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Operator Adapted Spectral Element Methods
I: Harmonic and Generalized Harmonic Polynomials

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Research Report No. 97-07
May 1997

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Abstract

The paper presents results on the approximation of functions which solve an elliptic differential equation by operator adapted systems of functions. Compared with standard polynomials, these operator adapted systems have superior local approximation properties. First, the case of Laplace's equation and harmonic polynomials as operator adapted functions is analyzed and sharp estimates for the approximation properties of harmonic polynomials (as measured in error in some Sobolev norm versus number of degrees of freedom) are given. In this measure harmonic polynomials are seen to be superior to standard polynomials. Special attention is paid to the approximation of singular functions that arise typically in corners. These results for harmonic polynomials are extended to general elliptic equations with analytic coefficients by means of the theory of Bergman and Vekua; the approximation results for Laplace's equation hold true verbatim, if harmonic polynomials are replaced with generalized harmonic polynomials. The Partition of Unity Method is used in a numerical example to construct an operator adapted spectral method for Laplace's equation that is based on approximating with harmonic polynomials locally.

Keywords: Finite element method, polynomial approximation in the complex domain

AMS Subject Classification: Primary: 65N30, Secondary: 30E10

1 Introduction

The Finite Element Method (FEM) is a widely used tool for the numerical approximation of solutions to partial differential equations. One of the reasons for its success is that, for second order strongly elliptic equations, the FEM is quasi-optimal, i.e., the FEM essentially minimizes the error (in the "energy norm") over the conforming finite element space. In such a setting, therefore, the performance of the FEM is completely determined by the choice of the conforming finite element space.

In the standard h, p, and hp versions of the FEM, the approximation properties of the finite element spaces are determined by the (local) approximation properties of polynomials. Thus, the classical FEM can be expected to perform well only when the exact solution can be approximated well locally by polynomials. In essence, this is a condition on the smoothness of the exact solution, i.e., on bounds on higher derivatives. However, for some parameter dependent problems such as the Helmholtz equation at high wave numbers or equations with rough coefficients arising, for example, in the modelling of problems with microstructure, the approximation properties of polynomials are poor. The poor performance of the standard FEM manifests itself typically as a "non-robustness" issue: The onset of the asymptotic behavior of the FEM depends on the smallest length scale of the solution and may not even be reachable by practicable values of the discretization parameter h or p. Due to this non-robustness, finite element calculations become more and more expensive when the wave number is increased or the scale of the microstructure is decreased. Similar situations are given by problems with boundary layers or singular solution.

For many of these problems the local regularity of the solution is available. The theory of homogenization for problems with (periodic) microstructure, asymptotic expansions for boundary layers, and Kondrat'ev's corner expansions are a few examples of mathematical techniques yielding knowledge about the local properties of the solution. This knowledge may be used to construct *local approximation spaces* which can capture the behavior of the solution much more accurately than the standard polynomials for a given number of degrees of freedom. Exploiting such information may therefore be much more efficient than the standard methods; in particular, for the above mentioned parameter dependent problems, incorporating this additional information into the method can greatly improve the robustness, [19, 23].

In [15, 24], the Partition of Unity Method (PUM) was proposed which provides a unified platform for constructing conforming ansatz spaces from user-provided local approximation spaces in such a way that the space constructed by the PUM inherits the approximation properties of the local, user-provided spaces. Therefore, the PUM can be used as a tool to create conforming ansatz spaces from operator adapted local approximation spaces capturing the local behavior of the solution. Let us briefly recall the main ingredients of the PUM, formulated in an H^1 setting, typical for second order strongly elliptic problems—other situations such as H^2 settings for fourth order problems, however, are completely analogous.

Definition 1.1. *Let an open cover $(\Omega_i)_{i=1}^N$ of $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be given which satisfies an overlap condition, i.e., there is $M \in \mathbb{N}$ such that*

$$\operatorname{card}\{i \mid x \in \Omega_i\} \leq M \qquad \forall x \in \Omega.$$

Let a Lipschitz continuous partition of unity $(\varphi_i)_{i=1}^N$ subordinate to that cover (i.e., $\operatorname{supp} \varphi_i \subset$

$\overline{\Omega_i}$) *be given which satisfies the following conditions: There are* C_G, $C_\infty > 0$ *such that*

$$\sum_i \varphi_i \equiv 1 \quad \text{on } \Omega, \qquad \|\varphi_i\|_{L^\infty(\mathbb{R}^d)} \le C_\infty, \quad \|\nabla\varphi_i\|_{L^\infty(\mathbb{R}^d)} \le \frac{C_G}{\operatorname{diam}(\Omega_i)}, \qquad i = 1, \ldots, N. \tag{1}$$

The sets Ω_i *are called* patches. *Let* $V_i \subset H^1(\Omega_i \cap \Omega)$, $i = 1, \ldots, N$, *be* local approximation spaces. *We define the* PUM space *by*

$$V_{PUM} := \sum_i \varphi_i V_i = \{\sum_i \varphi_i v_i \,|\, v_i \in V_i\} \subset H^1(\Omega). \tag{2}$$

Theorem 1.2. *Let a partition of unity and a collection of local approximation spaces be given as in Definition 1.1. Assume that the function* $u \in H^1(\Omega)$ *can be approximated locally on the patches, i.e., for each* Ω_i *there is* $v_i \in V_i$ *with*

$$\|u - v_i\|_{L^2(\Omega_i\cap\Omega)} \le \varepsilon_1(i), \qquad \|\nabla(u - v_i)\|_{L^2(\Omega_i\cap\Omega)} \le \varepsilon_2(i).$$

Then the global approximant $v := \sum_i \varphi_i v_i \in V_{PUM}$ *satisfies*

$$\begin{aligned} \|u - v\|_{L^2(\Omega)} &\le \sqrt{M} C_\infty \left(\sum_i \varepsilon_1(i)^2\right)^{1/2}, \\ \|\nabla(u - v)\|_{L^2(\Omega)} &\le \sqrt{2M} \left(\sum_i \left(\frac{C_G}{\operatorname{diam}(\Omega_i)}\right)^2 \varepsilon_1(i)^2 + C_\infty^2 \varepsilon_2(i)^2\right)^{1/2}. \end{aligned}$$

Proof. See [19, 15]. □

Theorem 1.2 states that the global approximation properties of the space V_{PUM} are as good as the approximation properties of the local spaces V_i permit. It is therefore sufficient to identify good local approximation spaces and analyze their approximation properties. In the present paper, we focus on the properties of a specific type of local approximation spaces, namely, *generalized harmonic polynomials*, which are well-suited for the approximation of solutions to scalar elliptic equations with analytic coefficients and homogeneous (or simple) right hand side. We analyze the approximation properties of generalized harmonic polynomials and give rates of convergence as the dimension of these local spaces (the "degree" of the generalized harmonic polynomials) is increased—in this sense, the methods presented here may be called "operator adapted spectral element methods". The results obtained here will be used in the second and third part of this series of papers in the analysis of operator adapted spectral element methods for the systems of the two dimensional elasticity equations, [17], and a more detailed analysis of Helmholtz's equation, [18]. An application to problems from homogenization will be given elsewhere.

The prototype of the equations under consideration here is Laplace's equation in two dimensions. It is well-known that harmonic functions, i.e., the solutions of Laplace's equation, can be represented as the real parts of holomorphic functions. A direct consequence is that harmonic functions can be approximated locally by harmonic polynomials of degree p, that is, span $\{\operatorname{Re} z^n, \operatorname{Im} z^n, 0 \le n \le p\}$ with $z = x + iy$.

The aim of Section 2 is to analyze the approximation properties of harmonic polynomials for the approximation of harmonic functions in terms of Sobolev-regularity of the function to be approximated and in terms of the geometry of the local patches. It is shown that the space of dimension $2p+1$ of harmonic polynomials of degree p has essentially the same approximation properties as the space of full polynomials of degree p, which has dimension $O(p^2)$. Special attention is placed on the approximation of singular functions arising typically in corners.
Section 3 extends the results for Laplace's equation to general elliptic equations with analytic coefficients. It is shown that the so-called Bergman-Vekua operator, which provides a bijection between solutions of Laplace's equation and the solutions of such equations, is bicontinuous in Sobolev norms. Thus, the approximation results for Laplace's equation can be transferred to the general case if harmonic polynomials are replaced with generalized harmonic polynomials, the images of harmonic polynomials under the Bergman-Vekua operator. It should be added that in general the Bergman-Vekua operator is not known explicitly. However, although the Bergman-Vekua operator was originally introduced as a global operator, it is possible to replace it with local variants thereby making it amenable to efficient numerical realizations by, e.g., series representations, [3, 22].
Section 4 concludes the paper with an example of an operator adapted spectral element method for Laplace's equation. In the framework of the PUM local spaces consisting of harmonic polynomials are used, and it is shown that this approach is superior to the classical p version in terms of error versus degrees of freedom. This is due to the fact that, locally, the space of dimension $2p+1$ of harmonic polynomials has very similar approximation properties for the approximation of harmonic functions as the full space of polynomials of dimension $O(p^2)$.

We close our introduction with a brief discussion of related operator adapted techniques. The PUM can be viewed as a unified approach for using custom-tailored ansatz spaces, and it is therefore a generalization of a variety of methodologies found in the literature, in particular, the "singular function method" (also known as "augmented Galerkin", "enriched spaces method") and methods known as "Trefftz method" or "boundary method". In the singular function method, [8], a standard finite element space is augmented by a few singular functions so as to resolve a corner singularity. Since the standard finite element hat functions form a partition of unity, this singular function method can be understood as a special case of the PUM with polynomial local approximation spaces away from the corners and local approximation spaces consisting of polynomials augmented by the appropriate singular functions in the vicinity of the corners. In the Trefftz and boundary methods the solution is approximated everywhere by operator adapted systems of functions. For example, [3, 22] approximate with operator adapted systems (in fact, generalized harmonic polynomials are used) and minimize the residual on the boundary in appropriately chosen points. In the classical Trefftz method (see [29] for an overview) the L^2 error on the boundary is minimized. These classical operator adapted methods are global in that they use operator adapted systems on the whole domain of interest. In order to localize the use of such systems, non-conforming methods have been proposed (e.g., [11, 13]). In these non-conforming methods, approximation with operator adapted systems is performed on subdomains and the continuity across subdomain boundaries is enforced in a weak sense through either the use of "mortar elements" or by minimizing the jump (or the jump of derivatives) across subdomain boundaries. In contrast to these non-conforming methods, using local operator adapted systems of functions in the framework of the PUM leads naturally to a conforming method. This is an attractive feature as the resulting discrete problem inherits many of the properties of the original continuous problem such as, e.g., coercivity and symmetry of coercive, symmetric problems. Nevertheless, the new approximation results obtained here can also be used independently from the PUM for an analysis of the non-conforming methods mentioned above.

Related to the idea of exploiting knowledge about the structure of the underlying differential equation is for example the "sparse grid" h version approach of [28]. There, extra regularity of the solution is exploited to choose judiciously subspaces of the classical piecewise polynomial spaces which share the approximation properties of the full spaces but have greatly reduced dimension. The approach discussed in Section 2 may be considered as a p version analog of this: Not the full space of polynomials of degree p is needed but merely the "sparse" set of those polynomials which also solve the differential equation is needed.

2 Approximation by Harmonic Polynomials

In this section we analyze the approximation properties of polynomials in the complex domain for the approximation of holomorphic functions which is closely related to the approximation of harmonic functions by harmonic polynomials. This question has been addressed in an L^∞ setting a long time ago. We mention the classical result by Runge,[14], on the density of polynomials in the set of holomorphic functions on a bounded domain and refer to [27, 20] for a detailed analysis in a Hölder space setting. In contrast to those classical results, we present approximation theory in a Sobolev space setting suitable in the context of the FEM. We will be particularly interested in H^1 estimates although the results generalize naturally to other Sobolev space settings. The main results are Theorems 2.9 and 2.11. Theorem 2.9 answers the question of approximability of holomorphic functions f in terms of the regularity of f (i.e., in which Sobolev space $H^k(\Omega)$ the function f lies) and in terms of the geometry of Ω (the angle of exterior cones). Theorem 2.11 addresses the question of approximating singular functions (e.g., of the form z^α) by polynomials. It is shown that for the approximation of such singlar functions on domains with a convex corner at the singularity, polynomials have even better approximation properties than those guaranteed by Theorem 2.9. The numerical examples of Section 2.6 indicate that the approximation results of Theorems 2.9, 2.11 are essentially sharp. In Section 2.5 finally, it is shown that additionally the approximating polynomials can be chosen to have some super-approximabiltiy properties on compact subsets.

It should be mentioned that polynomials are not the only choice for the approximation of holomorphic functions. On bounded domains, the set of rational functions, the set of functions $\{e^{nz} \,|\, n \in \mathbb{N}_0\}$, or $\{e^{az} \,|\, a \text{ in some dense subset of the unit circle}\}$ are dense (in Sobolev norms) in the set of holomorphic functions as well. An approximation-theoretical justification for considering polynomials is that they are optimal in the sense of n–width ([21]) for the approximation of rotationally invariant classes of holomorphic functions on discs, [24].

2.1 Properties of Conformal maps and Miscellaneous Lemmas

In this section, we consider bounded, simply connected Lipschitz domains $\Omega \subset \mathbb{C}$. We introduce the following semi norms on the space of holomorphic functions. For $k \in \mathbb{N}_0$, $s \in (0,1)$, and f holomorphic on Ω we define Sobolev norms of fractional order as follows:

$$\begin{aligned}
|f|_{k,\Omega} &= \left(\int_\Omega |f^{(k)}(z)|^2 dxdy \right)^{1/2}, \\
\|f\|_{k,\Omega} &= \left(\sum_{n=0}^{k} \int_\Omega |f^{(n)}(z)|^2 dxdy \right)^{1/2}, \\
|f|_{s,\Omega} &= \left(\int_{z\in\Omega} \int_{\zeta\in\Omega} \frac{|f(z)-f(\zeta)|^2}{|z-\zeta|^{2+2s}} \, d\xi d\eta \, dxdy \right)^{1/2}, \\
\|f\|_{k+s,\Omega} &= \left(\|f\|_{k,\Omega}^2 + |f^{(k)}|_{s,\Omega}^2 \right)^{1/2},
\end{aligned}$$

and we define the Sobolev spaces $\mathcal{H}^k(\Omega)$, $k \geq 0$, of holomorphic functions

$$\mathcal{H}^k(\Omega) := \{ f \text{ holomorphic on } \Omega \,|\, \|f\|_{k,\Omega} < \infty \}.$$

Definition 2.1. *Let Ω be a bounded, simply connected Lipschitz domain. Denote by φ_Ω the conformal map φ_Ω from the exterior of the unit ball $B_1(0)$ onto the exterior of Ω which satisfies $\varphi_\Omega(\infty) = \infty$ and $\varphi'_\Omega(\infty) > 0$. We introduce for $h > 0$ the* level–lines *L_h by*

$$L_h := \{ \varphi_\Omega(z) \in \mathbb{C} \,|\, |z| = 1 + h \}.$$

Since the level–lines are closed, non–selfintersecting, analytic curves, the set $\mathbb{C} \setminus L_h$ has two components of connectedness. Int(L_h) *denotes the bounded component of connectedness and* Ext(L_h) *the unbounded component.*
We say that Ω satisfies an exterior cone condition *with angle $0 < \lambda\pi < 2\pi$ if for any $z \in \mathbb{C} \setminus \Omega$ there is a cone C with vertex in z such that $C \subset \mathbb{C} \setminus \Omega$ and C is congruent to a fixed reference cone $C_0(\lambda\pi, \rho)$ of the form*

$$C_0(\lambda\pi, \rho) := \{ z \in \mathbb{C} \,|\, 0 < \arg z < \lambda\pi, |z| < \rho \}.$$

Similarly, we define an interior cone condition. *We call Ω* star-shaped *with respect to z_0, if for all $z \in \Omega$ the line segment $[z_0, z]$ is completely contained in Ω. We call Ω star-shaped with respect to the ball $B_r(z_0)$, if Ω is star-shaped with respect to every $z \in B_r(z_0)$.*

Since Ω is Lipschitz, the map φ_Ω has a continuous extension to $\partial\Omega$. Note that $\Omega \subset \text{Int}(L_h)$ for all $h > 0$. One essential tool for our approximation results are bounds on the distance of level–lines L_h to the boundary $\partial\Omega$.

