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Abstract

We consider the electric field integral equation on the surface of polyhedral domains and its Galerkin-discretization by means of divergence-conforming boundary elements. With respect to a Hodge-decomposition the continuous variational problem is shown to be coercive. However, this does not immediately carry over to the discrete setting, as discrete Hodge decompositions fail to possess essential regularity properties. Introducing an intermediate semidiscrete Hodge decomposition we can bridge the gap and come up with asymptotically optimal a-priori error estimates. Hitherto, those had been elusive, in particular for non-smooth boundaries.

Keywords: Electric field integral equation, Rumsey's principle, Raviart-Thomas elements, Hodge decomposition, discrete coercivity

Subject Classification: 65N12, 65N38, 78M15

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1 Introduction

One of the main tasks in computational electromagnetism is the computation of the scattering of electromagnetic waves at a perfectly conducting body $\Omega \subset \mathbb{R}^3$. It boils down to solving the time-harmonic Maxwell's equations for a fixed frequency, subject to vanishing tangential trace of the electric field on the surface of the scatterer and Silver-Müller radiation conditions at ∞. It is known that the exterior scattering problem for Maxwell's equations has a unique solution ([26, Ch. 6] and [22]). In most technical applications the boundary Γ of Ω will be only piecewise smooth.

Starting from the Stratton-Chu representation formulas [19, Sect. 3], an indirect method yields a boundary integral equation, featuring the jump of a magnetic field as principal unknown $\mathbf{j}$ [19, Sect. 4]. Cast in variational form this integral equation is known as Rumsey's principle and reads: Seek a complex amplitude $\mathbf{j} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ such that

$$\langle V_\varsigma \mathrm{div}_\Gamma \mathbf{j}, \mathrm{div}_\Gamma \mathbf{v} \rangle_{\frac{1}{2},\Gamma} - \varsigma^2 \langle \mathbf{A}_\varsigma \mathbf{j}, \mathbf{v} \rangle_{||,\Gamma} = f(\mathbf{v}) \quad \forall \mathbf{v} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \,. \tag{1}$$

Here, $\varsigma \in \mathbb{R}_+$ is the wave number, the continuous linear functional $f : \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \mapsto \mathbb{C}$ represents the excitation due to an incident wave, and $V_\varsigma, \mathbf{A}_\varsigma$ stand for scalar and vectorial single layer potential integral operators, respectively. All pairings are assumed to be sesqui-linear throughout the paper. Detailed explanations will be postponed to the following two sections.

Assumption 1.1 *A solution of* (1) *exists for all admissible right hand sides* f.

This assumption amounts to demanding that ς is bounded away from an interior Maxwell eigenvalue of Ω ([19, Thm. 4.4]).

Recalling the derivation of (1), the unknown $\mathbf{j}$ emerges as the jump of tangential traces $\mathbf{H} \times \mathbf{n}$ of magnetic field solutions for the full Maxwell equations in the interior and exterior of Ω. When stating Maxwell's equations in the language of differential forms [7, 14], which is doubtlessly the most concise formalism, the magnetic field is modeled by a twisted 1-form. The same will hold for its trace on Γ. This suggests that two-dimensional discrete twisted 1-forms built upon a triangulation of Γ should be used to approximate $\mathbf{j}$. Those are provided by the boundary element counterparts of the 2D Raviart-Thomas $\boldsymbol{H}(\mathrm{div}; \Omega)$-conforming finite elements. We could also reason in an entirely discrete setting: It is no longer a moot point that a suitable discretization of magnetic fields is provided by $\boldsymbol{H}(\mathbf{curl}; \Omega)$-conforming edge elements [42], which are discrete 1-forms in 3D . Taking a look at their tangential trace, again, we discover Raviart-Thomas elements mapped onto the surface [34]. Thus, we argue that the latter offer a "natural" boundary element discretization of (1).

This discretization of (1) is commonplace in engineering codes. The first theoretical examination was conducted by Bendali in [8, 9] based on a saddle point formulation and elliptic regularization, which is inherently confined to smooth surfaces. Using parametric variants of the Raviart-Thomas boundary elements, he could establish asymptotic a-priori convergence estimates. Yet, all attempts to adapt this approach to non-smooth surfaces have fizzled. Recently, Buffa, Costabel and Schwab succeeded in showing the convergence of a mixed discretization of (1), which, however, is different from the "natural" scheme.

Obstructions to convergence estimates on non-smooth surfaces are threefold: First, the correct function spaces and relevant surface differential operators have to be properly characterized. For smooth domains, using smooth charts and trace theorems for the entire scale of Sobolev spaces, this is not hard to do [2, 22]. It becomes a challenge in a non-smooth setting, as is vividly conveyed in the introduction of [20]. The first successes were achieved for piecewise smooth Γ by A. Buffa and P. Ciarlet, Jr. in [16–18]. Lipschitz-boundaries were tackled in [20]. We emphasize that only these results made possible the progress reported in the current paper.

Secondly, with (1) we recognize the typical difficulty faced when dealing with variational problems arising from Maxwell's equations: Owing to the large kernel of the surface divergence operator div_Γ, it becomes impossible to assign one term the role of a principal and, thus, the sesqui-linear form of (1) fails to be coercive. A remedy was first found in the case of the Maxwell differential equations [38, 40] and it is marked by the use of *Hodge decompositions*. Also for boundary integral equations the idea is fruitful and was exploited many times in order to recover coercive problems [3, 4, 19, 30].

Unfortunately, Hodge decompositions and the divergence conforming boundary elements do not match easily. This is the third obstacle and it is also faced in the analysis of $\boldsymbol{H}(\mathbf{curl}; \Omega)$-conforming finite element schemes. In that context a solution has been devised, relying on judiciously juggling discrete and continuous Hodge decompositions. This idea was successfully applied to the analysis of multigrid methods for edge elements [6, 35, 37] and to the investigations into "spurious free" discretizations of the Maxwell eigenproblem [10–12, 21, 41].

It is this idea that permits us to launch a successful attack on the discrete problem (1) on non-smooth domains. Yet, it took sophisticated adjustments to cope with the very poor regularity of the function spaces on Γ. Whereas for problems on domains $\subset \mathbb{R}^3$ all the fields are at least square integrable, here we find that surface vectorfields in $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ do not have this property. In this paper we aim to elucidate how to handle this difficulty. Upon finishing this paper we learned that S. Christiansen in [23] pursues a policy partly similar to ours, but with a different objective and confined to smooth domains.

The paper is organized as follows: In the next section we summarize important results about spaces of tangential vectorfields on polyhedra. The third section establishes the coercivity of the continuous variational problem with respect to a Hodge decomposition of $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Then we introduce divergence conforming boundary elements and review their main properties. In the fifth section we define and scrutinize mappings that create a link between discrete and continuous Hodge decompositions. The sixth section is dedicated to proving a discrete inf-sup condition and asymptotic a-priori error estimates arising from it.

It was our objective to keep the treatment as focused and self-contained as possible. To that end we forgo any generalizations and look at the simplest cases only. By and large, generalizations are straightforward. Numerical experiments can be skipped, since the popularity of the method gives ample evidence of its efficacy.

2 Spaces

We assume that Ω is a Lipschitz-polyhedron (cf. introduction of [29]). In particular, we assume that the Lipschitz boundary Γ is the union of a finite number of plane faces Γ_j, $j = 1, \ldots, N_\Gamma$, i.e. $\bar{\Gamma} = \bigcup_i \bar{\Gamma}_j$. For convenience only we assume further that Γ is simply connected (all assertions admit generalizations to the multiply connected case invoking suitably modified Hodge-decompositions), i.e. all its Betti numbers are zero. For each face Γ_j we find a constant unit normal vector $\mathbf{n}_i$ pointing into the exterior of Ω. These vectors can be blended into an exterior unit normal vector field $\mathbf{n} \in L^\infty(\Gamma)$, defined almost everywhere on Γ. In addition, we can fix two orthogonal unit vectors $\mathbf{e}_j^1, \mathbf{e}_j^2$ that span the tangential plane for Γ_i. It goes without saying that each Γ_j can be identified with a bounded subset of $\mathbb{R}^2$.

Then we can introduce two different tangential surface trace operators [20, Sect. 2]: The tangential components trace $\pi_\mathbf{t}$ is defined for $\mathbf{u} \in \boldsymbol{C}^\infty(\bar{\Omega})$ by $\pi_\mathbf{t}\mathbf{u}(\mathbf{x}) := \mathbf{n}(\mathbf{x}) \times (\mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}))$ for almost all $\mathbf{x} \in \Gamma$. Accordingly, the tangential surface trace $\gamma_\mathbf{t}$ can be computed through $\gamma_\mathbf{t}\mathbf{u}(\mathbf{x}) := \mathbf{u}(\mathbf{x}) \times \mathbf{n}(\mathbf{x})$. To begin with, they supply functions in

$$\boldsymbol{L}^2_\mathbf{t}(\Gamma) := \{\mathbf{u} \in (L^2(\Gamma))^3 \, , \mathbf{u} \cdot \mathbf{n} = 0\} \, .$$

The usual Sobolev spaces of scalar functions and related functionals $H^s(\Gamma)$, and $H^{-s}(\Gamma)$ can be defined in the classical fashion for $0 \le s \le 1$ [33, Sect. 1.3.3]. For larger indices $s > 1$ we resort to the piecewise definition

$$H^s(\Gamma) := \{u \in H^1(\Gamma), \, u_{|\Gamma_j} \in H^s(\Gamma_j), \, j = 1 \ldots, N_\Gamma\} \, .$$

This space is equipped with the natural graph norm

$$\|u\|^2_{H^s(\Gamma)} := \|u\|^2_{H^1(\Gamma)} + \sum_{j=1}^{N_\Gamma} \|u\|^2_{H^s(\Gamma_j)} \; .$$

Using the local coordinate systems introduced above, spaces of tangential vectorfields that feature certain Sobolev regularity in a piecewise sense, are readily available

$$\boldsymbol{H}^s_\mathbf{t}(\Gamma) := \{\mathbf{u} \in \boldsymbol{L}^2_\mathbf{t}(\Gamma), \, \mathbf{u}_{|\Gamma_j} \cdot \mathbf{e}^i_j \in H^s(\Gamma_j), \, j = 1, \ldots, N_\Gamma, \, i = 1, 2\} \, .$$

By localization to the Γ_j we can define the tangential surface gradient $\mathbf{grad}_\Gamma$ [20, Def. 3.1]. Its continuity as a mapping $H^{s+1}(\Gamma) \mapsto \boldsymbol{H}^s_\mathbf{t}(\Gamma)$, $s \ge 0$, is straightforward. The surface divergence is obtained as formal $\boldsymbol{L}^2_\mathbf{t}(\Gamma)$-adjoint

$\mathrm{div}_\Gamma : \boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \mapsto H^{-1}_*(\Gamma)$, where $H^{-s}_*(\Gamma) := \{\phi \in H^{-s}(\Gamma), \langle 1, \phi\rangle_{s,\Gamma} = 0\}$. The two operators can be used to define the surface Laplace-Beltrami operator $\Delta_\Gamma : H^1(\Gamma) \mapsto H^{-1}_*(\Gamma)$ by $\Delta_\Gamma := \mathrm{div}_\Gamma\, \mathbf{grad}_\Gamma$. It will be a key tool as theorem 5.3 of [19] reveals the following lifting property

