

# A Spectral Galerkin Method for Hydrodynamic Stability Problems

N.P. Hancke<sup>1</sup>, J.M. Melenk and C. Schwab

Research Report No. 98-06  
June 1998

Seminar für Angewandte Mathematik  
Eidgenössische Technische Hochschule  
CH-8092 Zürich  
Switzerland

---

<sup>1</sup>Institut für Mechanik, TU Darmstadt, D-64289 Darmstadt

# A Spectral Galerkin Method for Hydrodynamic Stability Problems

N.P. Hancke<sup>2</sup>, J.M. Melenk and C. Schwab

Seminar für Angewandte Mathematik  
Eidgenössische Technische Hochschule  
CH-8092 Zürich  
Switzerland

Research Report No. 98-06

June 1998

## Abstract

A spectral Galerkin method for calculating the eigenvalues of the Orr-Sommerfeld equation is presented. The matrices of the resulting generalized eigenvalue problem are sparse. A convergence analysis of the method is presented which indicates that a) no spurious eigenvalues occur and b) reliable results can only be expected under the assumption of *scale resolution*, i.e., that  $Re/p^2$  is small; here  $Re$  is the Reynolds number and  $p$  is the spectral order. Numerical experiments support that the assumption of scale resolution is necessary to obtain reliable results. Exponential convergence of the method is shown theoretically and observed numerically.

**Keywords:** Orr-Sommerfeld equation, hydrodynamic stability, eigenvalue problem, spectral method

**AMS Subject Classification:** 76E05, 65N25, 65N35

---

<sup>2</sup>Institut für Mechanik, TU Darmstadt, D-64289 Darmstadt

# 1 Introduction

The Orr-Sommerfeld equation, hereafter referred to as OSE, occurs in hydrodynamic stability theory of shear flows of viscous, Newtonian, incompressible fluids. The instability of such flows has been and continues to be one of the most constantly pursued topics in fluid mechanics. The above mentioned flows may exist under various physical conditions, for instance flows in a pipe or a channel, flows of superposed immiscible fluids, wakes, jets, plumes and free streams. These flows may be laminar or turbulent and the transition from the former to the latter is closely related to the above mentioned instability. The actual flow problem to be solved is physically a highly idealized one. The basic flow is assumed to be an exact solution to the steady state Navier Stokes equations, while it does not change in the flow direction and depends only on the distance from the walls. The disturbances imposed on the basic flow (with profile  $U$ ) have the form of travelling waves (with wavenumber  $a > 0$ ) whose amplification with respect to time, not with respect to the distance travelled in the flow direction, is investigated in the framework of (linearized) modelling equations. The task is to determine the complex eigenvalues  $\lambda = \Re\lambda + i\Im\lambda$  of the OSE since the real part of the temporal growth rate of the disturbances is given by  $e^{a\Im\lambda t}$  and amplifies perturbations, i.e., those with  $\Im\lambda > 0$ , become unbounded and make the flow unstable.

Both analytical (see, e.g., [5], [14], [19]) and numerical approaches have been made to solve the Orr-Sommerfeld eigenvalue problem. Finite Difference Methods (FDMs) were among the first by which the discretization of the OS eigenvalue problems were implemented (cf., e.g., [23], [7], [13]).

Spectral Methods implemented in the late 50's suffered from similar disadvantages as the FDM's did: the size of the matrices and the accuracy of the results were limited by computer speed and memory size (see, e.g., [3]). Moreover, so-called spurious modes, eigenvalues with large positive imaginary parts which are not at all related to the OSE, occurred even in regimes where the flow is known to be stable (see, e.g., [6], [15], [16]). Nevertheless, further investigations in linear instability theory using Spectral Methods were made, but seldom spectral orders  $p$  of several hundreds were employed. However, contrary to classical problems like the Bénard- or Taylor-Problem, where indeed only low to moderate spectral orders suffice to achieve excellent results, the Orr-Sommerfeld problem at high Reynolds numbers  $Re$  mandates the use of large spectral orders  $p$  (several hundreds to thousands) to guarantee *scale resolution* and reliable approximations of the eigenvalues. This need for scale resolution has been noted experimentally in, e.g., [4] and we shall give a rigorous proof below.

Almost all calculations aimed at finding the least stable eigenvalue for plane Poiseuille flow and the critical Reynolds number, and difficulties in performing calculations with sufficient accuracy are a commonly reported issue. In 1971, Orszag [18] used expansions in Chebyshev polynomials and a Tau-Method to transform the OSE into a matrix equation of the form  $\mathbf{B}\underline{x} = \lambda\mathbf{C}\underline{x}$  which was then solved by using the QR-algorithm. Orszag found his results due to the special properties of the Chebyshev polynomials to be more accurate than those obtained previously. Since then, most subsequent spectral techniques for the OSE employed the Tau discretization and Chebyshev polynomials, for which fast transform methods are available. The eigenvalue problem resulting from such a spectral discretization has the unfortunate property that the matrices  $\mathbf{B}$  and  $\mathbf{C}$  are relatively fully occupied and that  $\mathbf{C}$  is in general singular. The singularity of  $\mathbf{C}$  (which is due the way the boundary conditions are accounted for in the Tau-method) might account for the appearance of the spurious eigenvalues (see [4]).

The Galerkin Spectral Method proposed in the present paper avoids some of the difficulties Chebyshev-Tau methods encounter. Our main results are the following:

1. The use of a Galerkin formulation allows for a rigorous convergence analysis for the eigenvalues; we show in particular that reliable results can only be expected under the assumption of *scale resolution*, that is, that  $Re/p^2$  is sufficiently small.
2. New analytic regularity results for the eigenfunctions of the OSE (with analytic profile  $U$ ) are presented that are explicit in  $Re$  and  $\lambda$ ; these regularity results allow for a proof of exponential convergence.
3. No spurious eigenvalues occur and for the mass matrix  $\mathbf{C}$  there holds that  $-\mathbf{iC}$  is positive definite.
4. For plane Poiseuille flow, the stiffness and mass matrices  $\mathbf{B}$ ,  $\mathbf{C}$  are sparse with bandwidths 6 and 4, respectively; hence, iterative methods for the matrix eigenvalue problem may be very efficient and this opens to the way to the use of very large spectral orders, [12].
5. For polynomial profiles  $U$ , the stiffness and mass matrices  $\mathbf{B}$ ,  $\mathbf{C}$  can be calculated explicitly and stably thereby avoiding quadrature errors.
6. Numerical experiments with spectral orders up to  $p = 1000$  confirm our convergence analysis and illustrate the necessity of scale resolution to obtain reliable results.

## 1.1 Notation and Problem Formulation

We set  $\Omega = (-1, 1)$  and denote by  $L^2(\Omega)$  the Hilbert space of square integrable complex-valued functions with the usual inner product  $(\phi, \psi)_0 := \int_{-1}^1 \phi(x) \overline{\psi(x)} dx$ . The norm induced by  $(\cdot, \cdot)_0$  is denoted by  $\|\cdot\|_0$ . Next, we define for smooth functions  $\phi$  and  $k \in \mathbb{N}$  the following semi-norms and norms:

$$|\phi|_k^2 := \|D^k \phi\|_0^2, \quad \|\phi\|_k^2 := \sum_{n=0}^k |\phi|_n^2.$$

Here, the operator  $D$  denotes differentiation. The Sobolev spaces  $H^k(\Omega)$ ,  $H_0^k(\Omega)$  are then defined in the usual way (see, e.g., [1]). We recall here that  $\|\cdot\|_k$  and  $|\cdot|_k$  are equivalent norms on  $H_0^k(\Omega)$ .