Lemma 2.2. *Let Ω be a bounded Lipschitz domain satisfying an exterior cone condition with angle $\lambda\pi$ and an interior cone condition with angle $\Lambda\pi$ and let φ_Ω be the corresponding conformal map as in Definition 2.1. Then there are constants C_1, C_2, $C_3 > 0$ depending only on Ω such that*

$$\begin{aligned}
\text{dist}(L_h, L_{h'}) &\geq C_1 h |h - h'|, \qquad 0 < h < h' < 1, \\
C_2 h^{2-\Lambda} &\leq \text{dist}(L_h, \partial\Omega) \leq C_3 h^\lambda, \qquad 0 < h < 1, \\
|\varphi'_\Omega(z)| &\leq \varphi'_\Omega(\infty) \frac{|z|}{|z|-1} \qquad \forall\, |z| > 1.
\end{aligned}$$

Proof. These are classical estimates. The first may be proved using Koebe's distortion Theorem ([14]). The second estimates follow from a comparison of the level–lines of Ω with the level–lines of appropriately chosen polygons, for which the conformal maps are explicitly known owing to the Schwarz–Christoffel formulae. Finally, the last one follows from the area formula ([14]). $\square$

Lemma 2.3 (interior estimates)**.** *Let Ω be a domain. For $\epsilon > 0$ define $\Omega_\epsilon := \{z \in \Omega \mid B_\epsilon(z) \subset \Omega\}$. Then, for $f \in \mathcal{H}^k(\Omega)$, $0 \le k \le 1$,*

$$\begin{aligned} \|f\|_{L^\infty(\Omega_\epsilon)} &\le \pi^{-1/2}\epsilon^{-1}\|f\|_{0,\Omega}, && (3)\\ \|f'\|_{0,\Omega_\epsilon} &\le \epsilon^{-(1-k)}|f|_{k,\Omega}, && (4)\\ \|f^{(m)}\|_{0,\Omega_\epsilon} &\le m^{m-k}\epsilon^{-(m-k)}|f|_{k,\Omega}, \qquad m \in \mathbb{N}. && (5)\end{aligned}$$

Proof. The first estimate can be found in [14]. For the second one, we distinguish the cases $k = 0$ and $0 < k < 1$, the case $k = 1$ being obvious. For $k = 0$, we have by Cauchy's integral formula for every $0 < r < \epsilon$:

$$f'(z) = \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(t)}{(z-t)^2}dt = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z + re^{i\varphi})}{re^{i\varphi}}dt \qquad \forall z \in \Omega_\epsilon.$$

Taking squares on both sides and integrating in the z variable over Ω_ϵ, gives with the change of variables $\zeta = z + re^{i\varphi}$

$$\|f'\|^2_{0,\Omega_\epsilon} \le \frac{1}{r^2}\|f\|^2_{0,\Omega}.$$

Letting r tend to ϵ finishes the argument for the case $k = 0$. Let us now consider the case $0 < k < 1$. Cauchy's integral formula gives for $0 < r < \epsilon$

$$f'(z) = \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(t)}{(z-t)^2}dt = \frac{1}{2\pi i}\int_{\partial B_r(z)} \frac{f(t) - f(z)}{(z-t)^2}dt \qquad \forall z \in \Omega_\epsilon.$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned} |f'(z)|^2 &\le \frac{1}{4\pi^2}\int_{\partial B_r(z)} \frac{|f(t) - f(z)|^2}{|z-t|^{2+2k}}|dt| \int_{\partial B_r(z)} \frac{1}{|z-t|^{2-2k}}|dt| \\ &\le \frac{2\pi}{4\pi^2}r^{-(1-2k)}\int_{\partial B_r(z)} \frac{|f(t) - f(z)|^2}{|z-t|^{2+2k}}|dt|. \end{aligned}$$

Multiplying both sides of this inequality with r^{1-2k}, integrating over r from 0 to ϵ and then integrating in z over Ω_ϵ finishes the proof of the second estimate. The third inequality follows by applying the second one repeatedly: For $\delta = \epsilon/m$ define domains $\Omega_\epsilon = \Omega_{m\delta} \subset \Omega_{(m-1)\delta} \subset \cdots \subset \Omega_\delta \subset \Omega$ and use the second estimate to obtain

$$|f^{(m)}|_{0,\Omega_\epsilon} \le \delta^{-1}|f^{(m-1)}|_{0,\Omega_{(m-1)\delta}} \le \cdots \le \delta^{-(m-1)}|f'|_{0,\Omega_\delta} \le \delta^{-(m-1)}\delta^{-(1-k)}|f|_{k,\Omega}$$

which finishes the proof of the third inequality. $\square$

Lemma 2.4. *Let Ω be a bounded simply connected domain, $z_0 \in \Omega$, $f \in \mathcal{H}^0(\Omega)$. Then there is $C > 0$ depending only on the shape of Ω such that the function $F(z) = \int_{z_0}^z f(t)dt$ satisfies*

$$\|F\|_{0,\Omega} \le C\,\mathrm{diam}(\Omega)\|f\|_{0,\Omega}.$$

Proof. We will restrict ourselves to the case $\operatorname{diam}(\Omega) = 1$. The general case follows by a scaling argument. Choose $\epsilon > 0$ such that $B_{2\epsilon}(z_0) \subset \Omega$. By a classical Poincaré inequality there is a constant $C > 0$ such that

$$\|F\|_{0,\Omega} \leq C \left(|f|_{0,\Omega} + \|F\|_{0,B_\epsilon(z_0)} \right).$$

The term $\|F\|_{0,B_\epsilon(z_0)}$ can be bounded by the first estimate of Lemma 2.3. □

We finally need the following Bramble–Hilbert type lemma.

Lemma 2.5. *Let Ω be a bounded simply connected domain, $f \in \mathcal{H}^{k+s}(\Omega)$, $k \in \mathbb{N}_0$, $s \in [0,1)$, and $z_0 \in \Omega$. Denote T_k the k-th order Taylor polynomial of f about the point z_0 and set $T_{-1} \equiv 0$. Then*

$$\begin{aligned} \|f - T_k\|_{k,\Omega} &\leq C|f^{(k)}|_{s,\Omega} \qquad \text{if } s > 0, \\ \|f - T_{k-1}\|_{k,\Omega} &\leq C\|f^{(k)}\|_{0,\Omega} \qquad \text{if } s = 0 \end{aligned}$$

where $C > 0$ depends only on Ω, z_0, and k, s.

Proof. Let us first consider the case $s > 0$. Repeated application of Lemma 2.4 yields

$$|f - T_k|_{j,\Omega} \leq (C \operatorname{diam}(\Omega))^{k-j} \, |f^{(k)} - f^{(k)}(z_0)|_{0,\Omega} \qquad 0 \leq j \leq k.$$

For a fixed $K \subset\subset \Omega$ there is, by Poincaré, $C > 0$ such that

$$|g|_{0,\Omega} \leq C \left(|g|_{0,K} + |g|_{s,\Omega} \right) \qquad \forall g \in \mathcal{H}^s(\Omega).$$

On setting $g = f^{(k)} - f^{(k)}(z_0)$, we obtain

$$\begin{aligned} |f^{(k)} - f^{(k)}(z_0)|_{0,\Omega} &\leq C \left(|f^{(k)} - f^{(k)}(z_0)|_{0,K} + |f^{(k)}|_{s,\Omega} \right) \\ &\leq C \left(|f^{(k+1)}|_{0,K} + |f^{(k)}|_{s,\Omega} \right) \leq C|f^{(k)}|_{s,\Omega} \end{aligned}$$

where use made use of Lemmas 2.4, 2.3. Let us now turn to the case $s = 0$ and assume that $k \geq 1$ ($k = 0$ is trivial). Repeated application of Lemma 2.4 yields for $0 \leq j \leq k-1$

$$|f - T_{k-1}|_{j,\Omega} \leq (C \operatorname{diam}(\Omega))^{k-1-j} \, |f^{(k-1)} - f^{(k-1)}(z_0)|_{0,\Omega} \leq (C \operatorname{diam}(\Omega))^{k-j} \, |f^{(k)}|_{0,\Omega}.$$

As $T_{k-1}^{(k)} = 0$, we get $|f - T_{k-1}|_{k,\Omega} = |f^{(k)}|_{0,\Omega}$ and therefore obtain the desired bound for $\|f - T_{k-1}\|_{k,\Omega}$. □

2.2 Exponential Approximability of Harmonic Functions

We first address the question of polynomial approximation of functions which are holomorphic on domains properly containing the domain of interest. The rate of approximability in this case is exponential.

Theorem 2.6 (Szegö)**.** *Let Ω be a bounded, simply connected Lipschitz domain satisfying an interior and exterior cone condition. Let $f \in L^\infty(\operatorname{Int}(L_{2h}))$ be holomorphic on $\operatorname{Int}(L_{2h})$. Then there are C, $\alpha \geq 0$ depending only on Ω and a sequence $(P_p)_{p=0}^\infty$ of polynomials of degree p such that*

$$\|f - P_p\|_{L^\infty(\operatorname{Int}(L_{h/2}))} \leq Ch^{-\alpha}(1+h)^{-p}\|f\|_{L^\infty(\operatorname{Int}(L_{2h}))}.$$

Proof. See [20] or [16]. We will, however, sketch the proof. The polynomials P_p will be constructed explicitly by interpolating f in $p+1$ points.
Let φ be the conformal map of Definition 2.1. Define polynomials $\omega_p(z)$ of degree p by $\omega_p(z) = \prod_{n=0}^{p-1}(z - \varphi(\exp(2\pi i n/p)))$. We claim that there is $C > 0$ such that for h sufficiently small

$$h^C|\varphi'(\infty)|^p(1+h)^p \leq |\omega_p(z)| \leq h^{-C}|\varphi'(\infty)|^p(1+h)^p \qquad \forall z \in L_h. \tag{6}$$

We observe that $\ln|\omega_p(z)|^{1/p}$ can be interpreted as a Riemann sum of the integral $(2\pi)^{-1/2}\int_0^{2\pi}\ln|z-\varphi(e^i\theta)|d\theta$. On setting $w = \varphi^{-1}(z)$ for $z \in L_h$ this integral can be evaluated:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\ln\left|z-\varphi(e^{i\theta})\right|d\theta &= \operatorname{Re}\frac{1}{2\pi}\int_{|t|=1}\ln\frac{\varphi(w)-\varphi(t)}{w-t}\frac{dt}{it} + \operatorname{Re}\frac{1}{2\pi}\int_{|t|=1}\ln(w-t)\frac{dt}{it} \\ &= \operatorname{Re}\ln\varphi'(\infty) + \operatorname{Re}\ln w = \ln|\varphi'(\infty)\varphi^{-1}(z)|\end{aligned}$$

The difference between the Riemann sum and the integral can be bounded by $1/p$ times the variation of integrand which, for h sufficiently small, can be estimated by

$$\frac{1}{p}\int_{t\in\partial\Omega}\frac{1}{|z-t|}ds \leq \frac{C}{p}|\ln h|$$

where for the last estimate we used the fact that $\partial\Omega$ is Lipschitz and that $z \in L_h$. This proves (**??**).
Let P_p now be the polynomial which interpolates f in the $p+1$ points $\varphi(\exp(2\pi i n/p))$, $n = 0, \ldots, p-1$. Hermite's formula for the remainder (see [6]) gives

$$f(z) - P_p(z) = \frac{1}{2\pi i}\int_{L_{2h}}\frac{\omega_p(z)}{\omega_p(t)}\frac{f(t)}{z-t}dt \qquad \forall z \in \operatorname{Int}(L_{h/2}),$$

and the claim of Theorem 2.6 follows from Lemma 2.2 and (6). □

The classical formulation of Theorem 2.6 is concerned with L^∞ estimates. However, by Cauchy's integral theorem, estimates on higher derivatives are straightforward. We record this observation for further reference in

Corollary 2.7. *With the hypotheses of Theorem 2.6 and $f \in \mathcal{H}^0(\operatorname{Int}(L_{4h}))$ there is a sequence $(P_p)_{p=0}^\infty$ of polynomials of degree p such that*

$$|(f-P_p)^{(j)}|_{L^\infty(\Omega)} \leq C_j h^{-\beta_j}(1+h)^{-p}\|f\|_{0,\operatorname{Int}(L_{4h})}, \qquad j \in \mathbb{N}_0,$$

where C_j, $\beta_j \geq 0$ depend only on Ω, k, s, and j.

Proof. By Cauchy's formula for $(f-P_p)^{(j)}$, $j \in \mathbb{N}_0$, and with the estimate $Ch^{-(2-\Lambda)} \leq \operatorname{dist}(L_{h/2}, \partial\Omega)$ (Lemma 2.2), Theorem 2.6 implies that the polynomials P_p of Theorem 2.6 actually satisfy

$$\|(f-P_p)^{(j)}\|_{L^\infty(\Omega)} \leq C_j h^{-j(2-\Lambda)}h^{-\alpha}(1+h)^{-p}\|f\|_{L^\infty(\operatorname{Int}(L_{2h}))} \tag{7}$$

where the constants C_j depend only on j and Ω. Finally, the estimate on the distance of level-lines (Lemma 2.2) and the interior estimate (3) allow us to bound $\|f\|_{L^\infty(\operatorname{Int}(L_{2h}))}$ by $\|f\|_{0,\operatorname{Int}(L_{4h})}$ at the expense of some negative powers of h. □

2.3 Approximation in Sobolev Spaces

Let us first prove the density of polynomials in Sobolev spaces of holomorphic functions.

