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Abstract

In this paper we introduce new kind of nonuniform mesh, the so-called geometric mesh, and discuss the corresponding collocation method for Volterra integral equations of the second kind with proportional delay of the form qt ($0 < q < 1$). It will be shown that, in contrast to the uniform mesh, the iterated collocation solution associated with such a mesh exhibits almost optimal superconvergence at the mesh points, provided that collocation parameters are chosen as the Gauss points in $(0, 1)$.

Keywords: Delay integral equation, geometric mesh, collocation method, iterated collocation solution, superconvergence

Subject Classification (1991): 65R20

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1 Introduction

We consider the delay Volterra integral equation

$$y(t) = f(t) + \int_o^t k_1(t, s, y(s))ds + \int_0^{qt} k_2(t, s, y(s))ds, \quad t \in J := [0, T], \tag{1.1}$$

with $0 < q < 1$. This equation (1.1) represents the general form of a Volterra integral equation with proportional time delay, which includes the important case (set $k_1(\cdot, \cdot, \cdot) = -k_2(\cdot, \cdot, \cdot) =: k(\cdot, \cdot, \cdot)$)

$$y(t) = f(t) + \int_{qt}^t k(t, s, y(s))ds, \quad t \in J = [0, T].$$

It will always be assumed that (1.1) possesses a unique solution $y \in C^{2m}(J)$, where m is defined by the collocation space (Section 2). Regularity assumptions for the given functions f and k_i $(i = 1, 2)$ will be stated in Theorem 1 (see also [4]).

It is well known that for the classical Volterra integral equations ($k_2 \equiv 0$ in (1.1)) the iterated solution associated with piecewise $(m-1)$st degree polynomial spline collocation solution based on a uniform mesh possesses the optimal superconvergence order $2m$ at the nodes of the mesh, provided that the collocation parameters are chosen as the m Gauss points in $(0, 1)$. For Volterra integral equations with constant delay this property is preserved if the mesh is constrained (see [1, 3, 12]).

However, it has been shown in [4] and [14] that these superconvergence properties on uniform meshes do not carry over to equation (1.1) ($k_2 \not\equiv 0$). In fact, it can be seen from [4] and [14] that for this kind of delay integral equation the optimal (local) superconvergence order p^* is at most $p^* = 2m - 1$; for $q = 1/2$ and the Gauss points as the collocation parameters $\{c_i\}$ in $(0, 1)$, it is conjectured that $p^* = 2m$.

In the present paper we introduce, based on an important observation, a new kind of mesh, called *geometric mesh* (see [10]), in order to obtain local superconvergence results of order at least $2m - 1$. Theoretical results and numerical examples will show that when such a mesh is used, the corresponding iterated collocation solution for (1.1) will possess the superconvergence order $2m - \varepsilon$ at all nodes and for every $q \in (0, 1)$, provided the collocation parameters are chosen as the m Gauss points in $(0, 1)$. Here, ε is an arbitrarily small positive constant.

We note that nonuniform meshes similar to our geometric mesh have been employed in [13] and [2] for the analysis of asymptotic stability properties of the θ-method for the pantograph equation

$$y'(t) = ay(t) + by(qt), \quad t \geq 0 \quad (0 < q < 1),$$

with $\text{Re}(a) < 0$, $|b| < |a|$.

2 Main Result

For ease of notation, we consider the linear equation

$$y(t) = f(t) + \int_0^t K_1(t,s)y(s)ds + \int_0^{qt} K_2(t,s)y(s)ds, \quad t \in J, \tag{2.1}$$

where $0 < q <$ and where the given functions f and K_i are subject to the regularity assumptions stated in Theorem 1 below.

For given $N \in \mathbf{N}$, let $J_N: \ 0 = t_0 < t_1 < \cdots < t_N = T$ denote a partition (or mesh) for the given interval J, and set $e_n := [t_{n-1}, \ t_n], \ h_n := t_n - t_{n-1}$ $(n = 1, \cdots, N)$. In the following we shall be concerned with the finite-dimensional collocation spaces

$$S_{m-1}^{(-1)}(J_N) := \{v : \ v \mid_{e_n} \in P_{m-1} \ (n = 1, \cdots, N)\},$$

where $m \geq 1$ and P_{m-1} denotes the set of (real) polynomials of degree less than or equal to $m - 1$.

Definition 1.1: $\{J_N\}_{N \geq 2}$ is called a sequence of *geometric meshes* if the mesh points $\{t_n\} = \{t_n^{(N)}\}$ satisfy

$$t_n = t_n^{(N)} = d^{N-n}T, \quad n = 1, \cdots, N, \tag{2.2}$$

where d $(0 < d < 1;\ d$ is independent of $n)$ remains to be determined.

