HIERARCHIC MODELS OF HELMHOLTZ PROBLEMS ON THIN DOMAINS

Klaus Gerdes and Christoph Schwab

Seminar for Applied Mathematics
ETH Zürich
Rämistr. 101
CH-8092 Zürich, Switzerland

Abstract

The Helmholtz equation in a three-dimensional plate is approximated by a hierarchy of two-dimensional models. Computable a posteriori error estimators of the modelling error in exponentially weighted norms are derived, and sharp, computable estimates for their effectivity indices are also obtained. The necessity of including, besides polynomials, a certain number of trigonometric director functions into the Ansatz, in order to prevent pollution effects at high wave numbers is demonstrated both theoretically and computationally.

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Introduction

The dimensional reduction, i.e. the approximation of problems of mechanics on thin, three-dimensional domains by simplified, two- or one-dimensional "models" is a widely used and classical approach in computational mechanics. Recently, with the availability of accurate Finite Element (FE) approximations of such dimensionally reduced models, there has been an increased awareness of the so-called "modelling error" due to the dimensional reduction. As a consequence, the classical lower-dimensional models have been embedded into a hierarchy of higher order models and corresponding a-posteriori modelling error estimates have been developped with the aim of enabling the automatic and adaptive selection of the model from the hierarchy. [**?**, **?**, **?**, **?**]. All these works dealt with stationary elliptic problems; few results seem to be available for time-dependent problems or for their quasistatic approximations arising, e.g., in the description of vibrating bodies. We mention here, however, [**?**] where the Helmholtz equation was considered on an infinite domain and sharp a-priori estimates of the modelling error were obtained.

The analysis of hierarchic, dimensionally reduced models of the Helmholtz equation in thin, three-dimensional slabs and the derivation of a-posteriori modelling error estimates is the purpose of the present paper. Similar to [**?**, **?**, **?**, **?**], we propose a dimensional reduction of the problem based on a semidiscretization in the transverse direction. We obtain new a-priori and a-posteriori modelling error estimates in localized, exponentially weighted energy norms.

Our analysis shows that, unlike in the zero-wave number case [**?**, **?**], for high wave numbers, the dimensional reduction must be based on a proper combination of polynomial and trigonometric director functions in the transverse direction, rather than polynomials alone. We prove that the inclusion of a certain number of trigonometric director functions into the dimensional reduction process is necessary to ensure that all local (i.e. exponentially decaying) perturbations such as edge-effects and point- or line singularities of the data do not pollute the lower dimensional model. In addition, the local size of the residual is then an asymptotically exact indicator for the modelling error – a key step in the local adaptive hierarchic modelling. The number of trigonometric functions to be included into the models depends on wave number $\times$ thickness. Naturally, due to the high approximation order of polynomials, the inclusion of trigonometric functions introduces a certain redundancy and near linear dependence into the ansatz. We propose to cope with this by monitoring the angles between the spaces spanned by the polynomials and the trigonometric functions, respectively, with the aid of a generalized eigenvalue problem which is numerically diagonalized. This allows a) to decouple the hierarchy of models and b) to determine when a given trigonometric shape function is approximated to machine accuracy by the polynomials and can hence be dropped from the semidiscretization process. This is also confirmed by numerical results.

The outline of the paper is as follows: After introducing some notation and formulating the problem, we discuss the critical modes of our problem and state an existence result. We then describe the hierarchic modelling and identify the critical frequencies for the reduced model. We next prove a basic stability result of the bilinear form in exponentially weighted spaces and derive computable a-posteriori modelling error estimates in these weighted norms. We demonstrate that the estimator here obtained is asymptotically exact as the thickness of the structure and the wave number tends to zero. We show further that the effectivity index grows almost linearly with the wave number. We conclude with numerical experiments which confirm our asymptotic estimates and which underline in particular the necessity for including the trigonometric director functions.

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1 Notation and problem formulation

By $\omega \subset \mathbb{R}^2$ we denote a bounded domain with a piecewise smooth Lipschitz boundary γ. With ω and a positive thickness parameter d we associate the three-dimensional domain

$$\Omega = \omega \times (-d, d)$$

with lateral boundary

$$\Gamma = \gamma \times (-d, d)$$

and the faces

$$R_\pm = \{(x_1, x_2, y) \mid \boldsymbol{x} = (x_1, x_2) \in \omega,\ y = \pm d\}\,.$$

In Ω we consider the Helmholtz problem with prescribed Neumann data $f^\pm$ on the faces; i.e.

$$\begin{array}{rcll} Lu &=& 0 & \text{in } \Omega, \\ u &=& 0 & \text{on } \Gamma, \\ D_n u &=& f^\pm & \text{on } R_\pm, \end{array} \tag{1.1}$$

where the operator L is (in the sense of distributions) given by

$$Lu = \Delta u + k^2 u \tag{1.2}$$

where k is the wave number, Δ the Laplace operator and the operator $D_n u$ is the (distributional) exterior normal derivative on $R_\pm$.

To cast (1.1) into the weak form we introduce the Sobolev space

$$H := \{u \in H^1(\Omega) \mid \text{trace } u = 0 \text{ on } \Gamma\} \tag{1.3}$$

and define the bilinear form $B_{k^2}(\cdot,\cdot) : H \times H \to I\!R$ and the functional $F(\cdot) : H \to I\!R$ by

$$B_{k^2}(u,v) := \int_\Omega \Big(\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v - k^2 uv\Big)\ d\boldsymbol{x}\ dy \tag{1.4}$$

and

$$F(v) := \int_\omega \left(f^+(\boldsymbol{x})v(\boldsymbol{x},d) + f^-(\boldsymbol{x})v(\boldsymbol{x},-d)\right)\ d\boldsymbol{x} \tag{1.5}$$

respectively.

Then the weak form of (1.1) reads: Find $u \in H$ such that

$$B_{k^2}(u,v) = F(v) \qquad \forall\, v \in H. \tag{1.6}$$

We assume throughout that the data $f^\pm(\boldsymbol{x})$ satisfy

$$f^+,\ f^- \in L^2(\omega). \tag{1.7}$$

Remark 1.1 In (1.1) we assumed $u = 0$ on Γ. All results obtained below hold equally well under the more general edge condition $u = 0$ on $\Gamma_D \subset \Gamma$, $\partial u/\partial n = 0$ on Γ_N where $\Gamma_D = \gamma_D \times (-d,d)$, $\Gamma_N = \gamma_N \times (-d,d)$ are Dirichlet and Neumann parts, respectively, of the edge. In the same way, the analysis could be performed for other boundary conditions on $R_\pm$.

2 Properties of the solution

To determine the set Σ of resonance frequencies, the classical separation of variables approach is described for the case where the bounded domain ω is a 2D Lipschitz domain with a piecewise smooth boundary.

We wish to find the solution to (1.1) by making the Ansatz $u(\boldsymbol{x},y) = h(\boldsymbol{x})g(y)$. Formally substituting into (1.1) yields

$$-\frac{\partial^2 g}{\partial y^2}h - g\Delta_{\boldsymbol{x}} h - k^2 hg = 0 \tag{2.1}$$

or,

$$-\frac{\partial^2 g}{\partial y^2}\frac{1}{g} = \frac{\Delta_{\boldsymbol{x}} h}{h} + k^2 = \lambda^2 \tag{2.2}$$

where λ^2 is the separation constant. This gives the system of equations

$$\begin{aligned} -\frac{\partial^2 g}{\partial y^2} - \lambda^2 g &= 0 \quad \text{in } (-d,d),\ g'(\pm d) = 0, \\ \Delta_{\boldsymbol{x}} h + (k^2 - \lambda^2) h &= 0 \quad \text{in } \omega,\ h|_{\partial\omega} = 0. \end{aligned} \tag{2.3}$$

Denote by $h_n \in H_0^1(\omega)$ the eigenfunctions of the Laplacian $-\Delta_{\boldsymbol{x}}$ in ω with zero boundary conditions on γ, i.e.

$$-\Delta_{\boldsymbol{x}} h_n = \nu_n^2 h_n \text{ in } \omega, \quad h_n = 0 \text{ on } \gamma. \tag{2.4}$$

Then we have

$$\lambda_n^2 = k^2 - \nu_n^2, \quad n \geq 1. \tag{2.5}$$

The solutions in y are then given by

$$\begin{aligned} g_n^{(1)}(y) &= \cos\left(y\sqrt{k^2 - \nu_n^2}\right), \quad n \geq 1, \\ g_n^{(2)}(y) &= \sin\left(y\sqrt{k^2 - \nu_n^2}\right), \quad n \geq 1, \end{aligned} \tag{2.6}$$

and the solution to problem (1.1) by

$$u(\boldsymbol{x}, y) = \sum_{n=1}^{\infty} h_n(\boldsymbol{x}) \left(A_n g_n^{(1)}(y) + B_n g_n^{(2)}(y)\right) \tag{2.7}$$

where the coefficients A_n, B_n have to be determined according to the given boundary conditions.