Proposition 2.8. *Let Ω be a bounded simply connected Lipschitz domain. Then polynomials are dense in $\mathcal{H}^0(\Omega)$ and $\mathcal{H}^1(\Omega)$.*

Proof. The density in $\mathcal{H}^0(\Omega)$ is proved in [14]. For the density in $\mathcal{H}^1(\Omega)$, let $f \in \mathcal{H}^1(\Omega)$, $z_0 \in \Omega$, and $\epsilon > 0$ be given. By the density of polynomials in $\mathcal{H}^0(\Omega)$ there is a polynomial Q such that $\|f' - Q\|_{0,\Omega} \le \epsilon$. By Lemma 2.4 the polynomial $P(z) = f(z_0) + \int_{z_0}^{z} Q(t)dt$ satisfies

$$\|f - P\|_{0,\Omega} \le C\|f' - Q\|_{0,\Omega} \le C\epsilon.$$

This finishes the proof. □

The main theorem of this section is

Theorem 2.9. *Let Ω be a bounded Lipschitz domain, star-shaped with respect to a ball. Assume that Ω satisfies an exterior cone condition with angle $\lambda\pi$. Let $f \in \mathcal{H}^{k+s}(\Omega)$, $k \in \mathbb{N}_0$, $s \in [0, 1)$. Then there is a sequence (P_p) of polynomials of degree $p \ge k + s - 1$ such that*

$$|f - P_p|_{j,\Omega} \le C\,(\mathrm{diam}(\Omega))^{k+s-j} \left(\frac{\ln(p+2)}{p+2}\right)^{\lambda(k+s-j)} |f^{(k)}|_{s,\Omega}, \qquad j = 0, \ldots, k,$$

where the constant $C > 0$ depends only on k, s, and the shape of Ω.

The proof of Theorem 2.9 proceeds in two steps. In the first step, an approximation T_k is constructed which is holomorphic on a domain Ω^ϵ properly containing Ω. In the second step, T_k is approximated by polynomials using Corollary 2.7. The following lemma formulates the procedure of the first step.

Lemma 2.10. *Assume the hypotheses of Theorem 2.9 and assume that Ω is star-shaped with respect to the ball $B_\rho(0)$, $\rho > 0$. For $0 < \epsilon < \frac{1}{2}$ define*

$$\Omega^\epsilon = (1-\epsilon)^{-1}\Omega, \tag{8}$$

$$T_k(z) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}((1-\epsilon)z)(\epsilon z)^n. \tag{9}$$

Then T_k is holomorphic on Ω^ϵ and

$$C_1\epsilon \le \mathrm{dist}(\partial\Omega^\epsilon, \Omega) \le C_2\epsilon, \tag{10}$$

$$|f - T_k|_{j,\Omega} \le C_3\epsilon^{k+s-j}|f^{(k)}|_{s,\Omega}, \qquad j = 0, \ldots, k, \tag{11}$$

$$|T_k|_{0,\Omega^\epsilon} \le C_4\|f\|_{k,\Omega}, \tag{12}$$

where C_1, C_2, C_3, and C_4 depend only on Ω, k, and s.

Proof. By a simple geometrical consideration, we have

$$\frac{\rho}{1-\epsilon}\epsilon \leq \operatorname{dist}(\partial\Omega^\epsilon, \Omega) \leq \operatorname{diam}(\Omega)\frac{\epsilon}{1-\epsilon} \tag{13}$$

and therefore (10) follows. Estimate (12) follows by the change of variables $\zeta = (1-\epsilon)z$. Since T_k is the k-th order Taylor polynomial of f, the integral form of the remainder gives

$$\begin{aligned} f(z) - T_k(z) &= \frac{1}{k!}\int_{(1-\epsilon)z}^{z} f^{(k+1)}(t)(z-t)^k dt \\ &= \frac{1}{k!}\epsilon^{k+1}\int_{\tau=0}^{1} f^{(k+1)}\left((1-\epsilon\tau)z\right) z^{k+1}\tau^k d\tau. \end{aligned} \tag{14}$$

Let us first consider the case $j = 0$. If $0 = j = k+s$, then estimate (11) follows directly from the change of variables $\zeta = (1-\epsilon)z$. Let therefore $0 = j < k+s$. Choose $0 < \delta < 1/2$ such that $2(k+s+\delta-1) > -1$. (14) yields

$$\begin{aligned} |f - T_k|^2_{0,\Omega} &\leq \frac{1}{(k!)^2}\epsilon^{2(k+1)}\int_{z\in\Omega}\left|\int_{\tau=0}^{1} f^{(k+1)}\left((1-\epsilon\tau)z\right) z^{k+1}\tau^k d\tau\right|^2 dxdy \\ &\leq \frac{(\operatorname{diam}(\Omega))^{2(k+1)}}{(k!)^2(1-2\delta)}\epsilon^{2(k+1)}\int_{z\in\Omega}\int_{\tau=0}^{1}\left|f^{(k+1)}\left((1-\epsilon\tau)z\right)\right|^2\tau^{2(k+\delta)}d\tau dxdy \\ &\leq C\epsilon^{2(k+1)}\int_{\tau=0}^{1}\tau^{2(k+\delta)}\int_{z\in(1-\epsilon\tau)\Omega}\left|f^{(k+1)}(z)\right|^2 d\tau dxdy. \end{aligned}$$

Reasoning analogously to (13), we see that $\operatorname{dist}((1-\epsilon\tau)\Omega, \partial\Omega) \geq \rho\epsilon\tau$. Hence, we can apply estimate (4) of Lemma 2.3 to obtain

$$\begin{aligned} |f - T_k|^2_{0,\Omega} &\leq C\epsilon^{2(k+1)}\int_{\tau=0}^{1}\tau^{2(k+\delta)}(\epsilon\tau)^{-2(1-s)}|f^{(k)}|^2_{s,\Omega}d\tau \\ &\leq C\epsilon^{2(k+s)}|f^{(k)}|^2_{s,\Omega} \end{aligned}$$

where we used the fact that $2(k+s+\delta-1) > -1$. This concludes the proof of the case $j = 0$. Let us now consider the case $0 < j \leq k$. For $s = 0$ the claim follows directly from the Definition (9) by taking the jth derivative and then applying estimate (5) of Lemma 2.3 to each of the terms. For the remaining case $s > 0$, we see that by differentiating under the integral sign in the remainder formula (14) we have to estimate terms which are very similar to the ones in the case $j = 0$ above except that instead of $f^{(k+1)}$ higher derivatives $f^{(k+1+j')}$, $0 \leq j' \leq j$, are involved. *Mutatis mutandis*, the arguments of our analysis of the case $j = 0$ apply if we use (5) instead of (4) of Lemma 2.3 and choose $\delta \in (0, 1/2)$ such that $2(k+s+\delta-1-j') > -1$. □

We are now in position to prove Theorem 2.9.

Proof of Theorem 2.9: Without loss of generality, we may assume that Ω is star shaped with respect to a ball $B_\rho(0)$, $\rho > 0$. The factor $(\operatorname{diam}(\Omega))^{k+s-j}$ is obtained by the usual scaling argument. We may therefore restrict ourselves to the case $\operatorname{diam}(\Omega) = 1$.

For $\epsilon > 0$ to be chosen appropriately below, let T_k be defined as in Lemma 2.10. We have

$$|f - T_k|_{j,\Omega} \leq C\epsilon^{k+s-j}|f^{(k)}|_{s,\Omega}. \tag{15}$$

Let us now approximate T_k by polynomials P_p of degree $p \geq k+s-1$. To that end, we note that by (10) of Lemma 2.10 and Lemma 2.2 there is $C > 0$ such that the inclusion $\mathrm{Int}(L_{4h}) \subset \Omega^\epsilon$ holds for

$$h := C\epsilon^{1/\lambda}.$$

An application of Corollary 2.7 to the function $T_k \in \mathcal{H}^0(\mathrm{Int}(L_{4h}))$ gives the existence of polynomials P_p of degree p such that

$$|T_k - P_p|_{j,\Omega} \leq C_j h^{-\beta_j}(1+h)^{-p}\|T_k\|_{0,\mathrm{Int}(L_{4h})}, \qquad j \in \mathbb{N}_0.$$

Estimate (12) and the restriction $j \leq k$ imply

$$|T_k - P_p|_{j,\Omega} \leq Ch^{-\beta}(1+h)^{-p}\|f\|_{k,\Omega}, \qquad j \leq k,$$

for some C, $\beta \geq 0$. Thus, the total error is (recalling $h = C\epsilon^{1/\lambda}$)

$$\begin{aligned} |f - P_p|_{j,\Omega} &\leq |f - T_k|_{j,\Omega} + |T_k - P_p|_{j,\Omega} \\ &\leq C\left[\epsilon^{k+s-j} + \epsilon^{-\beta/\lambda}(1 + C\epsilon^{1/\lambda})^{-p}\right]\|f\|_{k+s,\Omega}, \qquad j = 0,\ldots,k. \end{aligned}$$

Choosing

$$\epsilon = \left(K\frac{\ln(p+2)}{p+2}\right)^\lambda \tag{16}$$

for sufficiently large K leads to the bound

$$|f - P_p|_{j,\Omega} \leq C\left(\frac{\ln(p+2)}{p+2}\right)^{\lambda(k+s-j)}\|f\|_{k+s,\Omega}, \qquad j = 0,\ldots,k. \tag{17}$$

Replacing the full $\|\cdot\|_{k+s,\Omega}$ by the appropriate semi norm on the right hand side follows now by a standard argument with the aid of Lemma 2.5: Let p_0 be the smallest integer such that $p_0 \geq k+s-1$. Applying estimate (17) to the function $f - \tilde{T}_{p_0}$ where $\tilde{T}_{p_0}$ is the Taylor polynomial of order p_0 of f about the origin $0 \in \Omega$, we get the existence of polynomials $\tilde{P}_p$, $p \geq p_0$ such that

$$\begin{aligned} |f - \tilde{P}_p|_{j,\Omega} &\leq C\left(\frac{\ln(p+2)}{p+2}\right)^{\lambda(k+s-j)}\|f - \tilde{T}_{p_0}\|_{k+s,\Omega} \\ &\leq C\left(\frac{\ln(p+2)}{p+2}\right)^{\lambda(k+s-j)}|f^{(k)}|_{s,\Omega}, \qquad j = 0,\ldots,k. \end{aligned}$$

2.4 Approximation of Singular Functions z^α, $z^\alpha \ln z$

In the preceding section, we analyzed the approximation properties of polynomials for the approximation of functions $f \in \mathcal{H}^{k+s}(\Omega)$. No assumptions on a possible continuation of f across (parts of) $\partial\Omega$ were made. Singular functions of the form z^α, $z^\alpha \ln z$, which are of practical importance as they arise typically as the solutions of Laplace's equation in domains with corners, permit analytic continuation across parts of $\partial\Omega$. This observation motivates the analysis of the approximation of functions $f \in \mathcal{H}^{k+s}(\Omega)$ which can be extended across $\partial\Omega$ except at a singularity $z_0 \in \partial\Omega$. The following theorem shows that for the approximation of such functions, the results of Theorem 2.9 can be improved provided that the singularity z_0 is located at a *convex* corner of the domain Ω.

Theorem 2.11. *Let $\Omega \subset \Omega_0$ be bounded Lipschitz domains with piecewise C^1 boundaries. Assume that Ω satisfies an exterior cone condition with angle $\lambda\pi > \pi$ at $z_0 \in \partial\Omega$, and assume that there is $\gamma > 0$ such that*

$$\operatorname{dist}(z, \partial\Omega_0) \geq \gamma |z - z_0| \qquad \forall z \in \partial\Omega. \tag{18}$$

Let $f \in \mathcal{H}^{k+s}(\Omega_0)$, $k \in \mathbb{N}_0$, $s \in [0, 1)$. Then there is $C > 0$ depending only γ and the shapes of Ω, Ω_0, and a sequence (P_p) of polynomials of degree $p \geq k + s - 1$ such that

$$|f - P_p|_{j,\Omega} \leq C \left(\operatorname{diam}(\Omega_0)\right)^{k+s-j} \left(\frac{\ln(p+2)}{p+2}\right)^{\lambda(k+s-j)} |f^{(k)}|_{s,\Omega_0}, \qquad j = 0, \ldots, k.$$

The geometric interpretation of condition (18) is that $\partial\Omega$ and $\partial\Omega_0$ can only meet in the point z_0 and they can only meet in a "non–tangential" way (cf. Fig. 1).

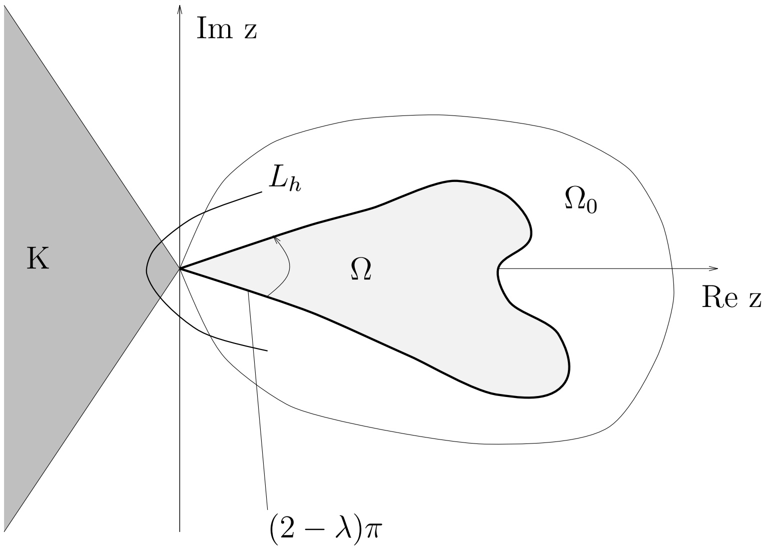

Figure 1: Setting of Theorem 2.11

Before proving Theorem 2.11, let us note

Corollary 2.12. *Let f be of the form z^α or $z^\alpha \ln^\beta z$ with α, $\beta \geq 0$. Let Ω be a bounded Lipschitz domain with exterior angle $\lambda\pi$ at $0 \in \partial\Omega$. Then there is a sequence $(P_p)_{p=0}^\infty$ of polynomials of degree p such that for all $\epsilon > 0$*

$$|f - P_p|_{j,\Omega} \leq C_\epsilon (p+1)^{-\lambda(\alpha+1-j)+\epsilon}, \qquad j = 0, 1,$$

where C_ϵ depends only on ϵ, Ω, and f.