Remark 1.1: Note that the mesh diameter h is given by $h_N = T(1-d)$. If we require that h possess the property $h := \max_{1 \leq n \leq N} h_n \to 0$ as $N \to \infty$), then $d \to 1$ $(N \to \infty)$. Therefore d will depend on N.

We are looking for $u \in S_{m-1}^{(-1)}(J_N)$ satisfying

$$u_n(t) = f(t) + \int_0^t K_1(t,s)u(s)ds + \int_0^{qt} K_2(t,s)u(s)ds, \quad t \in X_n(\ 1 \leq n \leq N), \tag{2.3}$$

where $u_n := u \mid_{e_n};\ X_n := \{t_{nj} := t_{n-1} + c_j h_n, \ 0 \leq c_1 < \cdots < c_m \leq 1 \ (n = 1, \cdots, N)\}$. The set $X(N) := \bigcup_{n=1}^N X_n$ will be referred to as the set of collocation points, which is completely determined by the given mesh J_N and the collocation parameters $\{c_j\}_{j=1}^m$.

The collocation equation (2.3) will define a unique approximation $u \in S_{m-1}^{(-1)}(J_N)$ whenever the mesh diameter h is sufficiently small. As for classical Volterra integral equations, this approximation u will be generated recursively by successive computation of its restrictions $u_1, \cdots, u_N$ to the subintervals $e_1, \cdots, e_N$ given by the mesh J_N (compare also [5]).

Once the collocation solution u has been found, we compute the corresponding iterated collocation solution u_{it} by

$$u_{it} := f(t) + \int_0^t K_1(t,s)u(s)ds + \int_0^{qt} K_2(t,s)u(s)ds, \ t \in J. \tag{2.4}$$

The following two assumptions are supposed to hold in the subsequent analysis:

H_1 : Let κ be the maximal natural number satisfying $q^{\frac{1}{\kappa}} \leq (1 - \frac{2m\ln N}{(m+1)N})$, namely,

$$\kappa := \left[\frac{\ln q}{\ln(1 - \frac{2m\ln N}{(m+1)N})} \right].$$

For a fixed $q \in (0, 1)$, we have $\kappa \geq 1$ when $N \to \infty$. For such κ, d in (2.2) is chosen as $d = q^{\frac{1}{\kappa}}$.

H_2 : The collocation parameters $\{c_j\}_{j=1}^m$ are chosen as the m Gauss points in $(0, 1)$ (that is, the zeros of the shifted Legendre polynomial $P_m(2s - 1)$).

Throughout this paper C will denote a generic positive constant which is independent of N and q but which will depend on the constant T and the given function f and K_i).

Theorem 2.1 *Let H_1 and H_2 hold. Assume that the functions f and K_i $(i = 1, 2)$ in (2.1) satisfy $f \in C^{2m}(J)$, $K_i \in C^{2m}(\Omega)$, where $\Omega_1 := \{(t, s) : \ 0 \leq s \leq t \leq T\}$ and $\Omega_2 := \{(t, s) : \ 0 \leq s \leq qt, \ t \in J\}$. If $u \in S_{m-1}^{(-1)}(J_N)$ denotes the collocation approximation determined by (2.3), and u_{it} is defined by (2.4), then the resulting error $e_{it} := u_{it} - y$ satisfies*

$$\max_{t \in Z_N} |e_{it}(t)| \leq CN^{-(2m-\varepsilon)}, \ N \to \infty, \tag{2.5}$$

where $Z_N := \{t_n : 1 \leq n \leq N\}$ and $\varepsilon = \varepsilon_N$ is an arbitrarily small positive constant satisfying $\lim_{N \to \infty} \varepsilon_N = 0$.

Remark 2.1: Theorem 1 indicates that a suitably chosen *geometric mesh* can, in contrast to the uniform mesh, generate iterated collocation solution possessing the almost optimal (local) superconvergence orderi $p^* = 2m - \varepsilon$ at all mesh points Z_N. This is in contrast to *uniform meshes*: it was shown in [4] that for such meshes the optimal order of local superconvergence of u_{it} satisfies $p^* \leq 2m - 1$ whenever $q \neq 1/2$; if $q = 1/2$ then it is conjectured (based on the order of u_{it} at $t_1 = h$ and on numerical evidence; compare also Table 2 in Section 5) that $p^* = 2m$.

3 Lemmas

In this section we state a number of lemmas which will be crucial for establishing the superconvergence result in Theorem 1 .

