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Abstract

The idea of the method of transport is introduced by means of non-linear conservation laws. The systems are rewritten in an advection form accounting for the characteristic propagation directions. A straightforward linearization of this advection form leads to a genuinely multi-dimensional method. Approximations using infinitely many or a finite number of propagation directions are shown.

1 Introduction

We want to solve numerically nonlinear systems of hyperbolic partial differential equations. For simplicity we assume that one of the independent variables is time while the others are space variables. Typical examples will be the unsteady Euler equations of gas dynamics, the shallow water equations and the Magneto-Hydrodynamic equations, MHD. We shall mainly be concerned with the Euler equations and indicate very briefly modifications needed for the shallow water equation and MHD. The main difference between hyperbolic and parabolic equations is that in the hyperbolic case information travels at finite speed while in the other case the speed is infinite. Thus, when it comes to solving the equations numerically, explicit methods are in general used in the hyperbolic case while time implicit schemes are needed for the parabolic case. Most numerical approaches start with the one-dimensional problem where one has finitely many propagation speeds, namely the Eigenvalues of the Jacobian of the flux function.

In more than one space dimension one generally introduces some cells, e.g. finite-volumes or finite-elements. Due to the mainly one dimensional solution operators the numerical propagation is performed, according to the geometry of the cells. Since cells usually consist of finitely many cell-vertices, this means that only finitely many transport directions are numerically accounted for. However for hyperbolic systems in several space dimensions one has infinitely many propagation direction, e.g. sound waves in a gas propagate in all space directions.

In this presentation we take the approach of Fey, [1, 2, 3, 5]. One rewrites the equations in what I shall call the advection form. This is a formulation of the equations where one can see more easily the transport directions. This form is independent of any discretization. Since infinitely many transport directions are involved this form will contain some integrals. This will be done in Section 2. Unfortunately, when one uses the advection form for designing a numerical scheme the computation of the integrals are rather time consuming. To reduce this computational effort Fey has introduced an **advection form** with finitely many advection directions. This will be presented in Section 3. Despite the fact that only finitely many advection directions are visible in this form the equation still represent the full set of equations. This will then be different in Section 4 where one does a first approximation by replacing the full set of equations by a finite number of linear advection equations where one **freezes the coefficients at time** t. This approximate system has a solution which differs at time $t + \Delta t$ from the exact solution by $O(\Delta t^2)$. Hence any discrete version of this approximate system will lead to a numerical scheme of first order only. In order to be able to develop a second order scheme we modify the approximate system such that its solution differs after a time step Δt by $O(\Delta t^3)$ from the exact solution. This modification is made for regions where the solution is smooth and Taylor series expansions can be used.

In Section 5 we explain the space discretization, and the overall scheme under the assumption that a transport scheme for the linear advection equation exists. Different realisation of the approximation of the state vector will then lead to an overall first order scheme formulated in Section 5 and an extended second order scheme described in Section 6. In Section 7 we briefly discuss the problem of avoiding oscillations near discontinuities using limiters. In the final section we give some conclusions.

2 Advection form of the Euler equation

We consider systems of hyperbolic conservation laws of the form

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \,, \tag{2.1}$$

where $\mathbf{U}$ is the state vector. The divergence acts on the rows of the flux matrix $\mathbf{F}(\mathbf{U})$. We give four examples:

i) **Euler equations of gas dynamics in $\mathbb{R}^N$**

Let $\mathbf{u} \in \mathbb{R}^N$ denote the velocity vector. Then the vector of state $\mathbf{U} \in \mathbb{R}^{N+2}$ and the flux matrix $\mathbf{F}(\mathbf{U}) \in \mathbb{R}^{(N+2)\times N}$ are given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{u} \\ E \end{pmatrix} \,, \quad \mathbf{F}(\mathbf{U}) = \mathbf{U}\,\mathbf{u}^T + c\,\mathbf{L}(\mathbf{U}) \tag{2.2a}$$

where

$$\mathbf{L}(\mathbf{U}) = \frac{p}{c} \begin{pmatrix} \mathbf{O}^T \\ \mathbf{I} \\ \mathbf{u}^T \end{pmatrix} \,. \tag{2.2b}$$

Here ρ is the density, E is the total energy, p the pressure, c is the speed of sound given by $c^2 = p\gamma/\rho$ and γ is the ratio of specific heats. $\mathbf{O}$ is the zero vector and $\mathbf{I}$ the identity matrix of appropriate dimensions. (2.1), (2.2) represent a system of $N+2$ nonlinear hyperbolic conservation laws and one has to close this system with the equation of state

$$p = (\gamma - 1)\Big(E - \rho\,\frac{\mathbf{u}^T\mathbf{u}}{2}\Big) \,. \tag{2.3}$$

ii) **Shallow water equations in $\mathbb{R}^N$**

Let again $\mathbf{u} \in \mathbb{R}^N$ denote the velocity vector. Then the vector of state $\mathbf{U} \in \mathbb{R}^{N+1}$ and the flux matrix $\mathbf{F}(\mathbf{U}) \in \mathbb{R}^{(N+1)\times N}$ are given by

$$\mathbf{U} = \begin{pmatrix} h \\ h\mathbf{u} \end{pmatrix} \,, \quad \mathbf{F}(\mathbf{U}) = \mathbf{U}\mathbf{u}^T + c\,\mathbf{L}(\mathbf{U}) \tag{2.4a}$$

where

$$\mathbf{L}(\mathbf{U}) = \frac{ch}{2} \begin{pmatrix} \mathbf{O}^T \\ \mathbf{I} \end{pmatrix} \,. \tag{2.4b}$$

