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Abstract

A stabilized hp-Finite Element Method (FEM) of Galerkin Least Squares (GLS) type is analyzed for the Stokes equations in polygonal domains. Contrary to the standard Galerkin FEM, this method admits equal-order interpolation in the velocity and the pressure, which is very attractive from an implementational point of view. In conjunction with geometrically refined meshes and linearly increasing approximation orders it is shown that the hp-GLSFEM leads to exponential rates of convergence for solutions exhibiting singularities near corners. To obtain this result a novel hp-interpolant is constructed that approximates pressure functions in certain weighted Sobolev spaces in an H^1-conforming way and at exponential rates of convergence on geometric meshes.

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1. Introduction

The Stokes equations describe the motion of incompressible fluids at moderate values of the Reynolds number and are form-identical to the equations of isotropic incompressible elasticity. Despite their linear structure they give the right setting for studying stability aspects encountered when discretizing the incompressibility constraint in fluid mechanics problems. Therefore, these equations have become an important model problem in CFD and the ability to solve Stokes type problems accurately and efficiently is an inalterable requirement in many engineering applications.

The classical mixed Galerkin Finite Element Method (FEM) for the Stokes equations leads to a saddle point problem for which stability is only guaranteed if the velocity and pressure FE-spaces satisfy the "Babuška-Brezzi" condition [8, 14]. Typically, this stability condition imposes the use of different polynomial orders for the velocity and the pressure interpolation (for low order pairs see [8, 14, 26] and the references there; for high order and spectral elements we refer to [4, 5, 25, 27] and the references there). Many attractive velocity-pressure combinations such as equal-order spaces are instable which results in wildely oscillating pressures in computations.

In [19], Hughes, Franca and their co-workers introduced a Galerkin Least Squares (GLS) approach which allows circumventing the Babuška-Brezzi condition and using equal order spaces. In this method the standard Galerkin form is perturbed by adding weighted residuals of the differential equation which results in enhanced stability properties of the scheme. In [10, 11], Franca, Hughes and Stenberg showed optimal h-version convergence rates for this GLS approach. The idea of stabilizing Finite Element Methods by appropriately chosen (and mesh-dependent) weights is by now widely used in the FE community and has been applied successfully in a variety of problems in fluid flow, elasticity and continuum mechanics. We mention here only [7, 9, 11, 19, 28] and the references there. We refer also to the survey article [10]. All these works are concerned with the h-version of the FEM where convergence is achieved by decreasing the meshwidth h at a fixed (typically) low approximation order.

Recently, there have been several attempts to extend stabilized methods to high order and spectral elements (see e.g. [6, 13, 21] and the references there). Boillat and Stenberg gave in [6] a complete hp-error analysis of the GLS Finite Element Method for Stokes flow. They derived error bounds for the pure p-version where convergence is obtained by increasing the polynomial approximation order on a fixed quasi-uniform mesh. However, these estimates give only algebraic rates of convergence and are restricted to smooth solutions which is unrealistic in domains with corners since there radial corner singularities are present (see e.g. [22]).

In the present work we extend the GLSFEM for Stokes flow discussed in [6, 10, 11] to hp-FEM on geometric meshes. The main novelties of our work are:

- We consider Stokes flow in non-smooth domains where the solutions exhibit singularities at conical boundary points. This solution regularity is described in terms of countably normed, weighted spaces from [17, 18]. To our best knowledge, we present the first error analysis for the GLSFEM valid for such singular solutions. The reduced regularity near corners and the continuous

pressure approximations impose several technical difficulties and require a careful treatment of the elements near vertices of the domain.

- To resolve the corner singularities we employ geometrically refined meshes combined with linearly increasing approximation orders and show that this hp-GLSFEM approach leads to exponential rates of convergence. This result indicates that the performance of the hp-GLSFEM is not downgraded by singular solution components despite the appearance of stabilization terms of higher order in the variational formulations.

- An hp-interpolant is constructed that approximates pressure functions in certain weighted Sobolev spaces in an H^1-conforming way and at exponential rates of convergence on geometric meshes. This result, proved along the lines of [2, 15, 16, 24], is important from an implementational point of view: In the hp-GLSFEM all field variables can be interpolated by the same H^1-conforming FE space without losing the exponential convergence property. We remark that this approximation theorem implies immediately corresponding results for the classical Galerkin FEM.

The theoretical results of this paper have been announced in the note [23] and confirmed numerically in [12]. Here, we give their detailed proof.

The outline of the paper is as follows: In Section 2 we review the Stokes problem and discuss the regularity of solutions near corners. The GLS discretization is presented in Section 3. We analyze the stability properties of the method and derive error bounds valid for solutions exhibiting corner singularities. In Section 4 we prove exponential rates of convergence in the hp-version of the GLSFEM on geometric meshes. In Section 4.4 we show that the same result holds true in the standard Galerkin FEM. The proof of these convergence properties is based on a novel hp-approximation result whose detailed derivation is presented in Section 5.

Subsequently, standard notations and conventions are followed. We denote by $C, C_1, C_2, \ldots$ generic constants independent of the polynomial degrees k_K, the meshwidths h_K and the stabilization parameter α.

2. The Stokes Problem

2.1. The Stokes Equations

Let $\Omega \subset \mathbb{R}^2$ be a polygonal and bounded domain with boundary $\Gamma = \partial\Omega$. The Stokes problem is to find a velocity field $\vec{u}$ and a pressure p such that

$$-\Delta\vec{u} + \nabla p = \vec{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \vec{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\vec{u} = \vec{0} \quad \text{on } \Gamma. \tag{2.3}$$

Here, the right-hand side $\vec{f} \in L^2(\Omega)^2$ is an exterior body force per unit mass. Introducing the spaces

$$\begin{aligned} H_0^1(\Omega)^2 &= \{\vec{u} \in H^1(\Omega)^2 : \vec{u}|_\Gamma = \vec{0} \text{ in the sense of trace}\}, \\ L_0^2(\Omega) &= \{p \in L^2(\Omega) : (p, 1) = 0\} \end{aligned}$$

with $(\cdot, \cdot)$ denoting the inner product in $L^2(\Omega)$, $L^2(\Omega)^2$ or $L^2(\Omega)^{2\times 2}$, the usual mixed formulation of (2.1)-(2.3) is:

Find $(\vec{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$B_0(\vec{u}, p; \vec{v}, q) = F_0(\vec{v}, q) \qquad \forall(\vec{v}, q) \in H_0^1(\Omega)^2 \times L_0^2(\Omega) \tag{2.4}$$

with

$$\begin{aligned} B_0(\vec{u}, p; \vec{v}, q) &= (\nabla\vec{u}, \nabla\vec{v}) - (\nabla \cdot \vec{v}, p) - (\nabla \cdot \vec{u}, q), \tag{2.5} \\ F_0(\vec{v}, q) &= (\vec{f}, \vec{v}). \tag{2.6} \end{aligned}$$

The system (2.4) has a unique solution $(\vec{u}, p)$ in $H_0^1(\Omega)^2 \times L_0^2(\Omega)$ (see [8, 14]).

2.2. Regularity near Corners

We assume that the right-hand side $\vec{f}$ in (2.1) is analytic in $\overline{\Omega}$. Then it follows from elliptic regularity theory that the exact solution $(\vec{u}, p)$ is analytic in $\overline{\Omega} \setminus \cup_{i=1}^M A_i$ where $\{A_i\}_{i=1}^M$ denotes the vertices of Ω. Near a vertex A_i there arise corner singularities which are solution components that are in polar coordinates (r, φ) at A_i essentially of the form

$$\vec{u}(r, \varphi) = r^\lambda \vec{\Phi}(r, \varphi), \qquad p(r, \varphi) = r^{\lambda-1}\Psi(r, \varphi) \tag{2.7}$$

for $\lambda \geq 0$ and with $\vec{\Phi}$, Ψ analytic. The value of the exponent λ depends only on the angle of the corner at A_i. Decompositions of the solutions into smooth parts and corner singularities have been given e.g. in [22].

For closely related elasticity and potential problems I. Babuška and B. Guo described in [1, 2, 17] elliptic regularity in terms of weighted Sobolev spaces. They showed that under the analyticity assumption on the data the solutions belong to the Gevrey-type countably normed $B_\beta^l(\Omega)$ spaces. They are defined as follows:

Denote by $A_1, \ldots A_M$ the vertices of the domain Ω. With each A_i we associate a weight $\beta_i \geq 0$ and store these numbers in the M-tuple $\beta = (\beta_1, \ldots, \beta_M)$. We define $\beta \pm k := (\beta_1 \pm k, \ldots, \beta_M \pm k)$ and use the shorthand $C_1 > \beta > C_2$ to mean $C_1 > \beta_i > C_2$ for $i = 1, \ldots, M$. Define a weight function $\Phi_\beta(x) := \sum_{i=1}^M r_i^*(x)^{\beta_i}$ where $r_i^*(x) = \min\{1, |x - A_i|\}$. For integers $m \geq l \geq 0$ introduce the seminorms

$$|u|_{H_\beta^{m,l}(\Omega)}^2 := \sum_{k=l}^m \||D^k u| \Phi_{\beta+k-l}\|_{L^2(\Omega)}^2.$$

Here, $|D^k u|^2 = \sum_{|\alpha|=k} |D^\alpha u|^2$. Let $H_\beta^{m,l}(\Omega)$, $m \geq l \geq 0$ integers, be the completion of the set of all infinitely differentiable functions with respect to the norm

$$\|u\|_{H_\beta^{m,l}(\Omega)}^2 := \|u\|_{H^{l-1}(\Omega)}^2 + |u|_{H_\beta^{m,l}(\Omega)}^2, \qquad l \geq 1,$$

$$\|u\|_{H_\beta^{m,0}(\Omega)}^2 := \sum_{k=0}^m \||D^k u| \Phi_{\beta+k}\|_{L^2(\Omega)}^2.$$

Definition 2.1 *Fix an M-tuple $\beta = (\beta_1, \ldots, \beta_M)$ and $l \geq 0$. The countably normed space $B_\beta^l(\Omega)$ consists of all functions u for which $u \in H_\beta^{m,l}(\Omega)$ for all $m \geq l$ and*

$$\||D^k u| \Phi_{\beta+k-l}\|_{L^2(\Omega)} \leq Cd^{(k-l)}(k-l)!, \qquad k = l, l+1, \ldots$$

for some constants $C > 0$, $d \geq 1$ independent of k.

We will also need the weighted spaces $H_\beta^{s,l}(\Omega)$ for noninteger $s = k + \theta$, $0 < \theta < 1$, defined by the K-method of interpolation as

$$H_\beta^{k+\theta,l}(\Omega) = (H_\beta^{k,l}(\Omega), H_\beta^{k+1,l}(\Omega))_{\theta,\infty}.$$

If $u \in B_\beta^l(\Omega)$, there holds for any $k \geq l$ that

$$\|u\|_{H_\beta^{k+\theta,l}(\Omega)} \leq Cd^{k+\theta-l}\Gamma(k + \theta - l + 1).$$

The $B_\beta^l(\Omega)$ spaces describe the singular behavior as in (2.7) near corners and the analytic one in the interior. For the Stokes problem (2.1)-(2.3) the corresponding regularity statement is then (see [18]):
There exist numbers $0 \leq \overline{\beta}_i < 1$ depending on the angles at A_i, $i = 1, \ldots, M$, such that for $\vec{f} \in B_\beta^0(\Omega)^2$, $\overline{\beta} < \beta < 1$, there holds

$$\vec{u} \in B_\beta^2(\Omega)^2, \qquad p \in B_\beta^1(\Omega). \tag{2.8}$$

Remark that in general $B_\beta^2(\Omega)^2 \not\subset H^2(\Omega)^2$ and $B_\beta^1(\Omega) \not\subset H^1(\Omega)$.
Throughout, we assume that $\vec{f} \in L^2(\Omega)^2 \cap B_\beta^0(\Omega)^2$ for $\overline{\beta} < \beta < 1$.