Lemma 3.1 *Assume that H_1 holds. Then, for $N \geq 2$:*

(i)

$$h_1 \leq CN^{-\frac{2m}{m+1}}; \tag{3.1}$$

(ii)

$$\sum_{n=2}^{N}(h_n)^{2m+1} \leq CN^{-(2m-\varepsilon)}. \tag{3.2}$$

Here, the positive number $\varepsilon = \varepsilon_N$ *satisfies* $\lim\limits_{N\to\infty} \varepsilon_N = 0.$

Proof (i) Noting that

$$(1 - \frac{2m\ln N}{(m+1)N})^{(m+1)N/2m\ln N} \leq e^{-1} \quad (N \geq 2),$$

we have

$$h_1 = t_1 = Td^{N-1} \leq Tq^{\frac{1}{k}} \leq T(1 - \frac{2m\ln N}{(m+1)N})^{N-1} \leq Ce^{-\frac{2m\ln N}{m+1}} = CN^{-\frac{2m}{m+1}}.$$

(ii) From H_1 we obtain

$$h_n = t_n - t_{n-1} = Td^{N-n}(1-d) \leq Cd^{N-n}\frac{2m\ln N}{(m+1)N} \qquad (n = 2, \cdots, N).$$

Thus,

$$\begin{aligned}\sum_{n=2}^{N}(h_n)^{2m+1} &\leq \frac{C}{(1-d^{2m+1})}\cdot\frac{(2m\ln N)^{2m+1}}{(m+1)^{2m+1}N}\cdot N^{-2m} \\ &= \frac{C(2m\ln N)^{2m+1}}{(m+1)^{2m+1}N(1-(1-\frac{2m\ln N}{(m+1)N})^{2m+1})}\cdot N^{-2m}.\end{aligned} \tag{3.3}$$

Since m is a constant,

$$\lim_{N\to\infty}\frac{2m\ln N}{(m+1)N} = 0.$$

Hence, we can assume that for sufficiently large N,

$$\frac{2m\ln N}{(m+1)N} < 1.$$

Using the standard inequality

$$(1-x)^{\alpha} \leq 1 - \alpha x + \frac{\alpha(\alpha-1)}{2}x^2, \quad 0 \leq x < 1,\ \alpha \geq 2,$$

we deduce that

$$\begin{aligned}1 - \left(1 - \frac{2m\ln N}{(m+1)N}\right)^{2m+1} &\geq \frac{2m(2m+1)\ln N}{(m+1)N} - \frac{4m^3(2m+1)(\ln N)^2}{(m+1)^2N^2} \\ &\geq \frac{m(2m+1)\ln N}{(m+1)N}, \quad N\to\infty\ ;\end{aligned}$$

by (3.3) this leads to

$$\sum_{n=2}^{N}(h_n)^{2m+1} \le \frac{C(2m\ln N)^{2m}}{(2m+1)(m+1)^{2m}}N^{-2m},\ N \to \infty. \tag{3.4}$$

Set

$$b := \frac{(2m\ln N)^{2m}}{(2m+1)(m+1)^{2m}}\ .$$

Then, using the identity $b = N^{log_N b}$, (3.4) can be written as

$$\sum_{n=2}^{N}(h_n)^{2m+1} \le CN^{-(2m-log_N b)},\ N \to \infty. \tag{3.5}$$

For a given constant m we thus obtain

$$\varepsilon_N = log_N b = \frac{\ln b}{\ln N} \to 0,\ N \to \infty,$$

and this, together with (3.5), yields the assertion (ii).

The following lemma reveals one of the key reasons for using geometric meshes: for a suitable choice of the integer κ (recall assumption H_1) the delay function $\theta(t) := qt$ maps a mesh points to some previous mesh point. Compare also [13, 2].

Lemma 3.2 *For $\kappa + 1 \le n \le N$, we have $qt_n = t_{n-\kappa} \in Z_N$. Here, κ is defined in assumption H_1.*

Proof: From the definitions of t_n and d (see (2.2) we have

$$qt_{\kappa+1} = Tqd^{N-(\kappa+1)} = Td^{\kappa}d^{N-(\kappa+1)} = Td^{N-1} = t_1$$

and

$$qt_n = Tqd^{N-n} = Td^{\kappa}d^{N-n} = Td^{N-(n-\kappa)} = t_{n-\kappa},\ \kappa + 2 \le n \le N.$$

Set

$$M_i := \max_{(t,s)\in\Omega_i} |K_i(t,s)|,\ (i = 1, 2).$$

Then

$$|\int_0^t K_1(t,s)y(s)ds| \le M_1 \int_0^t |y(s)|ds,\ t \in J, \tag{3.6}$$

and, since $q < 1$,

$$|\int_0^{qt} K_2(t,s)y(s)ds| \le M_2 \int_0^{qt} |y(s)|ds \le M_2 \int_0^t |y(s)|ds,\ t \in J. \tag{3.7}$$

Lemma 3.3 *Let the functions f and K_i $(i = 1, 2)$ in (2.1) satisfy the smoothness assumptions stated in Theorem 1. Then the equation (2.1) has a (unique) solution $y \in C^{2m}(J)$ (for any $q \in [0, 1]$). Moreover, its derivatives $y^{(j)}$ $(j = 1, \cdots, 2m)$ are uniformly bounded with respect to the parameter q (for any finite interval J).*

Proof: It can be verified by standard Picard iteration (see also [9, 7] and [6]) that the equation (2.1) has a unique solution $y \in C(J)$. Equation (2.1), together with (3.6) and (3.7), yields

$$|y(t)| \leq M + (M_1 + M_2) \int_0^t |y(s)| ds, \ t \in J,$$

where

$$M := \max_{t \in J} |f(t)|.$$

Thus, it follows by Gronwall's inequality that y is bounded with respect to q.