Proof. The corollary follows immediately from Theorem 2.11 for the case $\lambda\pi > \pi$ and from Theorem 2.9 for the case $\lambda\pi \leq \pi$ if we observe that $f \in \mathcal{H}^{\alpha+1-\epsilon}(\Omega)$ for all $\epsilon > 0$. ☐

Remark 2.13 This corollary shows more clearly the difference between Theorem 2.11 and Theorem 2.9. Whereas, in order to apply Theorem 2.9 one has to assume $\lambda \leq 1$, Theorem 2.11

allows us to infer *improved* rates of convergence if the singularity of the function is located in a convex corner of the domain. For the approximation of functions z^α in the H^1 norm, Corollary 2.12 yields – for the approximation in the H^1 norm – the rate $(p+1)^{-\lambda\alpha+\epsilon}$ where $\lambda > 1$ for a convex corner; Theorem 2.9 only yields $(p+1)^{-\alpha+\epsilon}$ for the approximation in the H^1 norm. This result for the approximation of singular functions is similar to the "doubling of the rate of convergence" in the standard p version: In [25] the approximation of functions of the form $g(r,\theta) = r^\alpha\Phi(\theta)$, ($\Phi$ smooth) by spaces of full polynomials on triangles T is considered. It is assumed that one vertex of T is located in the origin and that the angle at that vertex is less than $\pi/2$. Then there are polynomials Q_p of degree p and $C > 0$ such that

$$\|g - Q_p\|_{H^1(T)} \leq C(p+1)^{-2\alpha} \sim CN^{-\alpha}$$

where N stands for the number of degrees of freedom. On the other hand, approximating a harmonic function of the form $u = \operatorname{Im} z^\alpha$ on such a triangle T gives by Theorem 2.11 with $\lambda = 3/2$ the existence of harmonic polynomials u_p such that $\|u - u_p\|_{H^1(T)} \leq C(p+1)^{-\lambda\alpha+\epsilon} \sim CN^{-3/2\alpha+\epsilon}$.

Proof of Theorem 2.11: The proof of Theorem 2.11 is very similar to the proof of Theorem 2.9. Again, we will restrict ourselves to the case $\operatorname{diam}(\Omega_0) = 1$ and obtain the general case by a scaling argument. As in the proof of Theorem 2.9, we construct a function T_k with $f - T_k$ small and T_k being holomorphic on a domain properly containing Ω. Approximating T_k by polynomials then concludes the argument.
Without loss of generality let us assume that $z_0 = 0$ and that the exterior cone condition satisfied by Ω at the point z_0 is such that (for some $\rho > 0$)

$$\{z \in B_\rho(0) \,|\, |\arg z| > (2-\lambda)\pi/2\} \subset \mathbb{C} \setminus \Omega \tag{19}$$

The approximation T_k is then defined by

$$T_k(z) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(z+\epsilon)(-\epsilon)^n, \tag{20}$$

which is holomorphic on $\Omega - \epsilon$. In order to imitate the proofs of Lemma 2.10 and Theorem 2.9 we need the following two properties:

1. There is $C > 0$ such that
$$Ch^\lambda + \operatorname{Int}(L_h) \subset \Omega_0. \tag{21}$$

2. There are $\kappa > 0$, $\epsilon_0 > 0$ such that
$$\cup_{z\in\Omega} B_{\kappa\epsilon}(z+\epsilon) \subset \Omega_0 \qquad \forall\, 0 < \epsilon < \epsilon_0. \tag{22}$$

Let us postpone the proof of these two properties. Since T_k is the Taylor polynomial of f about the point $z+\epsilon$ evaluated at $z = (z+\epsilon) - \epsilon$ we have again the integral remainder formula:

$$\begin{aligned} f(z) - T_k(z) &= \frac{1}{k!} \int_{z+\epsilon}^{z} f^{(k+1)}(t)(z-t)^k dt \\ &= \frac{1}{k!}(-\epsilon)^{k+1} \int_{\tau=0}^{1} f^{(k+1)}(z+\epsilon\tau)\tau^k d\tau. \end{aligned}$$

Similar arguments as in the proof of Lemma 2.10 (here we need (22)) produce

$$|f - T_k|_{j,\Omega} \leq C\epsilon^{k+s-j}|f^{(k)}|_{s,\Omega_0}, \qquad j = 0, \ldots, k.$$

Let us now turn to the approximation of T_k by polynomials. (21) gives the existence of $C > 0$ such that for

$$h := C\epsilon^{1/\lambda}$$

the inclusion $\epsilon + \text{Int}(L_{4h}) \subset \Omega_0$ holds. Applying Corollary 2.7 to the approximation T_k gives the existence of polynomials P_p of degree p such that

$$|T_k - P_p|_{j,\Omega} \leq C_j h^{-\beta_j}(1+h)^{-p}|T_k|_{0,\text{Int}(L_{4h})}, \qquad j \in \mathbb{N}_0.$$

Observing that $\epsilon + \text{Int}(L_{4h}) \subset \Omega_0$, the change of variables $\zeta = z + \epsilon$ on the right hand side gives

$$|T_k - P_p|_{j,\Omega} \leq Ch^{-\beta}(1+h)^{-p}\|f\|_{k,\Omega_0}, \qquad j = 0, \ldots, k.$$

Just as in the proof of Theorem 2.9 setting $\epsilon = K\{\ln(p+2)/(p+2)\}^\lambda$ for sufficiently large K produces the estimate

$$|f - P_p|_{j,\Omega} \leq C\left(\frac{\ln(p+2)}{p+2}\right)^{\lambda(k+s-j)}\|f\|_{k+s,\Omega}, \qquad j = 0, \ldots, k.$$

Again, an application of Lemma 2.5 allows us to replace the full $\|\cdot\|_{k+s,\Omega}$ norm on the right hand side by the desired semi norm.
Let us now check the validity of assertions (21), (22). (22) follows from simple geometrical considerations: $\partial\Omega_0$ approaches Ω only in $z_0 = 0$ and $z_0 = 0$ is a convex corner of Ω.
Because the boundaries of Ω and Ω_0 are piecewise smooth, there are ρ', $\delta > 0$ such that (cf. Fig. 1)

$$\{z \in B_{\rho'}(0) \,|\, |\arg z| < (2-\lambda)\pi/2 + \delta\} \subset \Omega_0. \tag{23}$$

Define

$$K := \{z \in B_{\rho'}(0) \,|\, |\arg z| > (2-\lambda)\pi/2 + \delta\}.$$

Then it follows from the properties of conformal maps and the fact that Ω satisfies an exterior cone condition with angle $\lambda\pi$ that there is $C > 0$ such that

$$\text{dist}(z, 0) \leq Ch^\lambda \qquad \forall\, z \in L_h \cap K. \tag{24}$$

(21) follows now because for $z \in L_h \cap K$ (24) and (23) give $z + Ch^\lambda \in \Omega_0$ for C sufficiently large, and for $z \in L_h \setminus K$ either $|z| > \rho'$, in which case (21) follows for h sufficiently small by (18), or $|z| < \rho'$, in which case $z \in \Omega_0$ and thus $z + Ch^\lambda \in \Omega_0$.

Remark 2.14 Theorem 2.11 can be extended to the case of curvilinear polygons, star shaped with respect to a ball $B_\rho(z_0)$. Letting $z_1, \ldots, z_N$ be the corners of Ω and assuming that Ω satisfies a cone condition with angle $\lambda\pi$ at each corner and that

$$\text{dist}(z, \partial\Omega_0) \leq \gamma\prod_{j=1}^{N}|z - z_j|, \qquad z \in \partial\Omega,$$

the assertion of Theorem 2.11 still holds true. To see this, it is enough to set

$$T_k(z) = \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}((1-\epsilon)(z-z_0))(\epsilon(z-z_0))^n$$

and reasoning as in the proof of Theorem 2.11 yields the desired result.

2.5 Super-Approximability on Compact Subsets

In this subsection we will briefly discuss the question whether it is possible to construct approximating polynomials which have good approximation properties on the whole domain of interest and at the same time approximate the given function even better on compact subsets. In view of the fact that both the function to be approximated and the approximating functions are holomorphic, such a result may be expected from the maximum principle. We will discuss here the approach that makes use of "Fourier expansions" (i.e., expansions in terms of an orthonormal basis) of the given function. We demonstrate this tool for two simple cases only, viz., the case of a domain with an analytic boundary and the case of a rectangle. Further, we will outline how the proof of Theorem 2.11 can be modified to obtain super-approximability on compact subsets for singular functions.

Proposition 2.15. *Let Ω be a bounded, simply connected domain with analytic boundary. Assume that $f \in \mathcal{H}^{k+s}(\Omega)$, $k \in \mathbb{N}_0$, $s \in [0,1)$. Then there is a sequence (P_p) of polynomials of degree $p \geq k+s-1$ such that*

$$\forall \epsilon \geq 0 \qquad |f - P_p|_{j,\Omega_\epsilon} \leq C d^{k+s-j} p^{-(k+s-j)} e^{-\gamma p \epsilon / d} |f^{(k)}|_{s,\Omega}, \qquad j = 0, \ldots, k,$$

where

$$d = \operatorname{diam}(\Omega), \quad \Omega_\epsilon = \{z \in \Omega \,|\, \operatorname{dist}(z, \partial\Omega) > \epsilon\}, \quad 0 \leq \epsilon/d \leq \gamma_0,$$

and the constants C, γ, $\gamma_0 > 0$ depend only on the shape of Ω, k, and s.

Remark 2.16 Note that the polynomials P_p are independent of ϵ. Setting $\epsilon = 0$ (i.e., considering the whole domain Ω), we obtain an estimate which is essentially the same as an application of Theorem 2.9 with $\lambda = 1$ yields. However, the $\ln(p+2)$ term could be removed due to the assumption of analyticity of $\partial\Omega$.

Proof. Without loss of generality we may assume that $\operatorname{diam}(\Omega) = 1$. The general case is proved by a scaling argument. Let ψ be a conformal map from Ω onto $B_1(0)$. Let us first see that ψ can be extended to a neighborhood $\tilde{\Omega}$ of $\Omega \cup \partial\Omega$ and that $|\psi'| > 0$ on $\Omega \cup \partial\Omega$. The reflection principle ($\partial\Omega$, $\partial B_1(0)$ are analytic) allows us to extend ψ to a neighborhood of $\Omega \cup \partial\Omega$. Similarly, we can extend ψ^{-1} to a holomorphic function on a ball $B_\varrho(0)$, $\rho > 1$. Hence ψ is invertible on a neighborhood $\tilde{\Omega}$ of $\Omega \cup \partial\Omega$ and therefore $|\psi'| > 0$ on $\tilde{\Omega}$.
An L^2 orthonormal basis of $\mathcal{H}^0(\Omega)$ is given by $\{\sqrt{(n+1)/\pi}\psi^n\psi'\}$ as can be easily ascertained from the change of variables $w = \psi(z)$ (see, e.g., [14]). In fact, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n \sqrt{(n+1)/\pi}\, \psi^n(z) \psi'(z) \tag{25}$$

where the coefficients $a_n\sqrt{(n+1)\pi}$ are the Taylor coefficients of the function

$$\tilde{f}(w) = \sum_{n=0}^{\infty} a_n\sqrt{(n+1)/\pi}w^n = \frac{f(\psi^{-1}(w))}{\psi'(\psi^{-1}(w))} \in \mathcal{H}^{k+s}(B_1(0)).$$

Thus, we have

$$\sum_{n=0}^{\infty} |a_n|^2(n+1)^{2m} \leq C(m,\Omega)\|f\|^2_{m,\Omega}, \qquad 0 \leq m \leq k+s. \tag{26}$$

Let us further remark that, by the properties of conformal maps, there is $\kappa > 0$ such that

$$\{w = \psi(z) \,|\, z \in \Omega_\epsilon\} \subset B_{1-\kappa\epsilon}(0). \tag{27}$$

Let us split the sum in (25) in two parts

$$f(z) = \sum_{n=0}^{N-1} a_n\sqrt{(n+1)/\pi}\psi^n(z)\psi'(z) + \sum_{n=N}^{\infty} a_n\sqrt{(n+1)/\pi}\psi^n(z)\psi'(z) =: T_N + R_N$$

where the number $N \in \mathbb{N}$ will be chosen below depending on the polynomial degree p. Let $h > 0$ be so small that $\mathrm{Int}(L_{4h}) \subset \tilde{\Omega}$. By Corollary 2.7 applied to T_N there are polynomials P_p such that

$$|T_N - P_p|_{j,\Omega} \leq Ch^{-\beta}(1+h)^{-p}N\|\psi'\|_{L^\infty(\tilde{\Omega})}\|\psi\|^N_{L^\infty(\tilde{\Omega})}\|f\|_{0,\Omega}, \qquad j = 0,\ldots,k,$$

where we used estimate (26) with $m = 0$. Let us now turn to estimating R_N. The change of variables $w = \psi(z)$ (together with (27)) provides us with the bound

$$\begin{aligned} |R_N|^2_{0,\Omega_\epsilon} &\leq |\sum_{n=N}^{\infty} a_n\sqrt{(n+1)/\pi}w^n|^2_{0,B_{1-\kappa\epsilon}(0)} = \sum_{n=N}^{\infty} |a_n|^2(1-\kappa\epsilon)^{2n+2} \\ &\leq CN^{-2(k+s)}(1-\kappa\epsilon)^{2N+2}\|f\|^2_{k+s,\Omega} \end{aligned}$$

where we used estimate (26). Note that $C > 0$ is independent of ϵ. Similarly, we obtain for $0 \leq j \leq k$

$$|R_N|^2_{j,\Omega_\epsilon} \leq CN^{-2(k+s-j)}(1-\kappa\epsilon)^{2(N-j)+2}\|f\|^2_{k+s,\Omega}$$

where $C > 0$ depends on $\|\psi\|_{W^{k+1,\infty}(\tilde{\Omega})}$ and on $|\psi'|^{-1}_{L^\infty(\Omega)}$. We conclude that

$$|f - P_p|_{j,\Omega_\epsilon} \leq C\left(h^{-\beta}(1+h)^{-p}\|\psi\|^N_{L^\infty(\tilde{\Omega})}N + N^{-(k+s-j)}(1-\kappa\epsilon)^{N-j+1}\right)\|f\|_{k+s,\Omega}, \qquad j = 0,\ldots,k.$$

Choosing $N = \nu p$ with ν so small that $q := (1+h)^{-1}\|\psi\|^\nu_{L^\infty(\tilde{\Omega})} < 1$ results in

$$|f - P_p|_{j,\Omega} \leq C\left(p^{-(k+s-j)}e^{-\gamma\epsilon p} + pq^p\right)\|f\|_{k+s,\Omega}, \qquad j = 0,\ldots,k.$$

The term pq^p can be absorbed in the first term. Finally, the full $\|\cdot\|_{k+s,\Omega}$ norm can be replaced by the desired semi norm by the application of Lemma 2.5. □

The essential tool of Proposition 2.15 is the availability of an orthonormal basis of $\mathcal{H}^0(\Omega)$. For rectangular domains similar ideas can be used. We illustrate this in the next proposition for *harmonic* functions.