We observe that the h_n satisfy the Dirichlet boundary condition $h_n = 0$ on $\partial\omega$, and by the density of the $\{h_n\}$ in $L^2(\omega)$, the Neumann data on the faces $R_\pm$ can be expanded in a series of the form

$$f^{\pm}(\boldsymbol{x}) = \sum_{n=1}^{\infty} f_n^{\pm} h_n(\boldsymbol{x}). \tag{2.8}$$

Then the Neumann boundary condition translates into

$$\begin{aligned} \left.\frac{\partial u}{\partial n}\right|_{\pm d} &= \pm \left.\frac{\partial u}{\partial y}\right|_{\pm d} \\ &= \pm \sum_{n=1}^{\infty} h_n(\boldsymbol{x}) \left(-A_n \sin\left(\pm\sqrt{k^2 - \nu_n^2}\, d\right)\right. \\ &\qquad \left. + B_n \cos\left(\pm\sqrt{k^2 - \nu_n^2}\, d\right)\right) \sqrt{k^2 - \nu_n^2} \\ &= \sum_{n=1}^{\infty} f_n^{\pm} h_n(\boldsymbol{x}). \end{aligned} \tag{2.9}$$

The functions h_n are L^2-orthonormal, i.e.

$$\int_\omega h_n(\boldsymbol{x}) h_{\tilde{n}}(\boldsymbol{x})\, d\boldsymbol{x} = \begin{cases} 0 & n \neq \tilde{n}, \\ 1 & n = \tilde{n}. \end{cases}$$

Multiplying (2.9) by $h_{\tilde{n}}(\boldsymbol{x})$ and integrating over ω gives the following system for the coefficients A_n and B_n in (2.7)

$$\begin{aligned} \left(-A_n \sin\left(\sqrt{k^2-\nu_n^2}\, d\right) + B_n \cos\left(\sqrt{k^2-\nu_n^2}\, d\right)\right)\sqrt{k^2-\nu_n^2} &= f_n^+, \\ \left(A_n \sin\left(-\sqrt{k^2-\nu_n^2}\, d\right) - B_n \cos\left(-\sqrt{k^2-\nu_n^2}\, d\right)\right)\sqrt{k^2-\nu_n^2} &= f_n^-, \end{aligned} \tag{2.10}$$

which has solutions

$$\begin{aligned} A_n &= \frac{-(f_n^+ + f_n^-)}{2\sqrt{k^2-\nu_n^2}\sin\left(d\sqrt{k^2-\nu_n^2}\right)}, \\ B_n &= \frac{f_n^+ - f_n^-}{2\sqrt{k^2-\nu_n^2}\cos\left(d\sqrt{k^2-\nu_n^2}\right)}. \end{aligned} \tag{2.11}$$

From (2.11) we see that for certain values of k the solution of (1.1) will fail to exist for general data (resonance). These critical values of k are

$$k = \sqrt{\frac{l^2\pi^2}{4d^2} + \nu_n^2} \quad n \geq 1,\ l \geq 0. \tag{2.12}$$

The spectrum Σ of the three dimensional operator L in (1.1) is then given by

$$\Sigma = \left\{ k : k = \nu_n \text{ or } k = \sqrt{\frac{l^2\pi^2}{4d^2} + \nu_n^2};\ n, l \geq 1 \right\}. \tag{2.13}$$

We therefore have:

Proposition 2.1 *Assume (1.7) and that $k \notin \Sigma$. Then the problem (1.6) admits a unique weak solution $u \in H$.*

3 Hierarchical modeling

We will approximate the boundary value problem (1.3)-(1.7) by a sequence of two-dimensional problems on ω, the *hierarchy of dimensionally reduced models*, which we now define.

For nonnegative integers $q \geq 0$ and any dense sequence

$$\{\Phi_j(z)\}_{j=0}^{\infty} \subset H^1(-1, 1) \tag{3.1}$$

of linearly independent functions we define

$$S(q) := \left\{ u(\boldsymbol{x}, y) \ : \ u|_\omega = \sum_{j=0}^{q} U_j(\boldsymbol{x}) \Phi_j \left(\frac{y}{d} \right), \quad U_j(\boldsymbol{x}) \in H_0^1(\omega) \right\}. \tag{3.2}$$

Then $S(q) \subset H$ and the (q)-reduced model is the following two-dimensional boundary value problem: Find $u(q) \in S(q)$ such that

$$B_{k^2}(u(q), v) = F(v) \quad \forall\, v \in S(q) \tag{3.3}$$

i.e., $u(q)$ is the Galerkin projection of the weak solution u onto $S(q)$. Hence (3.3) constitutes an elliptic system of Helmholtz type on ω for the yet unknown coefficient functions $U_j(\boldsymbol{x})$ in (3.2).

The selection of the director functions Φ_i in (3.1) completely determines the (q)-model and will be discussed in the following. To do so, we go back and look at the separation of variables approach from a different point of view. Assume that

$$u(\boldsymbol{x}, y) = \sum_{i=0}^{\infty} a_i(\boldsymbol{x}) \tilde{\Phi}_i(y/d), \quad a_i(\boldsymbol{x}) = \left(u(\boldsymbol{x}, \cdot), \tilde{\Phi}_i \right)_{L^2(-d,d)}, \quad x \in \omega \tag{3.4}$$

where $\tilde{\Phi}_i$ are the eigenfunctions of

$$-\tilde{\Phi}_i'' = \alpha_i^2 \tilde{\Phi}_i \quad \text{in } (-1,1), \quad \tilde{\Phi}_i'(\pm 1) = 0. \tag{3.5}$$

This determines the $\tilde{\Phi}_i$ and α_i to be

$$\begin{aligned} \alpha_i &= \frac{i\pi}{2}, \\ \tilde{\Phi}_0(z) &= \frac{1}{\sqrt{2}}, \\ \tilde{\Phi}_{2i}(z) &= \cos(\alpha_{2i} z), \\ \tilde{\Phi}_{2i-1}(z) &= \sin(\alpha_{2i-1} z). \end{aligned} \tag{3.6}$$

We note that the $\{\tilde{\Phi}_i\}$ are dense in $L^2(-1,1)$ and that they are L^2-orthogonal.

The error analysis in the next sections will show that for localization of the modelling error (see also Remark 4.2 below) it is necessary and sufficient to choose

$$\Phi_i(y/d) = \frac{1}{\sqrt{d}} \tilde{\Phi}_i(y/d), \quad 0 \le i \le M. \tag{3.7}$$

For the approximation of the solution we also choose

$$\Phi_{M+i}(y/d) = L_i(y/d), \quad 1 \le i \le N, \tag{3.8}$$

where L_i are the Legendre polynomials of degree i on $(-1, 1)$ with the normalization $|L_i(1)| = 1$.

Next we derive explicitly the reduced 2D weak formulation to problem (1.1) by making the **Ansatz**

$$u(\boldsymbol{x}, y) = u^{M,N}(\boldsymbol{x}, y) = \sum_{m=0}^{N+M} \Phi_m(y/d) U_m(\boldsymbol{x}) \tag{3.9}$$

with certain yet unknown coefficient functions $U_m(\boldsymbol{x})$. To obtain a boundary value problem in ω for $U_m(\boldsymbol{x})$, we substitute (3.9) into the three dimensional weak formulation (1.6) and use as test function

$$v(\boldsymbol{x}, y) = V_{\tilde{m}}(\boldsymbol{x}) \Phi_{\tilde{m}}(y/d), \quad 0 \le \tilde{m} \le N + M. \tag{3.10}$$

This leads to the elliptic system

$$\begin{aligned} \int_\Omega \sum_{m=0}^{N+M} \Big\{ & \Big(\boldsymbol{\nabla_x} U_m(\boldsymbol{x}) \boldsymbol{\nabla_x} V_{\tilde{m}} - k^2 U_m(\boldsymbol{x}) V_{\tilde{m}}(\boldsymbol{x}) \Big) \Phi_m(\frac{y}{d}) \Phi_{\tilde{m}}(\frac{y}{d}) \\ & + \frac{\partial \Phi_m(y/d)}{\partial y} \frac{\partial \Phi_{\tilde{m}}(y/d)}{\partial y} U_m(\boldsymbol{x}) V_{\tilde{m}}(\boldsymbol{x}) \Big\} \, d\boldsymbol{x} \, dy \\ = \int_\omega & V_{\tilde{m}}(\boldsymbol{x}) \left(f^+(\boldsymbol{x}) \Phi_{\tilde{m}}(1) + f^-(\boldsymbol{x}) \Phi_{\tilde{m}}(-1) \right) \, d\boldsymbol{x}, \quad 0 \le \tilde{m} \le N + M, \end{aligned} \tag{3.11}$$

which can be rewritten in the form

$$\begin{aligned} \sum_{m=0}^{N+M} & \int_\omega \Big(\boldsymbol{\nabla_x} U_m(\boldsymbol{x}) \boldsymbol{\nabla_x} V_{\tilde{m}} - k^2 U_m(\boldsymbol{x}) V_{\tilde{m}}(\boldsymbol{x}) \Big) \, d\boldsymbol{x} \int_{-1}^{1} d \, \Phi_m(z) \Phi_{\tilde{m}}(z) \, dz \\ & + \int_\omega U_m(\boldsymbol{x}) V_{\tilde{m}}(\boldsymbol{x}) \, d\boldsymbol{x} \int_{-1}^{1} \frac{1}{d} \frac{\partial \Phi_m(z)}{\partial z} \frac{\partial \Phi_{\tilde{m}}(z)}{\partial z} \, dz \\ = & \int_\omega V_{\tilde{m}}(\boldsymbol{x}) \left(f^+(\boldsymbol{x}) \Phi_{\tilde{m}}(1) + f^-(\boldsymbol{x}) \Phi_{\tilde{m}}(-1) \right) \, d\boldsymbol{x}, \quad 0 \le \tilde{m} \le N + M. \end{aligned} \tag{3.12}$$

We define the $(N + M + 1) \times (N + M + 1)$ matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ by

$$\boldsymbol{A} = \int_{-1}^{1} \Phi_m(z) \Phi_{\tilde{m}}(z) \, dz \tag{3.13}$$

and

$$\boldsymbol{B} = \int_{-1}^{1} \frac{\partial \Phi_m(z)}{\partial z} \frac{\partial \Phi_{\tilde{m}}(z)}{\partial z} \, dz. \tag{3.14}$$