Theorem 2.1 *If $f \in H^s_*(\Gamma)$ for $s \geq -1$, the (unique) solution $u \in H^1(\Gamma)/\mathbb{R}$ of $-\Delta_\Gamma u = f$ belongs to $H^{1+r}(\Gamma)$ for $0 \leq r \leq \min\{s+1, s^*\}$, where $s^* > 0$ depends on the geometry of Γ in neighborhoods of vertices only.*

In other words, with $\tilde{C} = \tilde{C}(t, \Gamma)$ and $0 \leq r < \min\{1-s, s^*\}$

$$f \in H^s(\Gamma) \quad , \quad -\Delta_\Gamma u = f \quad \Rightarrow \quad \|u\|_{H^{r+1}(\Gamma)} \leq \tilde{C}\, \|f\|_{H^s(\Gamma)} \ . \tag{2}$$

We adopt the convention that C and c stand for generic positive constants, whose values might be different between different occurrences, but must not depend on any concrete function. When tagged with a tilde on top, they may only depend on ς, continuous function spaces, and the geometry of Γ. Hence, the space

$$H^{-\frac{1}{2}}(\Delta_\Gamma, \Gamma) := \{u \in H^1(\Gamma),\, \Delta_\Gamma u \in H^{-\frac{1}{2}}(\Gamma)\}$$

will actually be embedded in $H^{1+r}(\Gamma)$ for all $0 \leq r \leq \min\{\frac{3}{2}, s^*\}$. Based on div_Γ, we get the Hilbert spaces ($s \geq 0$)

$$\boldsymbol{H}^s(\mathrm{div}_\Gamma; \Omega) := \{\mathbf{u} \in \boldsymbol{H}^s_{\mathbf{t}}(\Gamma),\, \mathrm{div}_\Gamma \mathbf{u} \in H^s(\Gamma)\} \ .$$

Tangential traces of vectorfields in $\boldsymbol{H}^1_{\mathrm{loc}}(\Omega)$ form the spaces $\boldsymbol{H}^{\frac{1}{2}}_{||}(\Gamma)$ and $\boldsymbol{H}^{\frac{1}{2}}_{\perp}(\Gamma)$ which were characterized in [17, Prop. 1.6]. Loosely speaking, $\boldsymbol{H}^{\frac{1}{2}}_{||}(\Gamma)$ contains the tangential surface vectorfields that are in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma_i)$ for each smooth component Γ_i of Γ and feature a suitable "weak tangential continuity" across the edges of the Γ_i. A corresponding "weak normal continuity" is satisfied by surface vectorfields in $\boldsymbol{H}^{\frac{1}{2}}_{\perp}(\Gamma)$. For smooth Γ these spaces coincide with the spaces of tangential surface vectorfields in $\boldsymbol{H}^{\frac{1}{2}}(\Gamma)$. The associated dual spaces will be denoted by $\boldsymbol{H}^{-\frac{1}{2}}_{||}(\Gamma)$ and $\boldsymbol{H}^{-\frac{1}{2}}_{\perp}(\Gamma)$, respectively, where the duality pairings are taken with $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ as pivot space. We denote further by $\langle\cdot,\cdot\rangle_{||,\Gamma}$ and $\langle\cdot,\cdot\rangle_{\perp,\Gamma}$ the respective duality pairings. A fundamental result of [17] asserts that the tangential trace mapping $\pi_{\mathbf{t}} : \boldsymbol{H}^1_{\mathrm{loc}}(\Omega) \mapsto \boldsymbol{H}^{\frac{1}{2}}_{||}(\Gamma)$ is continuous, surjective and possesses a continuous right inverse (see proposition 1.7 in [17]).

One of the crucial insights gained in [17] and [20] was that the tangential surface gradient $\mathbf{grad}_\Gamma : H^1(\Gamma) \mapsto \boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$ can be both extended and restricted to continuous and *injective* linear operators

$$\mathbf{grad}_\Gamma : \tilde{\boldsymbol{H}}^{\frac{3}{2}}(\Gamma)/\mathbb{R} \mapsto \boldsymbol{H}^{\frac{1}{2}}_{||}(\Gamma) \quad , \quad \mathbf{grad}_\Gamma : H^{\frac{1}{2}}(\Gamma)/\mathbb{R} \mapsto \boldsymbol{H}^{-\frac{1}{2}}_{\perp}(\Gamma)$$

(cf. propositions 3.4 and 3.6 in [20]), where $\tilde{\boldsymbol{H}}^{\frac{3}{2}}(\Gamma)$ is the space of traces of functions in $H^2(\Omega)$. Consequently, div_Γ can also be read as continuous and *surjective* operator

$$\mathrm{div}_\Gamma : \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma) \mapsto \tilde{\boldsymbol{H}}_*^{-\frac{3}{2}}(\Gamma) \quad , \quad \mathrm{div}_\Gamma : \boldsymbol{H}_\perp^{\frac{1}{2}}(\Gamma) \mapsto H_*^{-\frac{1}{2}}(\Gamma) \ .$$

First, this is important for the definition of the space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ introduced in [17] by

$$\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) = \{\boldsymbol{\zeta} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma),\ \mathrm{div}_\Gamma \boldsymbol{\zeta} \in H^{-\frac{1}{2}}(\Gamma)\} \ .$$

It is endowed with the natural graph norm $\|\cdot\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}$.

The key role of *Hodge decompositions* was emphasized in the introduction. The following theorem reveals the nature of the Hodge decomposition that we will need. More details are given in [20, Sect. 5], [18], and [16].

Theorem 2.2 *The space $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ has the direct and stable decomposition*

$$\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) := \mathbf{grad}_\Gamma\, H^{-\frac{1}{2}}(\Delta_\Gamma, \Gamma) \oplus (\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \cap \mathrm{Ker}(\mathrm{div}_\Gamma)) \ .$$

Moreover, when restricted to $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \cap \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ the decomposition is $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$-orthogonal.

Any function in $\mathbf{grad}_\Gamma\, H^{-\frac{1}{2}}(\Delta_\Gamma, \Gamma) \cap \mathrm{Ker}(\mathrm{div}_\Gamma)$ must be the gradient of a function in the kernel of Δ_Γ on Γ. Since Γ was assumed to be simply connected, the latter only contains constants and therefore the decomposition is direct.

Next, pick some $\mathbf{v} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. Since $\mathrm{div}_\Gamma : \boldsymbol{H}_\perp^{\frac{1}{2}}(\Gamma) \mapsto H_*^{-\frac{1}{2}}(\Gamma)$ is surjective, we can find $\boldsymbol{\psi} \in \boldsymbol{H}_\perp^{\frac{1}{2}}(\Gamma)$ such that $\mathrm{div}_\Gamma \boldsymbol{\psi} = \mathrm{div}_\Gamma \mathbf{v} \in H_*^{-\frac{1}{2}}(\Gamma)$. Define $\varphi \in H^1(\Gamma)/\mathbb{R}$ by

$$(\mathbf{grad}_\Gamma\, \varphi, \mathbf{grad}_\Gamma\, \eta)_{0;\Gamma} = (\boldsymbol{\psi}, \mathbf{grad}_\Gamma\, \eta)_{0;\Gamma} \quad \forall \eta \in H^1(\Gamma)/\mathbb{R} \ ,$$

that is, the unique weak solution of $\Delta_\Gamma \varphi = \mathrm{div}_\Gamma \boldsymbol{\psi}$. This yields the decomposition

$$\mathbf{v} = \mathbf{grad}_\Gamma\, \varphi + (\boldsymbol{\psi} - \mathbf{grad}_\Gamma\, \varphi + \mathbf{v} - \boldsymbol{\psi}) \ ,$$

whose second part is readily seen to be divergence-free. By the open mapping theorem (div_Γ surjective!) $\boldsymbol{\psi}$ can be chosen such that

$$\|\boldsymbol{\psi}\|_{\boldsymbol{H}_\perp^{\frac{1}{2}}(\Gamma)} \leq \tilde{C}\, \|\mathrm{div}_\Gamma \boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\Gamma)} \ .$$

This implies

$$\|\mathbf{grad}_\Gamma\, \varphi\|_{\boldsymbol{L}^2(\Gamma)} \leq \|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(\Gamma)} \leq \|\boldsymbol{\psi}\|_{\boldsymbol{H}_\perp^{\frac{1}{2}}(\Gamma)} \leq \tilde{C}\, \|\mathrm{div}_\Gamma \boldsymbol{\psi}\|_{H^{-\frac{1}{2}}(\Gamma)} \ ,$$

which confirms the stability of the decomposition. For $\mathbf{v} \in \boldsymbol{L}^2_{\mathbf{t}}(\Gamma) \cap \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ the $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$-orthogonality is immediate from the definition of div_Γ.

In the sequel we write

$$\mathbf{X} := \mathbf{grad}_\Gamma\, H^{-\frac{1}{2}}(\Delta_\Gamma, \Gamma) \quad \text{and} \quad \mathbf{N} := \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma) \cap \mathrm{Ker}(\mathrm{div}_\Gamma).$$

From the stability of the Hodge decomposition we conclude that both $\mathbf{X}$ and $\mathbf{N}$ are closed subspaces of $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. If $\mathbf{v} \in \mathbf{X}$ satisfies $\mathrm{div}_\Gamma \mathbf{v} \in H^s(\Gamma)$ for some $s \geq -\frac{1}{2}$ then for all $0 \leq r \leq \min\{s+1, s^*\}$

$$\mathbf{v} \in \boldsymbol{H}^r_{\mathbf{t}}(\Gamma) \quad \text{and} \quad \|\mathbf{v}\|_{H^r(\Gamma)} \leq \tilde{C}\, \|\mathrm{div}_\Gamma \mathbf{v}\|_{\boldsymbol{H}^s_{\mathbf{t}}(\Gamma)} \ ,$$

with a constant $\tilde{C} = \tilde{C}(r,s)$. $\mathbf{v} \in \mathbf{X}$ means $\mathbf{v} = \mathbf{grad}_\Gamma\, \varphi$ for some $\varphi \in H^1(\Gamma)$. By definition of $\mathbf{X}$ we see $\Delta_\Gamma \varphi = \mathrm{div}_\Gamma \mathbf{v}$, and the assertion follows from theorem 2.1.

