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hp FEM for Reaction-Diffusion Equations I: Robust Exponential Convergence

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Abstract

A singularly perturbed reaction-diffusion equation in two dimensions is considered. We assume analyticity of the input data, i.e., the boundary of the domain is an analytic curve and the right hand side is analytic. We show that the *hp* version of the finite element method leads to *robust exponential convergence* provided that one layer of needle elements of width $O(p\varepsilon)$ is inserted near the domain boundary, that is, the rate of convergence is $O\big(\exp(-bp)\big)$ and independent of the perturbation parameter ε. We also show that the Spectral Element Method based on the use of a Gauss-Lobatto quadrature rule of order $O(p)$ for the evaluation of the stiffness matrix and the load vector retains the exponential rate of convergence.

Keywords: boundary layer, singularly perturbed problem, asymptotic expansions, error bounds

1 Introduction

Many boundary value problems (BVPs) arising in mechanics depend on a small or large parameter and are singularly perturbed. Frequently, this causes difficulties in the convergence of discretizations of such BVPs and requires especially designed schemes for their effective numerical solution (see, e.g., [1, 2, 3] and references therein). These difficulties are, roughly speaking, due to *stability problems* (especially in convection dominated fluid flow problems) and due to *boundary layers* which downgrade the *approximability* of the solution.
The Finite Element Method (FEM) is today the most widely used discretization technique for the numerical solution of BVPs. In recent years, in particular the *p*- and *hp*-FEM have emerged (see [4] and the references there), which achieve *exponential convergence* for elliptic problems with piecewise analytic solutions.
The aim of the present paper is to prove *robust exponential convergence* of the *hp*-FEM for a class of two dimensional singularly perturbed problems, i.e., the convergence is exponential and independent of the singular perturbation parameter ε.

1.1 Model Problem and Main Results

Consider

$$\begin{aligned} L_\varepsilon u_\varepsilon \equiv -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon &= f \qquad \text{on } \Omega \subset \mathbb{R}^2, \\ u_\varepsilon &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\partial\Omega$ is a closed, non-selfintersecting, analytic curve, f is analytic on $\overline{\Omega}$, g is analytic on $\partial\Omega$, and $\varepsilon \in (0,1]$ is a small parameter.
As usual, we denote by $L^2(\Omega)$ the square integrable functions on Ω and by $H^1(\Omega)$ those functions of $L^2(\Omega)$ whose (distributional) derivative is also in $L^2(\Omega)$. The trace operator maps $H^1(\Omega)$ onto the space $H^{1/2}(\partial\Omega)$ by restricting the elements of $H^1(\Omega)$ to the boundary $\partial\Omega$. $H_0^1(\Omega)$ denotes the kernel of the trace operator, that is, those functions in $H^1(\Omega)$ whose trace on $\partial\Omega$ is zero.
Assume $g = 0$. The weak formulation of (1.1) is: find $u_\varepsilon \in H_0^1(\Omega)$ such that

$$B_\varepsilon(u_\varepsilon, v) := \varepsilon^2 \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dxdy + \int_\Omega u_\varepsilon v \, dxdy = F(v) := \int_\Omega f v \, dxdy \qquad \forall v \in H_0^1(\Omega). \tag{1.2}$$

Associated with this problem is the notion of an "energy"

$$\|u\|_{\varepsilon,\Omega}^2 := B_\varepsilon(u,u) = \varepsilon^2 \|\nabla u\|_{L^2(\Omega)}^2 + \|u\|_{L^2(\Omega)}^2 \tag{1.3}$$

and an energy norm, being the square root of the energy. We have the a-priori estimate

$$\|u_\varepsilon\|_{\varepsilon,\Omega} \le \|f\|_{L^2(\Omega)} \tag{1.4}$$

independently of ε.
In the FEM a finite dimensional subspace $V_N \subset H_0^1(\Omega)$ of dimension $N = \dim V_N$ is chosen, and the finite element solution $u_N \in V_N$ is then given by

$$B_\varepsilon(u_N, v) = F(v) \qquad \forall v \in V_N. \tag{1.5}$$

By Céa's Lemma, the FE solution u_N is the best approximant of u_ε in the energy norm, i.e.,

$$\|u_\varepsilon - u_N\|_{\varepsilon,\Omega} = \inf_{v \in V_N} \|u_\varepsilon - v\|_{\varepsilon,\Omega}. \tag{1.6}$$

The question is therefore to choose the spaces V_N judiciously.

In [5] the one dimensional analog of (1.1) was analyzed, and it was shown that inserting one element of width $O(\varepsilon p)$ at the boundary points is sufficient for the p version of the FEM to resolve the boundary layer functions at a robust exponential rate. Hence, we may expect that in the two dimensional situation one layer of needle elements of width $O(\varepsilon p)$ should be introduced to obtain again robust exponential convergence. This was studied by Xenophontos in [6], who showed that robust convergence of arbitrary algebraic order can be obtained if a boundary fitted tensor product mesh is used that contains one layer of needle elements of width $O(p\varepsilon)$. The purpose of the present paper is to extend those results in several ways. Firstly, we prove in Section 3 that the introduction of one layer of needle elements of width $O(p\varepsilon)$ indeed leads to robust exponential convergence as conjectured and observed numerically in [6]. Secondly, we relax the restriction to boundary fitted tensor product meshes as we consider quite general quadrilateral as well as triangular meshes. In fact, contrary to the boundary fitted tensor product meshes, where all elements satisfy the maximum angle condition, the needle elements considered in this paper may violate the maximum angle condition (in a controlled way) as the perturbation parameter ε tends to zero.
In Section 4 we analyze a spectral element method, which is in effect a Galerkin FEM with a quadrature rule. We show that, under more restrictive assumptions on the mesh than for the Galerkin FEM, the spectral element method based on a Gauss-Lobatto quadrature rule of order $O(p)$ retains the robust exponential convergence of the Galerkin FEM.
The main tool in our proof of robust exponential approximability is, just like in the analysis of most schemes featuring robust algebraic convergence, the classical asymptotic expansion available for (1.1). However, whereas the analysis of schemes that converge at robust algebraic rates rests on asymptotic expansions of a fixed order, the expansion order in our analysis is variable. Thus, estimates of the remainder of the asymptotic expansion which are explicit in both the expansion order and the perturbation parameter ε are crucial to our analysis. Using the analyticity of the input data, such explicit estimates are proved in [7] (see also [8] for the simpler analysis of the one dimensional analog of (1.1)). We summarize the results of [7] in Section 2. Although we analyze here only the model problem (1.1), our spectral element mesh design principles can be applied whenever the length scale ε of the boundary layer and the spectral order p are known; this is often the case even without full asymptotics being available.

1.2 Notation

To define the asymptotic expansion of the exact solution, we introduce *boundary fitted coordinates*: Let $(X(\theta), Y(\theta))$, $\theta \in [0, L)$ be an analytic, L-periodic parametrization by arclength of the boundary $\partial\Omega$ such that the normal vector $(-Y'(\theta), X'(\theta))$ always points into the domain Ω. Introduce the notation $\kappa(\theta)$ for the curvature of the boundary curve and denote by $\mathbb{T}_L$ the one dimensional torus of length L, i.e., $\mathbb{R}/_{[0,L)}$ endowed with the usual topology. The functions X, Y, and hence also κ are analytic on $\mathbb{T}_L$ by the analyticity of $\partial\Omega$. For the remainder of this paper, let $\rho_0 > 0$ be fixed such that

$$0 < \rho_0 < \frac{1}{\|\kappa\|_{L^\infty([0,L))}}. \tag{1.7}$$

Then the mapping

$$\begin{array}{rcl} \psi : [0,\rho_0] \times \mathbb{T}_L & \to & \overline{\Omega} \\ (\rho,\theta) & \mapsto & (X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta)) \end{array} \tag{1.8}$$

is real analytic on $[0,\rho_0] \times \mathbb{T}_L$. The function ψ maps the rectangle $(0,\rho_0) \times [0,L)$ onto a tubular neighborhood Ω_0 of $\partial\Omega$. Furthermore by the choice of ρ_0, the inverse $\psi^{-1} : \overline{\Omega}_0 \to [0,\rho_0] \times \mathbb{T}_L$ exists and is also real analytic on the closed set $\overline{\Omega}_0$. For $\sigma > 0$ we introduce the *stretching map* s_σ via

$$\begin{array}{rcl} s_\sigma : (0,\infty) \times [0,L) & \to & \mathbb{R}^+ \times [0,L) \\ (\rho,\theta) & \mapsto & (\sigma\rho,\theta). \end{array} \tag{1.9}$$

The boundary layer expansion, i.e., the inner expansion, will be defined only in a neighborhood of the boundary $\partial\Omega$. Therefore, we introduce a cut-off function χ supported by a neighborhood of $\partial\Omega$. To this end, let

$$0 < \rho_1 < \rho_0 \tag{1.10}$$

be given and let χ be a smooth cut-off function, defined on $\overline{\Omega}$ satisfying

$$\chi(x,y) = \begin{cases} 1 & \text{for } 0 \leq \operatorname{dist}((x,y),\partial\Omega) \leq \rho_1 \\ 0 & \text{for } \operatorname{dist}((x,y),\partial\Omega) \geq (\rho_1+\rho_0)/2. \end{cases} \tag{1.11}$$

Finally, as f is assumed to be analytic on $\overline{\Omega}$ there is complex neighborhood $\tilde{\Omega} \subset \mathbb{C} \times \mathbb{C}$ of $\overline{\Omega}$ and a holomorphic extension of f (also denoted f) to $\tilde{\Omega}$. Therefore, we may assume that there are constants C_f, $\gamma_f > 0$ such that

$$\|D^\alpha f\|_{L^\infty(\tilde{\Omega})} \leq C_f \gamma_f^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2. \tag{1.12}$$

From this estimate, we can conclude with the aid of Cauchy's integral theorem for derivatives (after passing to a compact subset of $\tilde{\Omega}$ which we denote again by $\tilde{\Omega}$) the existence of $C_{\Delta f}$, $\gamma_{\Delta f} > 0$ such that

$$\|\Delta^{(i)} f\|_{L^\infty(\tilde{\Omega})} \leq C_{\Delta f} (2i)! \gamma_{\Delta f}^{2i} \qquad \forall i \in \mathbb{N}_0 \tag{1.13}$$

where $\Delta^{(i)}$ denotes the iterated Laplace operator, i.e., $\Delta^{(0)} = Id$, $\Delta^{(1)} = \Delta$, $\Delta^{(2)} = \Delta\Delta$, etc.

2 Regularity of the Solution

The aim of this section is to clarify the regularity properties of the solution u_ε of (1.1). More precisely, we are interested in the dependence of the higher derivatives of u_ε on the perturbation parameter ε. We will distinguish two cases:

(i) the asymptotic case where the order of the derivative is $> \varepsilon^{-1}$;

(ii) the pre-asymptotic case where the order of the derivatives is $\leq \varepsilon^{-1}$.

All results of this section are proved in the second part of this work, [7].

2.1 The Asymptotic Case

The growth of the derivatives of the solution u_ε can be estimated using the techniques of [9].

Theorem 2.1 *Let u_ε be the solution (1.1) with f, g, and $\partial\Omega$ analytic. Then there are constants C, $K > 0$ depending only on f, g, and Ω such that*

$$\|D^\alpha u_\varepsilon\|_{L^2(\Omega)} \leq CK^{|\alpha|} \max\left(|\alpha|, \varepsilon^{-1}\right)^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2.$$

We note that Theorem 2.1 yields estimates for the derivatives of u_ε which are independent of ε provided that $|\alpha| \geq c\varepsilon^{-1}$ for some $c > 0$. Roughly speaking, this means that derivatives of order $O(\varepsilon^{-1})$ "don't see" the boundary layers. Another observation is that, as the *asymptotic* behavior of the derivatives of u_ε can be controlled independently of ε, there is $\tilde{\Omega} \supset \overline{\Omega}$ such that u_ε is analytic on $\tilde{\Omega}$ independently of ε. Furthermore, it is not too hard to see that Theorem 2.1 yields robust exponential rates of convergence for the p version of the finite element method, provided that the polynomial degree p is at least $O(\varepsilon^{-1})$.

2.2 The Pre-asymptotic Case

The solution u_ε of (1.1) has boundary layer character when the parameter ε is small. This means that in a neighborhood of the boundary the behavior of u_ε in tangential direction differs substantially from its behavior in the normal direction. This "anisotropic" boundary layer behavior is not reflected in the results of Theorem 2.1 but can be described by means of the classical asymptotic expansions: For each $M \in \mathbb{N}_0$, the solution u_ε can be decomposed as

$$u_\varepsilon = w_M + \chi u_M^{BL} + r_M \tag{2.1}$$

where w_M is the *outer expansion*, u_M^{BL} is the *inner expansion*, χ is the cut-off function defined in (1.11), and r_M is the remainder.
For a given expansion order M, we define the outer expansion by

$$w_M := \sum_{i=0}^{M} \varepsilon^{2i} \Delta^{(i)} f = f + \varepsilon^2 \Delta f + \varepsilon^4 \Delta\Delta f + \cdots \tag{2.2}$$

where $\Delta^{(i)}$ denotes the iterated Laplacian. As

$$L_\varepsilon(u_\varepsilon - w_M) = \varepsilon^{2M+2} \Delta^{(M+1)} f \qquad \text{on } \Omega$$

we see that asymptotically, as ε tends to zero, the functions w_M satisfy the differential equation on Ω. However, the functions w_M do not satisfy the boundary conditions. Let us therefore introduce a correction function u^{BL} defined by

$$\begin{aligned} L_\varepsilon u^{BL} &= 0 && \text{on } \Omega, \\ u^{BL} &= g - w_M && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

The *inner expansion* is now an asymptotic expansion for this correction function u^{BL}. In order to define this expansion, we need to rewrite the differential operator L_ε in the boundary fitted coordinates (ρ, θ). If we introduce the curvature $\kappa(\theta)$ of $\partial\Omega$ and the function

$$\sigma(\rho, \theta) = \frac{1}{1 - \kappa(\theta)\rho}$$

we have (see, for example, [10])

$$\Delta u(\rho, \theta) = \partial_\rho^2 u - \kappa(\theta)\sigma(\rho, \theta)\partial_\rho u + \sigma^2(\rho, \theta)\partial_\theta^2 u + \rho\kappa'(\theta)\sigma^3(\rho, \theta)\partial_\theta u.$$

Expanding the function σ in a converging geometric series gives

$$\sigma(\rho, \theta) = \sum_{i=0}^{\infty} [\kappa(\theta)\rho]^i = \sum_{i=0}^{\infty} \varepsilon^i [\kappa(\theta)\widehat{\rho}]^i$$

where we introduced the *stretched variable* notation $\widehat{\rho} = \rho/\varepsilon$. Note that we chose $\rho_0 < \|\kappa\|_{L^\infty([0,L))}$ in (1.7) so that the power series expansion converges uniformly in $(\rho, \theta) \in [0, \rho_0] \times [0, L]$.
Recall that Ω_0 is the tubular neighborhood of $\partial\Omega$ which is the image of the rectangle $(0, \rho_0) \times [0, L)$ under the map ψ. In this tubular neighborhood Ω_0 the differential equation (2.3) takes the form

$$-\varepsilon^2 \left\{ \partial_\rho^2 u^{BL} + \sum_{i=0}^{\infty} \rho^i \left(a_1^i \partial_\rho u^{BL} + a_2^i \partial_\theta^2 u^{BL} + a_3^i \partial_\theta u^{BL} \right) \right\} + u^{BL} = 0 \qquad \text{in } \Omega_0 \tag{2.4}$$

where we introduced the abbreviations

$$a_1^i = -[\kappa(\theta)]^{i+1}, \quad a_2^i = (i+1)[\kappa(\theta)]^i, \quad a_3^i = \frac{i(i+1)}{2}[\kappa(\theta)]^{i-1}\kappa'(\theta). \tag{2.5}$$

For technical convenience let us also formulate (2.4) in terms of the stretched variable $\widehat{\rho}$:

$$-\partial_{\widehat{\rho}}^2 u^{BL} - \sum_{i=0}^{\infty} (\varepsilon\widehat{\rho})^i \left(\varepsilon a_1^i \partial_{\widehat{\rho}} u^{BL} + \varepsilon^2 a_2^i \partial_\theta^2 u^{BL} + \varepsilon^2 a_3^i \partial_\theta u^{BL} \right) + u^{BL} = 0. \tag{2.6}$$

