omega^c \cap \bar{B}_R}$ be defined as in (3.7). Let now $(\mathbf{m}, \mathbf{j}), (\tilde{\mathbf{m}}, \tilde{\mathbf{j}}) \in \mathbf{X}^2$ and let \mathbf{u}, \mathbf{v} respectively be the vector fields given by (3.15). Note that $\mathbf{v} \notin \mathbf{H}(\mathbf{curl}, \Omega^c)$; nevertheless, exploiting the *symmetry* of the bilinear forms Φ_{Ω} and $\Phi_{\Omega^c \cap B_R}$, we have:

$$\begin{aligned} 0 &= \Phi_{\Omega}(\mathbf{u}, \mathbf{v}) + \Phi_{\Omega^c \cap B_R}(\mathbf{u}, \mathbf{v}) - \Phi_{\Omega}(\mathbf{v}, \mathbf{u}) - \Phi_{\Omega^c \cap B_R}(\mathbf{v}, \mathbf{u}) \\ &= b([\gamma_D] \mathbf{v}, \{\gamma_N\} \mathbf{u}) + b(\{\gamma_D\} \mathbf{v}, [\gamma_N] \mathbf{u}) - b([\gamma_D] \mathbf{u}, \{\gamma_N\} \mathbf{v}) \\ &\quad - b(\{\gamma_D\} \mathbf{u}, [\gamma_N] \mathbf{v}) + \int_{S_R} (\gamma_D \mathbf{u} \cdot (\gamma_N \mathbf{v} \times \mathbf{n}) - \gamma_D \mathbf{v} \cdot (\gamma_N \mathbf{u} \times \mathbf{n})). \end{aligned}$$

Now, we apply Lemma 3.10 to deduce that

$$\begin{aligned} &b([\gamma_D] \mathbf{v}, \{\gamma_N\} \mathbf{u}) - b(\{\gamma_D\} \mathbf{u}, [\gamma_N] \mathbf{v}) = \\ &b([\gamma_D] \mathbf{u}, \{\gamma_N\} \mathbf{v}) - b(\{\gamma_D\} \mathbf{v}, [\gamma_N] \mathbf{u}) \end{aligned}$$

which coincides with (3.18) recalling the definition (3.16) of the operator A . \square

We can now state the main theorem of this section and to this end, we need the following definition:

Definition 3.11 *Let $\mathbf{u} \in \mathbf{X}$ be decomposed as $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathbf{v} \in \mathbf{V}$ and $\mathbf{w} \in \mathbf{W}$. We denote by $\Theta : \mathbf{X} \rightarrow \mathbf{X}$ the isomorphism associated with the mapping $\mathbf{u} = \mathbf{v} + \mathbf{w} \mapsto \overline{\mathbf{v} - \mathbf{w}}$.*

Theorem 3.12 (Inf-sup condition) *There exists a compact operator $T : \mathbf{X}^2 \rightarrow \mathbf{X}^2$ and $\alpha > 0$ such that for all $\mathbf{m}, \mathbf{j} \in \mathbf{X}$*

$$\operatorname{Re} B \left((A + T) \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix}, \begin{pmatrix} \Theta \mathbf{m} \\ \Theta \mathbf{j} \end{pmatrix} \right) \geq \alpha \left\| \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix} \right\|_{\mathbf{X}^2}^2. \quad (3.22)$$

In order to prove this theorem, we need the following compactness argument:

Proposition 3.13 (Compactness) *Let $b(\cdot, \cdot)$ be the bilinear form defined in (3.2) and \mathbf{V}, \mathbf{W} the spaces defined in Theorem 3.4, then*

$$\begin{aligned} &b(\cdot, M \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{C}, \quad b(\cdot, M \cdot) : \mathbf{W} \times \mathbf{W} \rightarrow \mathbb{C} \\ &\text{are compact bilinear forms.} \end{aligned} \quad (3.23)$$

Proof Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$. We know that $\mathbf{v}_i = \nabla_{\Gamma} p_i$, $p_i \in \mathcal{H}(\Gamma)$, $i = 1, 2$ and then $b(\mathbf{v}_1, \mathbf{v}_2) = \int_{\Gamma} \nabla_{\Gamma} p_1 \cdot (\nabla_{\Gamma} p_2 \times \mathbf{n}) \equiv 0$. Then we have:

$$b(\mathbf{v}_1, M\mathbf{v}_2) = b(\mathbf{v}_1, (-\gamma_D \Psi_M - I/2)\mathbf{v}_2) = b(\mathbf{v}_1, -\gamma_D \Psi_M \mathbf{v}_2).$$

Let \mathbf{V}_1 be the lifting of \mathbf{v}_1 in Ω provided by Lemma 3.6. Then,

$$\begin{aligned} b(\gamma_D \Psi_M \mathbf{v}_2, \mathbf{v}_1) &= \int_{\Omega} k^{-1} \mathbf{curl} \Psi \mathbf{v}_2 \cdot \mathbf{curl} \mathbf{V}_1 \\ &\quad - \int_{\Omega} k^{-1} \mathbf{curl} \mathbf{curl} \Psi \mathbf{v}_2 \cdot \mathbf{V}_1. \end{aligned}$$

Using the Maxwell equations, we have:

$$k^{-1} \mathbf{curl} \mathbf{curl} \Psi \mathbf{v}_2 = k \Psi \mathbf{v}_2 + k^{-1} \nabla \Psi \operatorname{div}_{\Gamma} \mathbf{v}_2.$$

Integrating by parts and recalling that $\operatorname{div} \mathbf{V}_1 = 0$, we have:

$$\begin{aligned} \int_{\Omega} k^{-1} \mathbf{curl} \mathbf{curl} \Psi \mathbf{v}_2 \cdot \mathbf{V}_1 &= k \int_{\Omega} \Psi \mathbf{v}_2 \cdot \mathbf{V}_1 \\ &\quad + k^{-1} \int_{\Gamma} \mathbf{V}_1|_{\Gamma} \cdot \mathbf{n} \Psi (\operatorname{div}_{\Gamma} \mathbf{v}_2). \end{aligned} \tag{3.24}$$

Since $\mathbf{v}_1 \in V_{\pi}^0$, Lemma 3.6 ensures that $\|\mathbf{V}_1|_{\Gamma} \cdot \mathbf{n}\|_0 \leq C \|\mathbf{v}_1\|_{V_{\pi}^0}$. Collecting terms and using the mapping properties of the single layer potential Theorem 3.8, we obtain:

$$\begin{aligned} b(\mathbf{v}_1, M\mathbf{v}_2) &\leq C((k^{-1} + k) \|\mathbf{v}_1\|_{\mathbf{X}} \|\mathbf{v}_2\|_{V_{\pi}'} \\ &\quad + k^{-1} \|\mathbf{v}_1\|_{V_{\pi}^0} \|\operatorname{div}_{\Gamma} \mathbf{v}_2\|_{-1}). \end{aligned} \tag{3.25}$$

We deduce the first statement using that $\mathbf{V} \hookrightarrow V_{\pi}^0 \hookrightarrow V_{\pi}'$ are compact injections. Let now $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{W}$. We use the splitting furnished by Theorem 3.4

$$\mathbf{w}_1 = \mathbf{w}_1^0 + \mathbf{h}_1, \quad \mathbf{w}_2 = \mathbf{w}_2^0 + \mathbf{h}_2 \quad \mathbf{h}_1, \mathbf{h}_2 \in \mathbb{H}, \quad \mathbf{w}_1^0, \mathbf{w}_2^0 \in \mathbf{W}_0.$$