A pair  $(\lambda, \phi) \in \mathbb{C} \times H_0^2(\Omega)$ ,  $\phi \neq 0$ , is an eigenpair of the Orr-Sommerfeld equation if

$$(D^2 - a^2)^2 \phi = \mathbf{i}aRe(U - \lambda)(D^2 - a^2)\phi - \mathbf{i}aRe(D^2 U)\phi. \quad (1)$$

Here,  $a > 0$  is the wavenumber,  $\mathbf{i} = \sqrt{-1}$ , and, since we are interested in large Reynolds numbers  $Re$ , we assume  $Re \geq 1$ . Concerning the flow profile  $U$ , a real-valued function, we make the assumption that  $U \in C^2(\overline{\Omega})$ . Multiplying (1) by a test function  $\psi \in H_0^2(\Omega)$  and integrating by parts yields the variational formulation of the eigenvalue problem (1):

$$\text{find } (\lambda, \phi) \in \mathbb{C} \times H_0^2(\Omega) \text{ s.t. } b(\phi, \psi) = \lambda \mathbf{i}aRe c(\phi, \psi) \quad \forall \psi \in H_0^2(\Omega). \quad (2)$$

Here, the continuous sesquilinear forms  $b$  and  $c$ , defined on  $H_0^2(\Omega) \times H_0^2(\Omega)$  are given by

$$\begin{aligned} b(\phi, \psi) &:= b_4(\phi, \psi) + \mathbf{i}a \operatorname{Re} b_2(\phi, \psi), \\ b_4(\phi, \psi) &:= ((D^2 - a^2)\phi, (D^2 - a^2)\psi)_0, \\ b_2(\phi, \psi) &:= ((D^2 U)\phi, \psi)_0 - (U(D^2 - a^2)\phi, \psi)_0, \\ c(\phi, \psi) &:= -((D^2 - a^2)\phi, \psi)_0. \end{aligned}$$

## 2 $p$ -FEM Galerkin discretization

To treat (2) numerically, the infinite dimensional space  $H_0^2(\Omega)$  is replaced by finite dimensional spaces  $V_N \subset H_0^2(\Omega)$  of dimension  $N \in \mathbb{N}$  giving rise to the discrete eigenvalue problem in variational form

$$\text{find } (\lambda, \phi) \in \mathbb{C} \times V_N \setminus \{0\} \text{ s.t. } b(\phi, \psi) = \lambda \mathbf{i}a \operatorname{Re} c(\phi, \psi) \quad \forall \psi \in V_N. \quad (3)$$

If  $\phi_1, \dots, \phi_N$  is a basis of  $V_N \subset H_0^2(\Omega)$ , then (3) can be formulated as a generalized matrix eigenvalue problem

$$\text{find } (\lambda, \underline{\phi}) \in \mathbb{C} \times (\mathbb{C}^N \setminus \underline{0}) \text{ s.t. } \mathbf{B}\underline{\phi} = \lambda \mathbf{C}\underline{\phi} \quad (4)$$

where the matrices  $\mathbf{B}, \mathbf{C} \in \mathbb{C}^{N \times N}$  are given by

$$\mathbf{B}_{ij} = b(\phi_j, \phi_i), \quad \mathbf{C}_{ij} = \mathbf{i}a \operatorname{Re} c(\phi_j, \phi_i), \quad 1 \leq i, j \leq N. \quad (5)$$

We note that (4) does have indeed  $N$  eigenvalues as the matrix  $-\mathbf{i}\mathbf{C}$  is positive definite (cf. Lemma 3.3 for the detailed argument) and hence the pair  $(\mathbf{B}, \mathbf{C})$  is a *regular* pair in the sense of [22].

Contrary to the classical Chebyshev-Tau spectral approach, our choice of the spaces  $V_N$  and the basis functions  $\{\phi_i\}_{i=1}^N$  will lead to *sparse* matrices  $\mathbf{B}$  and  $\mathbf{C}$  whose additional feature is that they are as well-conditioned as can be expected from discretizing a fourth-order equation. It should also be pointed out that the mass matrix  $\mathbf{C}$  is invertible as  $-\mathbf{i}\mathbf{C}$  is positive definite. This is in marked contrast to the classical Chebyshev-Tau spectral approach where the mass matrix is not invertible due to the way the boundary conditions are enforced; the appearance of spurious eigenvalues in the Chebyshev-Tau method is sometimes attributed to the fact that the mass matrix is singular, [4]. In agreement with our theory below, we do not observe any spurious eigenvalues.

### 2.1 The subspace $V_N$

We denote by  $\mathcal{P}^p(\Omega)$  the polynomials of degree  $p$  on  $\Omega$  and set

$$V_N := \mathcal{P}^p(\Omega) \cap H_0^2(\Omega).$$

The numerical properties such as conditioning and round-off sensitivity of the matrix eigenvalue problem (4) depend strongly on the choice of the basis for  $V_N$ .

We denote by  $L_i$ ,  $i \in \mathbb{N}_0$ , the Legendre polynomials on  $(-1, 1)$  normalized such that  $L_i(1) = 1$  (cf., e.g., [9]). For  $p \geq 4$  we set for  $i = 1, \dots, p-3$

$$\phi_i(z) := \sqrt{\frac{2i+3}{2}} \int_{-1}^z \int_{-1}^{\eta} L_{i+1}(z) \, dz \, d\eta = \sqrt{\frac{2i+3}{2}} \int_{-1}^z (z-\eta) L_{i+1}(\eta) \, d\eta. \quad (6)$$

It is easy to check that the functions  $\phi_i$ ,  $i = 1, \dots, p-3$  are linearly independent and span the space  $\mathcal{P}^p(\Omega) \cap H_0^2(\Omega)$ . Thus

$$V_N = \text{span}\{\phi_i\}, \quad i = 1, \dots, p-3, \quad \dim V_N = p-3. \quad (7)$$

## 2.2 Structure of the discrete problem

For convenience, we will first state some results concerning the Legendre polynomials. In the calculation of the stiffness and the mass matrix, essential use will be made of the following properties of the Legendre polynomials:

**Lemma 2.1** *The Legendre polynomials have the following properties:*

$$\begin{aligned} a) & (2i+1)L_i = D(L_{i+1} - L_{i-1}), \quad i = 1, 2, \dots, \\ b) & (L_i, L_j)_0 = \int_{-1}^1 L_i(z)L_j(z) dz = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \neq j, \end{cases} \\ c) & L_i(-1) = (-1)^i, \quad L_i(1) = 1, \\ d) & \int_{-1}^z L_{i+1}(\eta) d\eta = \frac{1}{2(i+1)+1} (L_{i+2}(z) - L_i(z)), \end{aligned}$$

**Lemma 2.2** *The shape functions  $\phi_i$  defined in (6) satisfy*

$$\begin{aligned} a) & \phi_i(z) = \sqrt{\frac{2i+3}{2}} \frac{1}{(2i+3)(2i+5)} (L_{i+3}(z) - L_{i+1}(z)) \\ & \quad - \sqrt{\frac{2i+3}{2}} \frac{1}{(2i+1)(2i+3)} (L_{i+1}(z) - L_{i-1}(z)), \\ b) & D\phi_i(z) = \frac{1}{\sqrt{2(2i+3)}} (L_{i+2}(z) - L_i(z)), \\ c) & D^2\phi_i(z) = \sqrt{\frac{2i+3}{2}} L_{i+1}(z). \end{aligned}$$

**Proof:** In all cases the results are obtained by straightforward calculation using integration by parts, Lemma 2.1, and the Leibniz rule.  $\triangleleft$

The stiffness matrix  $\mathbf{B}$  and the mass matrix  $\mathbf{C}$  are given by

$$\mathbf{B}_{ij} = b(\phi_j, \phi_i) = \mathbf{T}_{1ij} - 2a^2\mathbf{T}_{2ij} + a^4\mathbf{T}_{3ij} + \mathbf{iaRe} \mathbf{T}_{4ij} - \mathbf{iaRe} \mathbf{T}_{5ij} + \mathbf{ia}^3\mathbf{Re} \mathbf{T}_{6ij}, \quad (8)$$

$$\mathbf{C}_{ij} = \mathbf{iaRe} c(\phi_j, \phi_i) = -\mathbf{iaRe}(\mathbf{T}_{2ij} - a^2\mathbf{T}_{3ij}), \quad (9)$$

where, for shorthand,  $\mathbf{T}_{1ij} := (D^2\phi_j, D^2\phi_i)_0$ ,  $\mathbf{T}_{2ij} := (D^2\phi_j, \phi_i)_0$ ,  $\mathbf{T}_{3ij} := (\phi_j, \phi_i)_0$ ,  $\mathbf{T}_{4ij} := (\phi_j D^2U, \phi_i)_0$ ,  $\mathbf{T}_{5ij} := (UD^2\phi_j, \phi_i)_0$ ,  $\mathbf{T}_{6ij} := (U\phi_j, \phi_i)_0$ . Combining the orthogonality properties of the Legendre polynomials collected in Lemma 2.1 with the properties of the shape functions  $\phi_i$  listed in Lemma 2.2 it is easy to prove the following assertions concerning the bandwidths of the matrices  $\mathbf{T}_1, \dots, \mathbf{T}_6$  (cf. [11] for the detailed arguments).

**Proposition 2.3** *The matrix  $\mathbf{T}_1$  is the identity matrix,  $\mathbf{T}_2$  has bandwidth 2 and  $\mathbf{T}_3$  has bandwidth 4. If the profile  $U \in \mathcal{P}^k(\Omega)$  for some  $k \in \mathbb{N}$ , then the matrices  $\mathbf{T}_4$ ,  $\mathbf{T}_5$ , and  $\mathbf{T}_6$  are also banded with bandwidths  $k+2$ ,  $k+2$ , and  $k+4$ , respectively.*

**Remark 2.4** The bands of the matrices  $\mathbf{T}_2$  and  $\mathbf{T}_3$  are not even fully populated as every other diagonal is zero.