In order to obtain an equivalent, decoupled system of equations from (3.12) it is necessary to solve the generalized eigenvalue problem

$$\boldsymbol{B} q = \sigma^2 \boldsymbol{A} q \tag{3.15}$$

with the normalization $q^t \boldsymbol{A} q = \boldsymbol{1}$. Let $\boldsymbol{Q}$ denote the matrix whose columns are the eigenvectors of (3.15). Then the basis transformation

$$\boldsymbol{\Psi} = \boldsymbol{Q}^t \boldsymbol{\Phi} \tag{3.16}$$

will uncouple the reduced weak form (3.12) into the sequence of scalar problems

$$\int_\omega \left(-d^2\Delta_{\boldsymbol{x}} U_i + (\sigma_i^2 - k^2 d^2)U_i\right) V_i \, d\boldsymbol{x} = dF_i, \quad i = 0, \cdots, N+M \tag{3.17}$$

with the boundary condition

$$U_i = 0 \quad \text{on } \gamma \tag{3.18}$$

and

$$F_i = \int_\omega V_i(\boldsymbol{x}) \left(f^+(\boldsymbol{x})\Psi_i(1) + f^-(\boldsymbol{x})\Psi_i(-1)\right) \, d\boldsymbol{x}. \tag{3.19}$$

The eigenvectors of these scalar problems are the functions h_n from (2.4) which yields the eigenvalues $\nu_n^2 = k^2 - \sigma_i^2/d^2$. The set $\Sigma_{M,N}$ of critical values (resonance) for the dimensionally reduced problem is therefore given by

$$\Sigma_{M,N} = \left\{ k : k = \sqrt{\nu_n^2 + \frac{\sigma_i^2}{d^2}}; \ n \geq 1, \ 0 \leq i \leq N+M \right\}. \tag{3.20}$$

Proposition 3.1 *Assume (1.7) and that $k \notin \Sigma_{M,N}$. Then the dimensionally reduced problem (3.3) admits a unique weak solution $u(q) = u^{M,N}(\boldsymbol{x}, y) \in S(q)$.*

4 Some abstract results

Let H_1, H_2 be two reflexive Banach spaces furnished with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Further, let $B(u,v)$ be a bilinear form defined on $H_1 \times H_2$. We will call the bilinear form $(C, \hat{\gamma})$-*regular* if there exist constants $0 < C, \hat{\gamma} < \infty$ such that

$$|B(u,v)| \leq C\|u\|_1\|v\|_2, \tag{4.1}$$

$$\inf_{\|u\|_1=1} \sup_{\|v\|_2=1} |B(u,v)| \geq \hat{\gamma} \tag{4.2}$$

$$\text{for any } v \neq 0, v \in H_2, \sup_{\|u\|_1=1} |B(u,v)| > 0. \tag{4.3}$$

$(C, \hat{\gamma})$ regular bilinear forms have the following properties.

1. Let $F \in (H_2)'$ (i.e., F is a bounded, linear functional on H_2); then there exists exactly one $u \in H_1$ such that

$$B(u,v) = F(v) \quad \forall\, v \in H_2.$$

2. If

$$\sup_{\|v\|_2=1} |B(u,v)| = \sup_{\|v\|_2=1} |F(v)| \le A, \tag{4.4}$$

then

$$\|u\|_1 \le \frac{A}{\hat{\gamma}}.$$

Let us consider now some special cases that will be important later.

Let $0 < \varphi(\boldsymbol{x}) \in W^{1,\infty}(\omega)$ denote a weight function in ω. We define

$$H^L_\varphi = \{e \in H \mid e \text{ satisfies (4.6)}\} \tag{4.5}$$

$$\int_{-d}^{d} e(\boldsymbol{x}, y)\Phi_l(y/d)\, dy = 0, \quad l = 0, \cdots, L; \text{ a.e. } \boldsymbol{x} \in \omega \tag{4.6}$$

where Φ_l are as in (3.6) and furnish H^L_φ with the weighted H^1-norm defined by

$$\|e\|_\varphi = \left(\int_\omega \varphi^2(\boldsymbol{x}) \int_{-d}^{d} \left(|\boldsymbol{\nabla} e|^2 + |e|^2 \right) dy\, d\boldsymbol{x} \right)^{1/2}. \tag{4.7}$$

Further, $\|u\|_1^2 = B_{-1}(u,u)$ with $B_\rho(\cdot,\cdot)$ as in (1.4).

Theorem 4.1 *Let $0 < \varphi(\boldsymbol{x}) \in W^{1,\infty}(\omega)$ be a weight function and assume that*

1. $0 < d < 1$, $0 < \rho \le 1$ *and* $0 < \delta < 1$ *are given,*
2. $Q := \left\| \dfrac{\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi}{\varphi} \right\|_{L^\infty(\omega)}$ *satisfies* $Qd < 1$,
3. $$L + 1 \ge \frac{2d}{\pi} \sqrt{\frac{k^2 + Q/d + \delta}{1 - \delta}}. \tag{4.8}$$

Then the bilinear form $B_{k^2}(u,v)$ defined in (1.4) is $(1, \hat{\gamma})$ regular on $H^L_\varphi \times H^L_{1/\varphi}$, i.e.

$$\inf_{u \in H^L_\varphi} \sup_{v \in H^L_{1/\varphi}} \frac{B_{k^2}(u,v)}{\|u\|_\varphi \|v\|_{1/\varphi}} \ge \hat{\gamma} > 0 \tag{4.9}$$

with

$$\hat{\gamma} = \frac{\delta(1 - Qd)}{\sqrt{(1+\rho) + 4(1 + 1/\rho)Q^2}} \tag{4.10}$$

and

$$B_{k^2}(u,v) \le (1 + k^2)\|u\|_\varphi \|v\|_{1/\varphi} \tag{4.11}$$

Proof. Inequality (4.11) follows immediately from the Schwarz inequality. Let us now show (4.9). For $u \in H^L_\varphi$ define $v = \varphi^2 u$, which implies that $v \in H^L_{1/\varphi}$. Since the eigenfunctions $\tilde{\Phi}_i$ in (3.4)-(3.6) are dense in $L^2(-1,1)$ we know that u can be expanded as

$$u(\boldsymbol{x}, y) = \sum_{l=0}^{\infty} a_l(\boldsymbol{x})\tilde{\Phi}_l(y/d). \tag{4.12}$$

For v we get analogously

$$v(\boldsymbol{x}, y) = \sum_{l=0}^{\infty} b_l(\boldsymbol{x})\tilde{\Phi}_l(y/d) = \sum_{l=0}^{\infty} \varphi^2(\boldsymbol{x}) a_l(\boldsymbol{x})\tilde{\Phi}_l(y/d). \tag{4.13}$$

With this choice of u and v we will show

$$\|v\|_{1/\varphi} \leq c_1 \|u\|_\varphi \tag{4.14}$$

and

$$B_{k^2}(u, v) \geq c_2 \|u\|^2_\varphi \tag{4.15}$$

which will then imply (4.9) with $\gamma = c_2/c_1$. To prove (4.14) and (4.15), we first note that

$$\begin{aligned}
|a_l\varphi^2|^2 &= |a_l|^2\varphi^4, \\
\boldsymbol{\nabla}_{\boldsymbol{x}}(a_l\varphi^2) &= \varphi^2\boldsymbol{\nabla}_{\boldsymbol{x}} a_l + a_l \boldsymbol{\nabla}_{\boldsymbol{x}}\varphi^2, \\
|a+b|^2 &\leq (1+\rho)|a|^2 + (1+1/\rho)|b|^2, \\
|a_l\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi^2|^2 &= 4\varphi^2|a_l\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi|^2.
\end{aligned}$$

Using (4.7) and (4.12) we get

$$\|u\|^2_\varphi = \sum_{l=0}^{\infty} \int_\omega \varphi^2 \left(|\boldsymbol{\nabla}_{\boldsymbol{x}} a_l|^2 + (1+\alpha_l^2)|a_l|^2\right)\, d\boldsymbol{x} \tag{4.16}$$

and

$$\begin{aligned}
\|v\|^2_{1/\varphi} &= \sum_{l=0}^{\infty} \int_\omega \frac{1}{\varphi^2}\left(|\boldsymbol{\nabla}_{\boldsymbol{x}}(a_l\varphi^2)|^2 + (1+\alpha_l^2)|a_l\varphi^2|^2\right)\, d\boldsymbol{x} \\
&\leq \sum_{l=0}^{\infty} \int_\omega \frac{1}{\varphi^2}\left((1+\rho)\varphi^4|\boldsymbol{\nabla}_{\boldsymbol{x}} a_l|^2 + (1+1/\rho)|a_l\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi^2|^2 + (1+\alpha_l^2)\varphi^4|a_l|^2\right)\, d\boldsymbol{x} \\
&\leq (1+\rho)\|u\|^2_\varphi + \sum_{l=0}^{\infty} \int_\omega 4(1+1/\rho)|a_l\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi|^2\, d\boldsymbol{x} \\
&= (1+\rho)\|u\|^2_\varphi + \sum_{l=0}^{\infty} \int_\omega 4(1+1/\rho)\varphi^2|a_l\frac{\boldsymbol{\nabla}_{\boldsymbol{x}}\varphi}{\varphi}|^2\, d\boldsymbol{x} \\
&\leq (1+\rho)\|u\|^2_\varphi + \sum_{l=0}^{\infty} \int_\omega 4(1+1/\rho)\varphi^2 Q^2|a_l|^2\, d\boldsymbol{x} \\
&\leq (1+\rho+4(1+1/\rho)Q^2)\|u\|^2_\varphi
\end{aligned}$$

which implies that $c_1 = \sqrt{1+\rho+4(1+1/\rho)Q^2}$.