3. GLS Discretization

3.1. Finite Element Spaces

In order to discretize (2.1)-(2.3) we start with an affine and shape regular mesh $\mathcal{T}$ that partitions Ω into triangular and quadrilateral elements $\{K\}$ subject to the following standard assumptions: (i). $\overline{\Omega} = \cup_{K \in \mathcal{T}} \overline{K}$. (ii). Each element K is the image of the reference element $\hat{K}$ under an affine element mapping F_K, $K = F_K(\hat{K})$. $\hat{K}$ is either the unit square $\hat{Q} = (0,1)^2$ or the unit triangle $\hat{T} = \{(x,y) : 0 < x < 1, \ 0 < y < 1-x\}$. (iii). The mesh is regular, i.e. the intersection $\overline{K} \cap \overline{K'}$ of two elements K and K' is either empty, a single vertex or an entire side. (iv). There exist constants C_1, C_2 such that $C_1 h_K \leq \rho_K \leq C_2 h_K$ with h_K denoting the diameter of K and ρ_K the diameter of the largest circle that can be inscribed into K.
If (iii) is not satisfied, the mesh is called irregular and contains hanging nodes. The meshwidth h of $\mathcal{T}$ is $h = \max_{K \in \mathcal{T}}\{h_K\}$. $\mathcal{T}$ is quasi-uniform if there are constants $\kappa_1, \kappa_2 > 0$ such that $\kappa_1 h \leq h_K \leq \kappa_2 \rho_K$ for all $K \in \mathcal{T}$.
The space of polynomials of degree $\leq k$ on an interval $I = (a,b)$ is denoted $\mathcal{P}^k(I)$. In two dimensions we introduce the reference polynomial spaces

$$\begin{aligned}
\hat{Q}^k &= \operatorname{span}\{\hat{x}_1^{\kappa_1}\hat{x}_2^{\kappa_2} : 0 \leq \kappa_1, \kappa_2 \leq k\}, \\
\hat{P}^k &= \operatorname{span}\{\hat{x}_1^{\kappa_1}\hat{x}_2^{\kappa_2} : 0 \leq \kappa_1, \kappa_2, \kappa_1 + \kappa_2 \leq k\}.
\end{aligned}$$

For a (triangular or quadrilateral) element $K \in \mathcal{T}$ we set then

$$
\begin{aligned}
\mathcal{Q}^k(K) &= \{q : K \to \mathbb{R} : q \circ F_K = p|_{\hat{K}}, \ p \in \hat{\mathcal{Q}}^k\}, \\
\mathcal{P}^k(K) &= \{q : K \to \mathbb{R} : q \circ F_K = p|_{\hat{K}}, \ p \in \hat{\mathcal{P}}^k\}.
\end{aligned}
$$

We define further

$$
\mathcal{S}^k(K) = \begin{cases} \mathcal{Q}^k(K) & \text{if } K \text{ is a quadrilateral,} \\ \mathcal{P}^k(K) & \text{if } K \text{ is a triangle.} \end{cases}
$$

To define the FE spaces we associate with each element $K \in \mathcal{T}$ a polynomial degree k_K. These degrees are stored in the vector $\underline{k} = \{k_K : K \in \mathcal{T}\}$ and we set $|\underline{k}| := \max\{k_K : K \in \mathcal{T}\}$. The hp-FE spaces are then

$$
S^{\underline{k},l}(\mathcal{T}) = \{u \in H^l(\Omega) : u|_K \in \mathcal{S}^{k_K}(K) : K \in \mathcal{T}\}, \qquad l = 0, 1.
$$

If the polynomial degree $k_K = k$ throughout the mesh, we write simply $S^{k,l}(\mathcal{T})$.

3.2. GLSFEM with Equal-Order Spaces

Let $\underline{k}$ be a polynomial degree vector on $\mathcal{T}$. In the GLS method we approximate the velocities and the pressure by equal-order FE spaces $\vec{V}_N \subset H^1(\Omega)^2$ and $M_N \subset L^2(\Omega)$, respectively. These spaces are given by

$$
\vec{V}_N := S^{\underline{k},1}(\mathcal{T})^2, \qquad M_N := S^{\underline{k},l}(\mathcal{T}), \qquad l = 0, 1. \tag{3.1}
$$

The index l indicates whether the pressure $p \in M_N$ is approximated by L^2- or H^1-conforming FE-functions. Less formally, these spaces are often called "$\mathcal{S}^k \times \mathcal{S}^k$ elements". Throughout, we set $\vec{V}_{N,0} := \vec{V}_N \cap H_0^1(\Omega)^2$ and $M_{N,0} := M_N \cap L_0^2(\Omega)$. The spaces in (3.1) satisfy the following inverse estimates (see [4, 24]):

Lemma 3.1 *There exists a constant $C_{inv} > 0$ only depending on the shape regularity constants of the mesh such that for all $\vec{u} \in \vec{V}_N$ and $p \in M_N$*

$$
C_{inv} \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|D^2 \vec{u}\|_{L^2(K)}^2 \leq \|\nabla \vec{u}\|_{L^2(\Omega)}^2, \qquad C_{inv} \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|\nabla p\|_{L^2(K)}^2 \leq \|p\|_{L^2(\Omega)}^2.
$$

$$
\tag{3.2}
$$

Definition 3.2 (GLS Finite Element Method) *Let $\vec{V}_N$ and M_N be the equal order spaces in (3.1) with*

$$
k_K \geq 2, \qquad \forall K \in \mathcal{T}. \tag{3.3}
$$

Let α be a user-specified stabilization parameter in the range

$$
0 < \alpha \leq \frac{C_{inv}}{2} \qquad \text{with } C_{inv} \text{ defined in (3.2).} \tag{3.4}
$$

The Galerkin Least Squares discretization of (2.1)-(2.3) is then:
Find a discrete velocity field $\vec{u}_N \in \vec{V}_{N,0}$ and a discrete pressure $p \in M_{N,0}$ such that

$$
B_\alpha(\vec{u}_N, p_N; \vec{v}, q) = F_\alpha(\vec{v}, q) \qquad \forall (\vec{v}, q) \in \vec{V}_{N,0} \times M_{N,0}
$$

where the perturbed form B_α and the perturbed functional F_α are given by

$$
\begin{aligned}
B_\alpha(\vec{u}, p; \vec{v}, q) \;:=\; & (\nabla \vec{u}, \nabla \vec{v}) - (\nabla \cdot \vec{v}, p) - (\nabla \cdot \vec{u}, q) && (3.5) \\
& - \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} (-\Delta \vec{u} + \nabla p, -\Delta \vec{v} + \nabla q)_{L^2(K)},
\end{aligned}
$$

$$
F_\alpha(\vec{v}, q) \;:=\; (\vec{f}, \vec{v}) - \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} (\vec{f}, -\Delta \vec{v} + \nabla q)_{L^2(K)}. \tag{3.6}
$$

Remark 3.3 For $\alpha = 0$ the GLS method in Definition 3.2 coincides with the standard Galerkin approach which is considered in Section 4.4 ahead.

Remark 3.4 Let $(\vec{u}, p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ be the exact solution of (2.1)-(2.3). Since the right-hand side $\vec{f}$ belongs to $L^2(\Omega)^2$, we have

$$
(-\Delta \vec{u} + \nabla p)|_K = \vec{f}|_K \in L^2(K)^2, \qquad K \in \mathcal{T}. \tag{3.7}
$$

This implies immediately that the GLSFEM is fully consistent without further regularity assumptions on the exact solution, that is we have

$$
B_\alpha(\vec{u}, p; \vec{v}, q) = F_\alpha(\vec{v}, q) \qquad \forall (\vec{v}, q) \in \vec{V}_{N,0} \times M_{N,0}. \tag{3.8}
$$

Consequently, there holds the orthogonality property

$$
B_\alpha(\vec{u} - \vec{u}_N, p - p_N; \vec{v}, q) = 0 \qquad \forall (\vec{v}, q) \in \vec{V}_{N,0} \times M_{N,0}. \tag{3.9}
$$

3.3. Stability

The following stability proposition is proved in [6] for a constant polynomial degree distribution where $k_K = k$ for all $K \in \mathcal{T}$. The extension to variable degree distributions $\underline{k}$ is straightforward. Nevertheless, we present the proof for the sake of completeness.

Proposition 3.5 *Assume (3.3), (3.4) and set $\gamma(N)^2 := \alpha |\underline{k}|^{-2}$. Then for every $(\vec{u}, p) \in \vec{V}_{N,0} \times M_{N,0}$ there exists $(\vec{v}, q) \in \vec{V}_{N,0} \times M_{N,0}$ such that*

$$
\begin{aligned}
B_\alpha(\vec{u}, p; \vec{v}, q) \;\geq\; & C(\|\vec{u}\|_{H^1(\Omega)}^2 + \gamma(N)^2 \|p\|_{L^2(\Omega)}^2), \\
\|\vec{v}\|_{H^1(\Omega)}^2 + \gamma(N)^2 \|q\|_{L^2(\Omega)}^2 \;\leq\; & C(\|\vec{u}\|_{H^1(\Omega)}^2 + \gamma(N)^2 \|p\|_{L^2(\Omega)}^2).
\end{aligned}
$$

Remark 3.6 Although this seems to be the best stability estimate for the GLSFEM known at the moment, it is suboptimal with respect to the approximation order $\underline{k}$ since the constant $\gamma(N)$ enters into the estimate. Hence, error bounds for the pure p-version GLSFEM fall two powers of $\underline{k}$ short from being quasioptimal, at least for the pressure [6]. However, the inf-sup condition in Proposition 3.5 is sufficient to establish exponential rates of convergence in the hp-FEM.

In addition, it can be seen that the estimate deteriorates for $\alpha = 0$, in agreement with the fact that Galerkin FEM with equal-order elements are instable (see Section 4.4 ahead).

Proof: Fix $(\vec{u}, p) \in \vec{V}_{N,0} \times M_{N,0}$. We proceed in several steps:

Step 1: We have

$$B_\alpha(\vec{u}, p; \vec{u}, -p) \geq C\|\vec{u}\|^2_{H^1(\Omega)} + \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|\nabla p\|^2_{L^2(K)}. \qquad (3.10)$$

To see (3.10), we use (3.2) and get

$$
\begin{aligned}
B_\alpha(\vec{u}, p; \vec{u}, -p) &= \|\nabla \vec{u}\|^2_{L^2(\Omega)} - \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|\Delta \vec{u}\|^2_{L^2(K)} + \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|\nabla p\|^2_{L^2(K)} \\
&\geq (1 - \frac{\alpha}{C_{inv}})\|\nabla \vec{u}\|^2_{L^2(\Omega)} + \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|\nabla p\|^2_{L^2(K)}.
\end{aligned}
$$

The assertion (3.10) follows with (3.4) and Poincaré's inequality.

Step 2: There exists $\vec{w} \in \vec{V}_{N,0}$ such that

$$\|\vec{w}\|_{H^1(\Omega)} \leq C\|p\|_{L^2(\Omega)}, \qquad (3.11)$$

$$B_\alpha(\vec{u}, p; -\vec{w}, 0) \geq -C\|\vec{u}\|^2_{H^1(\Omega)} + C\|p\|^2_{L^2(\Omega)} - C \sum_{K \in \mathcal{T}} (\frac{h_K^2}{k_K^2} + \alpha \frac{h_K^2}{k_K^4})\|\nabla p\|^2_{L^2(K)}.$$

To prove (3.11), we apply Lemma 4.2 in [6] observing (3.3): There exists a $\vec{w} \in \vec{V}_{N,0}$ such that

$$
\begin{aligned}
\|\vec{w}\|_{H^1(\Omega)} &\leq C\|p\|_{L^2(\Omega)}, \\
(\nabla \cdot \vec{w}, p) &\geq \|p\|^2_{L^2(\Omega)} - C\|p\|_{L^2(\Omega)} \left(\sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^2} \|\nabla p\|^2_{L^2(K)} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, using the continuity of B_α, the properties of $\vec{w}$, the Cauchy-Schwarz inequality and (3.2), (3.4) we arrive at

$$
\begin{aligned}
B_\alpha(\vec{u}, p; -\vec{w}, 0) &= B_\alpha(\vec{u}, 0; -\vec{w}, 0) + B_\alpha(0, p; -\vec{w}, 0) \\
&\geq -C\|\vec{u}\|_{H^1(\Omega)}\|\vec{w}\|_{H^1(\Omega)} + (\nabla \cdot \vec{w}, p) - \alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} (\nabla p, \Delta \vec{w})_{L^2(K)} \\
&\geq -C\|\vec{u}\|_{H^1(\Omega)}\|p\|_{L^2(\Omega)} + C\|p\|^2_{L^2(\Omega)} - C\|p\|_{L^2(\Omega)} \left(\sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^2} \|\nabla p\|^2_{L^2(K)} \right)^{\frac{1}{2}} \\
&\quad - C\|p\|_{L^2(\Omega)} \left(\alpha \sum_{K \in \mathcal{T}} \frac{h_K^2}{k_K^4} \|\nabla p\|^2_{L^2(K)} \right)^{\frac{1}{2}}.
\end{aligned}
$$

Above, each of the negative terms is estimated with $ab \leq \frac{\varepsilon}{2}a^2 + \frac{b^2}{2\varepsilon}$ with suitably chosen weights $\varepsilon > 0$. This yields (3.11).