The application of Proposition 2.3 to the classical plane Poiseuille flow problem reads as follows.

**Corollary 2.5** *In the case of plane Poiseuille flow,  $U(z) = 1 - z^2$ , the stiffness matrix  $\mathbf{B}$  has bandwidth 6 and the mass matrix  $\mathbf{C}$  has bandwidth 4. Furthermore, every other diagonal in the band is zero.*

Using Lemmata 2.1, 2.2 all entries  $\mathbf{B}_{ij}$ ,  $\mathbf{C}_{ij}$  can be computed explicitly thereby avoiding quadrature errors, [11].

### 2.3 Conditioning of the stiffness and mass matrices

We equip the space  $\mathbb{C}^N$  with the norm  $\|\underline{\phi}\|_{l^2}^2 := \sum_{i=1}^N |\phi_i|^2$  for  $\underline{\phi} \in \mathbb{C}^N$ . This norm induces in the standard way the norm  $\|\cdot\|_{l^2}$  on the space of complex  $N \times N$  matrices via  $\|\mathbf{A}\|_{l^2} := \max\{\|\mathbf{A}\underline{\phi}\|_{l^2} \mid \underline{\phi} \in \mathbb{C}^N, \|\underline{\phi}\|_{l^2} = 1\}$ . With this notation, we can formulate the following proposition.

**Proposition 2.6** *Let the matrices  $\mathbf{B}$ ,  $\mathbf{C}$  be given by (8), (9) where the basis functions  $\varphi_i$  are given by (6). Then there is  $C > 0$  depending only on the parameter  $a$  and the profile  $U$  such that*

$$\|\mathbf{B}\|_{l^2} \leq CRe, \quad \|\mathbf{C}\|_{l^2} \leq CRe, \quad \|\mathbf{C}^{-1}\|_{l^2} \leq C \frac{p^4}{Re}.$$

**Proof:** For the sake of brevity, we will only prove the bound for  $\|\mathbf{C}^{-1}\|_{l^2}$  as the other estimates are proved analogously. We start by observing that Lemmata 2.2, 2.1 allow us to conclude that the map

$$\begin{aligned} F : (\mathbb{C}^N, \|\cdot\|_{l^2}) &\rightarrow (V_N, |\cdot|_2) \\ \underline{\psi} &\mapsto F(\underline{\psi}) := \sum_{i=1}^N \psi_i \phi_i \end{aligned}$$

is an isometric isomorphism, i.e.,  $|F(\underline{\psi})|_2 = \|\underline{\psi}\|_{l^2}$  for all  $\underline{\psi} \in \mathbb{C}^N$ . We have for any  $\underline{\psi} \in \mathbb{C}^N$

$$\|\mathbf{C}\underline{\psi}\|_{l^2} = \sup_{\underline{\theta} \in \mathbb{C}^N} \frac{|\underline{\theta}^T \mathbf{C}\underline{\psi}|}{\|\underline{\theta}\|_{l^2}} = \sup_{\theta \in V_N} \frac{|\mathbf{ia} \operatorname{Re} c(\psi, \theta)|}{|\theta|_2} \geq Re a \min\{1, a^2\} \frac{\|\psi\|_1^2}{|\psi|_2}$$

where we wrote  $\psi = F(\underline{\psi})$ ,  $\theta = F(\underline{\theta})$  and used Lemma 3.3 ahead. From the standard inverse estimate (cf., e.g., [20])

$$\|\psi\|_2 \leq Cp^2 \|\psi\|_1$$

valid for all polynomials  $\psi$  of degree  $p$ , we therefore obtain

$$\|\mathbf{C}\underline{\psi}\|_{l^2} \geq C \frac{Re |\psi|_2^2}{p^4 |\psi|_2} = C \frac{Re}{p^4} \|\underline{\psi}\|_{l^2}.$$

Whence the assertion  $\|\mathbf{C}^{-1}\|_{l^2} \leq Cp^4/Re$ . ◁

One way to solve the matrix eigenvalue problem (4) is to use the QZ algorithm. The QZ algorithm (cf., e.g., [8]) produces two orthogonal matrices  $\mathbf{Q}$ ,  $\mathbf{Z}$  such that  $\mathbf{Q}^T \mathbf{B} \mathbf{Z} = \mathbf{T}$ ,  $\mathbf{Q}^T \mathbf{C} \mathbf{Z} = \mathbf{S}$  where the matrices  $\mathbf{T}$ ,  $\mathbf{S}$  are upper triangular. The eigenvalues  $\lambda_i$  of the problem (4) are then given by  $\lambda_i = t_{ii}/s_{ii}$ . One source of round-off sensitivity is therefore the size of the diagonal entries  $s_{ii}$ . The final proposition of this section gives bounds on these entries:

**Proposition 2.7** *Let  $\mathbf{Q}$ ,  $\mathbf{Z}$  be orthogonal matrices such that  $\mathbf{S} = \mathbf{Q}^T \mathbf{C} \mathbf{Z}$  is upper triangular. There are  $C_1, C_2 > 0$  depending only the parameter  $a$  and the profile  $U$  such that the diagonal entries  $s_{ii}$  of  $\mathbf{S}$  satisfy*

$$C_1 \frac{Re}{p^4} \leq |s_{ii}| \leq C_2 Re, \quad i = 1, \dots, N.$$

**Proof:** For the upper bound, we observe that

$$|s_{ij}| \leq \|\mathbf{S}\|_{l^2} = \|\mathbf{C}\|_{l^2}, \quad 1 \leq i, j \leq N,$$

where the last equality follows from the fact that  $\mathbf{Q}$ ,  $\mathbf{Z}$  are orthogonal matrices. Proposition 2.6 now gives the upper bound. For the lower bound, we exploit the fact that if  $\mathbf{S}$  is an invertible upper triangular matrix, so is  $\mathbf{S}^{-1}$  and that the diagonal entries of  $\mathbf{S}^{-1}$  are given by  $1/s_{ii}$ . Therefore,

$$|1/s_{ii}| \leq \|\mathbf{S}^{-1}\|_{l^2} = \|\mathbf{C}^{-1}\|_{l^2}, \quad i = 1, \dots, N.$$

Hence appealing again to Proposition 2.6 allows us to conclude the argument.  $\triangleleft$

### 3 Convergence Analysis

The main theoretical result of the paper is

**Theorem 3.1** *Let the profile  $U \in C^2(\overline{\Omega})$ . Let  $\lambda \in \mathbb{C}$  be an eigenvalue of (1) whose algebraic multiplicity  $m \in \mathbb{N}$  coincides with the geometric multiplicity. Let  $\lambda_{N,j}$ ,  $j = 1, \dots, m$ , be the numerical eigenvalues that converge to  $\lambda$ . Then there are constants  $C_\lambda, C'_\lambda > 0$  depending on  $\lambda$ ,  $Re \geq 1$ ,  $a > 0$ , and  $U$  such that under the assumption of scale resolution*

$$\frac{Re}{p^2} \leq C_\lambda \tag{10}$$

there holds

$$|\lambda - \lambda_{N,j}| \leq C'_\lambda \left( \frac{Re(1 + |\lambda|)}{p^2} \right)^2, \quad j = 1, \dots, m. \tag{11}$$

If the profile  $U$  is analytic on  $\overline{\Omega}$ , then there are  $\gamma, \gamma' > 0$  depending only on  $U$  and  $a > 0$  such that under the assumption of scale resolution, (10), there holds

$$|\lambda - \lambda_{N,j}| \leq C'_\lambda \exp \left[ \sqrt{Re(\gamma' + |\lambda|)} - \gamma p \right], \quad j = 1, \dots, m. \tag{12}$$

It is worth stressing that the constants  $\gamma, \gamma'$  in (12) are independent of  $Re$  and  $\lambda$ .



**Remark 3.2** The assumption that the algebraic multiplicity coincides with the geometric multiplicity is made for convenience only. If the ascent of the eigenvalue is greater than 1, convergence of the numerical eigenvalues is still guaranteed, albeit at a reduced rate; cf. [21]. The constants  $C_\lambda$ ,  $C'_\lambda$  depend on the resolvent of the operator  $T$  (to be defined in the proof below) in the neighborhood of the eigenvalue  $\lambda$ . As  $T$  depends on  $Re$ , the constants  $C_\lambda$ ,  $C'_\lambda$  do depend on  $Re$  as well. Without additional *a priori* knowledge about the spectrum of the OSE, the general theory of [21] on which the present analysis is based does not lead to sharp estimates on these two constants.

The rest of this section is devoted to the proof of Theorem 3.1.