Now, in order to define the inner expansion, we make the formal ansatz $u^{BL} = \sum_{i=0}^{\infty} \varepsilon^i \widehat{U}_i(\widehat{\rho}, \theta)$ where the functions $\widehat{U}_i$ are to be determined. Inserting this ansatz in (2.4) and equating like powers of ε we obtain a recurrence relation for the functions $\widehat{U}_i$:

$$-\partial_{\widehat{\rho}}^2 \widehat{U}_i + \widehat{U}_i = \widehat{F}_i, \qquad i = 0, 1, \ldots,$$
$$\widehat{F}_i = \widehat{F}_i^1 + \widehat{F}_i^2 + \widehat{F}_i^3,$$
$$\widehat{F}_i^1 = \sum_{j=0}^{i-1} \widehat{\rho}^j a_1^j \partial_{\widehat{\rho}} \widehat{U}_{i-1-j}, \qquad \widehat{F}_i^2 = \sum_{j=0}^{i-2} \widehat{\rho}^j a_2^j \partial_\theta^2 \widehat{U}_{i-2-j}, \qquad \widehat{F}_i^3 = \sum_{j=0}^{i-2} \widehat{\rho}^j a_3^j \partial_\theta \widehat{U}_{i-2-j}$$

where we used the tacit convention that empty sums take the value zero. As we expect the boundary layer function u^{BL} to decay away from the boundary $\partial\Omega$ and as we want to satisfy the boundary conditions, we supplement these ODEs for the $\widehat{U}_i$ with the boundary conditions

$$\begin{aligned} \widehat{U}_i &\to 0 \qquad \text{as } \widehat{\rho} \to \infty, \\ [\widehat{U}_i]_{\partial\Omega} &= G_i := \begin{cases} g - [f]_{\partial\Omega} & \text{if } i = 0 \\ -[\Delta^{(i/2)} f]_{\partial\Omega} & \text{if } 0 < i \leq 2M \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We have

Theorem 2.2 *Let f, g, $\partial\Omega$ be analytic. Then there are constants K_1, K_2, and K_3 depending only f, g, and $\partial\Omega$ such that the functions $\widehat{U}_i$ are holomorphic on*

$$\mathbb{C} \times \{z \,|\, |\mathrm{Im}\, z| < K_1\}.$$

Additionally, for all $\alpha \in [0, 1)$, there are constants C_α depending only α, f, g, and Ω such that for all $i \in \mathbb{N}_0$

$$\left|\widehat{U}_i(\widehat{\rho} + z, \theta + \zeta)\right| \leq C_\alpha e^{-\alpha\hat{\rho}} e^{K_2|z|} \left(\frac{K_3}{1-\alpha}\right)^i i^i \frac{1}{K_1 - |\zeta|} \qquad (z, \zeta) \in \mathbb{C} \times \{z \,|\, |\mathrm{Im}\, z| < K_1\}.$$

The *inner expansion* of order $2M + 1$ is defined as the function

$$u_M^{BL}(\rho, \theta) := \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_i(\widehat{\rho}, \theta) = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_i(\rho/\varepsilon, \theta), \tag{2.7}$$

and it satisfies the boundary conditions

$$[u_M^{BL}]_{\partial\Omega} = g - \sum_{i=0}^{M} \varepsilon^{2i} [\Delta^{(i)} f]_{\partial\Omega}.$$

Remark 2.3: We defined u_M^{BL} as the inner expansion of order $2M + 1$ so that the first neglected term of the formal asymptotic expansion $\sum_{i=0}^{\infty} \varepsilon^i \widehat{U}_i$ is of order ε^{2M+2}. This is precisely the same power of ε as the first neglected term of the outer expansion $\sum_{i=0}^{\infty} \varepsilon^{2i} \Delta^{(i)} f$ truncated after the ε^{2M} term.

As the boundary fitted coordinates (ρ, θ) are only meaningful in a neighborhood of $\partial\Omega$ we restrict the approximation u_M^{BL} of u^{BL} to a tubular neighborhood of $\partial\Omega$ by the cut-off function χ defined in (1.11). Finally, we define r_M such that the following identity holds

$$u_\varepsilon = w_M + \chi u_M^{BL} + r_M. \tag{2.8}$$

Strictly speaking, the functions w_M and χ are defined in cartesian coordinates (x, y) whereas the function u_M^{BL} is defined in boundary fitted coordinates (ρ, θ) so that we should interpret u_M^{BL} as $u_M^{BL} \circ \psi^{-1}$ in the tubular neighborhood Ω_0 where the boundary fitted coordinates can be defined and we should understand χu_M^{BL} to vanish outside of Ω_0.
The following theorems clarify the derivative growth of the functions u_M^{BL} and r_M. Contrary to classical asymptotic expansions, the dependence on the perturbation parameter ε as well as the expansion order M is made explicit.

Theorem 2.4 *Let f, g, and $\partial\Omega$ be analytic, $\alpha \in [0,1)$ be fixed. Then the functions u_M^{BL} of (2.7) are analytic and there are constants K_1, K_2, K_3, $C > 0$ depending only on f, g, Ω, and α such that*

$$\begin{aligned} \sup_{\theta\in[0,L)} \left|\partial_\rho^p \partial_\theta^m u_M^{BL}(\rho,\theta)\right| &\leq Cm! K_2^m K_1^p \varepsilon^{-p} e^{-\alpha\rho/\varepsilon} \qquad \rho \geq 0, \quad p,m \in \mathbb{N}_0, \\ \sup_{\theta\in[0,L)} \|\partial_\rho^p \partial_\theta^m u_M^{BL}(\cdot,\theta)\|_{L^2(\rho,\infty)} &\leq Cm! K_2^m K_1^p \varepsilon^{1/2-p} e^{-\alpha\rho/\varepsilon} \qquad \rho \geq 0, \quad p,m \in \mathbb{N}_0 \end{aligned}$$

provided that ε and the expansion order M satisfy

$$0 < \varepsilon(2M+2) \leq K_3. \tag{2.9}$$

Theorem 2.4 shows that indeed the function u_M^{BL} and all its derivatives decay exponentially away from the boundary $\partial\Omega$. Let us now see under which conditions this asymptotic expansion has meaning, i.e., when the remainder r_M is indeed small. This is the object of the following theorem.

Theorem 2.5 *Let f, g, and $\partial\Omega$ be analytic. Then the remainder r_M of (2.8) vanishes on $\partial\Omega$ and for each $k \in \mathbb{N}_0$ there are C_k, $K > 0$ depending only on k, f, g, $\partial\Omega$ and χ such that*

$$\|r_M\|_{H^k(\Omega)} \leq C_k \varepsilon^{-k} \left(\varepsilon K(2M+2)\right)^{2M+2}, \qquad k = 0,1,2,\ldots$$

Note that Theorem 2.5 guarantees that the remainder r_M is indeed small provided that the expansion order M and the parameter ε satisfy a condition of the form $\varepsilon(2M+1)$ small. This is precisely condition (2.9) which was necessary to control u_M^{BL}. Theorem 2.5 suggests that in the complementary case, $\varepsilon(2M+2)$ not small, the asymptotic expansions lose their meaning.

3 *hp* Finite Element Approximation

In this section we will prove the robust exponential convergence of the Galerkin FEM. To that end, we introduce in Section 3.1 the notion of admissible boundary layer meshes which consist of quadrilaterals and have one layer of needle elements of width $O(\varepsilon p)$ at the boundary. In Section 3.2 we compile some results on the approximation properties of the Gauss-Lobatto interpolation operator on the unit square. In Section 3.3 we show that for admissible boundary layer meshes the difference between the exact solution u_ε and its piecewise Gauss-Lobatto interpolant is exponentially small (in the polynomial degree p) uniformly in ε. These approximation results are extended to meshes consisting of quadrilateral as well as triangular elements in Section 3.4.2. The admissible boundary layer meshes are essentially the "minimal" meshes that lead to robust exponential convergence. However, they do depend on ε as well as the spectral order p. In Section 3.4.3, therefore, we analyze fixed meshes which are refined geometrically towards the boundary $\partial\Omega$. With the proper number of layers, such fixed meshes have approximation properties similar to the "minimal", admissible boundary layer meshes.

3.1 Admissible Boundary Layer Meshes

Let us define regular meshes consisting of quadrilaterals Ω_i subject to the following standard restrictions. Denoting $S := [0,1] \times [0,1]$ the usual reference square, we associate with each quadrilateral Ω_i a differentiable, bijective element mapping

$$M_i : S \to \overline{\Omega}_i.$$

Furthermore, we assume as usual that

(M1) The elements Ω_i partition of the domain Ω, i.e., $\overline{\Omega} = \cup_i \overline{\Omega}_i$ and $\det M_i' > 0$ on S for all i.

(M2) For $i \neq j$, $\overline{\Omega}_i \cap \overline{\Omega}_j$ is either empty, or a vertex or an entire edge (vertices and edges are of course the images of the vertices and edges of the reference element under the maps M_i).

(M3) Common edges of neighboring elements Ω_i, Ω_j have the same parametrization "from both sides": Let $\gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ be the common edge with endpoints (vertices) P_1, P_2. Then for any point $P \in \gamma_{ij}$, we have $\mathrm{dist}(M_i^{-1}(P), M_i^{-1}(P_l)) = \mathrm{dist}(M_j^{-1}(P), M_j^{-1}(P_l))$, $l = 1, 2$.

Given such a mesh, we can define spaces S^p, S_0^p of piecewise mapped polynomials in the usual way:

$$S^p := \{u \in H^1(\Omega) \,|\, u|_{\Omega_i} = \varphi_p \circ M_i^{-1} \quad \text{for some } \varphi_p \in Q_p(S)\}, \tag{3.10}$$

$$S_0^p := S^p \cap H_0^1(\Omega) \tag{3.11}$$

where we used the notation $Q_p(S)$ to denote the set of all polynomials of degree p (in each variable) on the reference square S.

We indicated in the introduction that we would like to work with meshes which have needle elements of width $O(p\varepsilon)$ near the boundary where p is the polynomial degree. Since we want to achieve exponential rates of convergence it is necessary that the maps M_i be analytic and that the growth of the derivatives can be controlled in some way uniformly in i. This is the object of the next definition. It formalizes the assumptions on a mesh with needle elements of width (essentially) $\lambda p \varepsilon$ and whose remaining elements are of size $O(1)$.

Definition 3.1 (Admissible mesh family) *A three-parameter family of meshes $\left(\Omega_i(\lambda, p, \varepsilon)\right)$, $0 < \lambda \leq 1$, $p \in \mathbb{N}$, $0 < \varepsilon \leq 1$, satisfying the conditions (M1)—(M3) is called an* admissible boundary layer mesh with critical width c_0 *if there are constants λ_L, λ_U, C_1, C_2, $\gamma > 0$ independent of the three parameters λ, p, and ε such that the following conditions hold.*

If $\lambda p \varepsilon > c_0$ then all elements are regular elements*, i.e., the corresponding element maps M_i satisfy*

$$\|\,(M_i')^{-1}\,\|_{L^\infty(S)} \leq C_1 \tag{3.12}$$

$$\|D^\alpha M_i\|_{L^\infty(S)} \leq C_2 \gamma^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2. \tag{3.13}$$

If $\lambda p \varepsilon \leq c_0$, then we distinguish two kinds of elements:

1. *Ω_i abuts on the boundary: $\overline{\Omega}_i \cap \partial\Omega \neq \emptyset$. Then Ω_i is a* needle element*, that is, it satisfies*

$$\begin{aligned} \lambda_U c_0 &< \rho_0 && (3.14)\\ \operatorname{dist}(x,\partial\Omega) &\leq \lambda_U \lambda p\varepsilon, \qquad \forall x \in \Omega_i && (3.15)\\ \|\,(M_i')^{-1}\,\|_{L^\infty(S)} &\leq \frac{C_1}{\lambda p \varepsilon} && (3.16)\\ \|D^\alpha\left(s^{-1}_{\lambda p\varepsilon}\circ\psi^{-1}\circ M_i\right)\|_{L^\infty(S)} &\leq C_2\gamma^{|\alpha|}|\alpha|! \qquad \forall\alpha\in\mathbb{N}_0^2 && (3.17) \end{aligned}$$

where the stretching operator s is defined in (1.9).

2. *$\overline{\Omega}_i \cap \partial\Omega = \emptyset$. Then Ω_i $\operatorname{dist}(\Omega_i,\partial\Omega) \geq \lambda_L\lambda p\varepsilon$, and Ω_i is a regular element, i.e., the map M_i satisfies (3.12), (3.13).*

The notion of a regular element is the standard notion of "p version elements". Let us comment on the conditions imposed on needle elements, (3.14)–(3.17). (3.14), (3.15) stipulate that needle elements are completely contained in the tubular neighborhood Ω_0 of $\partial\Omega$ where the boundary fitted map ψ is invertible (cf. Section 1.2). This is merely a technical assumption to guarantee that (3.17) makes sense. Condition (3.17) is the crucial assumption, and it reflects the anisotropy of the needle elements. The map $s^{-1}_{\lambda p\varepsilon}\circ\psi^{-1}$ produces a stretching in the direction normal to the boundary $\partial\Omega$ by the factor $(\lambda p\varepsilon)^{-1}$. Therefore, the needle elements are mapped under this stretching to sets of size $O(1)$ whose element maps (i.e., the maps $s^{-1}_{\lambda p\varepsilon}\circ\psi^{-1}\circ M_i$) can be controlled independently of λ, p, ε, and i, which is (3.17).
A different approach to the definition of needle elements is to introduce "reference needle" elements, e.g., $R_i := (0,\lambda p\varepsilon)\times(0,1)$ and then to control the maps from R_i onto Ω_i:

Definition 3.2 (Regular admissible mesh family) *An admissible boundary layer mesh family with critical width c_0 is called* regular *if the needle elements satisfy the following additional condition: In the case $\lambda p\varepsilon \leq c_0$ there are C_1', C', γ' independent of λ, p, and ε such that the maps*

$$\begin{aligned} \widetilde{M}_i : R_i := (0,\lambda p\varepsilon)\times(0,1) &\to \Omega_i\\ (\xi,\eta) &\mapsto M_i(\xi/(\lambda p\varepsilon),\eta) \end{aligned}$$

satisfy

$$\begin{aligned} {C_1'}^{-1} &\leq \det\widetilde{M}_i' \leq C_1'\\ \|D^\alpha\widetilde{M}_i\|_{L^\infty(R_i)} &\leq C'{\gamma'}^{|\alpha|}|\alpha|! \qquad \forall\alpha\in\mathbb{N}_0^2. \end{aligned}$$

The additional conditions imposed on a regular admissible mesh guarantee that the maximal angles of the needle elements do not degenerate. In effect, regular admissible meshes satisfy the maximal angle condition known to be crucial for H^1 approximability (cf. [11]). However, as the energy norm is an ε-weighted H^1 norm, the maximal angle condition may be relaxed, and this is reflected in our notion of admissible mesh families (Definition 3.1). In admissible meshes, the maximal angle of the needle elements has merely to be greater than $\pi - c\lambda\varepsilon p$ for some fixed $c > 0$, i.e., the maximal angle of the needle elements is allowed to degenerate to π as ε tends to zero (cf. Fig. 3.1).

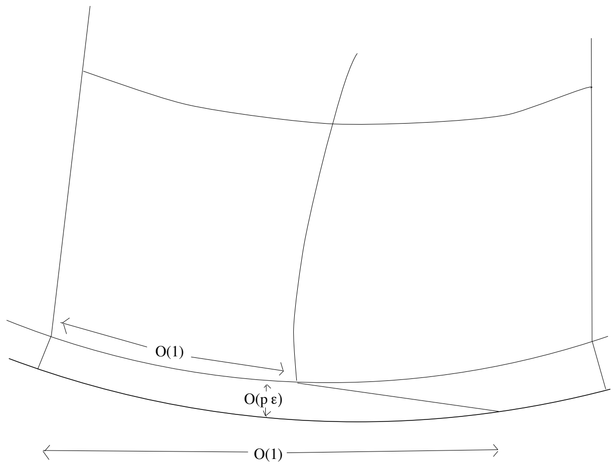

Figure 3.1: Example of admissible mesh

Remark 3.3: Note that for fixed ε, the needle elements become "fatter" as p increases. Asymptotically, i.e., if p is at least $O(\varepsilon^{-1})$, admissible meshes do not contain any needle elements and are just classical p version meshes. This ties in with our discussion of Theorem 2.1 where we saw that the p version of the FEM (on a coarse mesh) yields exponential convergence if p is at least $O(\varepsilon^{-1})$.

