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Error Estimates for Reconstruction using Thin Plate Spline Interpolants

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Abstract

Using thin plate spline interpolants we derive error bounds for the reconstruction of point values where the interpolation data is given by cell values. Sonar investigates in [IMA J. of Num. Analysis **16**, 1996, 549-581] numerical solutions of hyperbolic conservation laws where the local reconstruction step is performed using thin plate spline interpolants. We show that local reconstruction yields second order as in the global case but without assuming additional boundary conditions and smoothness requirements.

Keywords: Thin Plate Spline Reconstruction, Conservation Laws

AMS Subject Classification: 65D07, 65D15, 76M25, 76N10

1 Introduction

In the interpolation of multivariate scattered data, radial basis functions are by now a well established tool; we refer the reader to [6], [8], [9] for a general survey on this subject . In [12] and [13] Sonar applies radial basis functions for reconstruction algorithms in finite volume methods for the numerical solution of hyperbolic conservation laws. In reconstruction algorithms point values of the unknown solution have to be recovered from cell average data. Polynomial recovery is shown to be optimal only in trivial cases, while recovery with radial basis function interpolants is optimal on a native space assigned to the radial basis function, cf. [13], [5] and [12].

Further in [12] and [13] numerical examples for the solution of the linear advection equation and the Euler equations using radial basis function interpolants are given.

In the present paper we will focus on the approximation order of thin plate spline interpolants in the reconstruction setting. Thin plate spline interpolants are optimal in the Beppo Levi space

$$\mathrm{BL}^2(\mathbf{R}^2) = \left\{ f \in \mathcal{C}(\mathbf{R}^2),\ \frac{\partial^\alpha}{\partial x^\alpha} f \in L^2(\mathbf{R}^2) \text{ for } |\alpha| = 2 \right\}$$

equipped with the inner product

$$< f, g >_{\mathrm{BL}^2} = \int_{\mathbf{R}^2} \frac{\partial^2 f(x)}{\partial \xi^2}\frac{\partial^2 g(x)}{\partial \xi^2} + 2\frac{\partial^2 f(x)}{\partial \xi \partial \eta}\frac{\partial^2 g(x)}{\partial \xi \partial \eta} + \frac{\partial^2 f(x)}{\partial \eta^2}\frac{\partial^2 g(x)}{\partial \eta^2} dx,$$

where $x = (\xi, \eta)^T$. The inner product induces a semi norm on $\mathrm{BL}^2(\mathbf{R}^2)$ via $\|g\|^2 =< g, g >_{\mathrm{BL}^2}$ that represents the bending energy of an infinitely extended plate.

We investigate the recovery problem that reads as follows. Let $T_i \in \mathcal{C}'(\mathbf{R}^2),\ i = 1, \ldots, N$ be a set of $N \geq 3$ functionals. Here $\mathcal{C}'(\mathbf{R}^2)$ denotes the topological dual of linear space $\mathcal{C}(\mathbf{R}^2)$ that means the linear space of all distributions of order 0 with compact support on $\mathbf{R}^2$, cf. [11]. Note that the definition of the functionals is general to simplify some statements. In the main theorems we will restrict to the recovery of point values of a function from given cell average data.

Assume that these functionals $T_i,\ i = 1, \ldots, N$ are unisolvent on the linear space of bivariate polynomials

$$\mathcal{P}_1^2 = \{x^\alpha,\ |\alpha| \leq 1\},$$

i.e. $T_i p = 0$ implies $p = 0$ for all $p \in \mathcal{P}_1^2$.

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be a continuous function. Then the thin plate spline interpolant s subject to the interpolation conditions $T_i s(x) = T_i f(x),\ i = 1, \ldots, N$, has the form

$$s(x) = \sum_{i=1}^{N} \lambda_i T_i^y \phi(\|x - y\|_2) + \lambda_{N+1} + \lambda_{N+2}\xi + \lambda_{N+3}\eta \tag{1}$$

where $\phi : \mathbf{R}^{\geq 0} \to \mathbf{R}$ is the function

$$\phi(r) = r^2 \log(r),\ r \geq 0,$$

and $T_i^y \phi(\|x - y\|_2)$ represents the action of T_i on $\phi(\|x - y\|_2)$ regarded as a function of y. Moreover, the parameters λ_i, $i = 1, \ldots, N$, satisfy the additional constraints

$$\sum_{i=1}^{N} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{N} \lambda_i T_i x = 0,$$

i.e. the interpolation scheme reproduces linear polynomials. The problem is well posed, cf. [5], and following [12] the interpolant $s(x)$ is an abstract spline in $\mathrm{BL}^2(\mathbf{R}^2)$, therefore it is the solution of the variational problem

$$s = \min\{\|v\| \mid v \in \mathrm{BL}^2,\ T_i v = T_i f,\ i = 1, \ldots, N\}.$$

In the second section we will generalize the results of [7] to situations where the interpolation data are given by cell values on a cartesian grid instead of scattered point values. We prove the interpolation scheme to be first order accurate in this case and, using the ideas of [10], we show that even second order accuracy can be achieved under additional regularity assumptions. To be more precise, we consider a fixed compact set and assume the functions to be generalized differentiable up to the fourth order and to take homogenous boundary conditions in the second and third derivatives.