### 3.1 Preliminaries

For  $Re \geq 1$  it will be convenient to introduce the norm  $\|\cdot\|_{2,Re}$  on  $H^2(\Omega)$  which is equivalent to the standard norm  $\|\cdot\|_2$ :

$$\|\phi\|_{2,Re}^2 := |\phi|_2^2 + Re\|\phi\|_1^2.$$

We will denote by  $H_{0,Re}^2(\Omega)$  the space  $H_0^2(\Omega)$  equipped with the norm  $\|\cdot\|_{2,Re}$ .

**Lemma 3.3** *Let  $Re \geq 1$ . Then there are  $C_1, C_2, C_3 > 0$  depending only on  $a > 0$  and the profile  $U$  such that for all  $\phi, \psi \in H_0^2(\Omega)$ :*

$$\begin{aligned} \Re[b(\phi, \phi)] &\geq |\phi|_2^2 - \frac{1}{2}C_1 a Re \|\phi\|_1^2, \\ |b(\phi, \psi)| &\leq C_2 \|\phi\|_{2,Re} \|\psi\|_{2,Re}, \\ c(\phi, \phi) &= \|\phi\|_1^2 + a^2 \|\phi\|_0^2, \\ |c(\phi, \psi)| &\leq C_3 \|\phi\|_1 \|\psi\|_1. \end{aligned}$$

**Proof:** Follows immediately by some integration by parts and the Cauchy-Schwarz inequality.  $\triangleleft$

**Lemma 3.4** *With  $C_1$  of Lemma 3.3 there holds: The pair  $(\lambda, \phi) \in \mathbb{C} \times H_0^2(\Omega)$  is an eigenpair of (2) if and only if  $(\tilde{\lambda}, \phi) \in \mathbb{C} \times H_0^2(\Omega) \setminus \{0\}$  with  $\tilde{\lambda} = \lambda - \mathbf{i}C_1$  is an eigenpair of the eigenvalue problem*

$$\tilde{b}(\phi, \psi) := b(\phi, \psi) + C_1 a Re c(\phi, \psi) = \tilde{\lambda} \mathbf{i} a Re c(\phi, \psi) \quad \forall \psi \in H_0^2(\Omega). \quad (13)$$

*Furthermore, in the discrete case, the pair  $(\lambda, \phi) \in \mathbb{C} \times V_N \setminus \{0\}$  is an eigenpair of (3) if and only if  $(\tilde{\lambda}, \phi) \in \mathbb{C} \times V_N$  with  $\tilde{\lambda} = \lambda - \mathbf{i}C_1$  is an eigenpair of*

$$\tilde{b}(\phi, \psi) := b(\phi, \psi) + C_1 a Re c(\phi, \psi) = \tilde{\lambda} \mathbf{i} a Re c(\phi, \psi) \quad \forall \psi \in V_N. \quad (14)$$

*Moreover, there are  $C, C' > 0$  depending only on  $a > 0$  and  $U$  such that for all  $\phi, \psi \in H_0^2(\Omega)$*

$$\begin{aligned} \Re[\tilde{b}(\phi, \phi)] &\geq C \|\phi\|_{2,Re}^2, \\ |\tilde{b}(\phi, \psi)| &\leq C' \|\phi\|_{2,Re} \|\psi\|_{2,Re}. \end{aligned}$$

**Proof:** The first two assertions are obvious. The assertions about  $\tilde{b}$  follow from Lemma 3.3.  $\triangleleft$

Lemma 3.4 allows us to consider the eigenvalue convergence for the eigenvalue problem (13) instead of (2). As  $\tilde{b}$  satisfies an inf-sup condition by Lemma 3.4, the general approximation theory for eigenvalue problems of [21] can be applied. We will therefore perform the convergence analysis for (13) and (14) instead of (2), (3). The next subsection is devoted to the more detailed application of the general theory developed in [21] to the present Orr-Sommerfeld problem.

### 3.2 Abstract Convergence Results

In order to recast the variationally posed eigenvalue problem (13) in a more convenient form, we define the operator  $T : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$  and its “dual”  $T_* : H_0^2(\Omega) \rightarrow H_0^2(\Omega)$  by

$$\tilde{b}(T\phi, \psi) = \mathbf{i}aRe\,c(\phi, \psi) \quad \forall \psi \in H_0^2(\Omega), \quad (15)$$

$$\tilde{b}(\phi, T_*\psi) = \mathbf{i}aRe\,c(\phi, \psi) \quad \forall \phi \in H_0^2(\Omega). \quad (16)$$

The operators  $T, T_*$  are well-defined bounded linear operators and in fact compact:

**Lemma 3.5** *Let  $T, T_*$  be defined by (15), (16). Then there is  $C > 0$  depending only on  $a > 0$  and  $U$  such that for  $Re \geq 1$  and for all  $\phi \in H_0^2(\Omega)$*

$$\begin{aligned} \|T\phi\|_{2,Re} &\leq C\|\phi\|_{2,Re}, & |T\phi|_4 &\leq CRe\|\phi\|_{2,Re}, \\ \|T_*\phi\|_{2,Re} &\leq C\|\phi\|_{2,Re}, & |T_*\phi|_4 &\leq CRe\|\phi\|_{2,Re}. \end{aligned}$$

**Proof:** As the sesquilinear form  $\tilde{b}$  satisfies an inf-sup condition by Lemma 3.4, the operators  $T$  and  $T_*$  are well-defined bounded linear operators. For the bounds on these operators, we will only show the estimates for  $T$  as those for  $T_*$  are completely analogous. From Lemma 3.4 we obtain immediately

$$C'\|T\phi\|_{2,Re}^2 \leq |\tilde{b}(T\phi, T\phi)| = aRe\,|c(\phi, T\phi)| \leq aRe\|\phi\|_1\|T\phi\|_1$$

and hence the first estimate. For the second inequality, we integrate the defining equation for  $T$  by parts to discover that  $T\phi$  satisfies the following fourth order equation

$$(D^2 - a^2)^2 T\phi = \mathbf{i}aRe\,U(D^2 - a^2)T\phi - \mathbf{i}aRe\,(D^2 U)T\phi + C_1 aRe\,(D^2 - a^2)T\phi - \mathbf{i}aRe\,(D^2 - a^2)\phi$$

from which we can easily deduce the claim.  $\triangleleft$

As the operators  $T$  and  $T_*$  are compact their spectrum is discrete and only the origin is a possible point of accumulation. It is easy to see that  $(\tilde{\lambda}, \phi)$  is an eigenpair of (13) if and only if  $(\tilde{\lambda}^{-1}, \phi)$  is an eigenpair of the operator  $T$ . Furthermore,  $\tilde{\lambda}^{-1}$  is an eigenvalue of  $T$  iff it is an eigenvalue of  $T_*$ .

Next, we define the projection  $P_N : H_0^2(\Omega) \rightarrow V_N$  by

$$\tilde{b}(P_N\phi, \psi) = \tilde{b}(\phi, \psi) \quad \forall \psi \in V_N.$$

Again,  $P_N$  is well-defined by the fact that  $\tilde{b}$  satisfies an inf-sup condition. We observe that the discrete eigenvalue problem (14) is related to the operator  $T$  as follows: The pair

$(\tilde{\lambda}, \phi) \in \mathbb{C} \times V_N$  is an eigenpair of (14) if and only if  $(\tilde{\lambda}^{-1}, \phi)$  is an eigenpair of the compact operator

$$T_N := P_N T. \quad (17)$$

We are now in a position to quote from [21] a theorem that allows us quantify the difference between the eigenvalues of  $T$  and  $T_N$ :

**Theorem 3.6** *Let  $\tilde{\lambda}$  be an eigenvalue of (13) whose algebraic multiplicity  $m \in \mathbb{N}$  coincides with its geometric multiplicity. Define*

$$\begin{aligned} M(\tilde{\lambda}) &:= \{\phi \in H_0^2(\Omega) \mid \tilde{\lambda} T \phi = \phi, \|\phi\|_{2, Re} = 1\}, \\ M^*(\tilde{\lambda}) &:= \{\phi \in H_0^2(\Omega) \mid \tilde{\lambda} T_* \phi = \phi, \|\phi\|_{2, Re} = 1\}, \\ \varepsilon(\tilde{\lambda}) &:= \sup_{\phi \in M(\tilde{\lambda})} \inf_{v \in V_N} \|\phi - v\|_{2, Re}, \\ \varepsilon^*(\tilde{\lambda}) &:= \sup_{\phi \in M^*(\tilde{\lambda})} \inf_{v \in V_N} \|\phi - v\|_{2, Re}. \end{aligned}$$