Proposition 2.17. *Let Ω be a rectangle, $u : \Omega \to \mathbb{R}$ be harmonic, $u \in H^{k+s}(\Omega)$, $k \in \mathbb{N}$, $s \in [0,1)$. Then there is a sequence (u_p) of harmonic polynomials of degree $p \geq k+s-1$ such that*

$$\forall \epsilon > 0 \qquad \|u - u_p\|_{H^j(\Omega_\epsilon)} \leq C d^{k+s-j} p^{-(k+s-j)} e^{-\gamma p \epsilon/d} |u|_{k+s,\Omega}, \qquad j = 0, \ldots, k, \tag{28}$$

with

$$d = \mathrm{diam}(\Omega), \quad \Omega_\epsilon = \{z \in \Omega \,|\, \mathrm{dist}(z, \partial\Omega) > \epsilon\}, \quad 0 \leq \epsilon/d \leq \gamma_0,$$

where the constants C, γ, $\gamma_0 > 0$ depend only on the shape of Ω and k, s.

Proof. The proof is based on very similar ideas as the proof of Proposition 2.15. We will therefore merely sketch the main steps. For convenience's sake, let us assume that $\Omega = (0,\pi) \times (0,a)$ and that $s \neq 0$. After subtracting an appropriate harmonic polynomial, we may assume that all derivatives of u up to order $k-1$ vanish in the four vertices. Hence, we can write $u = u_1 + \cdots + u_4$, where the u_i are harmonic and vanish on three edges and are equal to u on the fourth edge. We may therefore approximate each of these four functions separately. Consider u_1, whose restriction to the boundary $\partial\Omega$ is supported by $(0,\pi) \times \{0\}$. Expanding $u_1(x,0)$ in a Fourier series gives

$$\begin{aligned} u_1(x,0) &= \sum_{n=1}^{\infty} c_n \sin nx, \\ \sum_{n=1}^{\infty} |c_n|^2 n^{2(k+s)-1} &\leq C\|u\|^2_{k+s,\Omega}. \end{aligned}$$

Thus, the function u_1 is given by

$$u_1(x,y) = \sum_{n=1}^{\infty} c_n \sin nx \frac{\sinh n(a-y)}{\sinh na}.$$

Noting that the functions $\{\sin nx \sinh n(a-y)\}$ are orthogonal on Ω and that the functions $\sinh n(a-y)/\sinh na$ decay very fast away from $y = 0$ we may infer the claim (28) by similar reasoning as in the proof of Proposition 2.15, viz., splitting the sum into two parts. □

If the function u to be approximated satisfies certain boundary conditions, it may be of interest to approximate with polynomials which also satisfy these boundary conditions.

Proposition 2.18. *Assume the same hypotheses of Proposition 2.17. Let Γ_1 be one of the four edges of Ω. Assume additionally that u satisfies $u|_{\Gamma_1} = 0$ ($\partial_n u|_{\Gamma_1} = 0$). Then the harmonic polynomials u_p in Proposition 2.17 may be chosen to be antisymmetric (symmetric) with respect to Γ_1 and assertion (28) holds with Ω_ϵ defined as*

$$\Omega_\epsilon = \{z \in \Omega \,|\, \mathrm{dist}(z, \partial\Omega \setminus \Gamma_1) > \epsilon\}.$$

Proof. We will only consider the case of u vanishing on Γ_1, the case of the $\partial_n u$ vanishing on Γ_1 being similar. Without loss of generality we may assume that $\Gamma_1 = (0,\pi) \times \{0\}$ and that $\Omega = (0,\pi) \times (0,a)$. By the reflection principle, the antisymmetric extension of u across Γ_1 is harmonic on $\tilde{\Omega} = (0,\pi) \times (-a,a)$ and is in $H^{k+s}(\tilde{\Omega})$. Hence Proposition 2.17 is applicable with the rectangle $\tilde{\Omega}$. Let u_p be the harmonic polynomials given by Proposition 2.17. Now, the harmonic polynomials $\tilde{u}_p(x,y) = -u_p(x,-y)$ satisfy the same error estimates as the functions u_p due to the antisymmetry of u. Hence, the average $(u_p + \tilde{u}_p)/2$ also satisfies the same error estimates and, additionally, is antisymmetric with respect to Γ_1. □

Let us now analyze the case of singular functions as they arise in corner singularities, that is, the case considered in Theorem 2.11. Since the function f is assumed to be holomorphic on Ω_0 which approaches Ω only in the point z_0 (the singularity), we may expect to get improved rates on compact subsets as soon as we stay away from z_0. This is formalized in the following theorem.

Theorem 2.19. *Assume the same hypotheses as Theorem 2.11. Assume additionally that there is a line g passing through z_0 such that $\partial\Omega \cap g = \{z_0\}$ (i.e., Ω is completely contained in one of the half planes determined by g). Then for each $K > 0$ there are polynomials (P_p) of degree $p \geq k+s-1$ such that for all $\delta > 0$*

$$|f - P_p|_{j,\Omega\setminus B_\delta(z_0)} \leq C_K d^{k+s-j} \left[\left(\frac{\ln p}{p}\right)^{\lambda(k+s-j)} e^{-\gamma_K p\delta/d} + p^{-K} \right] |f^{(k)}|_{s,\Omega_0}$$

for $j = 0,\ldots,k$ and $d = \operatorname{diam}(\Omega_0)$. The constants C_K, $\gamma_K > 0$ depend only on K, k, s, and the shapes of Ω, Ω_0.

Remark 2.20 If we set $\delta = 0$ in Theorem 2.19 we get the result of Theorem 2.11.

Proof. We may assume that $d = 1$. The general case follows from a scaling argument. The proof follows along the lines of the proof of Theorem 2.11. We will therefore merely delineate the main steps. Without loss of generality we may assume that $g = \{iy \,|\, y \in \mathbb{R}\}$, $z_0 = 0$ and $\Omega \subset \{z \in \mathbb{C} \,|\, \mathrm{Re}\, z > 0\}$. The essential idea of the proof of Theorem 2.11 is to produce an approximation T_k of f by an appropriate "shift" by ϵ. In order to get approximations which are better on compact subsets, we have to replace this "uniform" shift by a highly non–uniform one. In fact, we consider

$$\begin{aligned} T_k(z) &:= \sum_{n=0}^{k} \frac{1}{n!} f^{(n)}(z + \epsilon\zeta(z))(-\epsilon\zeta(z))^n, \\ \zeta(z) &= e^{-\kappa z/\epsilon}, \end{aligned}$$

where the parameter $\kappa > 0$ is chosen so small that the following two properties are satisfied:

1. There is $C > 0$ such that for $\epsilon := Ch^\lambda$

$$\{z + \epsilon\zeta(z) \,|\, z \in \mathrm{Int}(L_{4h})\} \subset \Omega_0.$$

2. There are ϵ_0, $\overline{\kappa} > 0$, such that for all $0 < \epsilon < \epsilon_0$

$$\cup_{z\in\Omega} B_{\overline{\kappa}\epsilon}\left(z + \epsilon\zeta(z)\right) \subset \Omega_0.$$

A careful analysis based on distinguishing the cases $|z| > \epsilon$ and $|z| \le \epsilon$ shows that these two conditions can be meet for sufficiently small $\kappa > 0$.
Taylor's formula for the remainder gives

$$f(z) - T_k(z) = \frac{1}{k!}\left(-\epsilon\zeta(z)\right)^{k+1}\int_{\tau=0}^{1} f^{(k+1)}(z + \tau\epsilon\zeta(z))\tau^k d\tau.$$

Observe now that

$$|\zeta^{(j)}(z)| \le C_j\epsilon^{-j}e^{-\gamma\delta/\epsilon} \qquad \forall z \in \Omega \setminus B_\delta(0), \qquad j \in \mathbb{N}_0,$$

for some $\gamma > 0$. This estimate together with the treatment of Taylor's remainder formula analogous to the proof of Theorem 2.11 yields

$$|f - T_k|_{j,\Omega\setminus B_\delta(0)} \le C\left(\epsilon e^{-\gamma\delta/\epsilon}\right)^{k+s-j}|f^{(k)}|_{s,\Omega_0}, \qquad j = 0, \dots, k.$$

On the other hand, according to Corollary 2.7 the function T_k can be approximated by polynomials such that

$$|T_k - P_p|_{j,\Omega} \le Ch^{-\beta}(1+h)^{-p}\|T_k\|_{0,\mathrm{Int}(L_{4h})}, \qquad j = 0, \dots, k.$$

Combining these two estimates and observing that $\epsilon = Ch^\lambda$, we arrive at

$$|f - P_p|_{j,\Omega} \le C\left[\left(h^\lambda e^{-\gamma\delta/h^\lambda}\right)^{k+s-j} + h^{-\beta}(1+h)^{-p}\right]\|f\|_{k+s,\Omega_0} \qquad j = 0, \dots, k. \qquad (29)$$

Choosing for $K > 0$ the parameter $h := K(\ln p)/p$, inserting this choice in (29), and using the fact that $\lambda > 1$ allows us to conclude the argument. An application of Lemma 2.5 replaces the full norm on the right hand side by the desired semi norm. □

Remark 2.21 In the proof of Theorem 2.19 we ultimately made the choice $h = K(\ln p)/p$. This leads to the term p^{-K}, algebraic of any desired order. Other choices may lead to sharper estimates. For example, choosing $h = Kp^{-(1-\mu)}\ln p$ for some $\mu \in [0, 1)$ allows us to get sharper estimates on the second term at the expense of the first one.

2.6 Numerical Example

The aim of the following numerical experiments is to illustrate the results of Theorems 2.9, 2.11 and to show numerically that the rates obtained there are essentially sharp. To that end, we consider the approximation of the harmonic functions $u_\alpha = \mathrm{Im}\, z^\alpha$ ($\alpha = 1/2$ and $\alpha = 3/2$) by harmonic polynomials of degree p on the sectors $S(\omega)$ of aperture ω at the origin given by

$$S(\omega) := \{z \in \mathbb{C} \mid |z| < 1 \text{ and } |\arg z| < \omega/2\}, \quad 0 < \omega < 2\pi.$$

Table 1: Theoretical and Numerical rates for various angles ω and values α

ω	$\pi/8$	$\pi/4$	$\pi/2$	$2\pi/3$	π	$3/2\pi$	$5/3\pi$
$\lambda = (2\pi - \omega)/\pi$	15/8	7/4	3/2	4/3	1	1/2	1/3
$\lambda\alpha$ for $\alpha = 1/2$	0.9375	0.875	0.75	0.667	0.5	0.25	0.1667
numerical rate; $\alpha = 1/2$	0.9313	0.8692	0.7450	0.6622	0.4967	0.2454	0.1565
$\lambda\alpha$; $\alpha = 3/2$	2.8125	2.625	2.25	2	1.5	0.75	0.5
numerical rate; $\alpha = 3/2$	2.7942	2.6080	2.2355	1.9872	1.4905	0.7428	0.4893

In our numerical examples we focus on the effect of ω on the following best approximation problem in the H^1 semi norm (the "energy norm"):

$$E(\omega, \alpha, p) := \min\{|\nabla(u_\alpha - v_p)|_{L^2(S(\omega))} \mid \ v_p \text{ harmonic polynomial of degree } p\}. \qquad (30)$$

For every ω, the functions u_α are in $H^{\alpha+1-\epsilon}(S(\omega))$ ($\epsilon > 0$ arbitrary). Theorems 2.9, 2.11 therefore yield bounds of the form $E(\omega, \alpha, p) \leq Cp^{-\lambda\alpha+\epsilon}$ with $\lambda = (2\pi - \omega)/\pi$. The computational results are depicted in Figs. 2 and 3 where we plot $E^2(\omega, \alpha, p)$ (the "energy") versus the polynomial degree p. As predicted by Theorem 2.9, the rate of the approximation deteriorates for $\omega \to 2\pi$, and it improves as the aperture $\omega \to 0$, the latter being in agreement with the results of Theorem 2.11. A quantitative analysis of these observations can be found in Table 1: For our choices of apertures ω we compare the bounds $Cp^{-\lambda\alpha}$ of Theorems 2.9, 2.11 with the rates of convergence observed numerically in Figs. 2, 3. We see that the numerically observed rates of convergence are very close to the rates predicted by Theorems 2.9, 2.11, which shows that the rates of convergence presented in these two theorems are essentially the best possible ones.

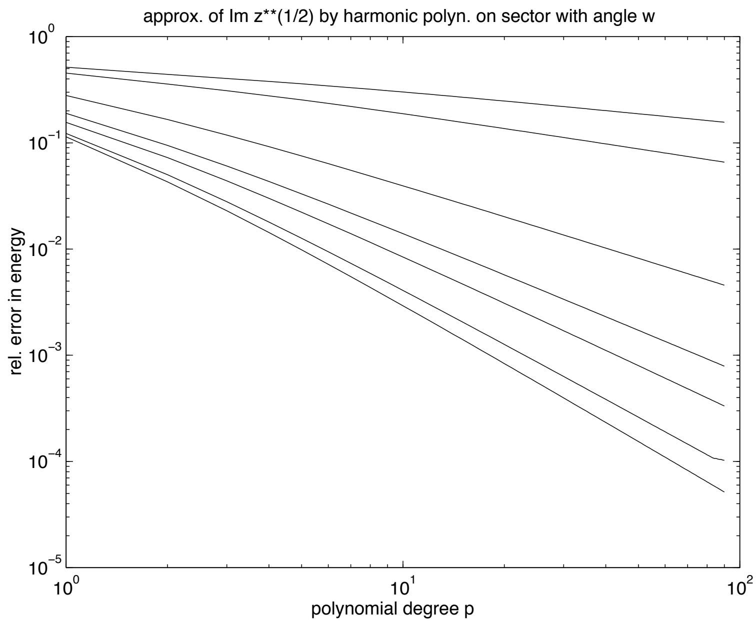

Figure 2: Approximation of $\operatorname{Im} z^{1/2}$ on $S(\omega)$ for $\omega = \pi/8$, $\pi/4$, $\pi/2$, $2\pi/3$, π, $3/2\pi$, $5/3\pi$ (in ascending order)

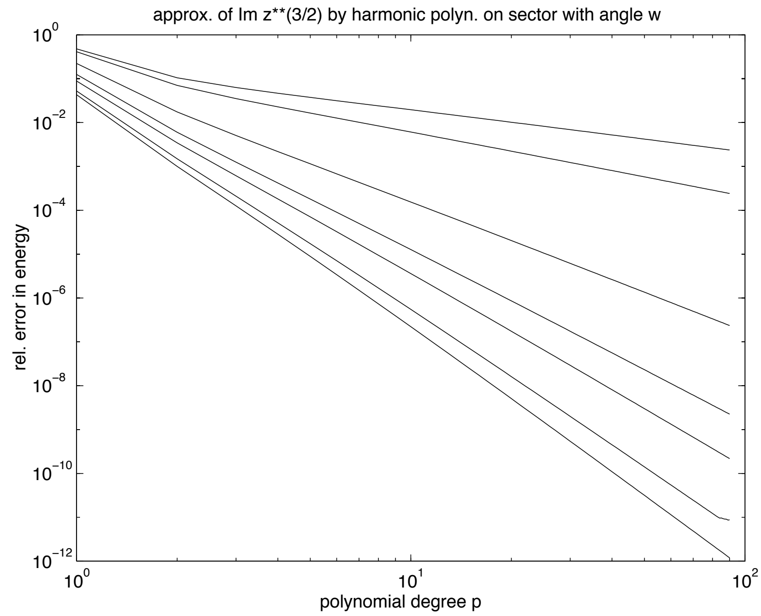

Figure 3: Approximation of $\operatorname{Im} z^{3/2}$ on $S(\omega)$ for $\omega = \pi/8$, $\pi/4$, $\pi/2$, $2\pi/3$, π, $3/2\pi$, $5/3\pi$ (in ascending order)

3 Generalized Harmonic Polynomials

3.1 Preliminaries

We consider now the general elliptic equation

$$Lu := -\Delta u + a(x,y)u_x + b(x,y)u_y + c(x,y)u = 0 \quad \text{on } D \subset \mathbb{R}^2 \tag{31}$$

where the functions a, b, and c are real analytic on D. We assume that $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain.