Step 3: We prove the assertion. Define $(\vec{v}, q) = (\vec{u} - \delta\vec{w}, -p) \in \vec{V}_{N,0} \times M_{N,0}$ where $\delta > 0$ is still at our disposal. Using Step 1 and Step 2 we obtain

$$
\begin{aligned}
B_\alpha(\vec{u}, p; \vec{v}, q) &\geq (C_1 - \delta C_2)\|\vec{u}\|^2_{H^1(\Omega)} + \delta C_3\|p\|^2_{L^2(\Omega)} \\
&\quad + \sum_{K \in \mathcal{T}} (\alpha \frac{h_K^2}{k_K^4} - C_4\delta(\frac{h_K^2}{k_K^2} + \alpha \frac{h_K^2}{k_K^4}))\|\nabla p\|^2_{L^2(K)}.
\end{aligned}
$$

Selecting $\delta = \min(\frac{C_1}{2C_2}, \frac{1}{C_4}, \frac{\alpha}{3C_4|\underline{k}|^2}) \simeq \frac{\alpha}{|\underline{k}|^2}$, we get

$$B(\vec{u}, p; \vec{v}, q) \geq C\|\vec{u}\|_{H^1(\Omega)}^2 + C\alpha|\underline{k}|^{-2}\|p\|_{L^2(\Omega)}^2. \tag{3.12}$$

Further,

$$\|\vec{v}\|_{H^1(\Omega)}^2 + \alpha|\underline{k}|^{-2}\|q\|_{L^2(\Omega)}^2 \leq C(\|\vec{u}\|_{H^1(\Omega)}^2 + \delta^2\|p\|_{L^2(\Omega)}^2 + \alpha|\underline{k}|^{-2}\|p\|_{L^2(\Omega)}^2). \tag{3.13}$$

Since $\delta^2 \leq \gamma(N)^2$, the assertion follows from from (3.12) and (3.13). $\qquad\square$

Remark 3.7 The GLS discretization in Definition 3.2 has a unique solution due to Proposition 3.5 and the continuity properties of B_α.

3.4. Error Analysis

To capture singular solution behaviour near the corners we partition the mesh $\mathcal{T}$ into two disjoint sets $\mathcal{T}_0$ and $\mathcal{T}_1$ where $\mathcal{T}_0$ consists of all the elements that abut at a corner of the domain, i.e.

$$\mathcal{T}_0 := \{K \in \mathcal{T} : \overline{K} \cap A_i \neq \emptyset \text{ for some corner } A_i \text{ of } \Omega\}, \qquad \mathcal{T}_1 := \mathcal{T} \setminus \mathcal{T}_0. \tag{3.14}$$

Throughout, let $\gamma(N)$ be the stability constant from Proposition 3.5, that is

$$\gamma(N)^2 := \alpha|\underline{k}|^{-2}. \tag{3.15}$$

Lemma 3.8 *Assume (2.8), (3.3) and (3.4). Let $\vec{u}_I$ and p_I be arbitrary approximations in $\vec{V}_N$ and M_N, correspondingly. Then we have*

$$B_\alpha(\vec{u} - \vec{u}_I, p - p_I; \vec{v}, q)^2 \leq C\gamma(N)^{-2}(\|\vec{v}\|_{H^1(\Omega)}^2 + \gamma(N)^2\|q\|_{L^2(\Omega)}^2)(E_1^2 + E_2^2 + E_3^2)$$

for all $(\vec{v}, q) \in \vec{V}_N \times M_N$. Here,

$$E_1^2 = \|\vec{u} - \vec{u}_I\|_{H^1(\Omega)}^2 + \|p - p_I\|_{L^2(\Omega)}^2, \tag{3.16}$$

$$E_2^2 = \sum_{K \in \mathcal{T}_1} \frac{h_K^2}{k_K^4}(\|\Delta(\vec{u} - \vec{u}_I)\|_{L^2(K)}^2 + \|\nabla(p - p_I)\|_{L^2(K)}^2), \tag{3.17}$$

$$E_3^2 = \sum_{K \in \mathcal{T}_0} \frac{h_K^2}{k_K^4}\|\vec{f} + \Delta\vec{u}_I - \nabla p_I\|_{L^2(K)}^2. \tag{3.18}$$

Remark 3.9 Due to (2.8), $\vec{u}$ and p behave analytically in the domain covered by $\mathcal{T}_1$. Therefore, the term E_2 containing second order derivatives of $\vec{u}$ and first order derivatives of p is well defined.

Proof: With the inequality of Cauchy-Schwarz we have

$$|B_\alpha(\vec{u} - \vec{u}_I, p - p_I; \vec{v}, q)|$$
$$\leq \|(\vec{u} - \vec{u}_I)\|_{H^1(\Omega)}\|\vec{v}\|_{H^1(\Omega)} + \sqrt{2}\|\vec{v}\|_{H^1(\Omega)}\|p - p_I\|_{L^2(\Omega)}$$
$$+ \sqrt{2}\|\vec{u} - \vec{u}_I\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}$$
$$+ \alpha \sum_{K \in \mathcal{T}_1} \frac{h_K}{k_K^2}\| - \Delta(\vec{u} - \vec{u}_I) + \nabla(p - p_I)\|_{L^2(K)} \frac{h_K}{k_K^2}\| - \Delta\vec{v} + \nabla q\|_{L^2(K)}$$
$$+ \alpha \sum_{K \in \mathcal{T}_0} \frac{h_K}{k_K^2}\|\vec{f} + \Delta\vec{u}_I - \nabla p_I\|_{L^2(K)} \frac{h_K}{k_K^2}\| - \Delta\vec{v} + \nabla q\|_{L^2(K)},$$

where we used the differential equation in the elements of $\mathcal{T}_0$ (see (3.7)). We apply again Cauchy-Schwarz and the inverse inequalities in (3.2) for the terms $\| - \Delta \vec{v} + \nabla q \|_{L^2(K)}$. Due to (3.4), this results in

$$|B_\alpha(\vec{u} - \vec{u}_I, p - p_I; \vec{v}, q)|^2 \leq C(\|\vec{v}\|^2_{H^1(\Omega)} + \|q\|^2_{L^2(\Omega)})(E_1^2 + E_2^2 + E_3^2).$$

Finally, we note that $\|\vec{v}\|^2_{H^1(\Omega)} + \|q\|^2_{L^2(\Omega)} \leq \gamma(N)^{-2}(\|\vec{v}\|^2_{H^1(\Omega)} + \gamma(N)^2\|q\|^2_{L^2(\Omega)})$, which completes the proof. □

Proposition 3.10 *Assume (2.8), (3.3) and (3.4). Let $(\vec{u}_N, p_N) \in \vec{V}_{N,0} \times M_{N,0}$ be the GLSFEM solution. Then we have for any $(\vec{u}_I, p_I) \in \vec{V}_{N,0} \times M_N$*

$$\|\vec{u} - \vec{u}_N\|_{H^1(\Omega)} + \gamma(N)\|p - p_N\|_{L^2(\Omega)} \leq C\gamma(N)^{-1}(E_1^2 + E_2^2 + E_3^2)^{\frac{1}{2}} \qquad (3.19)$$

with E_1, E_2, E_3 defined in (3.16), (3.17) and (3.18), respectively.

Remark 3.11 If $(\vec{u}, p) \in H^s(\Omega)^2 \times H^{s-1}(\Omega)$ for some $s \geq 2$, we can choose $\mathcal{T}_0 = \emptyset$ and have $E_3 = 0$. Proposition 3.10 then gives optimal convergence rates in the h-version GLSFEM (cf. [11]). However, in the p-version GLSFEM, the rates from Proposition 3.10 are suboptimal (cf. Remark 3.6): In the approximation of the velocity one power of $\underline{k}$ is lost, whereas the pressure approximation falls two powers of $\underline{k}$ short of being optimal. In [6], it is discussed how this p-version result can be improved in certain situations. Remark that in the presence of reentrant corners the assumption $(\vec{u}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ does not hold true anymore and (2.8) is the realistic regularity assumption in that case. This requires a careful treatment of the elements in $\mathcal{T}_0$ near the corners.

Proof: Let $(\vec{u}_I, p_I) \in \vec{V}_{N,0} \times M_N$. Denote by Π the L^2-projection of $L^2(\Omega)$ onto $L_0^2(\Omega)$ given by $\Pi p = p - \frac{1}{|\Omega|} \int_\Omega p dx$. We decompose p_I into $p_I = p_{1,I} + p_{2,I}$ where $p_{2,I} = \frac{1}{|\Omega|} \int_\Omega p_I dx$ and $p_{1,I} = \Pi p_I$. The inf-sup condition in Proposition 3.5 for $(\vec{u}_N - \vec{u}, p_N - p_{1,I})$ implies the existence of $(\vec{v}, q) \in \vec{V}_{N,0} \times M_{N,0}$ such that

$$(\|\vec{v}\|^2_{H^1(\Omega)} + \gamma(N)^2\|q\|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq C,$$
$$(\|\vec{u}_N - \vec{u}_I\|^2_{H^1(\Omega)} + \gamma(N)^2\|p_N - p_{1,I}\|^2_{L^2(\Omega)})^{\frac{1}{2}} \leq B_\alpha(\vec{u}_N - \vec{u}_I, p_N - p_{1,I}; \vec{v}, q).$$

We have $B_\alpha(\vec{u}_N - \vec{u}_I, p_N - p_{1,I}; \vec{v}, q) = B_\alpha(\vec{u} - \vec{u}_I, p - p_{1,I}; \vec{v}, q)$ due to the Galerkin orthogonality (3.9). Thus, Lemma 3.8 yields

$$\|\vec{u}_N - \vec{u}_I\|_{H^1(\Omega)} + \gamma(N)\|p_N - p_{1,I}\|_{L^2(\Omega)} \leq C\gamma(N)^{-1}(E_1^2 + E_2^2 + E_3)^{\frac{1}{2}}$$

with $p_{1,I}$ entering in the terms E_1, E_2 and E_3. Hence, the triangle inequality yields the desired bound (3.19) for $(\vec{u}_I, p_{1,I})$. Since

$$\|p - p_{1,I}\|_{L^2(\Omega)} = \|\Pi(p - p_I)\|_{L^2(\Omega)} \leq \|p - p_I\|_{L^2(\Omega)}, \qquad \nabla p_{1,I} = \nabla p_I,$$

$p_{1,I}$ can be replaced by p_I. □

Corollary 3.12 *Let $(\vec{u}, p)$ satisfy (2.8) with $\beta_{max} = \max\{\beta_i\} \in (0,1)$. Let $\mathcal{T}_0$ consist only of quadrilateral elements and assume (3.3), (3.4). Then we have for any $(\vec{u}_I, p_I) \in \vec{V}_{N,0} \times M_N$ the a-priori estimate*

$$\|\vec{u} - \vec{u}_N\|_{H^1(\Omega)} + \|p - p_N\|_{L^2(\Omega)} \le C\gamma(N)^{-2}(E_1^2 + E_2^2 + \mathcal{E}_3^2)^{\frac{1}{2}} \qquad (3.20)$$

with E_1, E_2 defined in (3.16), (3.17) and

$$\mathcal{E}_3^2 := \sum_{K \in \mathcal{T}_0} h_K^{2-2\beta_{max}}(\|\vec{f}\|_{L^2(K)}^2 + \|\vec{u}_I\|_{H_\beta^{2,2}(K)}^2 + \|p_I\|_{H_\beta^{1,1}(K)}^2).$$

Proof: Let $K \in \mathcal{T}$ be a quadrilateral element and let r denote the distance to one of its vertices. Then we claim for $\beta \ge 0$ the weighted inverse inequality

$$\|\pi\|_{L^2(K)}^2 \le C(\beta)\frac{k_K^{4\beta}}{h_K^{2\beta}}\int_K r^{2\beta}\pi^2 dx, \qquad \forall \pi \in \mathcal{S}^{k_K}(K). \qquad (3.21)$$

To prove (3.21), we may assume that $K = (-1,1)^2$ and consider the vertex $A = (-1,1)$, the general case follows by a scaling argument. Recall from [24] the one dimensional result that for all $\beta \ge 0$ holds

$$\int_{-1}^1 q^2(x)dx \le C(\beta)k^{2\beta}\int_{-1}^1 q^2(x)(1-x^2)^\beta dx, \qquad \forall q \in \mathcal{P}^k((-1,1)).$$

A tensor product argument yields then easily

$$\begin{aligned}\int_K \pi^2 dxdy &\le& Ck_K^{4\beta}\int_K \pi^2(1-x^2)^\beta(1-y^2)^\beta dxdy \\ &\le& Ck_K^{4\beta}(\int_K \pi^2(1-x^2)^{2\beta}dxdy + \int_K \pi^2(1-y^2)^{2\beta}dxdy). \quad (3.22)\end{aligned}$$

The distance from a point (x,y) to A is given by $r^2 = (1+x)^2 + (1+y)^2$. We have $(1-x^2)^{2\beta} \le C(1+x)^{2\beta} \le r^{2\beta}$ and the same estimate holds true for $(1-y^2)^{2\beta}$. Referring to (3.22) finishes the proof of (3.21).

Corollary 3.12 is now a direct consequence of Proposition 3.10 and (3.21). □

4. Exponential Rates of Convergence

4.1. Geometric Meshes

In order to resolve singular solution behaviour near corners we introduce meshes that are geometrically refined towards the vertices $\{A_i\}_{i=1}^M$. We define first the basic geometric meshes on $\hat{Q} = (0,1)^2$.