Here h is the total depth of the fluid, c is the celerity given by $c^2 = gh$ and g is the constant of gravity. Again $\mathbf{O}$ and $\mathbf{I}$ are the zero vector and the identity matrix of appropriate dimensions. Note that the isentropic Euler equations can be put in the same form.

iii) **Magneto-Hydro-Dynamic equations in $\mathbb{R}^N$**

Let $\mathbf{B} \in \mathbb{R}^N$ denote the magnetic field and $\mathbf{b} = \mathbf{B}/\sqrt{\rho}$ the propagation speed of the Alfvén waves. Let again $\mathbf{u} \in \mathbb{R}^N$ be the velocity vector. Then the vector of state $\mathbf{U} \in \mathbb{R}^{2N+2}$ and the flux matrix $\mathbf{F}(\mathbf{U}) \in \mathbb{R}^{(2N+2)\times N}$ can be given, using the notation of Fey, [7] by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\mathbf{u} \\ E \\ \mathbf{B} \end{pmatrix}, \quad \begin{aligned} \mathbf{F}(\mathbf{U}) \;=\; & \mathbf{A}_0\mathbf{u}^T + \\ & \mathbf{A}_1(\mathbf{u}+\mathbf{b})^T + \mathbf{A}_2(\mathbf{u}-\mathbf{b})^T \\ & + \widetilde{c}\,\mathbf{L}(\mathbf{U}) \end{aligned} \tag{2.5a}$$

where

$$\mathbf{A}_0 = \begin{pmatrix} \rho \\ \rho\mathbf{u} \\ E \\ 0 \end{pmatrix}, \quad \mathbf{A}_1 = \frac{\rho}{2}\begin{pmatrix} 0 \\ -\mathbf{b} \\ -\mathbf{u}^T\mathbf{b} \\ \frac{1}{\sqrt{\rho}}(\mathbf{b}-\mathbf{u}) \end{pmatrix}, \quad \mathbf{A}_2 = \frac{\rho}{2}\begin{pmatrix} 0 \\ \mathbf{b} \\ \mathbf{u}^T\mathbf{b} \\ \frac{1}{\sqrt{\rho}}(\mathbf{b}+\mathbf{u}) \end{pmatrix} \tag{2.5b}$$

and

$$\mathbf{L}(\mathbf{U}) = \frac{\widetilde{p}}{\widetilde{c}}\begin{pmatrix} \mathbf{O}^T \\ \mathbf{I} \\ \mathbf{u}^T \\ \mathbf{O} \end{pmatrix}. \tag{2.5c}$$

Here ρ is the density. $\widetilde{p}$ is related to the gas dynamical pressure p by

$$\widetilde{p} = p + \mathbf{B}^T\mathbf{B}/2$$

and $\widetilde{c}$ is the new speed of sound given by $\widetilde{c}^2 = \widetilde{p}\gamma/\rho$ where γ is the same as in the Euler equation. $\mathbf{O}$ is the zero vector or matrix and $\mathbf{I}$ the identity matrix of appropriate dimensions. We have to close the system given by (2.1), (2.5) with the equation of state

$$E = \frac{p}{\gamma-1} + \rho(\mathbf{u}^T\mathbf{u} + \mathbf{b}^T\mathbf{b})/2\,. \tag{2.6}$$

Note that if the magnetic field $\mathbf{B}$ is divergence free at the initial condition, then it will be divergence free for all times. It is the challenge to construct explicit finite volume methods that fulfill this property also in the numerical solution.

iv) **Wave equation in $\mathbb{R}^N$**

A simplified version of the Euler question, that includes lots of the multidimensional properties, i.e. infinitely many propagation directions, is the system of wave equations. They are a linear system with constant coefficients. In conservation form the

state vector $U \in \mathbb{R}^{N+1}$ and the flux matrix $F(u) \in \mathbb{R}^{(N+1)\times N}$ are given by

$$\mathbf{U} = \begin{pmatrix} \phi \\ \mathbf{u} \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = c\,\mathbf{L}(\mathbf{U})$$

with

$$\mathbf{L}(\mathbf{U}) = \begin{pmatrix} \mathbf{u}^T \\ \phi\,\mathbf{I} \end{pmatrix} .$$

Here ϕ is the pressure density perturbation and $\mathbf{u} \in \mathbb{R}^N$ the velocity perturbation.

The common feature of these four examples is that $\mathbf{F}(\mathbf{U})$ always has the structure

$$\mathbf{F}(\mathbf{U}) = \sum_{i=1}^{k} \mathbf{A}_i \mathbf{a}_i^T + c\,\mathbf{L} \tag{2.8}$$

with

$$\sum_{i=1}^{k} \mathbf{A}_i = \mathbf{U} . \tag{2.9}$$

Up to now we have just rewritten the flux function $\mathbf{F}$ and the state vector $\mathbf{U}$. Observe that the representation of $\mathbf{F}(\mathbf{U})$ as given in (2.8) is obviously not unique.