S. Bergman and I.N. Vekua derived independently an operator ReV, the so-called Bergman-Vekua operator, that maps holomorphic functions (normalized to be real in an arbitrary point) onto solutions of (31), [1, 26]. In fact, this operator is injective and onto. We show here that this operator is additionally bicontinuous (i.e., it is continuous and its inverse is also continuous) with respect to Sobolev norms $\|\cdot\|_{H^k}$.

In Section 2, we analyzed the approximation of holomorphic functions $f \in H^k$ by complex polynomials. The bicontinuity result of the Bergman-Vekua operator ReV implies that for each polynomial approximation result of Section 2, an analogous result for solutions $u \in H^k$ of (31) can be formulated for generalized harmonic polynomials, i.e., the functions ReV(1), ReV(z), ReV(iz), ReV(z^2)....

Before we embark on the proof the bicontinuity result, let us comment on the difference between the theories of Bergman and Vekua. Bergman constructed a series representation of the operator ReV directly from the data a, b, c. His approach therefore leads immediately to concrete realizations of the operator ReV, [2, 3, 22] (see also the discussion on localized versions of ReV in Section 3.3). In contrast to Bergman's definition of the operator ReV, Vekua introduced ReV in terms of the Riemann function G, which is the solution of an appropriate auxiliary problem. This abstract definition of the Bergman-Vekua operator is more suitable for our analysis, and we will use it in the following exposition. Nevertheless, the existence proof for G given by Vekua is constructive as it identifies G as the solution of an appropriate Volterra integral equation which can be solved by Picard iterations. This procedure may be imitated for constructing good approximations of $\mathrm{ReV}(z^k)$, $\mathrm{ReV}(iz^k)$ (see also Section 3.3).

Remark 3.1 The theory of Bergman and Vekua is restricted to elliptic equations in two dimensions. However, representation formulas for solutions of elliptic equations in three variables are available in terms of two holomorphic functions, [4, 5, 9].

Remark 3.2 We concentrate in the present section on the approximation properties of generalized harmonic polynomials, the images of (complex) polynomials under the Bergman-Vekua operator. However, it is worth noting that the Bergman-Vekua operator maps any system of functions which are dense in the set of holomorphic functions (in some Sobolev norm) onto a dense set of solutions of (31). It should be mentioned in this context that approaches for the construction of complete systems which are not based on the Bergman-Vekua operator have been been proposed in the literature. Typically, these approaches rely on the knowledge of a Green's function, [10].

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via $z = x + iy$ and introduce the two differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. Let $\mathcal{D} \subset \mathbb{C}$ be such that $D \cup \partial D \subset \mathcal{D}$. We assume that the real analytic functions a, b, and c permit analytic continuations to $\mathcal{D} \times \overline{\mathcal{D}}$ of the form

$$\begin{aligned} A(z,\zeta) &= \frac{1}{4}\left(a\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big) + ib\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big)\right), \\ B(z,\zeta) &= \frac{1}{4}\left(a\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big) - ib\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big)\right), \\ C(z,\zeta) &= \frac{1}{4}c\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big) \end{aligned}$$

such that the functions A, B, C are analytic on $\mathcal{D} \times \overline{\mathcal{D}}$. By [26], we have

Theorem 3.3 (Vekua). *The solution u of* (31) *can be continued analytically to a solution*

$$U(z,\zeta) = u\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big), \qquad (z,\zeta) \in D \times \overline{D},$$

of the equation

$$\mathcal{L}U \equiv \frac{\partial^2}{\partial z \partial \zeta}U + A(z,\zeta)\frac{\partial U}{\partial z} + B(z,\zeta)\frac{\partial U}{\partial \zeta} + C(z,\zeta)U = 0.$$

In order to define the operator ReV which is used for the characterization of the solutions of (31), we introduce the notion of the *Riemann function* G associated with the operator $\mathcal{L}$.

Definition 3.4. *The* Riemann *function*

$$\begin{aligned} G : \mathcal{D} \times \overline{\mathcal{D}} \times \mathcal{D} \times \overline{\mathcal{D}} &\to \mathbb{C} \\ (z,\zeta,t,\tau) &\mapsto G(z,\zeta,t,\tau) \end{aligned}$$

solves the problem

$$\begin{aligned}
\mathcal{L}^*G \equiv \frac{\partial^2 G}{\partial z\partial\zeta} - \frac{\partial AG}{\partial z} - \frac{\partial BG}{\partial \zeta} + CG &= 0, \\
G(t,\tau,t,\tau) &= 1, \qquad (t,\tau)\in\mathcal{D}\times\overline{\mathcal{D}}, \\
\frac{\partial}{\partial z}G(z,\tau,t,\tau) - B(z,\tau)G(z,\tau,t,\tau) &= 0, \qquad z\in\mathcal{D},\quad t,\tau\in\overline{\mathcal{D}}, \\
\frac{\partial}{\partial \zeta}G(t,\zeta,t,\tau) - A(t,\zeta)G(t,\zeta,t,\tau) &= 0, \qquad t\in\mathcal{D},\quad \zeta,\tau\in\overline{\mathcal{D}}.
\end{aligned}$$

The Riemann function G can be written as the solution of a Volterra integral equation which may be solved by Picard iterations. It follows that G is continuous on $\mathcal{D}\times\overline{\mathcal{D}}\times\mathcal{D}\times\overline{\mathcal{D}}$ and that it is holomorphic in each of the four variables ([26] for the details).

Example 3.5 For Laplace's equation, i.e., $a=b=c=0$, the Riemann function is $G\equiv 1$. For Helmholtz's equation, i.e., $a=b=0$, $c=k^2$, $k>0$, the Riemann function is $G(z,\zeta,t,\tau) = J_0\left(k\sqrt{(z-t)(\zeta-\tau)}\right)$ where J_0 is the Bessel function of the first kind of order 0. Similarly, in the case $a=b=0$, $c=-k^2$, $G(z,\zeta,t,\tau)=I_0\left(k\sqrt{(z-t)(\zeta-\tau)}\right)$, where I_0 is the modified Bessel functions of order 0.

Definition 3.6. *For $z_0\in\mathcal{D}$ introduce the integral operator $I[\varphi,z_0]$ acting on holomorphic functions φ by*

$$\begin{aligned}
I[\varphi,z_0](z,\zeta) &= \frac{1}{2}\Big(G(z,\overline{z}_0,z,\zeta)\varphi(z) + \int_{z_0}^{z}\varphi(t)H(t,\overline{z}_0,z,\zeta)\,dt + \\
&\qquad G(z_0,\zeta,z,\zeta)\overline{\varphi}(\overline{z}) + \int_{\overline{z}_0}^{\zeta}\overline{\varphi}(\overline{\tau})H^*(z_0,\tau,z,\zeta)\,d\tau\Big).
\end{aligned}$$

Here, the integrals are path-independent and the kernels H, H^* are given by

$$\begin{aligned}
H(t,\tau,z,\zeta) &= B(t,\tau)G(t,\tau,z,\zeta) - \frac{\partial G}{\partial t}(t,\tau,z,\zeta), \\
H^*(t,\tau,z,\zeta) &= A(t,\tau)G(t,\tau,z,\zeta) - \frac{\partial G}{\partial \tau}(t,\tau,z,\zeta).
\end{aligned}$$

Restricting $\zeta=\overline{z}$ in the function $I[\varphi,z_0](z,\zeta)$, we obtain (using the assumption that the coefficients a, b, c are real)

$$\mathrm{ReV}[\varphi,z_0](z) := I[\varphi,z_0](z,\overline{z}) = \mathrm{Re}\left\{G(z,\overline{z}_0,z,\overline{z})\varphi(z) + \int_{z_0}^{z}\varphi(t)H(t,\overline{z}_0,z,\overline{z})\,dt\right\}. \qquad (32)$$

Theorem 3.7 (Vekua)**.** *Let $D\subset\mathbb{C}$ be a simply connected, bounded Lipschitz domain. Fix $z_0\in D$ and let u be a solution of* (31)*. Then there is a unique holomorphic φ with $\varphi(z_0)$ real such that*

$$\begin{aligned}
u(x,y) &= \mathrm{ReV}[\varphi,z_0](z), \qquad z=x+iy, \\
U(z,\zeta) &= I[\varphi,z_0](z,\zeta), \qquad (z,\zeta)\in D\times\overline{D}.
\end{aligned}$$

Moreover, the function φ can be expressed in terms of U by

$$\varphi(z) = 2U(z,\overline{z}_0) - U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0).$$

Theorem 3.7 states essentially that the map I is a bijection between the holomorphic functions (which are real at the point z_0) and the (real) solutions of (31). The next section shows that this operator (and its inverse) are continuous in Sobolev norms.

3.2 Regularity of u vs. Regularity of φ

In [7], it was shown that if a solution u of (31) is in a Hölder space, then the holomorphic function φ corresponding to u by Theorem 3.7 is in the same Hölder space and *vice versa*. Theorems 3.8, 3.11 show that an analogous result holds true in a Sobolev space setting.

Theorem 3.8. *Let $D \subset \mathbb{C}$ be a simply connected, bounded Lipschitz domain. Fix $z_0 \in D$, $k \geq 1$. Then there is $C(L, z_0, D, k) > 0$ depending only on z_0, the differential operator L, the domain D, and k such that the following holds. Let $u \in H^k(D)$ satisfy* (31) *and let φ be the holomorphic function corresponding to u by Theorem 3.7. Then*

$$\|\varphi\|_{H^k(D)} \leq C(L, z_0, D, k)\|u\|_{H^k(D)}.$$

In order to prove Theorem 3.8, we need a few technical lemmas.

Lemma 3.9 (Interior estimates)**.** *Let D_1 be star shaped with respect to z_1 and denote $d =$* diam (D_1). *Assume that there is $\alpha \in (0, 1/2)$ such that $B_{2\alpha d}(z_1) \subset D_1 \subset B_d(z_1)$. Then there is $C(\alpha, L)$ depending only on α and the coefficients of L such that every $u \in H^1(D_1)$ with $Lu = 0$ on D_1 there holds*

$$\|u\|_{L^\infty(B_{\alpha d}(z_1))} + \|\nabla u\|_{L^\infty(B_{\alpha d}(z_1))} \leq C(\alpha, L)(1 + d^{-1})\|u\|_{H^1(D_1)}, \tag{33}$$

$$\|D^2 u\|_{L^\infty(B_{\alpha d}(z_1))} \leq C(\alpha, L)(1 + d^{-2})\|u\|_{H^1(D_1)}, \tag{34}$$

$$\int_{s=0}^{1} \|u\Big(s(z - z_1) + z_1\Big)\|^2_{L^2(D_1)}\, ds \leq C(\alpha, L)\left(d^2\|u\|^2_{L^\infty(B_{\alpha d}(z_1))} + \|u\|^2_{L^2(D_1)}\right), \tag{35}$$

$$\int_{s=0}^{1} \|\nabla u\Big(s(z - z_1) + z_1\Big)\|^2_{L^2(D_1)}\, ds \leq C(\alpha, L)(1 + d)^2\|u\|^2_{H^1(D_1)}. \tag{36}$$

Proof. The first two estimates are standard. Without loss of generality, we may assume that $z_1 = 0$. For (35), we split the integral into two parts by integrating from 0 to α and from α to 1. We have

$$\int_{s=0}^{\alpha} \|u(sz)\|^2_{L^2(D_1)}\, ds \leq \text{area}\,(D_1)\|u\|^2_{L^\infty(B_{\alpha d}(z_1))}.$$

For the second part, the change of variables $z' = sz$ implies

$$\int_{s=\alpha}^{1} \|u(sz)\|^2_{L^2(D_1)}\, ds = \int_{s=\alpha}^{1} \|u(z)\|^2_{L^2(sD_1)} \frac{ds}{s^2} \leq \frac{1-\alpha}{\alpha^2}\|u\|^2_{L^2(D_1)},$$

and (35) follows. For the last estimate, we apply (35) to the components of ∇u to arrive at

$$\int_{s=0}^{1} \|\nabla u(sz)\|^2_{L^2(D_1)}\, ds \leq C\left(d^2\|\nabla u\|^2_{L^\infty(B_{\alpha d}(z_1))} + \|\nabla u\|^2_{L^2(D_1)}\right)$$

and (36) follows by (33). □

Lemma 3.10. *Let D_1 be star shaped with respect to z_1, $d = \operatorname{diam}(D_1) \le R$ for some $R > 0$ and $\alpha \in (0, 1/2)$ such that $B_{\alpha d}(z_1) \subset D_1 \subset B_d(z_1)$. Let $M : D_1 \times D_1 \times \overline{D}_1 \to \mathbb{C}$ be Lipschitz continuous with Lipschitz constant M', i.e., $|M(z, t_1, \tau_1) - M(\zeta, t_2, \tau_2)| \le M'(|z - \zeta| + |t_1 - t_2| + |\tau_1 - \tau_2|)$. Let $u \in H^1(D_1)$ satisfy $Lu = 0$ and let U be the extension of u as in Theorem 3.7. Upon setting $\|M\|_{W^{1,\infty}} := \|M\|_{L^\infty} + dM'$ there is $C_1 = C_1(\alpha, L, R)$ such that*

$$\left\|\int_{z_1}^{z} U(t,\overline{t})M(z,t,\overline{t})\,dt\right\|_{L^2(D_1)} + \left\|\int_{z_1}^{z} \partial_1 U(t,\overline{t})M(z,t,\overline{t})\,dt\right\|_{L^2(D_1)} \le C_1\|M\|_{L^\infty} d\|u\|_{H^1(D_1)}.$$

If furthermore $u \in H^{1+k}(D_1)$, $k \in (0,1)$, there is $C_2 = C_2(\alpha, L, k, R)$ such that

$$\begin{aligned}
\left|\int_{z_1}^{z} U(t,\overline{t})M(z,t,\overline{t})\,dt\right|_{H^k(D_1)} &\le C_2\|M\|_{W^{1,\infty}}\left(d|u|_{H^k(D_1)} + d^{1-k}\|u\|_{H^1(D_1)}\right),\\
\left|\int_{z_1}^{z} \partial_1 U(t,\overline{t})M(z,t,\overline{t})\,dt\right|_{H^k(D_1)} &\le C_2\|M\|_{W^{1,\infty}}\left(d|\nabla u|_{H^k(D_1)} + d^{1-k}\|u\|_{H^1(D_1)}\right).
\end{aligned}$$

In all cases the path of integration is the straight line connecting z_1 with z.