Let  $\tilde{\lambda}_{N,1}, \dots, \tilde{\lambda}_{N,m}$  be eigenvalues of (14) converging to  $\tilde{\lambda}$ . Under the assumption

$$\|T - T_N\|_{2, Re} \leq C_{\tilde{\lambda}} \quad (18)$$

with  $C_{\tilde{\lambda}} > 0$  sufficiently small (depending only  $T$  and  $\tilde{\lambda}$ ) there is  $C > 0$  independent of  $p$  such that

$$|\tilde{\lambda} - \tilde{\lambda}_{N,j}| \leq C \varepsilon(\tilde{\lambda}) \varepsilon^*(\tilde{\lambda}), \quad j = 1, \dots, m.$$

**Proof:** See Chap. II, Sec. 8 of [21]. ◁

It is therefore important to obtain bounds for the best approximation problems  $\varepsilon(\tilde{\lambda})$ ,  $\varepsilon^*(\tilde{\lambda})$ . This will be done in Section 3.4. The remainder of this subsection is devoted to the analysis of  $\|T - T_N\|_{2, Re}$ . We start with a lemma about the approximation properties of the projector  $P_N$ :

**Lemma 3.7** *There is  $C > 0$  independent of  $Re \geq 1$ ,  $p$ , such that*

$$\|(I - P_N)u\|_{2, Re} \leq Cp^{-2} \left[ 1 + \frac{\sqrt{Re}}{p} \right] |u|_4 \quad \forall u \in H^4(\Omega) \cap H_0^2(\Omega).$$

**Proof:** From standard theory (cf., e.g., [2]) there holds with  $C, C'$  of Lemma 3.4 (which are independent of  $Re$ )

$$\|\phi - P_N \phi\|_{2, Re} \leq \left( 1 + \frac{C'}{C} \right) \inf_{\psi \in V_N} \|u - \psi\|_{2, Re}. \quad (19)$$

The result follows now by combining (19) with standard spectral approximation results as can be found in, e.g., [20]. ◁

We obtain therefore

**Theorem 3.8** *Let  $T, T_N$  be defined by (15), (17). Then there is  $C > 0$  depending only on  $a > 0, U$  such that for all  $Re \geq 1$  there holds*

$$\|T - T_N\|_{2, Re} \leq C \frac{Re}{p^2} \left[ 1 + \frac{\sqrt{Re}}{p} \right].$$

**Proof:** We observe that  $T - T_N = (I - P_N)T$ . Hence, for all  $u \in H_{0,Re}^2(\Omega)$  we get by combining Lemmata 3.5, 3.7

$$\|(I - P_N)Tu\|_{2,Re} \leq Cp^{-2} \left[ 1 + \frac{\sqrt{Re}}{p} \right] |Tu|_4 \leq Cp^{-2} \left[ 1 + \frac{\sqrt{Re}}{p} \right] Re\|u\|_{2,Re}$$

and hence the claim.  $\triangleleft$

We note at this point already that the assumption (10) together with Theorem 3.8 guarantees that (18) is satisfied. The next subsection clarifies the regularity properties of the eigenfunctions of  $T$  and  $T_*$ . These regularity results will then enable us to estimate  $\varepsilon(\tilde{\lambda})$ ,  $\varepsilon^*(\tilde{\lambda})$ , thereby allowing us to conclude the proof of Theorem 3.1 in Section 3.4.

### 3.3 Regularity of the Eigenfunctions

**Lemma 3.9** *Assume that  $U \in C^2(\overline{\Omega})$  and  $Re \geq 1$ . Then there is  $C > 0$  depending only on  $a > 0$  and the profile  $U$  such that for each eigenpair  $(\tilde{\lambda}^{-1}, \phi)$  of  $T$  and each eigenpair  $(\tilde{\lambda}^{-1}, \phi^*)$  of  $T_*$  there holds*

$$|\phi|_4 \leq CRe|\tilde{\lambda}|\|\phi\|_{2,Re}, \quad |\phi^*|_4 \leq CRe|\tilde{\lambda}|\|\phi^*\|_{2,Re}.$$

**Proof:** For eigenpairs  $(\tilde{\lambda}^{-1}, \phi)$  of  $T$  there holds  $\tilde{\lambda}T\phi = \phi$ . Appealing to Lemma 3.5 allows us to obtain the first estimate. The second one is obtained in the same manner.  $\triangleleft$

If the profile is analytic, we can estimate the growth of the derivatives:

**Lemma 3.10** *Assume that  $Re \geq 1$  and that  $U$  is analytic on  $\overline{\Omega}$ , i.e., there  $C_U, \gamma > 0$  such that*

$$\|D^n U\|_{L^\infty(\Omega)} \leq C_U \gamma^n n! \quad \forall n \in \mathbb{N}_0.$$

*Then each eigenfunction  $\phi$  of (13) is analytic on  $\overline{\Omega}$ . Moreover, there are  $C, K > 0$  depending only on the profile  $U$  and the parameter  $a > 0$  such that for all eigenpairs  $(\tilde{\lambda}, \phi)$  of (13) there holds*

$$\|D^{n+2}\phi\|_0 \leq CK^n \max\{n, R\}^n \|\phi\|_{2,Re} \quad \forall n \in \mathbb{N}_0, \quad (20)$$

where

$$R := \sqrt{Re(1 + |\tilde{\lambda}|)}. \quad (21)$$

**Proof:** Choosing  $C$  and  $K$  sufficiently large guarantees that (20) holds for  $n = 0$  and  $n = 2$  by Lemma 3.9. Let us now see that it holds for  $n = 1$  as well: We have

$$\|\phi\|_{H^1(\Omega)} \leq C [\|\phi\|_4 + \|\phi\|_2] \leq C(Re(1 + |\tilde{\lambda}|))\|\phi\|_{2,Re}$$

by Lemma 3.9. This estimate together with the interpolation estimate

$$\|\phi\|_3 \leq C [\varepsilon\|\phi\|_4 + \varepsilon^{-1}\|\phi\|_2] \quad \forall \varepsilon > 0$$

with  $\varepsilon = (Re(1 + |\tilde{\lambda}|))^{-1/2}$  yields the desired bounds for  $n = 1$ . We will now proceed by induction on  $n$ . For  $n \geq 3$  we therefore assume that (20) holds for  $0 \leq \nu \leq n - 1$ . We observe now that  $\phi$  satisfies the following equation:

$$D^4\phi = \left[ iaRe(U - \tilde{\lambda} - C_1\mathbf{i}) + 2a^2 \right] D^2\phi + \left[ -ia^3Re(U - \tilde{\lambda} - C_1\mathbf{i}) - iaReD^2U - a^4 \right] \phi \quad (22)$$

with  $C_1$  of Lemma 3.4. Introducing the shorthand

$$b_1 = \left[ \mathbf{i}a \operatorname{Re}(U - \tilde{\lambda} - C_1 \mathbf{i}) + 2a^2 \right], \quad b_2 = \left[ -\mathbf{i}a^3 \operatorname{Re}(U - \tilde{\lambda} - C_1 \mathbf{i}) - \mathbf{i}a \operatorname{Re} D^2 U - a^4 \right],$$

we obtain by differentiating (22)  $n - 2$  times from Leibniz's rule:

$$D^{n+2} \phi = \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} D^\nu b_1 D^{n-\nu} \phi + \sum_{\nu=0}^{n-2} \binom{n-2}{\nu} D^\nu b_2 D^{n-2-\nu} \phi. \quad (23)$$

The assumptions on  $U$  give the existence of  $C_B$ ,  $B > 0$  depending on  $C_U$ ,  $\gamma > 0$  such that

$$\begin{aligned} \|b_j\|_{L^\infty(\Omega)} &\leq C_B \operatorname{Re}(1 + |\tilde{\lambda}|), \quad j = 1, 2 \\ \|D^\nu b_j\|_{L^\infty(\Omega)} &\leq C_B \operatorname{Re} \nu! B^\nu, \quad \nu \geq 1, \quad j = 1, 2. \end{aligned}$$

The remainder of the proof follows the standard induction procedure—we refer to the proof of Theorem 1 of [17] where a similar calculation is carried out in detail.  $\triangleleft$

### 3.4 Proof of Theorem 3.1

We start by quoting the following lemma which can be extracted from the proof of Theorem 16 of [17]:

**Lemma 3.11** *Let  $\phi$  be analytic on  $\overline{\Omega}$  and satisfy, for some  $C$ ,  $K$ , and  $R > 0$*

$$\|D^n \phi\|_0 \leq CK^n \max(n, R)^n \quad \forall n \in \mathbb{N}_0.$$

*Then there is a polynomial  $v$  of degree  $p \geq 3$  with  $v(\pm 1) = \phi(\pm 1)$ ,  $v'(\pm 1) = \phi'(\pm 1)$ , such that*

$$\|\phi - v\|_2 \leq C' e^R e^{-\gamma p}$$

*for some  $C'$ ,  $\gamma > 0$  depending only on  $C$ ,  $K$ .*

Let us now turn to the proof of (11), (12). We observe that the assumption of scale resolution, (10), follows from combining the condition on  $\|T - T_N\|_{2, \operatorname{Re}}$  of Theorem 3.6 with Theorem 3.8. Next, if  $U \in C^2(\overline{\Omega})$ , we can bound by Lemmata 3.7, 3.9

$$\begin{aligned} \varepsilon(\tilde{\lambda}) \varepsilon^*(\tilde{\lambda}) &\leq \sup_{\phi \in M(\tilde{\lambda})} \|\phi - P_N \phi\|_{2, \operatorname{Re}} \sup_{\phi^* \in M^*(\tilde{\lambda})} \|\phi^* - P_N \phi^*\|_{2, \operatorname{Re}} \\ &\leq C \left( \frac{\operatorname{Re} |\tilde{\lambda}|}{p^2} \left[ 1 + \frac{\sqrt{\operatorname{Re}}}{p} \right] \right)^2 \|\phi\|_{2, \operatorname{Re}} \end{aligned}$$

which yields (11) after observing that  $\tilde{\lambda}$  and  $\lambda$  differ by the constant  $\mathbf{i}C_1$ . For (12), we observe that  $\varepsilon^*(\tilde{\lambda}) \leq 1$  and that from Lemmata 3.10, 3.11 we have

$$\varepsilon(\tilde{\lambda}) \leq C \sqrt{\operatorname{Re}} e^{\sqrt{\operatorname{Re}(1+|\tilde{\lambda}|)}} e^{-\gamma p}$$

for some  $\gamma > 0$  depending only  $U$  and  $a > 0$ . Whence (12).

## 4 Numerical Results

The aim of our numerical examples is to corroborate our convergence result Theorem 3.1 by considering the case of *plane Poiseuille flow*, i.e., the profile  $U$  is given by

$$U(z) = 1 - z^2 \quad \text{and we choose } a = 1 \quad (24)$$

in (1). The spaces  $V_N$  are given by (6); due to the structure of the profile  $U$ , the entries of  $\mathbf{B}$  and  $\mathbf{C}$  can be evaluated exactly. The matrix eigenvalue problem (4) is solved with the QZ algorithm in Matlab, i.e., the calculations are performed in Fortran double precision (16 digits/64 bit arithmetic).

The profile  $U$  is an analytic function and hence we have exponential rates of the convergence for the eigenvalues (cf. (12) in Theorem 3.1). This exponential convergence behavior is demonstrated in Fig. 1 where the convergence of the least stable eigenvalue, i.e., the eigenvalue with the largest (positive) imaginary part, is plotted against the polynomial degree  $p$ . As the value of the exact eigenvalue is unknown, a high order ( $p = 500$ ) approximation is taken as the reference value. A more detailed numerical convergence study of the behavior of the 5 least stable modes for  $Re = 1.E4$  and  $Re = 2.7E4$  can be found in [11]—in particular, the Spectral Galerkin Method was able to reproduce the 32 least stable eigenvalues listed in the literature, [18, 4].

(12) in Theorem 3.1 suggests that we can expect convergence of the eigenvalues of size  $|\lambda|$  only if the assumption of *scale resolution*,

$$\sqrt{Re(1 + |\lambda|)} - \gamma p \quad \text{is “sufficiently small”,} \quad (25)$$

is met, i.e., that  $Re(1 + |\lambda|)/p^2$  is sufficiently small. Our next experiments show that this assumption is indeed necessary. For  $Re = 2.7E4$  Fig. 2 shows the eigenvalue distribution for  $p = 200, 300, 400,$  and  $500$ . We notice that there is a “stem” (the so-called S-branch; cf. [5] for the standard notation of the branches of the eigenvalue distribution) whose elements have real part  $2/3$ . As  $p$  increases, this stem becomes longer. The S-branch consists of those modes which are “transported” with the mean bulk velocity, which for plane Poiseuille flow is evaluated exactly to  $2/3$  (cf., e.g., [10]). Eigenfunctions with numerical eigenvalues with large negative imaginary part and real part deviating considerably from  $2/3$  have therefore to be considered as underresolved. Hence, it is physically meaningful to take the “length” of the S-branch as a measure of the numerical scale resolution and we define the *numerical length of the S-branch* as

$$L(N) := -\min\{\Im\lambda_{N,i} \mid \Im\lambda_{N,i} \leq -1 \quad \text{and} \quad |\Re\lambda_{N,i} - 2/3| < \tau\}, \quad \tau = 0.01. \quad (26)$$

In Fig. 3 the numerical length  $L(N)$  of the S-branch is plotted versus the polynomial degree  $p$  for  $Re = 2.7E4$ . We observe in Fig. 3 a behavior

$$L(N) \sim p^2. \quad (27)$$

Next, we fix  $p = 500$  and compute the numerical length  $L(N)$  of the S-branch as a function of the Reynolds number  $Re$  in Fig. 4. There, we obtain

$$L(N) \sim Re^{-1}. \quad (28)$$

Combining (27), (28), we therefore have the empirical relation

$$\sqrt{Re L(N)} \sim p. \quad (29)$$

As the numerical length of the S-branch is essentially the magnitude of the largest “trust-worthy” eigenvalue, (29) coincides with (25). The assumption of scale resolution (25) is therefore an essential condition to guarantee the convergence of the discrete eigenvalue to the continuous eigenvalues.

It should be mentioned, however, that for  $p = 500$  the results for  $Re = 10^5$  do not seem very reliable: Although our technical definition of the numerical length of the S-branch yields a value of  $L(N)$  that fits the law (28), a comparison between the results for  $p = 500$  and  $Re = 10^5$  in Fig. 5 and those for  $p = 1000$  in Fig. 6 shows a significant discrepancy between these two numerical spectra.

It has been observed in the literature that Tau discretizations at large Reynolds numbers are very sensitive to finite precision arithmetic. In contrast, the present Spectral Galerkin Method seems to be more robust in this respect. To illustrate this, we compare our 64 bit arithmetic results for the two cases  $Re = 27000/p = 200$  and  $Re = 27000/p = 500$  in Figs. 7, 8 with those of [4] (cf. Figs. 2–5 of [4]). At 64 bit arithmetic, the Tau discretization produces—instead of a well-defined Branch-Point from which the S-branch emanates—a triangular region of numerical eigenvalues. This triangular region is entirely due to round-off problems as 128 bit calculations remove this triangular region and the expected well-defined Branch-Point emerges. In contrast to this behavior, the Spectral Galerkin Method resolves the Branch-Point with 64 bit arithmetic already (cf. Figs. 7, 8 with Figs. 2–5 of [4]).

## References

- [1] R. A. Adams. *Sobolev Spaces*. Academic Press, 1975.
- [2] A.K. Aziz and I.M. Babuška, editors. *Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*. Academic Press, New York, 1972.
- [3] C.L. Dolph and D.C. Lewis. On the application of infinite systems of ordinary differential equations to perturbations of plane Poiseuille flow. *Quarterly of applied mathematics*, 16:97–110, 1958.
- [4] J.J. Dongarra, B. Straughan, and D.W. Walker. Chebyshev Tau-QZ algorithm methods for calculating spectra of hydrodynamic stability problems. *Applied Numerical Mathematics*, 22:399–434, 1996.
- [5] P.G. Drazin and W.H. Reid. *Hydrodynamic Stability*. Cambridge University Press, 1981.
- [6] D.R. Gardner, S.A. Trogden, and R.W. Douglass. A modified tau spectral method that eliminates spurious eigenvalues. *J. of Comput. Physics*, 80:137–167, 1989.
- [7] J. Gary and R. Helgason. A matrix method for ordinary differential eigenvalue problems. *J. of Comput. Physics*, 5:169–187, 1970.
- [8] G. Golub and C. van Loan. *Matrix Computations*. Johns Hopkins, 3rd edition, 1996.
- [9] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series, and Products, corrected and enlarged edition*. Academic Press, New York, 1980.
- [10] C.E. Grosch and H. Salwen. The stability of steady and time-dependent plane Poiseuille flow. *J. Fluid Mechanics*, 34:177–194, 1968.