On the other hand, from (2.1) we have

$$\begin{aligned} y'(t) &= f'(t) + K_1(t,t)y(t) + qK_2(t,qt)y(qt) \\ &\quad + \int_0^t \frac{\partial}{\partial t} K_1(t,s)y(s)ds + \int_0^{qt} \frac{\partial}{\partial t} K_2(t,s)y(s)ds, \ t \in J. \end{aligned}$$

Therefore, the regularity of y' is the same that of the derivatives (with respect to t) of the given functions. The regularity of the higher derivatives of y is then proved recursively in an analogous way, for any $q \in [0, 1]$.

Set now

$$W^{k,\infty}(J) = L^\infty(J) \cap (\cap_{i=1}^N C^k(e_i)).$$

For a nonnegative integer number k, we define the norm $\| \cdot \|_{k,\infty}$ by

$$\|v\|_{k,\infty} := \left(\sum_{i=1}^N \|v\|_{k,e_i,\infty}^2 \right)^{\frac{1}{2}},$$

with

$$\|v\|_{k,e_i,\infty} := \max_{0 \leq j \leq k} \left(\max_{t \in e_i} |\frac{d^j}{dt^j} v(t)| \right).$$

For the sake of convenience, the norm $\| \cdot \|_{0,e_i,\infty}$ will be abbreviated by $\| \cdot \|_{e_i,\infty}$.

Let $\pi : C(J) \to S_{m-1}^{(-1)}(J_N)$ denote the sequence of interpolation operators such that $\pi v(t_{nj}) = v(t_{nj})$ $(n = 1, \cdots, N; \ j = 1, \cdots, m)$ for $v \in C(J)$. It is well known that

$$\|\pi v\|_{e_i,\infty} \leq C \|v\|_{e_i,\infty}, \ v \in C(J), \tag{3.8}$$

and

$$\|(\pi - I)v\|_{j,e_i,\infty} \leq Ch^{k-j}\|v\|_{k,e_i,\infty}, \quad 0 \leq j \leq k \leq m. \tag{3.9}$$

For ease of notation, we define the operator $K:\ L^\infty(J) \to L^\infty(J)$ by

$$Kg(t) := \int_0^t K_1(t,s)g(s)ds + \int_0^{qt} K_2(t,s)g(s)ds,\ t \in J.$$

Lemma 3.4 *Under the assumptions stated in Theorem 1 we have*

$$\|u - y\|_{e_1,\infty} \leq h_1^m \|y\|_{m,e_1,\infty} \tag{3.10}$$

and

$$\|u\|_{j,\infty} \leq C\|y\|_{m,\infty},\ 0 \leq j \leq 2m. \tag{3.11}$$

Proof: Since $u \in S_{m-1}^{(-1)}(J_N)$, it follows by the definition of π that $\pi u = u$. The equations (2.1) and (2.2) may be written in operator form as

$$y = Ky + f, \tag{3.12}$$

and

$$u = \pi K u + \pi f. \tag{3.13}$$

Set $e := u - y$ and let I denote the identity operator. Subtraction of (3.12) from (3.13) leads to

$$e = \pi K e + (\pi - I)(Ky + f).$$

Hence, by observing (3.12),

$$e = \pi K e + (\pi - I)y, \tag{3.14}$$

which, together with (3.6), (3.7), (3.8) and (3.9), yields

$$|e(t)| \leq (M_1 + M_2)\int_0^t |e(s)|ds + Ch_1^m\|y\|_{m,e_1,\infty},\ t \in e_1.$$

Thus, the inequality (3.10) is derived by employing Gronwall's inequality.
The inequality

$$\|e\|_{0,\infty} \leq Ch^m\|y\|_{m,\infty}. \tag{3.15}$$

can be proved in an analogous way. On the other hand, (3.14) can be as

$$e(t) = Ke(t) + (\pi - I)(Ke + y)(t),\ t \in J. \tag{3.16}$$

Furthermore,

$$\begin{aligned} e'(t) &= K_1(t,t)e(t) + qK_2(t,qt)e(qt) + \int_0^t \frac{\partial}{\partial t}K_1(t,s)e(s)ds \\ &+ \int_0^{qt} \frac{\partial}{\partial t}K_2(t,s)e(s)ds + \frac{d}{dt}(\pi - I)(Ke + y)(t),\ t \in J, \end{aligned} \tag{3.17}$$

which, by (3.15), leads to

$$\|e\|_{1,\infty} \le Ch^m \|y\|_{m,\infty} + \|(\pi - I)(Ke + y)\|_{1,\infty}. \tag{3.18}$$

