

# On $n$ -widths for Elliptic Problems

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## Abstract

The  $n$ -width of solution sets of a class of elliptic partial differential equations is calculated. In particular singularly perturbed equations of elliptic-elliptic type are considered.

**Keywords:**  $n$ -width, singularly perturbed problem, rough coefficients

**AMS Subject Classification:** 41A46, 35B25, 65N15

# 1 Introduction and Notation

In [1], Kolmogorov introduced the notion of  $n$ -widths which measure how accurately a given set of functions can be approximated by linear spaces of dimension  $n$  in a given norm. More precisely, for a normed linear space  $X$  (with norm  $\|\cdot\|_X$ ) and a subset  $A \subset X$  the  $n$ -width is given by

$$d_n(A, X) = \inf_{E_n} \sup_{f \in A} \inf_{g \in E_n} \|f - g\|_X \quad (1)$$

where the first infimum is taken over all subspaces  $E_n$  of  $X$  of dimension  $n \in \mathbb{N}$ .

In the context of numerical methods for partial differential equations, one application of the theory of  $n$ -widths is that it identifies a lower bound for the rate of convergence for any scheme and a given set of input data. In the present paper,  $n$ -widths of solutions sets of elliptic boundary value problems of the following type are considered:

$$Lu := -\nabla \cdot (A(x)\nabla u) + c(x)u = f \quad \text{on } \Omega \subset \mathbb{R}^d, \quad u|_{\partial\Omega} = 0. \quad (2)$$

where the matrix  $A \in L^\infty(\Omega)$  is assumed to be symmetric positive definite and  $c \in L^\infty(\Omega)$ ,  $c \geq 0$ . The space  $X$  is taken as the “energy space”, i.e., the space  $H_0^1(\Omega)$  equipped with the norm induced by the coercive operator  $L$ . For a fixed  $s \geq 0$ , the set  $A$  of (1) is taken as

$$A := \{u \in X \mid u \text{ solves (2) for some } f \text{ in the unit ball of } H^s(\Omega)\}.$$

In our main theorem, Theorem 3.1,  $d_n(A, X)$  is explicitly calculated. It is shown that  $d_n(A, X)$  can be controlled in terms of the regularity of the right hand side  $f$  (i.e.,  $s \geq 0$ ) and the upper and lower bounds on the eigenvalues of  $A$  and the upper and lower bounds on the coefficient  $c$ . In particular, the smoothness of the coefficients of  $L$  (and the smoothness of the domain  $\Omega$ ) is not relevant for the asymptotic behavior of  $d_n(A, X)$ . Theorem 3.1 covers the following two cases: The case of “rough” coefficients ( $A$  and  $c$  are merely in  $L^\infty(\Omega)$  but the “energy norm” is equivalent to the usual  $H^1(\Omega)$  norm) and the case of singularly perturbed equations of elliptic-elliptic type (the eigenvalues of  $A$  are small compared with  $c$ ). The  $n$ -widths in these two cases behave quite differently: In the first case, the  $n$ -width is (up to constants) the same as in the smooth case whereas in the second case the  $n$ -width deteriorates when the size of the eigenvalues of  $A$  tends to zero.

For the case of smooth coefficients, our result Theorem 3.1 is well-known and follows, as elliptic regularity theory yields that  $A \subset H^{s+2}(\Omega)$ , from [2] where the  $n$ -width of the unit ball of the Sobolev spaces  $H^{s+2}(\Omega)$  in some other Sobolev space, e.g., the energy space  $X$ , is calculated. For the singularly perturbed case, a full analysis seems to be available in one dimension for the constant coefficient case only, [3]. In order to obtain our result which makes minimal assumptions on the smoothness of the coefficients, methods different from those used above were necessary. The key observation in the present paper is the fact that the operator  $L$  induces an isomorphism between  $X$  and its dual space  $X'$ . It suffices therefore to calculate the  $n$ -width of the unit ball of  $H^s(\Omega)$  in  $X'$  and precise structural knowledge about the set  $A$  is not necessary.