Since \mathbb{H} is finite dimensional, the mappings $\mathbf{w}_i \mapsto \mathbf{h}_i$, $i = 1, 2$ are compact. Consequently, the bilinear form $b : \mathbf{W} \times \mathbb{H} \rightarrow \mathbb{C}$ is also compact. As a consequence, it remains to study the bilinear form $b : \mathbf{W}_0 \times \mathbf{W}_0 \rightarrow \mathbb{C}$. We observe that again $b(\mathbf{w}_1^0, \mathbf{w}_2^0) = 0$. Using Lemma 3.6, we know that there exists $\mathbf{W}_2^0 \in \mathbf{H}(\mathbf{curl}, \Omega)$ such that $\mathbf{curl} \mathbf{W}_2^0 = 0$, $\gamma_D \mathbf{W}_2^0 = \mathbf{w}_2^0$. Using the Maxwell equations as above, we deduce the identity:

$$k^{-1} \mathbf{curl} \mathbf{curl} \Psi \mathbf{w}_2^0 = k \Psi \mathbf{w}_2^0,$$

since $\operatorname{div}_\Gamma \mathbf{w}_2^0 = 0$. We then deduce further:

$$b(\mathbf{w}_1^0, M\mathbf{w}_2^0) = k \int_\Omega \Psi \mathbf{w}_1^0 \cdot \mathbf{W}_2^0 \leq k \|\Psi \mathbf{w}_1^0\|_0 \|\mathbf{W}_2^0\|_0.$$

We finish using the mapping properties of Ψ , Theorem 3.8, and the compactness of the injection $H^1(\Omega)^3 \hookrightarrow L^2(\Omega)^3$. \square

Proof of Theorem 3.12. Let Ψ_0 be the single layer potential associated with the kernel G_0 defined in (3.9) with $k = 0$. Correspondingly we define

$$\Psi_{E,0}\mathbf{j} := k\Psi_0\mathbf{j} + k^{-1}\nabla\Psi_0 \operatorname{div}_\Gamma \mathbf{j} \quad \Psi_{M,0}\mathbf{m} := \mathbf{curl} \Psi_0\mathbf{m}, \quad (3.26)$$

and also (compare with (3.17))

$$C_0 := -\{\gamma_D\}\Psi_{E,0}, \quad M_0 := -\{\gamma_D\}\Psi_{M,0}, \quad A_0 = \begin{pmatrix} M_0 & C_0 \\ C_0 & M_0 \end{pmatrix}.$$

Note that now $\{\gamma_D\}\Psi_{E,0} \neq \{\gamma_N\}\Psi_{M,0}$ and $\{\gamma_N\}\Psi_{E,0} \neq \{\gamma_D\}\Psi_{M,0}$, but the alternate choice (namely $C_0 := -\{\gamma_N\}\Psi_{M,0}$ and $M_0 := -\{\gamma_N\}\Psi_{E,0}$) would be completely equivalent for our purposes. First of all we need to prove that $C - C_0$ and $M - M_0$ are compact operators. To this end let B_R be a ball such that $\overline{\Omega} \subset B_R$. Because of the regularity of $G - G_0$, we obtain that $\Psi - \Psi_0$ is an operator of order -4 and is continuous from $H^1(B_R)'$ to $H^3(B_R)$. Let $\mathbf{u} \in \mathbf{X}$ and $\mathbf{V} \in H^1(B_R)^3$. Then there exists $\mathbf{U} \in \mathbf{H}(\mathbf{curl}, \Omega)$ with $\gamma_D \mathbf{U} = \mathbf{u}$ and we have

$$\int_\Gamma \mathbf{u} \cdot \mathbf{V}|_\Gamma = b(\gamma_D \mathbf{U}, \gamma_D \mathbf{V}) = \int_\Omega \mathbf{curl} \mathbf{V} \cdot \mathbf{U} - \mathbf{curl} \mathbf{U} \cdot \mathbf{V}$$

Therefore we can consider $\mathbf{u} \in \mathbf{X}$ as a distribution in $[H^1(B_R)']^3$ and we obtain that $\mathbf{u} \mapsto \gamma_D(\Psi - \Psi_0)\mathbf{u}$ is a compact mapping from \mathbf{X} to \mathbf{X} using

$$\mathbf{X} \hookrightarrow [H^1(B_R)']^3 \xrightarrow{\Psi - \Psi_0} H^3(B_R)^3 \xrightarrow{c} H^1(B_R)^3 \xrightarrow{\gamma_D} \mathbf{X}.$$

In the same way we obtain that the mappings $\gamma_D \nabla(\Psi - \Psi_0) \operatorname{div}_\Gamma$ and $\gamma_D \mathbf{curl}(\Psi - \Psi_0)$ are compact mappings from \mathbf{X} to \mathbf{X} using

$$\mathbf{X} \xrightarrow{\operatorname{div}_\Gamma} H^{-1/2}(\Gamma) \hookrightarrow H^1(B_R)' \xrightarrow{\nabla(\Psi - \Psi_0)} H^2(B_R)^3 \xrightarrow{c} H^1(B_R)^3 \xrightarrow{\gamma_D} \mathbf{X}$$

and

$$\mathbf{X} \hookrightarrow [H^1(B_R)']^3 \xrightarrow{\mathbf{curl}(\Psi - \Psi_0)} H^2(B_R)^3 \xrightarrow{c} H^1(B_R)^3 \xrightarrow{\gamma_D} \mathbf{X}.$$

Thus $C - C_0$ and $M - M_0$ are compact mappings from \mathbf{X} to \mathbf{X} . Hence the operator $A - A_0$ is compact and it is sufficient to prove (3.22) with A replaced by A_0 .

The symmetry of A (Theorem 3.9) implies that the bilinear form $B(A_0\xi, \tilde{\xi}) - B(A_0\tilde{\xi}, \xi): \mathbf{X}^2 \times \mathbf{X}^2 \rightarrow \mathbb{C}$ is compact.

Let $\xi := \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix}$. We decompose both \mathbf{m} and \mathbf{j} by means of Theorem 3.4. We set:

$$\mathbf{m} = \mathbf{m}^V + \mathbf{m}^W, \quad \mathbf{j} = \mathbf{j}^V + \mathbf{j}^W \quad (3.27)$$

for $\mathbf{m}^V, \mathbf{j}^V \in \mathbf{V}$ and $\mathbf{m}^W, \mathbf{j}^W \in \mathbf{W}$. Let then $\xi^V = \begin{pmatrix} \mathbf{m}^V \\ \mathbf{j}^V \end{pmatrix}$, $\xi^W = \begin{pmatrix} \mathbf{m}^W \\ \mathbf{j}^W \end{pmatrix}$ and $\theta\xi := \bar{\xi}^V - \bar{\xi}^W$. Then

$$\begin{aligned} \operatorname{Re} B(A_0\xi, \theta\xi) &= \operatorname{Re} B(A_0\xi^V, \bar{\xi}^V) - \operatorname{Re} B(A_0\xi^W, \bar{\xi}^W) \\ &\quad + \operatorname{Re} B(A_0\xi^W, \bar{\xi}^V) - \operatorname{Re} B(A_0\xi^V, \bar{\xi}^W). \end{aligned} \quad (3.28)$$