Next we show (4.15) and note that since $u \in H^L_\varphi$ and $v \in H^L_{1/\varphi}$ we have

$$B_{k^2}(u,v) = \sum_{l=L+1}^{\infty} \int_\omega \left(\varphi^2 \left(|\nabla_{\boldsymbol{x}} a_l|^2 + (\alpha_l^2 - k^2)|a_l|^2 \right) + a_l (\nabla_{\boldsymbol{x}} a_l \cdot \nabla_{\boldsymbol{x}} \varphi^2) \right) \, d\boldsymbol{x}. \tag{4.17}$$

We can then estimate

$$\begin{aligned} B_{k^2}(u,v) \;&\geq\; \sum_{l=L+1}^{\infty} \int_\omega \left(\varphi^2 \left(|\nabla_{\boldsymbol{x}} a_l|^2 + (\alpha_l^2 - k^2)|a_l|^2 \right) \right) \, d\boldsymbol{x} - \left| \int_\omega a_l (\nabla_{\boldsymbol{x}} a_l \cdot \nabla_{\boldsymbol{x}} \varphi^2) \, d\boldsymbol{x} \right| \\ &\geq\; \sum_{l=L+1}^{\infty} \int_\omega \varphi^2 \left((1-Q\varepsilon)|\nabla_{\boldsymbol{x}} a_l|^2 + (\alpha_l^2 - k^2 - \frac{Q}{\varepsilon})|a_l|^2 \right) \, d\boldsymbol{x}. \end{aligned}$$

In the above estimate we have used

$$|(\nabla_{\boldsymbol{x}} a_l) \cdot \nabla_{\boldsymbol{x}} (\varphi^2)| = |(\nabla_{\boldsymbol{x}} a_l) \cdot 2\varphi^2 \frac{\nabla_{\boldsymbol{x}} \varphi}{\varphi}| \leq |\nabla_{\boldsymbol{x}} a_l| 2\varphi^2 Q$$

which implies for any $\varepsilon > 0$ that

$$\begin{aligned} \left| \int_\omega a_l (\nabla_{\boldsymbol{x}} a_l \cdot \nabla_{\boldsymbol{x}} (\varphi^2)) \, d\boldsymbol{x} \right| \;&\leq\; 2Q \int_\omega \varphi^2 |a_l| |\nabla_{\boldsymbol{x}} a_l| \, d\boldsymbol{x} \\ &\leq\; Q \left(\frac{1}{\varepsilon} \sqrt{\int_\omega \varphi^2 |a_l|^2 \, d\boldsymbol{x}} + \varepsilon \sqrt{\int_\omega \varphi^2 |\nabla_{\boldsymbol{x}} a_l|^2 \, d\boldsymbol{x}} \right). \end{aligned}$$

We would like to further estimate

$$B_{k^2}(u,v) \geq c_2 \sum_{l=L+1}^{\infty} \int_\omega \varphi^2 \left(|\nabla_{\boldsymbol{x}} a_l|^2 + (1+\alpha_l^2)|a_l|^2 \right) \, d\boldsymbol{x} = c_2 \|u\|_\varphi^2. \tag{4.18}$$

In order to show (4.18) we require the following

1. $1 - Q\varepsilon > 0$,
2. $\alpha_l^2 - k^2 - Q/\varepsilon \geq \delta(1+\alpha_l^2)$, $l = L+1, L+2, \cdots$,
3. $\varepsilon = d$.

From (3.6) we recall that $\alpha_l = l\pi/(2d)$. Let $0 < \delta < 1$, then

$$\begin{aligned} \alpha_l^2 - k^2 - Q/d \;&\geq\; \delta(1+\alpha_l^2), \quad l = L+1, L+2, \cdots, \\ \Leftrightarrow \qquad \alpha_l^2 \;&\geq\; \frac{k^2 + Q/d + \delta}{1-\delta}, \quad l \geq L+1. \end{aligned}$$

Since the α_l are monotonically increasing, it suffices to require that

$$\alpha_{L+1}^2 \geq \frac{k^2 + Q/d + \delta}{1-\delta}, \tag{4.19}$$

which implies (4.8) and also that $c_2 = \delta(1 - Qd)$. This completes the proof. □

Remark 4.2 Examples for admissible weight functions $\varphi(\boldsymbol{x})$ are in particular exponential functions of the form

$$\varphi(\boldsymbol{x}) = \exp(\alpha|\boldsymbol{x} - \boldsymbol{x}_0|), \quad \alpha > 0 \tag{4.20}$$

for some $\boldsymbol{x}_0 \in \overline{\omega}$. In this case we have $Q = \alpha$ and Theorem 4.1 means that the form $B_{k^2}(\cdot,\cdot)$ is stable on exponentially decaying (off $\boldsymbol{x}_0$) functions times exponentially increasing test functions. This could also be interpreted as a *Saint-Venant Principle* for the Helmholtz equation in thin domains.

Our next goal is to show that the modelling error $e^{M,N} = u - u^{M,N}$ belongs to H^L_φ, where u is the solution to the full 3D problem and $u^{M,N}$ is the exact solution to the dimensionally reduced 2D problem. From (3.4)-(3.9) we know that u as well as $u^{M,N}$ can be expanded in terms of the director functions Φ_i defined in (3.7) and (3.8), that is

$$\begin{aligned} u(\boldsymbol{x}, y) &= \sum_{i=0}^{\infty} U_i(\boldsymbol{x})\Phi_i(y/d), \\ u^{M,N}(\boldsymbol{x}, y) &= \sum_{i=0}^{N+M} U_i^r(\boldsymbol{x})\Phi_i(y/d) \end{aligned}$$

and the $U_i^r(\boldsymbol{x})$ are exact solutions to the reduced problem (3.12). We know that the Φ_i, $0 \leq i \leq M$, are eigenfunctions by construction, see (3.7). By partial integration and (3.5) we obtain

$$\begin{aligned} \int_{-1}^{1} \Phi_i'(z)\Phi_j'(z)\,dz &= -\int_{-1}^{1} \Phi_i''(z)\Phi_j(z)\,dz + \Phi_i'(z)\Phi_j(z)\big|_{-1}^{1} \\ &= -\int_{-1}^{1} \left(-\alpha_i^2\Phi_i(z)\Phi_j(z)\right)\,dz \\ &= \alpha_i^2\delta_{ij}, \qquad 0 \leq i, j \leq M. \end{aligned}$$

This implies that the eigenvalue α_i^2 of (3.5) is also an eigenvalue of (3.15). On the interval $(-d, d)$ this gives

$$\alpha_i^2 = \frac{\sigma_i^2}{d^2}, \quad 0 \leq i \leq M. \tag{4.21}$$

Returning to the separation of variables approach, we see that

$$0 = -\Delta u - k^2 u = -\Delta_{\boldsymbol{x}} u - \frac{\partial^2 u}{\partial y^2} - k^2 u$$

yields for $u(\boldsymbol{x}, y) = U_i(\boldsymbol{x})\Phi_i(y)$ (compare (2.3) and (3.5))

$$-\Delta_{\boldsymbol{x}} U_i(\boldsymbol{x}) = (k^2 - \alpha^2)U_i(\boldsymbol{x}). \tag{4.22}$$

The same separation Ansatz was used to derive the reduced weak form and the sequence of decoupled weak forms (3.17). From (3.17) we get the eigenvalue problem in $\boldsymbol{x}$

$$-\Delta_{\boldsymbol{x}} U_i(\boldsymbol{x}) = \left(k^2 - \frac{\sigma_i^2}{d^2}\right) U_i(\boldsymbol{x}) \quad \text{in } \omega,\ U_i|_{\partial\omega} = 0. \tag{4.23}$$

Inserting (4.21) into (4.23) yields for $0 \le i \le M$

$$-\Delta_{\boldsymbol{x}} U_i(\boldsymbol{x}) = \left(k^2 - \alpha_i^2\right) U_i(\boldsymbol{x}) \quad \text{in } \omega,\ U_i|_{\partial\omega} = 0, \tag{4.24}$$

which admits a nonzero solution if $k \in \Sigma_{M,N}$. Then, $U_i(\boldsymbol{x})\Phi_i(y/d)$, $0 \le i \le M$, is an eigenfunction of the 3D problem which is exactly reproduced by the reduced 2D formulation corresponding to the same first $M+1$ eigenvalues α_i^2, $i = 0, \cdots, M$. We have proved the following

Lemma 4.3 *Let $k \notin \Sigma \cup \Sigma_{M,N}$ and let u be the solution to (1.1) and $u^{M,N}$ be the solution to (3.12) and let $0 \le L \le M$. Then*

$$e^{M,N}(\boldsymbol{x}, y) := u(\boldsymbol{x}, y) - u^{M,N}(\boldsymbol{x}, y) \in H_\varphi^L \tag{4.25}$$

for any weight function $\varphi(\boldsymbol{x})$ satisfying the assumptions of Theorem 4.1.

Remark 4.4 If we choose in (4.20) the parameter $\alpha = \overline{\alpha} d^{-1}$, $-1 < \overline{\alpha} < 0$ independent of d, Theorem 4.1 states that $B_{k^2}(\cdot,\cdot)$ is stable on $H_\varphi^L \times H_{1/\varphi}^L$ *provided* the condition (4.8) is satisfied, i. e. provided

$$L + 1 \ge \frac{2d}{\pi}\sqrt{\frac{k^2 + \overline{\alpha} + \delta}{1 - \delta}}. \tag{4.26}$$

Selecting δ close to zero, (4.26) gives immediately a lower bound for L, or, by Lemma 4.3, the minimal number $M+1$ of director functions $\tilde{\Phi}_i(z)$ in (3.6) to be included into the Ansatz (3.9) to ensure control of the modelling error in the exponentially weighted norm $\|\cdot\|_\varphi$. We will later see that this amounts to the absence of pollution.