Let us now explain what is meant by an **advection form**. Clearly the linear equation

$$\mathbf{U}_t + \nabla \cdot (\mathbf{U}\mathbf{a}^T) = 0 \tag{2.10}$$

describes the transport of the state vector $\mathbf{U}$ in direction of the vector $\mathbf{a}$. More generally one can introduce a superposition of several advections. To do this the state vector $\mathbf{U}$ is decomposed into different parts

$$\mathbf{U} = \sum_{i=1}^{k} \mathbf{A}_i \tag{2.11}$$

and each part is advected in the direction $\mathbf{a}_i$, i.e. the flux $\mathbf{F}(\mathbf{U})$ would be

$$\mathbf{F}(\mathbf{U}) = \sum_{i=1}^{k} \mathbf{A}_i \mathbf{a}_i^T . \tag{2.12}$$

Hence, with (2.11) and (2.12) the equation

$$\mathbf{U}_t + \nabla \cdot \mathbf{F}(\mathbf{U}) = 0 \tag{2.13}$$

can be written in the form

$$\sum_{i=1}^{k} ((\mathbf{A}_i)_t + \nabla \cdot (\mathbf{A}_i \mathbf{a}_i^T)) = 0 . \tag{2.14}$$

We shall say that (2.14) is the **advection form** of equation (2.13). Clearly the two equations (2.13) and (2.14) are identical. Only later when we shall use a numerical scheme, we shall replace at time t (2.13) by decomposing $\mathbf{U}(t)$ according to (2.11) and transport each part $\mathbf{A}_i(t)$ independently by the advection equation

$$(\mathbf{A}_i)_t + \nabla \cdot (\mathbf{A}_i \mathbf{a}^T) = 0 \,. \tag{2.15}$$

Because $\mathbf{F}(\mathbf{U})$ in (2.12) has only k transport directions and constant velocities one has

$$\mathbf{U}(t + \Delta t) = \sum_{i=1}^{k} \mathbf{A}_i(t + \Delta t) \,. \tag{2.16}$$

From now on we restrict ourselves to the Euler equations. Let $S = \{\mathbf{s} \in \mathbb{R}^N \Big| \, \|\mathbf{s}\|_2 = 1\}$ and $|S|$ denotes the area of S. $\mathbf{n}$ represents a unit vector. We decompose $\mathbf{U}$ as follows:

$$\mathbf{U} = \mathbf{R}_2 + \frac{1}{|S|} \int_S \mathbf{R}_1 \, ds + \frac{N}{|S|} \int_S \mathbf{L}\mathbf{n} \, ds \,. \tag{2.17}$$

$\mathbf{R}_1, \mathbf{R}_2$ are natural extensions of the right eigenvectors of the Jacobian of the flux function in one space dimension, see [1, 3, 5] and are given by

$$\mathbf{R}_1(\mathbf{U}) = \frac{1}{\gamma} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho H \end{pmatrix} \,, \quad \mathbf{R}_2(\mathbf{U}) = \frac{\gamma - 1}{\gamma} \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho \mathbf{u}^T \mathbf{u}/2 \end{pmatrix} \,. \tag{2.18}$$

Clearly

$$\frac{1}{|S|} \int_S ds = 1, \;\; \frac{1}{|S|} \int_S \mathbf{n} \, ds = 0, \;\; \frac{1}{|S|} \int_S \mathbf{n}\,\mathbf{n}^T \, ds = N \, \mathbf{I} \,. \tag{2.19}$$

Note, that if $N = 1$ one has to replace the integrals in an obvious fashion by a sum.

The decomposition in (2.17) is done according to different advections of the different terms. To each term the advection is described by the corresponding advection operator:

$$\phi_1(n) := \frac{\partial \mathbf{R}_1}{\partial t} + \nabla \cdot (\mathbf{R}_1(\mathbf{u}^T + c\,\mathbf{n}^T)), \;\; \text{for all} \;\; \mathbf{n}, \;\; \|\mathbf{n}\|_2 = 1 \tag{2.20a}$$

$$\phi_2 := \frac{\partial \mathbf{R}_2}{\partial t} + \nabla \cdot (\mathbf{R}_2(\mathbf{u}^T)) \tag{2.20b}$$

$$\phi_3(\mathbf{n}) := \frac{\partial (\mathbf{L}\mathbf{n})}{\partial t} + \nabla \cdot (\mathbf{L}\mathbf{n}(\mathbf{u}^T + c\,\mathbf{n}^T)), \;\; \text{for all} \;\; \mathbf{n}, \;\; \|\mathbf{n}\|_2 = 1 \,. \tag{2.20c}$$

We call these advection operators because they lead to advection equations if each by itself is set to 0, as one shall do it later. The full Euler equation can be written in what we call advection form

$$\phi_2 + \frac{1}{|S|} \int_S \phi_1(\mathbf{n}) \, ds + \frac{N}{|S|} \int_S \phi_3(\mathbf{n}) \, ds = 0 \,. \tag{2.21}$$

For a more detailed derivation, see [4, 5]. We call it advection form since one can see in the decomposition (2.17) of $\mathbf{U}$ the parts of $\mathbf{U}$ which are transported according to the advection operators by $\mathbf{u}$ and $\mathbf{u} + c\,\mathbf{n}$. Observe that there are infinitely many transport directions and that the advection form is not unique.

For the shallow water equations one can derive formulas which are basically of the same form. For the MHD equations the situation is more complicated. Moreover, for the numerical method of transport we can account only for finitely many transport directions and this can be done for the MHD equations also.

3 Advection form with finitely many directions

In order to be able to use the advection form (2.16), (2.20) one would have to compute the integral in (2.20) exactly. In Fey's first version of the transport scheme this was done [2, 3, 5]. It was possible to do it because in each computational cell, $\mathbf{U}$ was assumed to be constant. However the resulting scheme becomes computationally expensive. Hence one might replace the integrals in (2.16), (2.20) by quadrature formulas. This could be interpreted as integrating over the mach cone numerically. Since any such integration is affected by errors we might lose conservation. To avoid this Fey introduces an advection with a finite number of advection operators. The operators still add up to the full set of equations and hence conservation is guaranteed.