Due to the symmetry of A and the fact that G_0 is real, we have:

$$B(A_0\xi^W, \bar{\xi}^V) = \overline{B(A_0\xi^V, \bar{\xi}^W)} + \text{compact},$$

then the sum of the last two terms in (3.28) is a compact operator in \mathbf{X}^2 . We consider now the first term in (3.28).

$$\begin{aligned} B(A_0\xi^V, \bar{\xi}^V) &= -b(C_0\mathbf{j}^V, \bar{\mathbf{j}}^V) + b(\bar{\mathbf{m}}^V, C_0\mathbf{m}^V) + \\ &\quad -b(M_0\mathbf{j}^V, \bar{\mathbf{j}}^V) + b(\bar{\mathbf{m}}^V, M_0\mathbf{m}^V). \end{aligned} \quad (3.29)$$

Proposition 3.13 and the fact that $M - M_0$ is a compact operator ensure that the last two terms in (3.29) are compact. Using the anti-symmetry of the form b , we are then left with:

$$B(A_0\xi^V, \bar{\xi}^V) = b(\bar{\mathbf{j}}^V, C_0\mathbf{j}^V) + b(\bar{\mathbf{m}}^V, C_0\mathbf{m}^V) + \text{compact}. \quad (3.30)$$

Let us consider for example the first term. Using the definition of C_0 and integration by parts:

$$\begin{aligned} -b(C_0\mathbf{j}^V, \bar{\mathbf{j}}^V) &= -k b(\bar{\mathbf{j}}^V, \gamma_D \Psi_0 \mathbf{j}^V) \\ &\quad + k^{-1} \langle \operatorname{div}_\Gamma \bar{\mathbf{j}}^V, \gamma(\Psi_0 \operatorname{div}_\Gamma \mathbf{j}^V) \rangle_{-1/2, 1/2} \end{aligned} \quad (3.31)$$

where γ denotes the standard trace operator. Using the mapping properties of Ψ_0 , we have that (see [10] for more details):

$$\begin{aligned} b(\bar{\mathbf{j}}^V, \gamma_D \Psi_0 \mathbf{j}^V) &\geq c \|\mathbf{j}^V\|_{V_\pi'}^2 \\ \langle \operatorname{div}_\Gamma \bar{\mathbf{j}}^V, \gamma(\Psi_0 \operatorname{div}_\Gamma \mathbf{j}^V) \rangle_{-1/2, 1/2} &\geq C \|\operatorname{div}_\Gamma \mathbf{j}^V\|_{-1/2}^2. \end{aligned} \quad (3.32)$$

By the norm equivalence (3.4) and the compactness of the injection $\mathbf{V} \hookrightarrow V'_\pi$, we finally deduce that there exists $\alpha > 0$ such that:

$$\operatorname{Re} b(\bar{\mathbf{j}}^V, C_0 \mathbf{j}^V) \geq \alpha \|\mathbf{j}^V\|_{\mathbf{X}}^2 - \|T_0 \mathbf{j}^V\|_{\mathbf{X}}^2 \quad (3.33)$$

where $T_0 : \mathbf{X} \rightarrow \mathbf{X}$ is a compact operator. The same holds true for the second term in (3.30).

We consider now the second term in (3.28). Using again Proposition 3.13,

$$-B(A_0 \boldsymbol{\xi}^W, \bar{\boldsymbol{\xi}}^W) = -b(\bar{\mathbf{j}}^W, C_0 \mathbf{j}^W) - b(\bar{\mathbf{m}}^W, C_0 \mathbf{m}^W) + \text{compact}.$$

Since $\operatorname{div}_\Gamma \mathbf{w} = 0$ for all $\mathbf{w} \in \mathbf{W}$, using the definition of C_0 and integrating by parts, we have:

$$\begin{aligned} -B(A_0 \boldsymbol{\xi}^W, \bar{\boldsymbol{\xi}}^W) &= b(\bar{\mathbf{j}}^W, \gamma_D \Psi_0 \mathbf{j}^W) \\ &\quad + b(\bar{\mathbf{m}}^W, \gamma_D \Psi_0 \mathbf{m}^W) + \text{compact}. \end{aligned} \quad (3.34)$$

Now, collecting (3.30) and (3.34), and using (3.33) together with the ellipticity of $\gamma_D \Psi_0$, we obtain:

$$\operatorname{Re} B(A_0 \boldsymbol{\xi}, \Theta \boldsymbol{\xi}) \geq \alpha (\|\boldsymbol{\xi}^W\|_{\mathbf{X}^2} + \|\boldsymbol{\xi}^V\|_{\mathbf{X}^2}) - \|T_1 \boldsymbol{\xi}\|_{\mathbf{X}^2}$$

where $T_1 : \mathbf{X}^2 \rightarrow \mathbf{X}^2$ is again a compact operator. The claim follows now from the norm equivalence (3.4). \square

4 Discrete decompositions and convergence

We obtained an inf-sup condition for the operator A using the Hodge decomposition $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$. The statement of this inf-sup condition involves the isomorphism from Definition 3.11. For a discretization method we use Galerkin projection onto a sequence of finite dimensional subspaces $\mathbf{X}_h \subset \mathbf{X}$. But the Hodge decomposition of a function in \mathbf{X}_h gives in general functions which are no longer contained in \mathbf{X}_h . This prevents us from immediately applying Theorem 3.12 and standard results about conforming Galerkin discretizations on coercive variational problems. We therefore need to consider discrete decompositions $\mathbf{X}_h = \mathbf{V}_h \oplus \mathbf{W}_h$. If these discrete decompositions are in some sense close to the continuous decomposition we will be able to prove a discrete inf-sup condition and quasioptimal convergence.

We will first give an abstract formulation of this idea. Then we will show that the assumptions are satisfied for our Hodge decomposition $\mathbf{X} = \mathbf{V} \oplus \mathbf{W}$ and boundary elements of Raviart-Thomas and Brezzi-Douglas-Marini type.

4.1 Abstract convergence theorem

Consider a Hilbert space X with $X = V \oplus W$ so that we have for $u \in X$ a unique decomposition $u = v + w$ with $\|v\|_X + \|w\|_X \leq C \|u\|_X$. Then we can define the continuous mapping $\Theta: X \rightarrow X$ with $u = v + w \mapsto \overline{v - w}$.

Consider a sequence of closed subspaces $X_h \subset X$ with decompositions $X_h = V_h \oplus W_h$ which satisfy the following assumptions:

(A1) The family $\{X_h\}_h$ is dense in the space X , namely

$$\overline{\bigcup_h X_h} = X.$$

(A2) We have $W_h \subset W$ and

$$\text{For all } v_h \in V_h: \quad \inf_{v \in V} \|v_h - v\|_X \leq \delta_h \|v_h\|_X \quad (4.1)$$

with $\delta_h \rightarrow 0$ for $h \rightarrow 0$.

These assumptions turn out to be natural in our framework and were first used in [13], see also [7].

Theorem 4.1 *Assume that $A: X \rightarrow X'$ is continuous and that there exist a compact operator $T: X \rightarrow X'$ and a constant $\alpha > 0$ such that for all $u \in X$*

$$\text{Re} \langle (A + T)u, \Theta u \rangle \geq \alpha \|u\|_X^2$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between X' and X . Assume further that A is one-to-one. Let $\{X_h\}_h$ denote a sequence of subspaces of X satisfying (A1) and (A2).