Inspecting the proof of Theorem 4.1, i.e. selecting $\varepsilon < d/\overline{\alpha}$, we see that for L larger than the minimal value in (4.26) we can in fact choose $\overline{\alpha}$ in (4.26) larger, i.e. we have a corresponding stronger localization of the weights $\varphi(\boldsymbol{x})$ in (4.20). Precisely, we can choose $\overline{\alpha}$ such that

$$\overline{\alpha}^2 \le \frac{\pi^2}{4}(1-\delta)(L+1)^2 - (k^2 + \delta)d^2. \tag{4.27}$$

This will also be confirmed by the numerical experiments in Section 7 ahead.

5 A posteriori modelling error estimation

In this section we assume that $q = N + M$ and that the exact solution $u(q) = u^{M,N}(\boldsymbol{x}, y)$ is known. We will be interested in computable estimators for $\|e^{M,N}\|_\varphi$, the modelling error in the weighted norm. It is convenient to write $u = u_1 + u_2$, where

$$u_1(\boldsymbol{x}, y) = -u_1(\boldsymbol{x}, -y), \quad u_2(\boldsymbol{x}, y) = u_2(\boldsymbol{x}, -y) \tag{5.1}$$

and the u_i satisfy $u_i \in H_i$ such that

$$B_{k^2}(u_i, v) = F_i(v) \quad \forall\, v \in H_i, \quad i = 1, 2 \tag{5.2}$$

where

$$\begin{aligned} F_1(v) &= \int_\omega f_1(\boldsymbol{x})\, (v(\boldsymbol{x}, d) - v(\boldsymbol{x}, -d))\ d\boldsymbol{x}, \\ F_2(v) &= \int_\omega f_2(\boldsymbol{x})\, (v(\boldsymbol{x}, d) + v(\boldsymbol{x}, -d))\ d\boldsymbol{x} \end{aligned}$$

and

$$f_1(\boldsymbol{x}) = \frac{1}{2}(f^+ - f^-)(\boldsymbol{x}), \quad f_2(\boldsymbol{x}) = \frac{1}{2}(f^+ + f^-)(\boldsymbol{x}),$$

and $H_i = \{u \in H \mid u$ is antisymmetric (symmetric) in y for $i = 1$ $(i = 2)\}$. Obviously, the spaces H_1 and H_2 are orthogonal, i.e.

$$B_{k^2}(u, v) = 0 \quad \forall\, u \in H_1,\ \forall\, v \in H_2, \tag{5.3}$$

and $u(q) = u_1(q) + u_2(q)$, each of which can be obtained by projection of u_i onto

$$S_i(q) := S(q) \cap H_i, \quad i = 1, 2. \tag{5.4}$$

Further, from (5.3) we also get

$$\|e(q)\|_\varphi^2 = \|e_1(q)\|_\varphi^2 + \|e_2(q)\|_\varphi^2 \tag{5.5}$$

where $e_i(q) = u_i - u_i(q),\ i = 1, 2$.

Let us introduce some terminology for the analysis of the modelling error estimator. Our *a posteriori estimator* $\mathcal{E}$ for the modelling error (5.5) is of the form

$$\mathcal{E}(u_i(q)) = \sqrt{\int_\omega |\eta_i(\boldsymbol{x})|^2\ d\boldsymbol{x}}, \quad i = 1, 2. \tag{5.6}$$

Here $\eta_i(\boldsymbol{x})$ is called an *indicator function.*

Let $\|\cdot\|$ be any norm on H, and let $\mathcal{E}$ in (5.6) be an a posteriori error estimator for $\|e(q)\|$. Then we define the effectivity index Θ corresponding to $\mathcal{E}$ and $\|\cdot\|$ by

$$\Theta := \frac{\mathcal{E}(u(q))}{\|e(q)\|}. \tag{5.7}$$

We say that $\mathcal{E}$ is a *guaranteed upper estimator*, if $\Theta \geq 1$ for all u. The estimator $\mathcal{E}$ is called (κ_1, κ_2)-*proper* with respect to a class T of data, if

$$0 < \kappa_1 \leq \Theta \leq \kappa_2 < \infty \quad \forall\, f \in T. \tag{5.8}$$

Further, $\mathcal{E}$ is *asymptotically* exact on T if

$$\Theta \to 1 \text{ as } d \to 0^+ \quad \forall\, f \in T. \tag{5.9}$$

Finally, $\mathcal{E}$ is *locally asymptotically exact* on T, if (5.9) holds with the norm $\|\cdot\|_\varphi$ defined in (4.7), where the weight function $\varphi(\boldsymbol{x})$ is given by (4.20) and $\alpha = \overline{\alpha} d^{-1}$, $\overline{\alpha} > 0$.

We perform the analysis of the estimator $\mathcal{E}$ for the weighted norm $\|\cdot\|_\varphi$. Due to (5.5) we can derive the indicator functions η_i, $i = 1, 2$ separately. We begin by observing that, due to Lemma 4.3, the errors $e_i(q) \in H^L_\varphi$, $i = 1, 2$ defined in (5.5) satisfy

$$B_{k^2}(e_i(q), v) = R_i(v) \quad \forall\, v \in H \tag{5.10}$$

and

$$B_{k^2}(e_i(q), v) = 0 \quad \forall\, v \in S_i(q) \tag{5.11}$$

where

$$\begin{aligned} R_i(v) \quad &= \quad \int_\omega r_i(\boldsymbol{x}) \left(v(\boldsymbol{x}, d) \pm v(\boldsymbol{x}, -d)\right)\, d\boldsymbol{x} \\ &\quad + \int_\Omega v(\boldsymbol{x}, y) \left(\Delta u_i(q) + k^2 u_i(q)\right)\, d\boldsymbol{x}dy, \quad i = 1, 2, \end{aligned} \tag{5.12}$$

and the signs $-, +$ correspond to $i = 1, 2$, respectively. Here the computable residuals are

$$r_i(\boldsymbol{x}) = f_i(\boldsymbol{x}) - \frac{\partial u_i(q)}{\partial n}(\boldsymbol{x}, d), \quad i = 1, 2. \tag{5.13}$$

Based on (5.11) we calculate an explicit expression for $R_i(v)$, which we will use in the derivation of our modelling error estimator.

Lemma 5.1 *Let $i = 1$, $q = 2m + 1$ or $i = 2$, $q = 2m$, $m = 0, 1, 2, \cdots$, and $k \notin \Sigma \cup \Sigma_{0,q}$ and that the functions Φ_i are only the Legendre polynomials. Then*

$$R_i(v) \quad = \quad \int_\omega r_i(\boldsymbol{x}) \left(v(\boldsymbol{x}, d) \pm v(\boldsymbol{x}, -d) - \int_{-d}^{d} v(\boldsymbol{x}, y) \frac{\partial}{\partial y} \left(L_{q+1}(y/d)\right)\, dy \right)\, d\boldsymbol{x}. \tag{5.14}$$

Proof. Since $k \notin \Sigma_{0,q}$, $u_i(q)$ exists and hence $r_i(\boldsymbol{x})$ is well defined. Let $i = 1$, $q = 2m+1$. Then, since $k \notin \Sigma$, by assumption,

$$\Delta u_1(q) + k^2 u_1(q) = \sum_{j=0}^{2m+1} A_j^i(\boldsymbol{x}) L_j(y/d) \tag{5.15}$$

for some $A_j^i \in H^{-1}(\omega)$. To determine A_j^i, we use (5.11), i.e.

$$R_1(v) = 0 \quad \forall\, v \in S_1(q) \cup H_2.$$

We select in (5.12) the functions $v = V(\boldsymbol{x}) L_{2l}(y/d) \in H_2$ with arbitrary $V \in H_0^1(\omega)$ and get $A_j^1 = 0$ for even j. For $j = 2l+1$, $0 \le l \le m$, we find

$$0 = \int_\omega V(\boldsymbol{x}) \left(2r_1 + A_{2l+1}^1 d \int_{-1}^1 L_{2l+1}^2(z)\, dz \right) d\boldsymbol{x}.$$

Since $V \in H_0^1(\omega)$ is arbitrary, we get

$$A_{2l+1}^1 = -\frac{1}{d}(2(2l+1)+1) r_1, \quad l = 0, \cdots, m.$$

Hence

$$\begin{aligned} \Delta u_1(q) + k^2 u_1(q) &= -\frac{1}{d} r_1(\boldsymbol{x}) \sum_{l=0}^{m} (4l+3) L_{2l+1}(y/d) \\ &= -r_1(\boldsymbol{x}) \frac{\partial}{\partial y} \left(L_{2m+2}(y/d) \right). \end{aligned}$$

Inserting into (5.12) proves (5.14) for $i = 1$. For $i = 2$ and $q = 2m$, one proceeds analogously. □

In the general case, when span$\{\Phi_i\}$ contains both polynomials and trigonometric functions, we have