In finite volume methods for the solution of conservation laws, cf. [5], [12], one is interested in accurate *local* approximation schemes. Therefore the third section is concerned with local approximation. We show that the error improves to second order $\mathcal{O}(h^2)$ accuracy without additional boundary conditions or smoothness requirements on the function to be reconstructed. In particular this means that the finite volume method used in [5] is of second order accuracy in space.

2 Global Approximation

Lemma 2.1. *Let $T_i \in \mathcal{C}'(\mathbf{R}^2),\ i = 0, \ldots, \hat{N}$, be a set of $\hat{N}+1 \geq 3$ functionals with compact support and unisolvent on $\mathcal{P}_1^2$.*

If

$$\sum_{i=0}^{\hat{N}} \hat{\alpha}_i = 0 \quad \text{and} \quad \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i p = 0 \ \text{ for all } \ p \in \mathcal{P}_1^2,$$

then the functional $\hat{L} = \sum_{i=1}^{\hat{N}} \hat{\alpha}_i T_i$ can be bounded above by

$$|\hat{L}g| \leq \left[8\pi \|g\|^2 \sum_{i,j=0}^{\hat{N}} \hat{\alpha}_i \hat{\alpha}_j T_i^x T_j^y \phi(\|x-y\|_2) \right]^{1/2}, \tag{2}$$

for any $g \in \mathrm{BL}^2(\mathbf{R}^2)$ and $\phi(r) = r^2 \log(r),\ r \geq 0$.

Proof: The space BL^2 is equipped with a reproducing kernel $K(x, y)$. According to [12] we can write $K(x, y)$ in the form

$$\begin{aligned} K(x,y) &= \frac{1}{8\pi} \Bigg(\phi(\|x-y\|_2) - \sum_{k=0}^{2} p_k(x) T_k^x \phi(\|x-y\|_2) - \sum_{k=0}^{2} p_k(y) T_k^y \phi(\|x-y\|_2) \\ &\quad + \sum_{j,k=0}^{2} p_j(x) p_k(y) T_j^x T_k^y \phi(\|x-y\|_2) \Bigg), \end{aligned}$$

if $\{T_0, T_1, T_2\}$ is a unisolvent subset of $\{T_i,\ i = 0, \ldots, \hat{N}\}$ and $\{p_0, p_1, p_2\}$ a basis of $\mathcal{P}_1^2$ with $T_i p_k = \delta_i^k,\ i, k = 0, 1, 2$, where

$$\delta_i^k = \begin{cases} 1 & , \quad i = k, \\ 0 & , \quad i \neq k. \end{cases}$$

If we define $g_i(x) = T_i^y K(x,y)$, $i = 1,\dots,\hat{N}$, we obtain $T_i g =< g, g_i >_{\mathrm{BL}^2}$, $i = 1,\dots,\hat{N}$ and

$$|\hat{L}g| = |\sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i g| = |\sum_{i=0}^{\hat{N}} \hat{\alpha}_i < g, g_i >_{\mathrm{BL}^2} | \leq \|g\| \| \sum_{i=0}^{\hat{N}} \hat{\alpha}_i g_i \|. \tag{3}$$

In order to prove (2) it remains to show

$$\Big\| \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i^y \phi(\|x-y\|_2) \Big\|^2 = 8\pi \sum_{i,j=0}^{\hat{N}} \hat{\alpha}_i \hat{\alpha}_j T_i^x T_j^y \phi(\|x-y\|_2), \tag{4}$$

since the polynomials occuring in the kernel K are contained in the kernel of the semi norm on BL^2.

Following the lines of the proof of Lemma 2.1 in [7] the function

$$\hat{t}(x) = \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i^y \phi(\|x-y\|_2)$$

has the decay

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} \hat{t}(x) = \mathcal{O}(\|x\|_2^{1-|\alpha|}), \quad \|x\|_2 \to \infty,$$

for $|\alpha| < 4$.

This permits integration by parts, and we conclude with a regularization argument as in [4], Theorems 1.3.2 and 4.1.5

$$\begin{aligned}
\|\hat{t}(x)\|^2 &= \lim_{M\to\infty} \left[\int_{\xi=-M}^{M} \int_{\eta=-M}^{M} (\frac{\partial^2 \hat{t}(x)}{\partial \xi^2})^2 + 2(\frac{\partial^2 \hat{t}(x)}{\partial \xi \partial \eta})^2 + (\frac{\partial^2 \hat{t}(x)}{\partial \eta^2})^2 dx \right] \\
&= \lim_{M\to\infty} \left[\int_{\xi=-M}^{M} \int_{\eta=-M}^{M} \Delta^2 \hat{t}(x) \hat{t}(x) dx + \mathcal{O}(M^{-1}) \right] \\
&= \int_{\mathbf{R}^2} \sum_{i=0}^{\hat{N}} \hat{\alpha}_i \underbrace{T_i * \underbrace{\Delta^2 \phi}_{8\pi\delta}}_{8\pi T_i} \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i * \phi \, dx \\
&= 8\pi \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i (\sum_{j=0}^{\hat{N}} \hat{\alpha}_j T_j * \phi) \\
&= 8\pi \sum_{i,j=0}^{\hat{N}} \hat{\alpha}_i \hat{\alpha}_j T_i^x T_j^y \phi(\|x-y\|_2).
\end{aligned}$$

Here Δ^2 is the biharmonic operator, and we used that $\frac{1}{8\pi}\phi$ is the fundamental solution of this operator, cf. [11].