Then there exists $h_0 > 0$ such that for all $f \in X'$ and $h \leq h_0$ the Galerkin equation

$$\langle Au_h, \tilde{u}_h \rangle = \langle f, \tilde{u}_h \rangle \quad \text{for all } \tilde{u}_h \in X_h$$

has a unique solution u_h in X_h which converges quasioptimally, i.e.,

$$\|u - u_h\|_X \leq C \inf_{\tilde{u}_h \in X_h} \|u - \tilde{u}_h\|_X$$

where $u \in X$ satisfies $Au = f$.

Proof This proof is mainly inspired by [13], [10] and follows [7].

Since $A + T$ is invertible and A is one-to-one, A and its adjoint A' are invertible. As

$$\langle (A + T)u, \Theta u \rangle = \langle Au, (I + (A')^{-1}T')\Theta u \rangle$$

we have with $\tilde{\Theta} := (I + (A')^{-1}T')\Theta$ that

$$\forall u \in X : \quad \operatorname{Re} \langle Au, \tilde{\Theta}u \rangle \geq \alpha \|u\|_X^2.$$

Note that $\tilde{\Theta} - \Theta = (A')^{-1}T'\Theta$ is compact. Let $P_h: X \rightarrow X_h$ denote a uniformly bounded family of projection operators, i.e. $\|P_h\|_{X \rightarrow X} \leq C$ with C independent of h . Then

$$\left\| (I - P_h)\tilde{\Theta}u_h \right\|_X \leq \left\| (I - P_h)(\tilde{\Theta} - \Theta)u_h \right\|_X + \|(I - P_h)\Theta u_h\|_X$$

By assumption (A1) we have for all $U \in X$ that $\|(I - P_h)U\|_X \rightarrow 0$ as $h \rightarrow 0$. Since $\tilde{\Theta} - \Theta$ is compact we obtain that

$$\varepsilon_h := \left\| (I - P_h)(\tilde{\Theta} - \Theta) \right\|_{X \rightarrow X} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Let now $u_h \in X_h$ be arbitrary. Then u_h has the decomposition $u_h = v + w$ with $v \in V$, $w \in W$, and we have $\Theta u_h = \overline{v - w}$. There is also the decomposition $u_h = v_h + w_h$ with $v_h \in V_h$, $w_h \in W_h$. We have for $\Theta u_h = \overline{v - w}$

$$\begin{aligned} \|(I - P_h)\Theta u_h\|_X &= \left\| (I - P_h) \left(\Theta u_h - \overline{(v_h - w_h)} \right) \right\|_X \\ &\leq (1 + C) \|(v - w) - (v_h - w_h)\|_X \\ &\leq (1 + C)(\|v - v_h\|_X + \|w - w_h\|_X). \end{aligned}$$

Let $\Pi_V: X \rightarrow V$ denote the projection operator corresponding to the decomposition $X = V \oplus W$. Then $v = \Pi_V u_h = \Pi_V v_h$ (since $w_h \in W_h \subset W$). According to assumption (A2) there exists $\tilde{v} \in V$ such that $\|v_h - \tilde{v}\|_X \leq 2\delta_h \|v_h\|_X$ and we have

$$\begin{aligned} \|v_h - v\|_X &= \|(I - \Pi_V)v_h\|_X = \|(I - \Pi_V)(v_h - \tilde{v})\|_X \\ &\leq C\delta_h \|v_h\|_X \leq C\delta_h (\|v_h - v\|_X + \|v\|_X) \\ \|v_h - v\|_X &\leq \frac{C\delta_h}{1 - C\delta_h} \|v\|_X \leq 2C\delta_h \|u_h\|_X \end{aligned}$$

where we used $v = \Pi_V u_h$ and assume that h is sufficiently small so that $C\delta_h \leq \frac{1}{2}$. As $w - w_h = -(v - v_h)$ we obtain

$$\|v - v_h\|_X + \|w - w_h\|_X \leq C\delta_h \|u_h\|_X$$

and we obtain for all $u_h \in X_h$

$$\left\| (I - P_h)\tilde{\Theta}u_h \right\|_X \leq (\varepsilon_h + C\delta_h) \|u_h\|_X.$$

which implies that for sufficiently small h and for all $u_h \in X_h$

$$\operatorname{Re} \langle Au_h, P_h \tilde{\Theta}u_h \rangle \geq \operatorname{Re} \langle Au_h, \tilde{\Theta}u_h \rangle - C(\varepsilon_h + C\delta_h) \|u_h\|_X^2 \geq \alpha/2 \|u_h\|_X^2.$$

Since $P_h \tilde{\Theta} : X_h \rightarrow X_h$ is bounded independently of h , we have proved that there exists $\alpha > 0$ and $h_0 > 0$ such that for all $h < h_0$

$$\inf_{0 \neq u_h \in X_h} \sup_{0 \neq v_h \in X_h} \frac{|\langle Au_h, v_h \rangle|}{\|u_h\|_X \|v_h\|_X} \geq \frac{\alpha}{2}. \quad (4.2)$$

It is well known that this discrete inf-sup condition implies that the Galerkin equations have a unique solution and that the error is quasi-optimal. \square

4.2 Application to standard finite element families

Let $\{\mathbf{X}_h\}_h$ be a family of finite dimensional subspaces of \mathbf{X} . We set:

$$\begin{aligned} \mathbf{W}_h &:= \{\mathbf{w}_h \in \mathbf{X}_h : \operatorname{div}_\Gamma \mathbf{w}_h = 0\} \\ \mathbf{V}_h &:= \{\mathbf{v}_h \in \mathbf{X}_h : \int_\Gamma \mathbf{v}_h \cdot \mathbf{w}_h = 0 \ \forall \mathbf{w}_h \in \mathbf{W}_h\}. \end{aligned} \quad (4.3)$$

Note that by construction, $\mathbf{W}_h \subset \mathbf{W}$, but in general $\mathbf{V}_h \not\subset \mathbf{V}$. We now make the additional assumption that Ω is a polyhedron, possibly curvilinear.

Let now \mathcal{T}_h be a family of regular triangulations decomposing Γ . \mathcal{T}_h can be composed of triangles or quadrilaterals (or both), and no quasi-uniformity is required.

Both Raviart-Thomas (RT) and the Brezzi-Douglas-Marini (BDM) finite elements (see e.g., [22] and [4] respectively, for definitions and properties) can be defined on \mathcal{T}_h and are conforming approximations of the space $\mathbf{X}^0 := \{\mathbf{u} \in V_\pi^0 : \operatorname{div}_\Gamma \mathbf{u} \in L^2(\Gamma)\}$ endowed with the graph norm

$$\|\mathbf{u}\|_{\mathbf{X}^0} := \|\mathbf{u}\|_{V_\pi^0} + \|\operatorname{div}_\Gamma \mathbf{u}\|_0.$$

Note that the injection $\mathbf{X}^0 \subset \mathbf{X}$ is dense. We say that a family of finite elements is of *order* k when locally at each triangle or quadrilateral the polynomials of degree k are contained in the family. In particular, the lowest order RT element has *order* $k = 0$ and the lowest order BDM element has *order* $k = 1$.

In this section we denote by \mathbf{X}_h the approximation of \mathbf{X} generated either by RT or by BDM finite elements of order $k \geq 0$ ($k \geq 1$ respectively).