- [11] N.P. Hancke. Calculating large spectra in hydrodynamic stability: A  $p$ -FEM approach to solve the Orr-Sommerfeld equation. Master's thesis, Eidgenössische Technische Hochschule Zürich, 1998.
- [12] N.P. Hancke, J.M. Melenk, and C. Schwab. Computational aspects of the spectral-Galerkin FEM for the Orr-Sommerfeld equation. in preparation.
- [13] E.F. Kurtz and S.H. Crandall. Computer-aided analysis of hydrodynamic stability. *J. mathematics and physics*, 41:264–279, 1962.
- [14] C.C. Lin. *The theory of hydrodynamic stability*. Cambridge University Press, 1955.
- [15] K.A. Lindsay and R.R. Ogden. A practical implementation of spectral methods resistant to the generation of spurious eigenvalues of the Chebyshev tau spectral method. *Int. J. Numer. Meths. Fluids*, 15:1277–1294, 1992.
- [16] G.B. McFadden, B.T. Murray, and R.F. Boisvert. Elimination of spurious eigenvalues in the Chebyshev tau spectral method. *J. Comput. Physics*, 91:228–239, 1990.
- [17] J.M. Melenk. On the robust exponential convergence of  $hp$  finite element methods for problems with boundary layers. *IMA J. Numer. Anal.*, 17(4):577–601, 1997.
- [18] S.A. Orszag. Accurate solution of the Orr-Sommerfeld stability equation. *J. of Fluid Mechanics*, 50:684–703, 1971.
- [19] H. Schlichting. Entstehung der Turbulenz. In *Encyclopedia of Physics, Vol. VIII/1*. Springer, 1959.
- [20] C. Schwab. *Lecture Notes on  $p$  and  $hp$  FEM*. Oxford University Press, 1998.
- [21] I. Babuška and J.E. Osborn. Eigenvalue problems. In *Handbook of Numerical Analysis, Vol. II*, pages 641–789. North Holland, 1991.
- [22] G.W. Stewart and Ji Guang Sun. *Matrix Perturbation Theory*. Academic Press, 1990.
- [23] L.H. Thomas. The stability of plane Poiseuille flow. *Physical Review*, 90:780–783, 1953.



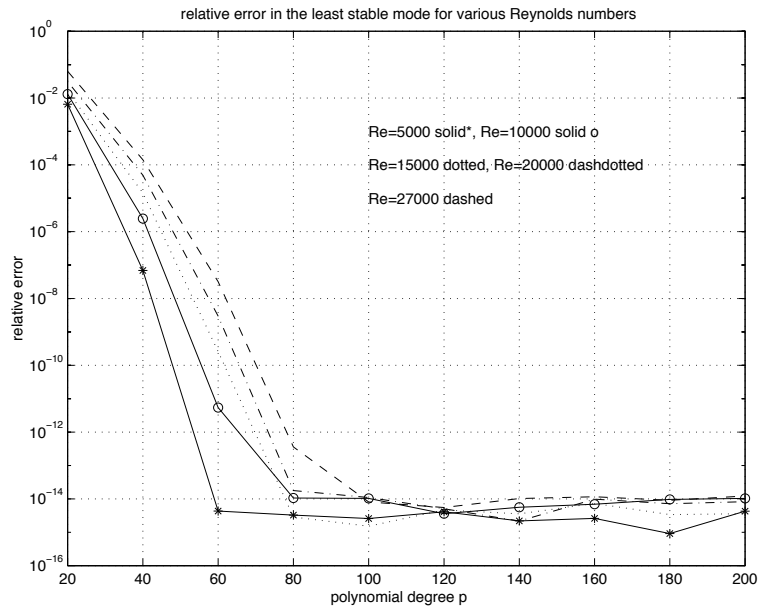


Figure 1:  $p$  convergence of the least stable eigenvalue for various Reynolds numbers.

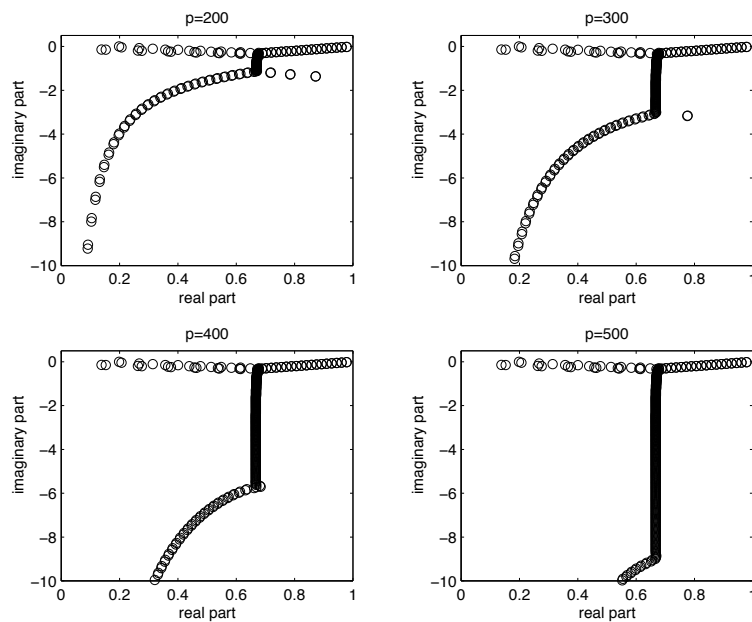


Figure 2: Eigenvalue distribution for  $Re = 27000$  and  $p = 200, 300, 400,$  and  $500$ .

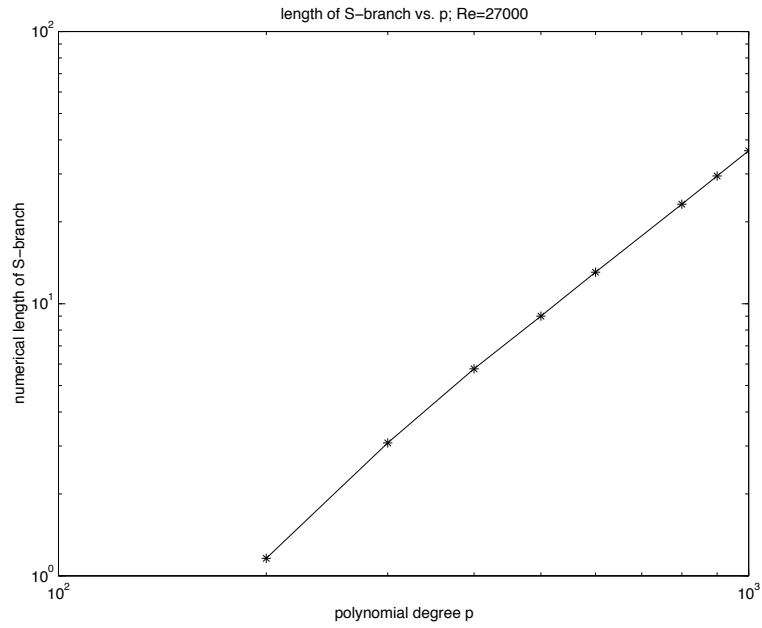


Figure 3: numerical length of S-branch vs.  $p$ ;  $Re = 27000$ .

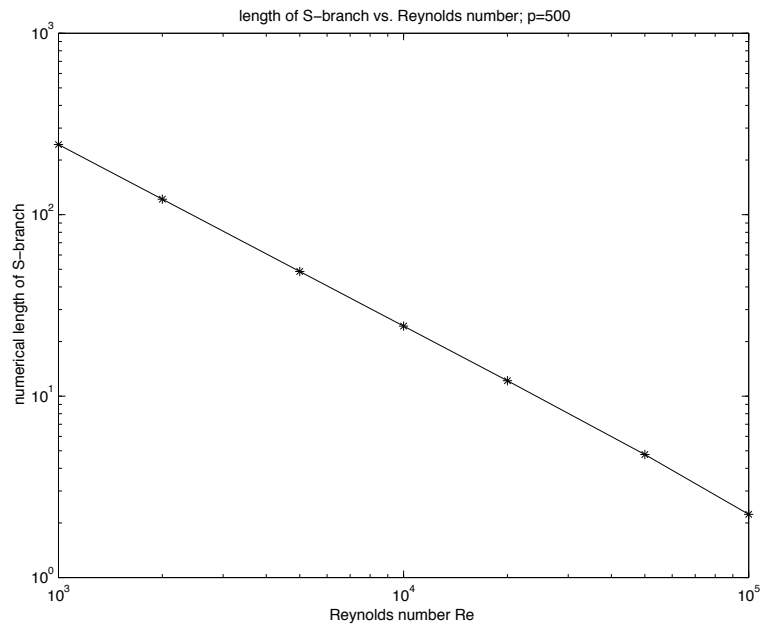


Figure 4: numerical length of S-branch vs.  $Re$ ;  $p = 500$ .