Lemma 2.1 enables us to estimate at a given point $\tilde{x} \in \mathbf{R}^2$ if the interpolation data are given by cell averages.

Corollary 2.2. *Let*

$$R_i = [v_i^1, v_i^1 + h] \times [v_i^2, v_i^2 + h], \quad v_i = \begin{pmatrix} v_i^1 \\ v_i^2 \end{pmatrix} \in R^2, \quad i = 1, \ldots, \hat{N}$$

be the cells assigned to the functionals T_i, $i = 1, \ldots, \hat{N}$ *defined by*

$$T_i = I_{R_i} \text{ with } I_{R_i}\varphi = \frac{1}{h^2} \int\limits_{R_i} \varphi(x)\, dx, \quad \varphi \in \mathcal{C}(\mathbf{R}^2), \quad i = 1, \ldots, \hat{N}.$$

Let $T_0 = \delta_{\tilde{x}}$ *be the point evaluation in* $\tilde{x}$ *and let* $\hat{\alpha}_i$, $i = 0, \ldots, \hat{N}$, *be given by*

$$\begin{aligned} &\hat{\alpha}_0 = -1, \\ &\hat{\alpha}_i = \alpha_i, \ i = 1, \ldots, \hat{N}, \quad \sum_{i=1}^{\hat{N}} \alpha_i = 1, \end{aligned} \tag{5}$$

such that

$$\tilde{x} - \begin{pmatrix} h/2 \\ h/2 \end{pmatrix} = \sum_{i=1}^{\hat{N}} \alpha_i v_i.$$

Then we obtain

$$|f(\tilde{x}) - s(\tilde{x})| \le \left[8\pi \|f\|^2 \Psi(\alpha)\right]^{1/2}$$

for all $f \in \mathrm{BL}^2$ *where* Ψ *is given by*

$$\Psi(\alpha) = \sum_{i,j=1}^{\hat{N}} \alpha_i \alpha_j I_{R_i}^x I_{R_j}^y \phi(\|x - y\|_2) - 2 \sum_{i=1}^{\hat{N}} \alpha_i I_{R_i}^y \phi(\|\tilde{x} - y\|_2),$$

and s *denotes the thin plate spline interpolant with respect to the data* $I_{R_i} f = I_{R_i} s$, $i = 1, \ldots, \hat{N}$.

Proof: Let $g = f - s$, $\hat{L}g = \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i g = s(\tilde{x}) - f(\tilde{x})$. By Lemma 2.1 we conclude

$$|f(\tilde{x}) - s(\tilde{x})| \le \left[8\pi \|g\|^2 \Psi(\alpha)\right]^{1/2}. \tag{6}$$

Using

$$\begin{aligned} \sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= -x + \sum_{i=1}^{\hat{N}} \frac{\hat{\alpha}_i}{h^2} \int\limits_{R_i} \begin{pmatrix} \xi \\ \eta \end{pmatrix} d\xi d\eta \\ &= -x + \sum_{i=1}^{\hat{N}} \frac{\hat{\alpha}_i}{2h} \begin{pmatrix} (v_i^1 + h)^2 - (v_i^1)^2 \\ (v_i^2 + h)^2 - (v_i^2)^2 \end{pmatrix} \\ &= -x + \sum_{i=1}^{\hat{N}} \hat{\alpha}_i v_i + \begin{pmatrix} h/2 \\ h/2 \end{pmatrix} \\ &= 0 \end{aligned}$$

and the fact that s is the solution to the variational problem

$$\min\{\|v\| \mid v \in \mathrm{BL}^2,\ T_i v = T_i f,\ i = 1, \ldots, \hat{N}\},$$

we get

$$\|g\|^2 = \|f - s\|^2 =< f - s, f - s >=< f, f > -2\underbrace{< s, f - s >}_{=0} - < s, s >\leq \|f\|^2.$$

This completes the proof.

The estimate in Lemma 2.2 is done in terms of a quadratic form whose argument α is connected to the original reconstruction problem in a weak way, namely, by the constraint (5). In the following theorems we will make use of the freedom to choose the coefficients α judiciously.

Remark: The generalization of Corollary 2.2 to data with unstructured geometries is much more complicated, since this would require to prove

$$\sum_{i=0}^{\hat{N}} \hat{\alpha}_i T_i x = 0.$$

In order to estimate the quadratic form $\Psi(\alpha)$, let us consider the special case where the interpolation data contain two rectangles R_i and R_j with a common edge that is divided by $\tilde{x}$.

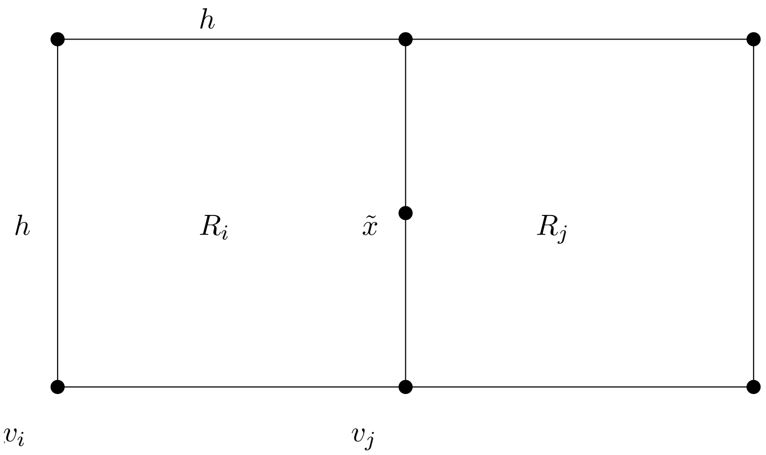

Figure 1

This means $v_i + \begin{pmatrix} h \\ 0 \end{pmatrix} = v_j$, and if we set $\alpha_i = 1/2 = \alpha_j$, we can rewrite Ψ :