Let I_h be the standard interpolation operator [4] from regular vectors on Γ onto X_h . We recall that this interpolation operator is obtained defining the degrees of freedom on the reference triangle (or square) \hat{T} and then transforming vectors by the standard Piola transform [4, Section III.1.3]. Moreover the moments up to order k of

the normal component to the edges are among the degrees of freedom. We refer to [21] (see also [13] or [24]) for a suitable definition on curved surfaces. The properties of this operator have been intensively studied, and we report here the ones we need:

(P1) For any $s > 0$, $\Pi_h : \mathbf{X}^0 \cap V_\pi^s \rightarrow X_h$ is linear and continuous (uniformly in h) [4, formula (3.40)] and there exists a function (depending on s) $\delta_s : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, such that $\delta_s(h) \rightarrow 0$ when $h \rightarrow 0$ and:

$$\|\mathbf{u} - \Pi_h \mathbf{u}\|_{\mathbf{X}^0} \leq C\delta_s(h) (\|\mathbf{u}\|_{\mathbf{X}^0} + \|\mathbf{u}\|_{V_\pi^s}).$$

Moreover, let $\tilde{\mathbf{V}}_h = \{\mathbf{v} \in \mathbf{X} : \operatorname{div}_\Gamma \mathbf{v} \in \operatorname{div}_\Gamma \mathbf{X}_h\}$, and $s > 0$, then:

$$\mathbf{u} \in \tilde{\mathbf{V}}_h \cap V_\pi^s, \quad \|\mathbf{u} - \Pi_h \mathbf{u}\|_{V_\pi^0} \leq C\delta_s(h) \|\mathbf{u}\|_{V_\pi^s}.$$

This statement comes from an argument due to V. Girault which was first used in [14, Proof of Lemma 4.1].

(P2) Let \mathcal{P}_h denote the L^2 -orthogonal projection from $L^2(\Gamma)$ onto the space $\operatorname{div}_\Gamma(\mathbf{X}_h)$. Note that for RT of order k and BDM of order $k - 1$ the space $\operatorname{div}_\Gamma(\mathbf{X}_h)$ consists of piecewise polynomials of degree k , [4]. Then, for any $s > 0$ [4, Proposition 3.7]:

$$\mathbf{u} \in \mathbf{X}^0 \cap V_\pi^s, \quad \operatorname{div}_\Gamma(\Pi_h \mathbf{u}) = \mathcal{P}_h \|\operatorname{div}_\Gamma \mathbf{u}\|_0.$$

The following statement has been proved in [7], see also [24] for partial results:

Theorem 4.2 *The RT and the BDM finite elements of any order k verify assumptions (A1) and (A2).*

Proof As far as (A1) is concerned, we refer again to [22] or [4]. Concerning (A2) we sketch the proof following verbatim [7] (see also [24]). Let $\mathbf{v}_h \in \mathbf{V}_h$ and $p \in H^1(\Gamma)/\mathbb{C}$ be the solution of the problem:

$$\operatorname{div}_\Gamma \nabla_\Gamma p = \operatorname{div}_\Gamma \mathbf{v}_h. \quad (4.4)$$

From [10, Theorem 5.3], we know that there exists a $s^* > 0$ depending on the geometry such that $\nabla_\Gamma p \in V_\pi^t$, $t < s^*$. Of course, $\nabla_\Gamma p \in \mathbf{X}^0 \cap V_\pi^\varepsilon$, for any fixed ε with $0 < \varepsilon < \min\{s^*, 1/2\}$. (P2) implies that $\Pi_h \nabla_\Gamma p - \mathbf{v}_h \in \mathbf{W}_h$; then, since $\mathbf{V}_h \perp \mathbf{W}_h$, we have

$$\|\mathbf{v}_h - \nabla_\Gamma p\|_{V_\pi^0} \leq \|\nabla_\Gamma p - \Pi_h \nabla_\Gamma p\|_{V_\pi^0}.$$

Now, using (P1), recalling that p solves (4.4), and $\nabla_\Gamma p \in \tilde{\mathbf{V}}_h \cap V_\pi^\varepsilon$, we have:

$$\|\mathbf{v}_h - \nabla_\Gamma p\|_{V_\pi^0} \leq C\delta_\varepsilon(h) \|\nabla_\Gamma p\|_{V_\pi^\varepsilon} \leq C\delta_\varepsilon(h) \|\operatorname{div}_\Gamma \mathbf{v}_h\|_{-1+\varepsilon, \Gamma}.$$

Since $\operatorname{div}_\Gamma(\mathbf{v}_h - \nabla_\Gamma p) = 0$, this implies (A2). \square

5 Electromagnetic scattering at a perfect conductor

As a first application of the results of the previous sections we consider the perfect conductor problem which is the Dirichlet problem for the Maxwell equations. The analogous Neumann problem can be solved with exactly the same boundary integral equation because of the symmetry of electric and magnetic quantities. However, the Neumann problem seems to be less relevant in practical applications.

The perfect conductor problem for non-smooth boundaries was already considered in [10], [24] and [7] where an indirect method was used. Here we use the so-called direct method where the unknown is the Neumann trace of the solution. Both approaches have the same operator C on the left hand side of the equation. The inf-sup condition for this operator follows as a special case from Theorem 3.12. For this case we actually do not need the compactness of the operator M in Proposition 3.13.

5.1 Definition of the problem

We assume that we have in the exterior domain Ω^c a material with constants $\mu, \varepsilon > 0$ and $k := \omega\sqrt{\mu\varepsilon}$. We assume that we have a perfect conductor in the bounded domain Ω . The normal vector \mathbf{n} on the boundary points from Ω to Ω^c .

We consider an incident field $\mathbf{u}^{\text{inc}} \in \mathbf{H}(\mathbf{curl}, \Omega^c)$ with $\mathbf{curl curl u}^{\text{inc}} - k^2 \mathbf{u}^{\text{inc}} = 0$. We want to find a scattered field \mathbf{u} in Ω^c such that

$$\mathbf{u} \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c), \quad (5.1a)$$

$$\mathbf{curl curl u} - k^2 \mathbf{u} = 0 \quad (5.1b)$$

$$\left| \mathbf{curl u}(\mathbf{r}) \times \frac{\mathbf{r}}{|\mathbf{r}|} - ik\mathbf{u} \right| = o\left(\frac{1}{|\mathbf{r}|}\right) \quad |\mathbf{r}| \rightarrow \infty \quad (5.1c)$$

Because of the properties of the perfect conductor the total field $\mathbf{u}^{\text{tot}} := \mathbf{u}^{\text{inc}} + \mathbf{u}$ in Ω^c should satisfy $\gamma_D^c \mathbf{u}^{\text{tot}} = 0$, i.e., we have with $\mathbf{m}_0 := -\gamma_D^c \mathbf{u}^{\text{inc}}$

$$\gamma_D^c \mathbf{u} = \mathbf{m}_0. \quad (5.1d)$$

We will assume that we are given data $\mathbf{m}_0 \in \mathbf{X}$ and want to find \mathbf{u} satisfying (5.1).

The uniqueness for the exterior problem is a direct consequence of Rellich's theorem (see [25] for a proof):

Theorem 5.1 *The perfect conductor problem (5.1) has at most one solution.*

The corresponding homogeneous interior problem can have nontrivial solutions.