Again we restrict ourselves to the Euler equation. Instead of admitting infinitely many advection directions $\mathbf{u}, \mathbf{u} + c\,\mathbf{n}$, for all unit vectors $\mathbf{n}$ we could restrict ourselves to finitely many, say $\mathbf{u}$ and $\mathbf{u} + c\,\mathbf{n}_i$, $i = 1, 2, \ldots, k$ where $\|\mathbf{n}_i\| = 1$, see e.g. [4]. We make here as in [4] directly the more general Ansatz for approximating the acoustic waves

$$\mathbf{R}_{an}(\alpha, \boldsymbol{\nu}) = \mathbf{R}_1 + \alpha\,\mathbf{L}\boldsymbol{\nu},\ \alpha \in \mathbb{R},\ \boldsymbol{\nu} \in \mathbb{R}^N\ . \tag{3.1}$$

The index an stands for **n**umerical **a**coustic wave. We choose k directions $\boldsymbol{\nu}_i$ and corresponding weight/factors α_i and w_i such that

$$\mathbf{R}_1 = \sum_{i=1}^{k} w_i\,\mathbf{R}_{an}(\alpha_i, \boldsymbol{\nu}_i)\ . \tag{3.2}$$

Clearly we will need that

$$\sum_{i=1}^{k} w_i = 1 \tag{3.3}$$

and

$$\sum_{i=1}^{k} w_i \alpha_i\,\mathbf{L}\,\boldsymbol{\nu}_i = 0 \tag{3.4}$$

because it is a discrete version of

$$\frac{N}{|S|}\int \mathbf{Ln}\, ds = 0\ . \tag{3.5}$$

To each of these numerical acoustic waves belongs an advection operator

$$\phi_{an}(\alpha_i,\boldsymbol{\nu}_i) := \frac{\partial \mathbf{R}_{an}(\alpha_i,\boldsymbol{\nu}_i)}{\partial t} + \nabla\cdot(\mathbf{R}_{an}(\alpha_i,\boldsymbol{\nu}_i)(\mathbf{u}^T + c\,\boldsymbol{\nu}_i^T)),\ \ i=1,2,\ldots,k\ , \tag{3.6}$$

i.e. one thinks of $R_{an}(\alpha_i,\boldsymbol{\nu}_i)$ to be advected by $\mathbf{u}+c\,\boldsymbol{\nu}_i$. In order to derive the advection form of the Euler equations with finitely many directions we decompose $\mathbf{U}$ as follows

$$\mathbf{U} = \mathbf{R}_2 + \sum_{i=1}^{k} w_i\,\mathbf{R}_{an}(\alpha_i,\boldsymbol{\nu}_i) \tag{3.7}$$

and request that

$$\phi_2 + \sum_{i=1}^{k} w_i\,\phi_{an}(\alpha_i,\boldsymbol{\nu}_i) = \mathbf{U}_t + \nabla\cdot\mathbf{F}(\mathbf{U}) = 0 \tag{3.8}$$

where $\mathbf{F}$ is the Euler flux matrix defined in (2.2). This leads to the conditions

$$\sum_{i=1}^{k} w_i\,\alpha_i\,\boldsymbol{\nu}_i = 0\ , \tag{3.9}$$

$$\sum_{i=1}^{k} w_i\,\alpha_i\,\boldsymbol{\nu}_i\,\boldsymbol{\nu}_i^T = \mathbf{I} \tag{3.10}$$

and

$$\sum_{i=1}^{k} w_i\,\boldsymbol{\nu}_i = 0\ . \tag{3.11}$$

Observe that up to now the equations (3.8) are still the Euler equations. No discretisation has been made, it is only that we have written the equations in a form, which from a formal point of view, gives a preference to certain directions. Of course we shall exploit this picture in our numerical approximation in the next section. Up to now the choice of $\boldsymbol{\nu}_i, \alpha_i, w_i$ is free, especially it does not depend on any space discretization. Only the equations (3.3), (3.9), (3.10), (3.11) have to be satisfied.

Just for an illustration we give two choices:

i) $k = 2^N$, $w_i = 1/k$, $\alpha_i = N$, $\boldsymbol{\nu}_i = \mathbf{n}_i$, where $\mathbf{n}_i$ are the unit vectors in coordinate directions, e.g. if $N = 2$

$$\mathbf{n}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ,\ \ \mathbf{n}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\ ,\ \ \mathbf{n}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}\ ,\ \ \mathbf{n}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}\ .$$

ii) $k = 2^N$, $w_i = 1/k$, $\alpha_i = 1$, $\boldsymbol{\nu}_i = \tilde{\mathbf{n}}_i$, where $\tilde{\mathbf{n}}_i$ are the vectors whose components are either 1 or -1, e.g. if $N = 3$

$$\tilde{\mathbf{n}}_1 = -\tilde{\mathbf{n}}_8 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \tilde{\mathbf{n}}_2 = -\tilde{\mathbf{n}}_7 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix},\ \tilde{\mathbf{n}}_3 = -\tilde{\mathbf{n}}_6 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},$$

$$\tilde{\mathbf{n}}_4 = -\tilde{\mathbf{n}}_5 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

Note that in the one dimensional case the natural choice is $i,\ k = 1$ and $\mathbf{n}_i = (1)$, $\mathbf{n}_2 = (-1)$. One obtains three transport speeds $u,\ u + c,\ u - c$ and the corresponding numerical scheme of Fey will reduce to the scheme of Steger and Warming.

Equation (3.8) has formally the same structure as (2.14). If we now solve each advection operator independently as in (2.15) which will be discribed in the next section, the sum of the solution will no longer be equal to the exact solution, i.e. (2.17) will not hold. Note, that even for linear systems that do not obey the structure in (2.12), e.g. the two dimensional wave equations, we will also not get equality in (2.16). This is related to the infinitely many propagation directions.