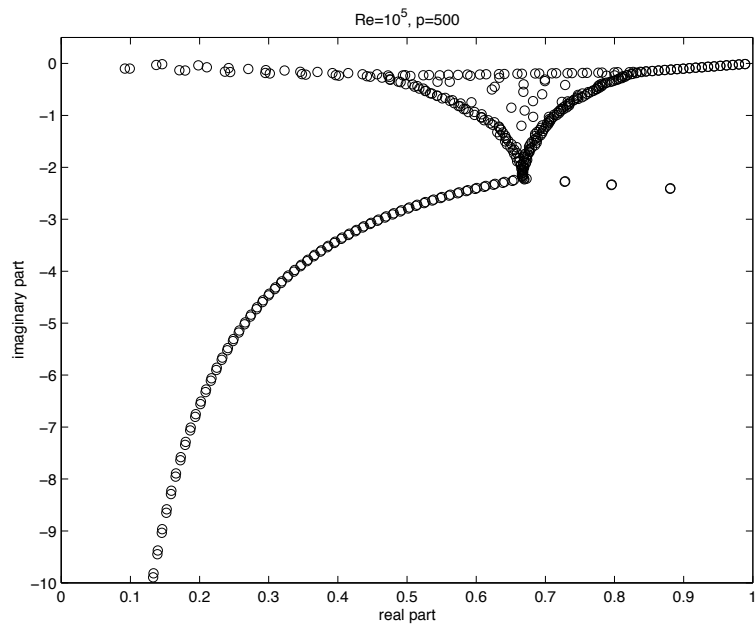


Figure 5: Eigenvalue distribution for  $Re = 10^5$ ,  $p = 500$ .

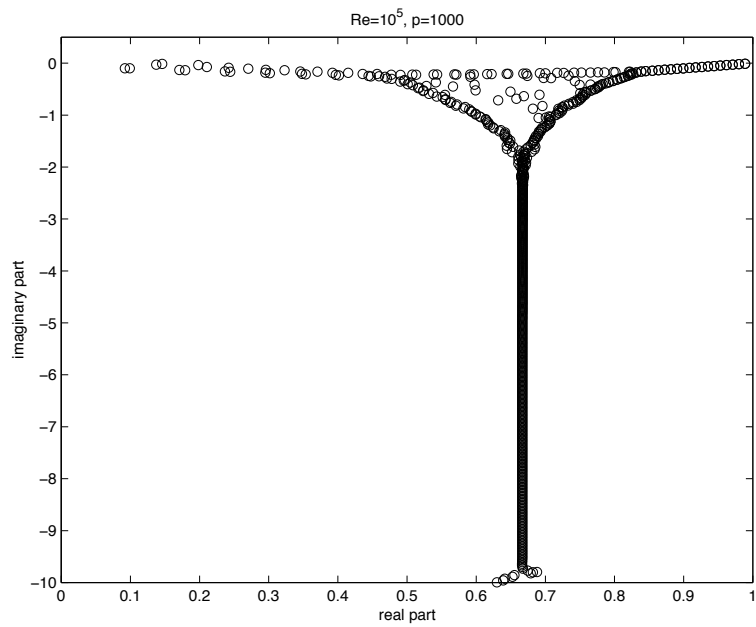


Figure 6: Eigenvalue distribution for  $Re = 10^5$ ,  $p = 1000$ .

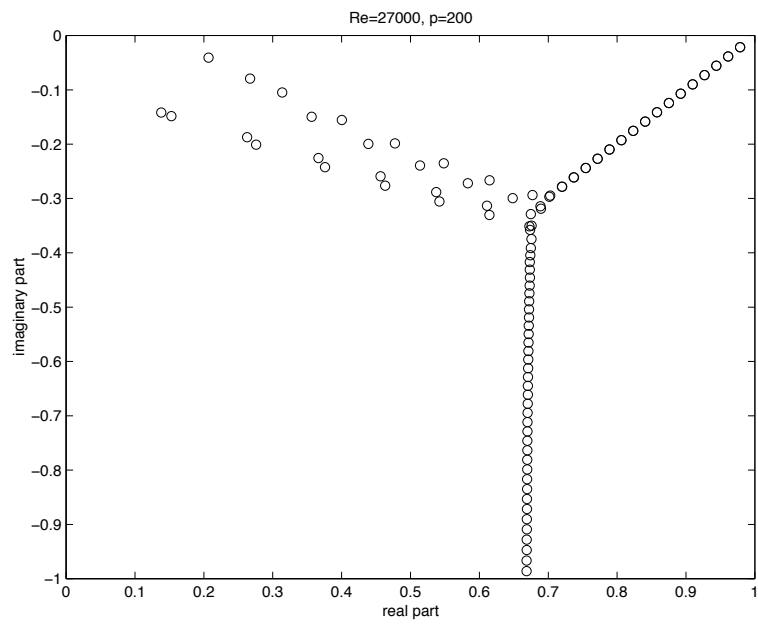


Figure 7: Eigenvalue distribution for  $Re = 27000$ ,  $p = 200$ .

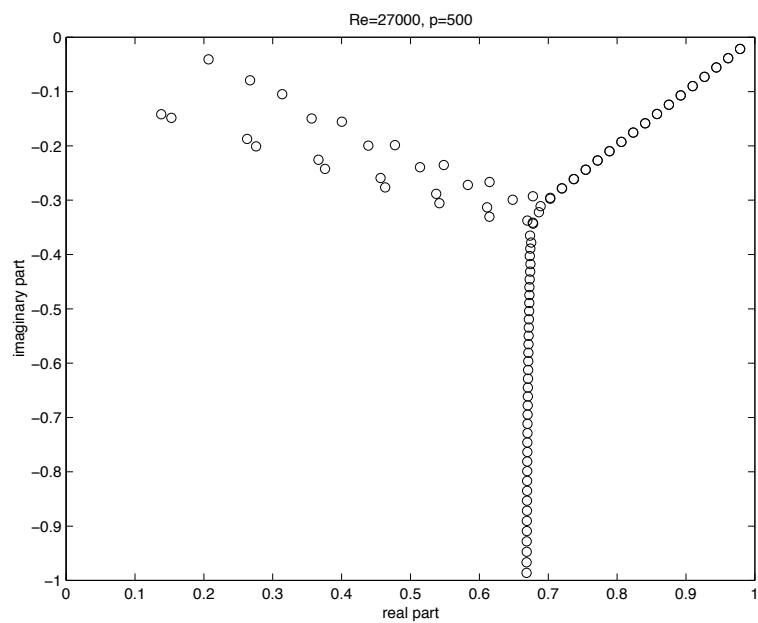


Figure 8: Eigenvalue distribution for  $Re = 27000$ ,  $p = 500$ .

# Research Reports

No.	Authors	Title
98-06	N.P. Hancke, J.M. Melenk, C. Schwab	A Spectral Galerkin Method for Hydrodynamic Stability Problems
98-05	J. Waldvogel	Long-Term Evolution of Coorbital Motion
98-04	R. Sperb	An alternative to Ewald sums, Part 2: The Coulomb potential in a periodic system
98-03	R. Sperb	The Coulomb energy for dense periodic systems
98-02	J.M. Melenk	On $n$ -widths for Elliptic Problems
98-01	M. Feistauer, C. Schwab	Coupling of an Interior Navier–Stokes Problem with an Exterior Oseen Problem
97-20	R.L. Actis, B.A. Szabo, C. Schwab	Hierarchic Models for Laminated Plates and Shells
97-19	C. Schwab, M. Suri	Mixed $hp$ Finite Element Methods for Stokes and Non-Newtonian Flow
97-18	K. Gerdes, D. Schötzau	$hp$ FEM for incompressible fluid flow - stable and stabilized
97-17	L. Demkowicz, K. Gerdes, C. Schwab, A. Bajer, T. Walsh	HP90: A general & flexible Fortran 90 $hp$ -FE code
97-16	R. Jeltsch, P. Klingenstein	Error Estimators for the Position of Discontinuities in Hyperbolic Conservation Laws with Source Terms which are solved using Operator Splitting
97-15	C. Lage, C. Schwab	Wavelet Galerkin Algorithms for Boundary Integral Equations
97-14	D. Schötzau, C. Schwab, R. Stenberg	Mixed $hp$ - FEM on anisotropic meshes II: Hanging nodes and tensor products of boundary layer meshes
97-13	J. Maurer	The Method of Transport for mixed hyperbolic - parabolic systems
97-12	M. Fey, R. Jeltsch, J. Maurer, A.-T. Morel	The method of transport for nonlinear systems of hyperbolic conservation laws in several space dimensions
97-11	K. Gerdes	A summary of infinite element formulations for exterior Helmholtz problems
97-10	R. Jeltsch, R.A. Renaut, J.H. Smit	An Accuracy Barrier for Stable Three-Time-Level Difference Schemes for Hyperbolic Equations
97-09	K. Gerdes, A.M. Matache, C. Schwab	Analysis of membrane locking in $hp$ FEM for a cylindrical shell