Theorem 5.2 *Assume that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \Omega)$ is a Maxwell solution in Ω with $\gamma_D \mathbf{u} = 0$. Then $\mathbf{u} = 0$ unless $k^2 \in S_\Omega$ where the set S_Ω of eigenvalues is countable and has no finite accumulation point.*

Proof Taking the divergence in the equation (5.1b), we deduce that $\operatorname{div} \mathbf{u} = 0$. Thus, it is easily proved that \mathbf{u} is solution of the problem: Find $\mathbf{u} \in \mathcal{V} := \{\mathbf{v} \in H_0(\mathbf{curl}, \Omega) : \operatorname{div} \mathbf{u} = 0\}$ such that

$$\int_{\Omega} \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} \mathbf{v} - k^2 \int_{\Omega} \mathbf{u} \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \in \mathcal{V}.$$

Since the injection $\mathcal{V} \hookrightarrow L^2(\Omega)^3$ is compact [2], the Fredholm alternative applies and this proves the statement. \square

5.2 Boundary integral formulation

We consider the unknown Neumann data

$$\mathbf{j} = \gamma_N^c \mathbf{u}. \quad (5.2)$$

Because of the properties of the Calderón projector we have that

$$\left(\frac{1}{2}I + A\right) \begin{pmatrix} \mathbf{m}_0 \\ \mathbf{j} \end{pmatrix} = 0 \quad (5.3)$$

if and only if there exists \mathbf{u} satisfying (5.1) and (5.2).

By just using the first row of (5.3) we get

$$C\mathbf{j} = -\left(\frac{1}{2}I + M\right)\mathbf{m}_0. \quad (5.4)$$

Theorem 5.3 *Assume $k^2 \notin S_\Omega$. Then the boundary integral equation (5.4) holds if and only if there exists \mathbf{u} satisfying (5.1) and (5.2).*

Proof Obviously (5.3) implies (5.4). To prove the other direction assume that (5.1) holds and let $\begin{pmatrix} \tilde{\mathbf{m}} \\ \tilde{\mathbf{j}} \end{pmatrix} := \left(\frac{1}{2}I + A\right)\begin{pmatrix} \mathbf{m}_0 \\ \mathbf{j} \end{pmatrix}$. Equation (5.4) states that $\tilde{\mathbf{m}} = 0$. Now $\begin{pmatrix} \tilde{\mathbf{m}} \\ \tilde{\mathbf{j}} \end{pmatrix}$ is in the range of the Calderón projector for Ω , i.e., $\begin{pmatrix} \tilde{\mathbf{m}} \\ \tilde{\mathbf{j}} \end{pmatrix}$ are Cauchy data $\begin{pmatrix} \gamma_D \\ \gamma_N \end{pmatrix} \tilde{\mathbf{u}}$ of a Maxwell solution $\tilde{\mathbf{u}}$ in Ω with $\gamma_D \tilde{\mathbf{u}} = 0$. Since we assume $k^2 \notin S_\Omega$ Theorem 5.2 implies that $\tilde{\mathbf{j}} = 0$, hence (5.3) holds. \square

Recall the mapping $\Theta: \mathbf{X} \rightarrow \mathbf{X}$ (Definition 3.11) which maps $\mathbf{u} \in \mathbf{X}$ with Hodge decomposition $\mathbf{u} = \mathbf{v} + \mathbf{w}$ to $\overline{\mathbf{v} - \mathbf{w}}$. Then we have the following coercivity property for C :

Theorem 5.4 *There is a compact operator $T: \mathbf{X} \rightarrow \mathbf{X}$ and $\alpha > 0$ such that for all $\mathbf{j} \in \mathbf{X}$*

$$\operatorname{Re} b(\Theta \mathbf{j}, (C + T)\mathbf{j}) \geq \alpha \|\mathbf{j}\|_{\mathbf{X}}^2.$$

Proof We use Theorem 3.12 with $\mathbf{m} = 0$ and obtain

$$\operatorname{Re} b(\Theta \mathbf{j}, C\mathbf{j}) = \operatorname{Re} B \left(A \begin{pmatrix} 0 \\ \mathbf{j} \end{pmatrix}, \begin{pmatrix} 0 \\ \Theta \mathbf{j} \end{pmatrix} \right) \geq \alpha \|\mathbf{j}\|_{\mathbf{X}}^2 - \|T_0 \mathbf{j}\|_{\mathbf{X}}^2$$

where $T_0: \mathbf{X} \rightarrow \mathbf{X}$ is compact. \square

Corollary 5.5 *Assume $k^2 \notin S_\Omega$. Then the boundary integral equation (5.4) has for every $\mathbf{m}_0 \in \mathbf{X}$ a unique solution $\mathbf{j} \in \mathbf{X}$.*

Proof By the previous theorem C plus a compact operator is invertible, hence C has index zero. By Theorem 5.3 a solution of (5.4) with $\mathbf{m}_0 = 0$ corresponds to a solution of problem (5.1) with zero data. Then Theorem 5.1 gives $\mathbf{j} = 0$. Hence $C: \mathbf{X} \rightarrow \mathbf{X}$ is invertible. \square

If $k^2 \in S_\Omega$ we still have that $\mathbf{j} = \gamma_N^c \mathbf{u}$ is a solution of (5.4). However, the general solution of (5.4) has the form $\mathbf{j} = \gamma_N^c \mathbf{u} + \gamma_N \tilde{\mathbf{u}}$ where $\tilde{\mathbf{u}}$ is a Maxwell solution in Ω with $\gamma_D \tilde{\mathbf{u}} = 0$. Using this \mathbf{j} and \mathbf{m}_0 in the representation formula then gives

$$\Psi_E \mathbf{j} + \Psi_M \mathbf{m}_0 = \begin{cases} \mathbf{u} & \text{in } \Omega^c \\ -\tilde{\mathbf{u}} & \text{in } \Omega \end{cases}$$

so we still obtain the correct result in Ω^c .

5.3 The discretized problem

We choose a family $\{\mathbf{X}_h\}_h$ of finite dimensional subspaces of \mathbf{X} satisfying Assumptions (A1) and (A2) of section 4.1 and use the Galerkin method: Find $\mathbf{j}_h \in \mathbf{X}_h$ such that

$$b(\tilde{\mathbf{j}}_h, C\mathbf{j}_h) = -b(\tilde{\mathbf{j}}_h, (\frac{1}{2}I + A)\mathbf{m}_0) \quad \text{for all } \tilde{\mathbf{j}}_h \in \mathbf{X}_h \quad (5.5)$$

We now apply Theorem 5.4 together with Theorem 4.1 and obtain

Theorem 5.6 *Assume $k^2 \notin S_\Omega$ and let \mathbf{j} denote the solution of (5.4). There exists $h_0 > 0$ and C_0 such that for all $h \leq h_0$ the discretized problem (5.5) has a unique solution $\mathbf{j}_h \in \mathbf{X}_h$. The Galerkin solution converges quasioptimally:*

$$\|\mathbf{j}_h - \mathbf{j}\|_{\mathbf{X}} \leq C_0 \inf_{\tilde{\mathbf{j}}_h \in \mathbf{X}_h} \|\tilde{\mathbf{j}}_h - \mathbf{j}\|_{\mathbf{X}}.$$

6 Electromagnetic scattering at a dielectric interface

We now consider the transmission problem between two dielectric media with different magnetic properties in the two domains Ω and Ω^c and derive a boundary integral formulation. The coercivity property of the operator in this formulation follows from the coercivity property of the operator $A = \begin{pmatrix} M & C \\ C & M \end{pmatrix}$ in Theorem 3.12. Here (unlike the case of the perfect conductor problem) we need the general case of this theorem which requires the compactness property of M in Proposition 3.13.