Definition 4.1 *Fix $n \in \mathbb{N}_0$ and $\sigma \in (0,1)$. On $\hat{Q}$, the (irregular) geometric mesh $\tilde{\Delta}_{n,\sigma}$ with $n+1$ layers and grading factor σ is created recursively as follows: If $n = 0$, $\tilde{\Delta}_{0,\sigma} = \{\hat{Q}\}$. Given $\tilde{\Delta}_{n,\sigma}$ for $n \ge 0$, $\tilde{\Delta}_{n+1,\sigma}$ is generated by subdividing that square K with $0 \in \overline{K}$ into four smaller rectangles by dividing its sides in a $\sigma : (1-\sigma)$ ratio. With each $\tilde{\Delta}_{n,\sigma}$ a (regular) geometric mesh $\Delta_{n,\sigma}$ can be associated by removing the hanging nodes by additional triangles.*

Examples of basic geometric meshes are shown in Figure 1. The elements $\{K_{ij}\}$ of $\widetilde{\Delta}_{n,\sigma}$ are numbered as indicated there. The elements K_{1j}, K_{2j} and K_{3j} constitute the layer j.
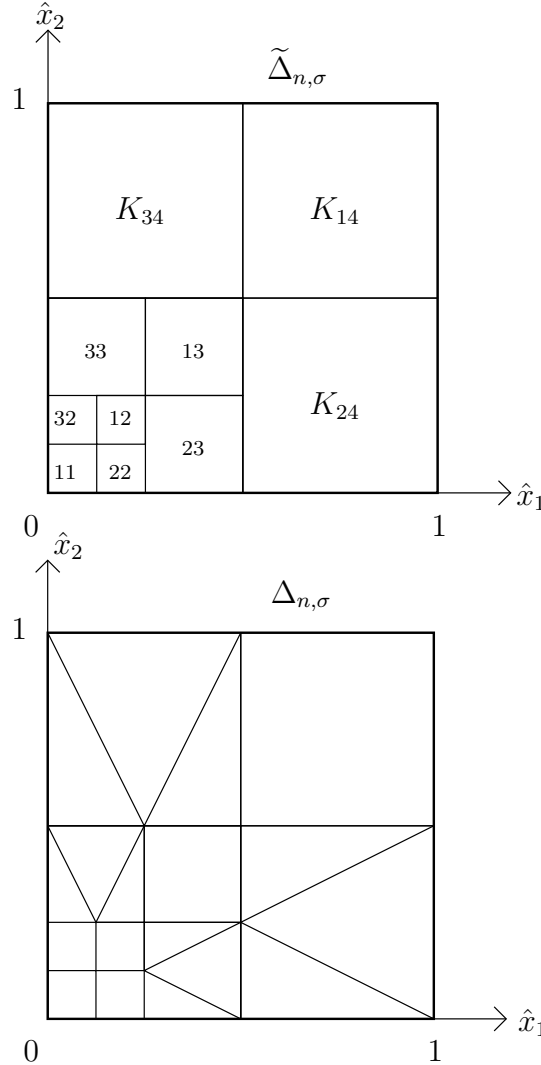

Figure 1: The geometric meshes $\widetilde{\Delta}_{n,\sigma}$ and $\Delta_{n,\sigma}$ with $n = 3$ and $\sigma = 0.5$.

Definition 4.2 *A geometric mesh $\mathcal{T}_{n,\sigma}$ in the polygon $\Omega \subset \mathbb{R}^2$ is obtained by mapping the basic geometric meshes $\Delta_{n,\sigma}$ from $\hat{Q}$ affinely to a vicinity of each convex corner of Ω. At reentrant corners three suitably scaled copies of $\Delta_{n,\sigma}$ are used (as shown in Figure 2). The remainder of Ω is subdivided with a fixed affine quasi-uniform and regular partition.*

In Figure 2 this local geometric refinement is illustrated. For ease of exposition we consider only mesh patches that are identically refined with a fixed σ and n, although different grading factors and numbers of layers may be used for the partition of each corner patch.
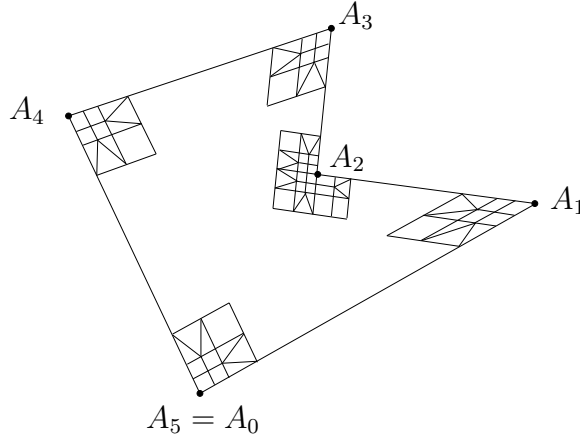
Figure 2: Local geometric refinement near vertices of Ω.

Definition 4.3 *A polynomial degree distribution $\underline{k}$ on a geometric mesh $\mathcal{T}_{n,\sigma}$ is called linear with slope $\mu > 0$ if the elemental polynomial degrees are layerwise constant in the geometric patches and given by $k_j := \max(2, \lfloor \mu j \rfloor)$ in layer j, $j = 1, \ldots, n+1$. In the interior of the domain the elemental polynomial degree is set constant to $\max(2, \lfloor \mu(n+1) \rfloor)$.*

4.2. An hp Approximation Result

On geometric meshes $\mathcal{T}_{n,\sigma}$ a function in $B_\beta^l(\Omega)$ with $l > 1$ can be approximated in $H^{l-1}(\Omega)$ at an exponential rate of convergence (see [15]). The same result holds true for $l = 1$, i.e. a pressure in $B_\beta^1(\Omega)$ can be approximated exponentially by a L^2-conforming FE-function in $S^{\underline{k},0}(\mathcal{T}_{n,\sigma})$. This has been proved for example in [20] in the context of boundary element methods for certain values of β. However, we are mainly interested in continuous pressure functions where $M_N = S^{\underline{k},1}(\mathcal{T}_{n,\sigma})$. Therefore, we construct an H^1-conforming interpolant that approximates $B_\beta^1(\Omega)$-functions at exponential rates of convergence in $L^2(\Omega)$. To do so, we modify the arguments given in [15, 16, 24] and in the elements abutting on a solution singularity we use a weighted Poincaré inequality (see [20] where the same inequality has been established for values of β in $(\frac{1}{2}, 1)$).

Our main result on the *hp*-approximation in $S^{\underline{k},1}(\mathcal{T}_{n,\sigma})$ is:

Theorem 4.4 *Let $l = 1, 2$ and $f \in B_\beta^l(\Omega)$ for some $\beta \in (0, 1)$. Let $\mathcal{T}_{n,\sigma}$ be a geometric mesh on Ω. Then there exists a $\mu_0 > 0$ such that for linearly increasing polynomial degree vectors $\underline{k}$ with slope $\mu \geq \mu_0$ there is an interpolant $\Psi^l \in S^{\underline{k},1}(\mathcal{T}_{n,\sigma})$ that satisfies*

$$\| f - \Psi^l \|_{H^{l-1}(\hat{Q})}^2 + \sum_{K \in (\mathcal{T}_{n,\sigma})_1} \frac{h_K^2}{k_K^4} |f - \Psi^l|_{H^l(K)}^2 \leq C \exp(-bN^{1/3})$$

with C and b independent of $N = \dim(S^{\underline{k},1}(\mathcal{T}_{n,\sigma}))$. On the first elements

$$K_A = \cup \{ K \in \mathcal{T}_{n,\sigma} : \overline{K} \cap A \neq \emptyset \}$$

near a vertex A the approximant $\Psi^2|_{K_A}$ is given by the (piecewise) bilinear nodal interpolant and $\Psi^1|_{K_A} = \frac{1}{|K_A|}\int_{K_A} f dx$. We have

$$\|\Psi^l\|_{H_\beta^{l,l}(K_A)} \leq C\|f\|_{H_\beta^{l,l}(K_A)}.$$

Additionally, if $f \in B_\beta^2(\Omega) \cap H_0^1(\Omega)$, the interpolant Ψ^2 can be chosen to satisfy the zero boundary conditions.

The proof of Theorem 4.4 is rather lengthy and technical. We postpone it to Section 5.

4.3. Convergence Results

The next result establishes exponential rates of convergence for the hp-GLSFEM, irrespective of whether the pressures are approximated by H^1- or L^2-conforming spaces.

Theorem 4.5 *Assume that the exact solution $(\vec{u}, p)$ of the Stokes equations satisfies (2.8) for some $\beta \in (0,1)$. Discretize these equations with the GLSFEM using the equal order spaces in (3.1) with $l = 0$ or $l = 1$ on a geometric mesh $\mathcal{T}_{n,\sigma}$ assuming (3.3), (3.4). Let $(\vec{u}_N, p_N) \in \vec{V}_{N,0} \times M_{N,0}$ be the discrete solution. Then there exists $\mu_0 = \mu_0(\sigma, \beta) > 0$ such that for linear degree vectors $\underline{k}$ with slope $\mu \geq \mu_0$ there holds the error estimate*

$$\|\vec{u} - \vec{u}_N\|_{H^1(\Omega)} + \|p - p_N\|_{L^2(\Omega)} \leq C \exp(-bN^{\frac{1}{3}})$$

with constants $C, b > 0$ independent of $N = \dim(\vec{V}_N) \approx \dim(M_N)$ (but depending on μ, σ, β). Moreover, the constant C depends on the stabilization parameter α as in (3.20).

Proof: Assume first that $M_N = S^{\underline{k},1}(\mathcal{T}_{n,\sigma})$. Insert the approximants $\vec{u}_I = \vec{\Psi}^2 = (\Psi^2, \Psi^2)$ and $p_I = \Psi^1$ from Theorem 4.4 into the abstract error bound (3.20). Using the interpolation properties yields

$$\|\vec{u} - \vec{u}_N\|_{H^1(\Omega)}^2 + \|p - p_N\|_{L^2(\Omega)}^2 \leq C \exp(-bN^{1/3})$$

$$+ C\sum_{i=1}^M \text{diam}(K_{A_i})^{2-2\beta_{max}}(\|\vec{f}\|_{L^2(K_{A_i})}^2 + \|\vec{u}\|_{H_\beta^{2,2}(K_{A_i})}^2 + \|p\|_{H_\beta^{1,1}(K_{A_i})}^2).$$

Here, $\{A_i\}_{i=1}^M$ are the vertices of Ω and $K_{A_i} := \{K \in \mathcal{T}_{n,\sigma} : \overline{K} \cap A_i \neq \emptyset\}$. The diameters of the sets K_{A_i} are exponentially small, i.e.

$$\text{diam}(K_{A_i})^{2-2\beta_{max}} \leq C\sigma^{2n(1-\beta_{max})} \leq C \exp(-bN^{1/3}).$$

This proves the assertion if $M_N = S^{\underline{k},1}(\mathcal{T}_{N,\sigma})$.
If now $M_N = S^{\underline{k},0}(\mathcal{T}_{n,\sigma})$, we have $S^{\underline{k},0}(\mathcal{T}_{n,\sigma}) \supset S^{\underline{k},1}(\mathcal{T}_{n,\sigma})$ and the approximant constructed in Theorem 4.4 can be employed to bound (3.20) in this case as well. $\quad\square$

Remark 4.6 If the polynomial degree is chosen to be constant throughout the mesh, i.e. $\vec{V}_N = S^{k,1}(\mathcal{T}_{n,\sigma})^2$ and $M_N = S^{k,l}(\mathcal{T}_{n,\sigma})$, $l = 0, 1$, exponential convergence is still obtained by choosing k proportionally to the number n of layers. This is due to the fact that for $k \sim n$ the interpolant with linearly increasing polynomial approximation order constructed in the proof of Theorem 4.4 can still be used in the error estimate (3.20).

Remark 4.7 The techniques applied in Section 5 to prove Theorem 4.4 allow us also to establish exponential rates of convergence for geometric meshes with hanging nodes where the local mesh refinement towards corners is obtained by mapping the irregular mesh patches $\tilde{\Delta}_{n,\sigma}$. Of course, other geometric refinement strategies are imaginable.