It follows by (3.9) and (3.15) that

$$\begin{aligned}
\|(\pi - I)(Ke + y)\|_{1,\infty} &\le \|(\pi - I)Ke\|_{1,\infty} + \|(\pi - I)y\|_{1,\infty} \\
&\le C\|Ke\|_{1,\infty} + Ch^{m-1}\|y\|_{m,\infty} \\
&\le C\|e\|_{0,\infty} + Ch^{m-1}\|y\|_{m,\infty} \\
&\le Ch^m\|y\|_{m,\infty} + Ch^{m-1}\|y\|_{m,\infty},
\end{aligned}$$

and hence, by (3.18),

$$\|e\|_{1,\infty} \le Ch^{m-1}\|y\|_{m,\infty}. \tag{3.19}$$

In a similar manner we can prove (using (3.17) and (3.19))

$$\|e\|_{2,\infty} \le Ch^{m-2}\|y\|_{m,\infty},$$

and we then successively obtain

$$\|e\|_{j,\infty} \le Ch^{m-j}\|y\|_{m,\infty},\ 0 \le j \le m.$$

Thus,

$$\|u\|_{j,\infty} \le \|y\|_{j,\infty} + \|e\|_{j,\infty} \le C\|y\|_{m,\infty},\ 0 \le j \le m.$$

Since $u^{(j)}(t) = 0$ $(j \ge m)$ on e_n $(n = 1, \cdots, N)$, the inequality (3.11) is also valid for $m \le j \le 2m$.

The last lemma is a standard result in the superconvergence theory of integral equations (compare [8]).

Lemma 3.5 *Assume that $\psi \in C^m(J)$ and $\varphi \in C^{2m}(J)$. If H_2 holds, then the following estimate is valid for all e_n:*

$$\left| \int_{e_n} \psi(t)(\pi - I)\varphi(t)dt \right| \le C(h_n)^{2m+1}\|\psi\|_{m,\infty} \cdot \|\varphi\|_{2m,\infty}. \tag{3.20}$$

4 Proof of Theorem 2.1

Since the resolvent operator of the global integral operator K is very complex (compare, for example, [6] for the case where $K_1 = 0$ in (2.1)), standard techniques of, e.g., [5, 3] cannot be employed to establish optimal local superconvergence results. Thus, in this section we propose a new approach using specially designed geometric meshes.

First, we prove inductively that, for all $\varphi \in C^m(e_i)$,

$$|\int_{e_i} \varphi(s)e(s)ds| \leq Ch_i(N^{-(2m-\varepsilon)}\|\varphi\|_{e_i,\infty} + h_i^{2m}\|\varphi\|_{m,e_i,\infty}) \quad (1 \leq i \leq N). \tag{4.1}$$

It follows by (3.14) and (3.8) that for any $\psi \in C^m(e_1)$ we have

$$|\int_{e_1} \psi(s)e(s)ds| \leq \int_{e_1} |\psi(s)| \cdot |Ke(s)|ds + |\int_{e_1} \psi(s)(I-\pi)y(s)ds|,$$

which, together with (3.10),(3.20) and (3.1), yields

$$\begin{aligned} |\int_{e_1} \psi(s)e(s)ds| &\leq C(h_1^2\|\psi\|_{e_1,\infty} \cdot \|e\|_{e_1,\infty} \\ &\quad + h_1^{2m+1}\|\psi\|_{m,e_1,\infty} \cdot \|y\|_{2m,e_1,\infty} \\ &\leq Ch_1(h_1^{m+1}\|\psi\|_{e_1,\infty} \cdot \|y\|_{m,e_1,\infty} \\ &\quad + h_1^{2m}\|\psi\|_{m,e_1,\infty} \cdot \|y\|_{2m,e_1,\infty}) \\ &\leq Ch_1(N^{-2m}\|\psi\|_{e_1,\infty} + h_1^{2m}\|\psi\|_{m,e_1,\infty}). \end{aligned} \tag{4.2}$$

We assume that the following inequality is valid for every $\varphi \in C^m(e_i)$ and $1 \leq i \leq n$:

$$|\int_{e_i} \varphi(s)e(s)ds| \leq Ch_i(N^{-2m}\|\varphi\|_{e_i,\infty} + h_i^{2m}\|\varphi\|_{m,e_i,\infty}). \tag{4.3}$$