In particular, we conclude

$$\|\mathbf{v}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq \|\mathbf{v}\|_{\boldsymbol{L}^2(\Gamma)} \leq \tilde{C}\, \|\mathrm{div}_\Gamma \mathbf{v}\|_{H^{-\frac{1}{2}}(\Gamma)} \ , \quad \forall \mathbf{v} \in \mathbf{X}\ . \tag{3}$$

3 Continuous variational problem

We recall the scalar single layer potential $\Psi^V_\varsigma : H^{-\frac{1}{2}}(\Gamma) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3)$ for the Helmholtz operator $-\Delta + \varsigma^2$ [39, Ch. 9]. Its relative, the vectorial Helmholtz single layer potential $\boldsymbol{\Psi}^{\mathbf{A}}_\varsigma(\mathbf{v})$ for $\mathbf{v} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$ is given by

$$\boldsymbol{\Psi}^{\mathbf{A}}_\varsigma(\mathbf{v})(\mathbf{x}) := \int_\Gamma \Phi_\varsigma(\mathbf{x},\mathbf{y})\mathbf{v}(\mathbf{y})\, dS(\mathbf{y}) \quad , \quad \Phi_\varsigma(\mathbf{x},\mathbf{y}) := \frac{\exp(i\varsigma|\mathbf{x}-\mathbf{y}|}{4\pi|\mathbf{x}-\mathbf{y}|}\ .$$

For every $\mathbf{v} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$ it defines a function in $\mathbf{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ and, as a consequence of the trace theorem, we can introduce the vectorial single layer boundary operator

$$\mathbf{A}_\varsigma : \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma) \mapsto \boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma) \quad , \quad \mathbf{A} := \pi_{\mathbf{t}} \circ \boldsymbol{\Psi}^{\mathbf{A}}_\varsigma$$

and the scalar single layer integral operator

$$V_\varsigma : H^{-1/2}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma) \quad , \quad V := \gamma \circ \Psi^V_\varsigma\ ,$$

where $\gamma : H^1_{\mathrm{loc}}(\mathbb{R}^3) \mapsto H^{\frac{1}{2}}(\Gamma)$ is the standard trace operator. In the case $\varsigma = 0$ these operators are coercive. The operators V_0 and $\mathbf{A}_0$ are continuous, selfadjoint and elliptic, i.e. there are constants $\tilde{c}_1, \tilde{c}_2 > 0$ only depending on Γ such that for all $\mu \in H^{-\frac{1}{2}}(\Gamma)$ and all $\boldsymbol{\mu} \in \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$

$$\langle V_0\mu, \mu\rangle_{\frac{1}{2},\Gamma} \geq \tilde{c}_1\, \|\mu\|^2_{H^{-\frac{1}{2}}(\Gamma)} \quad , \quad \langle \mathbf{A}_0\boldsymbol{\mu}, \boldsymbol{\mu}\rangle_{\frac{1}{2},\Gamma} \geq \tilde{c}_2\, \|\boldsymbol{\mu}\|^2_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)}\ .$$

See [27] or corollary 8.13 in [39], and theorem 6.2 in [36] or proposition 4.1 in [19].

Along with the following result this yields the coercivity of V_k and $\mathbf{A}_k$ (compare the proof of theorem 4.4 in [19]). The operators $\delta V_\varsigma := V_\varsigma - V_0 : H^{-\frac{1}{2}}(\Gamma) \mapsto H^{\frac{1}{2}}(\Gamma)$ and $\delta \mathbf{A}_\varsigma := \mathbf{A}_\varsigma - \mathbf{A}_0 : \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma) \mapsto \boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma)$ are compact. We denote by G_ς the Green's operator in $\mathbb{R}^3$ for the Helmholtz equation, defined by $(G_\varsigma \varphi)(\mathbf{x}) = \int_{\mathbf{y}\in\mathbb{R}^3} \Phi_\varsigma(\mathbf{x},\mathbf{y})\varphi(\mathbf{y})d\mathbf{y}$ for $\varphi \in C_0^\infty(\mathbb{R}^3)$. By the continuity $\gamma : H^s_{loc}(\mathbb{R}^3) \to H^{s-1/2}(\Gamma)$, $s \in (1/2, 3/2)$ of the trace map (Lemma 3.6 in [27]), we find $\delta V_\varsigma = \gamma \circ (G_\varsigma - G_0) \circ \gamma' : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ compactly, since the kernel $\Phi_\varsigma(\mathbf{x},\mathbf{y}) - \Phi_0(\mathbf{x},\mathbf{y})$ of the operator $G_\varsigma - G_0$ belongs to $C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. The vectorial case can be treated analogously.

The main tool in the analysis of the variational problem (1) are Hodge decompositions according to theorem 2.2 (cf. [19, Sect. 4.3]). We Hodge-decompose trial and test functions $\mathbf{j} := \mathbf{j}^\perp + \mathbf{j}^0$, $\mathbf{j}^\perp \in \mathbf{X}$, $\mathbf{j}^0 \in \mathbf{N}$, and $\mathbf{v} := \mathbf{u}^\perp + \mathbf{v}^0$, $\mathbf{v}^\perp \in \mathbf{X}$, $\mathbf{v}^0 \in \mathbf{N}$ in (1). This way, we end up with the equivalent variational problem: find $\mathbf{j}^\perp \in \mathbf{X}$, $\mathbf{j}^0 \in \mathbf{N}$ such that for all $\mathbf{v}^\perp \in \mathbf{X}$, $\mathbf{v}^0 \in \mathbf{N}$

$$\begin{aligned} \left\langle V_\varsigma \mathrm{div}_\Gamma \mathbf{j}^\perp, \mathrm{div}_\Gamma \mathbf{v}^\perp \right\rangle_{\frac{1}{2},\Gamma} - \zeta^2 \left\langle \mathbf{A}_\varsigma \mathbf{j}^\perp, \mathbf{v}^\perp \right\rangle_{||,\Gamma} - \quad & \zeta^2 \left\langle \mathbf{A}_\varsigma \mathbf{j}^0, \mathbf{v}^\perp \right\rangle_{||,\Gamma} &= f(\mathbf{v}^\perp) \ , \\ \zeta^2 \overline{\left\langle \mathbf{A}_\varsigma \mathbf{j}^\perp, \mathbf{v}^0 \right\rangle_{||,\Gamma}} \qquad + \quad & \zeta^2 \overline{\left\langle \mathbf{A}_\varsigma \mathbf{j}^0, \mathbf{v}^0 \right\rangle_{||,\Gamma}} &= \overline{f(\mathbf{v}^0)} \ . \end{aligned} \tag{4}$$

Here, $\langle .,. \rangle_{||,\Gamma}$ denotes the $\boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma) \times \boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$ duality pairing.

The natural setting for this formulation is the Hilbert space $\boldsymbol{\mathcal{G}} := \mathbf{X} \otimes \mathbf{N}$ which we endow with the graph norm

$$\left\| (\mathbf{v}^\perp, \mathbf{v}^0) \right\|^2_{\boldsymbol{\mathcal{G}}} := \left\| \mathbf{v}^\perp \right\|^2_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} + \left\| \mathbf{v}^0 \right\|^2_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \ , \quad (\mathbf{v}^\perp, \mathbf{v}^0) \in \boldsymbol{\mathcal{G}} \ .$$

Thanks to theorem 2.2 the space $\boldsymbol{\mathcal{G}}$ thus defined is isomorphic to $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ algebraically and topologically.

We denote by $a : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$ the sesqui-linear form related to (4) which is continuous

$$|a(\boldsymbol{\varphi}, \boldsymbol{\eta})| \leq \tilde{C}_a \ \|\boldsymbol{\varphi}\|_{\boldsymbol{\mathcal{G}}} \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{G}}} \quad \forall \boldsymbol{\varphi}, \boldsymbol{\eta} \in \boldsymbol{\mathcal{G}}$$

Then we can express the continuous variational problem (4) as: find $\boldsymbol{\iota} \in \boldsymbol{\mathcal{G}}$ such that

$$a(\boldsymbol{\iota}, \boldsymbol{\eta}) = f(\boldsymbol{\eta}) \quad \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{G}} \ , \tag{5}$$

where $f(\boldsymbol{\eta}) := f(\mathbf{v}^\perp) + \overline{f(\mathbf{v}^0)}$, $\boldsymbol{\eta} := (\mathbf{v}^\perp, \mathbf{v}^0)$. We point out that (5) is entirely equivalent to (1) in the sense that, if $\mathbf{j} \in \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$ is a solution of (1), then $\boldsymbol{\iota} := (\mathbf{j}^\perp, \mathbf{j}^0) \in \boldsymbol{\mathcal{G}}$ will solve (5). In particular, assertions on existence and uniqueness of solutions of (1) instantly carry over to (5).

To establish strong ellipticity of the form $a(\cdot,\cdot)$, we write $a = a_0 + k_0$, where $k_0 : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$ reads

$$\begin{aligned} k_0((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) := {} & \left\langle \delta V_\varsigma \mathrm{div}_\Gamma \mathbf{j}^\perp, \mathrm{div}_\Gamma \mathbf{v}^\perp \right\rangle_{\frac{1}{2},\Gamma} - \varsigma^2 \left\langle \delta \mathbf{A}_\varsigma \mathbf{j}^\perp, \mathbf{v}^\perp \right\rangle_{||,\Gamma} - \\ & - k^2 \left\langle \delta \mathbf{A}_\varsigma \mathbf{j}^0, \mathbf{v}^\perp \right\rangle_{||,\Gamma} + \varsigma^2 \overline{\left\langle \delta \mathbf{A}_\varsigma \mathbf{j}^\perp, \mathbf{v}^0 \right\rangle_{||,\Gamma}} + \varsigma^2 \overline{\left\langle \delta \mathbf{A}_\varsigma \mathbf{j}^0, \mathbf{v}^0 \right\rangle_{||,\Gamma}} \ , \end{aligned}$$

and where $a_0 : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$ emerges from a by replacing $V_k \to V_0$ and $\mathbf{A}_k \to \mathbf{A}_0$. The next lemma is crucial for establishing the strong ellipticity of the variational problem (5). The operator $L : \mathbf{X} \mapsto \mathbf{X}'$, defined by $L\mathbf{u}^\perp(\mathbf{z}^\perp) := \left\langle \mathbf{A}_0\mathbf{u}^\perp, \mathbf{z}^\perp \right\rangle_{||,\Gamma}$, for all $\mathbf{u}^\perp, \mathbf{z}^\perp \in \mathbf{X}$, is compact. Consider a bounded sequence $(\mathbf{u}_n^\perp)_{n\in\mathbb{N}}$ in $\mathbf{X}$. By Lemma 2 it is also bounded in $\boldsymbol{H}^t_{\mathbf{t}}(\Gamma)$. By Rellich's theorem we can find a subsequence, also designated by $(\mathbf{u}_n^\perp)_n$ that converges in $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$. Observe that due to the continuity of the vectorial single layer boundary integral operator

$$\left\| L\mathbf{z}^\perp \right\|_{\mathbf{X}'} = \sup_{\mathbf{v}\in\mathbf{X}} \frac{(L\mathbf{z}^\perp)(\mathbf{v})}{\|\mathbf{v}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)}} \leq \sup_{\mathbf{v}\in\mathbf{X}} \frac{\left\langle \mathbf{A}_0\mathbf{z}^\perp, \mathbf{v} \right\rangle_{||,\Gamma}}{\|\mathbf{v}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)}} \leq$$

$$\leq \left\| \mathbf{A}_0\mathbf{z}^\perp \right\|_{\boldsymbol{H}_{||}^{\frac{1}{2}}(\Gamma)} \leq \tilde{C} \left\| \mathbf{z}^\perp \right\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq \tilde{C} \left\| \mathbf{z}^\perp \right\|_{\boldsymbol{L}^2(\Gamma)} .$$

Thus $(L\mathbf{u}_n^\perp)_n$ will converge in $\mathbf{X}'$.