Lemma 5.2 *Let $i = 1$, $q = 2m+1$ or $i = 2$, $q = 2m$, $k \notin \Sigma \cup \Sigma_{M,N}$, $q = M + N$, and functions Φ_i be defined by (3.7) and (3.8). Further, let Ψ_j be defined by (3.15) and (3.16), i.e. $\Psi_j(z) = \sum_{i=0}^q Q_{ij} \Phi_i(z)$ and $\tilde{\sigma}_j = \int_{-1}^1 \Psi_j^2\, dz$. Then,*

$$R_i(v) = \int_\omega r_i(\boldsymbol{x}) \left(v(\boldsymbol{x}, d) \pm v(\boldsymbol{x}, -d) + \int_{-d}^{d} v(\boldsymbol{x}, y) \sum_{j=0}^{q} \frac{2\Psi_j(1)}{\tilde{\sigma}_j d} \delta_i(j) \Psi_j(y/d)\, dy \right) d\boldsymbol{x} \tag{5.16}$$

where

$$\delta_1(j) = \begin{cases} 0 & j \text{ even} \\ 1 & j \text{ odd} \end{cases}, \quad \delta_2(j) = \begin{cases} 0 & j \text{ odd} \\ 1 & j \text{ even} \end{cases}.$$

Proof. Since $k \notin \Sigma_{M,N}$, $u_i(q)$ exists and hence $r_i(\boldsymbol{x})$ is well defined. Next, we note that the matrix $\boldsymbol{Q}$ in (3.16) has block structure, Ψ_j is an even (odd) function for j even (odd). Let $i = 1$, $q = 2m + 1$. Then, since $k \notin \Sigma$,

$$\Delta u_1(q) + k^2 u_1(q) = \sum_{j=0}^{2m+1} A_j^i(\boldsymbol{x}) \Psi_j(y/d) \tag{5.17}$$

for some $A_j^i \in H^{-1}(\omega)$. To determine A_j^i, we use

$$R_1(v) = 0 \quad \forall\, v \in S_1(q) \cup H_2.$$

We select $v = V(\boldsymbol{x})\Psi_{2l}(y/d) \in H_2$ with arbitrary $V \in H_0^1(\omega)$ and get $A_j^1 = 0$ for even j. For $j = 2l + 1$, $0 \le l \le m$, we find

$$0 = \int_\omega V(\boldsymbol{x}) \left(2\Psi_{2l+1}(1) r_1 + A_{2l+1}^1 d \int_{-1}^1 \Psi_{2l+1}^2(z)\, dz \right) d\boldsymbol{x}.$$

Since $V \in H_0^1(\omega)$ is arbitrary, we get

$$2\Psi_{2l+1}(1) r_1 + A_{2l+1}^1 \tilde{\sigma}_{2l+1} d = 0$$

which yields

$$A_{2l+1}^1 = -\frac{2\Psi_{2l+1}(1)}{\tilde{\sigma}_{2l+1} d} r_1.$$

For $i = 2$ and $q = 2m$, one proceeds analogously. □

Next we derive the estimator $\mathcal{E}$ in the two cases of Lemma 5.1 and 5.2, with $M = 0$, $M > 0$, respectively. We start with the case $M = 0$, i.e. that no cosines are necessary in span$\{\Phi_l\}$.

Theorem 5.3 *Assume that f_i in (5.2) is square integrable over ω, $k \notin \Sigma \cup \Sigma_{0,N}$, $q = N$, d small enough so that $L = 0$ is admissible in (4.8) and that $\|\cdot\|_\varphi$ is as in (4.7). Then the error $\|e_i(q)\|_\varphi = \|e_i^{0,N}\|_\varphi$ for the hierarchical model of uniform order q (i.e. odd $q \ge 1$ for $i = 1$, even q for $i = 2$) can be estimated by*

$$\gamma^2 \|e_i(q)\|_\varphi^2 \le \frac{2d}{2q+3} \int_\omega \varphi^2 r_i^2 \, d\boldsymbol{x}, \tag{5.18}$$

where

$$r_i(\boldsymbol{x}) = f_i(\boldsymbol{x}) - \frac{\partial u_i(q)}{\partial y}(\boldsymbol{x}, d), \quad i = 1, 2.$$

Proof: We note that u_i and $u_i(q)$ exist since $k \notin \Sigma \cup \Sigma_{0,N}$. Further, with γ as in Theorem 4.1 and (5.10) we have

$$\gamma \|u_i - u_i(q)\|_\varphi \le \sup_{0 \ne \|v\|_{1/\varphi}} \frac{|B_{k^2}(u_i - u_i(q), v)|}{\|v\|_{1/\varphi}}, \quad i = 1, 2, \tag{5.19}$$

and

$$\gamma^2\|e_i(q)\|_\varphi^2 \le \sup_{0\neq\|v\|_{1/\varphi}} \frac{|R_i(v)|^2}{\|v\|_{1/\varphi}^2}. \tag{5.20}$$

The goal is now to estimate (5.20), that is to obtain an estimate of the form

$$|R_i(v)|^2 \le c\|v\|_{1/\varphi}^2, \tag{5.21}$$

i.e. to determine the constant in (5.21). Integration by parts with respect to y in (5.14) shows that

$$R_i(v) = \int_\omega r_i(\boldsymbol{x})\Upsilon_i[v](\boldsymbol{x})\, d\boldsymbol{x},$$

where

$$\Upsilon_i[v](\boldsymbol{x}) := \int_{-d}^{d} \frac{\partial v}{\partial y}(\boldsymbol{x},y)L_{q+1}(y/d)\, dy. \tag{5.22}$$

By the Schwarz inequality we have

$$|\Upsilon_i[v](\boldsymbol{x})|^2 \le \frac{2d}{2q+3}\int_{-d}^{d}\left(\frac{\partial v}{\partial y}(\boldsymbol{x},y)\right)^2 dy.$$

Hence,

$$\begin{aligned} |R_i(v)|^2 &\le \left(\int_\omega |r_i(\boldsymbol{x})||\Upsilon_i[v](\boldsymbol{x})|\, d\boldsymbol{x}\right)^2 \\ &\le \|\varphi r_i\|_{L^2(\omega)}^2 \int_\omega \varphi^{-2}(\boldsymbol{x})|\Upsilon_i[v](\boldsymbol{x})|^2\, d\boldsymbol{x} \\ &\le \frac{2d}{2q+3}\|\varphi r_i\|_{L^2(\omega)}^2 \int_\omega \varphi^{-2}(\boldsymbol{x})\int_{-d}^{d}\left(\frac{\partial v}{\partial y}\right)^2 dy\, d\boldsymbol{x} \\ &\le \frac{2d}{2q+3}\|\varphi r_i\|_{L^2(\omega)}^2\|v\|_{1/\varphi}^2. \end{aligned} \tag{5.23}$$

Referring to (5.20) completes the proof. □

In the general case of Lemma 5.2, i.e. for $L > 0$ in (4.8), we need the following estimate

Lemma 5.4 *Let $u \in \tilde{H}^1(-d,d)$, where $\tilde{H}^1(-d,d) = \{u \in H^1(-d,d) : \int_{-d}^{d} u(y)\, dy = 0\}$, and let $c_0 = \sqrt{2/3}$. Then we have the embedding*

$$\|u\|_{L^\infty(-d,d)} \le c_0\sqrt{d}\, \|u'\|_{L^2(-d,d)} \tag{5.24}$$

Proof. The proof of Lemma 5.4 can be found in the Appendix. □

Using Lemma 5.4 we can prove

Theorem 5.5 *Assume that f_i in (5.2) is square integrable over ω, $k \notin \Sigma \cup \Sigma_{M,N}$, $q = N + M$ and that $\varphi(\boldsymbol{x})$ is any admissible weight function in Theorem 4.1. Then the error $\|e_i(q)\|_\varphi =$*

$\|e_i^{M,N}\|_\varphi$ *for the hierarchical model of order* q *(i.e. odd* $q \geq 1$ *for* $i=1$*,* q *even for* $i=2$*) can be estimated by*

$$\gamma^2 \|e_i(q)\|_\varphi^2 \leq d \left(\frac{16}{3} + \frac{32}{\pi} \sum_{j=0}^{q} \Psi_j^2(1) \delta_i(j) \right) \int_\omega \varphi^2 r_i^2 \, d\boldsymbol{x}, \tag{5.25}$$

where

$$r_i(\boldsymbol{x}) = f_i(\boldsymbol{x}) - \frac{\partial u_i(q)}{\partial y}(\boldsymbol{x}, d), \quad i = 1, 2.$$

Proof: We can estimate with the representation (5.16) and Lemma 5.4

$$\begin{aligned}
|R_i(v)|^2 \leq & \int_\omega |r_i \varphi|^2 \, d\boldsymbol{x} \int_\omega \Bigg(4 \left| \frac{v(\boldsymbol{x}, d)}{\varphi} \right|^2 + 4 \left| \frac{v(\boldsymbol{x}, -d)}{\varphi} \right|^2 \\
& + 4 \int_{-d}^{d} \left| \frac{v(\boldsymbol{x}, y)}{\varphi} \right|^2 dy \int_{-d}^{d} \left| \sum_{j=0}^{q} \frac{2\Psi_j(1)}{d\tilde{\sigma}_j} \delta_i(j) \Psi_j(y/d) \right|^2 dy \Bigg) \, d\boldsymbol{x} \\
\leq & \|r_i \varphi\|_{L^2(\omega)}^2 \Bigg(\int_\omega \frac{d}{\varphi^2} \frac{16}{3} \left\| \frac{\partial v(\boldsymbol{x}, y)}{\partial y} \right\|_{L^2(-d,d)}^2 d\boldsymbol{x} \\
& + \int_\omega \frac{8}{\pi \varphi^2} d^2 \int_{-d}^{d} \left| \frac{\partial v(\boldsymbol{x}, y)}{\partial y} \right|^2 dy \left| \sum_{j=0}^{q} \frac{1}{d} (2\Psi_j(1))^2 \delta_i(j) \right| d\boldsymbol{x} \Bigg) \\
\leq & \|r_i \varphi\|_{L^2(\omega)}^2 \left(d \frac{16}{3} \|v\|_{1/\varphi}^2 + \frac{8}{\pi} d \|v\|_{1/\varphi}^2 4 \sum_{j=0}^{q} \delta_i(j) \Psi_j^2(1) \right) \\
= & \|r_i \varphi\|_{L^2(\omega)}^2 \|v\|_{1/\varphi}^2 d \left(\frac{16}{3} + \frac{32}{\pi} \sum_{j=0}^{q} \delta_i(j) \Psi_j^2(1) \right).
\end{aligned}$$