We need to prove that, for any $\psi \in C^m(e_{n+1})$,

$$|\int_{e_{n+1}} \psi(s)e(s)ds| \leq Ch_{n+1}(N^{-2m}\|\psi\|_{e_{n+1},\infty} + h_{n+1}^{2m}\|\psi\|_{m,e_{n+1},\infty}). \tag{4.4}$$

In fact, the equation (3.16) can be written as

$$e(t) = \int_0^t K_1(t,s)e(s)ds + A(t), \ t \in J,$$

with

$$A(t) = \int_0^{qt} K_2(t,s)e(s)ds + (\pi - I)(Ku + f)(t).$$

Let R_1 be the resolvent kernel of K_1; standard Volterra theory implies that it inherits the smoothness of the kernel K_1. Thus, by the classical theory of Volterra integral equations, we obtain

$$e(t) = A(t) + \int_0^t R_1(t,s)A(s)ds, \ t \in J.$$

Furthermore,

$$\begin{aligned}
\int_{e_{n+1}} \psi(t)e(t)dt &= \int_{e_{n+1}} \psi(t)A(t)dt + \int_{e_{n+1}} [\psi(t)\int_0^t R_1(t,s)A(s)ds]dt \\
&= \int_{e_{n+1}} \psi(t)(\pi - I)(Ku+f)(t)dt \\
&\quad + \int_{e_{n+1}} [\psi(t)\int_0^t R_1(t,s)(\pi - I)(Ku+f)(s)ds]dt \qquad (4.5) \\
&\quad + \int_{e_{n+1}} [\psi(t)\int_0^{qt} K_2(t,s)e(s)ds]dt \\
&\quad + \int_{e_{n+1}} [\psi(t)\int_0^t (R_1(t,s)\int_0^{qs} K_2(s,\tau)e(\tau)d\tau)ds]dt \\
&=: I_1 + I_2 + I_3 + I_4. \qquad (4.6)
\end{aligned}$$

The following inequality is a direct consequence of (3.20) and (3.11):

$$|I_1| \le Ch_{n+1}^{2m+1}\|\psi\|_{m,e_{n+1},\infty}. \qquad (4.7)$$

Changing the order of integration leads to

$$\begin{aligned}
I_2 &= \int_{t_n}^{t_{n+1}} [\int_s^{t_{n+1}} \psi(t)R_1(t,s)dt \cdot (\pi - I)(Ku+f)(s)]ds \\
&\quad + \int_0^{t_n} [\int_{t_n}^{t_{n+1}} \psi(t)R_1(t,s)dt \cdot (\pi - I)(Ku+f)(s)]ds.
\end{aligned}$$

Set

$$\psi_1(s) = \int_s^{t_{n+1}} \psi(t)R_1(t,s)dt$$

and

$$\psi_2(s) = \int_{t_n}^{t_{n+1}} \psi(t)r_1(t,s)dt.$$

Since the function R_1 possesses the same degree of regularity as the kernel K_1, it follows by (3.20) and (3.11) that (note that $h_i \le h_{n+1}$ for $1 \le i \le n$)

$$\begin{aligned}
& | \int_{t_n}^{t_{n+1}} [\int_s^{t_{n+1}} \psi(t)R_1(t,s)dt \cdot (\pi - I)(Ku+f)(s)]ds | \\
\le\ & Ch_{n+1}^{2m+1}\|\psi_1\|_{m,e_{n+1},\infty} \cdot \|(\pi - I)(Ku+f)\|_{2m,e_{n+1},\infty} \\
\le\ & Ch_{n+1}^{2m+1}\|\psi\|_{m,e_{n+1},\infty}
\end{aligned}$$

and

$$
\begin{aligned}
& \left| \int_0^{t_n} [\int_{t_n}^{t_{n+1}} \psi(t) r_1(t,s) dt \cdot (\pi - I)(Ku+f)(s)] ds \right| \\
\leq \;& \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} \psi_2(s)(\pi - I)(Ku+f)(s) ds \right| \\
\leq \;& C \sum_{i=1}^{n} h_i^{2m+1} \|\psi_2\|_{m,e_i,\infty} \cdot \|(\pi - I)(Ku+f)\|_{2m,e_i,\infty} \\
\leq \;& C \sum_{i=1}^{n} h_i h_{n+1}^{2m} h_{n+1} \|\psi\|_{m,e_{n+1},\infty} \\
\leq \;& C h_{n+1}^{2m+1} \|\psi\|_{m,e_{n+1},\infty}.
\end{aligned}
$$