4 First and second order time discretization

In Section 2 and 3 we have reformulated the system of conservation laws such that its form conveys infinitely many, or finitely many, transport operators. These transport operators are generally nonlinear since the velocities $\mathbf{u}$ and c depend on the state vector $\mathbf{U}$. One discretizes the equations in time as follows. At time t_0 the state vector $\mathbf{U}(\mathbf{x}, t_0)$ is decomposed according to (3.7), i.e.

$$\mathbf{U}(\mathbf{x}, t_0) = \mathbf{R}_2(\mathbf{U}(\mathbf{x}, t_0)) + \sum_{i=1}^{k} w_i\, \mathbf{R}_{an}(\mathbf{U}(\mathbf{x}, t_0),\ \alpha_i, \boldsymbol{\nu}_i)\ . \tag{4.1}$$

We transport each contribution of $\mathbf{U}$ with the corresponding transport equation where we have frozen the transport coefficients at time t_0, i.e.

$$\mathbf{a}_2(\mathbf{x}) := \mathbf{u}(\mathbf{U}(\mathbf{x}, t_0)) \tag{4.2}$$

$$\mathbf{a}_{an,i}(\mathbf{x}) := \mathbf{u}(\mathbf{U}(\mathbf{x}, t_0)) + c(\mathbf{U}(\mathbf{x}, t_0))\, \boldsymbol{\nu}_i,\ i = 1, 2, \ldots, k\ . \tag{4.3}$$

Hence the transported quantities $\mathbf{R}_2$ and $\mathbf{R}_{an}(\alpha_i, \boldsymbol{\nu}_i)$ will be approximated by

$$\mathbf{R}_2^{(1)}(\mathbf{x}, t) \ \text{ and } \ \mathbf{R}_{an,i}^{(1)}(\mathbf{x}, t) \tag{4.4}$$

and satisfy the following initial value problems

$$\phi_2 = \frac{\partial \mathbf{R}_2^{(1)}(\mathbf{x},t)}{\partial t} + \nabla \cdot (\mathbf{R}_2^{(1)}(\mathbf{x},t)\,\mathbf{a}_2^T(\mathbf{x})) = 0 \;\text{ for }\; t \geq t_0,\, \mathbf{x} \in \mathbb{R}^N \tag{4.5}$$

$$\mathbf{R}_2^{(1)}(\mathbf{x},t_0) = \mathbf{R}_2(\mathbf{U}(\mathbf{x},t_0)) \;\text{ for }\; \mathbf{x} \in \mathbb{R}^N\ . \tag{4.6}$$

Similarly one has for $i = 1, 2, \ldots, k$

$$\phi_{an}(\alpha_i, \boldsymbol{\nu}_i) = \frac{\partial \mathbf{R}_{an,i}^{(1)}(\mathbf{x},t)}{\partial t} + \nabla \cdot (\mathbf{R}_{an,i}^{(1)}(\mathbf{x},t)\,\mathbf{a}_{an,i}^T(\mathbf{x})) = 0 \;\text{ for }\; t \geq t_0,\; \mathbf{x} \in \mathbb{R}^N \tag{4.7}$$

$$\mathbf{R}_{an,i}^{(1)}(\mathbf{x},t_0) = \mathbf{R}_{an}(\mathbf{U}(\mathbf{x},t_0), \alpha_i, \boldsymbol{\nu}_i), \;\text{ for }\; \mathbf{x} \in \mathbb{R}^N\ . \tag{4.8}$$

We approximate then $\mathbf{U}(\mathbf{x}, t_0 + \Delta t)$ by

$$\mathbf{U}^{(1)}(\mathbf{x},t_0+\Delta t) := \mathbf{R}_2^{(1)}(\mathbf{x},t_0+\Delta t) + \sum_{i=1}^{k} w_i\, \mathbf{R}_{an,i}^{(1)}(\mathbf{x},t_0+\Delta t)\ . \tag{4.9}$$

If the solution is differentiable it can be shown, [2, 5] that

$$\mathbf{U}^{(1)}(\mathbf{x},t_0+\Delta t) - \mathbf{U}(\mathbf{x},t_0+\Delta t) = O(\Delta t^2)\ . \tag{4.10}$$

Independently of the numerical realization of (4.5), (4.6) and (4.7), (4.8), respectively, the overall numerical scheme will be of first order only. In order to obtain an approximation to the Euler equation which is of size $O(\Delta t^3)$, Fey showed that one has to replace in (4.1) and (4.2) $\mathbf{R}_{an,i}^{(1)}(\mathbf{x},t)$ by

$$\mathbf{R}_{an,i}^{(2)} = \mathbf{R}_{an}(\mathbf{U}(\mathbf{x},t),\, \alpha_i, \boldsymbol{\nu}_i) + \alpha_i\, \mathbf{K}\, \boldsymbol{\nu}_i\ .$$

Here $\mathbf{K}$ is a $\mathbb{R}^{(N+2)\times N}$ matrix with correction terms which depend on $\mathbf{U}(\mathbf{x},t)$, its derivatives, $\Delta t, \alpha_i$ and on $\boldsymbol{\nu}_i$, $i = 1, 2, \ldots, k$. If this replacement is done in (4.1), (4.7), (4.8) one then uses

$$\mathbf{U}^{(2)}(\mathbf{x},t_0+\Delta t) = \mathbf{R}_2^{(1)}(\mathbf{x},t_0+\Delta t) + \sum_{i=1}^{k} w_i\, \mathbf{R}_{an,i}^{(2)}(\mathbf{x},t+\Delta t)\ . \tag{4.11}$$

instead of (4.9) and obtains

$$\mathbf{U}^{(2)}(\mathbf{x},t_0+\Delta t) - \mathbf{U}(\mathbf{x},t_0+\Delta t) = O(\Delta t^3)\ . \tag{4.12}$$

For Euler equations these correction terms are given in [6] for $N = 2$. For the shallow water equation these correction terms are given in [8], $N = 2$. For the MHD equations the correction terms are given in [7] for $N = 1$.