Remark 3.4: Let us see that the conditions for admissible meshes imply the existence of constants C, $\gamma > 0$ such that

$$\|D^\alpha M_i\|_{L^\infty(S)} \leq C\gamma^{|\alpha|}|\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2 \quad \forall i.$$

Clearly, we have to check this condition only for the needle elements. Let Ω_i be a needle element with $\lambda p\varepsilon \leq c_0$. Upon setting $r = 1/c_0$ we have

$$M_i = (\psi \circ s_{\lambda p\varepsilon} \circ s_r) \circ \left(s_r^{-1} \circ s_{\lambda p\varepsilon}^{-1} \circ \psi^{-1} \circ M_i\right).$$

Note that $s_r^{-1} \circ s_{\lambda p\varepsilon}^{-1} \circ \psi^{-1} \circ M_i(S) \subset (0, \lambda_U c_0) \times \mathbb{T}_L$. The analyticity of ψ together with $\lambda p\varepsilon \leq c_0$ and $\lambda_U c_0 < \rho_0$ implies readily that

$$\|D^\alpha\left(\psi \circ s_{\lambda p\varepsilon} \circ s_r\right)\|_{L^\infty((0,\lambda_U c_0)\times\mathbb{T}_L)} \leq C\gamma^{|\alpha|}|\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2 \tag{3.18}$$

for some C, $\gamma > 0$ independent of λ, p, and ε. Assumption (3.17) together with Lemma 3.9 implies that

$$\|D^\alpha\left(s_r^{-1} \circ \left(s_{\lambda p\varepsilon}^{-1} \circ \psi^{-1} \circ M_i\right)\right)\|_{L^\infty(S)} \leq C'\gamma'^{|\alpha|}|\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2 \tag{3.19}$$

for some C', $\gamma' > 0$ independent of λ, p, and ε. Applying Lemma 3.9 again with the estimates (3.18), (3.19) implies the claim.

3.1.1 First family of admissible meshes

In this example we want to construct a family of admissible boundary layer meshes by defining the needle elements as “rectangles in boundary fitted coordinates”. These are essentially the boundary fitted tensor product meshes considered in [6]. To that end, let $c_0 < \rho_0$ be given and fix a partition $0 = \theta_1 < \theta_2 < \cdots < \theta_{n+1} = L$ (note that we can identify θ_{n+1} with θ_1 on $\mathbb{T}_L$).
For $\lambda p\varepsilon < c_0$ choose the needle elements $\Omega_1, \dots, \Omega_n$ as the images of the rectangles

$$(0, \lambda p\varepsilon) \times (\theta_i, \theta_{i+1}), \qquad i = 1, \dots, n$$

under the map ψ. Hence the needle elements Ω_i with corresponding maps M_i are given by

$$\Omega_i := \psi\left((0, \lambda p\varepsilon) \times (\theta_i, \theta_{i+1})\right), \qquad M_i(\xi, \eta) := \psi\big(\lambda p\varepsilon\xi, \theta_i + \eta(\theta_{i+1} - \theta_i)\big), \qquad i = 1, \dots, n.$$

The elements $\overline{\Omega}_1, \dots, \overline{\Omega}_n$ form a partition of $\psi\left((0, \lambda p\varepsilon) \times \mathbb{T}_L\right)$, and it is clear that they satisfy (3.14)–(3.17) with all constants depending only on ψ and the fixed partition $\theta_1, \dots, \theta_{n+1}$. Let us note that

$$s^{-1}_{\lambda p\varepsilon} \circ \psi^{-1} \circ M_i(\xi, \eta) = (\xi, \theta_i + \eta(\theta_{i+1} - \theta_i)), \qquad i = 1, \dots, n. \tag{3.20}$$

It is simple to add to these needle elements elements $\Omega_{n+1}, \dots \Omega_N$ of size $O(1)$ such that the total collection of elements $\Omega_1, \dots, \Omega_N$ satisfies (M1)–(M3) and the elements $\Omega_{n+1}, \dots, \Omega_N$ satisfy (3.12), (3.13) with constants independent of λ, p, and ε. Note that by construction

$$\operatorname{dist}(x, \partial\Omega) \geq \lambda p\varepsilon \qquad \forall x \in \Omega_i, \quad i = n+1, \dots, N.$$

Hence the meshes constructed in this way are admissible meshes in the sense of Definition 3.1 for the case $\lambda p\varepsilon \leq c_0$. For the case $\lambda p\varepsilon > c_0$ we simply take the mesh constructed for the case $\lambda p\varepsilon = c_0$ and this concludes the construction.
In order to see that the family of meshes obtained in this way is a family of regular admissible meshes in the sense of Definition 3.2, we consider for $\lambda p\varepsilon \leq c_0$ and $i = 1, \dots, n$

$$\begin{aligned} \widetilde{M}_i : R_i = (0, \lambda p\varepsilon) \times (0,1) &\to \Omega_i \\ (\xi, \eta) &\mapsto M_i(\xi/(\lambda p\varepsilon), \eta), \end{aligned}$$

which satisfies by (3.20)

$$s^{-1}_{\lambda p\varepsilon} \circ \psi^{-1} \circ M_i = \psi^{-1} \circ \widetilde{M}_i.$$

Hence the conditions of Definition 3.2 are seen to be satisfied owing to the analyticity of ψ.

3.1.2 Second family of admissible meshes

In this section, an admissible boundary layer mesh is constructed in a slightly different way. Again, the meshes designed here form a regular admissible family of meshes.
We start out with the asymptotic mesh, i.e., a fixed, coarse mesh of quadrilaterals $(\Omega_i)_{i=1}^N$ subject to (M1)–(M3) whose corresponding maps M_i satisfy

$$\|(M_i')^{-1}\|_{L^\infty(S)} \leq C_1, \qquad \|D^\alpha M_i\|_{L^\infty(S)} \leq C_2\gamma^{|\alpha|}|\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2, \quad i = 1, \dots, N \tag{3.21}$$

for some constants C_1, C_2, and γ independent of i. Without loss of generality let us assume that the quadrilaterals $\Omega_1, \dots, \Omega_n$, $n \leq N$, are the elements abutting on the boundary, i.e., $\overline{\Omega}_i \cap \partial\Omega \neq \emptyset$ for $i = 1, \dots, n$ and $\overline{\Omega}_i \cap \partial\Omega = \emptyset$ for $i > n$. For ease of exposition let us further assume that all the elements Ω_i, $i = 1, \dots, n$ abutting on the boundary are completely contained in the tubular neighborhood in which the boundary fitted coordinates can be defined, i.e.,

$$\mathrm{dist}(x, \partial\Omega) < \rho_0, \qquad \forall x \in \Omega_i, \quad i = 1, \dots, n.$$

Let us assume additionally that the maps M_i for these elements abutting on the boundary are such that the line $\xi = 0$ is mapped onto a subset of $\partial\Omega$ (and all the other parts of the boundary of the reference square S are mapped into Ω). This implies that for $i = 1, \dots, n$, the maps M_i satisfy

$$\left(\psi^{-1} \circ M_i\right)(\xi, \eta) = (\xi \mathcal{R}_i(\xi, \eta), \Theta_i(\xi, \eta)) \tag{3.22}$$

with $\mathcal{R}_i$, Θ_i analytic on the closed reference square S. Furthermore, there is a constant $C_3 > 0$, independent of i such that

$$0 < C_3 \leq \mathcal{R}_i(\xi, \eta) \leq C_3^{-1}, \qquad \forall (\xi, \eta) \in S. \tag{3.23}$$

For $\lambda > 0$ and $p \in \mathbb{N}$, we distinguish two cases.

1. $\lambda p \varepsilon \geq 1/2$. In this case we are in the asymptotic regime, and we use the coarse mesh defined above.

2. $\lambda p \varepsilon < 1/2$. In this regime, we need to define needle elements. This is done by splitting the elements Ω_i, $i = 1, \dots, n$ into two elements Ω_i^{needle} and Ω_i^{reg}. Split the reference square S into two elements

$$S^{needle} := (0, \lambda p \varepsilon) \times (0, 1) \quad \text{and} \quad S^{reg} := (\lambda p \varepsilon, 1) \times (0, 1)$$

and set

$$\begin{aligned} \Omega_i^{needle} &:= M_i(S^{needle}), & \Omega^{reg} &:= M_i(S^{reg}), \\ M_i^{needle}(\xi, \eta) &:= M_i(\lambda p \varepsilon \xi, \eta), & M_i^{reg}(\xi, \eta) &:= M_i(\lambda p \varepsilon + (1 - \lambda p \varepsilon)\xi, \eta). \end{aligned}$$

It is easy to see that the mesh $\Omega_1^{needle}, \dots, \Omega_n^{needle}, \Omega_1^{reg}, \dots, \Omega_i^{reg}, \Omega_{n+1}, \dots, \Omega_N$ satisfies the conditions (M1)–(M3). Furthermore, (3.22) and (3.23) imply that the needle elements satisfy (3.15) with $\lambda_U = C_3^{-1}$. The first estimate of (3.21) gives (3.16) for the needle elements. (3.22) together with the analyticity of $\mathcal{R}_i$, Θ_i yields (3.17). In order to conclude that the mesh constructed in this way is an admissible mesh, we have to see that the maps M_i^{reg} satisfy (3.12), (3.13), and stay away from $\partial\Omega$. (3.23) implies again that

$$\mathrm{dist}(x, \partial\Omega) \geq C_3 \lambda p \varepsilon, \qquad \forall x \in \Omega_i^{reg}$$

and the assumption $\lambda p \varepsilon \leq 1/2$ gives that the maps M_i^{reg} satisfy the desired conditions on the derivatives (3.13). (3.12) follows from the fact that $\lambda p \varepsilon \leq 1/2$ and that the maps M_i satisfy (3.12) already.

In order to see that the mesh constructed in this subsection is actually a regular admissible mesh, we note that the map $\widetilde{M}_i^{needle}$ here takes the form

$$\begin{array}{rcl} \widetilde{M}_i^{needle} : R_i := (0, \lambda p\varepsilon) \times (0,1) & \to & \Omega_i^{needle} \\ (\xi, \eta) & \mapsto & M_i^{needle}(\xi/(\lambda p \varepsilon), \eta) = M_i(\xi, \eta) \end{array}$$

and so the conditions for a regular admissible mesh are satisfied.

3.2 Polynomial Approximation Results

Lemma 3.5 *Let $I := [0,1]$ and $u \in C^\infty(I)$ satisfying*

$$\|D^p u\|_{L^2(I)} \leq C_1 p! \gamma^p \tag{3.24}$$

for some C_1, $\gamma > 0$. Then there is a sequence of polynomials $(P_p)_{p=0}^\infty$ of degree p such that

$$\|u - P_p\|_{L^\infty(I)} + \|\,(u - P_p)'\,\|_{L^\infty(I)} \leq C_2 C_1 e^{-\sigma p}$$

where the constants C_2, $\sigma > 0$ depend only on γ.

Proof: From Sobolev's imbedding theorem, we have that $\|D^p u\|_{L^\infty(I)} \leq C_1 C_1' p! \gamma'^p$ for some C_1', γ' depending only γ. Therefore, u is analytic on the closed set I and can be extended analytically to a complex neighborhood of I. The result follows from standard theory: For example, the polynomial P_p may be obtained by interpolating u in the Tschebyscheff points (see Chap. 4 of [12] for the details). □

For $p \geq 1$ define on the space $C(I)$ the operator i_p by interpolation in the $p+1$ Gauss–Lobatto points. By [13] we have the stability estimate

$$\|i_p u\|_{L^\infty(I)} \leq C_G (1 + \ln p) \|u\|_{L^\infty(I)}. \tag{3.25}$$

For the interpolation error in the Gauss–Lobatto points, we have

Lemma 3.6 *Let u satisfy the assumptions of Lemma 3.5. Then there are C, $\sigma > 0$ depending only on γ of Lemma 3.5 and C_G such that with C_1 in (3.24)*

$$\|u - i_p u\|_{L^\infty(I)} + \|\,(u - i_p u)'\,\|_{L^\infty(I)} \leq C C_1 e^{-\sigma p}.$$

Proof: Let P_p be the approximant of Lemma 3.5. As the interpolation operator i_p reproduces polynomials of degree p, we have

$$\begin{array}{rcl} \|u - i_p u\|_{L^\infty(I)} & \leq & \|u - P_p - i_p(u - P_p)\|_{L^\infty(I)} \leq [1 + C_G(1 + \ln p)]\, \|u - P_p\|_{L^\infty(I)} \\ \|(u - i_p u)'\|_{L^\infty(I)} & \leq & \|(u - P_p)'\|_{L^\infty(I)} + \|\,[i_p(u - P_p)]'\,\|_{L^\infty(I)} \\ & \leq & \|(u - P_p)'\|_{L^\infty(I)} + 2p^2 \|i_p(u - P_p)\|_{L^\infty(I)} \end{array}$$

where the estimate involving the factor $2p^2$ was obtained using Markov's inequality. Using the stability result (3.25) and appealing to Lemma 3.5 concludes the proof. □

On the unit square $S = I \times I$, we introduce the interpolation operator j_p as the tensor product of the two one dimensional Gauss–Lobatto interpolation operators i_p^x, i_p^y: $j_p = i_p^x \circ i_p^y = i_p^y \circ i_p^x$. From the one dimensional stability estimate and Markov's inequality in two dimensions, we obtain the following lemma.

Lemma 3.7 *Let $u \in C(S)$. Then*

$$\begin{aligned} \|j_p u\|_{L^\infty(S)} &\leq C_G^2(1+\ln p)^2\|u\|_{L^\infty(S)}, \\ \|\partial_x j_p u\|_{L^\infty(S)}, \|\partial_y j_p u\|_{L^\infty(S)} &\leq 2C_G^2 p^2(1+\ln p)^2\|u\|_{L^\infty(S)}. \end{aligned}$$

Lemma 3.8 *Let $u \in C^\infty(S)$ satisfy $\|D^\alpha u\|_{L^2(S)} \leq C_1|\alpha|!\gamma^{|\alpha|}$ for all multi-indices $\alpha \in \mathbb{N}_0^2$. Then there are constants C_2, $\sigma > 0$ depending only on γ such that*

$$\|u - j_p u\|_{L^\infty(S)} + \|\nabla(u - j_p u)\|_{L^\infty(S)} \leq C_2 C_1 e^{-\sigma p}$$

Proof: Again by Sobolev's imbedding theorem we may assume without loss of generality that the growth condition on the derivatives has the form

$$\|D^\alpha u\|_{L^\infty(S)} \leq C_1 C_1' |\alpha|! {\gamma'}^{|\alpha|}$$

with constants C_1', γ' depending only on γ. Hence u is analytic on the closed set S. We write

$$\begin{aligned} u - j_p u &= u - i_p^x u + i_p^x\left(u - i_p^y u\right) \\ \|u - j_p u\|_{L^\infty(S)} &\leq \sup_{y\in I}\sup_{x \in I}|u - i_p^x u| + C_G(1+\ln p)\sup_{x\in I}\sup_{y\in I}|u - i_p^y u| \end{aligned}$$

Consider now the first term. For each fixed y, we obtain from Lemma 3.6

$$\sup_{x\in I}|u - i_p^x u| \leq C C_1 e^{-\sigma p}$$

where C and $\sigma > 0$ depend only on C_1', γ' and are independent of y. The second term can be estimated similarly.
For the derivative, consider $\partial_x(u - j_p u)$, the y derivative being handled analogously. As ∂_x and i_p^y commute, we have

$$\begin{aligned} \|\partial_x(u - j_p u)\|_{L^\infty(S)} &\leq \|\partial_x u - i_p^y \partial_x u\|_{L^\infty(S)} + \|i_p^y\left[\partial_x(u - i_p^x)\right]\|_{L^\infty(S)} \\ &\leq \|\partial_x u - i_p^y\partial_x u\|_{L^\infty(S)} + C_G(1+\ln p)\|\partial_x(u - i_p^x)\|_{L^\infty(S)} \end{aligned}$$

For the first term, we note that the function $\partial_x u$ satisfies a similar growth estimate for the derivatives as the original function u and therefore the reasoning of the first part of the proof applies. The second term can be estimated again by Lemma 3.6. $\square$

The regularity results of Section 2 allow us to control the derivatives of the solution u_ε of (1.1) on the physical elements. However, as we formulated the approximation results above on the reference square S, we need to see that inserting the map from the reference square to the physical element does not affect the growth of higher derivatives adversely. This is shown in the next lemma.