Applying Theorem 4.1, similarly to (5.20), immediately yields (5.25). □

Based on (5.18) and (5.25) we note that the indicator functions

$$\eta_{iq}(\boldsymbol{x}) = \sqrt{\frac{2d}{2q+3}} \, \varphi(\boldsymbol{x}) r_i(\boldsymbol{x}), \quad i = 1, 2, \tag{5.26}$$

and

$$\eta_{iq}(\boldsymbol{x}) = \sqrt{d} \left(\frac{16}{3} + \frac{32}{\pi} \sum_{j=0}^{q} \Psi_j^2(1) \delta_i(j) \right)^{1/2} \varphi(\boldsymbol{x}) r_i(\boldsymbol{x}), \quad i = 1, 2, \tag{5.27}$$

respectively, and the estimator $\mathcal{E}(u_i(q))$ defined in (5.6) are, according to (5.18) and (5.25), respectively, guaranteed upper estimators for $\|e_i(q)\|_\varphi$. We also see that with Theorem 4.1 we have accomplished that $\kappa_2 \leq 1/\hat{\gamma}$. The value of κ_1 will be estimated in the next section.

6 Equivalence of the error estimator

In this section we will give an expression for the constant κ_1 in (5.8), which can be determined using the computed solution $u(q) = u^{M,N}(\boldsymbol{x}, y)$. We note that κ_2 was already given in the previous section. In addition, we will demonstrate that the estimator $\mathcal{E}$ is equivalent to $\|e\|_\varphi$ under stricter assumptions. First, we introduce some notation. Throughout, φ will denote the exponential weight function (4.20). Further,

$$|||r|||_{l,\varphi}^2 := \int_\omega |\boldsymbol{\nabla}_{\boldsymbol{x}}^l r|^2 \varphi^2 \, d\boldsymbol{x}, \quad l = 0, 1. \tag{6.1}$$

Finally, we introduce the class of data

$$T_\beta := \left\{ f \mid \text{either } r_i(f) = 0 \text{ or } |||r_i|||_{1,\varphi}^2 / |||r_i|||_{0,\varphi}^2 \leq \beta < \infty \right\}. \tag{6.2}$$

Because r_i was computed and used in the estimator, the value of β can be easily computed to determine in what class the solution belongs. The main result of this section is Theorem 6.1. But at first we note how the eigenvalues α_i in (3.6) behave with respect to the plate thickness parameter d. Since $\alpha_i = i\pi/(2d)$ we see that $\alpha_0 = 0$ and that for all other α_i, $i = 1, \cdots, \infty$ we have asymptotically

$$\lim_{d \to 0^+} \alpha_i = \infty, \quad i = 1, \cdots, \infty. \tag{6.3}$$

This means that for any fixed k, asymptotically $\alpha_i > k$ as $d \to 0$ for $i \geq 1$. We also note that the eigenfunction Φ_0 corresponding to α_0 can be represented by the Legendre polynomial L_0. In order to prove Theorem 4.1, i.e. the a-priori error estimates in exponentially weighted norms, we had to exclude the eigenfunctions Φ_i corresponding to eigenvalues $\alpha_i \leq k$ from the space H_φ^L. From (6.3) we see that asymptotically, as $d \to 0^+$, we do not have to consider the case in which eigenfunctions Φ_i, defined in (3.6) and (3.7), are excluded. This means that for the analysis of the asymptotic exactness as $d \to 0^+$, we can assume that we deal with the case where all functions Φ_i are defined by Legendre polynomials (3.8). In particular, for sufficiently small d, we also have Lemma 5.1.

Theorem 6.1 *Let $\Theta_i, i = 1, 2$ denote the effectivity indices (5.7) with respect to the weighted energy norm (4.7). Assume further that $\mathcal{P} = \{\omega\}$, i.e. the model order is uniform. Then for $i = 1, 2$ the following holds:*

1. *If $f_i \in L^2(\omega)$ we have that $\Theta_i \geq \kappa_{i1}$ where*

$$(\kappa_{i1})^2 := \delta^2 (1 - Qd)^2 / \Big((1 + \rho) + 4(1 + 1/\rho) Q^2 \Big) \tag{6.4}$$

and where ρ, δ, and Q are as in Theorem 4.1.

2. If $f_i \in T_\beta$, then $\Theta_i \leq \kappa_{i2}$ where

$$(\kappa_{i2})^2 := (1+k^2)\left(1+3d^2 D_q\left(Q^2+\beta^2\right)\right) \tag{6.5}$$

and

$$D_q = \frac{4}{(2q+3)^2-4}. \tag{6.6}$$

Proof.

1. The lower bound (6.4) follows immediately from Theorem 4.1 and Lemma 5.1.

2. To show the upper bound (6.5), we define

$$S := L^2(\omega, H^1(-d,d)) \cap \left\{ v \,\middle|\, \int_{-d}^{d} v\, dy = 0 \text{ a.e. } \boldsymbol{x} \in \omega \right\}. \tag{6.7}$$

We note that, since the functional Υ_i in (5.22) is strictly concave and upper semicontinuous on S, there exists a (unique) maximizing element $v_i^* \in S$, which satisfies the Euler-Lagrange equation

$$\frac{\partial^2 v_i^*}{\partial y^2} = \frac{\partial}{\partial y}\left(L_{q+1}(y/d)\right) \quad \text{in } (-d,d) \tag{6.8}$$

with

$$\left.\frac{\partial v_i^*}{\partial y}\right|_{\pm d} = \begin{cases} 1 & \text{if } i=1, \\ \pm 1 & \text{if } i=2. \end{cases}$$

Hence, we find that v_i^* is independent of $\boldsymbol{x}$ and is given by

$$v_i^* = d\frac{L_{q+2}(y/d) - L_q(y/d)}{2q+3}, \quad q \geq 0, \tag{6.9}$$

and

$$(\Upsilon_i[v_i^*])^2 = \frac{2d}{2q+3} =: dC_q. \tag{6.10}$$

Next, we select in (5.10)

$$v = \bar{v}\varphi^2 = v_i^*(y) r_i(\boldsymbol{x})\varphi^2$$

with v_i^* as in (6.9) and get with (6.10) that

$$R_i(v) = dC_q \int_\omega r_i^2 \varphi_i^2 \, d\boldsymbol{x} = B_{k^2}(e_i(q), v) \leq (1+k^2)\|e_i\|_\varphi \|v\|_{1/\varphi}. \tag{6.11}$$

Since

$$|\nabla_{\boldsymbol{x}} v|^2 \leq \varphi^2\left(3\varphi^2|\nabla_{\boldsymbol{x}}\bar{v}|^2 + 6|\bar{v}|^2|\nabla_{\boldsymbol{x}}\varphi|^2\right)$$

we find

$$\begin{aligned}
\|v\|_{1/\varphi}^2 &= \int_\Omega \varphi^{-2}\left(|\boldsymbol{\nabla}_{\boldsymbol{x}} v|^2 + \left(\frac{\partial v}{\partial y}\right)^2\right) d\boldsymbol{x}\, dy \\
&\le \int_\Omega \left(3\varphi^2 |\boldsymbol{\nabla}_{\boldsymbol{x}} \bar{v}|^2 + 6|\bar{v}|^2 |\boldsymbol{\nabla}_{\boldsymbol{x}} \varphi|^2 + \varphi^2 \left(\frac{\partial \bar{v}}{\partial y}\right)^2\right) d\boldsymbol{x}\, dy \\
&= \int_\Omega \left(|v_i^*|^2 \left(3\varphi^2 |\boldsymbol{\nabla}_{\boldsymbol{x}} r_i|^2 + 6|r_i|^2\, |\boldsymbol{\nabla}_{\boldsymbol{x}} \varphi|^2\right) + \varphi^2 |r_i|^2 \left(\frac{\partial v_i^*}{\partial y}\right)^2\right) d\boldsymbol{x}\, dy.
\end{aligned}$$

Since

$$\int_{-d}^{d} \left(\frac{\partial v_i^*}{\partial y}\right)^2 dy = dC_q$$

with C_q as in (6.10) and

$$\int_{-d}^{d} (v_i^*)^2\; dy = d^3 C_q D_q,$$

where D_q is as in (6.6), we get with $|\boldsymbol{\nabla}_{\boldsymbol{x}} \varphi|^2 \le Q^2\varphi^2$,

$$\|v\|_{1/\varphi}^2 \le dC_q |||r_i|||_{0,\varphi}^2 + 3d^3 C_q D_q \left(|||r_i|||_{1,\varphi}^2 + Q^2 |||r_i|||_{0,\varphi}^2\right). \tag{6.12}$$

For every $\varepsilon > 0$ we have from (6.11)

$$2dC_q |||r_i|||_{0,\varphi}^2 \le (1+k^2)\left(\varepsilon \|e_i\|_\varphi^2 + \varepsilon^{-1} \|v\|_{1/\varphi}^2\right).$$

If we select $\varepsilon_0 > 0$ so that

$$(1+k^2)\varepsilon_0^{-1} \|v\|_{1/\varphi}^2 \le dC_q |||r_i|||_{0,\varphi}^2 = (\mathcal{E}(u_i))^2, \tag{6.13}$$

we arrive at the desired (lower) bound

$$(\mathcal{E}(u_i(q)))^2 \le (1+k^2)\varepsilon_0 \|e_i(q)\|_\varphi^2, \quad i = 1, 2.$$

We estimate ε_0 by using (6.12) and (6.13).

$$\varepsilon_0 = \frac{(1+k^2)\|v\|_{1/\varphi}^2}{dC_q |||r_i|||_{0,\varphi}^2} \le (1+k^2)\left(1 + 3d^2 D_q \left(Q^2 + \frac{|||r_i|||_{1,\varphi}^2}{|||r_i|||_{0,\varphi}^2}\right)\right).$$

Using that $f_i \in T_\beta$ gives (6.5). □

Remark 6.2 From Theorem 6.1 we see that the a-posteriori error estimator (5.6) is asymptotically exact for $d \to 0^+$ if $k = 0$, $Q = 0$ ($\varphi(\boldsymbol{x}) = 1$) and $\delta \to 1^-$. In the case of $\delta \to 1^-$ we see from Theorem 4.1 that δ has to be coupled to d to obtain the stability estimate for γ with a finite L. For example $\delta = 1 - d$. Further, we see that we also get a similar result for d being finite but $q \to \infty$ in Theorem 6.1.