The important part in these correction terms is that the modification of (4.7) is an advection equation and thus one can use the same numerical method to solve the modified transport initial value problem as one used before. Conceptually one can extend this approach to even higher order approximations. In the case of the wave equation these terms are relatively simple but not zero as mentioned before. Thus, we computed them to fourth order to get a numerical method of $O(\Delta t^5)$. However, for non-linear problems the computational expenses will drastically rise. We are able to compute the third order correction terms for the Euler equations for $N = 2$.

5 The space discretisation and a first order overall scheme

In Section 4 we have approximated $\mathbf{U}(\mathbf{x}, t_0+\Delta t)$ by a first order approximation $\mathbf{U}^{(1)}(\mathbf{x}, t_0+\Delta t)$ and a second order approximation $\mathbf{U}^{(2)}(\mathbf{x}, t_0+\Delta t)$ under the assumption that at $t = t_0$ the approximated solutions are exactly equal to $\mathbf{U}(\mathbf{x}, t_0)$ for all $\mathbf{x}$. Clearly, it is enough to have a scheme which solves the scalar equation

$$u_t + \nabla \cdot (u\,\mathbf{a}^T) = 0 \tag{5.1}$$

exactly where $\mathbf{a}$ is the local advection velocity which is a function depending on the space variable only. This would give an overall first or second order scheme depending whether we use (4.9) or (4.11). Hence we don't have to solve (5.1) exactly, it is enough to solve it in an accuracy not to destroy the overall accuracy. Note, that the correction terms introduced in the previous section only eliminates the nonlinear decomposition error in the time integration. To get second order accuracy in space and time for fixed ratio of $\Delta t/\Delta x$, the approximate solution of (5.1) has to be of second order in time, too.

Schemes to solve (5.1) have been around for a long time, [10, 11, 12]. Here we follow the approach [13]. We shall now discretize the space domain $D \subset \mathbb{R}^N$. However in order that we don't have to deal with boundary conditions let us assume $D = \mathbb{R}^N$. D is divided into computational cells similar as in finite element or finite volume methods. Again for simplicity we restrict ourselves to $N = 2$ and to a cartesian grid with the step size Δx and Δy in x and y direction, respectively.

Let $x_i = i\Delta x$ and $y_j = j\Delta y$ and (x_i, y_j) is the center of the finite volume $V_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$. Let

$$u_{ij}^n = \frac{1}{|V_{ij}|} \int_{V_{ij}} u(x, y, n\Delta t)\, dxdy \tag{5.2}$$

be the average value of u over the cell V_{ij}. We assume now that in each control volume the solution is constant and has the value u_{ij}^n. If we assume that $\boldsymbol{a}(x, y)$ is also constant in this volume, e.g. has the value $\boldsymbol{a}(x_i, y_j)$, and we ignore effects from neighboring cell then we can say that the quantity u in V_{ij} is transported by $\Delta t\, \boldsymbol{a}(x_i, y_j)$, see Figure 1

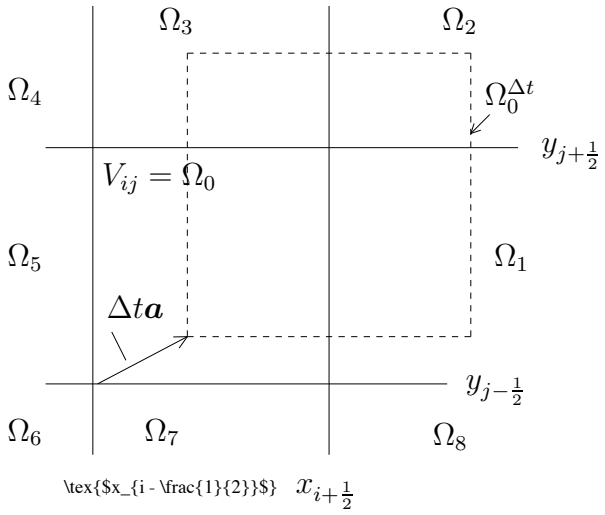

Figure 1: Movement of all points by $\Delta t\, \boldsymbol{a}(x_i, y_j)$

Let

$$\begin{aligned} \Omega_0^{\Delta t} &= \Omega_0 + \Delta t\, \boldsymbol{a}(x_i, y_j) \\ &= \left\{ (x,y) \in \mathbb{R}^2 \,\middle|\, (x,y) - \Delta t\, \boldsymbol{a}(x_i, y_j) \in \Omega_0 \right\}. \end{aligned} \tag{5.3}$$

Hence, the contribution of cell Ω_0 to cell Ω_j is

$$\begin{aligned} f_{\Omega_0\Omega_j} &= u_{ij}^n \int_{\Omega_0^{\Delta t} \cap \Omega_j} dxdy \\ &= u_j^n \left| \Omega_0^{\Delta t} \cap \Omega_j \right|. \end{aligned} \tag{5.4}$$

The notation $f_{\Omega_0\Omega_j}$ is still related to the idea of flux in hyperbolic equations. Note that $f_{\Omega_0\Omega_j}$ is in general different from zero even if there is no finite boundary between Ω_0 and Ω_j, i.e. contributions to diagonal neighbors.