Lemma 3.9 *Let f and g be real analytic functions defined on Ω and $S \subset \mathbb{R}^2$, respectively. Assume that* range $g \subset\subset \Omega$ *and that*

$$\sup_{x\in\Omega}|D^\alpha f| \leq C_f|\alpha|!\gamma_f^{|\alpha|}, \qquad \sup_{x\in S}|D^\alpha g| \leq C_g|\alpha|!\gamma_g^{|\alpha|} \qquad \forall\alpha \in \mathbb{N}_0^2.$$

Then there is C, $\gamma > 0$ depending only on C_g, γ_f, γ_g such that the function $f \circ g$ satisfies

$$\sup_{x\in S}|D^\alpha(f\circ g)| \leq C C_f|\alpha|!\gamma^{|\alpha|} \qquad \forall\alpha\in\mathbb{N}_0^2.$$

Proof: The proof follows immediately from Cauchy's integral representation of higher derivatives for analytic functions in several variables. $\square$

Lemma 3.10 *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set, $S = [0,1]^2$ the reference square. Assume that $g : S \to g(S) \subset\subset \Omega$ is analytic, injective, $0 < c_1 \leq \det g' \leq c_2 < \infty$ on S, and satisfies*

$$\|D^\alpha g\|_{L^\infty(S)} \leq C_g \gamma_g^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2.$$

Let $f : \Omega \to \mathbb{C}$ be analytic on Ω and satisfy

$$\|D^\alpha f\|_{L^2(\Omega)} \leq C_f \gamma_f^{|\alpha|} \max\left(|\alpha|, \varepsilon^{-1}\right)^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2$$

for some C_f, $\gamma_f > 0$, $\varepsilon \in (0,1]$. Then there are C, $\gamma > 0$ depending only on C_g, γ_g, γ_f, c_1, c_2 such that

$$\|D^\alpha (f \circ g)\|_{L^2(S)} \leq C_f C \gamma^{|\alpha|} \max\left(|\alpha|, \varepsilon^{-1}\right)^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2, \tag{3.26}$$

$$\|D^\alpha (f \circ g)\|_{L^2(S)} \leq C_f C \gamma^{|\alpha|} |\alpha|! e^{1/\varepsilon} \qquad \forall \alpha \in \mathbb{N}_0^2. \tag{3.27}$$

Proof: The proof of (3.26) can be found in Lemma 3.13 of [7]. (3.27) follows readily from (3.26) if we observe that

$$\max\left(|\alpha|, \varepsilon^{-1}\right)^{|\alpha|} \leq \max\left(|\alpha|^{|\alpha|}, |\alpha|! \varepsilon^{-|\alpha|}/|\alpha|!\right) \leq \max\left(|\alpha|^{|\alpha|}, |\alpha|! e^{1/\varepsilon}\right) \leq C|\alpha|! e^{|\alpha|} e^{1/\varepsilon} \qquad \forall \alpha \in \mathbb{N}_0^2$$

by Stirling's formula and then replace γ of (3.26) with γe. $\square$

Piecewise interpolation in the mapped Gauss–Lobatto points yields a global H^1 conforming interpolant with global approximation properties as good as the local approximations permit:

Proposition 3.11 *Let $(\Omega_i(\lambda, p, \varepsilon))$ be a family of admissible meshes in the sense of Definition 3.1 and let M_i be the corresponding element maps. Let $u \in C(\overline{\Omega})$ and assume that*

$$\begin{aligned} |u \circ M_i - j_p(u \circ M_i)|_{L^\infty(S)} &\leq \delta_1 \\ |\nabla(u \circ M_i - j_p(u \circ M_i))|_{L^\infty(S)} &\leq \delta_2 \begin{cases} \varepsilon^{-1} & \text{if } \Omega_i \text{ is a regular element} \\ 1 & \text{if } \Omega_i \text{ is a needle element.} \end{cases} \end{aligned} \tag{3.28}$$

Then the function π_p defined on each element Ω_i by local interpolation in the mapped Gauss-Lobatto points

$$\pi_p|_{\Omega_i} = j_p(u \circ M_i) \circ M_i^{-1}$$

is an element of $S^p \subset H^1(\Omega)$ and

$$\begin{aligned} \|u - \pi_p\|_{L^\infty(\Omega)} &\leq \delta_1 \\ \|\nabla(u - \pi_p)\|_{L^\infty(\Omega)} &\leq C_1 \frac{1}{\varepsilon} \max\left(1, 1/(\lambda p)\right) \delta_2 \end{aligned}$$

with C_1 of Definition 3.1. If additionally $u = 0$ on $\partial\Omega$, then $\pi_p \in S_0^p$.

Proof: To prove the claim that $\pi_p \in S^p$, it is enough to see that π_p is continuous across interelement edges. Let $\gamma_{ij} = \overline{\Omega}_i \cap \overline{\Omega}_j$ be the common edge of two neighboring elements Ω_i, Ω_j. Denote $\gamma_i = M_i^{-1}(\gamma_{ij})$, $\gamma_j = M_j^{-1}(\gamma_{ij})$ the sides of the reference square corresponding to the common edge. By construction, the pull-backs of the traces, $t_i := (\pi_p \circ M_i)|_{\gamma_i}$ and $t_j := (\pi_p \circ M_j)|_{\gamma_j}$, are polynomials of degree p on γ_i, γ_j respectively. Furthermore, as there are $p+1$ Gauss-Lobatto interpolation points on the sides γ_i, γ_j, the polynomials t_i, t_j are determined by the location of the $p+1$ Gauss-Lobatto points on the sides γ_i, γ_j and by their values there. As the Gauss-Lobatto points are distributed symmetrically with respect to the mid-point of the sides γ_i, γ_j, the $p+1$ sampling points for the polynomials t_i, t_j are the same. It suffices to show that $t_i = t_j$ in these $p+1$ sampling points. But this follows readily from the fact that the function t_i, t_j equal $u \circ M_i$, $u \circ M_j$ in the sampling points and from assumption (M3) of Section 3.1, which implies that these $p+1$ sampling points are mapped onto the same points in $\overline{\Omega}$ under the maps M_i and M_j.
Similarly, we see that, if $u = 0$ on $\partial\Omega$, then the interpolant π_p vanishes on $\partial\Omega$, i.e., $\pi_p \in S_0^p$.
It is enough to show now that $u - \pi_p$ satisfies the desired estimates on each element Ω_i. This follows readily from the assumptions on the element maps M_i. In fact, we even have

$$\begin{aligned} \|\nabla(u - \pi_p u)\|_{L^\infty(\Omega_i)} &\leq \|\nabla\left((u \circ M_i) - j_p(u \circ M_i)\right)\|_{L^\infty(S)} \| \left(M_i^{-1}\right)' \|_{L^\infty(\Omega_i)} \\ &\leq \begin{cases} C_1 \delta_2/\varepsilon & \text{if } \Omega_i \text{ is a regular element} \\ C_1/(\lambda p \varepsilon)\delta_2 & \text{if } \Omega_i \text{ is a needle element.} \end{cases} \end{aligned}$$

□

3.3 Main Result

Theorem 3.12 *Let f, g, and $\partial\Omega$ be analytic and let u_ε be the solution of (1.1). Let $(\Omega_i(\lambda, p, \varepsilon))$ be a family of admissible boundary layer meshes in the sense of Definition 3.1. Let the function π_p be defined on each element Ω_i by local interpolation of u_ε in the (mapped) Gauss–Lobatto points, i.e.,*

$$\pi_p|_{\Omega_i} = j_p\left(u_\varepsilon \circ M_i\right) \circ M_i^{-1}.$$

Then $\pi_p \in S^p$ and there are constants C, λ_0, λ_1, and $b > 0$ depending only on f, g, $\partial\Omega$, and the constants of Definition 3.1 such that for $0 < \lambda \leq \lambda_0$ and $\lambda p \geq \lambda_1$

$$\|u_\varepsilon - \pi_p\|_{L^\infty(\Omega)} + \varepsilon\|\nabla(u_\varepsilon - \pi_p)\|_{L^\infty(\Omega)} \leq C(1 + \ln p)^2 p^2 e^{-b\lambda p}.$$

Furthermore, if $g = 0$, then $\pi_p \in S_0^p$.

Before we proceed with the proof of this theorem, let us make a few comments and extract from it the exponential rate of convergence of the Galerkin FEM (1.5) based on piecewise polynomials on admissible meshes.
Remark 3.13: Under the assumption $0 < \lambda \leq \lambda_0$, $\lambda p \geq \lambda_1$, the constants C and b in the statement of Theorem 3.12 are independent of λ, p, and ε. In practical applications of Theorem 3.12, one has to make specific choices of the parameter λ. In Theorem 3.14 below, we choose $\lambda = \lambda_0$, but other choices are possible. For example, as Theorem 3.12 does not give a useful indication of

the size of the constants λ_0, λ_1, one choice for λ could be to take it as a function of p: $\lambda = \lambda(p)$; e.g., the specific choice $\lambda(p) := 1/\ln p$ for $p \geq 2$ guarantees that the conditions $\lambda \leq \lambda_0$ and $\lambda p \geq \lambda_1$ are met provided that p is sufficiently large. Therefore, we may conclude from Theorem 3.12 that there are constants C, $b > 0$ depending only on f, $\partial\Omega$, g, and the constants of Definition 3.1 such that

$$\|u_\varepsilon - \pi_p\|_{L^\infty(\Omega)} + \varepsilon\|\nabla(u_\varepsilon - \pi_p)\|_{L^\infty(\Omega)} \leq Ce^{-bp/\ln p} \qquad \forall p \geq 2.$$

Theorem 3.12 implies the robust exponential convergence of the *hp* Galerkin FEM (1.5):

Theorem 3.14 *Assume the hypotheses of Theorem 3.12 and assume additionally that $g = 0$ in (1.1) (see Section 3.4.1 ahead for $g \neq 0$). Furthermore, let the number of elements in $(\Omega_i(\lambda, p, \varepsilon))$ be bounded independently of λ, p, ε by $N_0 \in \mathbb{N}_0$. Then there are C, b, $\lambda_0 > 0$ independent of ε and p such that the choice $V_N := S_0^p$ based on the meshes $(\Omega_i(\lambda_0, p, \varepsilon))$ in the FEM (1.5) yields for the finite element solution u_N*

$$\|u_\varepsilon - u_N\|_{\varepsilon,\Omega} \leq Ce^{-b\sqrt{N}}$$

where $N = \dim V_N = \dim S_0^p \sim p^2$.

Proof: In view of (1.6), Theorem 3.12 guarantees the existence of C, b, λ_0, and λ_1 such that

$$\|u_\varepsilon - u_N\|_{\varepsilon,\Omega} \leq \|u_\varepsilon - u_N\|_{L^2(\Omega)} + \varepsilon\|\nabla(u_\varepsilon - u_N)\|_{L^2(\Omega)} \leq C(1 + \ln p)^2 p^2 e^{-b\lambda p}$$

provided that $0 < \lambda \leq \lambda_0$, $\lambda p \geq \lambda_1$. Fixing $\lambda = \lambda_0$ we see that the finite element solution u_N satisfies

$$\|u_\varepsilon - u_N\|_{\varepsilon,\Omega} \leq Ce^{-b'p}$$

for some C, $b' > 0$. As the number of elements in the family of meshes is bounded independently of λ, p, and ε, we have $\dim S_0^p \sim p^2$ with constants depending only on N_0; this concludes the proof. $\square$

Proof of Theorem 3.12: The basis of the proof is an application of Proposition 3.11. It suffices therefore to see that δ_1, δ_2 of the assumptions of Proposition 3.11 are exponentially small (in p), that is, we have to show that

$$\begin{aligned} \|(u_\varepsilon \circ M_i) - j_p(u_\varepsilon \circ M_i)\|_{L^\infty(S)} &\leq C(1 + \ln p)^2 p^2 e^{-b\lambda p} \qquad \forall i \\ \|\nabla((u_\varepsilon \circ M_i) - j_p(u_\varepsilon \circ M_i))\|_{L^\infty(S)} &\leq C(1 + \ln p)^2 p^2 e^{-b\lambda p} \begin{cases} \varepsilon^{-1} & \text{if } \Omega_i \text{ is a regular element} \\ 1 & \text{if } \Omega_i \text{ is a needle element} \end{cases} \end{aligned} \tag{3.29}$$

where the constants C, $b > 0$ are independent of λ, p, ε, and i.
The proof consists in considering the asymptotic case (i.e., $p\varepsilon$ large) and the pre-asymptotic case ($p\varepsilon$ small) separately.
The asymptotic case $\lambda p\varepsilon \geq c_0$. By Theorem 2.1 there are constants C, K independent of ε such that

$$\|D^\alpha u_\varepsilon\|_{L^2(\Omega)} \leq CK^{|\alpha|} \max(|\alpha|, \varepsilon^{-1})^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2.$$

For each element map M_i, we can apply Lemma 3.10

$$\|D^\alpha(u_\varepsilon \circ M_i)\|_{L^2(S)} \leq C\gamma^{|\alpha|}|\alpha|! e^{1/\varepsilon} \qquad \forall \alpha \in \mathbb{N}_0^2$$

where the constants C, $\gamma > 0$ are independent of ε and i. According to Lemma 3.6, the Gauss-Lobatto interpolant $j_p(u_\varepsilon \circ M_i)$ satisfies

$$\|(u_\varepsilon \circ M_i) - j_p(u_\varepsilon \circ M_i)\|_{L^\infty(S)} + \|\nabla\left((u_\varepsilon \circ M_i) - j_p(u_\varepsilon \circ M_i)\right)\|_{L^\infty(S)} \leq Ce^{-\sigma p + 1/\varepsilon}$$

where C, σ are independent of ε and i. Using the assumption $\lambda p \varepsilon \geq c_0$ we arrive at

$$\|(u_\varepsilon \circ M_i) - j_p(u_\varepsilon \circ M_i)\|_{L^\infty(S)} + \|\nabla\left((u_\varepsilon \circ M_i) - j_p(u_\varepsilon \circ M_i)\right)\|_{L^\infty(S)} \leq Ce^{-\sigma p + \lambda p/c_0}$$

which produces the desired local estimates (3.29) if we choose λ_0 so small that $\sigma - \lambda_0/c_0 > 0$.

The pre-asymptotic case $\lambda p\varepsilon < c_0$. In the pre-asymptotic case, we have to exploit more carefully the boundary layer structure of the solution u_ε which we analyzed in Section 2.2. We decompose u_ε as

$$u_\varepsilon = w_M + \chi u_M^{BL} + r_M$$

where the expansion order $M \in \mathbb{N}_0$ is now chosen in dependence on λp: We choose M such that

$$2M + 2 = \mu\lambda p \tag{3.30}$$

where the parameter $\mu > 0$ is fixed and satisfies

$$\mu\gamma_{\Delta f} c_0 < 1, \qquad \mu c_0 < K_3, \qquad \mu c_0 K =: q < 1 \tag{3.31}$$

where $\gamma_{\Delta f}$ was defined in (1.13), c_0 is the critical width of the family of meshes considered here, the constant K_3 is the constant K_3 of Theorem 2.4, and K is the constant K of Theorem 2.5. Let us remark at this point that, strictly speaking, we should take M as the integer part of $(\mu\lambda p - 2)/2$. However, for the sake of simplicity of notation, we will ignore this point for the remainder of the proof. In order for M to be non-negative (and for technical reasons below, we need that $2M + 2 \geq 3$), we impose on λ and p the condition

$$\lambda p \geq \lambda_1 := \frac{3}{\mu}. \tag{3.32}$$

Let us now see that with this choice of the expansion order M, each of the three terms in the decomposition of u_ε can be approximated by its Gauss-Lobatto interpolant with the desired exponential accuracy. Let us first consider w_M. By the definition of w_M we have

$$w_M = \sum_{i=0}^{M} \varepsilon^{2i}\Delta^{(i)} f.$$

By Cauchy's integral theorem for derivatives we obtain with the aid of estimate (1.13), the observation $2M = \mu\lambda p - 2 \leq \mu\lambda p$, and the assumption $\mu\lambda p\varepsilon \leq \mu c_0 < 1$

$$\begin{aligned}
\|D^\alpha w_M\|_{L^\infty(\Omega)} &\leq Cd^{|\alpha|}|\alpha|! \sum_{i=0}^{M} \varepsilon^{2i}\|\Delta^{(i)} f\|_{L^\infty(\tilde\Omega)} \leq Cd^{|\alpha|}|\alpha|! \sum_{i=0}^{M} \varepsilon^{2i}\gamma_{\Delta f}^{2i}(2i)! \\
&\leq Cd^{|\alpha|}|\alpha|! \sum_{i=0}^{M} \left(\varepsilon\gamma_{\Delta f} 2M\right)^{2i} \leq Cd^{|\alpha|}|\alpha|! \sum_{i=0}^{M} \left(\gamma_{\Delta f}\mu\lambda p\varepsilon\right)^{2i} \\
&\leq Cd^{|\alpha|}|\alpha|! \sum_{i=0}^{M} \left(\gamma_{\Delta f}\mu c_0\right)^{2i} \leq Cd^{|\alpha|}|\alpha|!
\end{aligned}$$

where the constants C, d depend only on f and $\mu c_0 < 1$. Hence, we may apply Lemma 3.9 (cf. also Remark 3.3) for the estimation of $w_M \circ M_i$, and using Lemma 3.6 we obtain

$$\|(w_M \circ M_i) - j_p\left(w_M \circ M_i\right)\|_{L^\infty(S)} + \|\nabla\left((w_M \circ M_i) - j_p\left(w_M \circ M_i\right)\right)\|_{L^\infty(S)} \leq Ce^{-\sigma p}$$

where the constants C, σ are independent of ε and i.
Let us now turn to the approximation of χu_M^{BL} by its Gauss-Lobatto interpolant. We will consider the two cases of needle elements and regular elements separately. Let us first consider the case of the needle elements. By the assumptions on the meshes (cf. Definition 3.1) we have that $\lambda_U c_0 < \rho_0$. Let us assume without loss of generality that the cut-off function χ of (1.11) is chosen such that $\rho_1 = \lambda_U c_0$, i.e., $\chi \equiv 1$ for $0 < \rho \leq \lambda_U c_0$. Therefore, for needle elements Ω_i we have $\chi u_M^{BL} \equiv u_M^{BL}$ and

$$\chi u_M^{BL} \circ \psi^{-1} \circ M_i = \left(u_M^{BL} \circ s_{\lambda p\varepsilon}\right) \circ \left(s_{\lambda p\varepsilon}^{-1} \circ \psi^{-1} \circ M_i\right)$$

where $s_{\lambda p\varepsilon}$ is the stretching map introduced in (1.9) and ψ the boundary fitted coordinate transformation of (1.8). Let us estimate now the growth of the derivatives of $u_M^{BL} \circ s_{\lambda p\varepsilon}$. We have by Theorem 2.4 (note that our choice of μ and M guarantees that $\varepsilon(2M+2) < K_3$)

$$\left|\partial_\rho^n \partial_\theta^m \left(u_M^{BL} \circ s_{\lambda p\varepsilon}\right)(\rho, \theta)\right| \leq Cm! K_2^m K_1^n (\lambda p)^n \qquad \forall n, m \in \mathbb{N}_0.$$