We need to introduce various spaces. For an open set  $\Omega \subset \mathbb{R}^d$  we denote by  $L^2(\Omega)$  the Hilbert space of square integrable functions equipped with the inner product  $(u, v) := \int_\Omega uv \, dx$  and norm  $\|u\|_{L^2(\Omega)} = |(u, u)|^{1/2}$ . For non-negative integers  $s \in \mathbb{N}_0$ , we can define the norm

$$\|u\|_{H^s(\Omega)} := \left\{ \sum_{|\alpha| \leq s} \|D^\alpha u\|_{L^2(\Omega)}^2 \right\}^{1/2}$$

where  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$  and  $|\alpha| = \sum_i \alpha_i$ . The Sobolev spaces  $H^s(\Omega)$  are then defined as the completion of  $C^\infty(\Omega)$  with respect to the norm  $\|\cdot\|_{H^s(\Omega)}$ . For  $s \geq 0$  we set  $[s] = \max\{n \in \mathbb{N}_0 \mid n \leq s\}$  and define the Sobolev spaces  $H^s(\Omega)$  via the usual interpolation (the  $K$ -method, see [4]) between the spaces  $H^{[s]}(\Omega)$  and  $H^{[s]+1}(\Omega)$ .

As we are interested in elliptic equation of second order which may be of singularly perturbed type, we introduce the following  $\varepsilon$ -weighted ‘‘energy’’ spaces and their duals. For  $\varepsilon \in (0, 1]$  we introduce the norm

$$\|u\|_{H_{0,\varepsilon}^1(\Omega)} := \left\{ \varepsilon^2 \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} |u|^2 dx \right\}^{1/2}. \quad (3)$$

The space  $H_{0,\varepsilon}^1(\Omega)$  is then defined as the completion under this norm of the space of infinitely differentiable, compactly supported functions, i.e., the space  $\{u \in C^\infty(\Omega) \mid \text{supp } u \subset \Omega\}$ . For  $\varepsilon = 1$  we write  $H_0^1(\Omega) = H_{0,1}^1(\Omega)$ .

Finally, we introduce the spaces  $H_\varepsilon^{-1}(\Omega)$  as the dual spaces of the spaces  $H_{0,\varepsilon}^1(\Omega)$  with pivot space  $L^2(\Omega)$ . In particular, denoting  $\langle \cdot, \cdot \rangle$  the duality pairing between a space and its dual space, the norm on  $H_\varepsilon^{-1}(\Omega)$  is given by

$$\|f\|_{H_\varepsilon^{-1}(\Omega)} = \sup_{v \in H_{0,\varepsilon}^1(\Omega)} \frac{|\langle f, v \rangle|}{\|v\|_{H_{0,\varepsilon}^1(\Omega)}}. \quad (4)$$

Again, we write  $H^{-1}(\Omega)$  for  $H_1^{-1}(\Omega)$  in the case  $\varepsilon = 1$ . Note that the spaces  $H_0^1(\Omega)$  and  $H_{0,\varepsilon}^1(\Omega)$  are the same space equipped with two (equivalent) norms; dually, the spaces  $H^{-1}(\Omega)$  and  $H_\varepsilon^{-1}(\Omega)$  can also be regarded as the same spaces equipped with two (equivalent) norms. We will use this fact to view the duality pairing  $\langle \cdot, \cdot \rangle$  at the same time as the pairing on  $H^{-1}(\Omega) \times H_0^1(\Omega)$  and on  $H_\varepsilon^{-1}(\Omega) \times H_{0,\varepsilon}^1(\Omega)$ .

Finally, for sequences  $(u_i)_{i=1}^\infty \subset \mathbb{R}$ , we use occasionally the shorthand  $\underline{u}$ . As usual, we denote by  $l^2$  the Hilbert space of square summable sequences,  $l^2 := \{\underline{u} \mid \|\underline{u}\|_{l^2}^2 := \sum_{i=1}^\infty |u_i|^2 < \infty\}$ .