To establish the strong ellipticity of the form $a(\cdot,\cdot)$, we further split $a_0(\cdot,\cdot)$ according to $a_0 = d - k_1$ where the sesqui-linear form $k_1 : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$ is defined by $k_1((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) := \zeta^2 \left\langle \mathbf{A}_0\mathbf{j}^\perp, \mathbf{v}^\perp \right\rangle_{||,\Gamma}$, and $d : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$ reads

$$d((\mathbf{j}^\perp, \mathbf{j}^0), (\mathbf{v}^\perp, \mathbf{v}^0)) := \left\langle V_0 \mathrm{div}_\Gamma \mathbf{j}^\perp, \mathrm{div}_\Gamma \mathbf{v}^\perp \right\rangle_{\frac{1}{2},\Gamma} - \varsigma^2 \left\langle \mathbf{A}_0\mathbf{j}^0, \mathbf{v}^\perp \right\rangle_{||,\Gamma} +$$

$$+ \varsigma^2 \overline{\left\langle \mathbf{A}_0\mathbf{j}^\perp, \mathbf{v}^0 \right\rangle_{||,\Gamma}} + \varsigma^2 \overline{\left\langle \mathbf{A}_0\mathbf{j}^0, \mathbf{v}^0 \right\rangle_{||,\Gamma}} .$$

Theorem 3.1 *The sesqui-linear form $a : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$ is coercive, that is, it can be written as the sum of a $\boldsymbol{\mathcal{G}}$-elliptic sesqui-linear form d and a compact sesqui-linear form $k : \boldsymbol{\mathcal{G}} \times \boldsymbol{\mathcal{G}} \mapsto \mathbb{C}$.*

Recall $a = a_0 + k_0$. Lemma 3 reveals that k_0 is a compact perturbation of a_0. Further, $a_0 = d - k_1$ and Lemma 3 implies that k_1 is a compact perturbation of d. From the ellipticity of the single layer boundary integral operators in lemma 3 we immediately get

$$|d((\mathbf{v}^\perp, \mathbf{v}^0), (\mathbf{v}^\perp, \mathbf{v}^0))| = | \left\langle V_0 \mathrm{div}_\Gamma \mathbf{v}^\perp, \mathrm{div}_\Gamma \mathbf{v}^\perp \right\rangle_{\frac{1}{2},\Gamma} + \varsigma^2 \overline{\left\langle \mathbf{A}_0\mathbf{v}^0, \mathbf{v}^0 \right\rangle_{||,\Gamma}} | \geq$$

$$\geq \tilde{c}_1 \left\| \mathrm{div}_\Gamma \mathbf{v}^\perp \right\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \tilde{c}_2 \varsigma^2 \left\| \mathbf{v}^0 \right\|^2_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)}$$

for all $(\mathbf{v}^\perp, \mathbf{v}^0) \in \boldsymbol{\mathcal{G}}$. Now, we can appeal to (3) and obtain

$$|d(\boldsymbol{\varphi}, \boldsymbol{\varphi})| \geq \tilde{c}_d \|\boldsymbol{\varphi}\|^2_{\boldsymbol{\mathcal{G}}} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}} .$$

Setting $k = k_0 - k_1$ yields $a = d + k$ with a principal part d which is positive on $\boldsymbol{\mathcal{G}}$ and a compact perturbation k, as claimed.

Since we take existence of solutions of (1) (and (5)) for granted, Fredholm theory also provides uniqueness and we infer the following inf-sup condition for $a(\cdot,\cdot)$

$$\sup_{\boldsymbol{\nu}\in\boldsymbol{\mathcal{G}}} \frac{|a(\boldsymbol{\varphi}, \boldsymbol{\nu})|}{\|\boldsymbol{\nu}\|_{\boldsymbol{\mathcal{G}}}} \geq \tilde{c}_a \|\boldsymbol{\varphi}\|_{\boldsymbol{\mathcal{G}}} \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}} . \tag{6}$$

4 Boundary element spaces

We equip Γ with a family of shape-regular, quasi-uniform triangulations $(\Gamma_h)_{h>0}$ [24] comprising only flat triangles. The parameter h designates the meshwidth, that is, the length of the longest edge. Let $\mathbb{H}$ stand for the collection of meshwidths occurring in $(\Gamma_h)_{h\in\mathbb{H}}$ and assume that $\mathbb{H} \subset \mathbb{R}^+$ forms a decreasing sequence converging to zero. The set $\mathcal{T}_h$ will include all triangles of Γ_h, and $\mathcal{E}_h$ stands for the set of edges of Γ_h.

Using the local coordinate systems on the faces Γ_j, $j = 1, \dots, N_\Gamma$, each $T \in \mathcal{T}_h$ can be embedded in $\mathbb{R}^2$. Then we can define the local spaces (cf. [43])

$$\boldsymbol{\mathcal{RT}}_0(T) := \{\mathbf{x} \mapsto \mathbf{a} + \beta\mathbf{x}, \mathbf{a} \in \mathbb{R}^2, \beta \in \mathbb{R}\} , \quad T \in \mathcal{T}_h .$$

They give rise to the global boundary element space

$$\boldsymbol{\mathcal{RT}}_0(\Gamma_h) := \{\mathbf{v} \in \boldsymbol{H}(\mathrm{div}_\Gamma; \Gamma), \mathbf{v}_{|T} \in \boldsymbol{\mathcal{RT}}_0(T) \, \forall T \in \mathcal{T}_h\} .$$

Keep in mind that this definition is based on a weak notion of div_Γ. So Green's formula applied to the surface triangles can be used to confirm that the "edge-normal" components of the tangential vectorfields in $\boldsymbol{\mathcal{RT}}_0(\Gamma_h)$ must be continuous across inter-element edges. This renders the following degrees of freedom well defined

$$\phi_e : \boldsymbol{\mathcal{RT}}_0(\Gamma_h) \mapsto \mathbb{C} \quad , \quad \phi_e(\mathbf{v}_h) := \int_e (\mathbf{v}_h \times \mathbf{n}_j) \cdot d\vec{s} , \quad e \in \mathcal{E}_h ,$$

where $\mathbf{n}_j$ is the normal of a face Γ_j in whose closure e is contained. Given the degrees of freedom we have nodal interpolation operators Π_h onto $\boldsymbol{\mathcal{RT}}_0(\Gamma_h)$ at our disposal that, to begin with, can be declared for $\{\Gamma_j\}$-piecewise continuous tangential surface vectorfields, whose edge-normal components are continuous, too. It turns out that this is not enough and we badly need to apply Π_h to less regular surface vectorfields. A first step is the following lemma (cf. formula (3.40) in [15]). For any $s > 0$ the local interpolation operator $\Pi_T : \boldsymbol{H}^s(T) \cap \boldsymbol{H}(\mathrm{div}; T) \mapsto \boldsymbol{\mathcal{RT}}_0(T)$, $T \in \mathcal{T}_h$, is continuous. Only the case $s \leq \frac{1}{2}$ is of interest. We consider a single degree of freedom on T: Pick an edge $e \subset \partial T$ and regard its characteristic function χ_e as an element in $W_q^{1-\frac{1}{q}}(e)$ for $q := 1+s$. As $1 < q < 2$ theorem 1.4.5.2 of [33] reveals that extension by zero of χ_e onto all of ∂T will provide a function $\tilde{\psi}$ in $W_q^{1-\frac{1}{q}}(\partial T)$. Then we can use the trace theorem [33, Thm. 1.5.1.3] to extend $\tilde{\psi}$ to a function $\psi \in W_q^1(T)$ in a continuous fashion. Using Green's formula, extended by continuity, we estimate for any smooth vectorfield $\mathbf{v}$

$$\int_e \mathbf{v} \cdot \mathbf{n}_e \, ds = \int_{\partial T} \tilde{\psi}\mathbf{v} \cdot \mathbf{n} \, ds = \int_T \mathbf{grad}\,\psi \cdot \mathbf{v} + \psi \,\mathrm{div}\,\mathbf{v} \, d\mathbf{x} \leq$$
$$\leq \|\mathbf{grad}\,\psi\|_{L^q(T)} \|\mathbf{v}\|_{L^p(T)} + \|\psi\|_{L^2(T)} \|\mathrm{div}\,\mathbf{v}\|_{\boldsymbol{L}^2(T)} ,$$

where p is the exponent conjugate to q, i.e. $p^{-1}+q^{-1}=2$. The Sobolev embedding theorem [1, Thm. 4.5] gives the continuous inclusions

$$W^1_q(T) \hookrightarrow \mathrm{L}^2(T) \quad , \quad \boldsymbol{H}^s(T) \hookrightarrow \boldsymbol{L}^p(T) \; .$$

This implies, with $\tilde{C}=\tilde{C}(s,T)$

$$\int\limits_e \mathbf{v}\cdot\mathbf{n}_e \, ds \leq \tilde{C}\left(\|\mathbf{grad}\,\psi\|^2_{L^q(T)}+\|\psi\|^2_{W^1_q(T)}\right)^{\frac{1}{2}}\left(\|\mathbf{v}\|^2_{\boldsymbol{H}^s(T)}+\|\mathrm{div}\,\mathbf{v}\|^2_{\boldsymbol{L}^2(T)}\right)^{\frac{1}{2}}$$

for all $\mathbf{v}\in\boldsymbol{H}^s(T)\cap\boldsymbol{H}(\mathrm{div};T)$ and the assertion of the theorem, since ψ is fixed.

The importance of the interpolation operator Π_h is due to the *commuting diagram property* [15, Prop. 3.7]:

$$\mathrm{div}_\Gamma\Pi_h\mathbf{v} = Q_h\mathrm{div}_\Gamma\mathbf{v} \quad \forall\mathbf{v}\in\boldsymbol{H}(\mathrm{div};\Gamma)\cap\mathrm{Dom}(\Pi_h)\; , \tag{7}$$

where Q_h is the $L^2(\Gamma)$-orthogonal projection onto the space

$$\mathcal{Q}_0(\Gamma_h) := \{\mu\in L^2(\Gamma),\ \mu_{|T}=\text{const.},\ \forall T\in\mathcal{T}_h\}\; .$$

Identity (7) is a simple consequence of the definition of the degrees of freedom and Gauß' theorem applied to elements. An important consequence is that

$$\mathrm{div}_\Gamma\mathbf{v}=0 \quad\wedge\quad \mathbf{v}\in\mathrm{Dom}(\Pi_h) \qquad\Rightarrow\quad \mathrm{div}_\Gamma(\Pi_h\mathbf{v})=0\; .$$

It also reveals that $\mathrm{div}_\Gamma\boldsymbol{\mathcal{RT}}_0(\Gamma_h)=\mathcal{Q}_0(\Gamma_h)$.

Remark. The reader should be aware that we have restricted ourselves to lowest order Raviart-Thomas elements only for the sake of simplicity. All other $\boldsymbol{H}(\mathrm{div};\Omega)$-conforming finite elements in 2D that provide valid discrete 1-forms could be used as well. A rich collection is offered in [15, Sect. III.3]. All arguments in the sequel will carry over to these elements with only slight alterations.

The Raviart-Thomas elements form an affine family of finite elements in the sense of [24] with respect to Piola's transformation [15, §III.1.3]

$$\mathfrak{P}_T : \boldsymbol{L}^2(\widehat{T}) \mapsto \boldsymbol{L}^2_{\mathbf{t}}(T)\,, \quad \mathfrak{P}_T(\widehat{\mathbf{v}}_h)(\mathbf{x}) := |\det D\boldsymbol{\Phi}_T|^{-1}D\boldsymbol{\Phi}_T\widehat{\mathbf{v}}_h(\boldsymbol{\Phi}_T^{-1}(\mathbf{x})),\ \mathbf{x}\in T\;,$$

where $\widehat{T}$ is the reference triangle $\widehat{T}:=\{\mathbf{x}\in\mathbb{R}^2,\ x_1,x_2>0,\ x_1+x_2<1\}$, $T\in\mathcal{T}_h$, and $\boldsymbol{\Phi}_T$ the unique affine mapping that takes $\widehat{T}$ to T. The Piola transform preserves the values of degrees of freedom. Shape-regularity and quasi-uniformity guarantee that $|\det D\boldsymbol{\Phi}_T|\asymp h^2$ and $\|D\boldsymbol{\Phi}_T\|\asymp h$ uniformly in $T\in\mathcal{T}_h$ and $h\in\mathbb{H}$. Here and in the sequel, we are using the symbol $\asymp$ to indicate equivalence up to constant that may depend on Γ and the shape regularity of $\{\Gamma_h\}_h$, but is independent of h. The same should be true for all generic constants unless they bear a tilde. Now, using standard affine equivalence techniques, the effect of Piola's transform on fractional Sobolev norms can be controlled: The Piola transform $\mathfrak{P}_T$, $T\in\mathcal{T}_h$, satisfies for $0\leq s\leq 1$

$$|\widehat{\mathbf{u}}|_{\boldsymbol{H}^s(\widehat{T})} \asymp h^s\,|\mathfrak{P}_T\widehat{\mathbf{u}}|_{\boldsymbol{H}^s(T)} \quad \forall\widehat{\mathbf{u}}\in\boldsymbol{H}^s(\widehat{T})\;,$$

with constants only depending on the shape-regularity of T. See lemma 3 in [43] for the cases $s=0$ and $s=1$. The rest follows by interpolation.