$$
\begin{aligned}
\Psi(\alpha) &= \frac{1}{2} I^x_{R_i} I^y_{R_j} \phi(\|x-y\|_2) + \frac{1}{4} I^x_{R_i} I^y_{R_i} \phi(\|x-y\|_2) + \frac{1}{4} I^x_{R_j} I^y_{R_j} \phi(\|x-y\|_2) \\
&\quad - I^y_{R_i} \phi(\|\tilde{x}-y\|_2) - I^y_{R_j} \phi(\|\tilde{x}-y\|_2) \\
&= \frac{1}{2} \left[I^x_{R_i} I^y_{R_j} \phi(\|x-y\|_2) + I^x_{R_i} I^y_{R_i} \phi(\|x-y\|_2) \right] - 2 I^y_{R_i} \phi(\|\tilde{x}-y\|_2) \\
&= \frac{1}{2h^4} \int\limits_{R_i} \int\limits_{R_i \cup R_j} \|x-y\|_2^2 \log \|x-y\|_2 dy dx - \frac{1}{h^2} \int\limits_{R_i \cup R_j} \|\tilde{x}-y\|_2^2 \log \|\tilde{x}-y\|_2 dy \\
&= \frac{h^2 \log h}{2} \int_0^1 \int_0^1 \int_0^1 \int_0^2 \left((x_1-y_1)^2 + (x_2-y_2)^2 \right) dy_1 dy_2 dx_1 dx_2 \\
&\quad + \underbrace{\frac{h^2}{2} \int_0^1 \int_0^1 \int_0^1 \int_0^2 \left([(x_1-y_1)^2 + (x_2-y_2)^2] \log[(x_1-y_1)^2 + (x_2-y_2)^2]^{1/2} \right) dy_1 dy_2 dx_1 dx_2}_{=c_1} \\
&\qquad - h^2 \log h \int_0^1 \int_0^2 \left((\frac{1}{2}-y_1)^2 + (1-y_2)^2 \right) dy_2 dy_1 \\
&\qquad \underbrace{- h^2 \int_0^1 \int_0^2 \left([(\frac{1}{2}-y_1)^2 + (1-y_2)^2] \log[(\frac{1}{2}-y_1)^2 + (1-y_2)^2]^{1/2} \right) dy_2 dy_1}_{=c_2} \\
&= h^2 \log h \underbrace{[\frac{1}{2}\frac{10}{6} - (\frac{2}{3} + \frac{1}{6})]}_{=0} + h^2 \underbrace{[\frac{c_1}{2} - c_2]}_{=c \geq 0}.
\end{aligned}
$$

We obtain

Theorem 2.3. *Let the given interpolation cells lie on a cartesian grid with mesh size h as depicted in Figure 1, and let x be the midpoint of an interior cell edge.*

Then we obtain first order accuracy of the thin plate spline interpolation scheme,

$$|f(x) - s(x)| \leq C \|f\| h,$$

where $C = (8\pi c)^{-1/2}$ and $f \in \mathrm{BL}^2$, and s denotes the thin plate spline interpolant to f.

Remark: The statement of the above Theorem is still valid if we replace the quadratic grid by a rectangular grid. In this case the side length h has to be replaced by $\sqrt{h_1 h_2}$.

A generalization of Theorem 2.3 is given by

Theorem 2.4. *Let us assume that the interpolation cells are given as in the previous theorem. Then we get for the reconstruction points $x \in K$, $K \subset \mathbf{R}^2$ compact*

$$|f(x) - s(x)| \leq C_K \|f\| h,$$

for all $f \in \mathrm{BL}^2$, and s denotes the thin plate spline interpolant to f. Here $C_K \geq 0$ depends only on the compact set K.

Proof: As in the proof of Theorem 2.3, we prove an upper bound for the quadratic form Ψ. Let v_0, v_1, v_2 be three unisolvent corners of the grid, such that the rectangles $R_i = [v_i^1, v_i^1+h] \times [v_i^2, v_i^2+h]$, $i = 0,1,2$ belong to the interpolation data, and such that the evaluation point $\tilde{x}$ is a point in the convex hull of v_0, v_1, v_2. We assume $v_0 = 0$ without loss of generality. We choose the parameters α in the way

$$\tilde{x} - \begin{pmatrix} h/2 \\ h/2 \end{pmatrix} = \sum_{i=0}^{2} \alpha_i v_i, \quad \sum_{i=0}^{2} \alpha_i = 1, \quad i = 0,1,2.$$

Note that $\alpha_i,\ i = 0,1,2,$ and $v_i,\ i = 1,2,$ may depend on h, but their values are bounded independently of h.