As $(\lambda p)^n \leq n! e^{\lambda p}$ we obtain

$$\|D^\alpha \left(u_M^{BL} \circ s_{\lambda p\varepsilon}\right)\|_{L^\infty((0,\infty)\times[0,L))} \leq C|\alpha|! \max\left(K_2, K_1\right)^{|\alpha|} e^{\lambda p} \qquad \forall \alpha \in \mathbb{N}_0^2$$

and by Lemma 3.9 with $f = u_M^{BL} \circ \psi^{-1} \circ s_{\lambda p\varepsilon}$ and $g = s_{\lambda p\varepsilon}^{-1} \circ \psi^{-1} \circ M_i$

$$\|D^\alpha \left(u_M^{BL} \circ \psi^{-1} \circ M_i\right)\|_{L^\infty(S)} \leq Ce^{\lambda p} K^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2$$

where C, K are independent of λ, p, and ε. Applying Lemma 3.6 yields the existence of C, $\sigma > 0$ independent of λ, p, and ε such that (for the remainder of the proof we write $u_M^{BL} \circ M_i$ instead of $u_M^{BL} \circ \psi^{-1} \circ M_i$, thus thinking of u_M^{BL} as being given in carteasian coordinates (x, y); a similar abuse of notation applies to χu_M^{BL} below)

$$\|(u_M^{BL} \circ M_i) - j_p(u_M^{BL} \circ M_i)\|_{L^\infty(S)} + \|\nabla\left((u_M^{BL} \circ M_i) - j_p(u_M^{BL} \circ M_i)\right)\|_{L^\infty(S)} \leq Ce^{\lambda p} e^{-\sigma p}.$$

These are the desired estimates for u_M^{BL} on the needle elements if $\lambda < \sigma$. Let us now consider the regular elements Ω. By assumption, in the case $\mathrm{dist}(x, \partial\Omega) \geq \lambda_L \lambda p\varepsilon$ for all $x \in \Omega_i$, Theorem 2.4 implies immediately

$$\begin{aligned}
\|\chi u_M^{BL} \circ M_i\|_{L^\infty(S)} &\leq C_\alpha e^{-\alpha\lambda_L\lambda p}, \\
\|\nabla\left(\chi u_M^{BL} \circ M_i\right)\|_{L^\infty(S)} &\leq C_\alpha \varepsilon^{-1} e^{-\alpha\lambda_L\lambda p}.
\end{aligned}$$

Appealing to Lemma 3.7 gives (after choosing $\alpha = 1/2$)

$$\begin{aligned}
\|(\chi u_M^{BL} \circ M_i) - j_p(\chi u_M^{BL} \circ M_i)\|_{L^\infty(S)} &\leq C(1 + \ln p)^2 e^{-\lambda_L\lambda p/2}, \\
\|\nabla\left((\chi u_M^{BL} \circ M_i) - j_p(\chi u_M^{BL} \circ M_i)\right)\|_{L^\infty(S)} &\leq C(1 + \ln p)^2 p^2 \varepsilon^{-1} e^{-\lambda_L\lambda p/2}
\end{aligned}$$

which produces the desired estimates (3.29).
Finally, let us consider the remainder r_M. By Theorem 2.5 and Sobolev's imbedding theorem we have

$$\begin{aligned} \|r_M\|_{L^\infty(\Omega)} \le C\|r_M\|_{H^2(\Omega)} &\le C\varepsilon^{-2}\left(\varepsilon K(2M+2)\right)^{2M+2}, \\ \|\nabla r_M\|_{L^\infty(\Omega)} \le C\|r_M\|_{H^3(\Omega)} &\le C\varepsilon^{-3}\left(\varepsilon K(2M+2)\right)^{2M+2} \end{aligned}$$

for some C, K independent of ε and M. As $\|M_i'\|_{L^\infty(S)} \le C$ with C independent of λ, p, ε, and i, we get

$$\begin{aligned} \|r_M \circ M_i\|_{L^\infty(S)} &\le C\varepsilon^{-2}\left(\varepsilon K(2M+2)\right)^{2M+2} \qquad \forall i, \\ \|\nabla\left(r_M \circ M_i\right)\|_{L^\infty(S)} &\le C\varepsilon^{-3}\left(\varepsilon K(2M+2)\right)^{2M+2} \qquad \forall i. \end{aligned}$$

An application of Lemma 3.7 gives

$$\begin{aligned} \|(r_M \circ M_i) - j_p(r_M \circ M_i)\|_{L^\infty(S)} &\le C\varepsilon^{-2}(1+\ln p)^2\left(\varepsilon K(2M+2)\right)^{2M+2} \qquad \forall i, \\ \|\nabla\left((r_M \circ M_i) - j_p(r_M \circ M_i)\right)\|_{L^\infty(S)} &\le C\varepsilon^{-3}(1+\ln p)^2 p^2\left(\varepsilon K(2M+2)\right)^{2M+2} \qquad \forall i. \end{aligned}$$

We observe that we can write

$$\varepsilon^{-\beta}\left(\varepsilon K(2M+2)\right)^{2M+2} = \left(K(2M+2)\right)^{\beta}\left(\varepsilon K(2M+2)\right)^{2M+2-\beta} \qquad \beta = 2,3.$$

Using now the basic assumption on μ that $\varepsilon K(2M+2) = K\mu\lambda p\varepsilon \le K\mu c_0 =: q < 1$ and the assumption $2M+2 \ge 3$ (cf. (3.32)) we can bound

$$\begin{aligned} \|(r_M \circ M_i) - j_p(r_M \circ M_i)\|_{L^\infty(S)} &\le C(1+\ln p)^2(\mu\lambda p)^2 q^{\mu\lambda p-2} \qquad \forall i, \\ \|\nabla\left((r_M \circ M_i) - j_p(r_M \circ M_i)\right)\|_{L^\infty(S)} &\le C(1+\ln p)^2 p^2(\mu\lambda p)^3 q^{\mu\lambda p-3} \qquad \forall i, \end{aligned}$$

and hence we see that the interpolation of the remainder r_M in the Gauss-Lobatto points also satisfies (3.29). $\square$

3.4 Extensions of the Main Result

3.4.1 Inhomogeneous Dirichlet Conditions

We considered in Theorem 3.14 the case of homogeneous Dirichlet data. For the FE formulation of (1.1) with $g \neq 0$ we proceed as usual: Let $\tilde{u} \in S^p$ be such that $\tilde{u}|_{\partial\Omega} = \pi_p$ on $\partial\Omega$ where π_p is the Gauss-Lobatto interpolant of Theorem 3.12. This is easily accomplished as each quadrilateral element Ω_i abutting on $\partial\Omega$ has $p+1$ Gauss-Lobatto interpolation points located on $\partial\Omega_i \cap \partial\Omega$. Then the finite element formation is

$$\text{find } u_{FE} \in S_0^p \text{ such that } B_\varepsilon(u_{FE}, v) = F(v) - B_\varepsilon(\tilde{u}, v) \qquad \forall v \in S_0^p.$$

The standard arguments then yield

$$\|u_\varepsilon - (\tilde{u} + u_{FE})\|_{\varepsilon,\Omega} \le \inf_{v \in S_0^p} \|u_\varepsilon - (\tilde{u} + v)\|_{\varepsilon,\Omega} \le \|u_\varepsilon - \pi_p\|_{\varepsilon,\Omega}$$

and hence robust exponential convergence for the case of inhomogeneous analytic boundary conditions.

3.4.2 Triangular Elements

We proved the approximation result Theorem 3.12 for meshes consisting of quadrilateral elements. Let us outline in this section how similar results can be obtained for meshes consisting of quadrilateral as well as triangular elements.
Denote T the reference triangle consisting of half the reference square S. Again, the maps M_i denote the bijective, analytic maps from the reference elements K_i (i.e., either the reference square S or the reference triangle T) to the physical elements Ω_i. We assume that the maps M_i satisfy (M1), (M2) of Section 3.1. As the edges of the reference triangle T have not all length 1, condition (M3) has to be replaced with

(M3′) (M3) of Section 3.1 holds with the condition $\operatorname{dist}(M_i^{-1}(P), M_i^{-1}(P_l)) = \operatorname{dist}(M_j^{-1}(P), M_j^{-1}(P_l))$ replaced with $\operatorname{dist}(M_i^{-1}(P), M_i^{-1}(P_l))/L_i = \operatorname{dist}(M_j^{-1}(P), M_j^{-1}(P_l))/L_j$ where L_i, L_j denote the lengths of the edges of the reference elements corresponding to γ_{ij}.

(M4) For triangular elements, the maps $M_i : T \to \Omega_i$ can be extended analytically to S.

For such meshes, we may define spaces of piecewise mapped polynomials by

$$\begin{aligned} T^p &:= \{u \in H^1(\Omega) \,|\, u|_{\Omega_i} = \varphi_p \circ M_i^{-1} \quad \text{for some } \Pi_p(K_i)\} \\ T_0^p &:= T^p \cap H_0^1(\Omega) \end{aligned}$$

where we write $\Pi_p(K_i)$ to denote $Q_p(S)$ if Ω_i is a quadrilateral and $\Pi_p(K_i) = P_p(T)$, the spaces of all polynomials of total degree p, if Ω_i is a triangle.
In complete analogy to Definition 3.1, we may introduce the notion of admissible triangulations.

Definition 3.15 *A three-parameter family of meshes $(\Omega_i(\lambda, p, \varepsilon))$, $0 < \lambda \le 1$, $p \in \mathbb{N}$, $\varepsilon \in (0, 1]$ consisting of quadrilaterals and triangles and which satisfy (M1), (M2), (M3′), (M4) is called an* admissible family of triangulations *with critical width c_0, if there are λ_L, λ_U, C_1, C_2, $\gamma > 0$ with $\lambda_U c_0 < \rho_0$ such that the following holds.*
If $\lambda p\varepsilon > c_0$ then all elements are regular elements and the corresponding maps M_i satisfy

$$\|\,(M_i')^{-1}\,\|_{L^\infty(K_i)} \le C_1, \qquad \|D^\alpha M_i\|_{L^\infty(S)} \le C_2\gamma^{|\alpha|}|\alpha|! \quad \forall \alpha \in \mathbb{N}_0^2. \tag{3.33}$$

If $\lambda p\varepsilon \le c_0$, then only the following two cases may occur.

1. *Ω_i is a needle element, i.e.,* $\operatorname{dist}(M_i(x), \partial\Omega) \le \lambda_U \lambda p\varepsilon$ *for all $x \in S$ and*

$$\|\,(M_i')^{-1}\,\|_{L^\infty(K_i)} \le \frac{C_1}{\lambda p\varepsilon}, \qquad \|D^\alpha\left(s_{\lambda p\varepsilon}^{-1} \circ \psi^{-1} \circ M_i\right)\|_{L^\infty(S)} \le C_2\gamma^{|\alpha|}|\alpha|! \quad \forall \alpha \in \mathbb{N}_0^2 \tag{3.34}$$

   where the stretching operator s is defined in (1.9).

2. *Ω_i is a regular element, i.e., it satisfies (3.33) and additionally* $\operatorname{dist}(M_i(x), \partial\Omega) \ge \lambda_L \lambda p\varepsilon$ *for all $x \in S$.*

Theorem 3.16 *Let f, g, $\partial\Omega$ be analytic and u_ε be the solution of (1.1). Let $(\Omega_i(\lambda, p, \varepsilon))$ be family of admissible triangulations in the sense of Definition 3.15. Then there are constants C, b, λ_0, $\lambda_1 > 0$ depending only on the data f, g, $\partial\Omega$ and the constants of Definition 3.15 such that for $0 < \lambda \leq \lambda_0$ and $\lambda p \geq \lambda_1$*

$$\inf_{v \in T^p} \|u_\varepsilon - v\|_{L^\infty(\Omega)} + \varepsilon \|\nabla(u_\varepsilon - v)\|_{L^\infty(\Omega)} \leq C(1 + \ln p)^2 p^6 e^{-b\lambda p}. \tag{3.35}$$

If $g = 0$, then the infimum in (3.35) may be taken over T_0^p.

Proof: The proof is very similar to the classical p version proof. It consists of finding first a discontinuous piecewise (mapped) polynomial approximation and then correct the interelement jumps with the aid of an appropriate lifting. Such a lifting may take the following shape: Let f_p be a polynomial of degree p defined on an edge γ of the reference square (triangle) K, and assume that f_p vanishes in the endpoints of γ. Then there is a lifted polynomial $F_p \in \Pi_p(K)$ which equals f_p on γ, vanishes on all the other edges of K, and satisfies

$$\|F_p\|_{L^\infty(K)} \leq Cp^2 \|f_p\|_{L^\infty(\gamma)}, \qquad \|\nabla F_p\|_{L^\infty(K)} \leq Cp^4 \|f_p\|_{L^\infty(\gamma)}$$

for some generic $C > 0$.
Let us now outline the proof. Without loss of generality we may assume that p is even. By assumption (M4) we may assume that the element maps M_i are always defined on S. Checking the proof of Theorem 3.12, we see that the assumptions of Definition 3.15 guarantee that (3.29) holds. Set the local (discontinuous) approximations $v_i = j_p(u_\varepsilon \circ M_i) \in Q_p(S)$ if Ω_i is a quadrilateral and choose $v_i = (j_{p/2}(u_\varepsilon \circ M_i))|_T \in P_p(T)$ if Ω_i is a triangle. Note that the vertices of the reference elements K_i are sampling points of the Gauss-Lobatto interpolation operators j_p, $j_{p/2}$. The above lifting allows us to conclude the proof of Theorem 3.16 just as in the standard p version proof. □

3.4.3 Meshes graded geometrically towards the boundary

Theorem 3.14 shows that the *hp*-FEM based on admissible meshes yields robust exponential convergence. However, the meshes depend on ε as well as p. In practice, it may be more convenient to fix a mesh and then increase the polynomial degree p until the desired accuracy is reached. Let us demonstrate in a simple setting how Theorem 3.12 can be applied in such a situations. The basic idea is to use a mesh that is refined geometrically (anisotropically) towards the boundary in such a way that the smallest element has width $O(\varepsilon)$. This produces a fixed mesh that essentially "contains" all the admissible meshes $(\Omega_i(\lambda_0, p, \varepsilon))$ for some λ_0 sufficiently small as stipulated in Theorem 3.12. Therefore approximation results similar to Theorem 3.12 hold true for that mesh as well. We will construct such a geometrically graded mesh as a variation of the construction of Section 3.1.2.
Let Ω_i, $i = 1, \ldots, N$ be the fixed coarse mesh of Section 3.1.2. Let us now create a mesh that is graded geometrically towards the boundary by subdividing the element Ω_i, $i = 1, \ldots, n$. Fix a grading factor $0 < \sigma < 1$ and a number $L \in \mathbb{N}$ of layers. Subdivide the square S into $L + 1$ rectangles as follows:

$$S^0 := (0, \sigma^L) \times (0, 1), \qquad S^l := (\sigma^{L-l+1}, \sigma^{L-l}) \times (0, 1), \qquad l = 1, \ldots, L$$

and then set for $i = 1, \dots, n$

$$\begin{array}{rclll} \Omega_i^0 & := & M_i(S^0), & M_i^0(\xi,\eta) := M_i(\sigma^L \xi, \eta) & \\ \Omega_i^l & := & M_i(S^l), & M_i^l(\xi,\eta) := M_i(\sigma^{L-l+1} + \xi\sigma^{L-l}, \eta), & l = 1, \dots, L \end{array}$$