## 2 $n$ -widths in $H_\varepsilon^{-1}$

**Lemma 2.1** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded domain with  $C^\infty$  boundary and  $s \geq 0$ . Set*

$$A := \{f \in H^s(\Omega) \mid \|f\|_{H^s(\Omega)} \leq 1\}.$$

*Then there is  $C > 0$  such that*

$$\forall n \in \mathbb{N}, \varepsilon \in (0, 1] \quad Cn^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}} \leq d_n(A, H_\varepsilon^{-1}(\Omega)).$$

**Proof:** Let  $(\varphi_i)_{i=1}^\infty \subset H_0^1(\Omega)$  be the eigenfunctions of  $-\Delta$  with eigenvalues  $\lambda_i^2 > 0$ , i.e.,

$$-\Delta \varphi_i - \lambda_i^2 \varphi_i = 0 \quad \text{on } \Omega, \quad \varphi_i = 0 \quad \text{on } \partial\Omega.$$

Without loss of generality, we may assume that the eigenvalues  $\lambda_i^2$  are non-decreasing and that the eigenfunctions  $\varphi_i$  form an orthonormal basis of  $L^2(\Omega)$  and are additionally orthogonal with respect to the  $(\nabla \cdot, \nabla \cdot)_{L^2(\Omega)}$  inner product of  $H_0^1(\Omega)$ . Denoting  $(\cdot, \cdot)$  the usual  $L^2(\Omega)$  inner product, it follows that

$$\|u\|_{L^2(\Omega)}^2 = \sum_{i=1}^\infty |(u, \varphi_i)|^2, \quad \|\nabla u\|_{L^2(\Omega)}^2 = \sum_{i=1}^\infty \lambda_i^2 |(u, \varphi_i)|^2$$

for functions  $u \in L^2(\Omega)$ ,  $u \in H_0^1(\Omega)$  respectively. Hence, the map

$$F : u \mapsto \underline{u} = (u_i)_{i=1}^\infty = ((u, \varphi_i))_{i=1}^\infty \quad (5)$$

induces an isometric isomorphism between the spaces  $L^2(\Omega)$  and  $l^2$  on the one hand and the spaces  $H_{0,\varepsilon}^1(\Omega)$  and

$$h_\varepsilon^1 := \{ \underline{u} = (u_i)_{i=1}^\infty \mid \|\underline{u}\|_{h_\varepsilon^1}^2 := \sum_{i=1}^\infty (1 + \lambda_i^2 \varepsilon^2) |u_i|^2 \} \quad (6)$$

on the other hand. The space

$$h_\varepsilon^{-1} := \{ \underline{u} = (u_i)_{i=1}^\infty \mid \|\underline{u}\|_{h_\varepsilon^{-1}}^2 := \sum_{i=1}^\infty (1 + \lambda_i^2 \varepsilon^2)^{-1} |u_i|^2 \} \quad (7)$$

is the dual space of  $h_\varepsilon^1$  (with respect to the pivot space  $l^2$ ) and therefore isometrically isomorphic to  $H_\varepsilon^{-1}(\Omega)$ . We conclude that

$$d_n(A, H_\varepsilon^{-1}(\Omega)) = d_n(F(A), h_\varepsilon^{-1}) \geq d_n(\tilde{a}, h_\varepsilon^{-1}) \quad \forall \tilde{a} \subset F(A). \quad (8)$$

In order to prove the lemma, we will now choose  $\tilde{a} \subset F(A)$  appropriately. To that end, we introduce the normed space

$$\tilde{h}^s := \{ \underline{u} = (u_i)_{i=1}^\infty \mid \|\underline{u}\|_{\tilde{h}^s}^2 := \sum_{i=1}^\infty \lambda_i^{2s} |u_i|^2 < \infty \}$$

and the linear map

$$\hat{F} : \tilde{h}^s \ni (u_i)_{i=1}^\infty \mapsto \sum_{i=1}^\infty u_i \varphi_i \in L^2(\Omega). \quad (9)$$