Thus, we get

$$\begin{aligned}
\Psi(\alpha) &= \sum_{i,j=0}^{2} \alpha_i \alpha_j I_{R_i}^x I_{R_j}^y \phi(\|x-y\|_2) - 2\sum_{i,j=0}^{2} \alpha_i I_{R_i}^y \phi(\|\tilde{x}-y\|_2) \\
&= \frac{1}{h^4}\left[(\alpha_0^2+\alpha_1^2+\alpha_2^2) \int\limits_{R_0}\int\limits_{R_0} + 2\alpha_0\alpha_1 \int\limits_{R_0}\int\limits_{R_1} + 2\alpha_0\alpha_2 \int\limits_{R_0}\int\limits_{R_2} + 2\alpha_1\alpha_2 \int\limits_{R_1}\int\limits_{R_2} \right] \phi(\|x-y\|_2)\,dy\,dx \\
&\quad -2\sum_{i=0}^{2} \alpha_i \frac{1}{h^2} \int\limits_{R_i} \phi(\|\tilde{x}-y\|_2)\,dy.
\end{aligned}$$

We can rewrite this representation for $\Psi(\alpha)$ using the following two equations.

$$\int\limits_{R_i}\int\limits_{R_j} \phi(\|x-y\|_2)\,dy\,dx =$$

$$h^6 \log h \left[\|\frac{1}{h}(v_i - v_j)\|_2^{\,2} + \frac{1}{3} \right] + h^6 \int\limits_{[0,1]^2}\int\limits_{[0,1]^2} \phi(\|x - y + \frac{1}{h}(v_i - v_j)\|_2)\,dy\,dx$$

$$\int\limits_{R_i} \phi(\|\tilde{x}-y\|_2)\,dy =$$

$$h^4 \log h \left[\frac{\tilde{x}_1 - v_i^1}{h}\frac{\tilde{x}_1 - v_i^1 - h}{h} + \frac{\tilde{x}_2 - v_i^2}{h}\frac{\tilde{x}_2 - v_i^2 - h}{h} + \frac{2}{3} \right] + h^4 \int\limits_{[0,1]^2} \phi(\|\tilde{x}-y\|_2)\,dy.$$

Because the integrals occuring on the right hand side of the equations are bounded by constants depending only on the compact set K, it is sufficient to show that the terms containing the factor $h^2 \log h$ cancel.

Using MATHEMATICA we calculate that the coefficient of $h^2 \log h$,

$$\begin{aligned}
&\frac{h^2}{3}(\alpha_0^2+\alpha_1^2+\alpha_2^2) \\
&\quad +2h^2 \sum_{\substack{i,j=0,1,2 \\ i\neq j}} \alpha_i\alpha_j \left(\frac{1}{h^2}\|v_i - v_j\|_2^{\,2} + \frac{1}{3}\right) - 2h^2 \sum_{i=0}^{2} \alpha_i \left((\tilde{x}-v_i)\cdot\left(\tilde{x} - v_i - \begin{pmatrix} h \\ h \end{pmatrix}\right)/h^2 + \frac{2}{3}\right),
\end{aligned}$$

vanishes for all values of v_1, v_2 and α_1, α_2.

Powell [7] and Wu, Schaback [14] get for the case of point interpolation first order accuracy. However, numerical results of [7] show that the order is between one and two, but that edge effects prevent the achievement of a higher order.

In [14] error bounds are proven for functions f of the native space $\mathcal{F}_{\mathbf{R}^2}$ that are characterised by the condition that the Fourier transform $\hat{f}$ be in some weighted L^2 space where the weight depends on the radial basis function. Although it is assumed that the native space of the thin plate splines coincides with the Beppo Levi space, it is only known that

$$H^2(\mathbf{R}^2) \subset \mathrm{BL}^2(\mathbf{R}^2) \supset \mathcal{F}_{\mathbf{R}^2}.$$

In [10] Schaback is able to double the approximation order on an open set $\Omega \subset \mathbf{R}^2$. He restricts to the space $\mathcal{H}_\Omega \subset \mathcal{F}_{\mathbf{R}^2}$ that contains functions f that are characterised in terms of a faster the decay of the Fourier transform $\hat{f}$ and that satisfy suitable boundary conditions. For more details we refer the reader to [14] and [10].

Here we will show that improved convergence rates hold for functions of the space

$$\mathrm{BL}^4_h(\Omega) = \left\{ f \in H^4(\Omega), \frac{\partial^\alpha}{\partial x^\alpha} f = 0 \text{ on } \partial\Omega \text{ for } |\alpha| = 2, 3 \text{ in the sense of trace} \right\}$$

where $\Omega \subset \mathbf{R}^2$ denotes an open, bounded set with a Lipschitz boundary such that integration by parts is possible. The space $H^4(\Omega)$ denotes the Sobolev space with respect to the L^2 inner product. By using Sobolevs imbedding theorem we get $H^4(\Omega) \subset \mathcal{C}^2_b(\bar{\Omega})$, therefore the boundary values are pointwise defined up to second order derivatives and in the trace sense for third order derivatives.

The space $\mathrm{BL}^4_h(\Omega)$ is equipped with the semi norm

$$\|f\|^2_{\mathrm{BL}^4(\Omega)} = \sum_{|\alpha|=4} \binom{4}{\alpha} \| \frac{\partial^4}{\partial x^\alpha} f(x) \|^2_{L^2(\Omega)}.$$

Before stating the theorem we need an extension result.