Thus,

$$
|I_2| \leq C h_{n+1}^{2m+1} \|\psi\|_{m,e_{n+1},\infty} \tag{4.8}
$$

When $1 \leq n \leq \kappa$, we have $qt_{n+1} \in (0, t_1]$ (see the proof of Lemma 3.2). Using (3.10) and (3.1) we obtain

$$
\begin{aligned}
|I_3| &\leq C h_{n+1} q t_{n+1} h_1^m \|\psi\|_{e_{n+1},\infty} \cdot \|y\|_{m,e_1,\infty} \\
&\leq C h_{n+1} h_1^{m+1} \|\psi\|_{e_{n+1},\infty} \\
&\leq C h_{n+1} N^{-2m} \|\psi\|_{e_{n+1},\infty}.
\end{aligned}
$$

We consider the case of $n \geq \kappa + 1$. Changing the order of integration leads to

$$
\begin{aligned}
I_3 &= \int_{qt_n}^{qt_{n+1}} [\int_{\frac{s}{q}}^{t_{n+1}} \psi(t) K_2(t,s) dt \cdot e(s)] ds \\
&\quad + \int_0^{qt_n} [\int_{t_n}^{t_{n+1}} \psi(t) K_2(t,s) dt \cdot e(s)] ds \\
&=: I_{31} + I_{32}.
\end{aligned}
$$

From Lemma 3.2 we have $qt_n = t_{n-\kappa}$ and $qt_{n+1} = t_{n+1-\kappa}$. Set

$$
\psi_3(s) = \int_{\frac{s}{q}}^{t_{n+1}} \psi(t) K_2(t,s) dt
$$

and

$$
\psi_4(s) = \int_{t_n}^{t_{n+1}} \psi(t) K_2(t,s) dt.
$$

Thus, it follows by the inductive assumption (4.2) that

$$
\begin{aligned}
|I_{31}| &\leq C q h_{n+1} (N^{-2m} \|\psi_3\|_{e_{n+1-\kappa},\infty} + (q h_{n+1})^{2m} \|\psi_3\|_{m,e_{n+1-\kappa},\infty}) \\
&\leq C q h_{n+1} (N^{-2m} h_{n+1} \|\psi\|_{e_{n+1},\infty} + (q h_{n+1})^{2m} q^{-m} \|\psi\|_{m,e_{n+1},\infty}) \\
&\leq C h_{n+1} (N^{-2m} \|\psi\|_{e_{n+1},\infty} + h_{n+1}^{2m} \|\psi\|_{m,e_{n+1},\infty})
\end{aligned}
$$

and

$$
\begin{aligned}
|I_{32}| &\leq \sum_{i=1}^{n-\kappa} | \int_{e_i} \psi_4(s)e(s)ds | \\
&\leq C \sum_{i=1}^{n-\kappa} h_i(N^{-2m}\|\psi_4\|_{e_i,\infty} + h_i^{2m}\|\psi_4\|_{m,e_i,\infty}) \\
&\leq C \sum_{i=1}^{n-\kappa} h_i(N^{-2m}h_{n+1}\|\psi\|_{e_{n+1},\infty} + h_i^{2m}h_{n+1}\|\psi\|_{e_{n+1},\infty}) \\
&\leq Ct_{n-\kappa}h_{n+1}(N^{-2m}\|\psi\|_{e_{n+1},\infty} + h_{n+1}^{2m}\|\psi\|_{m,e_{n+1},\infty}) \\
&\leq Ch_{n+1}(N^{-2m}\|\psi\|_{e_{n+1},\infty} + h_{n+1}^{2m}\|\psi\|_{m,e_{n+1},\infty}).
\end{aligned}
$$

Therefore,

$$|I_3| \leq Ch_{n+1}(N^{-2m}\|\psi\|_{e_{n+1},\infty} + h_{n+1}^{2m}\|\psi\|_{m,e_{n+1},\infty}). \tag{4.9}$$

In an analogous way we can prove that

$$|I_4| \leq Ch_{n+1}(N^{-2m}\|\psi\|_{e_{n+1},\infty} + h_{n+1}^{2m}\|\psi\|_{m,e_{n+1},\infty}),$$

and this estimate, together with (4.5)-(4.9), allows us to deduce (4.4). It then follows by the induction principle that the inequality (4.1) is valid.