It is easy to see that the element Ω_i^l, $i = 1, \dots, n$, $l = 0, \dots, L$, together with the elements Ω_i, $i = n+1, \dots, N$ satisfy the conditions (M1)–(M3).
Furthermore, let us assume that the number of layers L is chosen such that the smallest element has width $O(\varepsilon)$, i.e., let $L \in \mathbb{N}$ be such that

$$\sigma^L = c_L \varepsilon. \tag{3.36}$$

Let us now denote S^p_{geom} the ansatz space of the type (3.11) based on this geometric mesh and denote $S^p_I(\lambda, \varepsilon)$ the piecewise polynomial space of type (3.11) based on the mesh family described in Section 3.1.2. We observe that

$$S^p_{geom} \supset S^p_I(\lambda,\varepsilon) \quad \text{if there is } l \in \{1, \dots, L\} \text{ such that } \lambda p \varepsilon = \sigma^l.$$

By Theorem 3.12 there are C, b, λ_0, and λ_1 depending only on the input data f, g, $\partial\Omega$, and on the mesh family of Section 3.1.2 such that for $0 < \lambda \le \lambda_0$ and $\lambda p \ge \lambda_1 > 0$ the piecewise Gauss-Lobatto interpolant π_p of the solution u_ε satisfies

$$\|u_\varepsilon - \pi_p\|_{L^\infty(\Omega)} + \varepsilon \|\nabla(u_\varepsilon - \pi_p)\|_{L^\infty(\Omega)} \le C(1 + \ln p)^2 p^2 e^{-b\lambda p}. \tag{3.37}$$

To obtain estimates on the approximation properties of the spaces S^p_{geom} it suffices to show that for given p and ε, a judicious choice of $0 < \lambda \le \lambda_0$ yields $S^p_I(\lambda, \varepsilon) \subset S^p_{geom}$.
If the polynomial degree p satisfies $\lambda_0 p \varepsilon \ge 1/2$ then we may choose $\lambda = \lambda_0$, i.e., the spaces $S^p_I(\lambda_0, \varepsilon)$: By construction, for $\lambda_0 p \varepsilon \ge 1/2$ the space $S^p_I(\lambda_0, \varepsilon)$ consists just of all continuous piecewise (mapped) polynomials on the mesh $\Omega_1, \dots, \Omega_N$ and thus $S^p_I(\lambda_0, \varepsilon) \subset S^p_{geom}$ for all $L \in \mathbb{N}$.
Let us therefore concentrate on the case $\lambda_0 p \varepsilon < 1/2$. Let us assume additionally that the polynomial degree p is such that

$$\sigma \lambda_0 p \ge \max(\lambda_1, c_L) \tag{3.38}$$

Under these assumptions on p and ε, it is easy to see that there are $\lambda \in [\sigma\lambda_0, \lambda_0]$ and $l \in \{1, \dots, L\}$ such that

$$\lambda p \varepsilon = \sigma^l.$$

Hence, we may conclude with the same constants as in (3.37) and the assumption (3.38) on the polynomial degree p:

$$\inf_{v \in S^p_{geom}} \|u_\varepsilon - v\|_{L^\infty(\Omega)} + \varepsilon \|u_\varepsilon - v\|_{L^\infty(\Omega)} \le C(1 + \ln p)^2 p^2 e^{-b\sigma\lambda_0 p} \tag{3.39}$$

Thus, we obtain robust exponential rates of convergence of the FEM with fixed, geometrically graded meshes by merely increasing p on a fixed mesh. However, the number of elements of this mesh depends on ε as the number of layers in the geometric mesh refinement is linked to the perturbation parameter ε. Nevertheless, by (3.36) $L \sim |\ln \varepsilon|$, and thus this dependence is quite weak.

4 Spectral Element Method

In a practical implementation of (1.5) we have to evaluate the bilinear form $B_\varepsilon(u,v)$ and the right hand side $F(v)$ for functions $u,v \in S_0^p$ when creating the stiffness matrix and the load vector. As the elements of S_0^p are mapped polynomials with analytic mapping functions M_i, the integrands arising on the reference element are no longer polynomial. Therefore the integrals cannot (in general) be computed exactly, and we have to resort to some numerical quadrature scheme for the calculation of the stiffness matrix and the load vector as in the *spectral element method* [14]. The aim of this section is to demonstrate that the spectral element method, i.e., the use of a Gauss-Lobatto quadrature rule with $O(p)$ points (in each direction) preserves the exponential rate of convergence of the *hp*-FEM (Theorem 3.14).
We introduced two types of meshes in Section 3.1, admissible mesh families and regular admissible mesh families. For the approximation result Theorem 3.12 we merely needed an admissible mesh family. In these meshes, the maximal angle in needle elements is allowed to degenerate to π as ε tends to zero. For our analysis of the effect of the numerical quadrature, we exclude this case and consider a subclass of these meshes, namely, regular admissible meshes. Note that the examples of Sections 3.1.1, 3.1.2 are both regular admissible mesh families.

4.1 Preliminaries

On the reference square $S=(0,1)^2$, we denote by GL^{p+q} the Gauss-Lobatto quadrature rule with $p+q+1$ points in each direction, i.e.,

$$\int_S g\, d\xi d\eta \approx GL^{p+q}(g) := \sum_{n=0}^{p+q}\sum_{m=0}^{p+q} w_n w_m g(\xi_n,\xi_m).$$

Here the points ξ_n are given by $\xi_0 = 0$, $\xi_{p+q} = 1$ and for $1 \le n \le p+q-1$ the points ξ_n are the roots of the derivative of the Legrendre polynomial of order $p+q$ associated with the interval $I=(0,1)$. The weights w_n are all positive and chosen such that the Gauss-Lobatto quadrature rule is exact for polynomials of degree $2(p+q)-1$. For technical reasons, we will assume in all of Section 4 that $q \ge 1$ and this implies that

$$\|g\|^2_{L^2(S)} = GL^{p+q}(g^2) \qquad \text{for all polynomials } g \text{ of degree } p. \tag{4.1}$$

We will use Gauss-Lobatto quadrature rules for rectangles $R=(0,a)\times(0,b)$ and denote it by GL_R^{p+q}. Clearly, again polynomials of degree $2(p+q)-1$ are integrated exactly on R.

Lemma 4.1 *There is a generic constant $C>0$ such that for any rectangle R and any $p \ge 1$*

$$\text{area}(R)\|g_p\|^2_{L^\infty(R)} \le Cp^4\|g_p\|^2_{L^2(R)} \qquad \textit{for all polynomials } g_p \textit{ of degree } p.$$

Proof: The case of R being the unit square is standard. The case of a general rectangle follows by a change of variables argument. □

Lemma 4.2 *Let R be a rectangle and GL_R^{p+q} the Gauss-Lobatto rule of order $p+q$ with $q \geq 1$. Denote by j_{p+q} the Gauss-Lobatto interpolation operator on R as in Section 3.2. Then for all polynomials w_p of degree p and all functions $g \in C(\overline{R})$*

$$\left| \int_R g w_p \, dxdy - GL_R^{p+q}(g w_p) \right| \leq \|g - j_{p+q} g\|_{L^2(R)} \|w_p\|_{L^2(R)}.$$

Proof: As the function $(j_{p+q}g)w_p$ is a polynomial of degree $2(p+q)-1$ the Gauss-Lobatto quadrature rule is exact for $(j_{p+q}g)w_p$ by the assumption $q \geq 1$. Hence,

$$\begin{aligned} \left| \int_R g w_p \, dxdy - GL_R^{p+q}(g w_p) \right| &= \left| \int_R (g - j_{p+q}g) w_p \, dxdy - GL_R^{p+q}((j_{p+q}g - g) w_p) \right| \\ &\leq \|g - j_{p+q}g\|_{L^2(R)} \|w_p\|_{L^2(R)} \end{aligned}$$

where we made use of the observation that $j_{p+q}g - g = 0$ at the sampling points of the Gauss-Lobatto quadrature rule. □

Lemma 4.3 *There is a generic constant $C > 0$ such that the following holds true. Let R be any rectangle and denote by j_q the Gauss-Lobatto interpolation operator on R. Then for p, $q \geq 1$ and any polynomials v_p, w_p of degree p and any function $g \in C(\overline{R})$*

$$\left| \int_R v_p g w_p \, dxdy - GL_R^{p+q}(v_p g w_p) \right| \leq C(1 + \ln p)^2 p^2 \|g - j_q g\|_{L^\infty(R)} \|v_p\|_{L^2(R)} \|w_p\|_{L^2(R)}.$$

Proof: Applying Lemma 4.2, we obtain

$$\begin{aligned} \left| \int_R v_p g w_p \, dxdy - GL_R^{p+q}(v_p g w_p) \right| &\leq \|v_p g - j_{p+q}(v_p g)\|_{L^2(R)} \|w_p\|_{L^2(R)} \\ &\leq \sqrt{\text{area}(R)} \|v_p g - j_{p+q}(v_p g)\|_{L^\infty(R)} \|w_p\|_{L^2(R)} \\ &= \sqrt{\text{area}(R)} \|(v_p g - v_p j_q g) - j_{p+q}\left(v_p g - v_p j_q g\right)\|_{L^\infty(R)} \|w_p\|_{L^2(R)}. \end{aligned}$$

By Lemma 3.7 (Lemma 3.7 is formulated for the reference square S but the invariance of the L^∞ norm under transformations gives readily that the first estimate of Lemma 3.7 holds for any rectangle R) we can estimate

$$\|(v_p g - v_p j_q g) - j_{p+q}\left(v_p g - v_p j_q g\right)\|_{L^\infty(R)} \leq (1 + C_G^2)(1 + \ln p)^2 \|v_p g - v_p j_p g\|_{L^\infty(R)}.$$

Lemma 4.1 gives $\sqrt{\text{area}(R)} \|v_p\|_{L^\infty(R)} \leq C p^2 \|v_p\|_{L^2(R)}$ which allows us to conclude the proof. □

4.2 The spectral element method

For $u,\, v \in S_0^p$ we can write

$$\begin{aligned} B_\varepsilon(u,v) &= \sum_i \varepsilon^2 \int_{\Omega_i} \nabla u \cdot \nabla v \, dxdy + \int_{\Omega_i} uv \, dxdy \\ &= \sum_i \varepsilon^2 \int_S \nabla_{(\xi,\eta)} \hat{u}_i \cdot \hat{A}_i(\xi,\eta) \nabla_{(\xi,\eta)} \hat{v}_i \, d\xi d\eta + \int_S \hat{u}_i \hat{v}_i \det M_i' \, d\xi d\eta \\ F(v) &= \sum_i \int_{\Omega_i} fv \, dxdy = \sum_i \int_S \hat{f}_i \hat{v}_i \det M_i' \, d\xi d\eta \end{aligned} \tag{4.2}$$

where

$$\hat{u}_i = u \circ M_i, \qquad \hat{v}_i = v \circ M_i, \qquad \hat{f}_i = f \circ M_i, \tag{4.3}$$

$$\hat{A}_i = (M_i')^{-T} \cdot (M_i')^{-1} \, \det M_i'. \tag{4.4}$$

Note that the functions $\hat{u}_i$, $\hat{v}_i$ are polynomials of degree p as $u,\, v \in S_0^p$. Replacing all the integrals in the definition of B_ε and F by the Gauss-Lobatto quadrature rule of order $p+q$, we can define

$$\begin{aligned} B_\varepsilon^{GL}(u,v) &:= \sum_i \varepsilon^2 GL^{p+q} \left(\nabla_{(\xi,\eta)} \hat{u}_i \cdot \hat{A}_i \nabla_{(\xi,\eta)} \hat{v}_i \right) + GL^{p+q} \left(\hat{u}_i \hat{v}_i \det M_i' \right) \\ F^{GL}(v) &:= \sum_i GL^{p+q} \left(\hat{f}_i \hat{v}_i \det M_i' \right) \end{aligned}$$

for all $u,\, v \in S_0^p$. The spectral element method reads:

$$\text{find } u_{GL} \in S_0^p \text{ such that } B_\varepsilon^{GL}(u_{GL}, v) = F^{GL}(v) \qquad \forall v \in S_0^p. \tag{4.5}$$

Theorem 4.4 *Let f be analytic on $\overline{\Omega}$, $g = 0$, and u_ε be the exact solution of (1.1). Let $(\Omega_i(\lambda, p, \varepsilon))$ be a family of* regular *admissible meshes in the sense of Definition 3.2. Then there are C, $\sigma > 0$ independent of λ, p, ε, q such that the finite element solution u_{GL} of (4.5) satisfies*

$$\|u_\varepsilon - u_{GL}\|_{\varepsilon,\Omega} \le C \left(\inf_{v \in S_0^p} \|u_\varepsilon - v\|_{\varepsilon,\Omega} + (1 + \ln p)^2 p^2 e^{-\sigma q} \right).$$

As the proof of Theorem 4.4 is based on several lemmas, it is deferred to the end of this section. Theorem 4.4 shows that the use of Gauss-Lobatto quadrature rules of sufficiently high order does not destroy the exponential rate of convergence of the finite element method (1.5): By Theorem 3.14, the infimum can be bounded by $Ce^{-b'p}$ and hence choosing $q = \nu p$ with $\nu > 0$ allows us to conclude that the error of the finite element approximation with Gauss-Lobatto quadrature is exponentially small. The parameter q is a measure of "overintegration"; this overintegration is necessary as the integrals arising in the definition of B_ε are essentially weighted L^2 inner products of polynomials of degree p and the overintegration guarantees that the weight function is accounted for properly.

Remark 4.5: The proof of Theorem 4.4 shows that Theorem 4.4 holds true for other quadrature rules as well, such as Gaussian integration. The proof also shows that one could use a Gauss-Lobatto rule of order p (instead of $p+q$) for the integration of the load vectors F.

Remark 4.6: Let us stress that the conditions on the mesh are more restrictive than in Theorem 3.14 as we need a regular admissible mesh family rather than merely an admissible one (or an admissible triangulation). However, the two examples of boundary layer meshes considered in Section 3.1 are regular admissible meshes.