4.4. A Remark on Galerkin hp-FEM

For $\alpha = 0$ the GLSFEM in Definition 3.2 results in the Galerkin scheme:
Find $(\vec{u}_N, p_N) \in \vec{V}_{N,0} \times M_{N,0}$ such that

$$B_0(\vec{u}_N, p_N; v, q) = F_0(v, q) \qquad \forall (\vec{v}, q) \in \vec{V}_{N,0} \times M_{N,0} \tag{4.1}$$

(with B_0 and F_0 defined in (2.5), (2.6)).
However, for the equal-order spaces in (3.1) this approach is highly instable (cf. [5, 8, 14]), since the Babuška-Brezzi stability condition is not satisfied. To obtain stable discretizations different polynomial orders have to be chosen for the velocities and the pressure: In [25, 27], e.g., it is shown that "$\mathcal{S}^k \times \mathcal{S}^{k-2}$" elements are stable on geometric meshes with an inf-sup constant $\gamma(N)$ depending algebraically on the approximation order $\underline{k}$. These elements lead to quasioptimal error estimates and hence the approximant in Theorem 4.4 yields exponential convergence for this Galerkin approach as well:

Corollary 4.8 *Assume that the exact solution $(\vec{u}, p)$ of the Stokes equations satisfies (2.8) for some $\beta \in (0, 1)$. Discretize these equations with the Galerkin scheme (4.1) using "$\mathcal{S}^k \times \mathcal{S}^{k-2}$" elements, i.e.*

$$\vec{V}_N = S^{\underline{k},1}(\mathcal{T}_{n,\sigma})^2, \qquad M_N = S^{\underline{k}-2,l}(\mathcal{T}_{n,\sigma}), \quad l = 0, 1,$$

on a geometric mesh $\mathcal{T}_{n,\sigma}$ assuming (3.3). Let $(\vec{u}_N, p_N) \in \vec{V}_{N,0} \times M_{N,0}$ be the discrete solution. Then there exists $\mu_0 = \mu_0(\sigma, \beta) > 0$ such that for linear degree vectors $\underline{k}$ with slope $\mu \geq \mu_0$ there holds the error estimate

$$\|\vec{u} - \vec{u}_N\|_{H^1(\Omega)} + \|p - p_N\|_{L^2(\Omega)} \leq C \exp(-bN^{\frac{1}{3}})$$

with constants $C, b > 0$ independent of $N = \dim(\vec{V}_N) \approx \dim(M_N)$ (but depending on μ, σ, β).

If the polynomial degree is constant throughout the mesh, i.e. $\vec{V}_N = S^{k,1}(\mathcal{T}_{n,\sigma})^2$ and $M_N = S^{k-2,l}(\mathcal{T}_{n,\sigma})$, $l = 0, 1$, exponential convergence is still obtained by choosing k proportionally to the number n of layers.

5. Approximation of Functions in $\mathbf{B}_\beta^l(\Omega)$ for $l = 1, 2$

This section is devoted to the proof of Theorem 4.4. Section 5.1 collects some auxiliary facts about B_β^l-spaces. In Section 5.2 we show that B_β^l-functions can be approximated continuously at exponential rates of convergence on the basic geometric meshes. Finally, the proof of Theorem 4.4 is given in Section 5.3.

5.1. Auxiliary Results

In this subsection the spaces $H_\beta^{m,l}(\Omega)$ are equipped with the weight $\Phi_\beta(x) = r^\beta$ for some $\beta \in [0,1)$ with (r, θ) denoting polar coordinates at the origin. We assume the elements $\{K\}$ to be affine and shape regular.

We remark that $H_\beta^{2,2}(K) \hookrightarrow C(\overline{K})$ such that point evaluation is allowed for $H_\beta^{2,2}$-functions [3]. For the linear/bilinear interpolant there holds:

Proposition 5.1 *Let K be an element with vertex $A = 0$. Let $f \in H_\beta^{2,2}(K)$ for some $\beta \in [0,1)$. Then the linear/bilinear interpolant $I_K f$ of f at the vertices of K satisfies $\|f - I_K f\|_{H_\beta^{2,2}(K)}^2 \leq C|f|_{H_\beta^{2,2}(K)}^2$ and*

$$\|f - I_K f\|_{L^2(K)}^2 \leq C h_K^{2(2-\beta)} |f|_{H_\beta^{2,2}(K)}^2, \qquad \|f - I_K f\|_{H^1(K)}^2 \leq C h_K^{2(1-\beta)} |f|_{H_\beta^{2,2}(K)}^2.$$

Proof: The proof can be found in [16, 24]. $\qquad\square$

Subsequently, we construct an interpolant in $H_\beta^{1,1}(K)$ satisfying completely analogous estimates. To this end, we follow the lines of [16, 24], i.e. we establish a compactness property and then apply a Bramble-Hilbert type argument. We proceed in three propositions:

Proposition 5.2 *Let $\hat{K}$ be the reference element and let $f \in H_\beta^{1,1}(\hat{K})$ for some $\beta \in [0,1)$. Then we have $r^\beta f \in H^1(\hat{K})$ and $\|r^\beta f\|_{H^1(\hat{K})} \leq C\|f\|_{H_\beta^{1,1}(\hat{K})}$.*

Proof: If $\beta = 0$, this is trivial. Thus, let $\beta > 0$ and assume first $\hat{K} = \hat{T}$.
We set $g := r^\beta f$. Then $\|g\|_{L^2(\hat{T})} \leq C\|f\|_{H_\beta^{1,1}(\hat{T})}$ and it remains to bound $D^1 g$. Since

$$|D^1 g|^2 = (\beta r^{\beta-1} f + r^\beta f_r)^2 + \frac{1}{r^2}(r^\beta f_\theta)^2 \leq C\{r^{2(\beta-1)} f^2 + r^{2\beta}|D^1 f|^2\},$$

it is sufficient to show that $\|r^{\beta-1} f\|_{L^2(\hat{T})}^2 \leq C\|f\|_{H_\beta^{1,1}(\hat{T})}^2$. From the inequality of Hardy (see [3, Lemma 4.3]) follows the existence of a constant m such that

$$\int_{\hat{T}} r^{2(\beta-1)}|f - m|^2 dx \;\; \leq \;\; C \int_{\hat{T}} r^{2\beta}|D^1 f|^2 dx, \tag{5.1}$$

$$|m|^2 \;\; \leq \;\; C(\|f\|_{L^2(\hat{T})}^2 + \int_{\hat{T}} r^{2\beta}|D^1 f|^2 dx). \tag{5.2}$$

Let $S = \{(r, \theta) : 0 < r < R, \; 0 < \theta < \theta_0\}$ be a sector containing the triangle $\hat{T}$. We get with (5.1), (5.2)

$$
\begin{aligned}
\int_{\hat{T}} r^{2(\beta-1)} f^2 dx & \leq C \int_{\hat{T}} r^{2(\beta-1)} |f - m|^2 dx + C \int_{\hat{T}} r^{2(\beta-1)} |m|^2 dx \\
& \leq C \int_{\hat{T}} r^{2\beta} |D^1 f|^2 dx + C|m|^2 \int_S r^{2(\beta-1)} dx.
\end{aligned}
$$

The last integral $\int_S r^{2(\beta-1)} dx$ is bounded for $\beta > 0$, which proves the assertion for the reference triangle.

To obtain the same result for $\hat{Q}$, we split $\hat{Q}$ into $\hat{T}$ and a remainder T_1. The assertion follows now directly from the triangle proof and the fact that $\mathrm{dist}(0, T_1) > 0$. $\qquad \square$

Proposition 5.3 *Let $\hat{K}$ be the reference element. Then $H_\beta^{1,1}(\hat{K})$ is compactly imbedded into $L^2(\hat{K})$ for every $\beta \in [0, 1)$.*

Proof: Note that for $\beta = 0$ this is Rellich's theorem. Let $\{f_j\}$ be a bounded sequence in $H_\beta^{1,1}(\hat{K})$. Define $g_j := r^\beta f_j$. By Proposition 5.2, $\{g_j\}$ is bounded in $H^1(\hat{K})$. Choose $s > 1$ such that $2\beta s < 2$ (this is possible, since $\beta \in [0, 1)$) and let $1 \leq s' < \infty$ be such that $\frac{1}{s} + \frac{1}{s'} = 1$. By the theorem of Rellich, $H^1(\hat{K})$ is compactly imbedded into $L^r(\hat{K})$ for $r \in [1, \infty)$. So there exists a subsequence (again denoted by $\{g_j\}$) which converges to a function g in $L^{2s'}(\hat{K})$. Define $f := r^{-\beta} g$. With the inequality of Hölder we get

$$
\int_{\hat{K}} f^2 dx = \int_{\hat{K}} r^{-2\beta} g^2 dx \leq \Big(\int_{\hat{K}} r^{-2\beta s} dx\Big)^{\frac{1}{s}} \Big(\int_{\hat{K}} g^{2s'} dx\Big)^{\frac{1}{s'}} < \infty,
$$

since by construction $\int_{\hat{K}} r^{-2\beta s} dx < \infty$. Hence, $f \in L^2(\hat{K})$. Analogously, we conclude that $f_j \to f$ in $L^2(\hat{K})$, which finishes the proof. $\qquad \square$

Proposition 5.4 *Let K be an element with vertex $A = 0$. Let $f \in H_\beta^{1,1}(K)$ for some $\beta \in [0, 1)$ and $\overline{f} = \frac{1}{|K|} \int_K f dx$. Then there holds*

$$
\|f - \overline{f}\|_{L^2(K)}^2 \leq C h_K^{2(1-\beta)} |f|_{H_\beta^{1,1}(K)}^2, \qquad \|f - \overline{f}\|_{H_\beta^{1,1}(K)}^2 \leq C|f|_{H_\beta^{1,1}(K)}^2.
$$

Proof: Remark that for $\beta = 0$ this is Poincaré's inequality. First, let K be the reference element. We claim that there exists a constant $C > 0$ such that

$$
\|f\|_{H_\beta^{1,1}(K)} \leq C|f|_{H_\beta^{1,1}(K)} + C|\overline{f}|, \qquad f \in H_\beta^{1,1}(K). \tag{5.3}
$$

We prove (5.3) by contradiction: If (5.3) is not valid, there exists a sequence $\{f_j\}$ in $H_\beta^{1,1}(K)$ such that $\|f_j\|_{H_\beta^{1,1}(K)} = 1$ and

$$
|f_j|_{H_\beta^{1,1}(K)} + |\overline{f_j}| \to 0 \qquad \text{for } j \to \infty. \tag{5.4}
$$

By Proposition 5.3, $H_\beta^{1,1}(K)$ is compactly imbedded into $L^2(K)$. We can choose a subsequence (again denoted by $\{f_j\}$) which is a Cauchy sequence in $L^2(K)$. With

(5.4) follows that $\{f_j\}$ is also a Cauchy sequence in $H_\beta^{1,1}(K)$. $H_\beta^{1,1}(K)$ is complete and there exists $f \in H_\beta^{1,1}(K)$ with $f_j \to f$ in $H_\beta^{1,1}(K)$. Taking into account (5.4) we get

$$|f|_{H_\beta^{1,1}(K)} \le |f - f_j|_{H_\beta^{1,1}(K)} + |f_j|_{H_\beta^{1,1}(K)} \to 0, \qquad j \to \infty,$$

that is $|f|^2_{H_\beta^{1,1}(K)} = \int_K r^{2\beta} |D^1 f|^2 dx = 0$. Therefore, $|D^1 f| = 0$ and f is constant on K. Since $|\overline{f_j}| \to 0$ $(j \to \infty)$, it follows that $\overline{f} = 0$ and $f = 0$. But this is a contradiction to $1 = \lim_{j \to \infty} \|f_j\|_{H_\beta^{1,1}(K)} = \|f\|_{H_\beta^{1,1}(K)}$. We conclude that (5.3) must be valid.

Applying (5.3) to $f - \overline{f}$ yields the assertion on the reference element. A scaling argument finishes the proof. $\qquad\square$

Remark 5.5 Proposition 5.4 is a weighted Poincaré inequality. The same result has been obtained in [20] for values of β in the range $\beta \in (\frac{1}{2}, 1)$.

Remark 5.6 *Let $\mathcal{T}$ be a shape regular and affine mesh and $K_A = \cup\{K \in \mathcal{T} : \overline{K} \cap A \ne \emptyset\}$ be the union of elements abutting on the vertex $A = 0$. Setting $\overline{f} = \frac{1}{|K_A|}\int_{K_A} f dx$ we get with the same techniques*

$$\|f - \overline{f}\|_{L^2(K_A)} \le C \mathrm{diam}(K_A)^{2(1-\beta)} \|f\|^2_{H_\beta^{1,1}(K_A)}, \quad \|f - \overline{f}\|_{H_\beta^{1,1}(K_A)} \le C\|f\|^2_{H_\beta^{1,1}(K_A)}.$$

We need the following versions of weighted trace theorems:

Lemma 5.7 *Let K be an element with vertex $A = 0$ and $\beta \in [0, 1)$. Let γ be a side of K such that $0 \notin \overline{\gamma}$. We have:*

(i) *If $f \in H_\beta^{1,1}(K)$ and $\overline{f} = \frac{1}{|K|}\int_K f dx = 0$, then $\|f\|^2_{L^2(\gamma)} \le C h_K^{1-2\beta} |f|^2_{H_\beta^{1,1}(K)}$.*

(ii) *If $f \in H_\beta^{2,2}(K)$ and $f = 0$ at the vertices of K, then*

$$\|f\|^2_{L^2(\gamma)} \le C h_K^{3-2\beta} |f|^2_{H_\beta^{2,2}(K)}, \qquad \|f\|^2_{H^1(\gamma)} \le C h_K^{1-2\beta} |f|^2_{H_\beta^{2,2}(K)}.$$

If $\beta = 0$, γ can be an arbitrary side of K.