6.1 Definition of the problem

In the bounded domain Ω we have material constants $\mu_1, \varepsilon_1 > 0$, in the exterior domain Ω^c we have material constants $\mu_2, \varepsilon_2 > 0$. The circular frequency ω is the same in both domains, but we have $k_1 := \omega\sqrt{\mu_1\varepsilon_1}$, $k_2 := \omega\sqrt{\mu_2\varepsilon_2}$. The normal vector \mathbf{n} on the boundary points from Ω to Ω^c . Let $\hat{\gamma}_N \mathbf{u} := \mathbf{n} \times \mathbf{curl} \mathbf{u}|_T$ denote the Neumann trace without the factor k^{-1} .

We consider an incident field $\mathbf{u}_2^{\text{inc}} \in \mathbf{H}(\mathbf{curl}, \Omega^c)$ with $\mathbf{curl} \mathbf{curl} \mathbf{u}_2^{\text{inc}} - k_2^2 \mathbf{u}_2^{\text{inc}} = 0$. We want to find scattered fields \mathbf{u}_1 in Ω and \mathbf{u}_2 in Ω^c such that

$$\mathbf{u}_1 \in \mathbf{H}(\mathbf{curl}, \Omega), \quad \mathbf{u}_2 \in \mathbf{H}_{\text{loc}}(\mathbf{curl}, \Omega^c), \quad (6.1a)$$

$$\mathbf{curl} \mathbf{curl} \mathbf{u}_1 - k_1^2 \mathbf{u}_1 = 0, \quad \mathbf{curl} \mathbf{curl} \mathbf{u}_2 - k_2^2 \mathbf{u}_2 = 0 \quad (6.1b)$$

$$\left| \mathbf{curl} \mathbf{u}_2(\mathbf{r}) \times \frac{\mathbf{r}}{|\mathbf{r}|} - ik_2 \mathbf{u}_2 \right| = o\left(\frac{1}{|\mathbf{r}|}\right) \quad |\mathbf{r}| \rightarrow \infty \quad (6.1c)$$

For \mathbf{u}_1 in Ω and the total field $\mathbf{u}_2^{\text{inc}} + \mathbf{u}_2$ in Ω^c the tangential component of the electric field and the magnetic flux densities should be continuous across T , i.e., we have with $\mathbf{m}_0 := \gamma_D^c \mathbf{u}_2^{\text{inc}}$ and $\mathbf{j}_0 := \mu_2^{-1} \hat{\gamma}_N^c \mathbf{u}_2^{\text{inc}}$

$$\gamma_D \mathbf{u}_1 = \gamma_D^c \mathbf{u}_2 + \mathbf{m}_0, \quad \mu_1^{-1} \hat{\gamma}_N \mathbf{u}_1 = \mu_2^{-1} \hat{\gamma}_N^c \mathbf{u}_2 + \mathbf{j}_0. \quad (6.1d)$$

We will assume that we are given data $\mathbf{m}_0, \mathbf{j}_0 \in \mathbf{X}$ and want to find $\mathbf{u}_1, \mathbf{u}_2$ satisfying (6.1). We have a uniqueness result:

Theorem 6.1 *The dielectric problem (6.1) has at most one solution.*

Proof This proof is actually quite standard; we just sketch it here and we refer to [12] or [26] for details. Set $\mathbf{m}_0 = \mathbf{j}_0 = 0$ and suppose there exists a nontrivial solution $\mathbf{u}_1, \mathbf{u}_2$ of (6.1). Then, let B_R be a ball of radius R sufficiently large to ensure $\Omega \subset B_R$. Using the equations

(6.1b), multiplying the first one by $\bar{\mathbf{u}}_1$, the second one by $\bar{\mathbf{u}}_2$, and using (6.1d) and the integration by parts formula (3.6), we obtain:

$$\int_{\partial B_R} (\mathbf{curl} \mathbf{u}_2 \times \mathbf{n}_R) \cdot \bar{\mathbf{u}}_2 = \int_{B_R \setminus \Omega} |\mathbf{curl} \mathbf{u}_2|^2 - k_2 |\mathbf{u}_2|^2 + \int_{\Omega} |\mathbf{curl} \mathbf{u}_1|^2 - k_1 |\mathbf{u}_1|^2.$$

Thus, $\text{Im}(\int_{\partial B_R} (\mathbf{curl} \mathbf{u}_2 \times \mathbf{n}_R) \cdot \bar{\mathbf{u}}_2) = 0$. Using the radiation condition (6.1c) and Rellich theorem, we can easily deduce $\mathbf{u}_2 = 0$. Now, \mathbf{u}_1 solves (6.1a) and verifies: $\gamma_D \mathbf{u}_1 = \gamma_N \mathbf{u}_1 = 0$. Thus, $\mathbf{u}_1 = 0$. \square

6.2 Boundary integral formulation

We define the operators A_j, C_j, M_j for $j = 1, 2$ as in the previous section using $k = k_j$. We define

$$\hat{A}_j := \begin{pmatrix} 1 & 0 \\ 0 & k_j \mu_j^{-1} \end{pmatrix} A_j \begin{pmatrix} 1 & 0 \\ 0 & k_j^{-1} \mu_j \end{pmatrix} = \begin{pmatrix} M_j & k_j^{-1} \mu_j C_j \\ k_j \mu_j^{-1} C_j & M_j \end{pmatrix}$$

Note that Theorem 3.9 and Theorem 3.12 (with constants depending on μ_j) also hold for \hat{A}_j .

We now consider the Cauchy data defined by

$$\boldsymbol{\xi}_1 = \begin{pmatrix} \gamma_D \\ \mu_1^{-1} \hat{\gamma}_N \end{pmatrix} \mathbf{u}_1, \quad \boldsymbol{\xi}_2 = \begin{pmatrix} \gamma_D^c \\ \mu_2^{-1} \hat{\gamma}_N^c \end{pmatrix} \mathbf{u}_2 \quad (6.2)$$

and let $\boldsymbol{\xi}_0 := \begin{pmatrix} \mathbf{m}_0 \\ \mathbf{j}_0 \end{pmatrix}$. Because of the properties of the Calderón projector we have that

$$\left(\frac{1}{2}I - \hat{A}_1\right)\boldsymbol{\xi}_1 = 0, \quad \left(\frac{1}{2}I + \hat{A}_2\right)\boldsymbol{\xi}_2 = 0, \quad \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 = \boldsymbol{\xi}_0. \quad (6.3)$$

if and only if there exist $\mathbf{u}_1, \mathbf{u}_2$ satisfying (6.1) and (6.2). We can now obtain the boundary integral formulation in the same way as in [18]: Writing $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 + \boldsymbol{\xi}_0$ and subtracting the first equation from the second equation gives

$$(\hat{A}_1 + \hat{A}_2)\boldsymbol{\xi}_2 = \left(\frac{1}{2}I - \hat{A}_1\right)\boldsymbol{\xi}_0. \quad (6.4)$$

Theorem 6.2 *The boundary integral equation (6.4) holds if and only if there exist $\mathbf{u}_1, \mathbf{u}_2$ satisfying (6.1) and (6.2).*