The proof of Theorem 4.4 will be done in the framework of a lemma of Strang ([15], [16]):

Lemma 4.7 (Lemma of Strang) *Assume that the bilinear form B_ε^{GL} is coercive on S_0^p, i.e., it satisfies*

$$\beta_{p+q}\|u\|_{\varepsilon,\Omega}^2 \leq B_\varepsilon^{GL}(u,u) \qquad \forall u \in S_0^p$$

for some $\beta_{p+q} > 0$. Then problem (4.5) has a unique solution u_{GL} satisfying

$$\begin{aligned} \|u_\varepsilon - u_{GL}\|_{\varepsilon,\Omega} \;&\leq\; (1+\beta_{p+q}^{-1})\Big\{\inf_{v\in S_0^p}\left(\|u_\varepsilon - v\|_{\varepsilon,\Omega} + \sup_{w\in S_0^p}\frac{|B_\varepsilon(v,w) - B_\varepsilon^{GL}(v,w)|}{\|w\|_{\varepsilon,\Omega}}\right) \\ &\qquad + \sup_{w\in S_0^p}\frac{|F(w)-F^{GL}(w)|}{\|w\|_{\varepsilon,\Omega}}\Big\}. \end{aligned}$$

The proof of Theorem 4.4 follows immediately from Lemma 4.7 if we can show that the two consistency terms satisfy bounds of the form e^{-bq} and if we can show that the coercivity constants β_{p+q} can be bounded from below uniformly in the integration order $p+q$ and uniformly in the perturbation parameter ε.
For the rest of this section, we will make use of the fact that the meshes considered here are regular admissible. This implies that we can define "reference needle elements", and we define these reference elements R_i with corresponding element maps $\widetilde{M}_i : R_i \to \Omega_i$ by

$$\begin{aligned} R_i \;&:=\; \begin{cases} S & \text{if } \Omega_i \text{ is a regular element} \\ (0,\lambda p\varepsilon)\times(0,1) & \text{if } \Omega_i \text{ is a needle element,} \end{cases} \\ \widetilde{M}_i(\xi,\eta) \;&:=\; \begin{cases} M_i(\xi,\eta) & \text{if } \Omega_i \text{ is a regular element} \\ M_i(\xi/(\lambda p\varepsilon),\eta) & \text{if } \Omega_i \text{ is a needle element.} \end{cases} \end{aligned}$$

If we introduce the notation

$$\widetilde{u}_i = u\circ\widetilde{M}_i, \qquad \widetilde{v}_i = v\circ\widetilde{M}_i, \qquad \widetilde{f}_i = f\circ\widetilde{M}_i, \tag{4.6}$$

$$\widetilde{A}_i = \left(\widetilde{M}_i'\right)^{-T}\cdot\left(\widetilde{M}_i'\right)^{-1}\det\widetilde{M}_i' \tag{4.7}$$

we can write B_ε and F as in (4.2) with A_i, $\hat{f}_i$, M_i and S replaced with $\widetilde{A}_i$, $\widetilde{f}_i$, $\widetilde{M}_i$, and R_i respectively. The Gauss-Lobatto integrations read then

$$\begin{aligned} B_\varepsilon^{GL}(u,v) \;&=\; \sum_i \varepsilon^2 GL_{R_i}^{p+q}\left(\nabla_{(\xi,\eta)}\widetilde{u}_i\cdot\widetilde{A}_i\nabla_{(\xi,\eta)}\widetilde{v}_i\right) + GL_{R_i}^{p+q}\left(\widetilde{u}_i\widetilde{v}_i\det\widetilde{M}_i'\right), \\ F^{GL}(v) \;&=\; \sum_i GL_{R_i}^{p+q}\left(\widetilde{f}_i\widetilde{v}_i\det\widetilde{M}_i'\right) \end{aligned}$$

where we write $GL^{p+q}_{R_i}$ to denote the Gauss-Lobatto quadrature rule with $p+q+1$ points in each direction on the square R_i. Let us note that the functions $\widetilde{u}_i$, $\widetilde{v}_i$ are polynomials if u, $v \in S^p_0$ and that the Gauss-Lobatto formulas $GL^{p+q}_{R_i}$ integrate exactly polynomials of degree $2(p+q)-1$ on R_i.
In order to apply Lemma 4.7, we need to study the effect of the functions $\widetilde{A}_i$ and $\det \widetilde{M}'_i$ on the numerical quadrature. We have

Lemma 4.8 *Let $(\Omega_i(\lambda, p, \varepsilon))$ be a regular admissible family of meshes in the sense of Definition 3.2. Then there exist C_1, C_2, and $\gamma > 0$ independent of λ, p, ε, and i such that the symmetric matrices $\widetilde{A}_i$ and the Jacobians $\det \widetilde{M}'_i$ satisfy*

$$\begin{aligned} C_1^{-1} &\leq \widetilde{A}_i \leq C_1 \qquad \text{on } R_i \quad \forall i, && (4.8)\\ C_1^{-1} &\leq \det \widetilde{M}'_i \leq C_1 \qquad \text{on } R_i \quad \forall i, && (4.9)\\ \|D^\alpha \widetilde{A}_i\|_{L^\infty(R_i)} &\leq C_2 \gamma^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}^2_0, \quad \forall i, && (4.10)\\ \|D^\alpha \det \widetilde{M}'_i\|_{L^\infty(R_i)} &\leq C_2 \gamma^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}^2_0, \quad \forall i. && (4.11) \end{aligned}$$

The proof of this technical lemma is postponed until the end of this section.

Lemma 4.9 *Let $(\Omega_i(\lambda, p, \varepsilon))$ be a regular admissible family of meshes. Then the bilinear form B^{GL}_ε satisfies with the constant C_1 of Lemma 4.8*

$$C_1^{-2} \|u\|^2_{\varepsilon,\Omega} \leq B^{GL}_\varepsilon(u,u) \qquad \forall u \in S^p_0$$

for all p, $q \geq 1$.

Proof: For $u \in S^p_0$ we note that the function $\widetilde{u}_i$ is a polynomial of degree p in each variable. Furthermore, by the assumption that $q \geq 1$, we have that the Gauss-Lobatto rule on R_i integrates polynomials of degree $2p$ (in each variable) exactly. Hence we can estimate with the aid of Lemma 4.8

$$\begin{aligned} B^{GL}_\varepsilon(u,u) &= \sum_i \varepsilon^2 GL_{R_i}(\nabla_{(\xi,\eta)} \widetilde{u}_i \cdot \widetilde{A}_i \nabla_{(\xi,\eta)} \widetilde{u}_i) + GL_{R_i}(\widetilde{u}_i^2 \det \widetilde{M}'_i)\\ &\geq C_1^{-1} \left(\sum_i \varepsilon^2 GL_{R_i}(|\nabla_{(\xi,\eta)} \widetilde{u}_i|^2) + GL_{R_i}(|\widetilde{u}_i|^2) \right)\\ &= C_1^{-1} \left(\sum_i \varepsilon^2 \int_{R_i} \left|\nabla_{(\xi,\eta)} \widetilde{u}_i\right|^2 \, d\xi d\eta + \int_{R_i} |\widetilde{u}_i|^2 \, d\xi d\eta \right)\\ &\geq C_1^{-2} \left(\sum_i \varepsilon^2 \int_{R_i} \nabla_{(\xi,\eta)} \widetilde{u}_i \cdot \widetilde{A}_i \nabla_{(\xi,\eta)} \widetilde{u}_i \, d\xi d\eta + \int_{R_i} |\widetilde{u}_i|^2 \det \widetilde{M}'_i \, d\xi d\eta \right)\\ &= C_1^{-2} \|u\|^2_{\varepsilon,\Omega}. \end{aligned}$$

□

Lemma 4.10 *Let C_1 be as in Lemma 4.8. There exists a generic constant $C > 0$ depending only on the constants of Lemma 4.2, 4.3 such that for p, $q \geq 1$ the following holds true.*

$$\left| \int_{R_i} w_p \widetilde{f}_i \det \widetilde{M}_i' \, d\xi d\eta - GL_{R_i}^{p+q} \left(w_p \widetilde{f}_i \det \widetilde{M}_i' \right) \right| \leq c(f,i) \|w_p\|_{L^2(R_i)} \tag{4.12}$$

$$\left| \int_{R_i} v_p w_p \det \widetilde{M}_i' d\xi d\eta - GL_{R_i}^{p+q} \left(v_p w_p \det \widetilde{M}_i' \right) \right| \leq Cc(i)(1+\ln p)^2 p^2 \|v_p\|_{L^2(R_i)} \|w_p\|_{L^2(R_i)} \tag{4.13}$$

$$\left| \int_{R_i} \nabla v_p \cdot \widetilde{A}_i \nabla w_p d\xi d\eta - GL_{R_i}^{p+q} \left(\nabla v_p \cdot \widetilde{A}_i \nabla w_p \right) \right| \leq CC_1 c(\widetilde{A}_i, i)(1+\ln p)^2 p^2 \times \times \|v_p\|_{\widetilde{A}_i, R_i} \|w_p\|_{\widetilde{A}_i, R_i} \tag{4.14}$$

where

$$\begin{aligned}
c(f,i) &:= \|\widetilde{f}_i \det \widetilde{M}_i' - j_{p+q} \left(\widetilde{f}_i \det \widetilde{M}_i' \right) \|_{L^2(R_i)} \\
c(\widetilde{A}_i, i) &:= \max_{m,n=1,2} \{ \|(\widetilde{A}_i)_{m,n} - j_q(\widetilde{A}_i)_{m,n}\|_{L^\infty(R_i)} \} \\
c(i) &:= \|\widetilde{f}_i \det \widetilde{M}_i' - j_q \left(\widetilde{f}_i \det \widetilde{M}_i' \right) \|_{L^\infty(R_i)} \\
\|v\|_{\widetilde{A}_i, R_i} &:= \left(\int_{R_i} \nabla v \cdot \widetilde{A}_i \nabla v \, d\xi d\eta \right)^{1/2} \qquad \forall v \in H^1(R_i)
\end{aligned}$$

and where we used the standard notation $(\widetilde{A}_i)_{m,n}$ to denote the (m,n) entry of the matrix $\widetilde{A}_i$.

Proof: (4.12) follows directly from Lemma 4.2. The proof of (4.13) is analogous to that of (4.14), and we will therefore omit it. In order to prove (4.14), we write $(\xi, \eta) = (\xi_1, \xi_2)$ and decompose

$$\int_{R_i} \nabla v_p \cdot \widetilde{A}_i \nabla w_p d\xi d\eta - GL_{R_i}^{p+q} \left(\nabla v_p \cdot \widetilde{A}_i \nabla w_p \right) = \sum_{m,n=1}^{2} \int_{R_i} \partial_{\xi_n} v_p (\widetilde{A}_i)_{m,n} \partial_{\xi_m} w_p d\xi d\eta - GL_{R_i}^{p+q} \left(\partial_{\xi_n} v_p (\widetilde{A}_i)_{m,n} \partial_{\xi_m} w_p \right).$$

Each of the terms in this double sum may be estimated by Lemma 4.3, and we get using the definition of $c(\widetilde{A}_i, i)$

$$\left| \int_{R_i} \nabla v_p \cdot \widetilde{A}_i \nabla w_p d\xi d\eta - GL_{R_i}^{p+q} \left(\nabla v_p \cdot \widetilde{A}_i \nabla w_p d\xi d\eta \right) \right| \leq C(1+\ln p)^2 p^2 c(\widetilde{A}_i, i) \|\nabla v_p\|_{L^2(R_i)} \|\nabla w_p\|_{L^2(R_i)}.$$

Finally, Lemma 4.8 allows us to estimate

$$\|\nabla v_p\|_{L^2(R_i)} \leq C_1^{1/2} \|v_p\|_{\widetilde{A}_i, R_i}, \qquad \|\nabla w_p\|_{L^2(R_i)} \leq C_1^{1/2} \|w_p\|_{\widetilde{A}_i, R_i}$$

which concludes the proof. □

We are now in position to prove Theorem 4.4.

Proof of Theorem 4.4: The proof of Theorem 4.4 follows from Lemma 4.7, if we can bound the coercivity constant β_{p+q} from below and if we can control the consistency terms. Lemma 4.8 gives immediately that $\beta_{p+q} \geq C_1^{-1}$ for all p, $q \geq 1$. Let us therefore turn to the consistency terms. Lemma 4.10 implies that

$$\begin{aligned} \left|F(w) - F^{GL}(w)\right| &\leq C \max_i c(f,i) \|w\|_{L^2(\Omega)} \qquad \forall w \in S_0^p, \\ \left|B_\varepsilon(v,w) - B_\varepsilon^{GL}(v,w)\right| &\leq C C_1 (1+\ln p)^2 p^2 \max_i \{c(\widetilde{A}_i, i), c(i)\} \|v\|_{\varepsilon,\Omega} \|w\|_{\varepsilon,\Omega} \qquad \forall v, w \in S_0^p. \end{aligned}$$

We are therefore left with estimating $c(f,i)$, $c(i)$, and $c(\widetilde{A}_i, i)$. Let us just show how we will obtain the desired estimates for the $c(\widetilde{A}_i, i)$. By Lemma 4.8, we can control the derivatives of the entries of $\widetilde{A}_i$ uniformly in i. The functions

$$A_i(\xi,\eta) = \begin{cases} \widetilde{A}_i(\xi,\eta) & \text{if } \Omega_i \text{ is a regular element} \\ \widetilde{A}_i(\xi\lambda p\varepsilon, \eta) & \text{if } \Omega_i \text{ is a needle element} \end{cases}$$

are defined on the reference square S and satisfy growth estimates of the form required by Lemma 3.6 with constants independent of λ, p, ε, and i (note that for the case of needle elements, Definition 3.1 stipulates that $\lambda p \varepsilon \leq c_0$). Hence we obtain from Lemma 3.6 (using the scaling invariance of L^∞ norm)

$$\|(\widetilde{A}_i)_{m,n} - j_q(\widetilde{A}_i)_{m,n}\|_{L^\infty(R_i)} \leq C e^{-\sigma q}$$

where the constants C, $\sigma > 0$ are independent of λ, p, ε, and i. This completes the proof. ☐

Proof of Lemma 4.8: By the definition of regular admissible meshes, there are constants c_1, c_2, c_3, and γ independent of λ, p, ε such that

$$c_1 \leq \det \widetilde{M}_i' \leq c_2 \qquad \text{on } R_i \quad \forall i, \tag{4.15}$$

$$\|D^\alpha \widetilde{M}_i\|_{L^\infty(R_i)} \leq c_3 \gamma^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2 \quad \forall i. \tag{4.16}$$

(4.16) implies that there is $\delta > 0$ (depending only on γ) such that the functions $\widetilde{M}_i$ are holomorphic on

$$B_\delta := \{(x+z_1, y+z_2) \,|\, (x,y) \in R_i, z_1, z_2 \in \mathbb{C} \text{ with } |z_1|, |z_2| < \delta\}$$

and that there is $C > 0$ depending also only on γ and c_3 such that

$$\|\widetilde{M}_i\|_{L^\infty(B_\delta)} \leq C \qquad \forall i. \tag{4.17}$$

With the aid of Cauchy's integral formula for derivatives, we deduce from (4.17) that there are C, $\gamma > 0$ such that

$$\|D^\alpha \widetilde{M}_i\|_{L^\infty(B_{\delta/2})} \leq C \gamma^{|\alpha|} |\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2$$

where $B_{\delta/2}$ defined analogously to B_δ. This implies readily (4.11) and that the functions $\det \widetilde{M}_i'$ are uniformly Lipschitz continuous on $B_{\delta/4}$. (4.15) together with this uniform Lipschitz continuity gives the existence of $\delta' > 0$ (independent of λ, p, and ε) such that

$$c_1/2 \leq |\det \widetilde{M}_i'(x,y)| \leq 2c_2 \qquad \forall i \quad \forall (x,y) \in B_{\delta'}. \tag{4.18}$$

Estimates (4.17) and (4.18) together with Cramer's rule allow us to control the entries of $\widetilde{M}_i^{-1}$ on $B_{\delta'}$. Cauchy's integral theorem for derivatives gives the existence of C, $\gamma > 0$ such that

$$\|D^\alpha \widetilde{M}_i^{-1}\|_{L^\infty(R_i)} \leq C\gamma^{|\alpha|}|\alpha|! \qquad \forall \alpha \in \mathbb{N}_0^2 \tag{4.19}$$

and from this we can infer easily (4.10). It remains to see (4.8). For that, we have to get estimates for the eigenvalues $0 < \lambda_1 \leq \lambda_2$ of the symmetric positive definite matrices $\widetilde{A}_i = \widetilde{M}_i^{-T} \cdot \widetilde{M}_i^{-1} \det \widetilde{M}_i'$. Clearly, by (4.19), (4.18) we get uniform upper bounds for the eigenvalues, i.e., there is $C_1' > 0$ independent of i, (x, y) such that $\lambda_2 \leq C_1'$. For the lower estimates, we infer from (4.15)

$$c_2^{-1} \leq \det(\widetilde{M}_i')^{-1} = \det(\widetilde{M}_i')^{-T} \leq c_1^{-1} \qquad \text{on } R_i$$

and then conclude together with (4.15) that $c_2^{-2} c_1 \leq \det \widetilde{A}_i = \lambda_1\lambda_2$. As $\lambda_2 \leq C_1'$, this implies the desired uniform lower estimate for λ_1. □

5 Numerical Example

As we mentioned in the Introduction, robust exponential convergence was already observed and conjectured in [6] for boundary fitted tensor product meshes. The present paper has rigorously established this conjecture. We refer to [17] for a variety of examples where the introduction of only one layer of boundary fitted elements of width $O(p\varepsilon)$ is highly successful. Our results indicate moreover that strict boundary fitting is not necessary for robust exponential convergence. In fact, as mentioned in Section 3.1 the elements may violate in a controlled way minimal and maximal angle conditions. The purpose of our numerical examples is therefore to illustrate this insensitivity of the exponential convergence rate with respect to mesh distortion. To this end, consider the following quasi one dimensional model problem.

$$\begin{aligned} -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon &= 1 \qquad \text{on } S := (0,1)^2, \\ u_\varepsilon &= 0 \qquad \text{on } \Gamma_D := \{(x,y) \in \partial S \,|\, y = 0\}, \\ \partial_n u_\varepsilon &= 0 \qquad \text{on } \Gamma_N := \partial S \setminus \Gamma_D \end{aligned} \tag{5.1}$$

The solution of this problem, which has no singularities and a boundary layer only at Γ_D, is given by

$$u_\varepsilon(x,y) = 1 - \frac{\cosh((1-y)/\varepsilon)}{\cosh(1/\varepsilon)}. \tag{5.2}$$