We now claim that  $\hat{F}$  is in fact a continuous map into  $H^s(\Omega)$ : For integer  $s$ , this is a consequence of a repeated application of the shift theorem for  $-\Delta$ , valid due to the assumption that  $\partial\Omega$  is smooth (see, e.g., Chap. 3, Lemma 1 of [5] for the details). For  $s \notin \mathbb{N}_0$  the spaces  $\tilde{h}^s$  are isomorphic to those obtained by interpolation between  $\tilde{h}^{[s]}$ ,  $\tilde{h}^{[s]+1}$  via the  $K$ -method and hence interpolation allows us to conclude that  $\hat{F}$  maps continuously into  $H^s(\Omega)$ .

By the continuity of  $\hat{F}$  there is  $c > 0$  such that the set

$$\tilde{a} := \{ \underline{u} \mid \|\underline{u}\|_{\tilde{h}^s} \leq c \} \subset F(A).$$

To conclude the calculation of  $d_n(\tilde{a}, h_\varepsilon^{-1})$ , we observe that the map

$$e : (u_i)_{i=1}^\infty \mapsto \left( (1 + \varepsilon^2 \lambda_i^2)^{1/2} u_i \right)_{i=1}^\infty$$

is an isometric isomorphism between  $h_\varepsilon^{-1}$  and  $l^2$ . Therefore,

$$d_n(\tilde{a}, h_\varepsilon^{-1}) = d_n(e(\tilde{a}), l^2) = c \cdot d_n \left( \left\{ \underline{u} \mid \sum_{i=1}^\infty \lambda_i^{2s} (1 + \varepsilon^2 \lambda_i^2) |u_i|^2 \leq 1 \right\}, l^2 \right).$$

As the sequence  $(\lambda_i^s (1 + \varepsilon^2 \lambda_i^2)^{1/2})_{i=1}^\infty$  is non-decreasing, a theorem from Lorentz (Chap. 9.4 of [6], see also Cor. IV.2.6 of [7]) gives

$$d_n(\tilde{a}, h_\varepsilon^{-1}) = c \lambda_{n+1}^{-s} (1 + \varepsilon^2 \lambda_{n+1}^2)^{-1/2} \geq c \lambda_{n+1}^{-s} (1 + \varepsilon \lambda_{n+1})^{-1}.$$

Finally, by Weyl's formula for the eigenvalue distribution (cf., e.g., Chap. 3 of [8]) there is  $C > 0$  depending only on  $\Omega$  such that

$$\lambda_n \sim Cn^{1/d} \quad \text{as } n \rightarrow \infty.$$

Hence there is  $C > 0$  such that  $d_n(\tilde{a}, h_\varepsilon^{-1}) \geq Cn^{-s/d}(1 + \varepsilon n^{1/d})^{-1}$ , which proves the lemma.  $\square$

**Theorem 2.2** *Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain,  $s \geq 0$ . Set*

$$A := \{f \in H^s(\Omega) \mid \|f\|_{H^s(\Omega)} \leq 1\}. \quad (10)$$

*Then there are  $C_1, C_2 > 0$  such that*

$$\forall n \in \mathbb{N} \quad \forall \varepsilon \in (0, 1] \quad C_1 n^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}} \leq d_n(A, H_\varepsilon^{-1}(\Omega)) \leq C_2 n^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}}.$$

**Proof:** Let us first prove the lower bound. In order to apply Lemma 2.1, let  $\tilde{\Omega} \subset \Omega$  be a ball. By the smoothness of  $\partial\tilde{\Omega}$  there is a linear, continuous extension operator  $E : H^s(\tilde{\Omega}) \rightarrow H^s(\Omega)$  with norm  $\|E\|$  (see, e.g., [9]). Then the set

$$\tilde{A} := \{f \in H^s(\tilde{\Omega}) \mid \|f\|_{H^s(\tilde{\Omega})} \leq 1\}$$

satisfies  $E(\tilde{A}) \subset \|E\| \cdot A$ .