Lemma 2.5. *For*

$$\begin{aligned} \mathrm{F}_0 &= \{ f \in \mathrm{BL}^2(\mathbf{R}^2), f|_\Omega = 0 \}, \\ \mathrm{F}_1 &= \{ f|_\Omega, f \in \mathrm{BL}^2(\mathbf{R}^2) \}, \end{aligned}$$

let $\mathrm{F}' = \mathrm{BL}^2(\mathbf{R}^2) \ominus \mathrm{F}_0$ *be the complementary subspace with respect to* $< \, , \, >_{\mathrm{BL}^2(\mathbf{R}^2)}$.

Then for $f \in \mathrm{F}_1$ *there exists an unique extension* $f' \in \mathrm{F}'$ *such that*

$$f' = f + f_0, \quad f_0 \in F_0,$$

and f' *is characterized by the smallest semi norm among all functions in* $\mathrm{BL}^2(\mathbf{R}^2)$ *which have a restriction to* Ω *that is equal to* f.

Furthermore we get

$$< f', v >_{\mathrm{BL}^2(\mathbf{R}^2)} = < f, v >_{\mathrm{BL}^2(\Omega)}$$

for all $v \in \mathrm{F}'$.

For the proof cf. [1] Part I.5.

Corollary 2.6. *For* $f \in \mathrm{BL}^4_h(\Omega)$, *let* s *be the thin plate spline interpolant subject to the interpolation conditions* $T_i s(x) = T_i f(x)$ *where the support of* T_i *is assumed to be contained in* Ω, $i = 1, \ldots N$.

Then

$$< f', f' - s >_{\mathrm{BL}^2(\mathbf{R}^2)} \leq \|f\|_{\mathrm{BL}^4(\Omega)} \|f - s\|_{L^2(\Omega)}.$$

Proof: From the previous lemma, by using integration by parts and a regularization argument as in [2], Theorems 1.3.4 and 1.6.6, we obtain for $v = f' - s$

$$\begin{aligned} < f', v >_{\mathrm{BL}^2(\mathbf{R}^2)} &= \int\limits_{\Omega} \left(\frac{\partial^4}{\partial \xi^4} f(x) + 2 \frac{\partial^4}{\partial \xi^2 \partial \eta^2} f(x) + \frac{\partial^4}{\partial \eta^4} f(x) \right) v(x) dx \\ &+ \int\limits_{\partial\Omega} \left(\frac{\partial^2}{\partial \xi^2} f(x) \frac{\partial}{\partial \xi} v(x) \nu_1(x) + 2 \frac{\partial^2}{\partial \xi \partial \eta} f(x) \frac{\partial}{\partial \xi} v(x) \nu_2(x) + \frac{\partial^2}{\partial \eta^2} f(x) \frac{\partial}{\partial \eta} v(x) \nu_2(x) \right) dS(x) \\ &- \int\limits_{\partial\Omega} \left(\frac{\partial^3}{\partial \xi^3} f(x) v(x) \nu_1(x) + 2 \frac{\partial^3}{\partial \xi \partial \eta^2} f(x) v(x) \nu_1(x) + \frac{\partial^3}{\partial \eta^3} f(x) v(x) \nu_2(x) \right) dS(x), \end{aligned}$$

where $\nu(x)$ denotes the exterior normal vector to Ω at $x \in \partial\Omega$. This yields the statement in connection with the homogenous boundary conditions. It remains to show that $s \in \mathrm{F}'$.

Observe that

$$\|s\|_{\mathrm{BL}^2(\mathbf{R}^2)} = \min\{\|\tilde{s}\| \mid \tilde{s} \in \mathrm{BL}^2(\mathbf{R}^2),\ T_i \tilde{s} = T_i f,\ i = 1 \dots N\}.$$

In particular, if we consider some $s^\star \in \mathrm{BL}^2(\mathbf{R}^2)$ with $s^\star|_\Omega = s|_\Omega$, we get with $\operatorname{supp} T_i \subset\subset \Omega$

$$\|s\|_{\mathrm{BL}^2(\mathbf{R}^2)} \le \|s^\star\|_{\mathrm{BL}^2(\mathbf{R}^2)}.$$

It follows

$$\|s\|_{\mathrm{BL}^2(\mathbf{R}^2)} = \min\{\|\tilde{s}\| \mid \tilde{s} \in \mathrm{BL}^2(\mathbf{R}^2),\ \tilde{s}|_\Omega = s|_\Omega\},$$

and therefore we conclude $s = (s|_\Omega)' \in \mathrm{F}'$.

Proposition 2.7. *Let $f \in \mathrm{BL}_h^4(\Omega)$ and s be the thin plate spline interpolant subject to the interpolation conditions $I_{R_i} s(x) = I_{R_i} f(x)$ with $R_i \subset \Omega,\ i = 1, \dots N$.*

Using the same notation as in Corollary 2.2, we get

$$|f(\tilde{x}) - s(\tilde{x})| \le \left[8\pi \|f\|^2_{\mathrm{BL}^4(\Omega)} \Psi(\alpha) \|\Psi(\alpha)\|_{L^2(\Omega)} \right]^{1/2},$$

where the integration of $\Psi(\alpha)$ is performed with respect to $\tilde{x}$.