Proof Obviously (6.3) implies (6.4). To prove the other direction assume that (6.4) holds and let $\boldsymbol{\xi}_1 := \boldsymbol{\xi}_2 + \boldsymbol{\xi}_0$. Consider now $\tilde{\boldsymbol{\xi}}_1 := (\frac{1}{2}I - \hat{A}_1)\boldsymbol{\xi}_1$, $\tilde{\boldsymbol{\xi}}_2 := (\frac{1}{2}I + \hat{A}_2)\boldsymbol{\xi}_2$. Equation (6.4) states that $\tilde{\boldsymbol{\xi}}_1 = \tilde{\boldsymbol{\xi}}_2$. Now $\tilde{\boldsymbol{\xi}}_1$ is in the range of the Calderón projector for Ω^c with $k = k_1$, i.e., $\tilde{\boldsymbol{\xi}}_1$ are Cauchy data $(\frac{\gamma_D^c}{\mu_1^{-1}\gamma_N^c})\tilde{\mathbf{u}}_1$ of a Maxwell solution $\tilde{\mathbf{u}}_1$ in Ω^c with $k = k_1$. Similarly, $\tilde{\boldsymbol{\xi}}_2$ is in the range of the Calderón projector for Ω with $k = k_2$, i.e., $\boldsymbol{\xi}_2$ are Cauchy data $(\frac{\gamma_D^D}{\mu_2^{-1}\gamma_N^D})\tilde{\mathbf{u}}_2$ of a Maxwell solution $\tilde{\mathbf{u}}_2$ in Ω with $k = k_2$. Therefore $\tilde{\mathbf{u}}_2$ in Ω and $\tilde{\mathbf{u}}_1$ in Ω^c is the solution of a homogeneous dielectric problem with material constants ε_2, μ_2 in Ω and ε_1, μ_1 in Ω^c . By Theorem 6.1 (which holds for any material constants in Ω and Ω^c) we must have $\tilde{\boldsymbol{\xi}}_1 = \tilde{\boldsymbol{\xi}}_2 = 0$, hence (6.3) holds. \square

Again we have a coercivity property with the operator $\Theta: \mathbf{X} \rightarrow \mathbf{X}$ (Definition 3.11) which maps $\mathbf{u} \in \mathbf{X}$ with Hodge decomposition $\mathbf{u} = \mathbf{v} + \mathbf{w}$ to $\overline{\mathbf{v}} - \mathbf{w}$:

Theorem 6.3 *There is a compact operator $T: \mathbf{X}^2 \rightarrow \mathbf{X}^2$ and $\alpha > 0$ such that for all $\mathbf{m}, \mathbf{j} \in \mathbf{X}$*

$$\operatorname{Re} B \left((\hat{A}_1 + \hat{A}_2 + T) \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix}, \begin{pmatrix} \Theta \mathbf{m} \\ \Theta \mathbf{j} \end{pmatrix} \right) \geq \alpha \left\| \begin{pmatrix} \mathbf{m} \\ \mathbf{j} \end{pmatrix} \right\|_{\mathbf{X}^2}^2 \quad (6.5)$$

Proof We use that for \hat{A}_1 and \hat{A}_2 Theorem 3.12 holds. Therefore we have with $\tilde{\boldsymbol{\xi}} := \begin{pmatrix} \Theta \mathbf{m} \\ \Theta \mathbf{j} \end{pmatrix}$ that

$$\operatorname{Re} B((\hat{A}_1 + T_1 + \hat{A}_2 + T_2)\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}) \geq \alpha_1 \|\boldsymbol{\xi}\|_{\mathbf{X}}^2 + \alpha_2 \|\tilde{\boldsymbol{\xi}}\|_{\mathbf{X}}^2$$

yielding (6.5) with $T = T_1 + T_2$ and $\alpha = \alpha_1 + \alpha_2$. \square

Corollary 6.4 *The boundary integral equation (6.4) has a unique solution $\boldsymbol{\xi}_2 \in \mathbf{X}^2$ for every $\boldsymbol{\xi}_0 \in \mathbf{X}^2$.*

Proof By the previous theorem $Q := \hat{A}_1 + \hat{A}_2$ plus a compact operator is invertible, so Q has index zero. By Theorem 6.2 a solution of (6.4) with $\boldsymbol{\xi}_0 = 0$ corresponds to a solution of problem (6.1) with zero data. Then Theorem 6.1 gives $\boldsymbol{\xi}_2 = 0$. Hence $Q: \mathbf{X}^2 \rightarrow \mathbf{X}^2$ is invertible. \square

6.3 The discretized problem

We choose a family $\{\mathbf{X}_h\}_h$ of finite dimensional subspaces of \mathbf{X} satisfying Assumptions (A1) and (A2) of section 4.1 and use the Galerkin method: Find $\boldsymbol{\xi}_{2,h} \in \mathbf{X}_h^2$ such that

$$B((\hat{A}_1 + \hat{A}_2)\boldsymbol{\xi}_{2,h}, \tilde{\boldsymbol{\xi}}_h) = B((\frac{1}{2}I - \hat{A}_1)\boldsymbol{\xi}_0, \tilde{\boldsymbol{\xi}}_h) \quad \text{for all } \tilde{\boldsymbol{\xi}}_h \in \mathbf{X}_h^2 \quad (6.6)$$

We use Theorem 6.3 together with Theorem 4.1 and obtain

Theorem 6.5 *Let ξ_2 denote the solution of (6.4). There exists $h_0 > 0$ and C_0 such that for all $h \leq h_0$ the discretized problem (6.6) has a unique solution $\xi_{2,h} \in \mathbf{X}_h^2$. The Galerkin solution converges quasi-optimally:*

$$\|\xi_{2,h} - \xi_2\|_{\mathbf{X}^2} \leq C_0 \inf_{\tilde{\xi}_h \in \mathbf{X}_h^2} \|\tilde{\xi}_{2,h} - \xi_2\|_{\mathbf{X}^2}$$

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Research Reports

No.	Authors	Title
01-05	A. Buffa, R. Hiptmair, T. von Petersdorff, C. Schwab	Boundary Element Methods for Maxwell Equations in Lipschitz Domains
01-04	R. Hiptmair, C. Schwab	Natural BEM for the Electric Field Integral Equation on polyhedra
01-03	C. Schwab, A.-M. Matache	Generalized FEM for Homogenization Problems
01-02	A. Toselli, C. Schwab	Mixed hp -finite element approximations on geometric edge and boundary layer meshes in three dimensions
01-01	A. Buffa, M. Costabel, C. Schwab	Boundary element methods for Maxwell's equations on non-smooth domains
00-18	B.Q. Guo, C. Schwab	Analytic regularity of Stokes flow in polygonal domains
00-17	M.H. Gutknecht, S. Röllin	Variations of Zhang's Lanczos-Type Product Method
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00-15	G. Schmidlin, C. Schwab	Wavelet Galerkin BEM on unstructured meshes by aggregation
00-14	B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab	Local Discontinuous Galerkin methods for the Stokes system
00-13	O. Bröker, M.J. Grote, C. Mayer, A. Reusken	Robust Parallel Smoothing for Multigrid Via Sparse Approximate Inverses
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00-11	J. Liesen, M. Rozložník, Z. Strakoš	On Convergence and Implementation of Minimal Residual Krylov Subspace Methods for Unsymmetric Linear Systems
00-10	C. Schwab, O. Sterz	A scalar boundary integrodifferential equation for eddy current problems using an impedance boundary condition
00-09	M.H. Gutknecht, M. Rozložník	By how much can residual minimization accelerate the convergence of orthogonal error methods?
00-08	M. Rozložník, V. Simoncini	Short-term recurrences for indefinite preconditioning of saddle point problems
00-07	P. Houston, C. Schwab, E. Süli	Discontinuous hp -Finite Element Methods for Advection-Diffusion Problems