Proof. The proof is very similar to our procedure in Lemma 2.3. We will therefore merely sketch the proof of the last estimate. Without loss of generality, we may assume that $z_1 = 0$. We have to estimate

$$\int_{(z,\zeta)\in D_1\times D_1} |z-\zeta|^{-2-2k}\left|\int_{s=0}^{1} \partial_1 U(sz, s\overline{z})zM(z, sz, s\overline{z}) - \partial_1 U(s\zeta, s\overline{\zeta})\zeta M(\zeta, s\zeta, s\overline{\zeta})\right|^2.$$

Hence, after using Schwarz's inequality we have to bound

$$\begin{aligned}
I_1 &:= \int_{s=0}^{1}\int_{(z,\zeta)\in D_1\times D_1} |z-\zeta|^{-2-2k}|\partial_1 U(sz,s\overline{z}) - \partial_1 U(s\zeta, s\overline{\zeta})|^2 |zM(z,sz,s\overline{z})|^2\,ds,\\
I_2 &:= \int_{s=0}^{1}\int_{(z,\zeta)\in D_1\times D_1} |\partial_1 U(s\zeta, s\overline{\zeta})|^2|z-\zeta|^{-2-2k}|zM(z,sz,s\overline{z}) - \zeta M(\zeta, s\zeta, s\overline{\zeta})|^2\,ds.
\end{aligned}$$

Because $\partial_1 U(z,\overline{z}) = (u_x(\operatorname{Re} z, \operatorname{Im} z) - iu_y(\operatorname{Re} z, \operatorname{Im} z))/2$, we obtain for I_1:

$$I_1 \le Cd^2\|M\|^2_{L^\infty}\int_{s=0}^{1}\int_{(z,\zeta)} \frac{|\nabla u(sz) - \nabla u(s\zeta)|^2}{|z-\zeta|^{2+2k}}\,ds.$$

We split the integration over s into an integral from 0 to α and from α to 1. We observe for $s \in (0,\alpha)$ that $sD_1 \subset B_{\alpha d}(z_1) \subset D_1$ and thus $|\nabla u(sz) - \nabla u(s\zeta)| \le s|z-\zeta|\|D^2 u\|_{L^\infty(B_{\alpha d}(z_1))}$. For $s \in (\alpha, 1)$ we use the change of variables $z' = sz$, $\zeta' = s\zeta$ and the definition of the H^k semi norm to get

$$I_1 \le C(\alpha,k)d^2\|M\|^2_{L^\infty}\left[d^{2-2k}d^2\|D^2\|^2_{L^\infty(B_{\alpha d}(z_1))} + |\nabla u|^2_{H^k(D_1)}\right]$$

and thus I_1 can be bounded in the desired fashion by appealing to Lemma 2.3. For I_2, we use the Lipschitz continuity of M to get

$$|zM(z,sz,s\overline{z}) - \zeta M(\zeta, s\zeta, s\overline{\zeta})| \le C\|M\|_{W^{1,\infty}}|z-\zeta|.$$

Splitting again the s integration into an integral from 0 to α and from α to 1 allows us to conclude

$$I_2 \le Cd^{2-2k}\|M\|^2_{W^{1,\infty}}\left[d^2\|\nabla u\|^2_{L^\infty(B_{\alpha d}(z_1))} + \|\nabla u\|^2_{L^2(D_1)}\right]$$

and hence I_2 can also be bounded in the desired fashion using Lemma 2.3. □

Proof of Theorem 3.8: By [12], the extended function $U(z,\zeta)$ satisfies

$$U(z,\zeta) = U(\overline{\zeta},\zeta)G(\overline{\zeta},\zeta,z,\zeta)+ \int_{\overline{\zeta}}^{z} U(t,\overline{t})\left[\partial_2 G(t,\overline{t},z,\zeta) - A(t,\overline{t})G(t,\overline{t},z,\zeta)\right] d\overline{t} + \left[B(t,\overline{t})U(t,\overline{t}) + \partial_1 U(t,\overline{t})\right] G(t,\overline{t},z,\zeta)\,dt$$

where the integral is in fact path independent. Together with the formula for φ (cf. Theorem 3.7)

$$\varphi(z) = 2U(z,\overline{z}_0) - U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0)$$

we arrive at the representation

$$\begin{aligned}\varphi(z) = {} & U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0)+ \\ & 2\int_{z_0}^{z} U(t,\overline{t})\left[\partial_2 G(t,\overline{t},z,\overline{z}_0) - A(t,\overline{t})G(t,\overline{t},z,\overline{z}_0)\right] d\overline{t} + \\ & \left[B(t,\overline{t})U(t,\overline{t}) + \partial_1 U(t,\overline{t})\right] G(t,\overline{t},z,\overline{z}_0)\,dt.\end{aligned} \tag{37}$$

Let us first bound $\|\varphi\|_{L^2(D)}$. As D is assumed to be a bounded Lipschitz domain, there are finitely many points $(z_i)_{i=1}^N \subset D$ and corresponding domains $(D_i)_{i=1}^N \subset D$ star shaped with respect to the points z_i such that $\cup_{i=1}^N D_i = D$. For each i, we choose a path $p_i \subset D$ connecting z_0 with z_i. Using the path independence of (37), we write for φ on D_i:

$$\varphi(z) = U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0) + 2\int_{z_0}^{z_i} \quad +2\int_{z_i}^{z}.$$

Let us first consider the second integral. It is of the form considered in Lemma 3.10 and by the assumption $D \cup \partial D \subset \mathcal{D}$, the assumptions of Lemma 3.10 are satisfied. Hence, there is $C > 0$ with $\|\varphi\|_{L^2(D_i)} \le C\|u\|_{H^1(D_i)}$ where we exploited that $U(t,\overline{t}) = u(\mathrm{Re}\ t, \mathrm{Im}\ t)$, $\partial_1 U(t,\overline{t}) = (u_x(\mathrm{Re}\,t,\mathrm{Im}\,t) - iu_y(\mathrm{Re}\,t,\mathrm{Im}\,t))/2$. For the remaining terms, the interior estimates of Lemma 3.9 yield the existence of $C_i > 0$ such that $\|u\|_{L^\infty(p_i)}, \|\nabla u\|_{L^\infty(p_i)} \le C_i\|u\|_{H^1(D)}$. A compactness argument concludes the proof of the L^2 bound of φ. Let us now outline how the proof for higher derivatives of φ proceeds. Let $k = [k]+s$, $[k] \in \mathbb{N}$, $s \in [0,1)$. Differentiating (37) $[k]$ times, reveals that $\varphi^{([k])}(z)$ is of the form

$$\begin{aligned}\frac{d^{[k]}}{dz^{[k]}}\varphi(z) = {} & U(z_0,\overline{z}_0)\partial_3^{[k]}G(z_0,\overline{z}_0,z,\overline{z}_0) + \sum_{n=0}^{[k]} \partial_1^n U(z,\overline{z})C_n(z,\overline{z},z_0,\overline{z}_0)+ \\ & 2\int_{z_0}^{z} U(t,\overline{t})\left[\partial_3^{[k]}\partial_2 G(t,\overline{t},z,\overline{z}_0) - A(t,\overline{t})\partial_3^{[k]}G(t,\overline{t},z,\overline{z}_0)\right] d\overline{t} + \\ & \left[B(t,\overline{t})U(t,\overline{t}) + \partial_1 U(t,\overline{t})\right] \partial_3^{[k]}G(t,\overline{t},z,\overline{z}_0)\,dt\end{aligned}$$

for some analytic functions C_n which depend only on the analytic functions G, A, B, C and their derivatives up to order $[k]$. Hence, similar reasoning as above allows us to bound $\|\varphi\|_{H^k(D)}$ by $\|u\|_{H^k(D)}$.

Theorem 3.11. *Let $D \subset \mathbb{C}$ be a bounded, simply connected Lipschitz domain and fix $z_0 \in D$, $k \geq 0$. Then there is $C(z_0, L, D, k)$ such that for all $\varphi \in H^k(D)$ holomorphic on D,*

$$\| \operatorname{ReV}[\varphi, z_0]\|_{H^k(D)} \leq C(z_0, L, D, k)\|\varphi\|_{H^k(D)}.$$

Proof. The proof is very similar to the procedure of the proof of Theorem 3.8. The functions H, G appearing in the definition of ReV satisfy all the assumptions of Lemma 3.10 and the necessary interior estimates are provided by Lemma 2.3. □

3.3 Localization of the Bergman-Vekua Operator

In the preceding section, we fixed z_0 and were able to show that the Bergman-Vekua operator ReV is a bijection between the set of holomorphic functions (normalized to be real at z_0) and the solutions u of $Lu = 0$. Moreover, this operator is bicontinuous in Sobolev norms H^k, $k \geq 1$.

For fixed z_0, the operator ReV is a non-local operator; it is only a local operator in a neighborhood of z_0. For practical purposes, it is important to have local versions of the operator ReV as it not known explicitly in general. As mentioned in the introduction to this section, there are two approaches to approximating ReV. In Bergman's approach, ReV admits series expansions about the point z_0 which can be expected to converge rapidly in the neighborhood of z_0 (cf. [2, 3, 22] for global implementations ReV). Vekua showed the existence of G be means of Picard iterations of a Volterra integral equation which again can be expected to converge fast in the neighborhood of z_0.

The ensuing corollary shows that local versions of Theorems 3.8, 3.11 hold which eliminate the explicit dependence of the constants of Theorems 3.8, 3.11 on the point z_0.

Corollary 3.12. *Let D be a bounded domain, and let $D_0 \subset D$ be star shaped with respect to $z_0 \in D_0$, Denote $d = \operatorname{diam}(D_0) \leq \operatorname{diam}(D)$ and assume that there is $\alpha \in (0, 1/2)$ such that $B_{\alpha d}(z_0) \subset D_0 \subset B_d(z_0)$. Let $u \in H^k(D_0)$, $k \geq 1$ solve $Lu = 0$ on D_0 and let φ be such that $u = \operatorname{ReV}[\varphi, z_0]$. Then there are constants $C(\alpha, L, k)$ depending only on α, L and k such that*

$$\|\varphi\|_{H^k(D_0)} \leq C(\alpha, L, k)\|u\|_{H^k(D_0)}.$$

Furthermore, for every $k \geq 0$, there is $C(\alpha, L, k) > 0$ such that

$$\| \operatorname{ReV}[\varphi, z_0]\|_{H^k(D_0)} \leq C(\alpha, L, k)\|\varphi\|_{H^k(D_0)}$$

for all functions φ holomorphic on D_0

Proof. Follows immediately by tracking the dependence of the constants on the point z_0 in the proof of Theorem 3.8. □

3.4 Generalized Harmonic Polynomials

Definition 3.13 (Generalized Harmonic Polynomials)**.** *The set of* generalized harmonic polynomials *of degree p is given by*

$$\begin{aligned} G(p, z_0) := \operatorname{span}\{&\operatorname{ReV}[1, z_0], \operatorname{ReV}[z, z_0], \operatorname{ReV}[iz, z_0], \\ &\operatorname{ReV}[z^2, z_0], \operatorname{ReV}[iz^2, z_0], \ldots, \operatorname{ReV}[z^p, z_0], \operatorname{ReV}[iz^p, z_0]\}, \end{aligned} \tag{38}$$

and $\dim G(p, z_0) = 2p + 1$.

Example 3.14 For the case of Laplace's equation, $G \equiv 1$ by Example 3.5 and therefore the operator ReV reduces to taking the real part. $G(p, z_0)$ is therefore precisely the set of harmonic polynomials of degree p.
For the case of Helmholtz's equation, $a = b = 0$, $c = k^2$, a calculation shows that

$$\begin{aligned} \mathrm{ReV}[z^n, 0](x, y) &= n!(2/k)^n \cos(n\theta) J_n(kr), \\ \mathrm{ReV}[iz^n, 0](x, y) &= -n!(2/k)^n \sin(n\theta) J_n(kr), \end{aligned}$$

where we used polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The functions J_n are Bessel's functions of the first kind of order n.
Similarly, if $a = b = 0$, $c = -k^2$ above expressions for the generalized harmonic polynomials for Helmholtz's equation hold true with the Bessel functions J_n replaced with the modified Bessel functions I_n.

Remark 3.15 We mentioned at the outset of Section 2 that there are many systems of functions that are dense in the set of all holomorphic functions. Any such system generates by the continuity of the Bergman-Vekua operator ReV a dense system for the solutions of (31). For example, in the case of Helmholtz's equation, systems of plane waves are closely related to the holomorphic functions $\{e^{akz}\}$ where k is the wave number and the parameter $a \in \mathbb{C}$ is constrained to satisfy $|a| = 1$.

Theorem 3.16. *Let D be star shaped with respect to a ball and let the exterior angle of D be bounded from below by $\lambda\pi$ at each boundary point. Let $z_0 \in D$ be fixed and generalized harmonic polynomials be defined by (38). Assume that $u \in H^k(D)$, $k > 1$, satisfies (31). Then there are generalized harmonic polynomials $G_p \in G(p, z_0)$ of degree $p \geq k - 1$ such that*

$$\|u - G_p\|_{H^j(D)} \leq C(\mathrm{diam}(D))^{k-j} \left(\frac{\ln p}{p}\right)^{\lambda(k-j)} \|u\|_{H^k(D)}, \qquad j = 0, 1,$$

where $C > 0$ depends only on the shape of D, the relative position of z_0 within D, and the coefficients of the differential operator of (31).

Proof. For the fixed point $z_0 \in D$, let φ be the holomorphic function corresponding to u by Theorem 3.7. By Corollary 3.12, $\|\varphi\|_{H^k(D)} \leq C\|u\|_{H^k(D)}$. By Theorem 2.9 there are polynomials P_p of degree p such that

$$\|\varphi - P_p\|_{H^j(D)} \leq C(\mathrm{diam}(D))^{k-j} \left(\frac{\ln p}{p}\right)^{\lambda(k-j)} \|u\|_{H^k(D)}, \qquad j = 0, 1.$$

By the continuity of the operator $\mathrm{ReV}[\cdot, z_0]$ in Sobolev norms (Corollary 3.12) and the linearity of $\mathrm{ReV}[\cdot, z_0]$, we get with $u = \mathrm{ReV}[\varphi, z_0]$, $G_p := \mathrm{ReV}[P_p, z_0]$

$$\|u - G_p\|_{H^j(D)} \leq C(\mathrm{diam}(D))^{k-j} \left(\frac{\ln p}{p}\right)^{\lambda(k-j)} \|u\|_{H^k(D)}, \qquad j = 0, 1.$$

□

Remark 3.17 The bicontinuity of ReV allows us to obtain for each result of Section 2 an analogous result for the approximation with generalized harmonic polynomials. In particular, the analog of Theorem 2.11 holds. Furthermore, the results on super-approximability of Section 2.5 hold for generalized harmonic polynomials.