For our numerical calculations we chose $\varepsilon = 10^{-3}$ and used the commercial code STRESS CHECK, a p version code with highest polynomial degree $p_{max} = 8$. Our first numerical example is designed to illustrate the robustness with respect to mesh distortion. On a fixed quadrilateral mesh as depicted in Fig. 5.1 the tensor product spaces Q_p with p ranging from 1 to p_{max} are used. The relative error in energy (cf. (1.3)) versus square root of the number of degrees of freedom is reported in Fig. 5.4. In the case $b = 0.5$, the mesh is not a boundary fitted tensor product mesh but all quadrilaterals satisfy a maximum and minimum angle condition (even as ε tends to zero). For the case $b = 0.25$ the maximum angle is $\pi - O(\varepsilon)$ and the minimum angle is $O(\varepsilon)$, i.e., the

mesh is highly distorted. Nevertheless, the error curves in Fig. 5.4 are practically on top of each other showing the robustness with respect to mesh distortion of the approximation properties of admissible meshes. The situation is completely analogous for triangular meshes. Fig. 5.5 shows the performance of the p version on the triangular mesh of Fig. 5.2. Again, the convergence is not visibly affected by the use of highly distorted meshes in the boundary layer.
The needle elements should have width $O(p\varepsilon)$, i.e., the mesh should depend on ε as well as on p. However, for practical purposes, it is more convenient to fix a mesh and to increase p. The question arises then what the appropriate width of the needle elements is. If only one layer of needle elements is used, we advocate the use of needle elements of width $O(p_{max}\varepsilon)$ (however, cf. also the discussion in Section 3.4.3). In Fig. 5.6, we show the relative error in energy versus the number of degrees of freedom for the mesh of Fig. 5.3. Again, the robustness with respect to mesh distortion is clearly visible as the choice of the parameter b has practically no effect. However, we note that the error curves in Fig. 5.6 level off at about $O(10^{-7})$ corresponding to $p = 6$. Actually, already for $p = 5$, some deterioration of the rate of convergence is visible. This is due to the fact that the width of the needle elements is fixed at 4ε instead of $8\varepsilon = p_{max}\varepsilon$. In fact, the large elements are too close to Γ_D and dominate the global error reduction. The adverse effect of choosing the needle elements too small is more clearly visible in the following one dimensional analog of (5.1) which was studied in detail in [5]:

$$-\varepsilon^2 u_\varepsilon'' + u_\varepsilon = 1 \quad \text{on } (0,1), \qquad u_\varepsilon(0) = 0, \quad u_\varepsilon'(1) = 0. \tag{5.3}$$

The solution $u_\varepsilon(y)$ is given by the right hand side of (5.2). We consider the p version FEM based on a two element mesh determined by the points $0 = y_0 < y_1 = a\varepsilon < y_2 = 1$. The performance for $\varepsilon = 10^{-3}$ and various choices of the parameter a is reported in Fig. 5.7. For fixed a, we note the initial exponential convergence which deteriorates if p becomes large. In fact, the exponential rate of convergence is visible until $p \approx a$. For $p > a$, the large element (which is $a\varepsilon$ away from $y = 0$) dominates the overall possible error reduction. This can be seen as follows. As the boundary layer function in this particular case is essentially $e^{-y/\varepsilon}$, the function to be approximated on the large element is $e^{-a}e^{-(y-a\varepsilon)/\varepsilon}$. For small ε polynomial approximation of $e^{-(y-a\varepsilon)/\varepsilon}$ on the element $(a\varepsilon, 1)$ is quite poor (cf. [5] for sharp bounds for the case of interest $p << \varepsilon^{-1}$) and the factor e^{-a} is comparatively large if a is small (relative to p). However, if a is large ($a \geq p$, say), then the boundary layer function $e^{-y/\varepsilon}$ on the large element is exponentially small (in p), and thus the contribution of the large element to the total error as well. We conclude therefore that for fixed a, the error on the large element is negligible for $p < a$, and the global error reduction is controlled by the error on the small element. In the regime $p > a$, the error on the large element dominates the global error. The choice of a variable mesh, i.e., taking $a = p$ balances the two errors; we see in Fig. 5.7 that this choice allows us to obtain exponential convergence. Note that in our definition of admissible meshes, width $p\varepsilon$ of the needle elements corresponds to taking $a = p$ in this one dimensional model problem.

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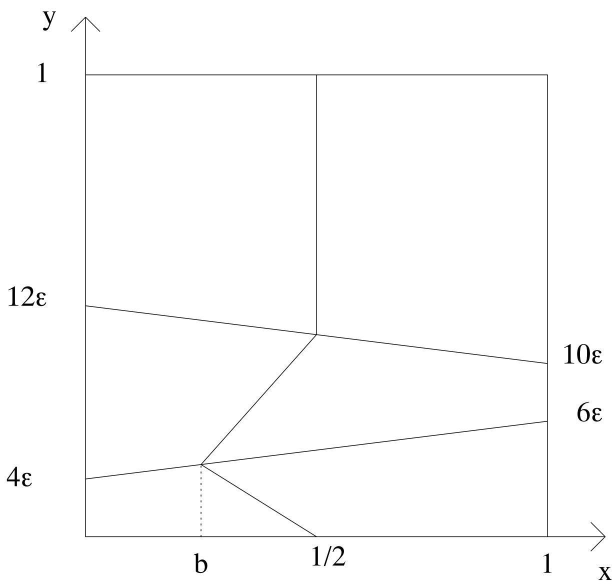

Figure 5.1: mesh (*not* drawn to scale)

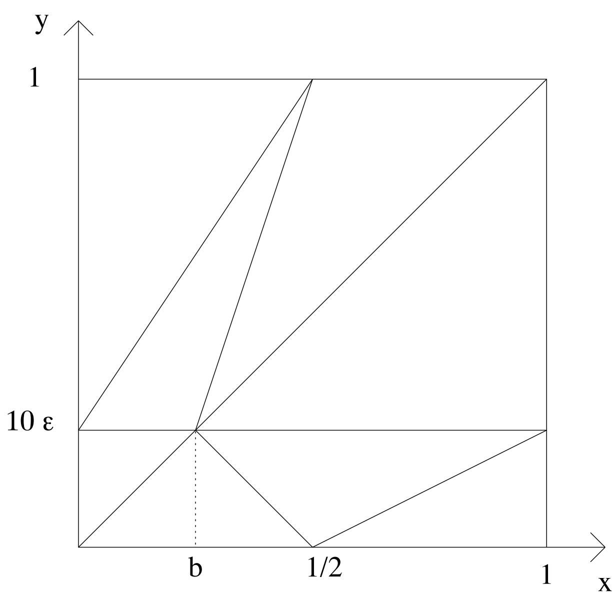

Figure 5.2: mesh (*not* drawn to scale)

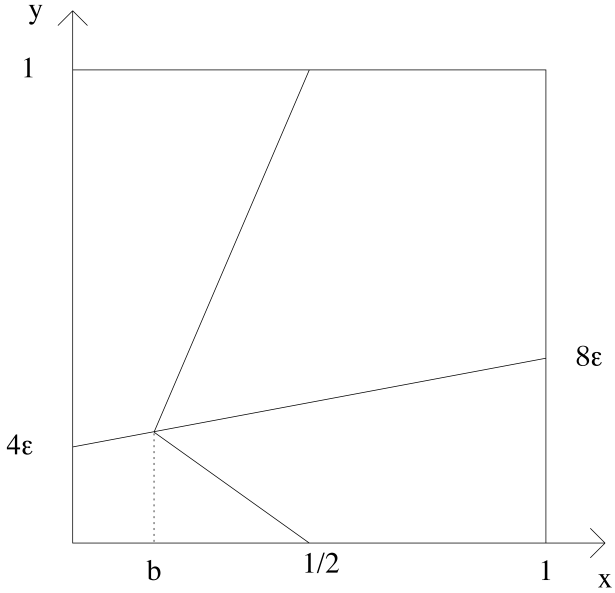

Figure 5.3: mesh (*not* drawn to scale)

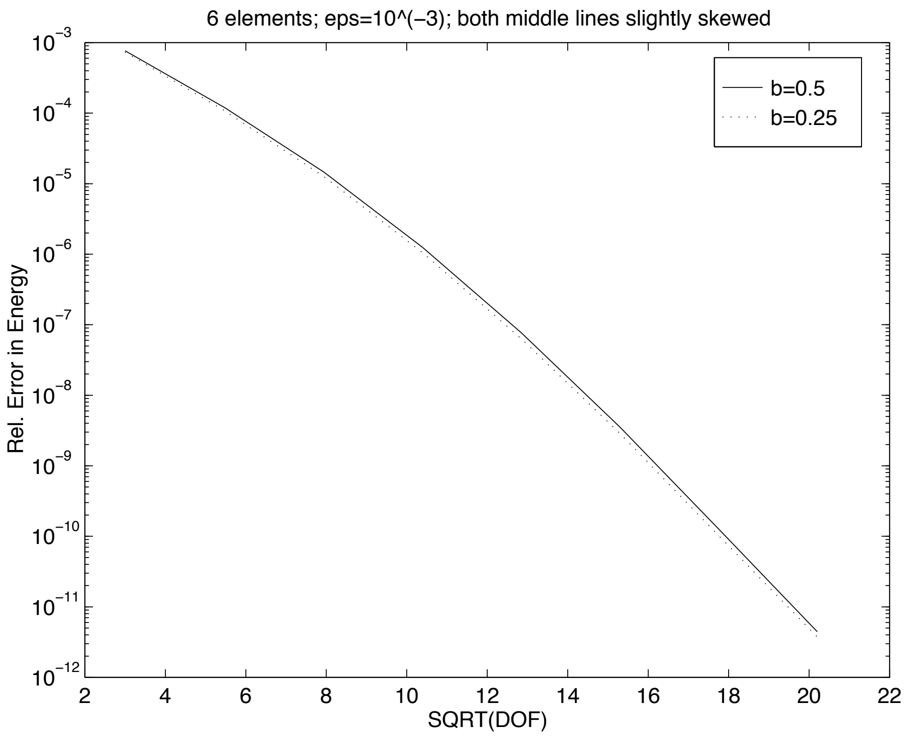

Figure 5.4: p version on mesh of Fig. 5.1; $\varepsilon = 10^{-3}$

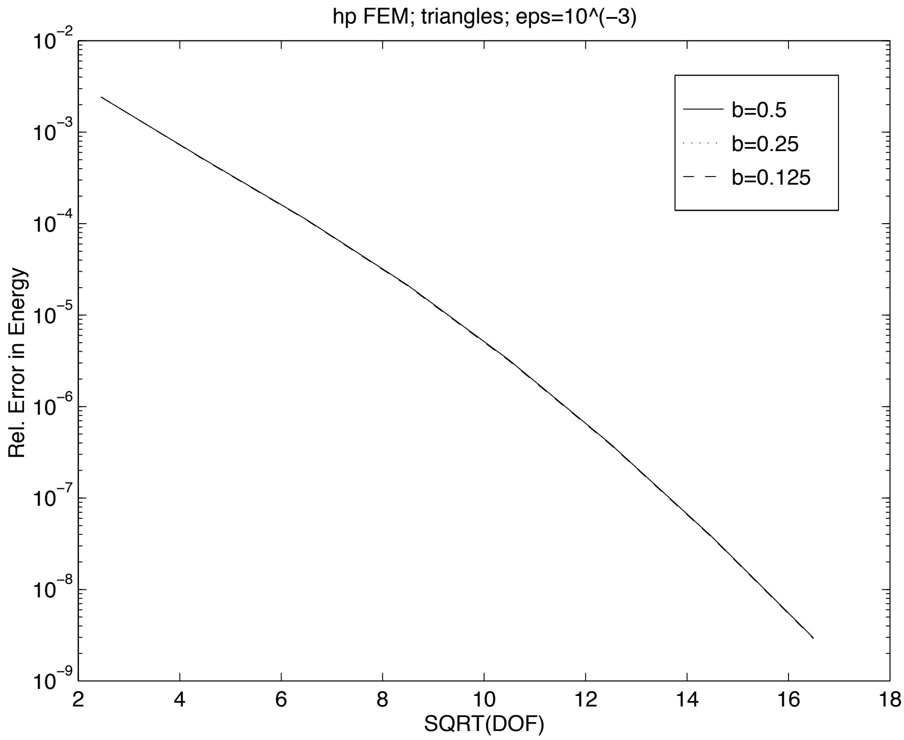

Figure 5.5: p version on mesh of Fig. 5.2; $\varepsilon = 10^{-3}$

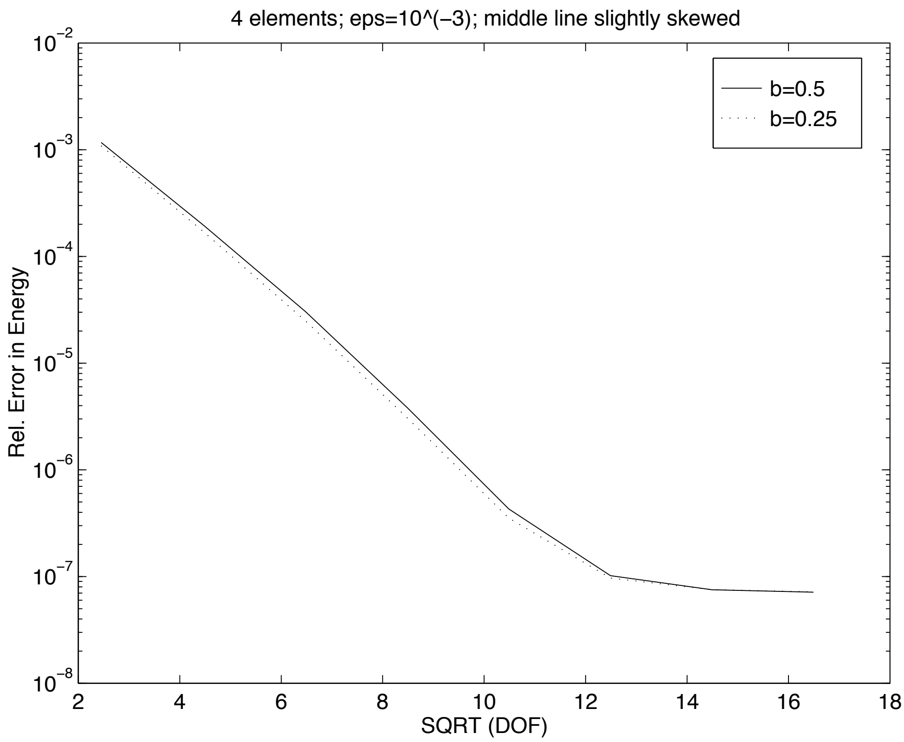

Figure 5.6: p version on mesh of Fig. 5.3; $\varepsilon = 10^{-3}$

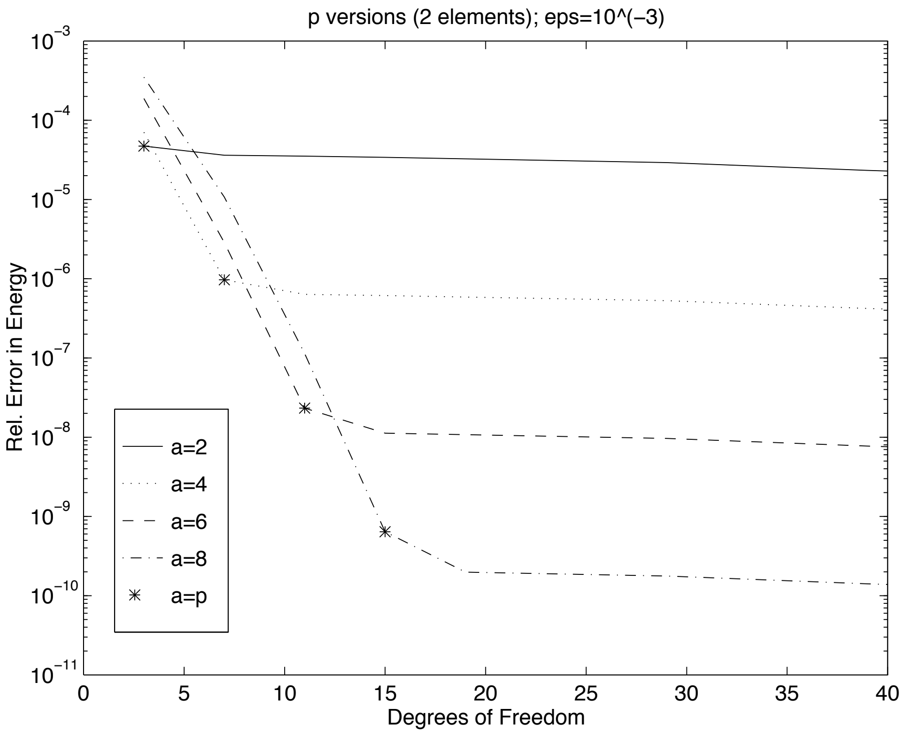

Figure 5.7: p version for 1D example and various values of a; $\varepsilon = 10^{-3}$

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