Collecting all these fluxes gives the final formula

$$\begin{aligned} u_{ij}^{n+1} &= u_{ij}^n - \frac{1}{|V_{ij}|} \sum_{j=1}^{8} (f_{\Omega_0\Omega_j} - f_{\Omega_j\Omega_0}) \\ &= \frac{1}{|V_{ij}|} \sum_{j=0}^{8} f_{\Omega_0\Omega_j}. \end{aligned} \tag{5.5}$$

Here Ω_0 denotes the control volume V_{ij} and Ω_j are the eight neighboring cells. Moreover we have assumed that Δt is restricted such that $\Omega_0^{\Delta t} \subset \bigcup_{j=0}^{8} \Omega_j$. Clearly, replacing u locally in space by a piecewise constant leads to a first order scheme.

At this point we should note that the numerical domain of influence in the case ii) of choosing the $\boldsymbol{\nu}_i$ in Section 3 includes the exact domain of influence. However the choice i) does not do this, see [8].

6 A second order overall scheme

To get an overall second order scheme we have to approximate $\mathbf{U}$ by piecewise linear functions. Again it is enough to do this just for the scalar equation (5.1). To obtain a second order overall scheme we have to do two steps.

I) **Transport**:

Solve (5.1) to second order if $u(x, y, t_0)$ and $\mathbf{a}(x, y)$ are piecewise linear. This includes computing fluxes of the form $f_{\Omega_0\Omega_j}$ and update the cell averages i.e. evaluate (5.5).

II) **Reconstruction**:

From the averages find a piecewise linear function that has the same cell average and which would approximate a smooth function to terms of second order, e.g. $O(\Delta x^3)$ if one assumes $\Delta x/\Delta y =$ constant.

Let us concentrate first on the **transport part**. Again we restrict ourselves to $N = 2$ but we use $\mathbf{x} = (x, y)$. Hence, (5.1) is

$$u_t(\mathbf{x}, t) + \nabla \cdot (u(\mathbf{x}, t)\, \mathbf{a}^T(\mathbf{x})) = 0 \; . \tag{6.1}$$

We rewrite this as

$$u_t + \mathbf{a}^T \nabla u = -u \nabla \cdot \mathbf{a}^T \; . \tag{6.2}$$

Let $\mathbf{z}(t)$ be the characteristic curve belonging to (6.2) through the point $\mathbf{x}_0$ i.e.

$$\dot{\mathbf{z}}(t) = \mathbf{a}(\mathbf{z}(t)), \; t \geq t_0 \;\; \text{with} \;\; \mathbf{z}(t_0) = \mathbf{x}_0 \; . \tag{6.3}$$

Then $u(\mathbf{z}(t), t)$ satisfies

$$\begin{aligned} \frac{d}{dt}\, u(\mathbf{z}(t), t) &= -u(\mathbf{z}(t), t)\, \nabla \cdot \mathbf{a}^T(z(t)) \;\; \text{for} \;\; t \geq t_0 \\ u(\mathbf{z}(t_0), t_0)) &= u(\mathbf{x}_0, t_0) \; . \end{aligned} \tag{6.4}$$

Observe first that even so $\mathbf{a}(\mathbf{x})$ is linear in a cell Ω_0, the resulting characteristics $\mathbf{z}(t)$ are not linear. However as we shall see further down for a fixed Δt the map $\mathbf{r}(\mathbf{x}_0)$ given by $\mathbf{x}_0 \longmapsto \mathbf{r}(\mathbf{x}_0) := \mathbf{z}(t_0 + \Delta t)$ is an affine map defined in Ω_0, where $\mathbf{z}(t_0 + \Delta t)$ is the solution of (6.3). The set Ω_0 is mapped into a general parallelogram, see Figure 2.

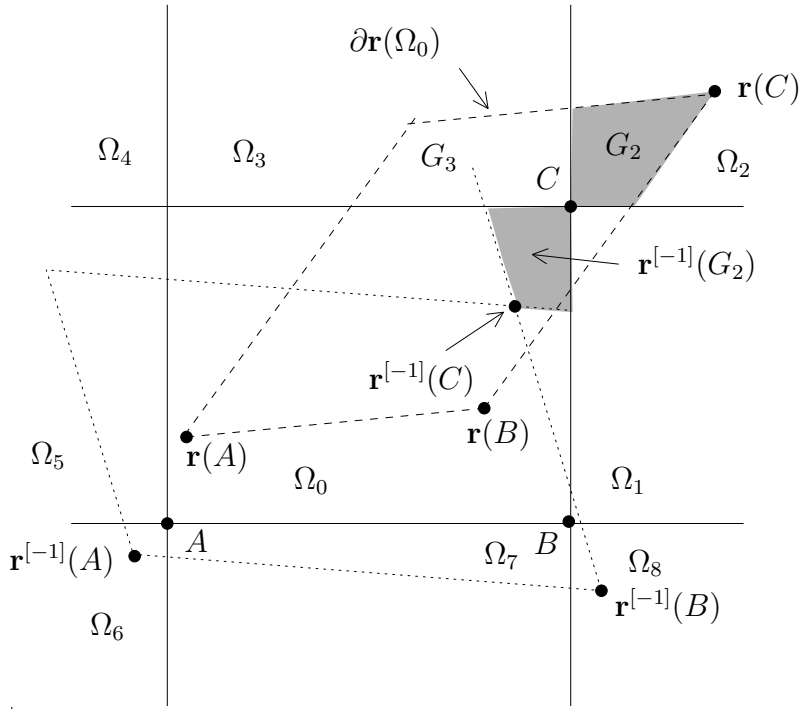

Fig. 2: Sketch of transformation $\mathbf{r}(\mathbf{x}_0)$ and its inverse. Solid lines represent the underlying grid. The dashed lines represent the boundary of the image of Ω_0 under the map $\mathbf{r}$. The dotted lines represent the underlying grid under the inverse transformation of $\mathbf{r}$.