Let  $E_n \subset H_\varepsilon^{-1}(\Omega)$  be a subspace of dimension  $n$ . Clearly  $E_n$  can also be viewed as a subspace of  $H_\varepsilon^{-1}(\tilde{\Omega})$  of dimension at most  $n$ . For each such  $E_n$  we estimate

$$\begin{aligned} \sup_{f \in A} \inf_{g \in E_n} \|f - g\|_{H_\varepsilon^{-1}(\Omega)} &= \sup_{f \in A} \inf_{g \in E_n} \sup_{v \in H_{0,\varepsilon}^1(\Omega)} \frac{|\langle f - g, v \rangle|}{\|v\|_{H_{0,\varepsilon}^1(\Omega)}} \\ &\geq \sup_{f \in A} \inf_{g \in E_n} \sup_{v \in H_{0,\varepsilon}^1(\tilde{\Omega})} \frac{|\langle f - g, v \rangle|}{\|v\|_{H_{0,\varepsilon}^1(\tilde{\Omega})}} = \sup_{f \in A} \inf_{g \in E_n} \|f - g\|_{H_\varepsilon^{-1}(\tilde{\Omega})} \\ &\geq \frac{1}{\|E\|} \sup_{f \in \tilde{A}} \inf_{g \in E_n} \|f - g\|_{H_\varepsilon^{-1}(\tilde{\Omega})} \geq \frac{1}{\|E\|} d_n(\tilde{A}, H_\varepsilon^{-1}(\tilde{\Omega})) \\ &\geq C_1 n^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}} \end{aligned}$$

where we appealed to Lemma 2.1 in the last step. Taking the infimum over all subspaces  $E_n \subset H_\varepsilon^{-1}(\Omega)$  of dimension  $n$  concludes the proof of the lower bound.

For the upper bound, let  $E_n$  be subspaces of  $L^2(\Omega)$  with the property that there is  $C > 0$  such that

$$\forall f \in H^s(\Omega) \quad \inf_{g \in E_n} \|f - g\|_{H^t(\Omega)} \leq Cn^{-(s-t)/d} \|f\|_{H^s(\Omega)}, \quad t \in \{-1, 0\}. \quad (11)$$

Such spaces  $E_n$  may be constructed as follows using standard finite element spaces: Let  $Q \subset \mathbb{R}^d$  be an open hyper cube such that  $\bar{\Omega} \subset Q$ . By the assumption that  $\Omega$  has a Lipschitz boundary, there is a linear, continuous extension operator  $E : H^s(\Omega) \rightarrow H^s(Q)$ , [9]. The restrictions to  $\Omega$  of the spaces  $S_h$  given by (discontinuous) piecewise polynomials of degree  $p \in \mathbb{N}_0$  with  $p + 1 \geq s$  on uniform meshes with meshwidth  $h$  on  $Q$  then have the desired

properties (see, e.g., Thms. 4.1.1, 4.1.6 of [10]). For  $E_n$  such that (11) is satisfied, we calculate for  $f \in H^s(\Omega)$ ,  $g \in E_n$ :

$$\|f - g\|_{H_\varepsilon^{-1}(\Omega)} = \sup_{v \in H_{0,\varepsilon}^1(\Omega)} \frac{|\langle f - g, v \rangle|}{\|v\|_{H_{0,\varepsilon}^1(\Omega)}} \leq \|f - g\|_{L^2(\Omega)}, \quad (12)$$

$$\|f - g\|_{H_\varepsilon^{-1}(\Omega)} = \sup_{v \in H_{0,\varepsilon}^1(\Omega)} \frac{|\langle f - g, v \rangle|}{\|v\|_{H_{0,\varepsilon}^1(\Omega)}} \leq \varepsilon^{-1} \|f - g\|_{H^{-1}(\Omega)}. \quad (13)$$