Remark. Using Piola's transform one easily constructs parametric divergence-conforming surface elements [9, 31] for piecewise smooth Γ. Thus, our approach can be instantly extended to curved Lipschitz-polyhedra.

5 Hodge mapping

Coercivity of the sesqui-linear form related to (1) could only be established in the split space $\boldsymbol{\mathcal{G}}$ arising from the Hodge decomposition. This means that, though the boundary element spaces $\boldsymbol{\mathcal{RT}}_0(\Gamma_h)$ perfectly fit (1), theorem 3.1 gives no immediate information about the convergence of the Galerkin discretization. The reason is that we needed conforming finite element subspaces of both $\mathbf{X}$ and $\mathbf{N}$ to apply the usual results (cf. [45, Sect. 2.3]) about the convergence of Galerkin schemes for coercive variational problems.

A discrete $\boldsymbol{L}^2_{\mathbf{t}}(\Gamma)$-orthogonal Hodge decomposition

$$\boldsymbol{\mathcal{RT}}_0(\Gamma_h) = \mathbf{X}_h \oplus \mathbf{N}_h \ , \quad \mathbf{N}_h := \mathrm{Ker}(\mathrm{div}_\Gamma) \cap \boldsymbol{\mathcal{RT}}_0(\Gamma_h) \ , \tag{8}$$

yields $\mathbf{N}_h \subset \mathbf{N}$, but we cannot expect $\mathbf{X}_h \subset \mathbf{X}$. In short, $\mathbf{X}_h$ provides only a *non-conforming* discretization of $\mathbf{X}$. On the other hand, no modification of the sesqui-linear form $a(\cdot,\cdot)$ is necessary, if we decided to pose the variational problem (5) over $\boldsymbol{\mathcal{G}}_h := \mathbf{X}_h \times \mathbf{N}_h$. This is simply due to the fact that everything remains perfectly conforming in $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$. In particular, $\boldsymbol{\mathcal{G}}_h$ can be equipped with the norm $\|\cdot\|_{\boldsymbol{\mathcal{G}}}$. However, embedding and regularity properties of $\mathbf{X}$ are crucial and the space $\mathbf{X}_h$ lacks them. We deal with this by introducing semi-discrete spaces arising from the *continuous* Hodge decomposition of the discrete boundary element space: we split $\mathbf{v}_h \in \boldsymbol{\mathcal{RT}}_0(\Gamma_h)$ in two ways

$$\mathbf{v}_h = \begin{cases} \mathbf{v}_h^\perp + \mathbf{v}_h^0 \ , & \mathbf{v}_h^\perp \in \mathbf{X}_h,\ \mathbf{v}_h^0 \in \mathbf{N}_h \ , \\ \mathbf{v}^\perp + \mathbf{v}^0 \ , & \mathbf{v}^\perp \in \mathbf{X},\ \mathbf{v}^0 \in \mathbf{N} \ . \end{cases}$$

The discrete field $\mathbf{v}_h^\perp$ is the one realized in the computation, the *semidiscrete* field $\mathbf{v}^\perp$ has desirable properties. We have labeled it semi-discrete because $\mathrm{div}_\Gamma \mathbf{v}^\perp = \mathrm{div}_\Gamma \mathbf{v}_h$ is still piecewise constant and hence $\mathbf{v}^\perp$ still depends on the triangulation. To bridge the gap between $\mathbf{v}_h^\perp$ and $\mathbf{v}^\perp$ we need the following device (cf. Def. 4.1 in [37]): We define the *Hodge mapping* $H_h : \boldsymbol{\mathcal{RT}}_0(\Gamma_h) \mapsto \mathbf{X}$ by

$$H_h \mathbf{v}_h \in \mathbf{X} : \quad \mathrm{div}_\Gamma H_h \mathbf{v}_h := \mathrm{div}_\Gamma \mathbf{v}_h \quad , \quad \mathbf{v}_h \in \boldsymbol{\mathcal{RT}}_0(\Gamma_h) \ .$$

Owing to (3) this is a valid definition the Hodge mappings are uniformly continuous with respect to $h \in \mathbb{H}$. The Hodge mapping creates the desired link between $\mathbf{X}_h$ and $\mathbf{X}$ (cf. Lemma 4.2 in [37]): For any $s \geq -\frac{1}{2}$ the Hodge mapping satisfies the estimate

$$\|\mathbf{v}_h - H_h\mathbf{v}_h\|_{\boldsymbol{L}^2(\Gamma)} \leq Ch^r \; \|\mathrm{div}_\Gamma \mathbf{v}_h\|_{H^s(\Gamma)} \quad \forall \mathbf{v}_h \in \mathbf{X}_h \; .$$

with $0 \leq r \leq \min\{s+1, 1, s^*\}$ and constants only depending on s, r, Γ, and the shape-regularity of the surface triangulations. We follow the proof of lemma 4.2 from [37], pick $\mathbf{u}_h \in \boldsymbol{\mathcal{RT}}_0(\Gamma_h)$, and focus on an single triangle $T \in \Gamma_h$. Take $H_h\mathbf{u}_{h|T}$ to the reference element and set $\widehat{\mathbf{w}} := \mathfrak{P}_T^{-1} H_h\mathbf{u}_h$ By (2) $\widehat{\mathbf{w}} \in \boldsymbol{H}^t(\widehat{T})$, so that the assumptions of lemma 4 are satisfied and we have for any eligible t

$$\|\hat{\Pi}\widehat{\mathbf{w}}\|_{\boldsymbol{L}^2(\widehat{T})} \leq \tilde{C}(r) \left(\|\widehat{\mathbf{w}}\|_{\boldsymbol{H}^r(\widehat{T})} + \|\mathrm{div}\,\widehat{\mathbf{w}}\|_{L^2(\widehat{T})}\right) ,$$

where $\hat{\Pi}$ is the local interpolation operator on $\widehat{T}$. Remember that $\mathrm{div}_\Gamma H_h\mathbf{u}_h$ is piecewise constant, which also renders $\mathrm{div}\,\widehat{\mathbf{w}}$ constant. Exploiting the equivalence of all norms on finite dimensional spaces, we can easily bound $\|\mathrm{div}\,\widehat{\mathbf{w}}\|_{L^2(\widehat{T})}$ and arrive at

$$\|\hat{\Pi}\widehat{\mathbf{w}}\|_{\boldsymbol{L}^2(\widehat{T})} \leq \tilde{C}(r) \, \|\widehat{\mathbf{w}}\|_{\boldsymbol{H}^r(\widehat{T})} \; .$$

Constant vectorfields on $\widehat{T}$ are preserved by the interpolation $\hat{\Pi}$. Thus, for any $\mathbf{p} \in \mathbb{R}^2$

$$\|\widehat{\mathbf{w}} - \hat{\Pi}\widehat{\mathbf{w}}\|_{\boldsymbol{L}^2(\widehat{T})} = \|\widehat{\mathbf{w}} - \mathbf{p} - \hat{\Pi}(\widehat{\mathbf{w}} - \mathbf{p})\|_{\boldsymbol{L}^2(\widehat{T})} \leq \|\widehat{\mathbf{w}} - \mathbf{p}\|_{\boldsymbol{L}^2(\widehat{T})} + \tilde{C}(r)\, \|\widehat{\mathbf{w}} - \mathbf{p}\|_{H^r(\widehat{T})} \; . \blacksquare$$

From the definition of the fractional Sobolev norm [33, Def. 1.3.2.1] and $0 \leq r \leq 1$, it is immediate that

$$\|\widehat{\mathbf{w}} - \mathbf{p}\|^2_{H^r(\widehat{T})} = \|\widehat{\mathbf{w}} - \mathbf{p}\|^2_{\boldsymbol{L}^2(\widehat{T})} + |\widehat{\mathbf{w}}|^2_{H^r(\widehat{T})} \; .$$

As, according to Prop. 6.1 in [32], a Bramble-Hilbert-type estimate of the form

$$\inf_{c \in \mathbb{R}} \|f - c\|_{L^2(\widehat{T})} \leq \tilde{C}(r)\, |f|_{H^r(\widehat{T})} \quad \forall f \in H^r(\widehat{T}) \; .$$

also holds in fractional Sobolev spaces, we end up with the estimate

$$\|\widehat{\mathbf{w}} - \hat{\Pi}\widehat{\mathbf{w}}\|_{\boldsymbol{L}^2(\widehat{T})} \leq \tilde{C}(r)\, |\widehat{\mathbf{w}}|_{H^r(\widehat{T})} \; .$$

Since interpolation and the Piola transform commute, we may use lemma 4 to pull the estimate back to element T

$$\|H_h\mathbf{u}_h - \Pi_h H_h\mathbf{u}_h\|_{\boldsymbol{L}^2(T)} \leq Ch^r \, \|H_h\mathbf{u}_h\|_{\boldsymbol{H}^r(T)} \; .$$

At this stage shape-regularity starts affecting the constants. Squaring and summing up over all elements yields

$$\|H_h\mathbf{u}_h - \Pi_h H_h\mathbf{u}_h\|_{\boldsymbol{L}^2(\Gamma)} \leq Ch^r \, \|H_h\mathbf{u}_h\|_{\boldsymbol{H}^r(\Gamma)} \; ,$$

which, in light of lemma 2, involves

$$\|H_h\mathbf{u}_h - \Pi_h H_h\mathbf{u}_h\|_{\boldsymbol{L}^2(\Gamma)} \leq Ch^r \|\mathrm{div}_\Gamma \mathbf{u}_h\|_{H^s(\Gamma)} \ . \tag{9}$$

By the commuting diagram property of Π_h we conclude from $\mathrm{div}_\Gamma(\mathbf{v}_h - H_h\mathbf{v}_h) = 0$ that also $\mathrm{div}_\Gamma(\mathbf{v}_h - \Pi_h H_h\mathbf{v}_h) = 0$. This means $\mathbf{v}_h - \Pi_h H_h\mathbf{v}_h \in \mathbf{N}_h$ and makes it possible for us to apply Nedelec's trick [42, Sect. 3.3]

$$\begin{aligned}\|\mathbf{u}_h - H_h\mathbf{u}_h\|^2_{\boldsymbol{L}^2(\Gamma)} &= (\mathbf{u}_h - H_h\mathbf{u}_h, \mathbf{u}_h - \Pi_h H_h\mathbf{u}_h + \Pi_h H_h\mathbf{u}_h - H_h\mathbf{u}_h)_{0;\Gamma} = \\ &= (\mathbf{u}_h - H_h\mathbf{u}_h, \Pi_h H_h\mathbf{u}_h - H_h\mathbf{u}_h)_{0;\Gamma} \ .\end{aligned}$$

Together with (9) this shows the assertion of the lemma.