7 Computational Aspects and Experiments

We derived in the previous sections computable a-posteriori modelling error estimates for dimensional reductions of the Helmholtz equation on thin domains. The main results, Theorem 5.3, 5.5 and 6.1, state that the residuals $r_i(\boldsymbol{x})$ in (5.13) are accurate indicators for the *local* contributions to the modelling error in a vicinity of $\boldsymbol{x}_0 \in \omega$, if the exponential weight functions (4.20) are used. In order to do so, however, the modelling error e must belong to H^L_φ which, by Theorem 4.1, requires that (4.6) holds, i.e. that the director functions $\Phi_l(y/d)$ in (3.7) are included into the Ansatz (3.9). If, however, the number N of polynomials in (3.9) is sufficiently large, the trigonometric director functions (3.7) will be very well approximated by these polynomials and become numerically linearly dependent (although strictly speaking they are always linearly independent of the polynomials). A quantitative measure of the amount of *numerical* linear dependence (and hence a criterion for when certain of the trigonometric basis functions can be dropped from (3.9) in a computation) is the *angle* between $\text{span}\{L_i(z)\}_{i=1}^N$ and $\text{span}\{\Phi_j(z)\}_{j=0}^M$ in (3.7).

7.1 Angle between Legendre polynomials and trigonometric functions

Due to the orthogonality (5.3) we can analyze the modelling error separately for the symmetric and antisymmetric part of the solution. Therefore, we assume for simplicity that the Neumann data in (1.1) are symmetric, i.e. $f^+ = f^-$, which means that the solution will be symmetric in y. All considerations that follow apply with the obvious changes for the antisymmetric case. Further, we choose k such that we have to include the first two even trigonometric functions $\Phi_i(z)$ into the Ansatz, i.e. from Theorem 4.1 we have for $Q = 0$, $\delta = 0$

$$L + 1 \geq k\frac{2d}{\pi}. \tag{7.1}$$

This yields for $L = 2$ the following interval for k such that $L = 1$ does not satisfy (7.1)

$$\frac{3\pi}{2d} \geq k \geq \frac{2\pi}{2d} \tag{7.2}$$

For the thickness of the plate $d = 0.2$, we see that k can be as large as 7.5π but should be at least 2.5π. The director functions for the hierarchical model are then given by

$$\begin{aligned} \Phi_0(z) &= \frac{1}{\sqrt{2d}}, \\ \Phi_1(z) &= \frac{1}{\sqrt{d}}\cos(\pi z), \\ \Phi_2(z) &= L_2(z), \\ &\vdots \\ \Phi_{N+1}(z) &= L_{2N}(z), \end{aligned} \tag{7.3}$$

depending on the model order $N+2$. Therefore, we are interested in determining the angle between the Legendre polynomials and the function $\cos(\pi z)$. In particular, we are looking at the finite dimensional space

$$V = \text{span}\{L_2(z), \cdots, L_{2N}(z), \cos(\pi z);\; z = y/d;\; -d \le y \le d\} \tag{7.4}$$

and the subspaces

$$\begin{aligned} F &= \text{span}\{L_2(z), \cdots, L_{2N}(z);\; z = y/d;\; -d \le y \le d\}, && (7.5)\\ G_1 &= \text{span}\{\cos(\pi z);\; z = y/d;\; -d \le y \le d\} && (7.6) \end{aligned}$$

to determine

$$\theta = \theta(1, N) = \angle(F, G_1). \tag{7.7}$$

The computation of the angle θ boils down to a generalized eigenvalue problem and follows [**?**], section 12.4.3. The space V is a Hilbert space with the inner product

$$(u, v)_{L^2(-d,d)} = \int_{-d}^{d} u(y)v(y)\; dy \tag{7.8}$$

and the norm

$$\|u\|^2_{L^2(-d,d)} = (u, u)_{L^2(-d,d)}. \tag{7.9}$$

Since the Legendre Polynomials are L^2-orthogonal, we can easily construct an orthonormal basis $\{v_i\}_{i=1,\cdots,N+1}$ for V by using

$$v_i(z) = \sqrt{\frac{4i+1}{2}}\; L_{2i}(z), \quad 1 \le i \le N, \tag{7.10}$$

and

$$v_{N+1}(z) = \frac{\tilde{v}_{N+1}(z)}{\|\tilde{v}_{N+1}(z)\|_{L^2(-1,1)}} \tag{7.11}$$

where

$$\tilde{v}_{N+1}(z) = \cos(\pi z) - \sum_{i=1}^{N} \Big(\cos(\pi z), \sqrt{2i+0.5}\; L_{2i}(z)\Big)_{L^2(-1,1)} \sqrt{2i+0.5}\; L_{2i}(z). \tag{7.12}$$

This allows to construct a $(N+1) \times N$ matrix $\boldsymbol{Q}_F$ and a $(N+1) \times 1$ matrix $\boldsymbol{Q}_G$, where the columns represent elements of V and a basis for F and G, respectively. Thus,

$$\boldsymbol{Q}_F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \\ 0 & \cdots & 0 \end{bmatrix}, \quad \boldsymbol{Q}_G = \begin{bmatrix} \Big(\cos(\pi z), \sqrt{2.5}\; L_2(z)\Big)_{L^2(-1,1)} \\ \vdots \\ \Big(\cos(\pi z), \sqrt{2N+0.5}\; L_{2N}(z)\Big)_{L^2(-1,1)} \\ \|\tilde{v}_{N+1}\|_{L^2(-1,1)} \end{bmatrix}. \tag{7.13}$$

The angle θ between the subspaces F and G_1 is then defined via the singular value of $\boldsymbol{Q}_F^t \boldsymbol{Q}_G$, that is

$$\cos(\theta) = \text{singular value of } \boldsymbol{Q}_F^t \boldsymbol{Q}_G. \tag{7.14}$$

Obviously,

$$\boldsymbol{Q}_F^t \boldsymbol{Q}_G = \begin{bmatrix} \left(\cos(\pi z), \sqrt{2.5}\, L_2(z)\right)_{L^2(-1,1)} \\ \vdots \\ \left(\cos(\pi z), \sqrt{2N+0.5}\, L_{2N}(z)\right)_{L^2(-1,1)} \end{bmatrix}. \tag{7.15}$$

From [**?**], section 8.3, we also know that

$$\cos^2(\theta) = (\boldsymbol{Q}_F^t \boldsymbol{Q}_G)^t\, \boldsymbol{Q}_F^t \boldsymbol{Q}_G = \sum_{i=1}^{N} (2i+0.5)\, (\cos(\pi z), L_{2i}(z))^2_{L^2(-1,1)}\,. \tag{7.16}$$

We note that, due to using the orthonormal basis in (7.10) and (7.11),

$$0 \leq \sum_{i=1}^{N} (2i+0.5)\, (\cos(\pi z), L_{2i}(z))^2_{L^2(-1,1)} \leq 1, \tag{7.17}$$

and therefore $\cos^2(\theta)$ is well defined and the angle θ is given by

$$\theta = \arccos\left(\sqrt{\sum_{i=1}^{N} (2i+0.5)\, (\cos(\pi z), L_{2i}(z))^2_{L^2(-1,1)}} \right). \tag{7.18}$$

Similarly, we can compute the angle $\theta(j,N)$ between F and $G_j = \text{span}\{\cos(j\pi z)\}$. The values for $\theta(j,N)$ in radians have been computed with the symbolic programming system Maple (Digits $= 60$) and are tabulated in Table **??** for $1 \leq N \leq 15$ and $1 \leq j \leq 4$. We observe that $\theta(j,N) \to 0$ as $N \to \infty$ extremely fast. In fact, we have

$$\theta(j,N) \leq \frac{c(j)}{N!}, \tag{7.19}$$

since

$$\inf_{p \in \Pi_N} \|\Phi_j - p\|_{L^\infty(-1,1)} \leq \frac{c(j)}{N!}. \tag{7.20}$$

Practically, i.e. in finite precision arithmetic, we cannot increase the model order N arbitrarily without dropping certain trigonometric basis functions, since the director functions exhibit numerical linear dependence as we can see from Table **??**. We propose to drop the trigonometric functions $\tilde{\Phi}_j(z)$ whenever the angle $\theta(j,N)$ becomes smaller than a prespecified multiple of the machine epsilon.

We finally remark that so far, we considered only the angle between the subspaces spanned by the director functions. In computational practice, however, the two dimensional problem will be further discretized by Finite Elements, for example. A small angle $\theta(j,N)$ in Table **??**, however, will entail in any case a corresponding ill-conditioning of the stiffness matrix.