Proof: The proof of (ii) can be found in [24]. (i) follows analogously, using Proposition 5.4. $\qquad\square$

The next results are concerned with the extension of given polynomial trace functions into a domain:

Lemma 5.8 *The following extensions are possible:*

(i) *Let K be a triangle or a quadrilateral and let γ be a side of K. Let v be a polynomial in $\mathcal{P}^k(\gamma)$ which vanishes at the endpoints of γ. Then there exists $V \in \mathcal{S}^k(K)$ such that $V \equiv v$ on γ, $V \equiv 0$ on the other sides of K, $\|V\|^2_{L^2(K)} \le C h_K \|v\|^2_{L^2(\gamma)}$ and $\|V\|^2_{H^1(K)} \le C h_K |v|^2_{H^1(\gamma)}$.*

(ii) Let K be a quadrilateral element and let γ be a side of K. Let v be a polynomial in $\mathcal{P}^k(\gamma)$. Then there exists $V \in \mathcal{Q}^k(K)$ such that $V \equiv v$ on γ, $V \equiv 0$ on the side opposite to γ and $\|V\|_{L^2(K)}^2 \leq Ch_K\|v\|_{L^2(\gamma)}^2$. Further, if γ_1 is a side adjacent to γ, then $\|V\|_{L^\infty(\gamma_1)} \leq \|V\|_{L^\infty(\gamma)}$.

(iii) Let K be a quadrilateral element, $\overline{\gamma}_1 = [A, B]$ and $\overline{\gamma}_2 = [B, C]$ two adjacent sides of K with common endpoint B. Let v_1 and v_2 be polynomials in $\mathcal{P}^k(\gamma_1)$ and $\mathcal{P}^k(\gamma_2)$, respectively. Let $v_1(A) = v_2(C) = 0$ and $v_1(B) = v_2(B)$. Then there exists a polynomial V in $\mathcal{Q}^k(K)$ such that $V \equiv v_1$ on γ_1, $V \equiv v_2$ on γ_2, $V \equiv 0$ on the other sides of K and $\|V\|_{L^2(K)}^2 \leq Ch_K k^2(\|v_1\|_{L^2(\gamma_1)}^2 + \|v_2\|_{L^2(\gamma_2)}^2)$.

Proof: (i) and (ii) are easy modifications of Lemma 4.55 in [24].
We verify (iii) on the reference square: Let $\gamma_1 = \{(0, y) : 0 < y < 1\}$ and $\gamma_2 = \{(x, 0) : 0 < x < 1\}$. Define $V(x, y) = v_1(y)(1 - x) + [v_2(x) - v_2(0)(1 - x)](1 - y)$. Surely, $V(0, y) = v_1(y)$ and $V(x, 0) = v_2(x)$. Further,

$$\int_0^1 \int_0^1 V(x, y)^2 dx dy \leq C \int_0^1 v_1^2(y) dy + C \int_0^1 v_2^2(x) dx + C v_2^2(0).$$

The last term can be bounded using the one-dimensional inverse inequality, i.e. $\|v_2\|_{L^\infty(\gamma_2)}^2 \leq Ck^2\|v_2\|_{L^2(\gamma_2)}^2$, which yields the desired result. $\square$

5.2. Approximation on the Basic Geometric Meshes

In this subsection we address the *continuous* approximation of a function f in $B_\beta^l(\hat{Q})$, $l = 1, 2$, with weight $\Phi_\beta(x) = |x|^\beta$ for some $\beta \in (0, 1)$ on a basic geometric mesh $\Delta_{n,\sigma}$ with underlying irregular mesh $\widetilde{\Delta}_{n,\sigma}$. Note that if $K \in \Delta_{n,\sigma}$ or $K \in \widetilde{\Delta}_{n,\sigma}$ belongs to layer j, we have

$$\begin{aligned} h_K &= \sqrt{2}\sigma^n, & j = 1, \\ C_1\sigma^{n+2-j} &\leq h_K \leq C_2\sigma^{n+2-j}, & 2 \leq j \leq n + 1. \end{aligned} \tag{5.5}$$

The distance to the origin is bounded by

$$\text{dist}(0, K) \leq C\sigma^{n+2-j}, \qquad 2 \leq j \leq n + 1. \tag{5.6}$$

Let the polynomial degree vector $\underline{k}$ on $\Delta_{n,\sigma}$ and $\widetilde{\Delta}_{n,\sigma}$ be layerwise constant, i.e. $\underline{k} = \{k_{K_{ij}} = k_{K_{ij}^l} =: k_j\}_{j=1}^{n+1}$ for some $k_j \geq 2$.

Lemma 5.9 *Let $l = 1, 2$, $f \in B_\beta^l(\hat{Q})$ for $\beta \in (0, 1)$ and $\{k_j\}$ be a layerwise constant polynomial degree distribution. On the elements $\{K_{ij}\}$ of $\widetilde{\Delta}_{n,\sigma}$ away from the origin there is a polynomial $\varphi_{K_{ij}}$ of degree k_j in each variable such that $f = \varphi_{K_{ij}}$ at the vertices of K_{ij} and*

$$|f - \varphi_{K_{ij}}|_{H^m(K_{ij})}^2 \leq C\sigma^{2(n+2-j)(l-m-\beta)}\frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j + 3 - 2m)}\left(\frac{\rho}{2}\right)^{2s_j}\|f\|_{H_\beta^{s_j+3,l}(K_{ij})}^2 \tag{5.7}$$

for any $1 \leq i \leq 3$, $2 \leq j \leq n + 1$, $0 \leq m \leq 2$ and $s_j \in [1, k_j]$. Above, $\rho = \max(1, \frac{1-\sigma}{\sigma})$.

Proof: For $l = 2$ this is proved in [16, 24]. The proof for $l = 1$ is analogous. $\quad\square$

Define the auxiliary space $Q^{\underline{k},1}(\Delta_{n,\sigma}) := \{f \in H^1(\Omega) : f|_K \in \mathcal{Q}^{k_K}(K), \ K \in \Delta_{n,\sigma}\}$.

Theorem 5.10 *Let $l = 1, 2$ and $f \in B^l_\beta(\hat{Q})$ with weight $\Phi_\beta(x) = |x|^\beta$ for $\beta \in (0, 1)$. Then for every layerwise constant polynomial degree distribution $\{k_j \geq 2\}_{j=1}^{n+1}$ on $\Delta_{n,\sigma}$ with $k_{j+1} \geq k_j$ there exists an interpolant $\Psi^l \in Q^{\underline{k},1}(\Delta_{n,\sigma})$ such that*

$$\|f - \Psi^l\|^2_{H^{l-1}(\hat{Q})} + \sum_{K \in (\Delta_{n,\sigma})_1} \frac{h_K^2}{k_K^4}|f - \Psi^l|^2_{H^l(K)} \leq Ck_2^4 \tag{5.8}$$

$$\cdot\left(\sigma^{2n(1-\beta)}\|f\|^2_{H^{l,l}_\beta(K_{11})} + \sum_{j=2}^{n+1} \sigma^{2(n+2-j)(1-\beta)}\frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j - 1)}\left(\frac{\rho}{2}\right)^{2s_j}\|f\|^2_{H^{s_j+3,l}_\beta(K_{ij})}\right)$$

where $\rho = \max(1, \frac{1-\sigma}{\sigma})$, $\gamma \geq 0$ and $s_j \in [1, k_j]$. On the first element K_{11} the approximant $\Psi^2|_{K_{11}}$ is given by the bilinear interpolant in Proposition 5.1 and $\Psi^1|_{K_{11}} = \frac{1}{|K_{11}|}\int_{K_{11}} f dx$. Additionally, if $f \in B^2_\beta(\hat{Q}) \cap H^1_0(\hat{Q})$, the interpolant Ψ^2 can be chosen to satisfy the zero boundary conditions. (In (5.8) we use the notation in (3.14) with respect to the vertex $A = 0$.)

The estimate (5.8) for $l = 1$ is the main novelty in the result of Theorem 5.10. For $l = 2$ estimates of this type can be found e.g. in [15, 16, 24]. We present the proof for $l = 1$ in full details and sketch the case $l = 2$ for the convenience of the reader.

Proof: We investigate the cases $l = 1$ and $l = 2$ separately.

1. Approximation of the pressure: We establish (5.8) for a pressure p in $B^1_\beta(\Omega)$: A first approximation of p is given by Lemma 5.9 on the irregular mesh $\widetilde{\Delta}_{n,\sigma}$: On the elements $\{K_{ij}\}$ away from the origin there exist polynomials $\varphi_{K_{ij}}$ of degree k_j on K_{ij} such that $p = \varphi_{K_{ij}}$ at the vertices of K_{ij} and such that (5.7) holds with $l = 1$. Observing (5.5) yields then

$$\|p - \varphi_{K_{ij}}\|^2_{L^2(K_{ij})} + h_{K_{ij}}^2|p - \varphi_{K_{ij}}|^2_{H^1(K_{ij})} + h_{K_{ij}}^4|p - \varphi_{K_{ij}}|^2_{H^2(K_{ij})}$$

$$\leq C\sigma^{2(n+2-j)(1-\beta)}\frac{\Gamma(k_j - s_j + 1)}{\Gamma(k_j + s_j - 1)}\left(\frac{\rho}{2}\right)^{2s_j}\|p\|_{H^{s_j+3,1}(K_{ij})}. \tag{5.9}$$

In the smallest element K_{11} at the origin we choose $\varphi_{K_{11}} := \frac{1}{|K_{11}|}\int_{K_{11}} p dx$. From Proposition 5.4 we get

$$\|p - \varphi_{K_{11}}\|^2_{L^2(K_{11})} \leq Ch_{K_{11}}^{2(1-\beta)}\|p\|^2_{H^{1,1}_\beta(K_{11})} = C\sigma^{2n(1-\beta)}\|p\|^2_{H^{1,1}_\beta(K_{11})}. \tag{5.10}$$

The regular geometric mesh $\Delta_{n,\sigma}$ is obtained from $\widetilde{\Delta}_{n,\sigma}$ by splitting the elements $K_{ij} \in \widetilde{\Delta}_{n,\sigma}$ for $i = 2, 3$ and $3 \leq j \leq n+1$ into three triangular elements K_{ij}^1, K_{ij}^2 and K_{ij}^3. We write $\varphi_{K_{ij}^l} = \varphi_{K_{ij}}|_{K_{ij}^l}$ and remark that $\varphi_{K_{ij}^l} \in \mathcal{Q}^{k_j}(K_{ij}^l)$. (In general, $\varphi_{K_{ij}^l} \notin \mathcal{P}^{k_j}(K_{ij}^l)$.) Note also that $C_1 h_{K_{ij}} \leq h_{K_{ij}^l} \leq C_2 h_{K_{ij}}$.

Obviously, the elementwise approximation $\{\varphi_K : K \in \Delta_{n,\sigma}\}$ is discontinuous over interelement boundaries and, generally, $\varphi_K(Q) \neq p(Q)$ at an irregular node Q of K considered as an element in $\widetilde{\Delta}_{n,\sigma}$. We remove the interelement discontinuities and construct in the following a continuous approximation $\{\Psi_K : K \in \Delta_{n,\sigma}\}$:

Assertion 5.11 *There exists* $\Psi = \{\Psi_K\}_{K \in \Delta_{n,\sigma}} \in Q^{\underline{k},1}(\Delta_{n,\sigma})$ *with*

$$\sum_{K \in \Delta_{n,\sigma}} \|p - \Psi_K\|^2_{L^2(K)} + \sum_{K \in (\Delta_{n,\sigma})_1} \frac{h_K^2}{k_K^4} |p - \Psi_K|^2_{H^1(K)} \tag{5.11}$$

$$\leq Ck_2^4 \big(h_{K_{11}}^{2(1-\beta)} \|p\|^2_{H^{1,1}_\beta(K_{11})} + \sum_{i=0}^{2} \sum_{K \in (\widetilde{\Delta}_{n,\sigma})_1} h_K^{2i} |p - \varphi_K|^2_{H^i(K)} \big).$$

The estimate (5.8) is then a direct consequence of Assertion 5.11, (5.9)-(5.10).

It remains to prove (5.11): We set $\Psi_{K_{11}} = \varphi_{K_{11}}$ and construct in a first step the approximant Ψ on the first two layers of $\Delta_{n,\sigma}$ near the origin using the notations in Figure 1:
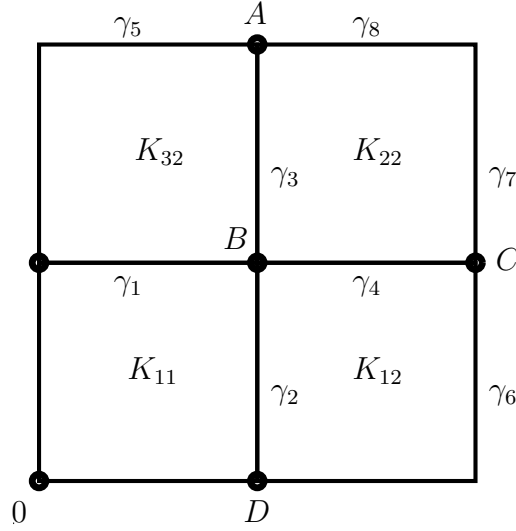
Figure 1: The first two layers of $\Delta_{n,\sigma}$ near the origin.