Proof: From Corollary 2.2 we conclude

$$\|f(\tilde{x}) - s(\tilde{x})\|^2_{L^2(\Omega)} \le 8\pi \|f' - s\|^2_{\mathrm{BL}^2(\mathbf{R}^2)} \|\Psi(\alpha)\|^2_{L^2(\Omega)}.$$

Using the previous corollary we obtain

$$\begin{aligned} \|f' - s\|^2_{\mathrm{BL}^2(\mathbf{R}^2)} &= < f', f' - s >_{\mathrm{BL}^2(\mathbf{R}^2)} \\ &\le \|f\|_{\mathrm{BL}^4(\Omega)} \|f - s\|_{L^2(\Omega)} \\ &\le \sqrt{8\pi}\, \|f\|_{\mathrm{BL}^4(\Omega)} \|f' - s\|_{\mathrm{BL}^2(\mathbf{R}^2)} \|\Psi(\alpha)\|_{L^2(\Omega)}, \end{aligned}$$

hence

$$\|f' - s\|_{\mathrm{BL}^2(\mathbf{R}^2)} \le \sqrt{8\pi}\, \|f\|_{\mathrm{BL}^4(\Omega)} \|\Psi(\alpha)\|_{L^2(\Omega)}.$$

Inserting this in (6) with $g = f' - s$ yields the desired result.

We conclude the desired result on the improved approximation order of the thin plate spline reconstruction.

Theorem 2.8. *Let us assume that the interpolation cells are given as in Theorem 2.3. Then we get for the reconstruction points $x \in K$, $K \subset \mathbf{R}^2$ compact*

$$|f(x) - s(x)| \leq C_{\Omega,K} \|f\| h^2,$$

for all $f \in \mathrm{BL}_h^4(\Omega)$, and s denotes the thin plate spline interpolant to f. Here, $C_{\Omega,K} \geq 0$ depends only on the compact set K and Ω.

Proof: The statement follows from the previous proposition together with the estimates on $\Psi(\alpha)$ given in the proof of Theorem 2.4.

Remarks:

- In the case of point interpolation, the statement of Theorem 2.8 can be viewed as a generalization of convergence properties of the univariate natural cubic splines to the two dimensional thin plate splines.
- In [3], point interpolation with thin plate splines on an infinite square grid of mesh size h is investigated. It is shown that thin plate splines generate a cardinal function. Interpolation with this cardinal functions reproduces the polynomial space $\mathcal{P}_3^2$ and yields a convergence order of $\mathcal{O}(h^4)$, provided that $f \in \mathcal{C}^4(\mathbf{R}^2)$.

3 Local Approximation

Definition 3.1.: *Let $N \in \mathbf{N}$ be a fixed number of functionals $T_i \in \mathcal{C}'(\mathbf{R}^2)$, $i = 1, \ldots, N$, which are independent of h. Define scaled functionals by*

$$T^x_{i,h,x_0} f(x) = T^x_i f(hx + x_0), \; i = 1, \ldots, N.$$

Let s^h denote any interpolant satisfying $T_{i,h,x_0} s^h = T_{i,h,x_0} f$, $i = 1, \ldots N$.

Then p is called the `local approximation order` *of s^h in $x_0 \in \mathbf{R}^2$, if*

$$|f(x_0 + hx) - s^h(x_0 + hx)| = \mathcal{O}(h^p), \; h \to 0$$

for x in some open ball around 0 independent of h.

Remarks:

- Even in the one dimensional reconstruction of cell values with continuous piecewise linear interpolants we obtain first order accuracy for global approximation, but second order in the local case.
- One reason for the improvement of the order is that edge effects mentioned in the previous section do not occur in the local context.

The application of Corollary 2.2 reduces again to an estimation of $\Psi(\alpha)$.

Proposition 3.2.: *Let all the functionals I_{R_i}, $i = 1, \ldots, N$ in Corollary 2.2 be scaled by the positive factor h, but the parameters α_i, $i = 1, \ldots, N$ be fixed. Then $\Psi(\alpha)$ is scaled with the factor h^2.*

Proof: We deduce from the definition of $\Psi(\alpha)$ that the scaled value is the expression

$$h^2\Psi(\alpha) + h^2 \log h\Big[\underbrace{\sum_{i,j=1}^{\hat{N}} \alpha_i\alpha_j I^x_{R_i} I^y_{R_j} \underbrace{\|x-y\|_2^2}_{=\|x\|_2^2-2x^Ty+\|y\|_2^2}}_{=2\sum_{i=1}^{\hat{N}}\alpha_i I^x_{R_i}\|x\|_2^2-2(\sum_{i=1}^{\hat{N}}\alpha_i I^x_{R_i}x)^T(\sum_{i=1}^{\hat{N}}\alpha_i I^y_{R_i}y)} -2\underbrace{\sum_{i=1}^{\hat{N}}\alpha_i I^y_{R_i} \underbrace{\|\tilde{x}-y\|_2^2}_{=\|\tilde{x}\|_2^2-2\tilde{x}^Ty+\|y\|_2^2}}_{=2\sum_{i=1}^{\hat{N}}\alpha_i I^x_{R_i}\|x\|_2^2-2\tilde{x}^T(\sum_{i=1}^{\hat{N}}\alpha_i I^y_{R_i}y)}\Big].$$

Using

$$\sum_{i=1}^{\hat{N}} \alpha_i I^y_{R_i} y = \tilde{x}$$

and by the assumption of Corollary 2.2 we get the desired result.