Now, we **fix** $t := \min\{\frac{1}{2}, s^*\}$ and keep it constant for the remainder of this paper. A legal choice for r in the previous lemma is $r = t$ for $s = -\frac{1}{2}$ and we denote the associated constant by C_3. The decomposition $\boldsymbol{\mathcal{RT}}_0(\Gamma_h) = \mathbf{X}_h \oplus \mathbf{N}_h$ is uniformly $\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)$-stable. For $\mathbf{u}_h \in \mathbf{X}_h$ we can use the Hodge projection and the previous lemma to estimate

$$\|\mathbf{u}_h\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq \|\mathbf{u}_h - H_h\mathbf{u}_h\|_{\boldsymbol{L}^2(\Gamma)} + \|H_h\mathbf{u}_h\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq C(h^t + 1) \|\mathrm{div}_\Gamma \mathbf{u}_h\|_{H^{-\frac{1}{2}}(\Gamma)} \ ,$$

as $H_h\mathbf{u}_h \in \mathbf{X}$. Since $\mathrm{div}_\Gamma H_h\mathbf{u}_h = \mathrm{div}_\Gamma \mathbf{u}_h$ and $\mathbb{H}$ is bounded, the proof is finished.

We shall also require the following right inverse of the Hodge mapping. We define the linear continuous mapping $T_h : \mathbf{X} \mapsto \mathbf{X}_h$ by

$$T_h\mathbf{w} \in \mathbf{X}_h : \quad \mathrm{div}_\Gamma T_h\mathbf{w} = Q_{-\frac{1}{2}} \mathrm{div}_\Gamma \mathbf{w} \quad \forall \mathbf{w} \in \mathbf{X} \ ,$$

where $Q_{-\frac{1}{2}} : H^{-\frac{1}{2}}(\Gamma) \mapsto \mathcal{Q}_0(\Gamma_h)$ is the $H^{-\frac{1}{2}}(\Gamma)$-orthogonal projection.

Note that only due to the preceding stability result this definition makes real sense. Besides, lemma 5 guarantees that the family of operators $(T_h)_{h\in\mathbb{H}}$ is uniformly continuous, as

$$\|T_h\mathbf{w}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq C \|\mathrm{div}_\Gamma T_h\mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C \|\mathrm{div}_\Gamma \mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} \ .$$

For fixed $\mathbf{w} \in \mathbf{X}$ we have

$$\lim_{h\to 0} \|\mathbf{w} - T_h\mathbf{w}\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma, \Gamma)} = 0 \ .$$

We resort to the same trick as in the proof of lemma 5 and use $H_hT_h\mathbf{w} - \mathbf{w} \in \mathbf{X}$

$$\begin{aligned}\|T_h\mathbf{w} - \mathbf{w}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} &\leq \|H_hT_h\mathbf{w} - T_h\mathbf{w}\|_{\boldsymbol{L}^2(\Gamma)} + \|H_hT_h\mathbf{w} - \mathbf{w}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \\ &\leq C_3 h^t \|\mathrm{div}_\Gamma T_h\mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} + \tilde{C} \|\mathrm{div}_\Gamma(H_hT_h\mathbf{w} - \mathbf{w})\|_{H^{-\frac{1}{2}}(\Gamma)} \\ &\leq Ch^t \|\mathrm{div}_\Gamma \mathbf{w}\|_{H^{-\frac{1}{2}}(\Gamma)} + \tilde{C} \inf_{\mu_h \in \mathcal{Q}_0(\Gamma_h)} \|\mathrm{div}_\Gamma \mathbf{w} - \mu_h\|_{H^{-\frac{1}{2}}(\Gamma)} \ .\end{aligned}$$

As $\bigcup_{h\in\mathbb{H}} \mathcal{Q}_0(\Gamma_h)$ is dense in $L^2(\Gamma)$, which is dense in $H^{-\frac{1}{2}}(\Gamma)$, the lemma holds true.

6 Stability of the Galerkin scheme

Galerkin discretization of (5) leads to the discrete variational problem: Seek $\boldsymbol{\iota}_h \in \boldsymbol{\mathcal{G}}_h$ such that

$$a(\boldsymbol{\iota}_h, \boldsymbol{\eta}_h) = f(\boldsymbol{\eta}_h) \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{\mathcal{G}}_h \ . \tag{10}$$

From theorem 3.1 we saw that problem (5) is strongly elliptic, i.e. $a = d + k$, with a $\boldsymbol{\mathcal{G}}$-elliptic sesqui-linear form d and a $\boldsymbol{\mathcal{G}}$-compact form k. Discretization of (5) by a dense family of finite dimensional subspaces would therefore imply quasioptimal asymptotic convergence of the approximate solutions. The problem here is that $\boldsymbol{\mathcal{G}}_h$ is a truly non-conforming approximation space, i.e $\boldsymbol{\mathcal{G}}_h \not\subset \boldsymbol{\mathcal{G}}$. Therefore, coercivity in the discrete setting must be established by a separate argument. For the proof, we draw on an idea of A. Schatz [44].

To get compact formulas, we replace bilinear forms by the associated Riesz operators. First, $A : \boldsymbol{\mathcal{G}} \mapsto \boldsymbol{\mathcal{G}}'$ is associatd to the sesqui-linear form a. Next, the operator $K : \boldsymbol{\mathcal{G}} \mapsto \boldsymbol{\mathcal{G}}'$ is associated with the sesqui-linear form k defined in the proof of theorem 3.1. Both operators are continuous from $\boldsymbol{\mathcal{G}} \mapsto \boldsymbol{\mathcal{G}}'$. However, since $\boldsymbol{\mathcal{G}}_h$ is non-conforming, these operators are not defined on $\boldsymbol{\mathcal{G}}_h$ a priori.

We will use Hodge mappings on $\boldsymbol{\mathcal{G}}_h$ which are defined through

$$\mathbf{H}_h : \boldsymbol{\mathcal{G}}_h \mapsto \boldsymbol{\mathcal{G}} \quad , \quad \mathbf{H}_h(\mathbf{v}_h^{\perp}, \mathbf{v}_h^0) := (H_h \mathbf{v}_h^{\perp}, \mathbf{v}_h^0) \in \boldsymbol{\mathcal{G}} \, , \quad (\mathbf{v}_h^{\perp}, \mathbf{v}_h^0) \in \boldsymbol{\mathcal{G}}_h \ .$$

Lemma 5 ensures the uniform boundedness in h of this family of operators. We also define the extension $\mathbf{T}_h : \boldsymbol{\mathcal{G}} \mapsto \boldsymbol{\mathcal{G}}_h$ of the right inverse of the Hodge map T_h, $h \in \mathbb{H}$ in Definition 5 to $\boldsymbol{\mathcal{G}}$:

$$\mathbf{T}_h(\mathbf{v}^{\perp}, \mathbf{v}^0) := (T_h \mathbf{v}^{\perp}, \mathbf{Q}_h^{-\frac{1}{2}} \mathbf{v}^0) \in \boldsymbol{\mathcal{G}}_h \quad (\mathbf{v}^{\perp}, \mathbf{v}^0) \in \boldsymbol{\mathcal{G}} \ ,$$

where $\mathbf{Q}_h^{-\frac{1}{2}}$ is the $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$-orthogonal projection $\mathbf{N} \mapsto \mathbf{N}_h$. The operator $\mathbf{T}_h$ is well defined, since $\mathbf{N}_h \subset \mathbf{N}$ and $\mathbf{N}$ is a closed subspace of $\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)$. Density of $\bigcup_{h \in \mathbb{H}} \mathbf{N}_h$ in $\mathbf{N}$ and lemma 5 confirm that

$$\lim_{h \to 0} \|\boldsymbol{\varphi} - \mathbf{T}_h \boldsymbol{\varphi}\|_{\boldsymbol{\mathcal{G}}} = 0 \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}} \ . \tag{11}$$

Next, we consider the operator $S : \boldsymbol{\mathcal{G}}' \mapsto \boldsymbol{\mathcal{G}}$ defined as the solution operator of the $\boldsymbol{\mathcal{G}}$-elliptic variational problem

$$d(S\boldsymbol{\eta}', \boldsymbol{\varphi}) = \boldsymbol{\eta}'(\boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}}, \quad \boldsymbol{\eta}' \in \boldsymbol{\mathcal{G}}' \ .$$

Continuity and ellipticity of the sesqui-linear form d give

$$\tilde{C}_d^{-1} \|\boldsymbol{\eta}'\|_{\boldsymbol{\mathcal{G}}'} \leq \|S\boldsymbol{\eta}'\|_{\boldsymbol{\mathcal{G}}} \leq \tilde{c}_d^{-1} \|\boldsymbol{\eta}'\|_{\boldsymbol{\mathcal{G}}'} \quad \forall \boldsymbol{\eta}' \in \boldsymbol{\mathcal{G}}' \ , \tag{12}$$

where $C_d := \|d\|$. Note that also the operator S is confined to the continuous setting. There is a function $b : \mathbb{H} \mapsto \mathbb{R}^+$ with $b(h) \to 0$ as $h \to 0$ such that

$$\|(\mathbf{T}_h - Id)SK\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{G}}} \leq b(h) \, \|\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{G}}} \quad \forall \boldsymbol{\eta} \in \boldsymbol{\mathcal{G}} \ .$$

Set $B_1(\boldsymbol{\mathcal{G}}) := \{\boldsymbol{\varphi} \in \boldsymbol{\mathcal{G}} : \|\boldsymbol{\varphi}\|_{\boldsymbol{\mathcal{G}}} \leq 1\}$. As $K : \boldsymbol{\mathcal{G}} \mapsto \boldsymbol{\mathcal{G}}'$ is compact, the set $KB_1(\boldsymbol{\mathcal{G}})$ is precompact in $\boldsymbol{\mathcal{G}}'$. Thanks to the continuity of S the closure w.r.t. the $\|\cdot\|_{\boldsymbol{\mathcal{G}}}$-norm

$$M := \overline{SKB_1(\boldsymbol{\mathcal{G}})}$$

is compact. Pick some $\epsilon > 0$ and write $B_\epsilon(\boldsymbol{\nu})$ for the ϵ-neighborhood of $\boldsymbol{\nu}$ in $\boldsymbol{\mathcal{G}}$. We can find finitely many $\boldsymbol{\nu}_1, \ldots, \boldsymbol{\nu}_L$, $L = L(\epsilon) \in \mathbb{N}$, in M such that $M \subset \bigcup_l B_\epsilon(\boldsymbol{\nu}_l)$. From (11) we learn that there is $h_0 = h_0(\epsilon) \in \mathbb{H}$ such that

$$\|\mathbf{T}_h \boldsymbol{\nu}_l - \boldsymbol{\nu}_l\|_{\boldsymbol{\mathcal{G}}} \leq \epsilon \quad \forall h < h_0,\ l = 1, \ldots, L\ .$$

For any $\boldsymbol{\eta} \in M$ there exists an $\boldsymbol{\nu}_l$ such that $\boldsymbol{\eta} \in B_\epsilon(\boldsymbol{\nu}_l)$. Hence

$$\|\mathbf{T}_h\boldsymbol{\eta} - \boldsymbol{\eta}\|_{\boldsymbol{\mathcal{G}}} \leq \|\mathbf{T}_h\boldsymbol{\eta} - \mathbf{T}_h\boldsymbol{\nu}_l\|_{\boldsymbol{\mathcal{G}}} + \|\mathbf{T}_h\boldsymbol{\nu}_l - \boldsymbol{\nu}_l\|_{\boldsymbol{\mathcal{G}}} + \|\boldsymbol{\nu}_l - \boldsymbol{\eta}\|_{\boldsymbol{\mathcal{G}}} \leq (\|\mathbf{T}_h\|_{\boldsymbol{\mathcal{G}}\mapsto\boldsymbol{\mathcal{G}}} + 2)\epsilon\ ,$$

if $h < h_0$. Undoing the substitutions, we get

$$\|(\mathbf{T}_h - Id)SK\boldsymbol{\eta}\|_{\boldsymbol{\mathcal{G}}} \leq (\|\mathbf{T}_h\|_{\boldsymbol{\mathcal{G}}\mapsto\boldsymbol{\mathcal{G}}} + 2)\epsilon \quad \forall \boldsymbol{\eta} \in B_1(\boldsymbol{\mathcal{G}}),\ h < h_0\ .$$

A homogeneity argument finishes the proof.