Remember that $\varphi_{K_{32}}(B) = \varphi_{K_{22}}(B) = \varphi_{K_{12}}(B) = p(B)$. Define:

$$w_{12} := (\varphi_{K_{12}} - \varphi_{K_{11}})|_{\gamma_2} \quad \text{polynomial of degree} \leq k_2 \text{ on } \gamma_2,$$
$$w_{32} := (\varphi_{K_{32}} - \varphi_{K_{11}})|_{\gamma_1} \quad \text{polynomial of degree} \leq k_2 \text{ on } \gamma_2.$$

By Lemma 5.8(ii) there exist polynomials W_{12} and W_{32} of degree k_2 in each variable such that

$$\|W_{12}\|^2_{L^2(K_{12})} \leq Ch_{K_{12}}\|w_{12}\|^2_{L^2(\gamma_2)}, \qquad \|W_{32}\|^2_{L^2(K_{32})} \leq Ch_{K_{32}}\|w_{32}\|^2_{L^2(\gamma_1)},$$

$W_{12}|_{\gamma_2} = w_{12}$, $W_{32}|_{\gamma_1} = w_{32}$, $W_{12} \equiv 0$ on γ_6 and $W_{32} \equiv 0$ on γ_5. By definition, there holds

$$\begin{aligned}
\|W_{12}\|^2_{L^2(K_{12})} &\leq Ch_{K_{12}}\|w_{12}\|^2_{L^2(\gamma_2)} \leq Ch_{K_{12}}\|\varphi_{K_{12}} - \varphi_{K_{11}}\|^2_{L^2(\gamma_2)} \\
&\leq Ch_{K_{12}}\|p - \varphi_{K_{12}}\|^2_{L^2(\gamma_2)} + Ch_{K_{12}}\|p - \varphi_{K_{11}}\|^2_{L^2(\gamma_2)}.
\end{aligned}$$

By Lemma 5.7(ii) with $\beta = 0$ we get

$$\|p - \varphi_{K_{12}}\|_{L^2(\gamma_2)}^2 \le Ch_{K_{12}}^3 |p - \varphi_{K_{12}}|_{H^2(K_{12})}^2.$$

Moreover, Lemma 5.7(i) gives

$$\|p - \varphi_{K_{11}}\|_{L^2(\gamma_2)}^2 \le h_{K_{11}}^{1-2\beta} \|p - \varphi_{K_{11}}\|_{H_\beta^{1,1}(K_{11})}^2 \le h_{K_{11}}^{1-2\beta} \|p\|_{H_\beta^{1,1}(K_{11})}.$$

Combining the above estimates and observing (5.5) result in

$$\|W_{12}\|_{L^2(K_{12})}^2 \le Ch_{K_{12}}^4 |p - \varphi_{K_{12}}|_{H^2(K_{12})}^2 + h_{K_{11}}^{2-2\beta} \|p\|_{H_\beta^{1,1}(K_{11})}^2. \tag{5.12}$$

The analogous estimate holds for W_{32}. Define now on the elements K_{12} and K_{32}:

$$\Psi_{K_{12}} := \varphi_{K_{12}} - W_{12}, \qquad \Psi_{K_{32}} := \varphi_{K_{32}} - W_{32}.$$

There holds $\Psi_{K_{12}}|_{\gamma_2} = \varphi_{K_{11}}|_{\gamma_2}$ and $\Psi_{K_{32}}|_{\gamma_1} = \varphi_{K_{11}}|_{\gamma_1}$. Hence, $\Psi_{K_{11}}$, $\Psi_{K_{12}}$ and $\Psi_{K_{32}}$ are continuous across γ_1 and γ_2 and $\varphi_{K_{11}}(B) = \Psi_{K_{32}}(B) = \Psi_{K_{12}}(B)$. The triangle inequality yields

$$\|p - \Psi_{K_{12}}\|_{L^2(K_{12})}^2 \le C\|p - \varphi_{K_{12}}\|_{L^2(K_{12})}^2 + C\|W_{12}\|_{L^2(K_{12})}^2.$$

Further, with inverse inequalities:

$$\frac{h_{K_{12}}^2}{k_{K_{12}}^4}|p - \Psi_{K_{12}}|_{H^1(K_{12})}^2 \le Ch_{K_{12}}^2 |p - \varphi_{K_{12}}|_{H^1(K_{12})}^2 + C\|W_{12}\|_{L^2(K_{12})}^2.$$

From (5.12) we get thus

$$\|p - \Psi_{K_{12}}\|_{L^2(K_{12})}^2 + \frac{h_{K_{12}}^2}{k_{K_{12}}^4}|p - \Psi_{K_{12}}|_{H^1(K_{12})}^2 \tag{5.13}$$

$$\le Ch_{K_{11}}^{2(1-\beta)} \|p\|_{H_\beta^{1,1}(K_{11})}^2 + C\sum_{i=0}^{2} h_{K_{12}}^{2i} |p - \varphi_{K_{12}}|_{H^i(K_{12})}^2.$$

The analogous estimate holds true for Ψ_{32}.
Next, we construct $\Psi_{K_{22}}$: Define

$$
\begin{aligned}
v_3 &:= (\Psi_{K_{32}} - \varphi_{K_{22}})|_{\gamma_3} \qquad \text{polynomial of degree} \le k_2 \text{ on } \gamma_3, \\
v_4 &:= (\Psi_{K_{12}} - \varphi_{K_{22}})|_{\gamma_4} \qquad \text{polynomial of degree} \le k_2 \text{ on } \gamma_4.
\end{aligned}
$$

We have by construction and the properties of $\{\varphi_{K_{ij}}\}$, W_{32} and W_{12}

$$v_3(A) = 0, \qquad v_4(C) = 0, \qquad v_3(B) = v_4(B) = \varphi_{K_{11}}(B) - \varphi_{K_{22}}(B).$$

Lemma 5.8(iii) can be applied and yields the existence of a polynomial $W_{22} \in \mathcal{Q}^{k_2}(K_{22})$ with $W_{22}|_{\gamma_3} = v_3$, $W_{22}|_{\gamma_4} = v_4$, $W_{22} \equiv 0$ on γ_7, γ_8 and

$$\|W_{22}\|_{L^2(K_{22})}^2 \le Ck_2^2 h_{K_{22}} \|v_3\|_{L^2(\gamma_3)}^2 + Ck_2^2 h_{K_{22}} \|v_4\|_{L^2(\gamma_4)}^2. \tag{5.14}$$

Define on K_{22}: $\Psi_{K_{22}} := \varphi_{K_{22}} + W_{22}$. We have $\Psi_{K_{22}}|_{\gamma_3} = \Psi_{K_{32}}|_{\gamma_3}$ and $\Psi_{K_{22}}|_{\gamma_4} = \Psi_{K_{12}}|_{\gamma_4}$, such that by construction $\Psi_{K_{11}}$, $\Psi_{K_{12}}$, $\Psi_{K_{22}}$ and $\Psi_{K_{32}}$ are a continuous approximation of p in the first two layers.

To estimate $\|p - \Psi_{K_{22}}\|_{L^2(K_{22})}$ we bound $\|W_{32}\|^2_{L^2(\gamma_3)}$ using Lemma 5.8(ii) and inverse inequalities:

$$
\begin{aligned}
\|W_{32}\|^2_{L^2(\gamma_3)} &\leq h_{K_{22}} \|W_{32}\|^2_{L^\infty(\gamma_3)} \leq h_{K_{22}} \|W_{32}\|^2_{L^\infty(\gamma_1)} \\
&\leq C h_{K_{22}} \frac{k_2^2}{h_{K_{22}}} \|W_{32}\|^2_{L^2(\gamma_1)} = C k_2^2 \|W_{32}\|^2_{L^2(\gamma_1)} = C k_2^2 \|w_{32}\|^2_{L^2(\gamma_1)} \\
&\leq C k_2^2 \|\varphi_{K_{32}} - \varphi_{K_{11}}\|^2_{L^2(\gamma_1)} \\
&\leq C k_2^2 \|\varphi_{K_{32}} - p\|^2_{L^2(\gamma_1)} + C k_2^2 \|p - \varphi_{K_{11}}\|^2_{L^2(\gamma_1)}.
\end{aligned}
$$

With Lemma 5.7(ii) ($\beta = 0$) and Lemma 5.7(i) we get

$$
\|W_{32}\|^2_{L^2(\gamma_3)} \leq C k_2^2 h_{K_{32}}^3 |p - \varphi_{K_{32}}|^2_{H^2(K_{32})} + C k_2^2 h_{K_{11}}^{1-2\beta} \|p\|^2_{H_\beta^{1,1}(K_{11})}. \tag{5.15}
$$

Using again Lemma 5.7(ii) with $\beta = 0$ and (5.15) gives

$$
\begin{aligned}
\|v_3\|^2_{L^2(\gamma_3)} &= \|\Psi_{K_{32}} - \varphi_{K_{22}}\|^2_{L^2(\gamma_3)} = \|\varphi_{K_{32}} - W_{32} - \varphi_{K_{22}}\|^2_{L^2(\gamma_3)} \\
&\leq C \|p - \varphi_{K_{32}}\|^2_{L^2(\gamma_3)} + C \|p - \varphi_{K_{22}}\|^2_{L^2(\gamma_3)} + \|W_{32}\|^2_{L^2(\gamma_3)} \\
&\leq C h_{K_{32}}^3 |p - \varphi_{K_{32}}|^2_{H^2(K_{32})} + C h_{K_{22}}^3 |p - \varphi_{K_{22}}|^2_{H^2(K_{22})} \\
&\quad + C k_2^2 h_{K_{32}}^3 |p - \varphi_{K_{32}}|^2_{H^2(K_{32})} + C k_2^2 h_{K_{11}}^{1-2\beta} \|p - \varphi_{K_{11}}\|^2_{H_\beta^{1,1}(K_{11})}.
\end{aligned}
$$

We see with (5.5) that

$$
\begin{aligned}
&k_2^2 h_{K_{22}} \|v_3\|^2_{L^2(\gamma_3)} \hspace{7cm} (5.16) \\
&\leq C k_2^4 \{ h_{K_{32}}^4 |p - \varphi_{K_{32}}|^2_{H^2(K_{32})} + h_{K_{22}}^4 |p - \varphi_{K_{22}}|^2_{H^2(K_{22})} + h_{K_{11}}^{2-2\beta} \|p\|^2_{H_\beta^{1,1}(K_{11})} \}.
\end{aligned}
$$

The analogous estimate holds true for $k_2^2 h_{K_{22}} \|v_4\|^2_{L^2(\gamma_4)}$. The desired bound for $\Psi_{K_{22}}$ follows now by the triangle inequality, inverse estimates, (5.14) and (5.16):

$$
\begin{aligned}
&\|p - \Psi_{K_{22}}\|^2_{L^2(K_{22})} + \frac{h_{K_{22}}^2}{k_{K_{22}}^4} |p - \Psi_{K_{22}}|^2_{H^1(K_{22})} \\
&\leq C \|p - \varphi_{K_{22}}\|^2_{L^2(K_{22})} + C h_{K_{22}}^2 |p - \varphi_{K_{22}}|^2_{H^1(K_{22})} + C \|W_{22}\|^2_{L^2(K_{22})} \\
&\leq C k_2^4 \Big\{ \|p - \varphi_{K_{22}}\|^2_{L^2(K_{22})} + h_{K_{22}}^2 |p - \varphi_{K_{22}}|^2_{H^1(K_{22})} \\
&\quad + \sum_{i=1}^3 h_{K_{i,2}}^4 |p - \varphi_{K_{i,2}}|^2_{H^2(K_{i,2})} + h_{K_{11}}^{2-2\beta} \|p - \varphi_{K_{11}}\|^2_{H_\beta^{1,1}(K_{11})} \Big\}. \hspace{1cm} (5.17)
\end{aligned}
$$

Note that

$$
\Psi_{K_{32}} = \varphi_{K_{32}} \text{ on } \gamma_5, \qquad \Psi_{K_{22}} = \varphi_{K_{22}} \text{ on } \gamma_7, \gamma_8, \qquad \Psi_{K_{12}} = \varphi_{K_{12}} \text{ on } \gamma_6.
$$

The construction of the continuous approximation Ψ in the elements away from the origin is analogous to [16]. We present the details for the sake of completeness:

Consider $3 \leq j \leq n+1$ and $i = 2, 3$. Let v_{ij}^l be linear on K_{ij}^l ($l = 1, 2, 3$) such that

$$
\begin{aligned}
v_{ij}^l(Q) &= p(Q) - \varphi_{K_{ij}}(Q) = p(Q) - \varphi_{K_{ij}^l}(Q) \quad \text{at the irregular node } Q \text{ (in } \widetilde{\Delta}_{n,\sigma}), \\
v_{ij}^l(P) &= 0 \qquad \text{at the other nodes } P \text{ of } K_{ij}^l.
\end{aligned}
$$