4 Numerical Example

In this section, we present an application of our approximation results for "operator-adapted" shape functions to Laplace's equation. An application to Helmholtz's equation will be presented in [18] (see also [16, 24]). We consider here the problem

$$\begin{aligned} -\Delta u &= 0 && \text{on } \Omega := (0,1)\times(0,1), \\ -\partial_n u &= g := \operatorname{Re}\left(\tfrac{1}{a^2+z^2} + \tfrac{1}{a^2-z^2}\right) && \text{on } \partial\Omega \qquad \text{where } a > 1. \end{aligned} \tag{39}$$

The solution of problem (39) is unique up to a constant. The weak formulation is given by: find $u \in H^1(\Omega)$ such that

$$B(u,v) := \int_\Omega \nabla u \cdot \nabla u \, dx = F(v) := \oint_{\partial\Omega} gv \, ds \qquad \forall v \in H^1(\Omega). \tag{40}$$

Again, the solution u is unique up to a constant which may be fixed imposing zero mean on u. Associated with this problem is the notion of an energy given by $B(u,u) = \|\nabla u\|^2_{L^2(\Omega)}$ and an "energy norm" being the square root of the energy, i.e., the H^1 semi norm. In the finite element method, a (finite dimensional) space $V_{FE} \subset H^1(\Omega)$ is chosen, and the FEM reads:

$$\text{find } u_{FE} \in V_{FE} \text{ such that } B(u_{FE}, v) = F(v) \qquad \forall v \in V_{FE}. \tag{41}$$

Céa's Lemma gives that

$$\|\nabla(u - u_{FE})\|_{L^2(\Omega)} = B(u - u_{FE}, u - u_{FE})^{1/2} = \inf_{v \in V_{FE}} \|\nabla(u - v)\|_{L^2(\Omega)}. \tag{42}$$

The quality of the finite element approximation u_{FE} is therefore completely determined by the approximability of the exact solution u in the space $V_{FE} \subset H^1(\Omega)$.
The approximation theory of Section 2 guarantees that, locally, harmonic polynomials have very good approximation properties for the approximation of the solution u of (39). The PUM, as presented in the Introduction allows us to construct a global, conforming ansatz space V_{PUM} from local spaces of harmonic polynomials with the aid of a partition of unity. One important class of examples of partition of unity that satisfy the conditions of Definition 1.1 is given by the classical finite element hat functions on shape-regular meshes (the shape-regularity implies that the conditions (1) on the partition of unity functions are satisfied). Our numerical examples are based on partitions of unity of that type.
Example 4.1 The classical bilinear finite element hat functions. Let $\Omega = (0,1)^2$, $n \in \mathbb{N}$ be given. Set $h = 1/n$ and subdivide Ω into n^2 squares. Denote (x_i, y_j), i, $j = 0, \ldots, n$ the nodes obtained in this way where $x_i = ih$, $y_j = jh$ and associate with each node the usual continuous, piecewise bilinear pyramid function $\varphi_{i,j}$, which takes the value 1 at node (x_i, y_j) and vanishes in all other nodes, i.e., is given by $\varphi_{i,j}(x,y) = \varphi\big((x - x_i)/h, (y - y_j)/h\big)$ with

$$\begin{aligned} &\varphi(x,y) = (1-x)(1-y) \qquad \text{for } (x,y) \in [0,1]\times[0,1] \\ &(1+x)(1-y) \qquad \text{for } (x,y) \in [-1,0]\times[0,1] \\ &(1+x)(1+y) \\ &\ \text{for } (x,y) \in [-1,0]\times[-1,0] \\ &(1-x)(1+y) \qquad \text{for } (x,y) \in [0,1]\times[-1,0] \\ &0 \qquad \text{elsewhere.} \end{aligned} \tag{43}$$

The patches, i.e., the supports of the functions φ_{ij}, are the closure of the sets $\Omega_{i,j} = \{(x,y) \mid |x - x_i| < h, |y - y_j| < h\}$, i, $j = 0, \ldots, n$.

The second example of a partition of unity is variation of the previous example where the partition of unity functions in the interior are kept and only the ones at the boundary are modified. For notational convenience, we will construct the partition of unity functions by a tensor product argument.

Example 4.2 First, construct a partition of unity in one dimension for $(0,1)$ as follows. Let $n \in \mathbb{N}$, $h = 1/n$, and denote $\widetilde{\varphi}_i(x)$ the usual piecewise linear hat functions associated with the nodes $x_i = ih$, $i = 0, \ldots, n$. Introduce now the partition of unity $(\varphi_i)_{i=1}^{n-1}$ by $\varphi_i = \widetilde{\varphi}_i$ if $2 \le i \le n-2$, $\varphi_1 = \widetilde{\varphi}_0 + \widetilde{\varphi}_1$, $\varphi_{n-1} = \widetilde{\varphi}_{n-1} + \widetilde{\varphi}_n$. For $\Omega = (0,1)^2$ define the two dimensional partition of unity $(\varphi_{ij})_{i,j=1}^{n-1}$ by $\varphi_{ij}(x,y) = \varphi_i(x)\varphi_j(y)$. Note that there are only $(n-1)^2$ partition of unity functions as opposed to the preceding example where there are $(n+1)^2$.

The ansatz space V_{FE} in (41) is taken as V_{PUM} of Definition 1.1 where the partition of unity is of the type described in Examples 4.1, 4.2, and the local approximation spaces V_j are chosen as spaces of harmonic polynomials of degree p, i.e., $V_j := \text{span}\,\{\text{Re}\,(x+iy)^n, \text{Im}\,(x+iy)^n, \mid n = 0, \ldots, p\}$. In the numerical examples, the parameter a of (39) is either $a = 1.05$ or $a = 1.5$. Hence, the approximation results of Section 2 apply. If the partition of unity is fixed and the size of the local spaces (i.e., the degree p of the harmonic polynomials) is increased, we may speak of the "p version" of the PUM and Theorem 2.6 together with Theorem 1.2 implies that

$$\inf_{v \in V_{PUM}} \|\nabla(u - v)\|_{L^2(\Omega)} \le C e^{-\sigma p}, \quad p = 0, 1, \ldots \tag{44}$$

where C, $\sigma > 0$ and independent of p. On the other hand, if the size of the local approximation spaces is fixed (i.e., the polynomial degree p is fixed) and the support of the partition of unity functions is varied, we may speak of an "h version" of the PUM. In this case, Theorem 2.9 together with Theorem 1.2 implies bounds of the form

$$\inf_{v \in V_{PUM}} \|\nabla(u - v)\|_{L^2(\Omega)} \le C h^p, \quad h = \frac{1}{n}, \tag{45}$$

with $C > 0$ independent of h.

Let us first consider the partition of unity of Example 4.1 and $a = 1.05$. In Fig. 4 the "p version" of the PUM is compared with two variants of the classical p version of the FEM, the spaces of tensor product polynomials Q_p and the trunk spaces (also called serendipity elements) Q'_p. The partition of unity is fixed ($n = 8$) and the polynomial degree p of the harmonic polynomials in the local spaces V_j is increased. The two classical p versions are based on the same 8×8 mesh. We see that the PUM, based on the operator adapted local approximation spaces, achieves the same accuracy as the two classical p versions with less degrees of freedom. The discrepancy between the methods increases as the accuracy requirement is increased. This is agreement with the approximation properties of polynomials and harmonic polynomials for this problem: In both cases we get exponential rates of convergence in terms of p; however, the number of degrees of freedom $N = O(p^2)$ for the classical p version and only $N = O(p)$ for the PUM based on harmonic polynomials. We refer to [15] for a detailed performance study concerning the three parameters p, n, and a.

In Figs. 5–7 the PUM is based on the partition of unity of Example 4.2. The numerical results of the "p version" of the PUM are presented in Figs. 5, 6. For $a = 1.05$ and each fixed

partition of unity ($n = 2, 4, 8, 16$), the degree p of the harmonic polynomials is increased. Fig. 6 confirms the exponential rate of convergence (in p) predicted by Theorem 2.6 (cf. (44)).
For the case $a = 1.5$, the results of the "h version" of the PUM are shown in Fig. 7. The local spaces are fixed as spaces of harmonic polynomials of degree p (p ranges from 0 to 4) and the support of the partition of unity functions is varied by increasing n from 2 to 32. In terms of degrees of freedom $N \sim pn^2$, the error bounds (45) are of the form $O(N^{-p/2})$ which are indeed obtained in Fig. 7.

Acknowledgement

Im deeply indebted to I. Babuška for suggesting this field of research and for his continual guidance and encouragement.

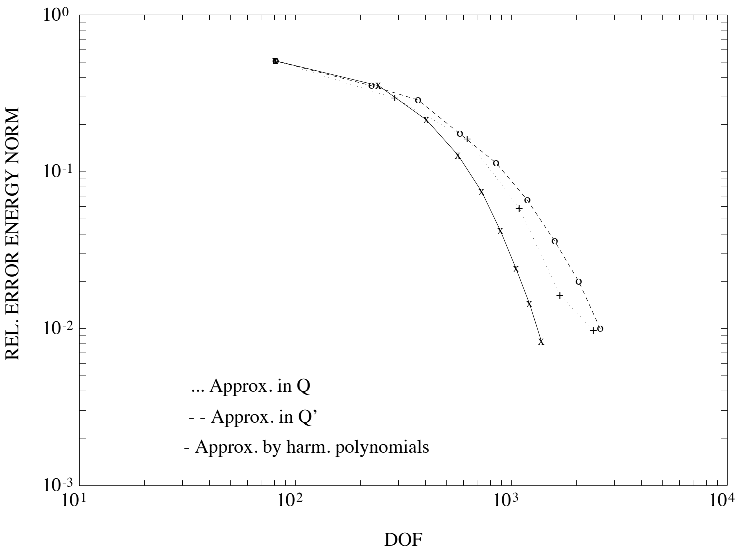

Figure 4: p version of PUM with part. of unity of Example 4.1 for $n = 8$, $a = 1.05$

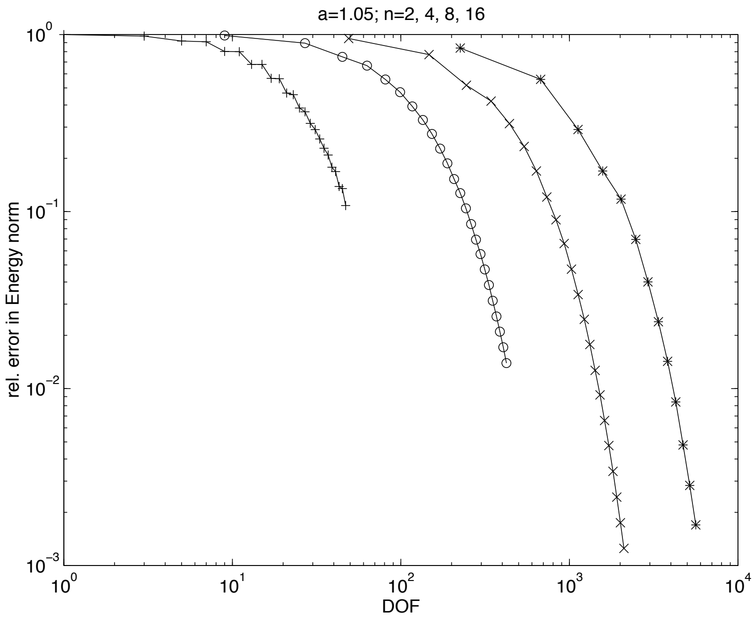

Figure 5: p version of PUM with part. of unity of Ex. 4.2 for $n = 2,\ 4,\ 8,\ 16$; $a = 1.05$

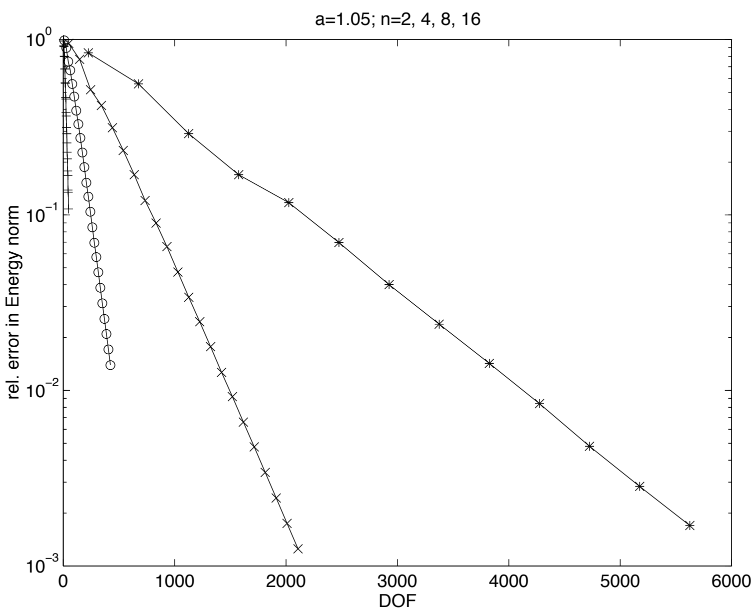

Figure 6: p version of PUM with part. of unity of Ex. 4.2 for $n = 2,\ 4,\ 8,\ 16$; $a = 1.05$

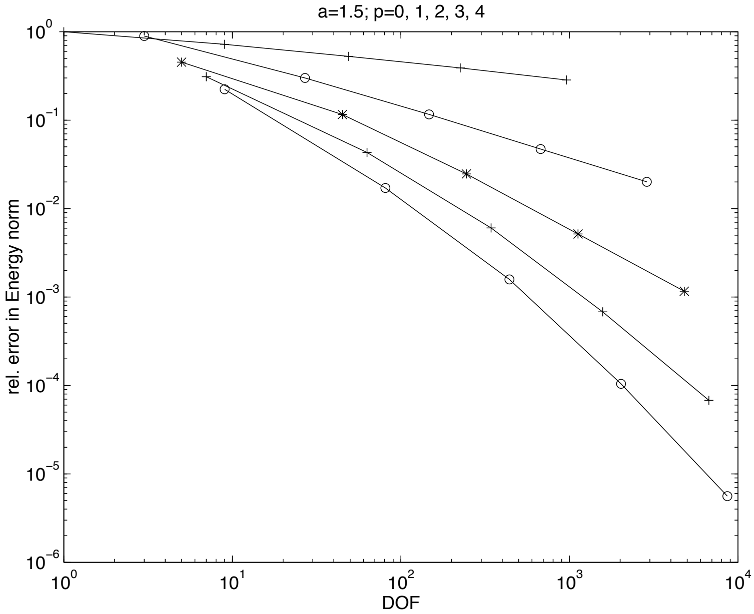

Figure 7: h version of PUM with part. of unity of Ex. 4.2 for $p = 0, 1, 2, 3, 4$; $a = 1.5$

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