We prove the discrete inf-sup condition for the form $a(\cdot,\cdot)$. For any $\boldsymbol{\eta}_h \in \boldsymbol{\mathcal{G}}_h$ we set

$$\boldsymbol{\varphi}_h := (Id - \mathbf{T}_h SK\mathbf{H}_h)\boldsymbol{\eta}_h \in \boldsymbol{\mathcal{G}}_h.$$

The uniform boundedness with respect to h of the operators involved ensures that there is $C_4 > 0$ independent of $h \in \mathbb{H}$ and $\boldsymbol{\eta}_h$ such that

$$\|\boldsymbol{\varphi}_h\|_{\boldsymbol{\mathcal{G}}} \leq C_4 \|\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{G}}}\ . \tag{13}$$

We therefore estimate

$$\begin{aligned}
|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)| &= |a(\boldsymbol{\eta}_h, (Id - \mathbf{T}_h SK\mathbf{H}_h)\boldsymbol{\eta}_h)| \\
&= |a(\boldsymbol{\eta}_h, ((Id - \mathbf{T}_h)(SK\mathbf{H}_h) + (Id - SK\mathbf{H}_h))\boldsymbol{\eta}_h)| \\
&\geq |a(\boldsymbol{\eta}_h, (Id - SK\mathbf{H}_h)\boldsymbol{\eta}_h)| - |a(\boldsymbol{\eta}_h, (Id - \mathbf{T}_h)SK\mathbf{H}_h\boldsymbol{\eta}_h)| \\
&\geq |a(\boldsymbol{\eta}_h, (Id - SK\mathbf{H}_h)\boldsymbol{\eta}_h)| - \tilde{C}_a \|\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{G}}} \|(Id - \mathbf{T}_h)SK\mathbf{H}_h\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{G}}} \\
&\geq |a(\boldsymbol{\eta}_h, (Id - SK\mathbf{H}_h)\boldsymbol{\eta}_h)| - b(h)\tilde{C}_a \|\boldsymbol{\eta}_h\|^2_{\boldsymbol{\mathcal{G}}}
\end{aligned}$$

the final inequality being a consequence of lemma 6. We further estimates the first term

$$\begin{aligned}
|a(\boldsymbol{\eta}_h, (Id - SK\mathbf{H}_h)\boldsymbol{\eta}_h)| &= |a(\boldsymbol{\eta}_h, ((Id - \mathbf{H}_h) + (Id - SK)\mathbf{H}_h)\boldsymbol{\eta}_h)| \\
&\geq |a(\boldsymbol{\eta}_h, (Id - SK)\mathbf{H}_h)\boldsymbol{\eta}_h)| - |a(\boldsymbol{\eta}_h, (Id - \mathbf{H}_h)\boldsymbol{\eta}_h)| \\
&\geq |a(\boldsymbol{\eta}_h, S(S^{-1} - K)\mathbf{H}_h\boldsymbol{\eta}_h)| - \tilde{C}_a \|\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{G}}} \|(Id - \mathbf{H}_h)\boldsymbol{\eta}_h\|_{\boldsymbol{\mathcal{G}}} \\
&\geq |a(\boldsymbol{\eta}_h, S(S^{-1} - K)\mathbf{H}_h\boldsymbol{\eta}_h)| - \tilde{C}_a C_3 h^t \|\boldsymbol{\eta}_h\|^2_{\boldsymbol{\mathcal{G}}}\ ,
\end{aligned}$$

by lemma 5.

Now we note that $\boldsymbol{\psi} := S(S^{-1}-K)\boldsymbol{\lambda} \in \boldsymbol{\mathcal{G}}$, $\boldsymbol{\lambda} \in \boldsymbol{\mathcal{G}}$, satisfies !!! VORZEICHEN : $a = d + k$ Kann durch geeignete Wahl von φ_h erreicht werden ohne Argument zu aendern.

$$d(\boldsymbol{\psi}, \boldsymbol{\nu}) = \left\langle (S^{-1} - K)\boldsymbol{\lambda}, \boldsymbol{\nu} \right\rangle = d(\boldsymbol{\lambda}, \boldsymbol{\nu}) - k(\boldsymbol{\lambda}, \boldsymbol{\nu}) = a(\boldsymbol{\lambda}, \boldsymbol{\nu})$$

for all $\boldsymbol{\nu} \in \boldsymbol{\mathcal{G}}$. In short, $S(S^{-1} - K) = SA$. This enables us to continue the estimates

$$\begin{aligned} |a(\boldsymbol{\eta}_h, S(S^{-1} - K)\mathbf{H}_h\boldsymbol{\eta}_h)| &= |a(\boldsymbol{\eta}_h - \mathbf{H}_h\boldsymbol{\eta}_h + \mathbf{H}_h\boldsymbol{\eta}_h, SA\mathbf{H}_h\boldsymbol{\eta}_h)| \\ &\geq |a(\mathbf{H}_h\boldsymbol{\eta}_h, SA\mathbf{H}_h\boldsymbol{\eta}_h)| - \tilde{C}_a \left\|(Id - \mathbf{H}_h)\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}} \left\|SA\mathbf{H}_h\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}} \\ &\geq |d(SA\mathbf{H}_h\boldsymbol{\eta}_h, SA\mathbf{H}_h\boldsymbol{\eta}_h)| - \tilde{C}_a^2\tilde{c}_d^{-1}C_3h^t \left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 . \end{aligned}$$

For the last time we target the first term

$$\begin{aligned} |d(SA\mathbf{H}_h\boldsymbol{\eta}_h, SA\mathbf{H}_h\boldsymbol{\eta}_h)| &\geq \tilde{c}_d \left\|SA\mathbf{H}_h\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 \geq \tilde{c}_d\tilde{C}_d^{-1} \left\|A\mathbf{H}_h\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}'}^2 \\ &\geq \tilde{c}_d\tilde{C}_d^{-1}\tilde{c}_a \left\|\boldsymbol{\eta}_h - (Id - \mathbf{H}_h)\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 \\ &\geq \tilde{c}_4(\left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 - \left\|(Id - \mathbf{H}_h)\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2) \\ &\geq \tilde{c}_4 \left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 - \tilde{c}_4C_3h^t \left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 , \end{aligned}$$

with $\tilde{c}_4 := \tilde{c}_d\tilde{C}_d^{-1}\tilde{c}_a$. Summing up, we have obtained

$$|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)| \geq \left(\tilde{c}_4 - (\tilde{c}_4 + \tilde{C}_a^2\tilde{c}_d^{-1} + \tilde{C}_a)C_3h^t - \tilde{C}_a b(h)\right) \left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 .$$

If $h < h_*$ with $(\tilde{c}_4 + \tilde{C}_a^2\tilde{c}_d^{-1} + \tilde{C}_a)C_3h_*^t + \tilde{C}_a b(h_*) < \frac{1}{2}\tilde{c}_4$, we obtain the lower bound

$$|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)| \geq \frac{1}{2}\tilde{c}_4 \left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}}^2 \quad \forall h < h_* .$$

This is valid for any $\boldsymbol{\eta}_h$. Recalling (13), an immediate consequence is the discrete inf-sup condition

$$\sup_{\boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{G}}_h} \frac{|a(\boldsymbol{\eta}_h, \boldsymbol{\varphi}_h)|}{\left\|\boldsymbol{\varphi}_h\right\|_{\boldsymbol{\mathcal{G}}}} \geq \frac{\tilde{c}_4}{2C_4} \left\|\boldsymbol{\eta}_h\right\|_{\boldsymbol{\mathcal{G}}} \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{\mathcal{G}}_h,\ h < h_*$$

Based on this discrete stability condition, the stability of the continuous problem and the continuity of the bilinear forms involved, we obtain quasioptimality of the sequence of Galerkin solutions.

Theorem 6.1 *There exists a constant $C > 0$ only depending on Γ, ς, and the shape-regularity of the triangulations Γ_h such that the discrete problem (10) has a unique solution $\boldsymbol{\iota}_h$ and the family $\{\boldsymbol{\iota}_h\}_h$ converges quasioptimally:*

$$\left\|\boldsymbol{\iota} - \boldsymbol{\iota}_h\right\|_{\boldsymbol{\mathcal{G}}} \leq C \inf_{\boldsymbol{\varphi}_h \in \boldsymbol{\mathcal{G}}_h} \left\|\boldsymbol{\iota} - \boldsymbol{\varphi}_h\right\|_{\boldsymbol{\mathcal{G}}} ,$$

provided that $h < h_*$ with a sufficiently small h_*.