Define $\tilde{\varphi}_{K_{ij}^l} := \varphi_{K_{ij}^l} + v_{ij}^l$. $\tilde{\varphi}_{K_{ij}^l}$ is equal to p at all the vertices of K_{ij}^l. Proposition 5.1 with $\beta = 0$ yields

$$
\begin{aligned}
\|p - \tilde{\varphi}_{K_{ij}^l}\|_{L^2(K_{ij}^l)}^2 &\leq Ch_{K_{ij}}^4 |p - \varphi_{K_{ij}^l}|_{H^2(K_{ij}^l)}^2, & (5.18) \\
\|p - \tilde{\varphi}_{K_{ij}^l}\|_{H^1(K_{ij}^l)}^2 &\leq Ch_{K_{ij}}^2 |p - \varphi_{K_{ij}^l}|_{H^2(K_{ij}^l)}^2, & (5.19) \\
\|p - \tilde{\varphi}_{K_{ij}^l}\|_{H^2(K_{ij}^l)}^2 &\leq C|p - \varphi_{K_{ij}^l}|_{H^2(K_{ij}^l)}^2. & (5.20)
\end{aligned}
$$

We have

$$
\begin{aligned}
\varphi_{K_{ij}} &= p \qquad \text{at the vertices of } K_{ij}, & i = 1, \ 3 \leq j \leq n+1, \\
\tilde{\varphi}_{K_{ij}^l} &= p \qquad \text{at the vertices of } K_{ij}^l, & i = 1, 2, \ 3 \leq j \leq n+1, \ l = 1, 2, 3.
\end{aligned}
$$

The jumps over a common edge of two neighbouring elements can be corrected without affecting the polynomials $\{\varphi_{K_{ij}}\}$ or $\{\tilde{\varphi}_{K_{ij}^l}\}$ on the remaining sides. By symmetry, we have to consider the 5 correction cases shown in Figure 2 . We present the details only for the cases 2 and 3, the remaining ones are completely analogous.
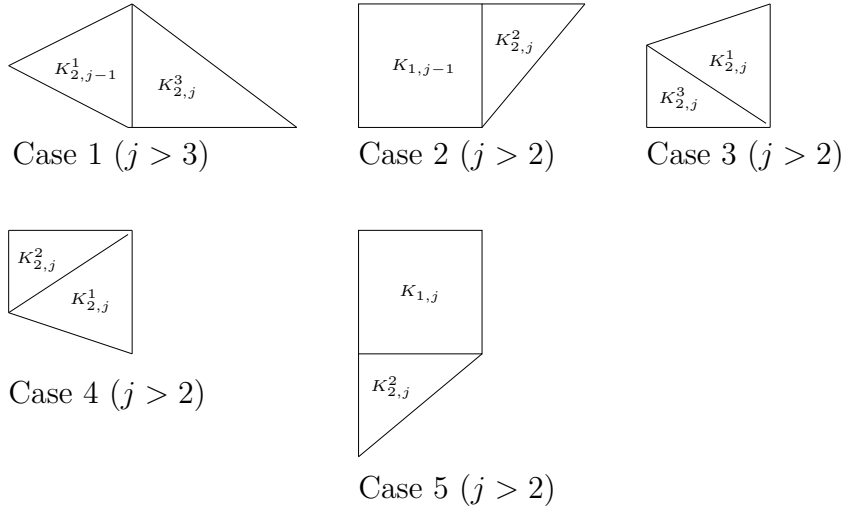

Figure 2: The 5 basic correction cases.

Case 2: $\varphi_{K_{1,j-1}}$ and $\tilde{\varphi}_{K_{2j}^2}$ coincide with p at the vertices of $K_{1,j-1}$ and K_{2j}^2, respectively. Define

$$
w := (\varphi_{K_{1,j-1}} - \tilde{\varphi}_{K_{2j}^2})|_\gamma
$$

where $\gamma = \overline{K_{1,j-1}} \cap \overline{K_{2j}^2}$. w is a polynomial of degree $\leq k_j$ on γ. By Lemma 5.8(i) there is a polynomial W in $\mathcal{P}^{k_j}(K_{2j}^2) \subseteq \mathcal{Q}^{k_j}(K_{2j}^2)$ such that $W \equiv w$ on γ, $W \equiv 0$

on the other sides of K^2_{2j} and $\|W\|^2_{L^2(K^2_{2j})} \leq Ch_{K_{2j}}\|w\|^2_{L^2(\gamma)}$. We get using Lemma 5.7(ii) with $\beta = 0$ and (5.20)

$$
\begin{aligned}
\|W\|^2_{L^2(K^2_{2j})} &\leq Ch_{K_{2j}}\|\varphi_{K_{1,j-1}} - \tilde{\varphi}_{K^2_{2j}}\|^2_{L^2(\gamma)} \\
&\leq Ch_{K_{2j}}\|p - \varphi_{K_{1,j-1}}\|^2_{L^2(\gamma)} + Ch_{K_{2j}}\|\tilde{\varphi}_{K^2_{2j}} - p\|^2_{L^2(\gamma)} \\
&\leq Ch_{K_{2j}}h^3_{K_{1,j-1}}|\varphi_{K_{1,j-1}} - p|^2_{H^2(K_{1,j-1})} + Ch_{K_{2j}}h^3_{K_{2j}}|\tilde{\varphi}_{K^2_{2j}} - p|^2_{H^2(K^2_{2j})}.
\end{aligned}
$$

We conclude with (5.5)

$$
\|W\|^2_{L^2(K^2_{2j})} \leq Ch^4_{K_{1,j-1}}|\varphi_{K_{1,j-1}} - p|^2_{H^2(K_{1,j-1})} + Ch^4_{K_{2j}}|\varphi_{K_{2j}} - p|^2_{H^2(K_{2j})}. \qquad (5.21)
$$

Define

$$
\Psi_{K^2_{2j}} := \tilde{\varphi}_{K^2_{2j}} + W \text{ in } K^2_{2j}, \qquad \Psi_{K_{1,j-1}} := \varphi_{K_{1,j-1}} \text{ in } K_{1,j-1}.
$$

Clearly, $\Psi_{K^2_{2j}} = \Psi_{K_{1,j-1}}$ on γ and we get from (5.18)-(5.20) and (5.21) with the triangle inequality and inverse inequalities

$$
\begin{aligned}
&\|p - \Psi_{K^2_{2j}}\|_{L^2(K^2_{2j})} + \frac{h^2_{K^2_{2j}}}{k^4_{K^2_{2j}}}|p - \Psi_{K_{2j}}|^2_{H^1(K^2_{2j})} \\
&\leq C\|p - \tilde{\varphi}_{K^2_{2j}}\|^2_{L^2(K^2_{2j})} + h^2_{K^2_{2j}}|p - \tilde{\varphi}_{K^2_{2j}}|_{H^1(K^2_{2j})} + C\|W\|^2_{L^2(K^2_{2j})} \\
&\leq Ch^4_{K_{2j}}|p - \varphi_{K^2_{2j}}|^2_{H^2(K^2_{2j})} + Ch^4_{1,j-1}|\varphi_{K_{1,j-1}} - p|^2_{H^2(K_{1,j-1})} \\
&\quad + Ch^4_{K_{2j}}|\varphi_{K_{2j}} - p|^2_{H^2(K_{2j})}. \qquad (5.22)
\end{aligned}
$$

Case 3: Here, $\tilde{\varphi}_{K^1_{2j}} = \tilde{\varphi}^3_{K_{2j}} = p$ at the endpoints of $\gamma = \overline{K^1_{2j}} \cap \overline{K^3_{2j}}$. Define $w := (\tilde{\varphi}_{K^1_{2j}} - \tilde{\varphi}_{K^3_{2j}})|_\gamma$. Since $w = v^1_{2j} - v^2_{3j}$ is linear, $w \equiv 0$ on γ and the piecewise polynomial $\{\tilde{\varphi}_{K^l_{2j}}\}_{l=1,3}$ is continuous across γ. With $\Psi_{K^1_{2j}} = \tilde{\varphi}_{K^1_{2j}}$ and $\Psi_{K^3_{2j}} = \tilde{\varphi}_{K^3_{2j}}$ there holds (cf. (5.18))

$$
\begin{aligned}
&\|p - \Psi_{K^1_{2j}}\|^2_{L^2(K^2_{2j})} + \|p - \Psi_{K^3_{2j}}\|^2_{L^2(K^3_{2j})} \\
&+ \frac{h^2_{K^1_{2j}}}{k^4_{K^1_{2j}}}|p - \Psi_{K^1_{2j}}|^2_{H^1(K^1_{2j})} + \frac{h^2_{K^3_{2j}}}{k^4_{K^3_{2j}}}|p - \Psi_{K^3_{2j}}|^2_{H^1(K^3_{2j})} \\
&\leq Ch^4_{K_{ij}}|p - \varphi_{K_{2j}}|^2_{H^2(K_{2j})}. \qquad (5.23)
\end{aligned}
$$

Referring to (5.13), (5.17), (5.22) and (5.23) proves (5.11).

2. Approximation of the velocity: We establish (5.8) for a function $u \in B^2_\beta(\hat{Q})$: In [16], Guo and Babuška constructed an interpolant $\Psi \in Q^{\underline{k},1}(\Delta_{n,\sigma})$ that satisfies

$$
\|u - \Psi\|^2_{H^1(\hat{Q})} \leq \text{r.h.s. of (5.8).} \qquad (5.24)
$$

We refer also to [24]. The proof of (5.24) is again based on Lemma 5.9, which provides a discontinuous interpolant on the irregular mesh $\widetilde{\Delta}_{n,\sigma}$. Considering the correction cases in Figure 2 a C^0-conforming approximation Ψ is constructed on

$\Delta_{n,\sigma}$. On the first element K_{11} near the vertex Ψ is the bilinear interpolant in Proposition 5.1. Moreover, if $u \in H_0^1(\hat{Q})$, Ψ can be constructed in such a way that the zero boundary conditions are satisfied. The additional terms $\sum_{K \in (\Delta_{n,\sigma})_1} \frac{h_K^2}{k_K^4} |u - \Psi|_{H^2(K)}^2$ in the left hand side of (5.8) can be controlled with the same techniques used in the approximation of the pressure.

This finishes the proof of Theorem 5.10. □

An immediate consequence of Theorem 5.10 is (see [15, 16, 24]):

Corollary 5.12 *Let $l = 1, 2$ and $f \in B_\beta^l(\hat{Q})$ for some $\beta \in (0,1)$. Then there exists a $\mu_0 > 0$ such that for linearly increasing polynomial degree vectors $\underline{k}$ with slope $\mu \geq \mu_0$ there is an interpolant $\Psi^l \in S^{\underline{k},1}(\Delta_{n,\sigma})$ that satisfies*

$$\|f - \Psi^l\|_{H^{l-1}(\hat{Q})}^2 + \sum_{K \in (\Delta_{n,\sigma})_1} \frac{h_K^2}{k_K^4} |f - \Psi^l|_{H^l(K)}^2 \leq C \exp(-bN^{1/3})$$

with C and b independent of $N = \dim(S^{\underline{k},1}(\Delta_{n,\sigma}))$. On the first element K_{11} near the origin $\Psi^2|_{K_{11}}$ is given by the bilinear interpolant in Proposition 5.1 and we have $\Psi^1|_{K_{11}} = \frac{1}{|K_{11}|} \int_{K_{11}} f dx$. Additionally, if $f \in B_\beta^2(\hat{Q}) \cap H_0^1(\hat{Q})$, the interpolant Ψ^2 can be chosen to satisfy the zero boundary conditions.

5.3. Approximation on Polygons: Proof of Theorem 4.4

Proof of Theorem 4.4: $\mathcal{T}_{n,\sigma}$ is obtained by mapping affinely up to three geometric mesh patches $\Delta_{n,\sigma}$ to a neighborhood of each corner (cf. Figure 2). Locally, we can construct Ψ in each of these parallelogram patches according to Corollary 5.12 (a generalization of Theorem 5.10 or Corollary 5.12 to parallelograms can be established straightforwardly, cf. [15, 16, 24]). Near a reentrant corner A the approximation Ψ is constructed over three geometric patches with the techniques presented in the proof of Theorem 5.10 (the analog of Corollary 5.12 holds true in that case). Here, the continuous pressure approximation is in the first elements $K_A = \cup\{K : \overline{K} \cap A \neq \emptyset\}$ near A chosen as $\Psi = \frac{1}{|K_A|} \int_{K_A} f \, dx$, $f \in B_\beta^1(\Omega)$ (see also Remark 5.6). In the interior Ω_{int} of the domain the polynomial approximation order is set to $k = \lfloor \mu(n+1) \rfloor$ on a fixed quasi-uniform partition $\mathcal{T}_q$. B_β^l-functions behave analytically away from the vertices and it follows thus from standard approximation theory that

$$\inf_{\Psi \in S^{k,1}(\mathcal{T}_q)} (\|f - \Psi\|_{H^{l-1}(\Omega_{int})}^2 + \sum_{K \in \mathcal{T}_q} \frac{h_K^2}{k_K^4} |f - \Psi|_{H^l(K)}^2) \leq C \exp(-bk).$$

Joining continuously together the local and the interior approximations as in the proof of Theorem 5.10 gives the desired approximant Ψ.

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