Hence, we get  $\|f - g\|_{H_\varepsilon^{-1}(\Omega)} \leq \min\{\|f - g\|_{L^2(\Omega)}, \varepsilon^{-1} \|f - g\|_{H^{-1}(\Omega)}\}$  and therefore together with (11)

$$\begin{aligned} \inf_{g \in E_n} \|f - g\|_{H_\varepsilon^{-1}(\Omega)} &\leq C \min\{n^{-s/d}, \varepsilon^{-1} n^{-(s+1)/d}\} \|f\|_{H^s(\Omega)} \\ &\leq C n^{-s/d} \min\{1, \varepsilon^{-1} n^{-1/d}\} \|f\|_{H^s(\Omega)} \leq C n^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}} \|f\|_{H^s(\Omega)}. \end{aligned}$$

□

### 3 $n$ -widths of solution sets of elliptic equations

We will now apply the  $n$ -width result Theorem 2.2 to calculate the  $n$ -widths of solution set of elliptic partial differential equations. Let  $\Omega \subset \mathbb{R}^d$  be a bounded Lipschitz domain and consider elliptic partial differential equation of the form

$$Lu := -\nabla \cdot (A(x)\nabla u) + c(x)u = f \in H^{-1}(\Omega) \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (14)$$

where the symmetric matrix  $A(x) = (a_{ij}(x))_{i,j=1}^d$  and the coefficient  $c(x)$  are in  $L^\infty(\Omega)$ . Solutions of (14) are understood in the weak sense, i.e.,  $u \in H_0^1(\Omega)$  solves (14) if

$$B(u, v) = \int_\Omega \nabla u \cdot A(x)\nabla v + c(x)uv \, dx = \langle f, v \rangle \quad \forall v \in H_0^1(\Omega). \quad (15)$$

Concerning  $A$  and  $c$  we assume furthermore that the operator  $L$  is  $H_0^1(\Omega)$ -elliptic and that in fact

$$\exists \underline{\alpha}, \bar{\alpha} > 0, \varepsilon \in (0, 1] \quad \underline{\alpha} \|u\|_{H_{0,\varepsilon}^1(\Omega)}^2 \leq B(u, u) \leq \bar{\alpha} \|u\|_{H_{0,\varepsilon}^1(\Omega)}^2 \quad \forall u \in H_0^1(\Omega). \quad (16)$$

Under this assumption, the Lax-Milgram Lemma ([11]; see also Thm. 5.2.1 of [10]) guarantees the unique solvability of (15) for each  $f \in H^{-1}(\Omega)$ ; in fact, the solution operator

$$\begin{aligned} S : H_\varepsilon^{-1}(\Omega) &\rightarrow H_{0,\varepsilon}^1(\Omega) \\ f &\mapsto Sf = u = \text{solution of (15) corresponding to right hand side } f \end{aligned}$$

is an isomorphism between  $H_\varepsilon^{-1}(\Omega)$  and  $H_{0,\varepsilon}^1(\Omega)$  and satisfies

$$\underline{\alpha} \|Sf\|_{H_{0,\varepsilon}^1(\Omega)} \leq \|f\|_{H_\varepsilon^{-1}(\Omega)} \leq \bar{\alpha} \|Sf\|_{H_{0,\varepsilon}^1(\Omega)} \quad \forall f \in H_\varepsilon^{-1}(\Omega). \quad (17)$$

For  $s \geq 0$  we have  $H^s(\Omega) \subset H_\varepsilon^{-1}(\Omega)$  and we may therefore define the solution set  $\mathcal{U}^s$  of solutions  $u$  of (15) with  $f \in A := \{f \in H^s(\Omega) \mid \|f\|_{H^s(\Omega)} \leq 1\}$  by

$$\mathcal{U}^s := S(A) = \{u \in H_0^1(\Omega) \mid u \text{ solves (15) for some } f \in H^s(\Omega) \text{ with } \|f\|_{H^s(\Omega)} \leq 1\}. \quad (18)$$

For the solution set  $\mathcal{U}^s$  we have the following result:

**Theorem 3.1** *Let  $\mathcal{U}^s$  be given by (18) and assume that (16) holds. Then there are  $C_1, C_2$  depending only on  $\Omega, s \geq 0$ , and the constants  $\underline{\alpha}, \bar{\alpha}$  such that*

$$\forall n \in \mathbb{N} \quad C_1 n^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}} \leq d_n(\mathcal{U}^s, H_{0,\varepsilon}^1(\Omega)) \leq C_2 n^{-s/d} \frac{1}{1 + \varepsilon n^{1/d}}.$$

**Proof:** Let  $A = \{f \in H^s(\Omega) \mid \|f\|_{H^s(\Omega)} \leq 1\}$ . (17) implies that

$$\begin{aligned} \frac{1}{\bar{\alpha}} d_n(A, H_\varepsilon^{-1}(\Omega)) &\leq d_n(S(A), H_{0,\varepsilon}^1(\Omega)) d_n(\mathcal{U}^s, H_{0,\varepsilon}^1(\Omega)) = \\ d_n(\mathcal{U}^s, H_{0,\varepsilon}^1(\Omega)) &= d_n(S(A), H_{0,\varepsilon}^1(\Omega)) \leq \frac{1}{\underline{\alpha}} d_n(A, H_\varepsilon^{-1}(\Omega)). \end{aligned}$$

Appealing to Theorem 2.2 allows us to conclude the argument.  $\square$

It is noteworthy that the assumptions on the coefficients  $A$  and  $c$  and the domain  $\Omega$  are essentially the weakest possible that still lead to a meaningful variational formulation of (14). Theorem 3.1 covers in particular the following cases:

1. The case of ‘‘rough coefficients’’: the symmetric matrix  $A$  and the coefficient  $c$  are in  $L^\infty(\Omega)$  and there is  $\underline{\alpha} > 0$  such that

$$0 < \underline{\alpha} \leq A \quad \text{a.e. on } \Omega, \quad 0 \leq c \quad \text{a.e. on } \Omega.$$

In that case, one may take  $\varepsilon = 1$  and the constants  $C_1, C_2$  in Theorem 3.1 depend on  $\underline{\alpha}, \|A\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}, s$ , and  $\Omega$ .

2. The case of a singularly perturbed problem of elliptic-elliptic type: There are  $\underline{\alpha}, \bar{\alpha}, \underline{c} > 0$  such that for some  $\varepsilon \in (0, 1]$  there holds

$$\underline{\alpha} \varepsilon^2 \leq A \leq \bar{\alpha} \varepsilon^2 \quad \text{a.e. on } \Omega, \quad 0 < \underline{c} \leq c \quad \text{a.e. on } \Omega.$$

In that case, the constants  $C_1, C_2$  of Theorem 3.1 depend only on  $\underline{\alpha}, \bar{\alpha}, \underline{c}$ , and again on  $\|A\|_{L^\infty(\Omega)}, \|c\|_{L^\infty(\Omega)}, s$ , and  $\Omega$ .

**Remark 3.2** In the case of smooth coefficients and under the assumption that  $\varepsilon = 1$ , the finite element method based on *standard* conforming finite element spaces of piecewise polynomials of degree  $p \in \mathbb{N}, p + 1 \geq s$  lead to the optimal rate of convergence  $O(n^{-(s+1)/d})$ . In the case of ‘‘rough’’ coefficients (in the sense above) standard finite element spaces perform poorly. However, non-polynomial spaces can be constructed which recover the optimal rate of convergence, [12, 13]. In the singularly perturbed case, i.e., when  $\varepsilon$  is small (with respect to 1), the singular perturbation character of the equation manifests itself in a reduced pre-asymptotic  $n$ -width: pre-asymptotically, i.e., when  $\varepsilon n^{1/d}$  is small, only a rate of  $O(n^{-s/d})$  can be attained instead of  $O(n^{-(s+1)/d})$  in the asymptotic regime (i.e., when  $\varepsilon n^{1/d}$  is large). The design of finite element methods for singularly perturbed problems that lead to the optimal or at least near optimal rate of convergence has so far been successful only in the case of smooth coefficients.

**Remark 3.3** This paper is concerned with the calculation of the  $n$ -width in the natural ‘‘energy norm’’. The techniques employed here, however, are not restricted to this setting.