The flux $f_{\Omega_0\Omega_j}$ from Ω_0 into Ω_j in the time interval $[t_0, t_0 + \Delta t]$ is given by

$$f_{\Omega_0\Omega_j} = \int_{\mathbf{r}(\Omega_0)\cap\Omega_j} u(\mathbf{z}(t_0 + \Delta t),\, t_0 + \Delta t)\, d\mathbf{x}\ . \tag{6.5}$$

This integral is difficult to evaluate since for each $\mathbf{x} \in \mathbf{r}(\Omega_0) \cap \Omega_j$ we have to compute $\mathbf{x}_0 = \mathbf{r}^{[-1]}(\mathbf{x})$ and then solve the initial value problem (6.4) to obtain $u(\mathbf{z}(t_0+\Delta t), t_0+\Delta t)$. We shall show further down that $\mathbf{r}$ is bijective on Ω_0 and hence $\mathbf{r}^{[-1]}$ exist on $\mathbf{r}(\Omega_0)$. To avoid the complicated integral (6.5) we map the integration variable back by $\mathbf{r}^{[-1]}$. Let $G_j = \mathbf{r}(\Omega_0) \cap \Omega_j$ then we obtain

$$f_{\Omega_0\Omega_j} = \int_{\mathbf{r}^{[-1]}(G_j)} u(\mathbf{x}, t_0)\, d\mathbf{x}\ , \tag{6.6}$$

see e.g. [9]. This integral can easily and exactly be computed since $\mathbf{r}^{[-1]}(G_j) \subset \Omega_0$ and $u(\mathbf{x}, t_0)$ is linear in Ω_0. It remains to show that $\mathbf{r}(\mathbf{x})$ is an affine map which can be inverted. Since we assume $\mathbf{a}(\mathbf{x})$ to be a linear reconstruction in all Ω_0 we can write

$$\mathbf{a}(\mathbf{z}) = \mathbf{a}_0 + \mathbf{A}\mathbf{z} \tag{6.7}$$

where $\mathbf{A}$ is a constant 2×2 matrix. Hence the right hand side of (6.7) satisfies a Lipschitz condition and then the solution of the corresponding initial value problem (6.3) is unique.

hence the map $\mathbf{r}(\mathbf{x}_0)$ is bijective and affine and can be inverted, for details see [9]. $\mathbf{r}^{[-1]}$ is only computed to an accuracy to guarantee that the overall scheme is second order. To obtain $\mathbf{r}^{[-1]}(G_j)$ for $j = 0, 1, \ldots, 8$ we just compute $\mathbf{r}^{[-1]}$ for the grid points A, B, C. With this we get easily the inverse map of the coordinate lines and these define then the regions $\mathbf{r}^{[-1]}(G_j)$, see Figure 2. With these regions we can compute the fluxes and obtain the new cell averages u_{ij}^{n+1} using formula (5.5). Note that the integration of the characteristic takes care of the time accuracy. Thus, the method is second order in time, too, even if the time integration in (5.5) formally looks like a simple Euler step.

II. Reconstruction. Given the cell averages u_{ij}^n we have to replace $\mathbf{U}$ and therefore u in each cell by a linear function in order that the overall scheme becomes second order. The crucial point is that we have to get approximations to the first partial derivatives. Following the by now classical ENO upwind idea for nonlinear hyperbolic conservation laws one is lead to the following problem. While for problems in one space dimension there are only three different stencils to second order approximations to the first derivative the number of stencils increases dramatically in two and three space dimensions. For an early analysis of these possibilities, see [14]. Following the classical approach one introduces a limiter which turns off the second order terms if one is close to a discontinuity or a critical point. The second strategy is to just use everywhere central differences. The limiting process is then based on the non-oscillating character of the first order solution. Hence one calculates first in the whole computational domain the first order solution $(u_{ij}^{n+1})^1$ and the second order solution $(u_{ij}^{n+1})^2$. One checks now in each cell $V_{ij} = \Omega_0$, whether the following holds:

$$\min_{j \in V} (u_j^{n+1})^1 \leq (u_j^{n+1})^2 \leq \max_{j \in V} (u_j^{n+j})^1 \; . \tag{6.8}$$

Here u_j^{n+1} is the value in the cell Ω_j and $V = \{0, 1, 2, \ldots, 8\}$. If (6.8) is satisfied then the second order solution is used, otherwise the first order is used. To keep the method conservative the fluxes have to be adapted, for details see [9].

7 Conclusions and outlook

We have presented the method of transport of Fey which is of first and second order. The scheme is genuinely multidimensional in the sense that the transport is done grid independent. However the usual limiting process of the state variable does depend on the grid. The first order is basically non-oscillating and for the second order a new limiter is used which lifts this property to the second order solution. A basic underlying philosophy in the construction of the method is that one always tries to use properties of the solution itself rather than artificially introduced or derived quantities. For example, using the characteristic directions instead of cell normals, similarly the limiter is not based on slopes but on state variables.

This concept makes the algorithm robust in the sense that physically positive values remain positive even in areas where these are close to zero. The scheme is explicit and can be easily used for parallel computers. Since hanging nodes can be treated in a simple fashion one can use mesh adaptation and this is currently done by T. Gutzmer of our

group. The approach can also be used to deal with the viscosity terms in the Navier-Stokes equations. This task is currently done by J. Maurer of our group. H. Forrer is investigating the treatment of non cartesian boundary cells. The problem is that small cells introduced by the geometry should not restrict the stepsize nor the second order of the scheme. Numerical examples for the Euler equations in gas dynamics, the shallow-water equations and the MHD equations will be given.

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