Corollary 3.3. *Reconstruction on a cartesian grid with mesh size h using thin plate spline interpolants yields local approximation of first order on $\mathrm{BL}^2(\mathbf{R}^2)$ and of second order on $\mathrm{BL}^4_h(\Omega)$.*

Proof: Corollary 2.2 and Proposition 2.7.

Using the cardinal representation of the thin plate spline interpolant gives even second order of local approximation on $\mathcal{C}^2$ without the requirement of boundary conditions.

Theorem 3.4.: *Let f be $\mathcal{C}^2$ in a region containing x_0. Then the local approximation order of thin plate splines is two, i.e.,*

$$\big|f(x_0+hx) - s^h(x_0+hx)\big| = \mathcal{O}(h^2),\ h \to 0.$$

Here s^h denotes the thin plate spline interpolant of the data $I_{R_i,h,x_0}s^h = I_{R_i,h,x_0}f,\ i=1,\dots N$.

Proof: Without loss of generality we take $x_0 = 0$. We can describe $s^h(hx)$ by a sum of cardinal functions $u_i^h(hx),\ i=1,\dots N$,

$$s^h(hx) = \sum_{i=1}^{N} u_i^h(hx) I^x_{R_i} f(hx),$$

which are the unique solution of the linear system

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix}\begin{pmatrix} \big(u_i^h(hx)\big)_{i=1,\dots,N} \\ \big(v_k^h(hx)\big)_{k=1,2,3} \end{pmatrix} = \begin{pmatrix} \Big(I^y_{R_j}\phi(\|hx-hy\|_2)\Big)_{j=1,\dots,N} \\ \big(p_k(hx)\big)_{k=1,2,3} \end{pmatrix}, \tag{7}$$

$$A = \Big(I^x_{R_i}I^y_{R_j}\phi(\|hx-hy\|_2)\Big)_{i,j=1,\dots,N},$$

$$P = \big(I^x_{R_i}p_k(hx)\big)_{i=1,\dots N,k=0,1,2}, \quad p_0(x)=1,\ p_1(x)=\xi,\ p_2(x)=\eta.$$

If we choose for p^h the linear part of f

$$p^h(y) = f(hx) + \nabla f(hx)^T(y-hx)$$

we get

$$f(hx) - s^h(hx) = \sum_{i=1}^{N} u_i^h(hx) \underbrace{\left[I_{R_i}^x p^h(hx) - I_{R_i}^x f(hx) \right]}_{\mathcal{O}(h^2)}.$$

Obviously the vector $\left(u_i^1(x)\right)_{i=1,\ldots,N}$ reproduces the scaled polynomials

$$\sum_{i=1}^{N} u_i^1(x) I_{R_i}^x p_k(hx) = p_k(hx),\ k = 0, 1, 2,$$

and if we take $\tilde{v}(x)$ as the solution of

$$\sum_{i=1}^{N} u_i^1(x) I_{R_i}^x \|x\|_2^2 + \tilde{v}(x) = \|x\|_2^2,$$

we conclude from (7) for $j = 1, \ldots, N$

$$\sum_{i=1}^{N} u_i^1(x)(h^2 I_{R_i}^x I_{R_j}^y \phi(\|x-y\|_2) + h^2 \log(h) I_{R_i}^x \|x\|_2^2) + h^2 \sum_{k=1}^{3} v_k^1(x) I_{R_j}^x p_k(x) + h^2 \log(h)\tilde{v}(x) =$$
$$h^2 I_{R_j}^y \phi(\|x-y\|_2) + h^2 \log(h) \|x\|_2^2.$$

Adding $h^2 \log(h)(I_{R_j}^x \|x\|_2^2 - 2(I_{R_j}^x x)^T x)$ on both sides of the equation, we see that

$$u_i^h(hx) = u_i^1(x),\ i = 1, \ldots, N.$$

This means the Lebesgue constants $\sum_{i=1}^{N} |u_i^h(hx)|$ are bounded independent of h which completes the proof.

Remark: Note that for $f \in \mathcal{C}^1$ we achieve $\mathcal{O}(h)$ for the local approximation order using the same arguments as in the proof above. If f is continuous the approximation is at least consistent, i.e. $|f(x_0 + hx) - s^h(x_0 + hx)| \to 0,\ h \to 0$.

4 Conclusions

We showed that first order error bounds for point interpolation using thin plate splines can be generalized if the interpolation data is given by cell values. Even second order of global approximation can be achieved if the reconstructed function is sufficiently smooth and satisfies homogenous boundary conditions in second and third order derivatives.

For local approximation we showed second order accuracy if the reconstructed function is in $\mathcal{C}^2$. We observe from [12] that linear polynomials have local approximation order two, and we can expect an order r only if the number of interpolation data is greater than $\dim \mathcal{P}_r^2 = \binom{r+2}{r}$. The advantage of thin plate spline reconstruction in the local case manifests itself in the stability on unstructured grids. For a fixed polynomial basis the local reconstruction leads to a singular linear equation system if the position of the grid points does not match. Furthermore thin plate spline reconstruction is consistent, that is, constants are reproduced. This is a basic property required in the reconstruction step of finite volume methods. For more details on the efficacy of thin plate spline reconstruction for the numerical solution of hyperbolic conservation laws cf. [5] and [12].

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