Now, we can readily prove Theorem 2.1. By (4.1) and Lemma 3.1, we obtain

$$| \int_0^{t_n} \varphi(t)e(t)dt | \leq CN^{-(2m-\varepsilon)}\|\varphi\|_{m,[0,t_n],\infty}, \text{ for all } \varphi \in C^m[0,t_n],\ 1 \leq n \leq N.$$

In particular, we find that

$$| \int_0^{t_n} K_1(t_n,s)e(s)ds | \leq CN^{-(2m-\varepsilon)},\ 1 \leq n \leq N. \tag{4.10}$$

and

$$| \int_0^{qt_n} K_2(t_n,s)e(s)ds | \leq CN^{-(2m-\varepsilon)},\ \kappa+1 \leq n \leq N. \tag{4.11}$$

When $n \leq \kappa$, we have $qt_n \leq t_1$. Hence, using (3.10) and (3.1) we are led to

$$
\begin{aligned}
| \int_0^{qt_n} K_2(t_n,s)e(s)ds | &\leq Ct_1h_1^m\|y\|_{m,e_1,\infty} \\
&\leq Ch_1^{m+1} \leq CN^{-2m},\ 1 \leq n \leq \kappa. \qquad (4.12)
\end{aligned}
$$

On the other hand, subtraction of (2.1) from (2.3) yields

$$e_{it} = u_{it} - y = Ke,$$

which, together with (4.10), (4.11) and (4.12), gives the desired result.

Remark 4.1: In the proof of Theorem 2.1, the use of the inductive method to prove the inequality (4.1) is the key technique. This technique is, in essence, similar to the recursive method introduced in [11] and [12].

Remark 4.2: The fact that we have $qt_n \in Z_N$ when $qt_n \geq t_1$ ($t_n \in Z_N$) is important in the proof of Theorem 1. This just reflects our motivation for introducing the "geometric" meshes.

5 Numerical examples

For the numerical verification of the result stated in Section 2, we consider

$$y(t) = f(t) - \int_0^t y(s)ds + \frac{1}{2}\int_0^{qt} y(s)ds, \ t \in [0, T], \tag{5.13}$$

where the function f is chosen as $f(t) = \frac{1}{2}(1 + e^{-qt})$, so that the exact solution is $y(t) = e^{-t}$; the delay parameter q is chosen to have the values $q = 0.9$, $q = 0.5$, and $q = 0.2$. We set $T = 10$. The equation (5.1) is solved by three collocation methods using the space $S_1^{(-1)}(J_N)$ $(m = 2)$.

The first method (M1) is based on the *geometric meshes* introduced in Section 2 and the Gauss collocation parameters: $c_1 = (3 - \sqrt{3})/6$, $c_2 = (3 + \sqrt{3})/6$; the second method (M2) is based on the *uniform meshes* and the Gauss collocation parameters c_1 and c_2; and the third method (M3) uses *uniform meshes* and the *$q-$Gauss collocation parameters* $\bar{c}_1 := qc_1$ and $\bar{c}_2 := qc_2$ (see [4] and [14]).

The resulting nodal errors are given in Table 1 ($q = 0.9$), Table 2 ($q = 0.5$), and Table 3 ($q = 0.2$).

Table 1

$\max_{t \in Z_N} |e_{it}(t)|$ $(q = 0.9)$

N $(h = 10/N)$	100	200	400	800	1600
M1	7.28D-8	4.57D-9	1.17D-10	1.11D-11	8.75D-13
M2	3.49D-6	4.67D-7	6.04D-8	7.68D-9	9.68D-10
M3	2.55D-5	6.03D-6	1.46D-6	3.61D-7	8.97D-8

Table 2

$\max_{t \in Z_N} |e_{it}(t)|$ $(q = 0.5)$

N $(h = 10/N)$	100	200	400	800	1600
M1	1.94D-7	1.49D-8	1.45D-9	1.40D-10	1.23D-11
M2	2.95D-7	1.90D-8	1.21D-9	7.62D-11	4.78D-12
M3	1.86D-4	4.45D-5	1.09D-5	2.69D-6	6.69D-7

Table 3

$$\max_{t \in Z_N} |e_{it}(t)| \ (q = 0.2)$$

N $(h = 10/N)$	100	200	400	800	1600
M1	1.70D-7	1.78D-8	1.72D-9	1.60D-10	1.47D-11
M2	3.91D-6	4.98D-7	6.25D-8	7.82D-9	9.77D-10
M3	4.18D-4	1.00D-4	2.44D-5	6.03D-6	1.50D-6

These tables confirm that our method (M1) is very effective for all parameter $q \in (0, 1)$, in that it generates iterated collocation solutions possessing the almost optimal superconvergence rate when $N \to \infty$. Moreover, the new method (M1) is better than the standard method (M2) except for the particular case of $q = \frac{1}{2}$. As we have mentioned before, it is conjectured that the iterated collocation solution based on $S_{m-1}^{(-1)}(J_N)$, with uniform mesh J_N and collocation at the Gauss points, exhibits the optimal order of local superconvergence, $p^* = 2m$. The results for (M2) in Table 2, and numerous other numerical examples, clearly show that $p^* = 2m = 4$ holds.

The numerical results also show that, when $K_1 \neq 0$, the $q-$Gauss collocation points leads to a lower rate of convergence at the nodal points (it equals the global convergence rate on J, $p = m$).

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