7 Convergence Rates

Eventually, we are interested in getting a convergence estimate depending on the smoothness of the continuous solution $\mathbf{j}$ of (1) only. The next lemma is a first step towards this goal. If $\mathbf{j} \in \boldsymbol{H}^{\sigma}(\mathrm{div}_{\Gamma};\Omega)$, $0 < \sigma$, and $h < h_*$, then

$$\|\boldsymbol{\iota} - \boldsymbol{\iota}_h\|_{\boldsymbol{\mathcal{G}}} \leq C\Big(\|\mathbf{j} - \Pi_h \mathbf{j}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} + h^r \|\mathrm{div}_{\Gamma}\mathbf{j}\|_{H^{\sigma}(\Gamma)} \Big)$$

with $r = \min\{1, s^*\}$. Lemma 4 tells us that $\mathbf{j}$ is sufficiently regular to render $\Pi_h\mathbf{j}$ well defined. Then we take a look at the Hodge decompositions

$$\mathbf{j} = \mathbf{j}^{\perp} + \mathbf{j}^{0} \ , \quad \mathbf{j}^{\perp} \in \mathbf{X}, \mathbf{j}^{0} \in \mathbf{N} \quad \text{and} \quad \Pi_h\mathbf{j} = \mathbf{v}_h^{\perp} + \mathbf{v}_h^{0} \ , \quad \mathbf{v}_h^{\perp} \in \mathbf{X}_h, \ \mathbf{v}_h^{0} \in \mathbf{N}_h \ .$$

The commuting diagram property (7) yields, with a constant merely depending on the shape regularity of $\{\Gamma_h\}_h$,

$$\begin{aligned}\left\|\mathrm{div}_{\Gamma}\mathbf{j}^{\perp} - \mathrm{div}_{\Gamma}\mathbf{v}_h^{\perp}\right\|_{H^{-\frac{1}{2}}(\Omega)} = \|\mathrm{div}_{\Gamma}\mathbf{j} - \mathrm{div}_{\Gamma}\Pi_h\mathbf{j}\|_{H^{-\frac{1}{2}}(\Omega)} &= \|(Id - Q_h)\mathrm{div}_{\Gamma}\mathbf{j}\|_{H^{-\frac{1}{2}}(\Omega)} \\ &\leq \ Ch^{\min\{\frac{3}{2},\frac{1}{2}+\sigma\}} \|\mathrm{div}_{\Gamma}\mathbf{j}\|_{H^{\sigma}(\Gamma)} \ .\end{aligned}$$

This is a consequence of approximation estimates for the $L^2(\Gamma)$-orthogonal projections Q_h in negative norms, which can be verified by duality techniques. We rely on the above estimate and lemma 5 to get

$$\begin{aligned}\left\|\mathbf{j}^{\perp} - \mathbf{v}_h^{\perp}\right\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} &\leq \left\|\mathbf{j}^{\perp} - H_h\mathbf{v}_h^{\perp}\right\|_{\boldsymbol{L}^2(\Gamma)} + \left\|H_h\mathbf{v}_h^{\perp} - \mathbf{v}_h^{\perp}\right\|_{\boldsymbol{L}^2(\Gamma)} \\ &\leq \tilde{C} \left\|\mathrm{div}_{\Gamma}\mathbf{j}^{\perp} - \mathrm{div}_{\Gamma}\mathbf{v}_h^{\perp}\right\|_{H^{-\frac{1}{2}}(\Omega)} + Ch^r \|\mathrm{div}_{\Gamma}\mathbf{j}\|_{H^{\sigma}(\Gamma)} \\ &\leq \ Ch^r \|\mathrm{div}_{\Gamma}\mathbf{j}\|_{H^{\sigma}(\Gamma)} \ .\end{aligned}$$

In the course of the estimates both (3) and lemma 5 have been used. By the triangle inequality

$$\begin{aligned}\left\|\mathbf{j}^{0} - \mathbf{v}_h^{0}\right\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} &\leq \|\mathbf{j} - \Pi_h\mathbf{j}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} + \left\|\mathbf{j}^{\perp} - \mathbf{v}_h^{\perp}\right\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \\ &\leq \|\mathbf{j} - \Pi_h\mathbf{j}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} + Ch^r \|\mathrm{div}_{\Gamma}\mathbf{j}\|_{H^{\sigma}(\Gamma)} \ .\end{aligned}$$

Taking into account theorem 6.1 and the definition of the graph norm $\|\cdot\|_{\boldsymbol{\mathcal{G}}}$ the assertion follows.

Unfortunately, plain interpolation error estimates based on affine equivalence techniques and a Bramble-Hilbert type result (cf. proof of lemma 5) leave us with the suboptimal estimate

$$\|\mathbf{j} - \Pi_h\mathbf{j}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq \ h^{\min\{1,\sigma\}} \|\mathbf{j}\|_{\boldsymbol{H}^{\sigma}(\Gamma)} \ ,$$

because interpolation thwarts duality estimates. An improvement is only possible, if we can lift interpolation off the boundary. This by no means wishful thinking, because $\mathbf{j}$ is the jump $[\gamma_{\mathbf{t}}\mathbf{H}]_{\Gamma}$ of the tangential traces of magnetic field solutions of Maxwell boundary value problems in Ω and its complement $\Omega' := \mathbb{R}^3 \setminus \Omega$. If the source term (incident wave) has has minimal smoothness, regularity theory for solutions of Maxwell's equations [28] shows that

$$\mathbf{H} \in \boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega \cup \Omega') \quad , \quad \mathbf{curl}\,\mathbf{H} \in \boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega \cup \Omega') \ , \qquad (14)$$

for some $\sigma > 0$, where

$$\boldsymbol{H}^{\sigma}(\mathbf{curl}; \Omega) := \{\mathbf{V} \in \boldsymbol{H}^{\sigma}_{\mathrm{loc}}(\Omega),\ \mathbf{curl}\,\mathbf{V} \in \boldsymbol{H}^{\sigma}_{\mathrm{loc}}(\Omega)\} \ .$$

Prerequisite to exploiting the information about $\mathbf{H}$ is an "extension" of the surface triangulations. We call the family of surface meshes $\{\Gamma_h\}_{h\in\approx}$ *extensible*, if there is $R > 0$ such that Ω is contained in a cube B_R with diameter R and a family of tetrahedral meshes $\{\Omega_h\}_{h\in\mathbb{H}}$ covering B_R that satisfy

- $\{\Omega_h\}_h$ is shape-regular and quasi-uniform, and h retains its meaning as the meshwidth of Ω_h.
- Γ_h is composed of those simplices of Ω_h that are located on Γ.

Extensibility is not far-fetched, considering practical ways to obtain a family $\{\Gamma_h\}_h$: Whenever the meshes Γ_h are created by regular refinement of some coarse initial mesh, an extensible family will naturally emerge. This property makes it possible to switch to three-dimensional interpolation by means of Ndélec's edge elements [42] temporarily. If $\{\Gamma_h\}_{h\in\mathbb{H}}$ is extensible and $\mathbf{j} := \gamma'_{\mathbf{t}}\mathbf{H}' - \gamma_{\mathbf{t}}\mathbf{H}$, for magnetic fields $\mathbf{H} \in \boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega)$, $\mathbf{H}' \in \boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; B_R \setminus \Omega)$, $0 < \sigma \leq \frac{1}{2}$, then

$$\|\mathbf{j} - \Pi_h\mathbf{j}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \leq Ch^{\frac{1}{2}+\sigma}\Big(\|\mathbf{H}\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl};\Omega)} + \|\mathbf{H}'\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl};B_R\setminus\Omega)}\Big)$$

We are going to write Ω_h for the meshes obtained by extending Γ_h into Ω, and Ω'_h for the exterior extensions into $\Omega'_R := \Omega' \cap B_R$. Recall Nédélecs lowest order **curl**-conforming elements on tetrahedral, the so-called edge elements. We refer to [13, 25, 42] for the definition of the local spaces. The resulting global spaces have feature tangential continuity across inter-element faces and will be denoted by $\boldsymbol{\mathcal{ND}}_1(\Omega_h)$ and $\boldsymbol{\mathcal{ND}}_1(\Omega'_h)$. Suitable degrees of freedom are given by path integrals along the edges of the triangulations. This defines nodal interpolation operators $\Theta_h : \boldsymbol{C}^{\infty}(\Omega) \mapsto \boldsymbol{\mathcal{ND}}_1(\Omega_h)$ and $\Theta'_h : \boldsymbol{C}^{\infty}(\Omega)\Omega'_R \mapsto \boldsymbol{\mathcal{ND}}_1(\Omega'_h)$. As explained in [34], there holds

$$\gamma_{\mathbf{t}}\Theta_h\mathbf{H} = \Pi_h\gamma_{\mathbf{t}}\mathbf{H} \ , \quad \forall \mathbf{H} \in \boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega) \ . \qquad (15)$$

Next, Lemma 4.7 from [5] teaches that edge element interpolation is well defined for vectorfields in $\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega)$ and $\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl}; \Omega'_R)$. A fundamental trace theorem for $\boldsymbol{H}(\mathbf{curl}; \Omega)$ (Theorem 4.1 of [20]) states that $\gamma_{\mathbf{t}}$:

$\boldsymbol{H}(\mathbf{curl};\Omega\cup\Omega'_R) \mapsto \boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)$, is continuous with a continuous right inverse. Combined with $\mathbf{j} = \gamma'_{\mathbf{t}}\mathbf{H}' - \gamma_{\mathbf{t}}\mathbf{H}$ and (15), this means

$$\begin{aligned}\|\mathbf{j}-\Pi_h\mathbf{j}\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} &= \|\gamma'_{\mathbf{t}}\mathbf{H}' - \gamma_{\mathbf{t}}\mathbf{H} - (\gamma'_{\mathbf{t}}\Theta'_h\mathbf{H}' - \gamma_{\mathbf{t}}\Theta_h\mathbf{H})\|_{\boldsymbol{H}_{||}^{-\frac{1}{2}}(\Gamma)} \\ &\leq C\left(\|\mathbf{H}-\Theta_h\mathbf{H}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} + \|\mathbf{H}-\Theta'_h\mathbf{H}\|_{\boldsymbol{H}(\mathbf{curl};\Omega'_R)}\right) .\end{aligned}$$

The last step is based on an estimate for the interpolation error in edge element space [25, Lemma 3.2]

$$\|\mathbf{H}-\Theta_h\mathbf{H}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} \leq Ch^{\frac{1}{2}+\sigma}\,\|\mathbf{H}\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl};\Omega)} .$$

It is applied to both Ω and Ω'_R.

Recall that according to [20, Formula (29)]

$$\mathrm{div}_\Gamma(\mathbf{j}) = \mathbf{curl}\,\mathbf{H}\cdot\mathbf{n} - \mathbf{curl}\,\mathbf{H}'\cdot\mathbf{n} .$$

Now, we appeal to (14) and standard trace theorems for Sobolev spaces, and see that $\mathrm{div}_\Gamma\mathbf{j} \in H^\sigma(\Gamma)$. Thus, we can merge Lemmata 7 and 7 into the final convergence result.

Theorem 7.1 *Assume the a σ-regularity, $\sigma > 0$, according to* (14) *for the interior and exterior magnetic field solutions of Maxwell's equations subject to some excitation. The family of triangular surface meshes $\{\Gamma_h\}_{h\in\mathbb{H}}$ with meshwidths h is to be shape-regular, quasi-uniform, and extensible. Then there is $h_* > 0$ such that for $r := \min\{1, s^*\}$ and all $h < h_*$*

$$\begin{aligned}\|\mathbf{j}-\mathbf{j}_h\|_{\boldsymbol{H}^{-\frac{1}{2}}(\mathrm{div}_\Gamma,\Gamma)} \leq C\Big(& h^{\min\{1,\frac{1}{2}+\sigma\}}\big(\|\mathbf{H}\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl};\Omega)} + \|\mathbf{H}'\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\mathbf{curl};\Omega'_R)}\big) + \\ & h^r\big(\|\mathbf{curl}\,\mathbf{H}\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\Omega)} + \|\mathbf{curl}\,\mathbf{H}'\|_{\boldsymbol{H}^{\frac{1}{2}+\sigma}(\Omega'_R)}\big)\Big) , \qquad \blacksquare\end{aligned}$$

with $C > 0$ depending on Γ, ς, and the shape-regularity of the surface and volume meshes.

However, this estimate is optimal in the sense that the spread of Sobolev scales occurs as the exponent of h. What spoils any attempt to raise the exponent further is the presence of the $\boldsymbol{L}^2(\Gamma)$-norm in the estimate of lemma 5. Thus, in various estimates we are forced to trade a negative norm for the $\boldsymbol{L}^2(\Gamma)$-norm without compensation. Yet, we cannot avoid this, because interpolation estimates for Π_h are not available in negative norms. Unfortunately, Π_h is indispensable due to the commuting diagram property.

Remark. In [19] an equivalent mixed formulation of (1) is proposed that takes the variational problem into classical Sobolev spaces. There duality techniques are available that really provide an optimal asymptotic convergence, provided that s^* is sufficiently large.

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