For example, for (14) with  $A = \varepsilon^2 I$ ,  $c \equiv 1$  and smooth boundary  $\partial\Omega$  there holds for  $\mathcal{U}^s$  as defined in (18):

$$C_1 n^{-s/d} \frac{1}{1 + \varepsilon^2 n^{2/d}} \leq d_n(\mathcal{U}^s, L^2(\Omega)) \leq C_2 n^{-s/d} \frac{1}{1 + \varepsilon^2 n^{2/d}} \quad (19)$$

for some  $C_1, C_2$  independent of  $n, \varepsilon$ . The proof is very similar to the reasoning in this paper and therefore merely sketched. We introduce the space  $H_\varepsilon^2(\Omega) := \{u \in H_{0,\varepsilon}^1(\Omega) \mid -\Delta u \in L^2(\Omega)\}$  equipped with the norm

$$\|u\|_{H_\varepsilon^2(\Omega)}^2 := \|u\|_{L^2(\Omega)}^2 + \varepsilon^2 \|\nabla u\|_{L^2(\Omega)}^2 + \varepsilon^4 \|\Delta u\|_{L^2(\Omega)}^2,$$

and its dual space  $H_\varepsilon^{-2}(\Omega)$ . The very weak formulation of (14) reads: find  $u \in L^2(\Omega)$  such that

$$\int_\Omega u(-\varepsilon^2 \Delta v + v) dx = \langle f, v \rangle \quad \forall v \in H_\varepsilon^2(\Omega). \quad (20)$$

The assumptions on the smoothness of  $\partial\Omega$  guarantee that the shift-theorem for  $-\Delta$  holds true. It is therefore easy to check that the Babuška-Brezzi condition (see, e.g., Thm. 5.2.1 of [10]) is satisfied for (20) and to conclude that the solution operator  $S : H_\varepsilon^{-2}(\Omega) \rightarrow L^2(\Omega)$  for (20) is an isomorphism with constants independent of  $\varepsilon$ . It suffices therefore to calculate  $d_n(A, H_\varepsilon^{-2}(\Omega))$  where  $A$  is given by (10). This can be done as in Section 2. A straightforward adaptation of Lemma 2.1 together with the assumption that  $\partial\Omega$  is smooth gives the lower bound in (19). In order to obtain the upper bound, one can modify the proof of Theorem 2.2 as follows. Introducing the space  $H^{-2}(\Omega)$  as the dual space of  $H^2(\Omega)$  one can check that (11) holds for  $t = -2$  as well. Next, we observe that the shift theorem for  $-\Delta$  gives that the norm  $\|\cdot\|_{H_\varepsilon^2(\Omega)}$  is equivalent to the norm  $\|\cdot\|_{H_{0,\varepsilon}^1(\Omega)} + \varepsilon^2 \|\cdot\|_{H^2(\Omega)}$  with constants independent of  $\varepsilon$ . Hence, we can estimate for some  $C > 0$  and for all  $f, g \in L^2(\Omega)$ :

$$\|f - g\|_{H_\varepsilon^{-2}(\Omega)} = \sup_{v \in H_\varepsilon^2(\Omega)} \frac{|\langle f - g, v \rangle|}{\|v\|_{H_\varepsilon^2(\Omega)}} \leq C \varepsilon^{-2} \|f - g\|_{H^{-2}(\Omega)}.$$

Similarly to (12) we have  $\|f - g\|_{H_\varepsilon^{-2}(\Omega)} \leq \|f - g\|_{L^2(\Omega)}$  for  $f, g \in L^2(\Omega)$ . Combining these two estimates for we obtain

$$\|f - g\|_{H_\varepsilon^{-2}(\Omega)} \leq C \min \{ \|f - g\|_{L^2(\Omega)}, \varepsilon^{-2} \|f - g\|_{H^{-2}(\Omega)} \}.$$

Appealing to (11) with  $t = 0$  and  $t = -2$  as in the proof of Theorem 2.2 concludes the argument.

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